

# Groups, Analysis, and Geometry Seminars:

## Harmonic Analysis of $\mathcal{SU}(2)$

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### 1 Introduction

Suppose  $f$  is  $2\pi$ -periodic, complex valued, integrable over  $[0, 2\pi)$ , then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \quad (1)$$

with  $n \in \mathbb{Z}$ , is the Fourier transform of  $f$ . Question: What is  $e^{inx}$ ? Why is  $n \in \mathbb{Z}$ ? Answer:  $e^{inx}$  is the character of circle group, denoted by  $\mathbb{T}$  (i.e.  $e^{ix} \in \mathbb{T}$ ).

A character is a continuous homomorphism from a locally compact Abelian group  $G$  to  $\mathbb{T}$ :  $\chi : G \rightarrow \mathbb{T}$  where

$$\chi(gh) = \chi(g)\chi(h) \quad (2)$$

for  $g, h \in G$ . Let's work out  $\chi : \mathbb{R} \rightarrow \mathbb{T}$  first, where  $\mathbb{R}$  is the group  $(\mathbb{R}, +)$ . Since  $\chi(0) = 1$  (identity to identity) and  $\chi$  is continuous, then  $\exists a > 0$  such that

$$\int_0^a \chi(y) dy. \quad (3)$$

Let  $\xi = \int_0^a \chi(y) dy$ , then

$$\chi(x)\xi = \int_0^a \chi(x+y) dy = \int_x^{a+x} \chi(t) dt \quad (4)$$

so

$$\chi(x) = \xi^{-1} \int_x^{a+x} \chi(t) dt \quad (5)$$

and

$$\begin{aligned} \chi'(x) &= \xi^{-1} (\chi(a+x) - \chi(x)) \\ &= \xi^{-1} \chi(x) (\chi(a) - 1) \\ &= c\chi(x). \end{aligned} \quad (6)$$

We have an ODE

$$\chi'(x) = c\chi(x) \quad (7)$$

where solving the equation gives us

$$\chi(x) = e^{cx}. \quad (8)$$

Solving it gives us  $\chi(x) = e^{cx}$ . Since  $|\chi| = 1$ , then  $c = i\lambda$  with  $\lambda \in \mathbb{R}$ . Thus  $\chi(x) = e^{i\lambda x}$  and we have characters of  $\mathbb{R}$ , all the  $\chi_\lambda$  form a dual group of  $\mathbb{R}$ , denoted by  $\widehat{\mathbb{R}}$ .

Since we identify each  $\chi_\lambda$  with  $\lambda \in \mathbb{R}$ , then

$$\widehat{\mathbb{R}} \cong \mathbb{R}. \quad (9)$$

To work out  $\widehat{\mathbb{T}}$ , notice that

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \quad (10)$$

i.e. each element in  $[0, 2\pi)$  is a representative of the cosets of  $\mathbb{R}/2\pi\mathbb{Z}$ . Suppose  $x, y \in \mathbb{R}/2\pi\mathbb{Z}$  and  $x + y = 2\pi$ , then

$$\chi(x + y) = \chi(0) = 1 = e^{i\lambda(x+y)} \quad (11)$$

we know  $\lambda \in \mathbb{R}$ , but the only way  $e^{i\lambda(x+y)} = 1$  is that  $\lambda \in \mathbb{Z}$ . So all the  $\chi_n(x) = e^{inx}$  for the dual group  $\widehat{\mathbb{T}}$ , and

$$\widehat{\mathbb{T}} \cong \mathbb{Z}. \quad (12)$$

Similarly, we have  $\mathbb{R}^n \cong \mathbb{R}^n$  and  $\widehat{\mathbb{T}} \cong \mathbb{Z}^n$ .

**Theorem 1.1:** If  $G$  is compact,  $\widehat{G}$  is discrete.

In addition,  $\{e^{inx} : n \in \mathbb{Z}\}$  form an orthonormal basis for the Hilbert space  $L^2(\mathbb{T})$ , with respect to its inner product, i.e.

$$\begin{aligned} \langle e^{imx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} dx \\ &= \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{aligned} \quad (13)$$

If  $G$  is non-Abelian, the analogy generalising the characters is called the *irreducible unitary representation* of  $G$ :

$$\sigma : G \rightarrow U(\mathcal{H}) \quad (14)$$

for  $\mathcal{H}$ , some Hilbert space.

$$\begin{aligned} \sigma(gh) &= \sigma(g)\sigma(h) \\ \sigma(e) &= \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1}) = \sigma(g)\sigma(g)^* \\ \langle \sigma(g)u, \sigma(g)v \rangle &= \langle u, v \rangle \end{aligned} \quad (15)$$

for  $u, v \in \mathcal{H}$ . We'll look at the irreducible representations of  $SU(2)$ .  $SU(2)$  is the first compacy and non-abelian group we normally look at in Harmonic Analysis.

