Fundamentals of linear algebra

- Linear dependence or independence
- Linear combinations of vectors
- Vector spaces and subspaces
- Basis of a vector space
- Coordinate systems

Friday, 20 April 2018

Brief revision

- Properties of vectors in $\mathbb{R}^n$, and linear combinations.
- Equivalence between $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$:

$$x_1\mathbf{a}_1 + \ldots x_n\mathbf{a}_n = \mathbf{b} \qquad \Leftrightarrow \qquad \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mathbf{x} = \mathbf{b}$$

- Homogeneous $\mathbf{A}\mathbf{x} = \mathbf{0}$ and inhomogeneous $\mathbf{A}\mathbf{x} = \mathbf{b}$ systems:

  Relation $\mathbf{w} = \mathbf{v}_0 + \mathbf{p}$ between the solutions.

Linear independence

Definitions:

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is *linearly independent* if

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

  has only the trivial solution (with all $c_i = 0$).

- A set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is *linearly dependent* if there are weights $c_1, c_2, \ldots, c_m$, not all equal to zero, such that

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

  The above relation is a linear dependence relation.

Quite obviously, a set of vectors is linearly dependent if and only if it is not linearly independent (and vice versa).

Linear independence

Example. Determine if the set $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is linearly dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To do so, we need to solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ and check if there is a nontrivial solution.

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to REF as

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear independence

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has x_1, x_2 as basic, and x_3 as a free variable.

Now we can obtain the linear dependence equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ in an explicit form, by solving

$$\left\{ \begin{array}{rcrcl} x_1 & - & 2x_3 & = & 0 \\ x_2 & + & x_3 & = & 0 \\ & & 0 & = & 0 \end{array} \right. \quad \left| \quad \begin{array}{rcl} x_1 & = & 2x_3 \\ x_2 & = & -x_3 \\ x_3 & \in & \mathbb{R} \end{array} \right.$$

Each non-zero value for x_3 yields a nontrivial solution; e.g. $x_3 = 1$:

$$2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

So the vectors $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ are linearly dependent.

Any other coefficients satisfying the above relations are suitable.

Linear independence and Ax=0

- A matrix equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ is equivalent to the vector form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

- Any linear dependence relation between the columns of $\mathbf{A}$ corresponds to a nontrivial solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$.
- So, the columns of matrix $\mathbf{A}$ are linearly independent if and only if the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- A set of only one vector $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$ (as the equation $c\,\mathbf{v} = \mathbf{0}$ has only the trivial solution for $\mathbf{v} \neq \mathbf{0}$).
- The zero vector is linearly dependent as the equation $c \cdot \mathbf{0} = \mathbf{0}$ has infinite number of non-trivial solutions.

Example.

Check if the columns of this matrix are linearly independent:

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$$

The EF indicates that there are no free variables

So the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$ has only the trivial solution, implying that vectors $\mathbf{a}_i$ are linearly independent.

In other words, each column has a pivot, so the $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution and columns of $\mathbf{A}$ are linearly independent.

Linear independence for two vectors

Example 1: $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

We can see that $\mathbf{v}_2 = 2\mathbf{v}_1$, so $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ which implies that the set of $\mathbf{v}_1$ and $\mathbf{v}_2$ is linearly dependent.

Example 2: $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Suppose there are non-zero scalars c, d such that $c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}$. Then $\mathbf{v}_1 = (-d/c)\mathbf{v}_2$, implying them to be multiples of each other. That is not the case, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an independent set.

Example 3: $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}.$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other. They span the (x_1, x_2) plane in $\mathbb{R}^3$.

Linear independence for many vectors

It is straightforward to check if two vectors are linearly dependent:

It is sufficient to find out whether they are multiples of each other.

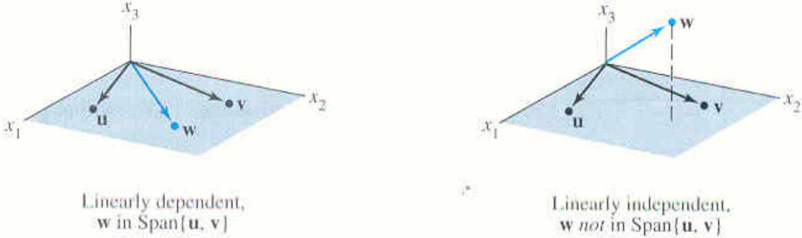

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

For many vectors, a more formal consideration is required.

