Fundamentals of linear algebra

- Brief revision of the previous lecture

- Properties of vectors in $\mathbb{R}^n$
- Linear combinations of vectors
- Spanning sets
- Homogeneous and inhomogeneous linear systems

Wednesday, 10 April 2018

Revision: iteration methods

Jacobi:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right)$$

Gauss-Seidel:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_i^{(k)}\right)$$

Relaxation (with $0 < \omega < 2$):

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right)$$

Revision: Pivoting procedures

- At each step of row reduction, numerical algorithms compare pivots in a given column and swap the rows (if necessary) to put the *row with the largest pivot first*.
- This procedure is called *elimination with partial pivoting* and this is one of the basic algorithms of numerical linear algebra.
- In contrast to the *partial pivoting* strategy, with a *full pivoting* or *complete pivoting* strategy, both rows and columns are swapped (this leads to swapping of the order of unknowns).
- Full pivoting is more reliable but slower then partial pivoting. Partial pivoting is quite adequate for many applications.

Revision: Pivoting procedures, example

Take 3 significant digits rounding, like $0.435 + 0.00132 = 0.436$.

$$\begin{bmatrix} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.0001 & 1 & 1 \\ 0 & -9999 & -9998 \end{bmatrix}$$

$$x_2 = 9998/9999 \approx 1$$

$$0.0001x_1 + x_2 = 1$$

$$\Rightarrow \quad x_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0.0001 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 9999 & 9998 \end{bmatrix}$$

$$x_2 = 9998/9999 \approx 1$$

$$x_1 + x_2 = 2$$

$$\Rightarrow \quad x_1 = 1$$

The exact solution is

$$x_1 = \frac{10000}{9999}, \qquad x_2 = \frac{9998}{9999}$$

Revision: Sensitive matrices

Consider two linear systems with the augmented matrices

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1.0001 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1.0001 & 2.0001 \end{bmatrix}$$

The solution of the first one is $x_1 = 2$ and $x_2 = 0$,
whereas the solution of the second is $x_1 = 1$ and $x_2 = 1$.

A slight change leads to a substantial change to the solution.

- Some matrices are extremely sensitive to small changes. These matrices are called *ill-conditioned matrices*.
- The determinants of such matrices are usually either very small or very large (in this example, $\det = 10^{-4}$).
- Special "balancing" (pre-conditioning) numerical techniques need to be applied to avoid numerical instabilities.
- Even balanced matrices can be sensitive for numerics.

Fundamentals of linear algebra

- Properties of vectors in $\mathbb{R}^n$
- Linear combinations of vectors
- Spanning sets
- Homogeneous and inhomogeneous linear systems

Vectors in $\mathbb{R}^n$

Numbers can be organised as scalars, vectors, matrices, as well as higher-order tensors.

Scalar a is a single number. Vector $\mathbf{a}$ ($\equiv \vec{a}$) is a list of numbers.

We consider vectors in two-, three-, ... n-dimensional space:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3, \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

For example,

$$\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$$

Vectors in $\mathbb{R}^n$

Equality: if all the corresponding components are equal

$$\mathbf{v} = \mathbf{u} \qquad \text{if} \qquad v_i = u_i \qquad \forall i = 1 \ldots n$$

Multiplication by a scalar: each component is multiplied

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad c\,\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

Addition: corresponding components are added

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

Vectors in $\mathbb{R}^2$: Examples

Equality: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

Multiplication: $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \; c = 2,$ then $c\,\mathbf{u} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}.$

Addition: $\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix},$ then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}.$

Example: $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix},$ then

$$4\mathbf{u} - 3\mathbf{v} = 4\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 3\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} - \begin{bmatrix} -9 \\ -3 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.$$

Vectors in $\mathbb{R}^2$

Consider a 2D Cartesian coordinate system.

Each point is defined by an ordered pair of numbers (a, b).

We can identify a point (a, b) with the vector

$$\begin{bmatrix} a \\ b \end{bmatrix}.$$

Figure 2 shows geometric representations of

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

For vectorial visualisation an arrow is drawn from the coordinate origin towards the point defined by the vector.

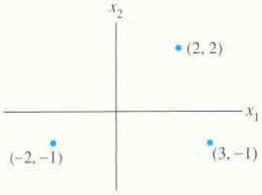


FIGURE 1
Vectors as points.