## 2 Aspects of $\mathcal{SU}(2)$

We begin by looking at  $\mathcal{U}(2)$ , a group of unitary transformations of  $\mathbb{C}^2$ .

$$AA^* = A^*A = I, A \in \mathcal{U}(2) \quad (16)$$

$\mathcal{SU}(2) \subset \mathcal{U}(2)$  where elements of  $\mathcal{SU}(2)$  have determinant 1.

$$\begin{cases} \det(AB) = \det(A) \det(B), & A, B \in \mathcal{SU}(2) \\ \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1}) \end{cases} \quad (17)$$

suppose

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathcal{U}(2) \quad (18)$$

then,  $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\eta|^2 = 1$  and  $\alpha\bar{\gamma} + \beta\bar{\eta} = 0$ . So,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \quad (19)$$

are unit vectors and orthogonal. Thus,

$$\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}, \quad \theta \in \mathbb{R}. \quad (20)$$

Hence we have

$$A = \begin{pmatrix} \alpha & \beta \\ -e^{i\theta}\bar{\beta} & e^{i\theta}\bar{\alpha} \end{pmatrix} \in \mathcal{U}(2) \quad (21)$$

with  $\det(A) = e^{i\theta}$  and  $\det(A^*) = e^{-i\theta}$ . If  $A \in \mathcal{SU}(2)$ , then

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{SU}(2). \quad (22)$$

$$\mathcal{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \quad (23)$$

where  $\alpha, \beta \in \mathbb{C}$ . If  $\alpha = a + ib$  and  $\beta = c + id$  then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 &= 1 \\ \left| \sqrt{a^2 + b^2} \right|^2 + \left| \sqrt{c^2 + d^2} \right|^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1 \end{aligned} \quad (24)$$

and so  $\mathcal{SU}(2) \cong S^3$ , the 3-sphere.  $\mathbb{T} \subset \mathcal{SU}(2)$

$$\mathbb{T} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha| = 1 \right\} \quad (25)$$

**Theorem 2.1:** Given  $A \in \mathcal{SU}(2)$ ,  $\exists V \in \mathcal{SU}(2)$ , such that

$$VAV^{-1} = T \quad (26)$$

for some  $T \in \mathbb{T}$ .

**Proof 2.2:** If  $A \in \mathcal{U}(2)$ ,  $A$  is normal, then by the *spectral theorem*,  $\exists V \in \mathcal{U}(2)$  such that

$$VAv^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad (27)$$

with  $|\alpha| = |\beta| = 1$ . □

If  $A \in \mathcal{SU}(2)$ , then  $\beta = \bar{\alpha}$ . So,  $V \in \mathcal{U}(2)$ ,  $A \in \mathcal{SU}(2)$ , we have

$$VAV^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}.$$

To make  $V$  special unitary, let  $\tilde{V} = \det(V)^{1/2} V$  such that  $\det(\tilde{V}) = 1$  and we have  $\tilde{V}A\tilde{V}^{-1} = T$ .

$\mathcal{SU}(2)$  is a compact, connected, Lie group, (?) we have a theorem to generalise this conjugation relation.

**Theorem 2.3:** If  $G$  is a compact, connected, Lie group,  $\mathbb{T}$  is its maximal torus, then for  $\chi \in G$ ,  $\exists g \in G$ , such that

$$g\chi g^{-1} = t \quad : \quad t \in \mathbb{T}. \quad (28)$$

**Theorem 2.4:** (proposition?) Define  $O_x = \{g\chi g^{-1} : g \in G\}$ , then for  $\mathcal{SU}(2)$ , every  $O_x$  intersects  $\mathbb{T}$  at exactly two points.

**Proof 2.5:** We have  $g\chi g^{-1} = t$ , let

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$\begin{aligned} (g_1 g) \chi (g_1 g)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned}$$

□

Now, we describe the irreducible representations of  $\mathcal{SU}(2)$ .  $\mathcal{SU}(2)$  naturally acts on  $\mathbb{C}^2$  ( $\sigma$  is just the identity mapping) i.e.

