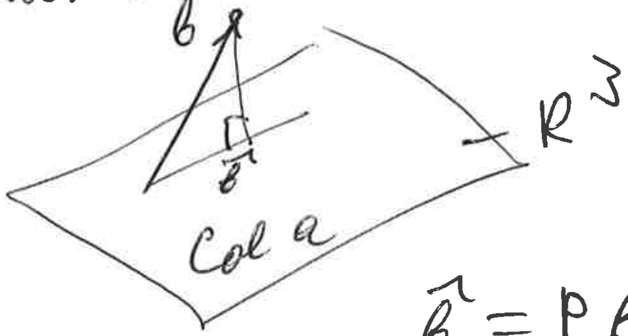
Q1 Find the projection matrix $\mathbb{P}$ spanned by $\vec{a}$

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$




$$\hat{b} = Pb$$

$P = ?$

$$\boxed{\mathbb{P} = A(A^TA)^{-1}A^T}$$

$$\mathbb{P} = a(a^Ta)^{-1}a^T$$

$a^Ta$

$$a^T = (1 \;\; 1 \;\; -1); \quad \underbrace{a^Ta}_{a \cdot a} = (1 \;\; 1 \;\; -1)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} =$$

$$= 1^2 + 1^2 + (-1)^2 = \underline{3}$$

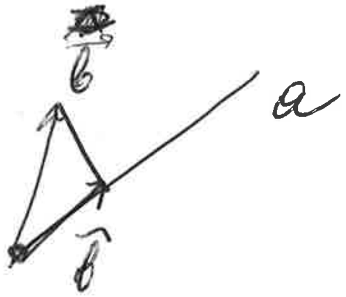
$$(a^Ta)^{-1} = \frac{1}{3}$$

$$P = a\frac{1}{3}a^T = \frac{1}{3}aa^T = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}[1 \;\; 1 \;\; -1] =$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$$

$P \nearrow$

$$\hat{b} = Pb$$




$$b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$Pb = b$$

Q2. Find the singular values of A

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^T A \rightarrow \lambda_i \rightarrow \sigma_i = \sqrt{\lambda_i}$$

$$A^T = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) = 0$$

$\lambda_1 = 3$

$\lambda_2 = 2$

$$\boxed{\begin{aligned} \sigma_1 &= \sqrt{3} \\ \sigma_2 &= \sqrt{2} \end{aligned}}$$

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$Ax = 0$

$\sigma = 0 \Rightarrow x \neq 0$

Q3 Orthogonaly diagonalize matrix A.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$A = PDP^T$ $\quad \lambda_i \to v_i$

First find eigenvalues

① $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \quad + (-0)\begin{vmatrix} 0 & 1 \\ 0 & 1-\lambda \end{vmatrix} + (-0)$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$(1-\lambda)\left[\underset{a}{(1-\lambda)}^2 - \underset{b}{1}^2\right] = 0$$

$$(1-\lambda)\left[(1-\cancel{8}\lambda-1)(1-\lambda+1)\right] = 0$$

$$(1-\lambda)(-\lambda)(-\lambda+2) = 0 \qquad \lambda(\lambda-1)(\lambda-2) = 0$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues are ⇒ $\boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{matrix}}$

$(A - \lambda_i I)x = 0$ Eigenvector for $\lambda = 0$

$\lambda = 0$ $\quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $A$ $\quad R_3 \to R_3 - R_1$

$Ax = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = 0$ $\quad x_1 + x_3 = 0$

$x_1 = -x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Normalized eigenvector $u_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{2}} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

(2) $\begin{pmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{pmatrix}$ Eigensystem for $\lambda = 1$

$\boxed{Bx = 0}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$B$

$x_1 = 0$

$x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

③ $$\underbrace{\begin{pmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 0 \\ 1 & 0 & 1-2 \end{pmatrix}}_{C} \leftarrow \text{Eigensystem for } \lambda = 2$$

$\underline{Cy = 0}$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad R_3 \to R_3 + R_1$$

$$\begin{array}{c} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \qquad\qquad x_3 \end{array}$$

$-x_2 = 0$

$-x_1 + x_3 = 0$

$x_1 = x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Normalized eigenvector for $\lambda = 2$ $\rightarrow$ $$u_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$A = PDP^T$$

$$P = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Spectral decomposition of A

$$\underline{A} = \cancel{\underbrace{\lambda_1}_{=0} u_1 u_1^T} + \underbrace{\lambda_2}_{=1} u_2 u_2^T + \underbrace{\lambda_3}_{=2} u_3 u_3^T =$$

$$= u_2 u_2^T + 2 u_3 u_3^T =$$

$$= 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

A

Q4 Given the basis B

$$B = \left\{ \underset{b_1}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{b_3}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \right\}.$$

find $\to x$ $\to$ if $[x]_B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\quad x = ?$

$$x = 1\, b_1 + (-1)\, b_2 + 2\, b_3$$

$$[d]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad E: \left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}.$$

$$d = 1 \cdot e_1 + 2e_2 + 3e_3 =$$

$$= 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x = 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

$[y]_B = ?$ Find $[y]_B$ if $y = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$[y]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ?$$

$$x_1 b_1 + x_2 b_2 + x_3 b_3 = y$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad R_2 \to R_2 + R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$x_1 + \underset{=1}{x_2} = 2 \qquad x_1 = 1$