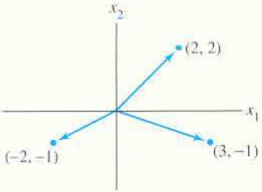


FIGURE 2
Vectors with arrows.

Vectors in $\mathbb{R}^2$

Parallelogram rule for vectorial addition

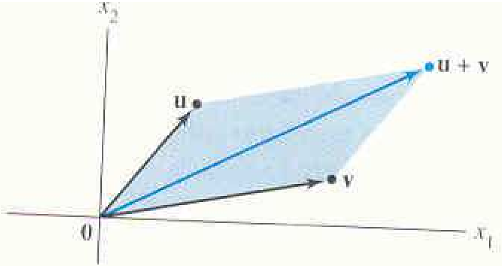


FIGURE 3 The parallelogram rule.

For vectors $\mathbf{u}$ and $\mathbf{v}$, the sum $\mathbf{u} + \mathbf{v}$ correspond to the fourth vertex of a parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$.

Vectors in $\mathbb{R}^2$

Example: geometric addition ofvectors $\mathbf{u}$ and $\mathbf{v}$ below:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, \qquad \mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

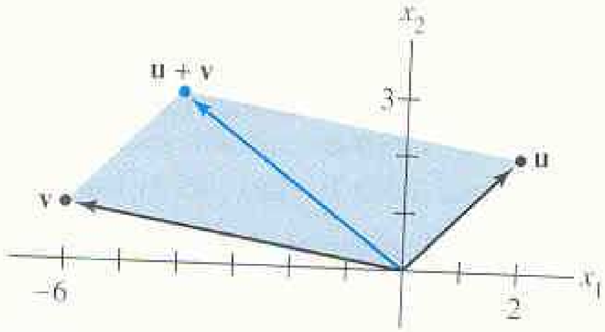
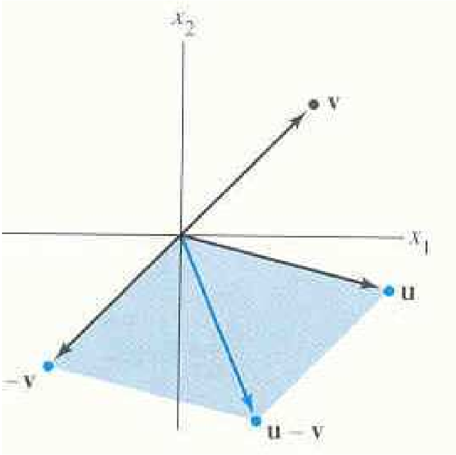


FIGURE 4

Subtraction of vectors: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$

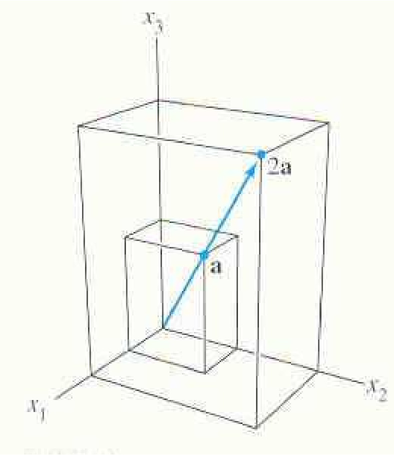


Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ have 3 entries, e.g.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$



(Figure shows $\mathbf{a} = \mathbf{u}$ and $2\mathbf{a} = \mathbf{v}$)

Vectors in $\mathbb{R}^n$

Vectors in $\mathbb{R}^n$ space have n entries:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Vector with all entries equal to 0 is called a *zero vector*, $\mathbf{0}$:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(all components are zero)}$$

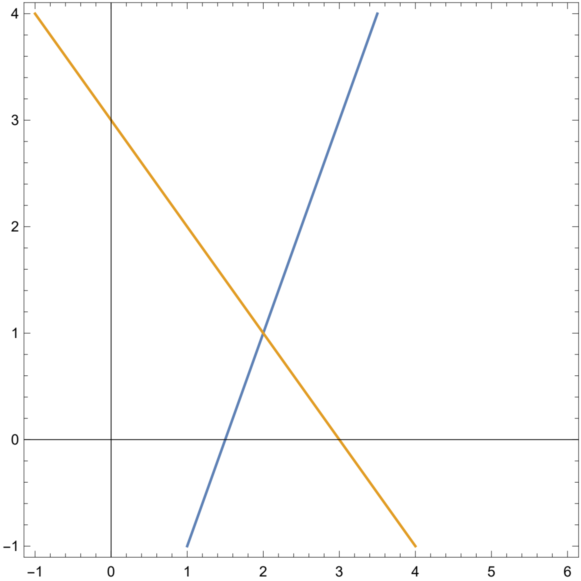
Equality of vectors, addition, scalar multiplication of vectors in $\mathbb{R}^n$ follow the same rules as for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Geometric interpretation of linear systems

