Groups, Analysis, and Geometry Seminars: Harmonic Analysis of $\mathcal{SU}(2)$

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20th August, 2020

1 Introduction

Suppose f is 2π-periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\ dx \tag{1}$$

with $n \in \mathbb{Z}$, is the Fourier transform of f. Question: What is e^{inx}? Why is $n \in \mathbb{Z}$? Answer: e^{inx} is the character of circle group, denoted by $\mathbb{T}$ (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group G to $\mathbb{T}$: $\chi : G \to \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \tag{2}$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \to \mathbb{T}$ first, where $\mathbb{R}$ is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and χ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y)\ dy. \tag{3}$$

Let $\xi = \int_0^a \chi(y)\ dy$, then

$$\chi(x)\xi = \int_0^a \chi(x+y)\ dy = \int_x^{a+x} \chi(t)\ dt \tag{4}$$

so

$$\chi(x) = \xi^{-1}\int_x^{a+x} \chi(t)\ dt \tag{5}$$

and

$$\begin{aligned}\chi'(x) &= \xi^{-1}\left(\chi(a+x) - \chi(x)\right)\\ &= \xi^{-1}\chi(x)\left(\chi(a) - 1\right)\\ &= c\chi(x).\end{aligned} \tag{6}$$

We have an ODE

$$\chi'(x) = c\chi(x) \tag{7}$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \tag{8}$$

Solving it gives us $\chi(x) = e^{cx}$. Since $|\chi| = 1$, then $c = i\lambda$ with $\lambda \in \mathbb{R}$. Thus $\chi(x) = e^{i\lambda x}$ and we have characters of $\mathbb{R}$, all the χ_λ form a dual group of $\mathbb{R}$, denoted by $\widehat{\mathbb{R}}$.

Since we identify each χ_λ with $\lambda \in \mathbb{R}$, then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \tag{9}$$

To work out $\widehat{\mathbb{T}}$, notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \tag{10}$$

i.e. each element in $[0, 2\pi)$ is a representative of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$. Suppose $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ and $x + y = 2\pi$, then

$$\chi(x+y) = \chi(0) = 1 = e^{i\lambda(x+y)} \tag{11}$$

we know $\lambda \in \mathbb{R}$, but the only way $e^{i\lambda(x+y)} = 1$ is that $\lambda \in \mathbb{Z}$. So all the $\chi_n(x) = e^{inx}$ for the dual group $\widehat{\mathbb{T}}$, and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \tag{12}$$

Similarly, we have $\mathbb{R}^n \cong \mathbb{R}^n$ and $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$.

Theorem 1.1: If G is compact, $\widehat{G}$ is discrete.

In addition, $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$, with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} \, dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \tag{13}$$

If G is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of G:

$$\sigma : G \to U(\mathcal{H}) \tag{14}$$

for $\mathcal{H}$, some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \tag{15}$$

for $u, v \in \mathcal{H}$. We'll look at the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ is the first compacy and non-abelian group we normally look at in Harmonic Analysis.

2 Aspects of $\mathcal{SU}(2)$

We begin by looking at $\mathcal{U}(2)$, a group of unitary transformations of $\mathbb{C}^2$.

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \tag{16}$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$ where elements of $\mathcal{SU}(2)$ have determinant 1.

$$\begin{cases} \det(AB) = \det(A)\det(B), \ A, B \in \mathcal{SU}(2) \\ \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) \end{cases} \tag{17}$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \tag{18}$$

then, $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$ and $\alpha\overline{\gamma} + \beta\overline{\eta} = 0$. So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \tag{19}$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta}\begin{pmatrix} -\overline{\beta} \\ \overline{\alpha} \end{pmatrix}, \ \theta \in \mathbb{R}. \tag{20}$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\overline{\beta} & e^{i\theta}\overline{\alpha} \end{pmatrix} \in \mathcal{U}(2) \tag{21}$$

with $\det(A) = e^{i\theta}$ and $\det(A^*) = e^{-i\theta}$. If $A \in \mathcal{SU}(2)$, then

