Final Exam Revision for Advanced Mathematics and Physics

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- Bohr Model of the atom – be able to prove the diameter of an atom is approximately an amstrong (Å$= 10^{-10}$m)
- Solving the heat equation with boundary conditions

Recommended Resources

Split by Physics/Mathematics and then listed in order of importance:

Physics

- Revision lecture
- Lecture Notes' Additional Notes – learn the derivations mentioned in the revision lecture.
- Learn the rest of the derivations.

Mathematics

- Examples in the lecture notes
- Tutorial problems
- *Vector Calculus* by Susan J. Colley
  (there are solutions/instructor's manuals for this, which gives a good feedback loop).
- *Problems and Solutions for Complex Analysis* by Rami Shakarchi

Radioactive Decay

Derive the formula for the number of atoms present during radioactive decay, $N(t)$, given that:

$$-\frac{dN}{dt} \propto N \tag{1}$$

$$-\frac{dN}{dt} = \lambda N \tag{2}$$

Where:
$N \equiv$ the number of atoms present
$\lambda \equiv$ rate of radioactive decay (specific to the element)
$N_0 \equiv$ is the number of atoms present at $t = 0$

Answer

$$-\frac{dN}{dt} = \lambda N \tag{3}$$

$$-\frac{dN}{N} = \lambda dt \tag{4}$$

$$\int -\frac{dN}{N} = \int \lambda dt \tag{5}$$

$$\ln N = -\lambda t + C \tag{6}$$

$$N = e^{-\lambda t + C} \tag{7}$$

$$e^C = N_0 \tag{8}$$

$$N = N_0 e^{-\lambda t} \tag{9}$$

$$N(t) = N_0 e^{-\lambda t} \tag{10}$$

Gauss'/Divergence Theorem

Consider the field:

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{11}$$

And the surface:

$$z = 9 - x^2 - y^2 \text{ for } z \geq 0 \tag{12}$$

Using Gauss'/Divergence Theorem:

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \ dV \tag{13}$$

Calculate:

$$\oiint_{\partial S} \mathbf{F} \cdot d\mathbf{S} \tag{14}$$

Answer

First thing we're going to do is choose to do the triple integral, as it's easier.

$$\iiint_D \nabla \cdot \mathbf{F} \ dV \tag{15}$$

To do this, we need to calculate the divergence of our vector field.

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \tag{16}$$

Let's now transform dV so that we can do the integral in cylindrical coordinates.

$$dV = dz \ dx \ dy = dz \ r \ dr \ d\theta \tag{17}$$

Substituting into our integral:

$$\iiint 3 \ dz \ r \ dr \ d\theta \tag{18}$$

Great! Now all we need are our bounds.
Recall that we should do the most dependent thing first, which will be z
So we need to express z in terms of r and θ.
It doesn't seem to be dependent on θ, only r

$$z = 9 - x^2 - y^2 \tag{19}$$

In cylindrical coordinates:

$$x^2 + y^2 = r^2 \qquad \Rightarrow \qquad z = 9 - r^2 \tag{20}$$

Based on the shape, that function will be the upper bound and the lower bound will be zero.

$$\therefore z \in [\ 0,\ \ 9 - r^2] \tag{21}$$

For the radius, the shape gives us a range of zero to 3 which we can see by looking at the shape when $z = 0$ and $z = 9$

$$\left(z = 9 - r^2\right)\Big|_{z=0} \quad \Rightarrow \quad 9 = r^2 \quad \Rightarrow \quad r = 3 \tag{22}$$

$$\left(z = 9 - r^2\right)\Big|_{z=9} \quad \Rightarrow \quad 0 = r^2 \quad \Rightarrow \quad r = 0 \tag{23}$$

$$\therefore r \in [0, 3] \tag{24}$$

And as the shape is cylindrically symmetric its angle will range from $0 \rightarrow 2\pi$