Consider an example

$$\begin{cases} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{cases}$$

By plotting the above equations in $\{x_1, x_2\}$ plane, we find the unique solution of this system as:
$x_1 = 2$ and $x_2 = 1$.



Geometric interpretation of linear systems

However, we can rewrite this linear system as follows:

$$\left\{ \begin{array}{rcrcr} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{array} \right. \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Multiplication of the matrix by the vector $\mathbf{x}$ yields

$$\begin{bmatrix} 2x_1 & - & x_2 \\ x_1 & + & x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Splitting the variables on left-hand side we get

$$\begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

So the system can be rewritten in the form

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Note the first vector is the first column, and the second vector is the second column of the matrix of the linear system.

Geometric interpretation of linear systems

So, the original system was rewritten in the form

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which can be represented as $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

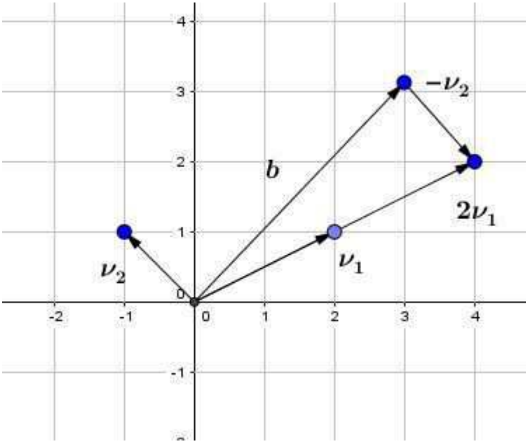
- Multiplication of a matrix by a vector from the right can be regarded as a linear combination of its columns.
- Solving a linear system is therefore equivalent to finding weights x_1 and x_2 for vectors $\mathbf{v}_1$, $\mathbf{v}_2$ to produce vector $\mathbf{b}$.
- In other words, solving $\mathbf{Ax} = \mathbf{b}$ is finding a linear combination of column vectors from $\mathbf{A}$ to produce vector $\mathbf{b}$.

Geometric interpretation of linear systems

Graphical visualisation for the unique solution $x_1 = 2$ and $x_2 = 1$.

$$\left\{ \begin{array}{ccccc} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{array} \right.$$

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$$



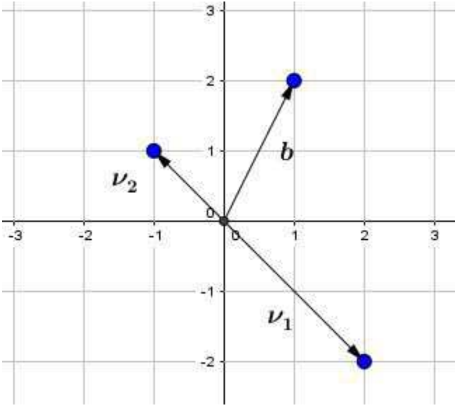
$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Geometric interpretation of linear systems

Another example:

$$\begin{cases} 2x_1 & - & x_2 & = & 1 \\ -2x_1 & + & x_2 & = & 2 \end{cases}$$

Here, no solutions.



$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = x_1\begin{bmatrix} 2 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}$$

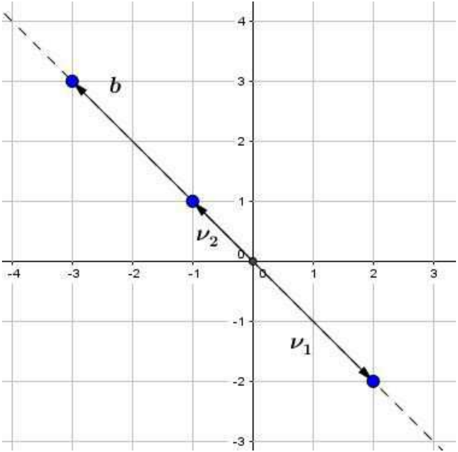
Geometric interpretation of linear systems

