Groups, Analysis, and Geometry Seminars: Harmonic Analysis of $\mathcal{SU}(2)$

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1 Introduction

Suppose f is 2π-periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \tag{1}$$

with $n \in \mathbb{Z}$, is the Fourier transform of f. Question: What is e^{inx}? Why is $n \in \mathbb{Z}$? Answer: e^{inx} is the character of circle group, denoted by $\mathbb{T}$ (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group G to $\mathbb{T}$: $\chi : G \to \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \tag{2}$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \to \mathbb{T}$ first, where $\mathbb{R}$ is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and χ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y) \, dy. \tag{3}$$

Let $\xi = \int_0^a \chi(y) \, dy$, then

$$\chi(x)\xi = \int_0^a \chi(x+y) \, dy = \int_x^{a+x} \chi(t) \, dt \tag{4}$$

so

$$\chi(x) = \xi^{-1} \int_x^{a+x} \chi(t) \, dt \tag{5}$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1} \left(\chi(a+x) - \chi(x) \right) \\ &= \xi^{-1} \chi(x) \left(\chi(a) - 1 \right) \\ &= c\chi(x). \end{aligned} \tag{6}$$

We have an ODE

$$\chi'(x) = c\chi(x) \tag{7}$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \tag{8}$$

Solving it gives us $\chi(x) = e^{cx}$. Since $|\chi| = 1$, then $c = i\lambda$ with $\lambda \in \mathbb{R}$. Thus $\chi(x) = e^{i\lambda x}$ and we have characters of $\mathbb{R}$, all the χ_λ form a dual group of $\mathbb{R}$, denoted by $\widehat{\mathbb{R}}$.

Since we identify each χ_λ with $\lambda \in \mathbb{R}$, then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \tag{9}$$

To work out $\widehat{\mathbb{T}}$, notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \tag{10}$$

i.e. each element in $[0, 2\pi)$ is a representative of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$. Suppose $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ and $x + y = 2\pi$, then

$$\chi(x+y) = \chi(0) = 1 = e^{i\lambda(x+y)} \tag{11}$$

we know $\lambda \in \mathbb{R}$, but the only way $e^{i\lambda(x+y)} = 1$ is that $\lambda \in \mathbb{Z}$. So all the $\chi_n(x) = e^{inx}$ for the dual group $\widehat{\mathbb{T}}$, and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \tag{12}$$

Similarly, we have $\mathbb{R}^n \cong \mathbb{R}^n$ and $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$.

Theorem 1.1: If G is compact, $\widehat{G}$ is discrete.

In addition, $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$, with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx}\ dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \tag{13}$$

If G is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of G:

$$\sigma : G \to U(\mathcal{H}) \tag{14}$$

for $\mathcal{H}$, some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \tag{15}$$

for $u, v \in \mathcal{H}$. We'll look at the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ is the first compacy and non-abelian group we normally look at in Harmonic Analysis.

2 Aspects of $\mathcal{SU}(2)$

We begin by looking at $\mathcal{U}(2)$, a group of unitary transformations of $\mathbb{C}^2$.

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \tag{16}$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$ where elements of $\mathcal{SU}(2)$ have determinant 1.

$$\begin{cases} \det(AB) = \det(A)\det(B), \ A, B \in \mathcal{SU}(2) \\ \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) \end{cases} \tag{17}$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \tag{18}$$

then, $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$ and $\alpha\overline{\gamma} + \beta\overline{\eta} = 0$. So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \tag{19}$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\overline{\beta} \\ \overline{\alpha} \end{pmatrix}, \ \theta \in \mathbb{R}. \tag{20}$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\overline{\beta} & e^{i\theta}\overline{\alpha} \end{pmatrix} \in \mathcal{U}(2) \tag{21}$$

with $\det(A) = e^{i\theta}$ and $\det(A^*) = e^{-i\theta}$. If $A \in \mathcal{SU}(2)$, then

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \tag{22}$$

$$\mathcal{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \tag{23}$$

where $\alpha, \beta \in \mathbb{C}$. If $\alpha = a + ib$ and $\beta = c + id$ then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 &= 1 \\ \left|\sqrt{a^2 + b^2}\right|^2 + \left|\sqrt{c^2 + d^2}\right|^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1 \end{aligned} \tag{24}$$

and so $\mathcal{SU}(2) \cong S^3$, the $3-$sphere. $\mathbb{T} \subset \mathcal{SU}(2)$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} : |\alpha| = 1 \right\} \tag{25}$$

Theorem 2.1: Given $A \in \mathcal{SU}(2)$, $\exists V \in \mathcal{SU}(2)$, such that

$$VAV^{-1} = T \tag{26}$$

for some $T \in \mathbb{T}$.

Proof 2.2: If $A \in \mathcal{U}(2)$, A is normal, then by the *spectral theorem*, $\exists V \in \mathcal{U}(2)$ such that

$$VAv^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{27}$$

with $|\alpha| = |\beta| = 1$. □

If $A \in \mathcal{SU}(2)$, then $\beta = \overline{\alpha}$. So, $V \in \mathcal{U}(2)$, $A \in \mathcal{SU}(2)$, we have

$$VAV^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}.$$

To make V special unitary, let $\tilde{V} = \det(V)^{1/2} V$ such that $\det\left(\tilde{V}\right) = 1$ and we have $\tilde{V}A\tilde{V}^{-1} = T$.

$\mathcal{SU}(2)$ is a compact, connected, Lie group, (?) we have a theorem to generalise this conjugation relation.