$$\therefore \theta \in [0, 2\pi] \tag{25}$$

Hence:

$$\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} 3\ dz\ r\ dr\ d\theta \tag{26}$$

$$\int_0^{2\pi} \int_0^3 3\ (9 - r^2)\ r\ dr\ d\theta \tag{27}$$

$$\int_0^{2\pi} \int_0^3 (27r - 3r^3)\ dr\ d\theta \tag{28}$$

$$\int_0^{2\pi} (27(3) - 3(3)^3)\ dr\ d\theta \tag{29}$$

$$\int_0^{2\pi} (27(3) - 3(27))\ dr\ d\theta = 0 \tag{30}$$

Parameterised Surface

Consider the field:

$$\mathbf{X}(s,t) = \begin{bmatrix} s \\ s+t \\ t \end{bmatrix} \tag{31}$$

Where $0 \leq s \leq 1$ and $0 \leq t \leq 2$

$$\int\int_{\mathbf{X}} \left(x^2 + y^2 + z^2\right) dS \tag{32}$$

Ask yourself, what is visually happening here?

Answer

We didn't do this in the U:PASS session, it was mainly included to make sure everyone knows about parameterisations.

Answer can be found in the instructor's manual for Susan J. Colley's *Vector Calculus*, 7.2 Exercises, Exercise 1.

Computing the Roots of Complex Numbers

Find the expressions for all unique roots of the following complex numbers:

$$\sqrt{-i} \tag{33}$$

$$\left(82,3543\ e^{i\frac{\pi}{3}}\right)^{\frac{1}{7}} \tag{34}$$

Answer

Using formulas provided in the lecture notes titled *Functions of a Complex Variable*

$$\phi = \frac{\theta + 2\pi k}{n} \tag{35}$$

$$\rho = r^{\frac{1}{n}} \tag{36}$$

The answer to equation 33 would be:

$$-i = e^{j\frac{3\pi}{2}} \tag{37}$$

$$\sqrt{-i} = \left(e^{j\frac{3\pi}{2}}\right)^{\frac{1}{2}} \tag{38}$$

$$n = 2 \qquad k \in [0, n-1] \quad = \quad k \in [0,1] \qquad \theta = \frac{3\pi}{2} \tag{39}$$

Giving us:

$$\rho = 1 \qquad \phi_1 = \frac{\frac{3\pi}{2}}{2} \qquad \phi_2 = \frac{\frac{3\pi}{2} + 2\pi}{2} \tag{40}$$

Contour Integration

Find:

$$\int_C \left(x^2 + iy^2\right) dz \tag{41}$$

Where C is the parabola $y = x^2$ from $(0, 0)$ to $(2, 2)$

Then:
Determine if $x^2 + iy^2$ is analytic and calculate:

$$\oint_C \left(x^2 + iy^2\right) dz \tag{42}$$

Where C has now changed to a circle with radius 1 around the origin of the complex plane.

Answer

Compton Scattering

Compton scattering is described by:

$$\lambda^{'} - \lambda = \frac{h}{m_e c}\left(1 - \cos\theta\right) \tag{43}$$

(i) If a photon hits a stationary electron and refracts with an angle of 85^o, what is the change in wavelength of the photon?

(ii) Draw a diagram of this.

(iii) Would the colour change be visible to the human eye?

Answer

$$\lambda^{'} \equiv \text{ New wavelength} \tag{44}$$

$$\lambda \equiv \text{ Old wavelength} \tag{45}$$

$$\theta \equiv \text{ Angle of refraction} \tag{46}$$

$$\theta \equiv \text{ Angle of refraction} \tag{47}$$

$$m_e \equiv \text{ Mass of an electron} \tag{48}$$

$$h \equiv \text{Planck's constant} \tag{49}$$

$$c \equiv \text{ Speed of light} \tag{50}$$

Plug in values into equation 43 and (i) is answered.