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \alpha z + \beta w \\ -\bar{\beta} z + \bar{\alpha} w \end{pmatrix} \quad (29)$$

We are interested in  $SU(2)$  acting on the vector spaces which are built out of  $\mathbb{C}^2$ , e.g.

$$L^2(SU(2)) \cong L^2(S^3) \quad (30)$$

made up of functions

$$f : \mathbb{C}^2 \rightarrow \mathbb{C} \quad (31)$$

and let  $P$  be the space of all polynomials with two complex variables, i.e.  $f \in P$  such that

$$f(z, w) = \sum_{j,k} a_{jk} z^j w^k \quad (32)$$

and

$$P \subset L^2(S^3) \quad (33)$$

where both spaces are infinite dimensional. But  $SU(2)$  is compact, so  $d_\sigma < \infty$ . This means we want to find subspaces of  $P$  which are:

- finite dimensional
- invariant under the action of  $SU(2)$

Thus, we pick  $P_m$ , the space of homogeneous polynomials of degree  $m$ . For  $f \in P_m$

$$f(z, w) = \sum_{j=0}^m a_j z^{m-j} w^j \quad (34)$$

and let

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad (35)$$

we define the regular representation of  $\sigma(A_{\alpha,\beta})$  on  $f$ :

$$\begin{aligned} \sigma(A_{\alpha,\beta})f(z, w) &= f(A_{\alpha,\beta}(z, w)) \\ &= f(\bar{\alpha}z - \beta w, \bar{\beta}z + \alpha w) \end{aligned} \quad (36)$$

**Theorem 2.6:**  $P_m$  is invariant under  $\sigma(A_{\alpha,\beta})$ .

**Proof 2.7:** Suppose monomials  $z^{m-j}w^j \in P_m$  then

$$f(\bar{\alpha}z - \beta w, \bar{\beta}z + \alpha w) = (\bar{\alpha}z - \beta w)^{m-j}(\bar{\beta}z + \alpha w)^j \in P_m. \quad (37)$$

So,  $P_m$  is invariant under  $\sigma$ . □

Now, we want to define an inner product, i.e.

$$\langle f, g \rangle = \int_{S^3} f \bar{g} \, d\mu \quad (38)$$

where  $\mu(S^3) = 1$  is the surface measure of  $S^3$  which is normalised to 1.

So, we'll show monomials  $z^{m-j}w^j$  are orthogonal in  $P$  with respect to this inner product. To achieve this, we use polar coordinates.

Suppose  $Z = (z, w) \in \mathbb{C}^2$  and  $Z = r\tilde{z}$ , where  $\tilde{z} \in S^3$ , and  $r = |z| = \sqrt{|z|^2 + |w|^2}$ . We define Lebesgue measure on  $\mathbb{C}^2$  as

$$\begin{aligned} dz &= dz \, dw \\ &= d_a \, d_b \, d_c \, d_d \\ &= r^3 \, dr \, d\tilde{\mu}(\tilde{z}) \quad : \quad (\text{un-normalised}) \\ &= 2\pi^2 r^3 \, dr \, d\mu(\tilde{z}) \quad : \quad (\text{normalised}) \end{aligned} \tag{39}$$

To see this, we have  $\varphi_1, \varphi_2 \in [0, \pi]$ ,  $\theta \in [0, 2\pi)$ .

$$\begin{aligned} a &= r \cos(\varphi_1) \\ b &= r \sin(\varphi_1) \sin(\varphi_2) \\ c &= r \sin(\varphi_1) \sin(\varphi_2) \cos(\theta) \\ d &= r \sin(\varphi_1) \sin(\varphi_2) \sin(\theta) . \end{aligned} \tag{40}$$

The Jacobian matrix is then

$$J = \begin{bmatrix} \frac{\partial a}{\partial r} & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & \frac{\partial d}{\partial \theta} \end{bmatrix} \tag{41}$$

giving

$$\det(J) = r^3 \sin^2(\varphi_1) \sin(\varphi_2) . \tag{42}$$

The surface measure of  $S^3$  is exactly

$$\begin{aligned} \mu(S^3) &= \int_0^{2\pi} \int_0^\pi \int_0^\pi \sin^2(\varphi_1) \sin(\varphi_2) \, d\varphi_1 \, d\varphi_2 \, d\theta(?) \\ &= 2\pi^2 \end{aligned} \tag{43}$$

