37233 Linear Algebra

Problem Sheet 4 Solutions

Question 1

Here we need to reduce the augmented matrix

```
Clear[h, vh];
MatrixForm[vh = {{1, -2, h}, {0, 1, -3}, {2, 7, -5}}]
```

$$\begin{pmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 2 & 7 & -5 \end{pmatrix}$$

to get

```
MatrixForm[{{1, -2, h}, {0, 1, -3}, {0, 11, -5 - 2 h}}]
```

$$\begin{pmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 11 & -5-2h \end{pmatrix}$$

and finally

```
MatrixForm[{{1, -2, h}, {0, 1, -3}, {0, 0, 28 - 2 h}}]
```

$$\begin{pmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 28-2h \end{pmatrix}$$

So the system is consistent provided that h=14.

Question 2

$$x1 = -9\,x2 + 4\,x3$$

$$\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} -9\,x2 + 4\,x3 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} -9\,x2 \\ x2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4\,x3 \\ 0 \\ x3 \end{pmatrix} = x2 \begin{pmatrix} -9 \\ 1 \\ 0 \end{pmatrix} + x3 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

This is a plane passing through the origin and spanned by the vectors $\begin{pmatrix} -9 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$.

$x1 = -9\,x2 + 4\,x3 + 4$

$$\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} -9\,x2 + 4\,x3 + 4 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -9\,x2 \\ x2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4\,x3 \\ 0 \\ x3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x2 \begin{pmatrix} -9 \\ 1 \\ 0 \end{pmatrix} + x3 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

This is a plane spanned by the vectors $\begin{pmatrix} -9 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and shifted by a vector $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$.

Question 3

```
RowReduce[{{1, 2, 2, 1}, {2, 4, 5, 4}}] // MatrixForm
```

$$\begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The basic variables x1 and x3 and x2 is free variable

$x1 = -3 - 2\,x2;\ x3 = 2;$

$$\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} -3 - 2\,x2 \\ x2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2\,x2 \\ x2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + x2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

The particular solution is $p = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ and the general solution for $Ax = b$ is

$$w = x2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Question 4

To answer this, solve Ax = 0 :

```
MatrixForm[a = {{1, 0, 2, 0, -1}, {0, 1, 0, 0, 5}, {3, 3, 6, 1, 14}, {0, -1, 0, -2, -9}}]
```

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 3 & 3 & 6 & 1 & 14 \\ 0 & -1 & 0 & -2 & -9 \end{pmatrix}$$

```
MatrixForm[b = {0, 0, 0, 0}]
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
GaussianReduce[a, b]
```

Number of rows = 4, number of columns = 5, coefficient matrix = $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 3 & 3 & 6 & 1 & 14 \\ 0 & -1 & 0 & -2 & -9 \end{pmatrix}$, RHS = $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 3 & 3 & 6 & 1 & 14 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 3 & 3 & 6 & 1 & 14 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 3 & 0 & 1 & 17 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 3 & 0 & 1 & 17 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -4 & 0 \end{pmatrix}$

Pivot position in row 3, column 4

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot columns are: {1, 2, 4}

Row echelon form is $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Back-substitute for solution:

x_5 is a free variable

$x_4 = -2\,x_5$

x_3 is a free variable

$x_2 = -5\,x_5$

$x_1 = -2\,x_3 + x_5$

Reduced row echelon form is $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

General solution is $\mathbf{x} = \begin{pmatrix} -2\,x_3 + x_5 \\ -5\,x_5 \\ x_3 \\ -2\,x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2\,x_3 \\ 0\text{"} \\ x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \\ -5\,x_5 \\ 0 \\ -2\,x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ -5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

Question 5

Part (a)

True: choose **x** to be the vector of coefficients in the linear combination, and A to be the matrix whose columns are the corresponding vectors.

Part (b)

True: every solution is a linear combination of the columns of A summing to **b**, and vice-versa.

Part (c)

True: for the same reason.

Part (d)

True: there are no zero rows (ie redundant rows) in the RREF for A, so no possibility of inconsistency.

Part (e)

True: otherwise every sysytem A**x** = **b** (ie for every **b**) would have some solution.

Question 6

```
Clear[a]
```

```
MatrixForm[a = {{0, 0, 4}, {1, -3, -1}, {-2, 8, -5}}]
```

$$\begin{pmatrix} 0 & 0 & 4 \\ 1 & -3 & -1 \\ -2 & 8 & -5 \end{pmatrix}$$

```
MatrixForm[RowReduce[Transpose[{{0, 0, 4}, {1, -3, -1}, {-2, 8, -5}}]]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Yes, these vectors Span R3.

Question 7

```
Clear[a]
```

```
MatrixForm[a = {{1, 3, -2, b1}, {2, 0, -3, b2}, {0, 12, -2, b3}, {3, 3, 4, b4}}]
```

$$\begin{pmatrix} 1 & 3 & -2 & b1 \\ 2 & 0 & -3 & b2 \\ 0 & 12 & -2 & b3 \\ 3 & 3 & 4 & b4 \end{pmatrix}$$

Use Gaussian ellimination. No these vectors do not Span R4.

```
MatrixForm[RowReduce[Transpose[{{1, 3, -2}, {2, 0, -3}, {0, 12, -2}, {3, 3, 4}}]]]
```

$$\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Yes, these vectors Span R3.

Question 8

```
Clear[a]
```

```
MatrixForm[a = {{1, 1, 2, 2}, {2, 3, -1, 5}, {3, 4, 1, h}}]
```

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & h \end{pmatrix}$$

After Gaussian ellimination

```
MatrixForm[{{1, 1, 2, 2}, {0, 1, -5, 1}, {0, 0, 0, h - 7}}]
```

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & -7+h \end{pmatrix}$$

So h = 7. Otherwise no solution.