Schrodinger's Wave Equation

Prove that the following is a solution to Schrodinger's wave equation.

$$\Psi(x,t) = Ae^{i(kx-wt)} \tag{51}$$

Where we're working with a single dimensional version of the wave equation with no potential field, which has the form:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \tag{52}$$

Hint: Think about how "proving this is a solution" is done, otherwise you might get to the end and not realise it.

Answer

To prove we have a solution, if we get a physical law out, this implies that the solution is correct.

First step: Get the derivatives.

We need:

$$\frac{\partial^2\Psi}{\partial x^2} \quad \text{and} \quad \frac{\partial\Psi}{\partial t}$$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(Ae^{i(kx-\omega t)}\right)$$

$$\frac{\partial^2\Psi}{\partial x^2} = Ae^{-\omega t}\frac{\partial^2}{\partial x^2}\left(e^{i(kx)}\right)$$

$$\frac{\partial^2\Psi}{\partial x^2} = Ae^{-\omega t}ik\frac{\partial}{\partial x}\left(e^{i(kx)}\right)$$

$$\frac{\partial^2\Psi}{\partial x^2} = Ae^{-\omega t}(ik)^2e^{i(kx)}$$

$$\frac{\partial^2\Psi}{\partial x^2} = (ik)^2Ae^{i(kx-\omega t)}$$

Woohoo! Now for the time derivative:

$$\frac{\partial\Psi}{\partial t} = \frac{\partial}{\partial t}\left(Ae^{i(kx-\omega t)}\right)$$

$$\frac{\partial\Psi}{\partial t} = Ae^{ikx}\frac{\partial}{\partial t}\left(e^{-i\omega t}\right)$$

$$\frac{\partial\Psi}{\partial t} = Ae^{ikx}(-i\omega)e^{-i\omega t}$$

$$\frac{\partial \Psi}{\partial t} = (-i\omega)Ae^{i(kx-\omega t)}$$

Second step: Substitute our derivatives and the wave equation into the Schrodinger Wave Equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

$$i\hbar(-i\omega)Ae^{i(kx-wt)} = -\frac{\hbar^2}{2m}(ik)^2 Ae^{i(kx-\omega t)}$$

Factor out $Ae^{i(kx-\omega t)}$

$$i\hbar(-i\omega) = -\frac{\hbar^2}{2m}(ik)^2$$

Simplify our imaginary numbers:

$$\hbar\omega = \frac{\hbar^2}{2m}(k)^2$$

And we know that:

$$\omega = 2\pi f \quad \text{and} \quad p = hk \quad \text{and} \quad \hbar = \frac{h}{2\pi}$$

$$E = hf = \frac{(\hbar k)^2}{2m}$$

Because $E = hf$ pops out, a physical law, this implies that our equation is a solution.

Partial Differential Equations

The heat equation is given by:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{53}$$

With boundary conditions given by:

$$u(x, 0) = f(x) \qquad u(0, t) = 0 \qquad u(L, t) = 0 \tag{54}$$

There exists a rod of length L.
At $t = 0$ it has a heat distribution of:

$$f(x) = 12 \sin\left(\frac{9\pi x}{L}\right) - 9 \sin\left(\frac{4\pi x}{L}\right) \tag{55}$$

By inspection (or by the *Lecture 5 Extra Notes*) what is the final form of $u(x, t)$?

Note: for the final form, assume κ is yet to be known and t is a variable.

Answer

To save my own time typing, this one comes straight from the *Lecture 5 Extra Notes* where it is well explained with all relevant equations.

Stoke's Theorem

Stoke's Theorem is:

$$\int\int_{\mathbf{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} \tag{56}$$

For a surface:

$$S: \qquad z = 9 - x^2 - y^2 \qquad \text{for } z \geq 0 \tag{57}$$

And a field:

$$\mathbf{F} = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k} \tag{58}$$

Calculate:

$$\int\int_{\mathbf{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{59}$$

Answer