37233 Linear Algebra

Questions Sheet 5 Solutions

Question 1

To answer this, solve Ax = 0 :

```
MatrixForm[a = {{1, 0, 2, 0, -1}, {0, 1, 0, 0, 5}, {3, 3, 6, 1, 14}, {0, -1, 0, -2, -9}}]
```

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 3 & 3 & 6 & 1 & 14 \\ 0 & -1 & 0 & -2 & -9 \end{pmatrix}$$

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MatrixForm[b = {0, 0, 0, 0}]
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$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
GaussianReduce[a, b]
```

Number of rows = 4, number of columns = 5, coefficient matrix = $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 3 & 3 & 6 & 1 & 14 \\ 0 & -1 & 0 & -2 & -9 \end{pmatrix}$, RHS = $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 3 & 3 & 6 & 1 & 14 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 3 & 3 & 6 & 1 & 14 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 3 & 0 & 1 & 17 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 3 & 0 & 1 & 17 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & -2 & -9 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -4 & 0 \end{pmatrix}$

Pivot position in row 3, column 4

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot columns are: {1, 2, 4}

Row echelon form is $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Back-substitute for solution:

x_5 is a free variable

$x_4 = -2 x_5$

x_3 is a free variable

$x_2 = -5 x_5$

$x_1 = -2 x_3 + x_5$

Reduced row echelon form is $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

General solution is $\mathbf{x} = \begin{pmatrix} -2 x_3 + x_5 \\ -5 x_5 \\ x_3 \\ -2 x_5 \\ x_5 \end{pmatrix}$

A particular solution is $\mathbf{x}_p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

A basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

{{0, 0, 0, 0, 0}, {{1, -5, 0, -2, 1}, {-2, 0, 1, 0, 0}},
{{1, 0, 2, 0, -1, 0}, {0, 1, 0, 0, 5, 0}, {0, 0, 0, 1, 2, 0}, {0, 0, 0, 0, 0, 0}}}

So the columns are linearly dependent, with all linear dependence relations being completely determined by

$-2\, \mathbf{a}_1 + \mathbf{a}_3 = \mathbf{0}$ and $1\, \mathbf{a}_1 - 5\, \mathbf{a}_2 - 2\, \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{0}$.

Question 2

Assume that for some index j we have

$\mathbf{v}_j = c_1\, \mathbf{v}_1 + \ldots\, c_{j-1}\, \mathbf{v}_{j-1}$

with not all coefficients being zero. Then it follows that

$-c_1\, \mathbf{v}_1 - c_2\, \mathbf{v}_2 - \ldots - c_{j-1}\, \mathbf{v}_{j-1} + \mathbf{v}_j + 0\, \mathbf{v}_{j+1} + \ldots + 0\, \mathbf{v}_n = \mathbf{0}$

with not all coefficients being zero. Thus S is linearly dependent.

Now assume that S is linearly dependent. Then there exist some index values in the set {1, ..., n} such that the corresponding coefficients in the linear combination $\mathbf{0} = c_1'\, \mathbf{v}_1 + \ldots\, c_n'\, \mathbf{v}_n$ are nonzero. Let j be the largest such index value. Then

$c_1'\, \mathbf{v}_1 + c_2'\, \mathbf{v}_2 + \ldots + c_{j-1}'\, \mathbf{v}_{j-1} + c_j'\, \mathbf{v}_j + 0\, \mathbf{v}_{j+1} + \ldots + 0\, \mathbf{v}_n = \mathbf{0}$

Hence, solving for $\boldsymbol{v}_j$:

$$\boldsymbol{v}_j = -\frac{c_1'}{c_n'}\boldsymbol{v}_1 - \ldots - \frac{c_{j-1}'}{c_j'}\boldsymbol{v}_{j-1},$$

that is, $\boldsymbol{v}_j$ is a linear combination of the preceding elements of S. This completes the proof.

Question 3

```
MatrixForm[a = {{1, 3, -2}, {2, 0, -3}, {0, 12, -2}, {3, 3, 4}}]
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$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \\ 0 & 12 & -2 \\ 3 & 3 & 4 \end{pmatrix}$$

```
MatrixForm[RowReduce[a]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are no free variables, the columns of A are independent. However, applying this to A^T gives

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MatrixForm[RowReduce[Transpose[a]]]
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$$\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and x_3 is a free variable since the third column is a non-pivot colunm. Hence the columns of A are linearly dependent. In general, if a matrix has more columns than rows, its columns must be linearly dependent.