Linear Algebra, Assignment 6

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May 17, 2019

Question 1

Part (a)

If we define $\mathbf{x}$ as the unit square, $\mathbf{y}$ as the image of $\mathbf{x}$ and coordinates to be given as: $\begin{bmatrix} x-\text{coordinate} \\ y-\text{coordinate} \end{bmatrix}$ then the mapping is doing the following:

$$\mathbf{x} \coloneqq \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{T}(\mathbf{x}) = \mathbf{y}$$

$$\mathbf{y} \coloneqq \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

We know that a transformation of $\mathcal{T}$ can be expressed as a matrix by:

$$\mathbf{T} = \left[\mathcal{T}(\mathbf{e_1}) \mid \mathcal{T}(\mathbf{e_2}) \right]$$

Fortunately, we have these given to us in our question:

$$\mathbf{T} = \left[\mathcal{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \middle| \mathcal{T}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

Part (b)

Verifying the image of the top right corner:

$$\mathbf{T} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Which is as expected.

Please turn over for parts (c) and (d).

Part (c) and (d)

Now we are to apply the transformation to $\mathbf{Tx}$, (note that $\mathbf{Tx}$ is the image of $\mathbf{x}$) leading to:

$$\mathbf{TTx} = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{T}^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

This is a dilation, a stretching of the original unit square to three times its dimension, depicted below.

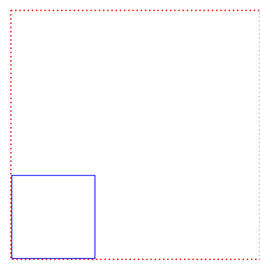

Figure 1: A dilation of the unit square by a factor of 3, using $\mathbf{T}^2$.

Question 2

Part (a)

If a 2×2 matrix has two pivots, then it spans $\mathbb{R}^2$.
Putting

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 \ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix}$$

Then:

$$\text{rref}(\mathbf{P}_{\mathcal{B}}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Showing us that $\mathbf{P}_{\mathcal{B}}$ spans $\mathbb{R}^2$.
Similarly:

$$\mathbf{P}_{\mathcal{C}} = \begin{bmatrix} \mathbf{c}_1 \ \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}$$

Again:

$$\text{rref}(\mathbf{P}_{\mathcal{C}}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ The two bases span $\mathbb{R}^2$.

Part (b)

$$\mathbf{x} = \mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \quad \Rightarrow \quad [\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}}^{-1}\ \mathbf{x}$$

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_{\mathcal{B}}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{7}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Consequently:

$$\begin{bmatrix} \frac{2}{3} & \frac{7}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Part (c)

We know that

$$\mathbf{x} = \mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

And that

$$\mathbf{x} = \mathbf{P}_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$$

Hence

$$\mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$$

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}}{}^{-1}\mathbf{P}_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$$

$$\left(\mathbf{P}_{\mathcal{B}}{}^{-1}\mathbf{P}_{\mathcal{C}}\right)[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$$

So we have in conclusion:

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$$

Where $\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ is a matrix that maps from basis $\mathcal{C}$ to basis $\mathcal{B}$.
We of course want the exact inverse of that mapping:

$$(\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}})^{-1} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \left(\mathbf{P}_{\mathcal{C}}{}^{-1}\mathbf{P}_{\mathcal{B}}\right)$$

Another thing we know is that:

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix}$$

$$\mathbf{P}_{\mathcal{C}} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}$$

Hence:

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

$$[\mathbf{x}_{\mathcal{C}}] = \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Part (d)

Again:

$$\mathbf{y} = \mathbf{P}_{\mathcal{C}}[\mathbf{y}]_{\mathcal{C}}$$

$$[\mathbf{y}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\mathbf{y}$$

$$[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Part (e)

$$\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \mathbf{P}_{\mathcal{B}}^{-1}\mathbf{P}_{\mathcal{C}}$$

$$[\mathbf{y}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Question 3

Part (a)

$$\mathbf{b}_1 = -\mathbf{c}_1 + 3\mathbf{c}_2 = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\mathbf{b}_2 = 2\mathbf{c}_1 - 4\mathbf{c}_2 = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \left[\begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \middle| \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right]$$

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2\end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} = \mathbf{P}_{\mathcal{C}} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\mathbf{P}_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\mathbf{P}_{\mathcal{B}}\,\mathbf{I} = \mathbf{P}_{\mathcal{C}} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \mathbf{I}$$

Setting $[\mathbf{x}]_{\mathcal{B}} := \mathbf{I}$ we get:

$$\mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$$

So this must mean that our mystery matrix is the change of basis matrix: $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$

$$\mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}}(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}$$

So we have a way to use one as a basis and transform between them.
Hence if $\mathcal{C}$ is a basis, then $\mathcal{B}$ must also be.

Part (b)

$$\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2 = \mathbf{P}_{\mathcal{B}} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\mathbf{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

From the previous question we have:

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

Hence:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Part (c)

$$\mathbf{y} = 3\mathbf{c}_1 - 5\mathbf{c}_2 = \mathbf{P}_{\mathcal{C}} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \mathbf{P}_{\mathcal{C}}[\mathbf{y}]_{\mathcal{C}}$$

$$[\mathbf{y}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}\mathbf{P}_{\mathcal{C}}[\mathbf{y}]_{\mathcal{C}} = (\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\mathbf{P}_{\mathcal{C}}[\mathbf{y}]_{\mathcal{C}}$$

From the previous question we have:

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

Hence:

$$[\mathbf{y}]_{\mathcal{B}} = (\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\mathbf{P}_{\mathcal{C}}[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -54 \\ -\frac{77}{2} \end{bmatrix}$$

Question 4

Part (a)

$$\mathcal{Q} = \left[\ \mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3\ \mathbf{q}_4\ \right] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{rref}(\mathcal{Q}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{Q}$ has 4 pivots, meaning it must span $\mathbb{R}^4$ and be linearly independent, meeting the criteria for a basis.

Part (b)

$$\mathbf{P}_{\mathcal{Q}\leftarrow\mathcal{H}} = \mathbf{P}_{\mathcal{Q}}^{-1}\mathbf{P}_{\mathcal{H}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & -12 \\ 1 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 & 12 \\ 0 & 2 & -4 & -12 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Part (c)

$$[\mathbf{r}]_{\mathcal{H}} = \mathbf{P}_{\mathcal{H}}^{-1}\mathbf{r}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & -12 \\ 1 & 0 & -2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{11}{4} \\ \frac{1}{2} \\ \frac{3}{8} \end{bmatrix}$$

Part (d)

$$[\mathbf{r}]_{\mathcal{Q}} = \mathbf{P}_{\mathcal{Q}\leftarrow\mathcal{H}}[\mathbf{r}]_{\mathcal{H}} = \begin{bmatrix} 1 & -2 & -2 & 12 \\ 0 & 2 & -4 & -12 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{11}{4} \\ \frac{1}{2} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$