Groups, Analysis, and Geometry Seminars:
Harmonic Analysis of $\mathcal{SU}(2)$

Tim Gou

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1 Introduction

Suppose f is 2π-periodic, complex valued, integrable over $[0, 2\pi)$, then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\ dx \tag{1}$$

with $n \in \mathbb{Z}$, is the Fourier transform of f. Question: What is e^{inx}? Why is $n \in \mathbb{Z}$? Answer: e^{inx} is the character of circle group, denoted by $\mathbb{T}$ (i.e. $e^{ix} \in \mathbb{T}$).

A character is a continuous homomorphism from a locally compact Abelian group G to $\mathbb{T}$: $\chi : G \to \mathbb{T}$ where

$$\chi(gh) = \chi(g)\chi(h) \tag{2}$$

for $g, h \in G$. Let's work out $\chi : \mathbb{R} \to \mathbb{T}$ first, where $\mathbb{R}$ is the group $(\mathbb{R}, +)$. Since $\chi(0) = 1$ (identity to identity) and χ is continuous, then $\exists a > 0$ such that

$$\int_0^a \chi(y)\ dy. \tag{3}$$

Let $\xi = \int_0^a \chi(y)\ dy$, then

$$\chi(x)\xi = \int_0^a \chi(x+y)\ dy = \int_x^{a+x} \chi(t)\ dt \tag{4}$$

so

$$\chi(x) = \xi^{-1}\int_x^{a+x} \chi(t)\ dt \tag{5}$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1}\left(\chi(a+x) - \chi(x)\right) \\ &= \xi^{-1}\chi(x)\left(\chi(a) - 1\right) \\ &= c\chi(x). \end{aligned} \tag{6}$$

We have an ODE

$$\chi'(x) = c\chi(x) \tag{7}$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \tag{8}$$

Solving it gives us $\chi(x) = e^{cx}$. Since $|\chi| = 1$, then $c = i\lambda$ with $\lambda \in \mathbb{R}$. Thus $\chi(x) = e^{i\lambda x}$ and we have characters of $\mathbb{R}$, all the χ_λ form a dual group of $\mathbb{R}$, denoted by $\widehat{\mathbb{R}}$.

Since we identify each χ_λ with $\lambda \in \mathbb{R}$, then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \tag{9}$$

To work out $\widehat{\mathbb{T}}$, notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \tag{10}$$

i.e. each element in $[0, 2\pi)$ is a representative of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$. Suppose $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ and $x + y = 2\pi$, then

$$\chi(x+y) = \chi(0) = 1 = e^{i\lambda(x+y)} \tag{11}$$

we know $\lambda \in \mathbb{R}$, but the only way $e^{i\lambda(x+y)} = 1$ is that $\lambda \in \mathbb{Z}$. So all the $\chi_n(x) = e^{inx}$ for the dual group $\widehat{\mathbb{T}}$, and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \tag{12}$$

Similarly, we have $\mathbb{R}^n \cong \mathbb{R}^n$ and $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$.

Theorem 1.1: If G is compact, $\widehat{G}$ is discrete.

In addition, $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$, with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx}\, dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \tag{13}$$

If G is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of G:

$$\sigma : G \to U(\mathcal{H}) \tag{14}$$

for $\mathcal{H}$, some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \tag{15}$$

for $u, v \in \mathcal{H}$. We'll look at the irreducible representations of $\mathcal{SU}(2)$. $\mathcal{SU}(2)$ is the first compacy and non-abelian group we normally look at in Harmonic Analysis.

2 Aspects of $\mathcal{SU}(2)$

We begin by looking at $\mathcal{U}(2)$, a group of unitary transformations of $\mathbb{C}^2$.

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \tag{16}$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$ where elements of $\mathcal{SU}(2)$ have determinant 1.

$$\begin{cases} \det(AB) = \det(A)\det(B)\,, \;\; A, B \in \mathcal{SU}(2) \\ \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) \end{cases} \tag{17}$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \tag{18}$$

then, $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$ and $\alpha\overline{\gamma} + \beta\overline{eta} = 0$. So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \tag{19}$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\overline{\beta} \\ \overline{\alpha} \end{pmatrix}, \;\; \theta \in \mathbb{R}. \tag{20}$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\overline{\beta} & e^{i\theta}\overline{\alpha} \end{pmatrix} \in \mathcal{U}(2) \tag{21}$$

with $\det(A) = e^{i\theta}$ and $\det(A^*) = e^{-i\theta}$. If $A \in \mathcal{SU}(2)$, then

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \tag{22}$$