37233 Linear Algebra

Problem Set 3 Solutions

Question 1

Part (a)

```
x = {3, -4, 0, 3/2}
```

$\{3, -4, 0, \frac{3}{2}\}$

$|x| = |3| + |-4| + |0| + |3/2| = 17/2$

```
Norm[x, 1]
```

$\frac{17}{2}$

$|x| = \sqrt{3^2 + (-4)^2 + 0^2 + (3/2)^2} = \frac{\sqrt{109}}{2}$

```
Norm[x, 2]
```

$\frac{\sqrt{109}}{2}$

```
Norm[x, Infinity]
```

4

$|x| = \text{max of } \{|3|, |-4|, |0|, |3/2|\}$

Part (b)

```
x = {Sin[k], Cos[k], 2^k}
```

$\{\mathrm{Sin}[k], \mathrm{Cos}[k], 2^k\}$

```
Refine[Norm[x, 2], Assumptions → k > 1] // Simplify
```

$\sqrt{4^k + \mathrm{Abs}[\mathrm{Cos}[k]]^2 + \mathrm{Abs}[\mathrm{Sin}[k]]^2}$

or, more simply, $\sqrt{4^k + 1}$

```
Norm[x, Infinity]
```

$\mathrm{Max}\left[2^{\mathrm{Re}[k]}, \mathrm{Abs}[\mathrm{Cos}[k]], \mathrm{Abs}[\mathrm{Sin}[k]]\right]$

or, more simply,

```
Refine[Norm[x, Infinity], Assumptions → k > 1] // Simplify
```

2^k

Question 2

```
Limit[{1/k, 1 - E^(1 - k), (-2)/(k^2)}, k → Infinity]
```

```
{0, 1, 0}
```

Question 3

Make x_i the subject of the i' th equation :

$x_1^{(k+1)} == 1/3\, x_2^{(k)} - 1/3\, x_3^{(k)} + 1/3;$

$x_2^{(k+1)} == -1/2\, x_1^{(k)} - 1/3\, x_3^{(k)};$

$x_3^{(k+1)} == -3/7\, x_1^{(k)} - 3/7\, x_2^{(k)} + 4/7;$

To find $\boldsymbol{x}^{(1)}$ we substitute k=0 and the zero vector $\boldsymbol{x}^{(0)}=\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ into iterative system above and obtain the first iteration

$$\boldsymbol{x}^{(1)}=\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}=\begin{pmatrix} 1/3 \\ 0 \\ 4/7 \end{pmatrix}$$

To find $\boldsymbol{x}^{(2)}$ we substitute k=1 and the values of the obtained vector $\boldsymbol{x}^{(1)}=\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}=\begin{pmatrix} 1/3 \\ 0 \\ 4/7 \end{pmatrix}$

into iterative system above and obtain the second iteration

$$\boldsymbol{x}^{(2)}=\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix}=\begin{pmatrix} -4/21+1/3 \\ -1/6-4/21 \\ -3/21+4/7 \end{pmatrix}=\begin{pmatrix} 1/7 \\ -5/14 \\ 3/7 \end{pmatrix}$$

Question 4

Make x_i the subject of the i ' th equation :
$x_1^{(k+1)} == 1/10\, x_2^{(k)} + 9/10;$
$x_2^{(k+1)} == 1/10\, x_1^{(k+1)} + 2/10\, x_3^{(k)} + 7/10;$
$x_3^{(k+1)} == 2/10\, x_2^{(k+1)} + 6/10;$

To find the first iteration vector $x^{(1)}$ we substitute k = 0 and obtain the value for the first component of vector $x_1^{(1)} = 9/10$ from the the first equation above.

Now we use this value and $x_3^{(0)} = 0$ in the second equation to get $x_2^{(1)} = 1/10 * 9/10 + 7/10 = 79/100$.
Now substitute the obtained value for $x_2^{(1)}$ into the third equation to obtain $x_3^{(1)} = 758/1000$.
So the first iteration vector is

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} 9/10 \\ 79/100 \\ 758/1000 \end{pmatrix}.$$

To find $x^{(2)}$ we substitute k = 1 and the value for $x_2^{(1)}$ into the first equation to get
$x_1^{(2)} == 1/10\, x_2^{(1)} + 9/10 = 1/10 * 79/100 + 9/10 = 979/1000.$

From the second equation we obtain $x_2^{(2)} == 1/10\, x_1^{(2)} + 2/10\, x_3^{(1)} + 7/10 = 979/10\,000 + 2/10 * 758/1000 + 7/10 = 1899/2000.$
From the third equation we obtain $x_3^{(2)} == 2/10\, x_2^{(2)} + 6/10 = 2/10 * 1899/2000 + 6/10 = 7899/10\,000.$

So the second iteration vector is $x^{(2)} =$

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} -4/21 + 1/3 \\ -1/6 - 4/21 \\ -3/21 + 4/7 \end{pmatrix} = \begin{pmatrix} 979/1000 \\ 1899/2000 \\ 7899/10\,000 \end{pmatrix}$$

Question 5

The matrix

```
MatrixForm[a = {{3, 1, -1}, {0, 2, 1}, {1, 2, -4}}]
```

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 2 & -4 \end{pmatrix}$$

is strictly diagonally dominant, so Jacobi’s method will converge.