Another example:

$$\begin{cases} 2x_1 & - & x_2 & = & -3 \\ -2x_1 & + & x_2 & = & 3 \end{cases}$$

Infinite number of solutions

$(x_2 = 2x_1 + 3 \quad \forall x_1)$



$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = x_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + (2x_1+3) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \mathbf{b}$$

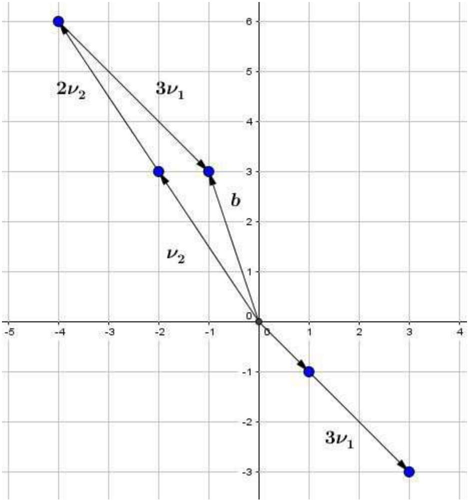
Geometric interpretation of linear systems

One more example:

$$\begin{cases} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{cases}$$

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Solution is $x_1 = 3$, $x_2 = 2$



Vector space in $\mathbb{R}^n$

- We can add vectors and multiply vectors by a scalars in $\mathbb{R}^n$. This results in a vector within the same space $\mathbb{R}^n$.

- Vector operations in $\mathbb{R}^n$ satisfiy the following conditions: For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ and any scalars c, d:

$$
\begin{array}{rl}
\text{(i)} & \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} \\
\text{(ii)} & (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) \\
\text{(iii)} & \mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u} \\
\text{(iv)} & \mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0} \\
\text{(v)} & c\,(\mathbf{u}+\mathbf{v})=c\,\mathbf{u}+c\,\mathbf{v} \\
\text{(vi)} & (c+d)\mathbf{u}=c\,\mathbf{u}+d\,\mathbf{u} \\
\text{(vii)} & c(d\mathbf{u})=(cd)\mathbf{u} \\
\text{(viii)} & 1\mathbf{u}=\mathbf{u}, \quad (-\mathbf{u})=(-1)\mathbf{u}
\end{array}
$$

- The satisfying set of vectors is called a *vector space*

Linear combinations

Definition: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n \in \mathbb{R}^n$ and given scalars $c_1, c_2, \ldots c_n$, ($c_i \in \mathbb{R}$ — any real numbers including zero)

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n$$

is called a *linear combination* of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$
w ith coefficients $\{c_1, c_2 \ldots c_n\}$.

Example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Linear combinations

Example: integer linear combinations of

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

have form $\mathbf{u} = n_1\mathbf{v}_1 + n_2\mathbf{v}_2$, where n_1 and n_2 are any integers:

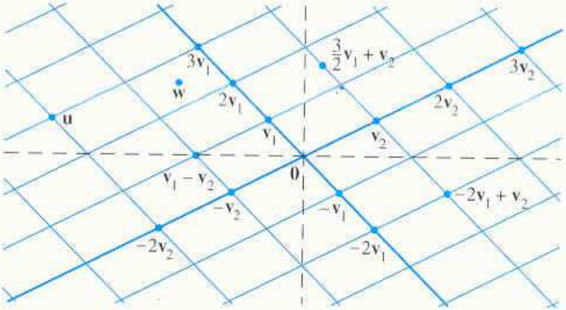


FIGURE 8 Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Linear combinations

Example:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Can $\mathbf{b}$ be represented as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$?
That is, are there scalars x_1, x_2 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

Solution: Rewrite the vector equation:

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad \Leftrightarrow$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & + & 5x_2 & = & 4 \\ -5x_1 & + & 6x_2 & = & -3 \end{array}$$

Linear combinations

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

Reduce to REF with $\mathsf{R}_2 \to \mathsf{R}_2 + 2\mathsf{R}_1$ and then $\mathsf{R}_3 \to \mathsf{R}_3 + 5\mathsf{R}_1$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is $x_1 = 3$, $x_2 = 2$.

