Fields & Waves – Write Up

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Contents

Problem Sheet 1

Griffiths, Chapter 3, Problem 54

Note

Positioning used is that present in the original question, **not** placement of the pipe's corner on $z = 0$.

Working

Using separation of variables and the fact that the voltage function obeys Laplace's Equation, we get:

$$V(x, y) = (Ae^{ky} + Be^{-ky})\,(C \sin(kx) + D \cos(kx))$$

	Boundary Values
(i)	$V(x, 0) = 0$
(ii)	$V(x, a) = V_0$
(iii)	$V(b, y) = 0$
(iv)	$V(-b, y) = 0$

First thing is that there's a common approach of "it's symmetric across the x-axis" and so the coefficient C for $C \cdot \sin(kx)$
We'll abuse that convention to simplify the calculation.
Now we have:

$$V(x, y) = (Ae^{ky} + Be^{-ky})D \cos(kx)$$

Using limit (i):

$$V(x, 0) = (A + B)D \cos(kx) = 0$$

If we assume $D \neq 0$:

$$\therefore\ A = -B$$

We now have:

$$(-Be^{ky} + Be^{-ky})D \cos(kx)$$

Let's simplify the y-function.

$$-B(e^{ky} - e^{-ky})D \cos(kx)$$

$$\frac{B'}{2} = -B$$

Meaning our y-function is:

$$B' \sinh(ky)$$

And our voltage function is:

$$B' \sinh(ky) D \cos(kx)$$

Let's combine our coefficients:

$$A' = B'D \qquad \Rightarrow \qquad V(x,y) = A' \sinh(ky)\cos(kx)$$

Using boundary (iii):

$$V(b,y) = 0 \qquad \Rightarrow \qquad A' \sinh(ky)\cos(kb) = 0 \qquad \Rightarrow \qquad \cos(kb) = 0$$

So we actually have infinite solutions for kb.

$$kb = \sum_{n=1}^{\infty} \pm \frac{n\pi}{2}$$

But b is a constant value of a certain measurement of our pipe.

$$\therefore\ k = \sum_{n=1}^{\infty} \frac{(2n-1)\pi}{2b}$$

So now we have a many different natural frequencies that satisfy the conditions.
Let's say that the n-th term of k is α_n.

$$V(x,y) = \sum_{n=1}^{\infty} A'_n \sinh(\alpha_n y)\cos(\alpha_n x)$$

Using the V_0 boundary condition:

$$V_0 = \sum_{n=1}^{\infty} A'_n \sinh(\alpha_n y)\cos(\alpha_n x)$$

Multiplying $\cos(\alpha_n x)$ on both sides of the equation and integrating from b to $-b$ gives:

$$V_0 \cos(\alpha_{n'} x) = \sum_{n=1}^{\infty} A'_n \sinh(\alpha_n y)\cos(\alpha_{n'} x)\cos(\alpha_n x)$$

$$V_0 \int_{-b}^{b} \cos(\alpha_{n'} x)\, dx = \sum_{n=1}^{\infty} A'_n \sinh(\alpha_n y) \int_{-b}^{b} \cos(\alpha_{n'} x)\cos(\alpha_n x)\, dx$$

$$\because\ \sum_{n=1}^{\infty} A'_n \sinh(\alpha_n y) \int_{-b}^{b} \cos\left(\frac{(2n-1)\pi}{2b} x\right)\cos\left(\frac{(2n-1)\pi}{2b} x\right)\, dx = A'_n \sinh(\alpha_n y)\, b \left(1 - \frac{\sin(2n\pi)}{(2n-1)\pi}\right) = A'_n \sinh(\alpha_n y)\, b$$

$$\therefore A'_n = \frac{2V_0}{b} \frac{\sin(\alpha_n b)}{\alpha_n \sinh(\alpha_n)\, a} \cos(\alpha_n x)$$

$$V(x,y) = \sum_{n=1}^{\infty} \frac{2V_0}{b} \frac{\sin(\alpha_n b)\sinh(\alpha_n y)}{\alpha_n \sinh(\alpha_n)\, a} \cos(\alpha_n x)$$

Griffiths, Chapter 3, Problem 40

Problem Sheet 2

$$\times\mathbf{F} = 0$$

FAW Notes - Chapter 1

FAW Notes - Chapter 2

Other Topics

Fourier Series

Differential Equations

Flux

Symmetry

Proof of Cauchy-Riemann Equations

Finite Fourier Sine Transform

On Learning