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \tag{22}$$

$$\mathcal{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \tag{23}$$

where $\alpha, \beta \in \mathbb{C}$. If $\alpha = a + ib$ and $\beta = c + id$ then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 &= 1 \\ \left|\sqrt{a^2+b^2}\right|^2 + \left|\sqrt{c^2+d^2}\right|^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1 \end{aligned} \tag{24}$$

and so $\mathcal{SU}(2) \cong S^3$, the $3-$sphere. $\mathbb{T} \subset \mathcal{SU}(2)$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} : |\alpha| = 1 \right\} \tag{25}$$

Theorem 2.1: Given $A \in \mathcal{SU}(2)$, $\exists V \in \mathcal{SU}(2)$, such that

$$VAV^{-1} = T \tag{26}$$

for some $T \in \mathbb{T}$.

Proof 2.2: If $A \in \mathcal{U}(2)$, A is normal, then by the *spectral theorem*, $\exists V \in \mathcal{U}(2)$ such that

$$VAv^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{27}$$

with $|\alpha| = |\beta| = 1$. $\square$

If $A \in \mathcal{SU}(2)$, then $\beta = \overline{\alpha}$. So, $V \in \mathcal{U}(2)$, $A \in \mathcal{SU}(2)$, we have

$$VAV^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}.$$

To make V special unitary, let $\tilde{V} = \det(V)^{1/2} V$ such that $\det\left(\tilde{V}\right) = 1$ and we have $\tilde{V}A\tilde{V}^{-1} = T$.

$\mathcal{SU}(2)$ is a compact, connected, Lie group, (?) we have a theorem to generalise this conjugation relation.

Theorem 2.3: If G is a compact, connected, Lie group, $\mathbb{T}$ is its maximal torus, then for $\chi \in G$, $\exists g \in G$, such that

$$g\chi g^{-1} = t \quad : \quad t \in \mathbb{T}. \tag{28}$$

Theorem 2.4: (proposition?) Define $O_x = \{g\chi g^{-1} : g \in G\}$, then for $\mathcal{SU}(2)$, every O_x intersects $\mathbb{T}$ at exactly two points.

Proof 2.5: We have $g\chi g^{-1} = t$, let

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$\begin{aligned} (g_1 g)\chi(g_1 g)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned}$$

$\square$

Now, we describe the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ naturally acts on $\mathbb{C}^2$ (σ is just the identity mapping) i.e.

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \alpha z + \beta w \\ -\overline{\beta} z + \overline{\alpha} w \end{pmatrix} \tag{29}$$

We are interested in $\mathcal{SU}(2)$ acting on the vector spaces which are built out of $\mathbb{C}^2$, e.g.

$$L^2(\mathcal{SU}(2)) \cong L^2(S^3) \tag{30}$$

made up of functions

$$f : \mathbb{C}^2 \to \mathbb{C} \tag{31}$$

and let P be the space of all polynomials with two complex variables, i.e. $f \in P$ such that

$$f(z, w) = \sum_{j,k} a_{jk} z^j w^k \tag{32}$$

and

$$P \subset L^2(S^3) \tag{33}$$

where both spaces are infinite dimensional. But $\mathcal{SU}(2)$ is compact, so $d_\sigma < \infty$. This means we want to find subspaces of P which are:

- finite dimensional
- invariant under the action of $\mathcal{SU}(2)$

Thus, we pick P_m , the space of homogeneous polynomials of degree m. For $f \in P_m$

$$f(z, w) = \sum_{j=0}^{m} a_j z^{m-j} w^j \tag{34}$$

and let

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{35}$$

we define the regular representation of $\sigma(A_{\alpha,\beta})$ on f:

$$\begin{aligned} \sigma(A_{\alpha,\beta}) f(z, w) &= f(A_{\alpha,\beta}(z, w)) \\ &= f(\overline{\alpha} z - \beta w, \overline{\beta} z + \alpha w) \end{aligned} \tag{36}$$

Theorem 2.6: P_m is invariant under $\sigma(A_{\alpha,\beta})$.