Note: the original vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix $\left[\, \mathbf{a}_1 \; \mathbf{a}_2 \,|\, \mathbf{b} \,\right]$

Linear combinations

So, the solution was $x_1 = 3$, $x_2 = 2$, which means

$$\mathbf{b} = 3\mathbf{a}_1 + 2\mathbf{a}_2,$$

and thus $\mathbf{b}$ can be presented as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.
(vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$ form the the augmented matrix $\left[\mathbf{a}_1\ \mathbf{a}_2 \,\middle|\, \mathbf{b}\right]$)

Generally, a vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the system of linear equations represented by an augmented matrix

$$\left[\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n \,\middle|\, \mathbf{b}\right]$$

Linear combinations and span

One of the key concepts in linear algebra is a set of all vectors that are generated as a linear combination of a fixed set of vectors.

Definition: For vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p \in \mathbb{R}^n$, the set of all their linear combinations is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ *spanned* (generated) by $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$.

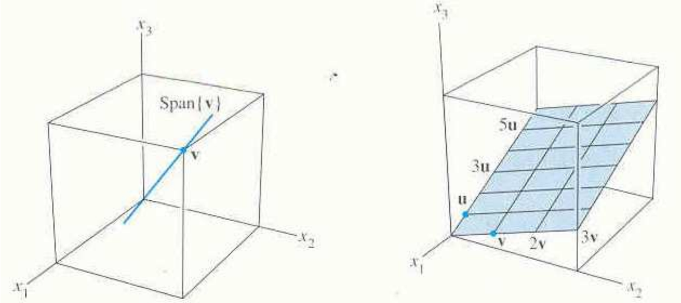
Thus, $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with arbitrary scalars $c_1, c_2, \dots, c_p$ (possibly including zeros).

Geometric interpretation of span

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$.
Then $\mathrm{Span}\{\mathbf{v}\} = c\,\mathbf{v}$ is the set of all scalar multiples of $\mathbf{v}$.
This can be visualised as the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.



If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are nonzero vectors with $\mathbf{u}$ not a multiple of $\mathbf{v}$, then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = c_1\mathbf{u} + c_2\mathbf{v}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}, \mathbf{v}, \mathbf{0}$.

Linear combinations and span

Consider vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b} \in \mathbb{R}^3$.

$\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin and vectors $\mathbf{a}_1$, $\mathbf{a}_2$.

Three equivalent statements:

- $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$
- $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$
- Equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has a solution.

Linear combinations and span

Example:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

We can check whether $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ by solving the system.

Row-reduce the augmented matrix $[\mathbf{a}_1\, \mathbf{a}_2 \,|\, \mathbf{b}]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The bottom row in the REF shows an inconsistency $0 \overset{!}{=} -2$, so there are no solutions and therefore $\mathbf{b} \notin \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$, that is, vector $\mathbf{b}$ is not a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Linear combinations and span

If $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{a}_1 \ldots \mathbf{a}_n$ and if $\mathbf{x} \in \mathbb{R}^n$ then the product $\mathbf{Ax}$ is the linear combination of the columns of $\mathbf{A}$:

$$\mathbf{Ax} = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n$$

In general, $\mathbf{Ax} = \mathbf{b}$ denotes the linear combination

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n = \mathbf{b},$$

which can be regarded as a system of linear equations for x_i.

This system can be solved by row-reducing the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{array}\right]$$

Linear combinations and span

Example:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 1 & + & 3 \cdot 2 & + & 7 \cdot (-1) \\ 4 \cdot 0 & + & 3 \cdot (-5) & + & 7 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Example: Express $-3\mathbf{v}_1 + 2\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix product:

$$-3\mathbf{v}_1 + 2\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}$$

Linear combinations and span

Comments:

$\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.

We have already considered the equivalence of this to
$\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n\}$

An equivalent question is whether $\mathbf{Ax} = \mathbf{b}$ is a consistent system.

A tricky question is whether $\mathbf{Ax} = \mathbf{b}$ has a solution for any $\mathbf{b}$.

Linear combinations and span

Example: Check if the system $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$, given

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \qquad \begin{array}{l} \mathsf{R}_2 \to \mathsf{R}_2 + 4\mathsf{R}_1 \\ \mathsf{R}_3 \to \mathsf{R}_3 + 3\mathsf{R}_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right] \qquad \mathsf{R}_3 \to \mathsf{R}_3 - \frac{1}{2}\mathsf{R}_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{1}{2}(-4b_1 - b_2) + b_3 \end{array}\right]$$

which tells that the system is not consistent for every $\mathbf{b}$.

