Groups, Analysis, and Geometry Seminars: Harmonic Analysis of $\mathcal{SU}(2)$

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Suppose f is 2π-periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \tag{1}$$

with $n \in \mathbb{Z}$, is the Fourier transform of f. Question: What is e^{inx}? Why is $n \in \mathbb{Z}$? Answer: e^{inx} is the character of circle group, denoted by $\mathbb{T}$ (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group G to $\mathbb{T}$: $\chi : G \to \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \tag{2}$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \to \mathbb{T}$ first, where $\mathbb{R}$ is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and χ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y) \, dy. \tag{3}$$

Let $\xi = \int_0^a \chi(y) \, dy$, then

$$\chi(x)\xi = \int_0^a \chi(x+y) \, dy = \int_x^{a+x} \chi(t) \, dt \tag{4}$$

so

$$\chi(x) = \xi^{-1} \int_x^{a+x} \chi(t) \, dt \tag{5}$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1} \left(\chi(a+x) - \chi(x) \right) \\ &= \xi^{-1} \chi(x) \left(\chi(a) - 1 \right) \\ &= c\chi(x). \end{aligned} \tag{6}$$

We have an ODE

$$\chi'(x) = c\chi(x) \tag{7}$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \tag{8}$$