

Groups, Analysis, and Geometry Seminars:

Harmonic Analysis of $\mathcal{SU}(2)$

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1 Introduction

Suppose f is 2π -periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \quad (1)$$

with $n \in \mathbb{Z}$, is the Fourier transform of f . Question: What is e^{inx} ? Why is $n \in \mathbb{Z}$? Answer: e^{inx} is the character of circle group, denoted by \mathbb{T} (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group G to \mathbb{T} : $\chi : G \rightarrow \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \quad (2)$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \rightarrow \mathbb{T}$ first, where \mathbb{R} is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and χ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y) dy. \quad (3)$$

Let $\xi = \int_0^a \chi(y) dy$, then

$$\chi(x)\xi = \int_0^a \chi(x+y) dy = \int_x^{a+x} \chi(t) dt \quad (4)$$

so

$$\chi(x) = \xi^{-1} \int_x^{a+x} \chi(t) dt \quad (5)$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1} (\chi(a+x) - \chi(x)) \\ &= \xi^{-1} \chi(x) (\chi(a) - 1) \\ &= c\chi(x). \end{aligned} \quad (6)$$

We have an ODE

$$\chi'(x) = c\chi(x) \quad (7)$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \quad (8)$$

Solving it gives us $\chi(x) = e^{cx}$. Since $|\chi| = 1$, then $c = i\lambda$ with $\lambda \in \mathbb{R}$. Thus $\chi(x) = e^{i\lambda x}$ and we have characters of \mathbb{R} , all the χ_λ form a dual group of \mathbb{R} , denoted by $\widehat{\mathbb{R}}$.

Since we identify each χ_λ with $\lambda \in \mathbb{R}$, then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \quad (9)$$

To work out $\widehat{\mathbb{T}}$, notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \quad (10)$$

i.e. each element in $[0, 2\pi)$ is a representative of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$. Suppose $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ and $x + y = 2\pi$, then

$$\chi(x + y) = \chi(0) = 1 = e^{i\lambda(x+y)} \quad (11)$$

we know $\lambda \in \mathbb{R}$, but the only way $e^{i\lambda(x+y)} = 1$ is that $\lambda \in \mathbb{Z}$. So all the $\chi_n(x) = e^{inx}$ for the dual group $\widehat{\mathbb{T}}$, and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \quad (12)$$

Similarly, we have $\mathbb{R}^n \cong \mathbb{R}^n$ and $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$.

Theorem 1.1: If G is compact, \widehat{G} is discrete.

In addition, $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$, with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \quad (13)$$

If G is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of G :

$$\sigma : G \rightarrow U(\mathcal{H}) \quad (14)$$

for \mathcal{H} , some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \quad (15)$$

for $u, v \in \mathcal{H}$. We'll look at the irreducible representations of $SU(2)$. $SU(2)$ is the first compact and non-abelian group we normally look at in Harmonic Analysis.

2 Aspects of $\mathcal{SU}(2)$

We begin by looking at $\mathcal{U}(2)$, a group of unitary transformations of \mathbb{C}^2 .

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \quad (16)$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$ where elements of $\mathcal{SU}(2)$ have determinant 1.

$$\begin{cases} \det(AB) = \det(A) \det(B), & A, B \in \mathcal{SU}(2) \\ \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1}) \end{cases} \quad (17)$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \quad (18)$$

then, $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$ and $\alpha\bar{\gamma} + \beta\bar{\eta} = 0$. So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \quad (19)$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}, \quad \theta \in \mathbb{R}. \quad (20)$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\bar{\beta} & e^{i\theta}\bar{\alpha} \end{pmatrix} \in \mathcal{U}(2) \quad (21)$$

with $\det(A) = e^{i\theta}$ and $\det(A^*) = e^{-i\theta}$. If $A \in \mathcal{SU}(2)$, then

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \quad (22)$$

$$\mathcal{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \quad (23)$$

where $\alpha, \beta \in \mathbb{C}$. If $\alpha = a + ib$ and $\beta = c + id$ then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 &= 1 \\ \left| \sqrt{a^2 + b^2} \right|^2 + \left| \sqrt{c^2 + d^2} \right|^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1 \end{aligned} \quad (24)$$

and so $\mathcal{SU}(2) \cong S^3$, the 3-sphere. $\mathbb{T} \subset \mathcal{SU}(2)$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha| = 1 \right\} \quad (25)$$

Theorem 2.1: Given $A \in \mathcal{SU}(2)$, $\exists V \in \mathcal{SU}(2)$, such that

$$VAV^{-1} = T \quad (26)$$

for some $T \in \mathbb{T}$.