Linear combinations and span

So, the EF of the augmented matrix upon the reduction is:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right]$$

The system is consistent if: $b_1 - (1/2)b_2 + b_3 = 0$.

So the right-hand side must have the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + (1/2)b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

i.e.

$$\mathbf{b} \in \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

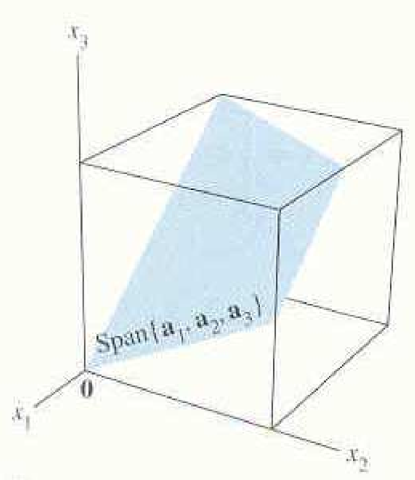
Linear combinations and span

Geometrically:

columns of $\mathbf{A}$ span the plane

$$x_1 - \frac{1}{2}x_2 + x_3 = 0$$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$



Linear combinations and span

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n \in \mathbb{R}^m$ spans $\mathbb{R}^m$

$$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$$

if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

In particular, the columns of $\mathbf{A}$ span $\mathbb{R}^m$ if every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of $\mathbf{A}$.

Theorem: For $m \times n$ matrix $\mathbf{A}$, the following statements are logically equivalent (either all true or all false for a given matrix):

- For each $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{Ax} = \mathbf{b}$ has a solution,
- The columns of $\mathbf{A}$ span $\mathbb{R}^m$
- $\mathbf{A}$ has a pivot position in every row.

Note: the latter statement refers to the coefficients matrix $\mathbf{A}$, not the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$.

Linear combinations and span

Returning to the previous example

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right]$$

note that the matrix only has two pivots and the system is consistent only if $b_1 - (1/2)b_2 + b_3 = 0$.

Any vector spanned by the columns of $\mathbf{A}$ must have the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{where} \qquad b_1 - \frac{1}{2}b_2 + b_3 = 0$$

Linear combinations and span

In particular, each column of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$$

satisfies the same constraint as $b_1 - b_2/2 + b_3 = 0$, as follows:

First column:	$1 - (-4)/2 - 3 = 0$
Second column:	$3 - 2/2 - 2 = 0$
Third column:	$4 - (-6)/2 - 7 = 0$

Linear combinations and span

Furthermore, the vector space spanned by the columns of $\mathbf{A}$ is generated by only two vectors (since there are only two pivots).

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + (1/2)b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

So any vector $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ is consistent belongs to the span of these vectors:

$$\mathbf{b} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

Linear combinations and span

For this example, the statements of the previous theorem are false:

- " For each $\mathbf{b} \in \mathbb{R}^3$, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution "

  False: System only has a solution if $b_1 - b_2/2 + b_3 = 0$.

- " The columns of $\mathbf{A}$ span $\mathbb{R}^3$ "

  False: The columns of $\mathbf{A}$ span a plane (an $\mathbb{R}^2$ area within $\mathbb{R}^3$)

- " $\mathbf{A}$ has a pivot position in every row "

  False: $\mathbf{A}$ has only 2 pivots, not 3.

The two pivots correspond to the plane spanned by the two vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Linear combinations and span

Another example: check if these vectors span $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}.$$

Applying the theorem, we will perform a reduction on $\begin{bmatrix}\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\end{bmatrix}$ and see if we get a pivot in every row.

$$\mathsf{R}_3 \to \mathsf{R}_3 + \mathsf{R}_1 \qquad\qquad \mathsf{R}_3 \to \mathsf{R}_3 - 2\mathsf{R}_2$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ -1 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

So there are three pivots and thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

Homogeneous linear systems

A linear system of equations is *homogeneous* if it has form

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

This system always has at least one solution $\mathbf{x} = \mathbf{0}$.
This solution is called a *trivial solution*.

For a given homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ an important question is whether there is a nontrivial solution $\mathbf{x} \neq \mathbf{0}$.