Linear independence

$\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

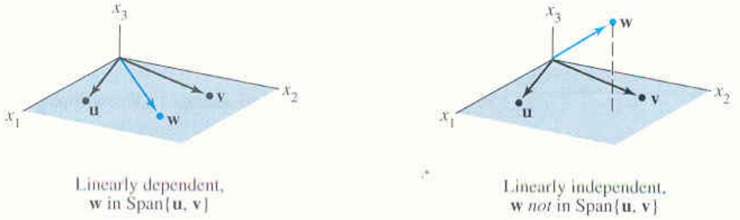

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

If $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$, which can be rewritten as $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ for non-trivial c_1, c_2, and $c_3 = -1$. Therefore, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Conversely, if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set, then $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ for some non-trivial c_1, c_2, c_3, therefore $\mathbf{w} = -(c_1/c_3)\mathbf{u} - (c_2/c_3)\mathbf{v}$, implying $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Linear independence: Example

Example: Set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v}_6\}$ given by the columns of

$$\mathbf{V} = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

We wish to determine which vectors are linearly independent.

$$\mathbf{V} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So pivots are found in columns 1, 2, 4, 6.

Therefore $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_6$ are linearly independent.

Linear independence: Example

Let us determine the linear dependence explicitly:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 & - & 2x_5 \\ 3x_3 & + & 2x_5 \\ x_3 & & \\ & & x_5 \\ & & x_5 \\ & 0 & \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Linear independence: Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 & - & 2x_5 \\ 3x_3 & + & 2x_5 \\ x_3 & & \\ & & x_5 \\ & & x_5 \\ & 0 & \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Using this solution in the form

$$\begin{cases} x_1 = -2x_3 - 2x_5 \\ x_2 = 3x_3 + 2x_5 \end{cases} \quad \text{and} \quad \begin{cases} x_4 = x_5 \\ x_6 = 0 \end{cases}$$

we can rewrite the vector equation as follows:

$$\begin{aligned} x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 + x_5\mathbf{v}_5 + x_6\mathbf{v}_6 &= \mathbf{0} \\ (-2x_3 - 2x_5)\mathbf{v}_1 + (3x_3 + 2x_5)\mathbf{v}_2 + x_3\mathbf{v}_3 + x_5\mathbf{v}_4 + x_5\mathbf{v}_5 + 0 \cdot \mathbf{v}_6 &= \mathbf{0} \\ x_3(-2\mathbf{v_1} + 3\mathbf{v}_2 + \mathbf{v}_3) + x_5(-2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5) &= \mathbf{0} \end{aligned}$$

Linear independence: Example

So: $\quad x_3(-2\mathbf{v_1} + 3\mathbf{v}_2 + \mathbf{v}_3) + x_5(-2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5) = \mathbf{0}$

This relation must be true for any x_3 and x_5, therefore

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v}_5 = 2\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4$$

This can be also seen directly from the reduced matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dependence coefficients appear in the non-pivot columns.

Thus, $\mathbf{v}_3$ depends on $\{\mathbf{v}_1, \mathbf{v}_2\}$, and $\mathbf{v}_5$ depends on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$.

Linear independence: Theorems

Theorem:

- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of two or more vectors in $\mathbb{R}^n$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.
- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.

Theorem:

- Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$. That is, if a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

Compose $\mathbf{A} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, an $n \times p$ matrix. Then $\mathbf{Ax} = \mathbf{0}$ corresponds to n equations with p unknowns. If $p > n$, there are more variables than equations so there must be free variables and non-trivial solutions, so the columns of $\mathbf{A}$ are linearly dependent.

Linear independence: Example

Example: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

must be linearly dependent because there are only two entries in each vector ($n = 2$) but there are 3 vectors ($p = 3$).

Indeed, if we compose the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, upon REF reduction the corresponding matrix is

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The basic variables are x_1, x_2 and the free variable is x_3.