**Theorem 2.8:** If  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  satisfies  $f(aZ) = a^m f(z)$  for  $a > 0$ , then

$$\int_{S^3} f(\tilde{z}) \, d\mu(\tilde{z}) = \frac{1}{\pi^2 \Gamma(\frac{m+4}{2})} \int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} \, dZ. \tag{44}$$

**Proof 2.9:** We integrate in polar coordinates:

$$\begin{aligned} \int_{\mathbb{C}^2} f(Z) e^{-|Z|^2} \, dZ &= 2\pi^2 \int_0^\infty \int_{S^3} f(r\tilde{z}) e^{-r^3} r^3 \, d\mu(\tilde{z}) \, dr \\ &= 2\pi^2 \int_0^\infty e^{-r^3} (?) r^3 \int_{S^3} f(r\tilde{z}) \, d\mu(\tilde{z}) \, dr \\ &= 2\pi^2 \frac{1}{2} \Gamma\left(\frac{m+4}{2}\right) \int_{S^3} f(\tilde{z}) \, d\mu(\tilde{z}) \end{aligned} \tag{45}$$

□

**Theorem 2.10:** If  $p, q, r, s \in \mathbb{Z}$ , then

$$\int_{S^3} z^p \bar{z}^q w^r \bar{w}^s d\mu(z, w) = \begin{cases} 0 & \text{if } p \neq q \text{ or } r \neq s \\ \frac{p!r!}{(p+r+1)!} & \text{if } p = q \text{ and } r = s \end{cases} \quad (46)$$

**Proof 2.11:** By Prop ??,

$$\int_{S^3} z^p \bar{z}^q w^r \bar{w}^s d\mu(z, w) = \frac{1}{\pi^2 \Gamma(\frac{p+q+r+s+4}{2})} \int_{\mathbb{C}} z^p \bar{z}^q e^{-|z|^2} dz \cdot \int_{\mathbb{C}} w^r \bar{w}^s dw \quad (47)$$

Let  $m = p + q + r + s$ . Since  $f(z) = 1^m f(z)$ :

$$\begin{aligned} \int_{\mathbb{C}} z^p \bar{z}^q e^{-|z|^2} dz &= \int_0^\infty \int_0^{2\pi} e^{i(p-q)\theta} r^{p+q+1} e^{-r^2} d\theta dr \\ &= \int_0^\infty r^{p+q+1} e^{-r^2} dr \cdot \int_0^{2\pi} e^{i(p-q)\theta} d\theta \end{aligned} \quad (48)$$

If  $p \neq q$ , then the integral with respect to  $\theta$  is 0, and if  $p = q$ , then

$$\int_0^{2\pi} e^{i(p-q)\theta} d\theta = 2\pi \frac{1}{2} \Gamma\left(\frac{2p+2}{2}\right) = \pi \Gamma(p+1). \quad (49)$$

Similarly for

$$\int_{\mathbb{C}} w^p \bar{w}^q e^{-|w|^2} dw = \begin{cases} 0 & \text{if } r \neq s \\ \pi r! & \text{if } r = s. \end{cases} \quad (50)$$

Also,

$$\Gamma\left(\frac{p+q+r+s+4}{2}\right) = (p+r+1)! \quad (51)$$

if  $p = q$  and  $r = s$ . □

**Theorem 2.12:** The subspaces  $P_m$  are mutually orthogonal in  $L^2(S^3)$ , and

$$\left\{ \sqrt{\frac{(m+1)!}{(m-j)!j!}} z^{m-j} w^j : 0 \leq j \leq m \right\}. \quad (52)$$

**Proof 2.13:** By the previous theorem ??. □

For each  $P_m$ , we can define the representation  $\sigma_m$  acting on it, and we can quickly access the information of  $\sigma_m$  by looking at its (trace) characters. Let  $A_{\alpha, \sigma} \in \mathbb{T}$ , then

$$\sigma_m(A_{\alpha, 0(?)}) (z^{m-j} w^j) = e^{i(m-j)\theta} z^{m-j} w^j. \quad (53)$$

Therefore, each orthogonal vector  $z^{m-j}w^j$  of  $P_m$  is an eigenvector of  $\sigma_m(A_{\alpha,0(?)})$ . This means we can write

$$\sigma_m(A_{\alpha,0}) = \begin{bmatrix} e^{im\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{i(m-2)\theta} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & e^{-i(m-2)\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-im\theta} \end{bmatrix} \quad (54)$$