Example:

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Homogeneous linear systems

$$\begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So x_1 and x_2 are basic variables, and x_3 is a free variable.

$$\begin{aligned} x_1 - (4/3)x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The solution set is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

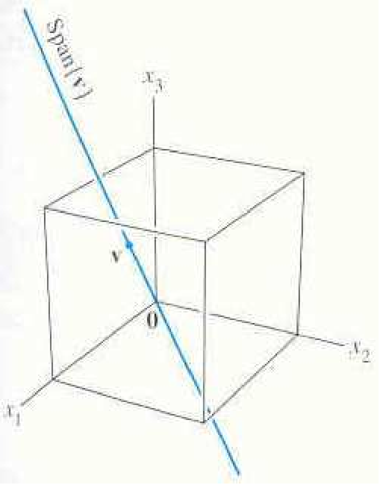
Homogeneous linear systems

So every solution is scalar multiple $t\mathbf{v}$ of a vector $\mathbf{v}$, $\forall\, t \in \mathbb{R}$.

$$\mathbf{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \equiv t\mathbf{v}$$

Geometrically, this solution set represents a line though $\mathbf{0}$ in $\mathbb{R}^3$.

A homogeneous system has a non-trivial solution if and only if there is at least one free variable.



Homogeneous linear systems

Example. Describe all the solutions of the homogeneous equation:

$$10x_1 - 3x_2 - 2x_3 = 0$$

A general solution is $x_1 = 0.3x_2 + 0.2x_3$. In a vector form, that is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

which is

$$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$$

where

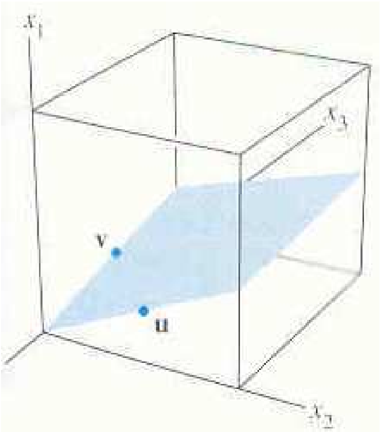
$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

Homogeneous linear systems

This solution $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$ is a *parametric equation* of a plane through the origin, defined by

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, so the solution set is $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.



For any homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, solution set can be written as $\mathrm{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ (for some vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$).

If the only solution is zero vector, then the solution set is $\mathrm{Span}\{\mathbf{0}\}$.

Inhomogeneous systems

Now consider $\mathbf{A}\mathbf{x} = \mathbf{b}$ (same $\mathbf{A}$) with the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system becomes

$$\begin{aligned} x_1 - (4/3)x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

The solution is $x_1 = (4/3)x_3 - 1$, $x_2 = 2$ and x_3 is a free variable.

Inhomogeneous systems

In a vector form, the solution can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

which is $\quad \mathbf{x} = \mathbf{p} + x_3\mathbf{v} \quad$ with $\quad \mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$

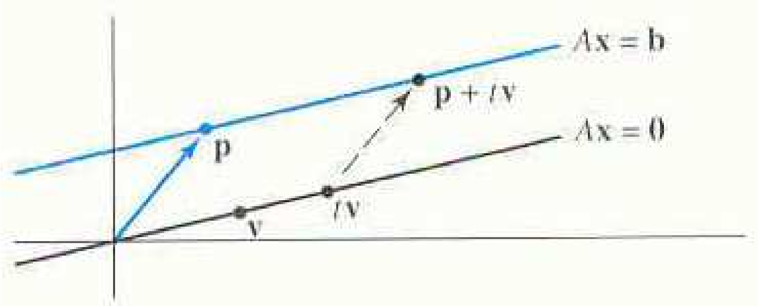
Recall the solution of the homogeneous system was $\mathbf{x} = x_3\mathbf{v}$.

Thus the solutions of inhomogeneous system $\mathbf{Ax} = \mathbf{b}$ are obtained by adding $\mathbf{p}$ to the solution of homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Vector $\mathbf{p}$ is a *particular solution* of $\mathbf{Ax} = \mathbf{b}$ for $x_3 = 0$.