Proof 2.2: If $A \in \mathcal{U}(2)$, A is normal, then by the *spectral theorem*, $\exists V \in \mathcal{U}(2)$ such that

$$VAv^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad (27)$$

with $|\alpha| = |\beta| = 1$. □

If $A \in \mathcal{SU}(2)$, then $\beta = \bar{\alpha}$. So, $V \in \mathcal{U}(2)$, $A \in \mathcal{SU}(2)$, we have

$$VAV^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}.$$

To make V special unitary, let $\tilde{V} = \det(V)^{1/2} V$ such that $\det(\tilde{V}) = 1$ and we have $\tilde{V}A\tilde{V}^{-1} = T$.

$\mathcal{SU}(2)$ is a compact, connected, Lie group, (?) we have a theorem to generalise this conjugation relation.

Theorem 2.3: If G is a compact, connected, Lie group, \mathbb{T} is its maximal torus, then for $\chi \in G$, $\exists g \in G$, such that

$$g\chi g^{-1} = t \quad : \quad t \in \mathbb{T}. \quad (28)$$

Theorem 2.4: (proposition?) Define $O_x = \{g\chi g^{-1} : g \in G\}$, then for $\mathcal{SU}(2)$, every O_x intersects \mathbb{T} at exactly two points.

Proof 2.5: We have $g\chi g^{-1} = t$, let

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$\begin{aligned} (g_1 g) \chi (g_1 g)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned}$$

□

Now, we describe the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ naturally acts on \mathbb{C}^2 (σ is just the identity mapping) i.e.

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \alpha z + \beta w \\ -\bar{\beta} z + \bar{\alpha} w \end{pmatrix} \quad (29)$$

We are interested in $SU(2)$ acting on the vector spaces which are built out of \mathbb{C}^2 , e.g.

$$L^2(SU(2)) \cong L^2(S^3) \quad (30)$$

made up of functions

$$f : \mathbb{C}^2 \rightarrow \mathbb{C} \quad (31)$$

and let P be the space of all polynomials with two complex variables, i.e. $f \in P$ such that

$$f(z, w) = \sum_{j,k} a_{jk} z^j w^k \quad (32)$$

and

$$P \subset L^2(S^3) \quad (33)$$

where both spaces are infinite dimensional. But $SU(2)$ is compact, so $d_\sigma < \infty$. This means we want to find subspaces of P which are:

- finite dimensional
- invariant under the action of $SU(2)$

Thus, we pick P_m , the space of homogeneous polynomials of degree m . For $f \in P_m$

$$f(z, w) = \sum_{j=0}^m a_j z^{m-j} w^j \quad (34)$$

and let

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad (35)$$

we define the regular representation of $\sigma(A_{\alpha,\beta})$ on f :

$$\begin{aligned} \sigma(A_{\alpha,\beta})f(z, w) &= f(A_{\alpha,\beta}(z, w)) \\ &= f(\bar{\alpha}z - \beta w, \bar{\beta}z + \alpha w) \end{aligned} \quad (36)$$

Theorem 2.6: P_m is invariant under $\sigma(A_{\alpha,\beta})$.

Proof 2.7: Suppose monomials $z^{m-j}w^j \in P_m$ then

$$f(\bar{\alpha}z - \beta w, \bar{\beta}z + \alpha w) = (\bar{\alpha}z - \beta w)^{m-j}(\bar{\beta}z + \alpha w)^j \in P_m. \quad (37)$$

So, P_m is invariant under σ . □

Now, we want to define an inner product, i.e.

$$\langle f, g \rangle = \int_{S^3} f \bar{g} \, d\mu \quad (38)$$

where $\mu(S^3) = 1$ is the surface measure of S^3 which is normalised to 1.

So, we'll show monomials $z^{m-j}w^j$ are orthogonal in P with respect to this inner product. To achieve this, we use polar coordinates.