Proof 2.7: Suppose monomials $z^{m-j} w^j \in P_m$ then

$$f(\overline{\alpha} z - \beta w, \overline{\beta} z + \alpha w) = (\overline{\alpha} z - \beta w)^{m-j} (\overline{\beta} z + \alpha w)^j \in P_m. \tag{37}$$

So, P_m is invariant under σ. □

Now, we want to define an inner product, i.e.

$$\langle f, g \rangle = \int_{S^3} f \overline{g} \; d\mu \tag{38}$$

where $\mu(S^3) = 1$ is the surface measure of S^3 which is normalised to 1.

So, we'll show monomials $z^{m-j}w^j$ are orthogonal in P with respect to this inner product. To achieve this, we use polar coordinates.

Suppose $Z = (z, w) \in C^2$ and $Z = r\tilde{z}$, where $\tilde{z} \in S^3$, and $r = |z| = \sqrt{|z|^2 + |w|^2}$. We define Lebesgue measure on $\mathbb{C}^2$ as

$$\begin{aligned} dz &= dz\ dw \\ &= d_a\ d_b\ d_c\ d_d \\ &= r^3\ dr\ d\tilde{\mu}(\tilde{z})\ :\ \text{(un-normalised)} \\ &= 2\pi^2 r^3\ dr\ d\mu(\tilde{z})\ :\ \text{(normalised)} \end{aligned}. \tag{39}$$

To see this, we have $\varphi_1, \varphi_2 \in [0, \pi]$, $\theta \in [0, 2\pi)$.

$$\begin{aligned} a &= r\cos(\varphi_1) \\ b &= r\sin(\varphi_1)\sin(\varphi_2) \\ c &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\theta) \\ d &= r\sin(\varphi_1)\sin(\varphi_2)\sin(\theta). \end{aligned} \tag{40}$$

The Jacobian matrix is then

$$J = \begin{bmatrix} \frac{\partial a}{\partial r} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \frac{\partial d}{\partial \theta} \end{bmatrix} \tag{41}$$

giving

$$\det(J) = r^3 \sin^2(\varphi_1)\sin(\varphi_2). \tag{42}$$

The surface measure of S^3 is exactly

$$\begin{aligned} \mu(S^3) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin^2(\varphi_1)\sin(\varphi_2)\, d\varphi_1\ d\varphi_2\ d\theta(?) \\ &= 2\pi^2 \end{aligned} \tag{43}$$

Theorem 2.8: If $f : \mathbb{C}^2 \to \mathbb{C}$ satisfies $f(aZ) = a^m f(z)$ for $a > 0$, then

$$\int_{S^3} f(\tilde{z})\ d\mu(\tilde{z}) = \frac{1}{\pi^2\Gamma(\frac{m+4}{2})}\int_{\mathbb{C}^2} f(Z)e^{-|Z|^2}\ dZ. \tag{44}$$

Proof 2.9: We integrate in polar coordinates:

$$\begin{aligned} \int_{\mathbb{C}^2} f(Z)e^{-|Z|^2}\ dZ &= 2\pi^2\int_0^{\infty}\int_{S^3} f(r\tilde{z})e^{-r^3}r^3\ d\mu(\tilde{z})\ dr \\ &= 2\pi^2\int_0^{\infty} e^{-r^3}(?)r^3\int_{S^3} f(r\tilde{z})\ d\mu(\tilde{z})\ dr \\ &= 2\pi^2\frac{1}{2}\Gamma(\frac{m+4}{2})\int_{S^3} f(\tilde{z})\ d\mu(\tilde{z}) \end{aligned} \tag{45}$$

$\square$

Theorem 2.10: If $p, q, r, s \in \mathbb{Z}$, then

$$\int_{S^3} z^p \overline{z}^q w^r \overline{w}^s \; d\mu(z,w) = \begin{cases} 0 & \text{if } p \neq q \text{ or } r \neq s \\ \frac{p!r!}{(p+r+1)!} & \text{if } p = q \text{ and } r = s \end{cases} \tag{46}$$