$x_2 + x_3 = 2 \qquad x_2 + 1 = 2 \mid \Rightarrow x_2 = 1$

$\Rightarrow -2x_3 = -2 \cdot 1 \mid x_3 = 1$

$$[y]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad y = 1 \cdot b_1 + 1 \cdot b_2 + 1 \cdot b_3$$

Q5 Find the best linear fit for points $(1, 1)$, $(2, 3)$, $(3, 4)$

$$\underset{x_1}{1},\ \underset{y_1}{1} \quad \underset{x_2}{2},\ \underset{y_2}{3} \quad \underset{x_3}{3},\ \underset{y_3}{4}$$

$$\beta_0 + \beta_1 x = y$$

$$\beta_0 + \beta_1 x_1 = y_1$$

$$\beta_0 + \beta_1 = 1$$
$$\beta_0 + 2\beta_2 = 3$$
$$\beta_0 + 3\beta_3 = 4$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$Ax = b$$
$$A^T A x = A^T b$$
$$x = (A^T A)^{-1} A^T b$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 19 \end{pmatrix}$$

$$X = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \frac{1}{3 \cdot 14 - 36} \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 19 \end{pmatrix}$$

$$(A^T A)^{-1} \cdot A^T b$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$X = \frac{1}{6} \begin{pmatrix} 14 \cdot 8 - 6 \cdot 19 \\ -48 + 57 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -2 \\ 9 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 3/2 \end{pmatrix}$$

$$\beta_0 = -\frac{1}{3}, \quad \beta_1 = 3/2$$

$$y = -\frac{1}{3} + \frac{3}{2} x$$

Q6. Prove that a vector product

$T(x) = \underline{\vec{a} \times \vec{x}}$ is a linear transformation

$\vec{a} = const$ Find the matrix of this transformation

$a, x \in \mathbb{R}^3$

$$\begin{cases} T(x+y) = T(x) + T(y) \\ T(\lambda x) = \lambda T(x) \end{cases}$$

$$T(x+y) = \vec{a} \times (\vec{x} + \vec{y}) = \underbrace{\vec{a} \times \vec{x}}_{T(x)} + \underbrace{\vec{a} \times \vec{y}}_{T(y)}$$

$$T(\lambda x) = \vec{a} \times (\lambda \vec{x}) = \lambda \underbrace{\vec{a} \times x}_{T(x)} = \lambda T(x)$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$$

$$T(e_1) = \vec{a} \times \vec{e_1} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix} = 0 \cdot e_1 - e_2(-a_3) + e_3(-a_2)$$

$$= e_2 a_3 - e_3 a_2$$

$$T(e_1) = \begin{pmatrix} 0 \\ a_3 \\ -a_2 \end{pmatrix}$$

$$T(e_2) = \vec{a} \times \vec{e_2} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$= e_1(-a_3) + e_3 a_1 \qquad T(e_2) = \begin{pmatrix} -a_3 \\ 0 \\ a_1 \end{pmatrix}$$

$$T(e_3) = \vec{a} \times \vec{e_3} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 1 \end{vmatrix} = e_1 a_2 - e_2 a_1$$

$$T(e_3) = \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

$A$

$\vec{a} \times \vec{b}$ $\quad \forall \vec{b} \parallel \vec{a} \to 0$

$A b$

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$R_2 \leftrightarrow R_1$

$A[m \times n]$ $\quad R^n \to R^m$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This ~~trans~~ transformation is not onto ~~bend~~ and it is not one-to-one

Q7 Solve the linear system

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & -1 \\ 2 & -4 & -2 & 5 & 1 \\ 2 & -4 & -1 & 4 & 1 \\ -1 & 2 & 3 & 2 & 6 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 + R_1$

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$R_3 \to R_3 - 2R_2$

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & \phi \end{array}\right)$$

$R_4 \to R_4 + R_3$

$$\begin{array}{c} \left(\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 1 & -1 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{0} & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \phantom{0}\quad x_2 \end{array}$$

$x_4 = 1$

$x_3 + 2x_4 = 3$

$x_3 = 1$

$x_1 - 2x_2 - x_3 + x_4 = -1$

$x_1 - 2x_2 \cancel{-1+1} = -1$

$x_1 = -1 + 2x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1+2x_2 \\ x_2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} =$$

$Ax = 0$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 \in R$

~~Solve the linear System~~

$Ax = b$

$$A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & -4 & -2 & 5 \\ 2 & -4 & -1 & 4 \\ -1 & 2 & 3 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

$$B \sim \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row(A)

Basis for Col A $\rightarrow$ $\boxed{\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -8 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 4 \\ 2 \end{pmatrix}}$

② $Ax = 0$

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Nul } A = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

All solutions for $Ax = 0$

③ Row A

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\text{rank } A = \dim \text{Col}(A) = 3$

$\dim \mathrm{Nul}(A) = 1$

$\underline{Ax = 0}$

$x_1 \underline{a_1} + x_2 \underline{a_2} + x_3 \underline{a_3} + x_4 \underline{a_4} = 0$

(1)

$x_3 = 0$
$x_4 = 0$
$x_2$
$x_1 = 2x_2$

$2x_2 a_1 + x_2 a_2 + 0 \cdot a_3 + 0 \cdot a_4 = 0$

$x_2(2a_1 + a_2) = 0$

$$\boxed{2a_1 + a_2 = 0}$$

$\underline{a_2 = -2a_1}$

$R^4 \to R^4$

$A: R^4 \to R^4$

This transformation is not one-to-one and it is not onto

Columns of A are linearly dependent

pivot is missing and Col A does not span $R^4$