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \quad \left| \quad \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \end{array} \right.$$

So the vector equation becomes $-x_3\mathbf{v}_1 + x_3\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ which is then $x_3(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$, and the explicit form of the linear dependence is: $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear independence: Summary

- If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \in \mathbb{R}^n$ contains $\mathbf{0}$, then the set is linearly dependent (suppose $\mathbf{v}_1 = \mathbf{0}$, then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_m = \mathbf{0}$ shows the linear dependence).
- A single vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.
- A set of two non-zero vectors $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one is a multiple of the other.
- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.
- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.
- Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Vector space: Definition

A *vector space* in $\mathbb{R}^n$ is a non-empty set V of vectors, on which are defined two operations, addition and multiplication by real scalars, subject to ten axioms:

(i) $(\mathbf{u}+\mathbf{v}) \in V$

(ii) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(iii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iv) $\exists\, \mathbf{0}$ such that $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$

(v) $\forall\, \mathbf{u}\ \exists\, (-\mathbf{u})$ such that $\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}$

(vi) $c\,\mathbf{u} \in V$

(vii) $c\,(\mathbf{u}+\mathbf{v}) = c\,\mathbf{u}+c\,\mathbf{v}$

(viii) $(c+d)\mathbf{u} = c\,\mathbf{u}+d\,\mathbf{u}$

(ix) $c\,(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$

(x) $1\mathbf{u} = \mathbf{u}, \quad (-\mathbf{u}) = (-1)\mathbf{u}$

These rules must hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for any $c, d \in \mathbb{R}$.

Vector spaces: Examples

1. An $\mathbb{R}^n$ space itself is a vector space.

2. A set of all arrows (directed line segments) in 2D, with

- multiplication $c\mathbf{v}$ defined to produce an arrow with the length $|c|$ times the length of $\mathbf{v}$ and pointing in the same direction as $\mathbf{v}$ for $c > 0$ or in the opposite direction for $c < 0$ (Fig. 1);
- addition defined by parallelogram rule shown in Fig. 2;
  e.g. axiom $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ is verified in Fig. 3.

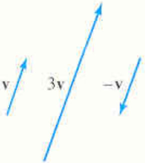

FIGURE 1

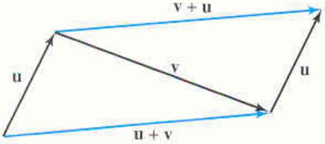

FIGURE 2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

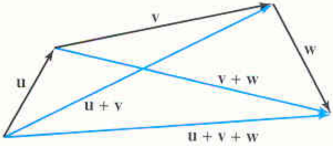

FIGURE 3 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

Vector spaces: Examples

3. Let $\mathbb{S}$ be the space of "double-infinite" sequences of numbers:

$$\mathfrak{Y} = \langle y_k \rangle = \{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$$

If $\mathfrak{Z} = \langle z_k \rangle$ is another element of $\mathbb{S}$, the sum is $\mathfrak{Y} + \mathfrak{Z} = \langle y_k + z_k \rangle$.

The scalar multiple is defined by $c \cdot \mathfrak{Y} = \langle c y_k \rangle$.

For $\mathbb{S}$, the ten axioms can be verified, so this is a vector space.

Such elements arise in engineering when a signal (such as electrical, optical or mechanical) is measured at discrete times.

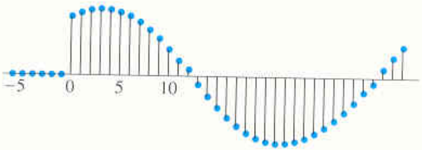

FIGURE 4 A discrete-time signal.

Vector spaces: Examples

4. For $n \geqslant 0$ let $\mathbb{P}_n$ be a set of polynomials of a degree up to n

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where variable t and the coefficients $a_0 \ldots a_n$ are real numbers.

The degree of $\mathbf{p}$ is the highest power of t with non-zero coefficient (for $\mathbf{p}(t) = a_0 \neq 0$ it is zero). The *zero polynomial* has all $a_i = 0$.

Given $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n$, the sum is defined as

$$\big[\mathbf{p} + \mathbf{q}\big](t) = \mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + \ldots + (a_n + b_n)t^n$$

The scalar multiple $c\,\mathbf{p}$ is the polynomial defined by

$$c\,\mathbf{p}(t) = ca_0 + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n$$

The axioms of a vector space are satisfied so $\mathbb{P}_n$ is a vector space.