Suppose $Z = (z, w) \in \mathbb{C}^2$ and $Z = r\tilde{z}$, where $\tilde{z} \in S^3$, and $r = |z| = \sqrt{|z|^2 + |w|^2}$. We define Lebesgue measure on \mathbb{C}^2 as

$$\begin{aligned} dz &= dz \, dw \\ &= d_a \, d_b \, d_c \, d_d \\ &= r^3 \, dr \, d\tilde{\mu}(\tilde{z}) \quad : \quad (\text{un-normalised}) \\ &= 2\pi^2 r^3 \, dr \, d\mu(\tilde{z}) \quad : \quad (\text{normalised}) \end{aligned} \tag{39}$$

To see this, we have $\varphi_1, \varphi_2 \in [0, \pi]$, $\theta \in [0, 2\pi]$.

$$\begin{aligned} a &= r \cos(\varphi_1) \\ b &= r \sin(\varphi_1) \sin(\varphi_2) \\ c &= r \sin(\varphi_1) \sin(\varphi_2) \cos(\theta) \\ d &= r \sin(\varphi_1) \sin(\varphi_2) \sin(\theta) . \end{aligned} \tag{40}$$

The Jacobian matrix is then

$$J = \begin{bmatrix} \frac{\partial a}{\partial r} & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & \frac{\partial d}{\partial \theta} \end{bmatrix} \tag{41}$$

giving

$$\det(J) = r^3 \sin^2(\varphi_1) \sin(\varphi_2) . \tag{42}$$

The surface measure of S^3 is exactly

$$\begin{aligned} \mu(S^3) &= \int_0^{2\pi} \int_0^\pi \int_0^\pi \sin^2(\varphi_1) \sin(\varphi_2) \, d\varphi_1 \, d\varphi_2 \, d\theta(?) \\ &= 2\pi^2 \end{aligned} \tag{43}$$

Theorem 2.8: If $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies $f(aZ) = a^m f(z)$ for $a > 0$, then

$$\int_{S^3} f(\tilde{z}) \, d\mu(\tilde{z}) = \frac{1}{\pi^2 \Gamma(\frac{m+4}{2})} \int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} \, dZ. \tag{44}$$

Proof 2.9: We integrate in polar coordinates:

$$\begin{aligned} \int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} \, dZ &= 2\pi^2 \int_0^\infty \int_{S^3} f(r\tilde{z}) e^{-r^3} r^3 \, d\mu(\tilde{z}) \, dr \\ &= 2\pi^2 \int_0^\infty e^{-r^3} (?) r^3 \int_{S^3} f(r\tilde{z}) \, d\mu(\tilde{z}) \, dr \\ &= 2\pi^2 \frac{1}{2} \Gamma\left(\frac{m+4}{2}\right) \int_{S^3} f(\tilde{z}) \, d\mu(\tilde{z}) \end{aligned} \tag{45}$$

□

Theorem 2.10: If $p, q, r, s \in \mathbb{Z}$, then

$$\int_{S^3} z^p \bar{z}^q w^r \bar{w}^s d\mu(z, w) = \begin{cases} 0 & \text{if } p \neq q \text{ or } r \neq s \\ \frac{p!r!}{(p+r+1)!} & \text{if } p = q \text{ and } r = s \end{cases} \quad (46)$$

Proof 2.11: By Prop ??,

$$\int_{S^3} z^p \bar{z}^q w^r \bar{w}^s d\mu(z, w) = \frac{1}{\pi^2 \Gamma(\frac{p+q+r+s+4}{2})} \int_{\mathbb{C}} z^p \bar{z}^q e^{-|z|^2} dz \cdot \int_{\mathbb{C}} w^r \bar{w}^s dw \quad (47)$$

Let $m = p + q + r + s$. Since $f(z) = 1^m f(z)$:

$$\begin{aligned} \int_{\mathbb{C}} z^p \bar{z}^q e^{-|z|^2} dz &= \int_0^\infty \int_0^{2\pi} e^{i(p-q)\theta} r^{p+q+1} e^{-r^2} d\theta dr \\ &= \int_0^\infty r^{p+q+1} e^{-r^2} dr \cdot \int_0^{2\pi} e^{i(p-q)\theta} d\theta \end{aligned} \quad (48)$$

If $p \neq q$, then the integral with respect to θ is 0, and if $p = q$, then

$$\int_0^{2\pi} e^{i(p-q)\theta} d\theta = 2\pi \frac{1}{2} \Gamma\left(\frac{2p+2}{2}\right) = \pi \Gamma(p+1). \quad (49)$$

Similarly for

$$\int w^p \bar{w}^q e^{-|w|^2} dw = \begin{cases} 0 & \text{if } r \neq s \\ \pi r! & \text{if } r = s. \end{cases} \quad (50)$$

Also,

$$\Gamma\left(\frac{p+q+r+s+4}{2}\right) = (p+r+1)! \quad (51)$$

if $p = q$ and $r = s$. □