

# Groups, Analysis, and Geometry Seminars:

## Harmonic Analysis of $\mathcal{SU}(2)$

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### 1 Introduction

Suppose  $f$  is  $2\pi$ -periodic, complex valued, integrable over  $[0, 2\pi)$ , then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \quad (1)$$

with  $n \in \mathbb{Z}$ , is the Fourier transform of  $f$ . Question: What is  $e^{inx}$ ? Why is  $n \in \mathbb{Z}$ ? Answer:  $e^{inx}$  is the character of circle group, denoted by  $\mathbb{T}$  (i.e.  $e^{ix} \in \mathbb{T}$ ).

A character is a continuous homomorphism from a locally compact Abelian group  $G$  to  $\mathbb{T}$ :  $\chi : G \rightarrow \mathbb{T}$  where

$$\chi(gh) = \chi(g)\chi(h) \quad (2)$$

for  $g, h \in G$ . Let's work out  $\chi : \mathbb{R} \rightarrow \mathbb{T}$  first, where  $\mathbb{R}$  is the group  $(\mathbb{R}, +)$ . Since  $\chi(0) = 1$  (identity to identity) and  $\chi$  is continuous, then  $\exists a > 0$  such that

$$\int_0^a \chi(y) dy. \quad (3)$$

Let  $\xi = \int_0^a \chi(y) dy$ , then

$$\chi(x)\xi = \int_0^a \chi(x+y) dy = \int_x^{a+x} \chi(t) dt \quad (4)$$

so

$$\chi(x) = \xi^{-1} \int_x^{a+x} \chi(t) dt \quad (5)$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1} (\chi(a+x) - \chi(x)) \\ &= \xi^{-1} \chi(x) (\chi(a) - 1) \\ &= c\chi(x). \end{aligned} \quad (6)$$

We have an ODE

$$\chi'(x) = c\chi(x) \quad (7)$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \quad (8)$$

Solving it gives us  $\chi(x) = e^{cx}$ . Since  $|\chi| = 1$ , then  $c = i\lambda$  with  $\lambda \in \mathbb{R}$ . Thus  $\chi(x) = e^{i\lambda x}$  and we have characters of  $\mathbb{R}$ , all the  $\chi_\lambda$  form a dual group of  $\mathbb{R}$ , denoted by  $\widehat{\mathbb{R}}$ .

Since we identify each  $\chi_\lambda$  with  $\lambda \in \mathbb{R}$ , then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \quad (9)$$

To work out  $\widehat{\mathbb{T}}$ , notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \quad (10)$$

i.e. each element in  $[0, 2\pi)$  is a representative of the cosets of  $\mathbb{R}/2\pi\mathbb{Z}$ . Suppose  $x, y \in \mathbb{R}/2\pi\mathbb{Z}$  and  $x + y = 2\pi$ , then

$$\chi(x + y) = \chi(0) = 1 = e^{i\lambda(x+y)} \quad (11)$$

we know  $\lambda \in \mathbb{R}$ , but the only way  $e^{i\lambda(x+y)} = 1$  is that  $\lambda \in \mathbb{Z}$ . So all the  $\chi_n(x) = e^{inx}$  for the dual group  $\widehat{\mathbb{T}}$ , and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \quad (12)$$

Similarly, we have  $\mathbb{R}^n \cong \mathbb{R}^n$  and  $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$ .

**Theorem 1.1:** If  $G$  is compact,  $\widehat{G}$  is discrete.

In addition,  $\{e^{inx} : n \in \mathbb{Z}\}$  form an orthonormal basis for the Hilbert space  $L^2(\mathbb{T})$ , with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \quad (13)$$

If  $G$  is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of  $G$ :

$$\sigma : G \rightarrow U(\mathcal{H}) \quad (14)$$

for  $\mathcal{H}$ , some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \quad (15)$$

for  $u, v \in \mathcal{H}$ . We'll look at the irreducible representations of  $SU(2)$ .  $SU(2)$  is the first compacy and non-abelian group we normally look at in Harmonic Analysis.

## 2 Aspects of $\mathcal{SU}(2)$

We begin by looking at  $\mathcal{U}(2)$ , a group of unitary transformations of  $\mathbb{C}^2$ .