5. Let $\mathcal{F}$ be the space of all real-valued functions defined on a number space $\mathbb{D}$.

Addition is defined as the function $\mathbf{f}+\mathbf{g}$ with the value equal to $\mathbf{f}(t)+\mathbf{g}(t)\ \forall\, t \in \mathbb{D}$.

Scalar multiplication by c is defined as the function $c\,\mathbf{f}$ with the value $c \cdot \mathbf{f}(t)\ \forall\, t \in \mathbb{D}$.

Two functions are equal if their values are equal $\forall\, t \in \mathbb{D}$.

The zero vector in $\mathcal{F}$ is $\mathbf{f}_0(t) \equiv 0\ \forall\, t \in \mathbb{D}$ and the negative of $\mathbf{f}$ is $-\mathbf{f}$.

FIGURE 5
The sum of two vectors (functions).

Subspaces

Definition: A **subspace** H of a vector space V is a subset of vectors with the following properties:

- H includes the zero vector of V.
- H is closed under vector addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\ \ \mathbf{u}+\mathbf{v} \in H$.
- H is closed under multiplication by scalars:
  $\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R},\ c\mathbf{v} \in H$.

Every subspace is a vector space and satisfies the ten axioms.

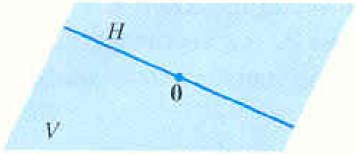

FIGURE 6

A subspace of V.

Subspaces: Examples

1. The set consisting of only the zero vector of a vector space V is a subspace of V and is called *zero subspace* $\{\mathbf{0}\}$.
2. Consider space $\mathbb{P}$ of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for real-valued functions. Then $\mathbb{P}$ is a subspace of the space $\mathcal{F}$ of all real-valued functions operating on $\mathbb{R}$, whereas a space $\mathbb{P}_n$ is the subspace of $\mathbb{P}$.
3. A line within $\mathbb{R}^2$, not passing through the origin, is not a subspace of $\mathbb{R}^2$, as it does not contain the $\mathbf{0}$ vector of $\mathbb{R}^2$.
4. A plane within $\mathbb{R}^3$, not including the origin, is not a subspace of $\mathbb{R}^3$ because this plane does not contain the $\mathbf{0}$ vector of $\mathbb{R}^3$.

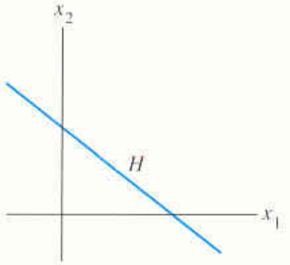

FIGURE 8
A line that is not a vector space.

Subspaces: Examples

5. The entire vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. Vectors in $\mathbb{R}^3$ have three entries whereas vectors in $\mathbb{R}^2$ have two. However, the set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}, \quad (s,t) \in \mathbb{R}$$

is a subset of $\mathbb{R}^3$ that looks like $\mathbb{R}^2$.

Indeed, this subset includes the zero vector of $\mathbb{R}^3$, and the set is closed: Any multiplication by a scalar or any addition of two vectors, produces a vector from this subset (because the third component is always zero).

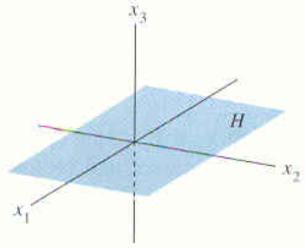

FIGURE 7
The x_1x_2-plane as a subspace of $\mathbb{R}^3$.

Subspaces: Examples

6. Consider $H = \mathrm{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$ where vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in V$.

(a) The zero vector is in H because

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

(b) Any two vectors in H can be written as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$$

therefore their sum $\mathbf{u} + \mathbf{w} \in H$ because

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \\ &= (c_1 + s_1)\mathbf{v}_1 + (c_2 + s_2)\mathbf{v}_2 \end{aligned}$$

(c) For any $c \in \mathbb{R}$ vector $c\mathbf{u} \in H$ because

$$c\mathbf{u} = c(c_1\mathbf{v}_1 + c_2\mathbf{v_2}) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v_2}$$

Thus H is a subspace of V.