Inhomogeneous systems

Visualisation of $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ in $\mathbb{R}^2$:



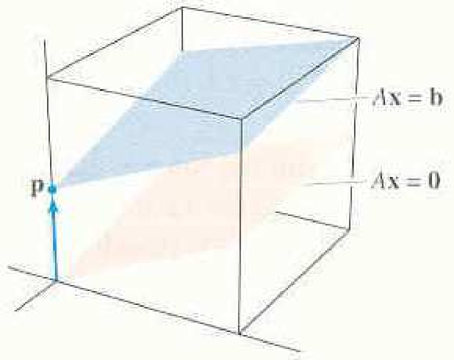
Solutions to a homogeneous system form a line through $\mathbf{0}$, and to the inhomogeneous system — a parallel line, shifted by $\mathbf{p}$.

Inhomogeneous systems

Theorem:

Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ and let $\mathbf{p}$ be a solution.

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ is a set of all vectors in the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$, where $\mathbf{v}_0$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Indeed: $\mathbf{Aw} = \mathbf{A}(\mathbf{p} + \mathbf{v}_0) = \mathbf{Ap} + \mathbf{Av}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Inhomogeneous systems

Summary:

Step 1: Row-reduce the augmented matrix to REF

Step 2: Express each basic variable (variables of the pivot columns) in terms of any free variables

Step 3: Write a typical solution $\mathbf{x}$ in the vector form

Step 4: Decompose $\mathbf{x}$ into a linear combination of vectors using the free variables as scalars.

Solutions to a homogeneous system form a sub-space through $\mathbf{0}$.

Solutions to the corresponding inhomogeneous system form a parallel sub-space, shifted by vector $\mathbf{p}$ of a particular solution.

Inhomogeneous systems

Example:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Basic variables are x_1, x_2, x_5 and free variables are x_3, x_4.

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 - 24 \\ x_2 &= 2x_3 - 2x_4 - 7 \\ x_5 &= 4 \end{aligned}$$

General solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix}$$

Inhomogeneous systems

General solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix}$$

Therefore

$$\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v} + \mathbf{p}$$

where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Inhomogeneous systems

$\mathbf{u}$ and $\mathbf{v}$ are solutions to the corresponding homogeneous system

$$\begin{array}{rcrcrcrcrl} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = 0 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 9x_5 & = 0 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = 0 \end{array}$$

Check by substituting $\mathbf{u} = (2;\ 2;\ 1;\ 0;\ 0)$:

$$\begin{array}{rcrcrcrcrl} & & 3\cdot 2 & - & 6\cdot 1 & + & 0 & + & 0 & = 0 \\ 3\cdot 2 & - & 7\cdot 2 & + & 8\cdot 1 & - & 0 & + & 0 & = 0 \\ 3\cdot 2 & - & 9\cdot 2 & + & 12\cdot 1 & - & 0 & + & 0 & = 0 \end{array}$$

Check by substituting $\mathbf{v} = (-3;\ -2;\ 0;\ 1;\ 0)$:

$$\begin{array}{rcrcrcrcrl} & & 3\cdot(-2) & - & 0 & + & 6\cdot 1 & + & 0 & = 0 \\ 3\cdot(-3) & - & 7\cdot(-2) & + & 0 & - & 5\cdot 1 & + & 0 & = 0 \\ 3\cdot(-3) & - & 9\cdot(-2) & + & 0 & - & 9\cdot 1 & + & 0 & = 0 \end{array}$$

Inhomogeneous systems

Once again: $\mathbf{u}$ and $\mathbf{v}$ are solutions to the homogeneous system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= 0 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 &= 0 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 0 \end{aligned}$$

The general solution is: $\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v}$, which is:
$\mathbf{x} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Then, $\mathbf{p}$ is a particular solution to the inhomogeneous system, which is obtained by specifying $x_3 = x_4 = 0$

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

The general solution here is: $\mathbf{x} = \mathbf{p} + x_3\mathbf{u} + x_4\mathbf{v}$.

Summary

- Properties of vectors in $\mathbb{R}^n$, and linear combinations
- Equivalence between $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mathbf{x} = \mathbf{b} \qquad \Leftrightarrow \qquad x_1\mathbf{a}_1 + \ldots x_n\mathbf{a}_n = \mathbf{b}$$

- Homogeneous $\mathbf{Ax} = \mathbf{0}$ and inhomogeneous $\mathbf{Ax} = \mathbf{b}$ systems: relation $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$ between the solutions

Next lecture

See you next Wednesday

17 April 2019

Assignment 3 is due this week

Assignment 4 is due week 7 (on 1–3 May)

No tutorial classes next week (17–19 April)