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \quad (16)$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$  where elements of  $\mathcal{SU}(2)$  have determinant 1.

$$\begin{cases} \det(AB) = \det(A) \det(B), & A, B \in \mathcal{SU}(2) \\ \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1}) \end{cases} \quad (17)$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \quad (18)$$

then,  $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$  and  $\alpha\bar{\gamma} + \beta\bar{\eta} = 0$ . So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \quad (19)$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}, \quad \theta \in \mathbb{R}. \quad (20)$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\bar{\beta} & e^{i\theta}\bar{\alpha} \end{pmatrix} \in \mathcal{U}(2) \quad (21)$$

with  $\det(A) = e^{i\theta}$  and  $\det(A^*) = e^{-i\theta}$ . If  $A \in \mathcal{SU}(2)$ , then

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \quad (22)$$

$$\mathcal{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \quad (23)$$

where  $\alpha, \beta \in \mathbb{C}$ . If  $\alpha = a + ib$  and  $\beta = c + id$  then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 &= 1 \\ \left| \sqrt{a^2 + b^2} \right|^2 + \left| \sqrt{c^2 + d^2} \right|^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1 \end{aligned} \quad (24)$$

and so  $\mathcal{SU}(2) \cong S^3$ , the 3-sphere.  $\mathbb{T} \subset \mathcal{SU}(2)$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha| = 1 \right\} \quad (25)$$

**Theorem 2.1:** Given  $A \in \mathcal{SU}(2)$ ,  $\exists V \in \mathcal{SU}(2)$ , such that

$$VAV^{-1} = T \quad (26)$$

for some  $T \in \mathbb{T}$ .

**Proof 2.2:** If  $A \in \mathcal{U}(2)$ ,  $A$  is normal, then by the *spectral theorem*,  $\exists V \in \mathcal{U}(2)$  such that

$$VAv^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad (27)$$

with  $|\alpha| = |\beta| = 1$ . □

If  $A \in \mathcal{SU}(2)$ , then  $\beta = \bar{\alpha}$ . So,  $V \in \mathcal{U}(2)$ ,  $A \in \mathcal{SU}(2)$ , we have

$$VAV^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}.$$

To make  $V$  special unitary, let  $\tilde{V} = \det(V)^{1/2} V$  such that  $\det(\tilde{V}) = 1$  and we have  $\tilde{V}A\tilde{V}^{-1} = T$ .

$\mathcal{SU}(2)$  is a compact, connected, Lie group, (?) we have a theorem to generalise this conjugation relation.

**Theorem 2.3:** If  $G$  is a compact, connected, Lie group,  $\mathbb{T}$  is its maximal torus, then for  $\chi \in G$ ,  $\exists g \in G$ , such that

$$g\chi g^{-1} = t \quad : \quad t \in \mathbb{T}. \quad (28)$$

**Theorem 2.4:** (proposition?) Define  $O_x = \{g\chi g^{-1} : g \in G\}$ , then for  $\mathcal{SU}(2)$ , every  $O_x$  intersects  $\mathbb{T}$  at exactly two points.

**Proof 2.5:** We have  $g\chi g^{-1} = t$ , let

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$\begin{aligned} (g_1 g) \chi (g_1 g)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned}$$

□

Now, we describe the irreducible representations of  $\mathcal{SU}(2)$ .  $\mathcal{SU}(2)$  naturally acts on  $\mathbb{C}^2$  ( $\sigma$  is just the identity mapping) i.e.

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \alpha z + \beta w \\ -\bar{\beta} z + \bar{\alpha} w \end{pmatrix} \quad (29)$$

We are interested in  $SU(2)$  acting on the vector spaces which are built out of  $\mathbb{C}^2$ , e.g.

$$L^2(SU(2)) \cong L^2(S^3) \quad (30)$$

made up of functions

$$f : \mathbb{C}^2 \rightarrow \mathbb{C} \quad (31)$$

and let  $P$  be the space of all polynomials with two complex variables, i.e.  $f \in P$  such that

$$f(z, w) = \sum_{j,k} a_{jk} z^j w^k \quad (32)$$

and

$$P \subset L^2(S^3) \quad (33)$$

where both spaces are infinite dimensional. But  $SU(2)$  is compact, so  $d_\sigma < \infty$ . This means we want to find subspaces of  $P$  which are:

- finite dimensional
- invariant under the action of  $SU(2)$

Thus, we pick  $P_m$ , the space of homogeneous polynomials of degree  $m$ . For  $f \in P_m$

$$f(z, w) = \sum_{j=0}^m a_j z^{m-j} w^j \quad (34)$$

and let

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad (35)$$

we define the regular representation of  $\sigma(A_{\alpha,\beta})$  on  $f$ :

$$\begin{aligned} \sigma(A_{\alpha,\beta})f(z, w) &= f(A_{\alpha,\beta}(z, w)) \\ &= f(\bar{\alpha}z - \beta w, \bar{\beta}z + \alpha w) \end{aligned} \quad (36)$$