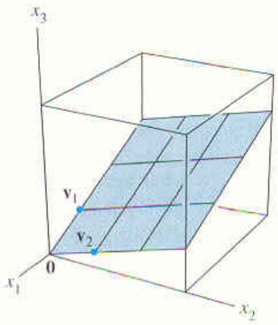

FIGURE 9
An example of a subspace.

Subspaces spanned by a set

Theorem:

For $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in V$, $\quad \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

This is called a subspace spanned (generated) by $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$.

Example 1: Let H be the set of all vectors of the form $\big[(a-3b);\ (b-a);\ a;\ b\big]$ where a, b are arbitrary scalars.

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \equiv a\mathbf{v}_1 + b\mathbf{v}_2.$$

This rearrangement demonstrates that $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Therefore, H is a subspace of $\mathbb{R}^4$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Subspaces spanned by a set: Example

Example 2: Find h such that $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$, if

$$\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: vector $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if $\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

This vector equation corresponds to the augmented matrix

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

which is only consistent if $h = 5$, so then $\mathbf{y} = [-4; 3; 5]$.

Subspaces spanned by a set: Example

Continue reduction towards REF, taking into account $h = 5$:

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is now equivalent to the system

$$\begin{cases} x_1 + 7x_3 = 1 \\ x_2 - 2x_3 = -1 \end{cases} \qquad \left| \qquad \begin{array}{l} x_1 = 1 - 7x_3 \\ x_2 = -1 + 2x_3 \end{array} \right.$$

We may choose the free variable $x_3 = 0$, so $x_1 = 1$ and $x_2 = -1$.

Thus $\mathbf{y} = 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$, which is easily checked:

$$\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

Linearly independent sets

To identify subsets that span a vector space V or its subspace H, we use linear independence defined in the same way as in $\mathbb{R}^n$.

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \in V$ is *linearly independent* if the vector equation

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

  has only the trivial solution ($c_1 = 0, \ldots, c_p = 0$).

  Conversely, the set is *linearly dependent* if there is a nontrivial solution (there are $c_1, \ldots, c_p$ not all equal to zero, such that the above vector equation holds).

  The relation $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ is a *linear dependence relation* between vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

Linearly independent sets

Theorem:

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of $p \geqslant 2$ vectors with $\mathbf{v}_1 \neq \mathbf{0}$ is linearly dependent if and only if some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Note: The main difference from the analogous definitions in the entire $\mathbb{R}^n$ space is that the linear independence relation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0$$

in general, *cannot* be written as a linear system $\mathbf{Ax} = \mathbf{0}$.

Linearly independent sets: Examples

- **Example 1:** Given $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$, this polynomial set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent in $\mathbb{P}$, because $\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$, that is, $4\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 0$ ("vectors" are linearly dependent with weights 4, -1, -1).

- **Example 2:** The set $\{\sin t, \cos t\}$ is linearly independent in space $\mathcal{C}_{[0,\, 2\pi]}$ (continuous functions on $0 \leqslant t \leqslant 2\pi$ interval).

  The relation $c_1 \sin t + c_2 \cos t = 0$ only has the trivial solution:

  There is no scalar c such that $\cos t = c \sin t \quad \forall\, t \in [0,\, 2\pi]$, so functions $\sin t$ and $\cos t$ are not multiples of each other.

- **Example 3:** The set $\{\sin t \cos t,\, \sin 2t\}$ is linearly dependent in $\mathcal{C}_{[0,\, \pi]}$, because $\sin 2t = 2 \sin t \cos t \;\; \forall\, t \in [0,\, \pi]$, and the functions are linearly dependent: $2 \sin t \cos t - \sin 2t = 0$.

  The weights in this linear dependence are 2 and -1.

Basis

Key definition: Let H be a subspace of a vector space V.

An indexed set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** in H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) $H = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$

Notes:

- The definition of a basis also applies for $H = V$, because any vector space is a subspace of itself.
- So, a basis of V is a linearly independent set that spans V.
- When $H \neq V$ condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p \in H$ because $\mathrm{Span}\,\mathcal{B}$ contains all these vectors.

Coordinate systems

- An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to impose a coordinate system on V.
- If $\mathcal{B}$ for V contains n vectors, then a coordinate system will make V behave like $\mathbb{R}^n$.

Theorem: (the *unique representation theorem*)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V.

