# **Groups, Analysis, and Geometry Seminars**: Harmonic Analysis of $\mathcal{SU}(2)$

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Suppose $f$ is $2\pi$-periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\ dx \tag{1}$$

with $n \in \mathbb{Z}$, is the Fourier transform of $f$. Question: What is $e^{inx}$? Why is $n \in \mathbb{Z}$? Answer: $e^{inx}$ is the character of circle group, denoted by $\mathbb{T}$ (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group $G$ to $\mathbb{T}$: $\chi : G \to \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \tag{2}$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \to \mathbb{T}$ first, where $\mathbb{R}$ is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and $\chi$ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y)\ dy. \tag{3}$$

Let $\xi = \int_0^a \chi(y)\ dy$, then

$$\chi(x)\xi = \int_0^a \chi(x + y)\ dy = \int_x^{a+x} \chi(t)\ dt \tag{4}$$

so

$$\chi(x) = \xi^{-1}\int_x^{a+x} \chi(t)\ dt \tag{5}$$

and

$$\begin{aligned}\chi'(x) &= \xi^{-1}\left(\chi(a + x) - \chi(x)\right)\\ &= \xi^{-1}\chi(x)\left(\chi(a) - 1\right)\\ &= c\chi(x).\end{aligned} \tag{6}$$

We have an ODE

$$\chi'(x) = c\chi(x) \tag{7}$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \tag{8}$$

Solving it gives us $\chi(x) = e^{cx}$. Since $|\chi| = 1$, then $c = i\lambda$ with $\lambda \in \mathbb{R}$. Thus $\chi(x) = e^{i\lambda x}$ and we have characters of $\mathbb{R}$, all the $\chi_\lambda$ form a dual group of $\mathbb{R}$, denoted by $\widehat{\mathbb{R}}$.

Since we identify each $\chi_\lambda$ with $\lambda \in \mathbb{R}$, then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \tag{9}$$

To work out $\widehat{\mathbb{T}}$, notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \tag{10}$$

i.e. each element in $[0, 2\pi)$ is a representative of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$. Suppose $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ and $x + y = 2\pi$, then

$$\chi(x + y) = \chi(0) = 1 = e^{i\lambda(x+y)} \tag{11}$$

we know $\lambda \in \mathbb{R}$, but the only way $e^{i\lambda(x+y)} = 1$ is that $\lambda \in \mathbb{Z}$. So all the $\chi_n(x) = e^{inx}$ for the dual group $\widehat{\mathbb{T}}$, and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \tag{12}$$

Similarly, we have $\mathbb{R}^n \cong \mathbb{R}^n$ and $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$.

**Theorem 0.1**: If $G$ is compact, $\widehat{G}$ is discrete.

In addition, $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$, with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} \, dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \tag{13}$$