Proof 2.11: By Prop ??,

$$\int_{S^3} z^p \overline{z^q} w^r \overline{w}^s \; d\mu(z,w) = \frac{1}{\pi^2 \Gamma(\frac{p+q+r+s+4}{2})} \int_{\mathbb{C}} z^p \overline{z^q} e^{-|z|^2} \; dz \cdot \int_{\mathbb{C}} w^r \overline{w}^s \; dw \tag{47}$$

Let $m = p + q + r + s$. Since $f(z) = 1^m f(z)$:

$$\begin{aligned} \int_{\mathbb{C}} z^p \overline{z^q} e^{-|z|^2} \; dz &= \int_0^\infty \int_0^{2\pi} e^{i(p-q)\theta} r^{p+q+1} e^{-r^2} \; d\theta \; dr \\ &= \int_0^\infty r^{p+q+1} e^{-r^2} \; dr \cdot \int_0^{2\pi} e^{i(p-q)\theta} \; d\theta \end{aligned} \tag{48}$$

If $p \neq q$, then the integral with respect to θ is 0, and if $p = q$, then

$$\int_0^{2\pi} e^{i(p-q)\theta} \; d\theta = 2\pi \frac{1}{2} \Gamma(\frac{2p+2}{2}) = \pi \Gamma(p+1). \tag{49}$$

Similarly for

$$\int w^p \overline{w}^q e^{-|w|^2} \; dw = \begin{cases} 0 & \text{if } r \neq s \\ \pi r! & \text{if } r = s. \end{cases} \tag{50}$$

Also,

$$\Gamma\left(\frac{p+q+r+s+4}{2}\right) = (p+r+1)! \tag{51}$$

if $p = q$ and $r = s$. $\square$

Theorem 2.12: The subspaces P_m are mutually orthogonal in $L^2(S^3)$, and

$$\left\{ \sqrt{\frac{(m+1)!}{(m-j)!j!}} z^{m-j} w^j \;\; : \;\; 0 \leq j \leq m \right\}. \tag{52}$$

Proof 2.13: By the previous theorem ??. $\square$

For each P_m, we can define the representation σ_m acting on it, and we can quickly access the information of σ_m by looking at its (trace) characters. Let $A_{\alpha,\sigma} \in \mathbb{T}$, then

$$\sigma_m(A_{\alpha,0(?)})(z^{m-j} w^j) = e^{i(m-zj)\theta} z^{m-j} w^j. \tag{53}$$

Therefore, each orthogonal vector $z^{m-j}w^j$ of P_m is an eigenvector of $\sigma_m(A_{\alpha,0(?)})$. This means we can write

$$\sigma_m(A_{\alpha,0}) = \begin{bmatrix} e^{im\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{i(m-2)\theta} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & e^{-i(m-2)\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-im\theta} \end{bmatrix} \tag{54}$$

So,

$$\begin{aligned} \operatorname{Tr}\left(\sigma_m(A_{\alpha,0})\right) &= \sum_{j=0}^{m} e^{i(m-2j)\theta} \\ &= \frac{e^{i(m+2)\theta} - e^{-im\theta}}{e^{2i\theta} - 1} \\ &= \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} \\ &= \frac{\sin\left((m+1)\theta\right)}{\sin(\theta)} \end{aligned} \tag{55}$$

By theorem ??, every $\chi \in \mathcal{SU}(2)$ can be written as

$$\chi = g^{-1}tg \;:\; \exists g \in \mathcal{SU}(2), t \in \mathbb{T}. \tag{56}$$

So, by the cyclic property of trace we have

$$\begin{aligned} \operatorname{Tr}\left(\sigma_m(g^{-1}tg)\right) &= \operatorname{Tr}\left(\sigma_m(g)^{-1}\sigma_m(t)\sigma_m(g)\right) \\ &= \operatorname{Tr}\left(\sigma_m(t)\right). \end{aligned} \tag{57}$$

Hence, we could conclude that every σ_m looks like $\frac{\sin((m+1)\theta)}{\sin(\theta)}$.

