University of Technology Sydney
Department of Mathematical and Physical Sciences

37233 Linear Algebra
Problem Set 7 – Solutions

Question 1.

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{pmatrix}.$$

Find bases for:

(a) Row (A)

Solution: After the row reduction

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Row}(\mathbf{A}): \text{ basis } = \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 9 \\ -16 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 5 \end{pmatrix} \right\}.$$

(b) Col (A)

Solution: $\text{Col}(A)$: basis $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} \right\}.$

(c) Nul (A)

Solution: $\text{Nul}(A)$: basis $= \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}.$

(d) Nul (A^T)

Solution: $\text{Nul}(A^T)$: basis $= \left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} \right\}.$

Question 2.

Let

$$A = \begin{pmatrix} 3 & 0 & -1 & 1 \\ 3 & 0 & -1 & 2 \\ 3 & 0 & 5 & 3 \end{pmatrix}.$$

Find bases for:

(a) Row (A)

Solution:

$$A = \begin{pmatrix} 3 & 0 & -1 & 1 \\ 3 & 0 & -1 & 2 \\ 3 & 0 & 5 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\text{Row}(A)\text{: basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(b) Col (A)

Solution: $\text{Col}(A)\text{: basis} = \left\{ \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$

(c) Nul (A)

Solution: $\text{Nul}(A)\text{: basis} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$

(d) Nul (A^T)

Solution: $\text{Nul}(A^T)$: no basis, since $\text{Nul}(A^T) = \{\mathbf{0}\}$.

Question 3.

Suppose a system of nine equations in ten unknowns has a solution for all possible right hand sides of the equations. Is it possible to find two non-zero solutions of the associated homogeneous system that are not multiples of one another?

Solution: A is 9×10 and $\dim \text{col}(A) = 9 \implies \dim \text{nul}(A) = 1$. Hence: no, it is not possible to find two non-zero solutions to the homogeneous system that are linearly independent.

Question 4.

A homogeneous system of twelve equations in eight unknowns has two fixed solutions that are not multiples of one another, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than 12 homogeneous equations? If so, how many?

Solution: A is 12×8 and $\dim \text{nul}(A) = 2$. Hence $\dim \text{col}(A) = 8 - 2 = 6$ and $\dim \text{row}(A) = 6$. Thus we need only 6 equations.

Question 5.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

(a) Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution:

$$V_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -1 & 5 \\ 4 & -3 \end{pmatrix}.$$

(b) Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

Solution:

$$[\mathbf{x}]_{\mathcal{C}} = V_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -1 & 5 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}.$$

Question 6.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space $\mathbb{R}^2$. Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

$$\mathbf{b}_1 = \left(\begin{array}{c} 7 \\ -2 \end{array}\right), \ \mathbf{b}_2 = \left(\begin{array}{c} 2 \\ -1 \end{array}\right), \quad \mathbf{c}_1 = \left(\begin{array}{c} 4 \\ 1 \end{array}\right), \ \mathbf{c}_2 = \left(\begin{array}{c} 5 \\ 2 \end{array}\right).$$

Solution: To find $V_{\mathcal{B}\leftarrow\mathcal{C}}$:

$$\left(\begin{array}{cc|cc} 7 & 2 & 4 & 5 \\ -2 & -1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2/7 & 4/7 & 5/7 \\ 0 & -3/7 & 15/7 & 24/7 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -8 \end{array}\right),$$

so

$$V_{\mathcal{B}\leftarrow\mathcal{C}} = \left(\begin{array}{cc} 2 & 3 \\ -5 & -8 \end{array}\right) \text{ and } V_{\mathcal{C}\leftarrow\mathcal{B}} = V_{\mathcal{B}\leftarrow\mathcal{C}}^{-1} = \left(\begin{array}{cc} 8 & 3 \\ -5 & -2 \end{array}\right).$$

Question 7.

The Legendre polynomials are a family of polynomial functions that have many applications. The first five Legendre polynomials are given by

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_2(x) &= \frac{1}{2}(-1+3x^2), \\ p_3(x) &= \frac{1}{2}(-3x+5x^3), \\ p_4(x) &= \frac{1}{8}(3-30x^2+35x^4). \end{aligned}$$

(a) Write down the coordinate vectors of each of these Legendre polynomials with respect to the standard basis in $\mathbb{P}_4$ (that is, the basis of power functions $\{1, x, x^2, x^3, x^4\}$).

Solution: Let $\mathcal{E}$ denote the standard basis of $\mathbb{P}_4$. Then

$$[p_0]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ [p_1]_{\mathcal{E}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ [p_2]_{\mathcal{E}} = \begin{pmatrix} -1/2 \\ 0 \\ 3/2 \\ 0 \\ 0 \end{pmatrix}, \ [p_3]_{\mathcal{E}} = \begin{pmatrix} 0 \\ -3/2 \\ 0 \\ 5/2 \\ 0 \end{pmatrix}, \ [p_4]_{\mathcal{E}} = \begin{pmatrix} 3/8 \\ 0 \\ -30/8 \\ 0 \\ 35/8 \end{pmatrix}.$$

(b) Use the coordinate mapping to show that the set $\mathcal{L} = \{p_0, \dots, p_4\}$ is a basis for $\mathbb{P}_4$

Solution: The matrix

$$P = \begin{pmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & -30/8 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}$$

is clearly invertible since its determinant is nonzero. Hence, by the invertible matrix theorem, its columns form a basis for $\mathbb{R}^5$. Since the coordinate mapping is an isomorphism, the corresponding set of vectors $\mathcal{L} = \{p_0, p_1, p_2, p_3, p_4\}$ in $\mathbb{P}_4$ is a basis for $\mathbb{P}_4$.

(c) Let f be the Maclaurin polynomial of degree 4 for e^x. Find $[f]_{\mathcal{L}}$. Hint: find the change of coordinates matrix from $\mathcal{E}$ to $\mathcal{L}$.

Solution: We have $P_{\mathcal{L}\leftarrow\mathcal{E}} = P^{-1}$ and the Maclaurin polynomial is $f(x) = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$, so

$$[f]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 1 \\ 1/2 \\ 1/6 \\ 1/24 \end{pmatrix}.$$

Hence

$$\begin{aligned} [f]_{\mathcal{L}} = P_{\mathcal{L}\leftarrow\mathcal{E}}[f]_{\mathcal{E}} &= P^{-1}[f]_{\mathcal{E}} \quad \text{(and can find } P^{-1} \text{ in Mathematica)} \\ &= \begin{pmatrix} 47/40 \\ 11/10 \\ 5/14 \\ 1/15 \\ 1/105 \end{pmatrix}. \end{aligned}$$