Then $\forall\, \mathbf{x} \in V$ there exists a *unique* set of scalars $x_1, \ldots, x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n.$$

Coordinate systems

Proof: As $V = \operatorname{Span} \mathcal{B}$ there exists a set of scalars $\{c_i\}^n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

Suppose another set $\{d_i\}^n$ also satisfies $\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$.

Then we can write

$$\begin{aligned}\mathbf{0} &= \mathbf{x} - \mathbf{x} \\ &= c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n - d_1\mathbf{b}_1 - \ldots - d_n\mathbf{b}_n \\ &= (c_1 - d_1)\mathbf{b}_1 + \ldots + (c_n - d_n)\mathbf{b}_n.\end{aligned}$$

However, because $\mathcal{B}$ is a linearly independent set, the $(c_i - d_i)$ coefficients must be zero $\forall\, i$:

$$c_i = d_i \qquad 1 \leqslant i \leqslant n.$$

Therefore, representation $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ is unique.

Coordinate systems

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V, and $\mathbf{x} \in V$. The *coordinates* of $\mathbf{x}$ *relative to basis* $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\mathbf{x}$) are the coefficients $x_1, \ldots, x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n.$$

If $x_1, \ldots, x_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is the *coordinate vector* of $\mathbf{x}$ (relative to $\mathcal{B}$),
or the $\mathcal{B}$*-coordinate vector* of $\mathbf{x}$.

Mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping defined by $\mathcal{B}$.

Coordinate systems

Example 1: Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{b}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \qquad \mathbf{b}_2 = \left[\begin{array}{c} 1 \\ 2 \end{array} \right].$$

Let $\mathbf{x} \in \mathbb{R}^2$ have the following $\mathcal{B}$-coordinate vector:

$$[\mathbf{x}]_{\mathcal{B}} = \left(\begin{array}{c} -2 \\ 3 \end{array} \right).$$

The $\mathcal{B}$-coordinates of $\mathbf{x}$ directly produce $\mathbf{x}$ from the vectors of $\mathcal{B}$:

$$\mathbf{x} = -2\mathbf{b_1} + 3\mathbf{b_2} = -2 \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + 3 \left[\begin{array}{c} 1 \\ 2 \end{array} \right] = \left[\begin{array}{c} 1 \\ 6 \end{array} \right].$$

Coordinate systems: Extra examples

Example 1a: The same basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and a vector $\mathbf{x} \in \mathbb{R}^2$:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \mathbf{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

To find the $\mathcal{B}$-coordinates for a vector $\mathbf{x}$, we need to solve

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = 4 \\ x_2 = 2 \end{cases} \qquad \text{thus} \quad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

This can be easily verified: $\mathbf{x} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$

Coordinate systems: Extra examples

Example 1b: Coordinates of $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ in the basis

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

are obvious, but can be formally found from

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = 4 \\ x_2 = 5 \\ x_3 = 6 \end{cases} \quad \text{so} \quad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

Bases

Example 2: Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Solution: Check that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad c_i = 0 \ \forall\, i$$

and spans $\mathbb{R}^3$, so $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ is consistent $\forall\, \mathbf{b} \in \mathbb{R}^3$.

We therefore need to check if there are pivots in every column of

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

There are pivots in every column, therefore the homogeneous system only has the trivial solution, and the inhomogeneous system has a unique solution for every $\mathbf{b}$. Thus, $\{\mathbf{v}_i\}$ is a basis for $\mathbb{R}^3$.

Bases

Example 3: Set $\mathcal{S} = \{1,\, t,\, t^2,\, \ldots,\, t^n\}$ is a basis for $\mathbb{P}_n$.
This basis is called the *standard basis* for polynomial space $\mathbb{P}_n$.

Proof: It is obvious that any polynomial of degree at most n can be written as a combination of the members of $\mathcal{S}$.

Suppose that coefficients $c_0, \ldots, c_n$ satisfy

$$c_0 \cdot 1 + c_1 t + c_2 t^2 + \ldots + c_n t^n = \mathbf{0}(t) \qquad \forall\, t$$

However a polynomial of degree n has at most n zeros.

Therefore the above relation can only be satisfied if $c_i = 0 \;\; \forall\, i$, which means that the set $\mathcal{S}$ is linearly independent.