The trace character simply does not give enough information about σ_m, so we still need to work out the matrix coefficients of σ_m. Suppose

$$e_j(z,w) = \sqrt{\frac{(m+1)!}{(m-j)!j!}}\, z^{m-j}w^j \tag{58}$$

and let

$$\sigma_m(\alpha,\beta) = \sigma_m(A_{\alpha,\beta}). \tag{59}$$

Then the matrix coefficient $\sigma_m^{jk}(\alpha,\beta)$ can be recovered by

$$\sigma_m^{jk}(\alpha,\beta) = \langle \sigma_m(\alpha,\beta)e_k, e_j \rangle \tag{60}$$

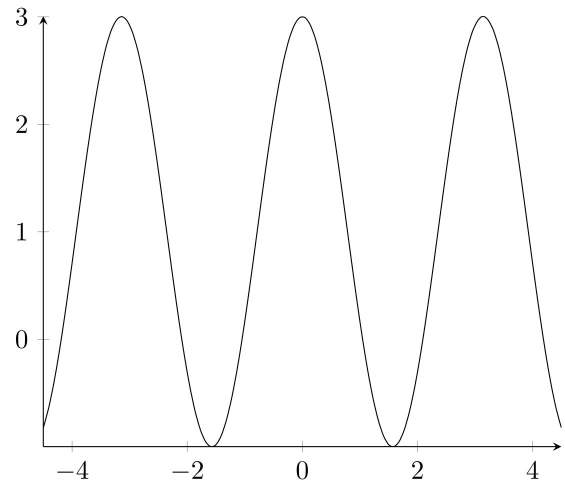

Figure 1: An example of $\frac{\sin((m+1)\theta)}{\sin(\theta)}$ when $m = 2$.

where the right-hand-side is the inner product on $L^2(S^3)$. Following this we have

$$\begin{aligned}\sigma_m(\alpha,\beta)\cdot e_k(z,w) &= \sqrt{\frac{(m+1)!}{(m-k)!k!}}(\overline{\alpha}z-\beta w)^{m-k}(\overline{\beta}z+\alpha w)^k \\ &= \sum_{j=0}^{m}\sigma_m^{jk}(\alpha,\beta)e_j(z,w) \\ &= \sum_{j=0}^{m}\sqrt{\frac{(m+1)!}{(m-j)!j!}}\sigma_m^{jk}(\alpha,\beta)z^{m-j}w^j.\end{aligned} \tag{61}$$

Thus, we have

$$\sum_{j=0}^{m}\sqrt{\frac{(m+1)!}{(m-j)!j!}}\sigma_m^{jk}(\alpha,\beta)z^{m-j}w^j = (\overline{\alpha}z-\beta w)^{m-k}(\overline{\beta}z+\alpha w)^k. \tag{62}$$

We can solve this by multiplying out the right hand side and matching the coefficients of $z^{m-j}w^j$. However, observe and realise that if we let $z = 1$, $w = e^{ix}$, then we have

$$\sum_{j=0}^{m}\sigma_m^{jk}(\alpha,\beta)e^{ij\chi} = \underbrace{(\overline{\alpha}-\beta e^{i\chi})^{m-k}(\overline{\beta}+\alpha e^{i\chi})^k}_{\in L^2(\mathbb{T})(?)} \tag{63}$$

Thus, we have

$$\sigma_m^{jk}(\alpha,\beta) = \sqrt{\frac{(m-j)!j!}{(m-k)!k!}}\cdot\frac{1}{2\pi}\int_0^{2\pi}(\overline{\alpha}-\beta e^{i\chi})^{m-k}(\overline{\beta}+\alpha e^{i\chi})^k e^{-ij\chi}\ d\chi \tag{64}$$

Let us write down σ_m for a few m.