Theorem 2.3: If G is a compact, connected, Lie group, $\mathbb{T}$ is its maximal torus, then for $\chi \in G$, $\exists g \in G$, such that

$$g\chi g^{-1} = t \quad : \quad t \in \mathbb{T}. \tag{28}$$

Theorem 2.4: (proposition?) Define $O_x = \{g\chi g^{-1} : g \in G\}$, then for $\mathcal{SU}(2)$, every O_x intersects $\mathbb{T}$ at exactly two points.

Proof 2.5: We have $g\chi g^{-1} = t$, let

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$\begin{aligned} (g_1 g)\chi(g_1 g)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned}$$

□

Now, we describe the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ naturally acts on $\mathbb{C}^2$ (σ is just the identity mapping) i.e.

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \alpha z + \beta w \\ -\overline{\beta} z + \overline{\alpha} w \end{pmatrix} \tag{29}$$

We are interested in $\mathcal{SU}(2)$ acting on the vector spaces which are built out of $\mathbb{C}^2$, e.g.

$$L^2(\mathcal{SU}(2)) \cong L^2(S^3) \tag{30}$$

made up of functions

$$f : \mathbb{C}^2 \to \mathbb{C} \tag{31}$$

and let P be the space of all polynomials with two complex variables, i.e. $f \in P$ such that

$$f(z,w) = \sum_{j,k} a_{jk} z^j w^k \tag{32}$$

and

$$P \subset L^2(S^3) \tag{33}$$

where both spaces are infinite dimensional. But $\mathcal{SU}(2)$ is compact, so $d_\sigma < \infty$. This means we want to find subspaces of P which are:

- finite dimensional
- invariant under the action of $\mathcal{SU}(2)$

Thus, we pick P_m , the space of homogeneous polynomials of degree m. For $f \in P_m$

$$f(z,w) = \sum_{j=0}^{m} a_j z^{m-j} w^j \tag{34}$$

and let

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{35}$$

we define the regular representation of $\sigma(A_{\alpha,\beta})$ on f:

$$\begin{aligned} \sigma(A_{\alpha,\beta}) f(z,w) &= f(A_{\alpha,\beta}(z,w)) \\ &= f(\overline{\alpha} z - \beta w, \overline{\beta} z + \alpha w) \end{aligned} \tag{36}$$

Theorem 2.6: P_m is invariant under $\sigma(A_{\alpha,\beta})$.

Proof 2.7: Suppose monomials $z^{m-j} w^j \in P_m$ then

$$f(\overline{\alpha} z - \beta w, \overline{\beta} z + \alpha w) = (\overline{\alpha} z - \beta w)^{m-j} (\overline{\beta} z + \alpha w)^j \in P_m. \tag{37}$$

So, P_m is invariant under σ. □

Now, we want to define an inner product, i.e.

$$\langle f, g \rangle = \int_{S^3} f \overline{g} \, d\mu \tag{38}$$

where $\mu(S^3) = 1$ is the surface measure of S^3 which is normalised to 1.

So, we'll show monomials $z^{m-j}w^j$ are orthogonal in P with respect to this inner product. To achieve this, we use polar coordinates.

Suppose $Z = (z, w) \in C^2$ and $Z = r\tilde{z}$, where $\tilde{z} \in S^3$, and $r = |z| = \sqrt{|z|^2 + |w|^2}$. We define Lebesgue measure on $\mathbb{C}^2$ as

$$\begin{aligned} dz &= dz\ dw \\ &= d_a\ d_b\ d_c\ d_d \\ &= r^3\ dr\ d\tilde{\mu}(\tilde{z})\ :\ \text{(un-normalised)} \\ &= 2\pi^2 r^3\ dr\ d\mu(\tilde{z})\ :\ \text{(normalised)} \end{aligned}. \tag{39}$$

To see this, we have $\varphi_1, \varphi_2 \in [0, \pi]$, $\theta \in [0, 2\pi)$.

$$\begin{aligned} a &= r\cos(\varphi_1) \\ b &= r\sin(\varphi_1)\sin(\varphi_2) \\ c &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\theta) \\ d &= r\sin(\varphi_1)\sin(\varphi_2)\sin(\theta)\,. \end{aligned} \tag{40}$$

The Jacobian matrix is then

$$J = \begin{bmatrix} \frac{\partial a}{\partial r} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \frac{\partial d}{\partial \theta} \end{bmatrix} \tag{41}$$

giving

$$\det(J) = r^3 \sin^2(\varphi_1)\sin(\varphi_2)\,. \tag{42}$$

The surface measure of S^3 is exactly

$$\begin{aligned} \mu(S^3) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin^2(\varphi_1)\sin(\varphi_2)\,d\varphi_1\ d\varphi_2\ d\theta(?) \\ &= 2\pi^2 \end{aligned} \tag{43}$$

Theorem 2.8: If $f : \mathbb{C}^2 \to \mathbb{C}$ satisfies $f(aZ) = a^m f(z)$ for $a > 0$, then

$$\int_{S^3} f(\tilde{z})\ d\mu(\tilde{z}) = \frac{1}{\pi^2\Gamma(\frac{m+4}{2})}\int_{\mathbb{C}^2} f(Z)e^{-|Z|^2}\ dZ. \tag{44}$$

Proof 2.9: We integrate in polar coordinates:

$$\begin{aligned} \int_{\mathbb{C}^2} f(Z)e^{-|Z|^2}\ dZ &= 2\pi^2\int_0^{\infty}\int_{S^3} f(r\tilde{z})e^{-r^3}r^3\ d\mu(\tilde{z})\ dr \\ &= 2\pi^2\int_0^{\infty} e^{-r^3}(?)r^3\int_{S^3} f(r\tilde{z})\ d\mu(\tilde{z})\ dr \\ &= 2\pi^2\frac{1}{2}\Gamma(\frac{m+4}{2})\int_{S^3} f(\tilde{z})\ d\mu(\tilde{z}) \end{aligned} \tag{45}$$

□