$$m = 0 \Rightarrow \sigma_0 = \begin{bmatrix} 1 \end{bmatrix}$$

$$m = 1 \Rightarrow \sigma_1 = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

$$m = 2 \Rightarrow \sigma_2 = \begin{bmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\overline{\beta} & \alpha\overline{\alpha} - \beta\overline{\beta} & \sqrt{2}\beta\overline{\alpha} \\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{bmatrix}$$

$$m = 3 \Rightarrow \sigma_3 = \begin{bmatrix} \alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3 \\ -\sqrt{3}\alpha^2\overline{\beta} & -2\alpha\beta\overline{\beta} + \alpha^2\overline{\alpha} & 2\alpha\beta\overline{\alpha} - \beta^2\overline{\beta} & \sqrt{3}\beta^2\overline{\alpha} \\ \sqrt{3}\alpha\overline{\beta}^2 & -2\alpha\overline{\alpha}\overline{\beta} + \beta\overline{\beta}^2 & \alpha\overline{\alpha}^2 - 2\beta\overline{\alpha}\overline{\beta} & \sqrt{3}\beta\overline{\alpha}^2 \\ -\overline{\beta}^3 & \sqrt{3}\overline{\alpha}\overline{\beta}^2 & -\sqrt{3}\overline{\alpha}^2\overline{\beta} & \overline{\alpha}^3 \end{bmatrix}.$$

Interestingly, every σ_m is irreducible. We demonstrate this using the concept of the Lie algebra, whose details are out of the scope of this talk. For now at least, we have:

Theorem 2.14: σ_m is irreducible for each $m \geq 0$.

Also, we have

Theorem 2.15: (*Dual*)

$$\widehat{\mathcal{SU}(2)} = \{[\sigma_m] : m \geq 0\} \tag{65}$$

Note that $[\cdot]$ denotes the equivalence class of σ_m. It turns out that σ_m is equivalent to its contragradient $\overline{\sigma_m}$.

$$\sigma_m^{*\,(g)} = \sigma(g^{-1})^T = \overline{\sigma_m(g)}. \tag{66}$$

Now, let's examine the Schur orthogonality relation.

Theorem 2.16: Let σ and σ' be irreducible representations of G, $\mathcal{E}_\sigma$ and $\mathcal{E}_{\sigma'}$ are the linear span of the matrix elements of σ and σ', respectively.

1. If $[\sigma] \neq [\sigma']$, then $\mathcal{E}_\sigma \perp \mathcal{E}_{\sigma'}$.
2. If $\{e_j\}$ is any orthonormal basis for $\mathcal{H}_\sigma$ and $\sigma_{ij} = \langle \sigma e_j, e_i \rangle$, then

$$\left\{ \sqrt{d_\sigma}\sigma_{ij} : i, j = 1, \ldots, d_\sigma \right\}$$

is an orthonormal basis for $\mathcal{E}_\sigma$.

Proof 2.17: See Folland - Abstract Harmonic Analysis pg.139. □

Finally, we come to the major result of representations of compact groups.

Theorem 2.18: (*Peter-Weyl Theorem*) Let G be a compact group, then

$$L^2(G) = \bigoplus_{[\sigma]\in\widehat{G}} \mathcal{E}_\sigma \tag{67}$$

and if $\sigma_{ij} = \langle \sigma e_j, e_i \rangle$, then

$$\left\{ \sqrt{d_\sigma}\sigma_{ij} : i, j = 1, \ldots, d_\sigma, [\sigma] \in \widehat{G} \right\} \tag{68}$$

is an orthonormal basis for $L^2(G)$.

According to the Peter-Weyl theorem, if $f \in L^2(G)$, then

$$f = \sum^{[\sigma]\in\widehat{G}} \sum_{i,j=1}^{d_\sigma} a_{ij}^\sigma \sigma_{ij}, \tag{69}$$

and

$$a_{ij}^\sigma = d_\sigma \int_G f(\chi)\overline{\sigma_{ij}(\chi)}\, d\chi. \tag{70}$$

The drawback is we have an orthonormal basis for each $\mathcal{H}_\sigma$. If $f \in L^1(G)$, $[\sigma] \in \widehat{G}$, and σ is the representative of class $[\sigma]$, we define the Fourier transform of f at σ to be

$$\widehat{f}(\sigma)(?) = \int_G f(\chi)\sigma(\chi^{-1})\, d\chi. \tag{71}$$