Question 6

```
Clear[x1, x2, x3];
```

Equations are :

```
3 x1 - x2 + x3 == 1;
```

```
3 x1 + 6 x2 + 2 x3 == 0;
```

```
3 x1 + 3 x2 + 7 x3 == 4;
```

Make x_i the subject of the i' th equation :

```
x1 == 1/3 x2 - 1/3 x3 + 1/3;
```

```
x2 == -1/2 x1 - 1/3 x3;
```

```
x3 == -3/7 x1 - 3/7 x2 + 4 / 7;
```

Express the corresponding iteration equations in matrix form:

```
Clear[x1, x2, x3];
MatrixForm[{x1, x2, x3}]^"(k+1)" ==
 MatrixForm[{{0, 1/3, -1/3}, {-1/2, 0, -1/3}, {-3/7, -3/7, 0}}]
   MatrixForm[{x1, x2, x3}]^"(k)" + MatrixForm[{1/3, 0, 4 / 7}]
```

$$\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}^{\text{"(k+1)"}} == \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{3} \\ -\frac{3}{7} & -\frac{3}{7} & 0 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}^{\text{"(k)"}}$$

Question 7

We have

$$x_1{}^{(k+1)} = -\frac{1}{a_{11}}\left(0 + a_{12}\, x_2{}^{(k)} + a_{13}\, x_3{}^{(k)} + \ldots + +a_{1\,n}\, x_n{}^{(k)}\right) + \frac{b_1}{a_{11}}$$

$$x_2{}^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}\, x_1{}^{(k+1)} + 0 + a_{23}\, x_3{}^{(k)} + \ldots + +a_{2\,n}\, x_n{}^{(k)}\right) + \frac{b_2}{a_{22}}$$

$$\ldots$$

$$x_n{}^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}\, x_1{}^{(k+1)} + a_{n2}\, x_3{}^{(k+1)} + \ldots + +a_{n,n-1}\, x_{n-1}{}^{(k+1)} + 0\right) + \frac{b_n}{a_{nn}}$$

or

$$\boldsymbol{x}^{(k+1)} = \begin{pmatrix} 0 - \frac{a_{12}}{a_{11}} x_2{}^{(k)} - \frac{a_{13}}{a_{11}} x_3{}^{(k)} - \ldots - \frac{a_{1n}}{a_{11}} x_n{}^{(k)} \\ -\frac{a_{21}}{a_{22}} x_1{}^{(k+1)} + 0 - \frac{a_{23}}{a_{22}} x_3{}^{(k)} - \ldots - \frac{a_{2n}}{a_{22}} x_n{}^{(k)} \\ \ldots \\ -\frac{a_{n1}}{a_{nn}} x_1{}^{(k+1)} - \frac{a_{n2}}{a_{nn}} x_2{}^{(k+1)} - \ldots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}{}^{(k+1)} + 0 \end{pmatrix} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \ldots \\ \frac{b_n}{a_{nn}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 + 0 + 0 + \ldots + 0 \\ -\frac{a_{21}}{a_{22}} x_1{}^{(k+1)} + 0 + 0 + \ldots + 0 \\ \ldots \\ -\frac{a_{n1}}{a_{nn}} x_1{}^{(k+1)} - \frac{a_{n2}}{a_{nn}} x_2{}^{(k+1)} - \ldots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}{}^{(k+1)} + 0 \end{pmatrix} + \begin{pmatrix} 0 - \frac{a_{12}}{a_{11}} x_2{}^{(k)} - \frac{a_{13}}{a_{11}} x_3{}^{(k)} - \ldots - \frac{a_{1n}}{a_{11}} x_n{}^{(k)} \\ 0 + 0 - \frac{a_{23}}{a_{22}} x_3{}^{(k)} - \ldots - \frac{a_{2n}}{a_{22}} x_n{}^{(k)} \\ \ldots \\ 0 + 0 + 0 + \ldots + 0 \end{pmatrix} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \ldots \\ \frac{b_n}{a_{nn}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ -\frac{a_{21}}{a_{22}} & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \ldots & -\frac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix} \begin{pmatrix} x_1{}^{(k+1)} \\ x_2{}^{(k+1)} \\ \ldots \\ x_n{}^{(k+1)} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & \ldots & -\frac{a_{1,n-1}}{a_{11}} & -\frac{a_{1n}}{a_{11}} \\ 0 & 0 & \ldots & -\frac{a_{2,n-1}}{a_{22}} & -\frac{a_{2n}}{a_{22}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1{}^{(k)} \\ x_2{}^{(k)} \\ \ldots \\ x_n{}^{(k)} \end{pmatrix} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \ldots \\ \frac{b_n}{a_{nn}} \end{pmatrix}$$

$= L\boldsymbol{x}^{(k+1)} + U\boldsymbol{x}^{(k)} + \hat{\boldsymbol{c}}$, say.

Now,

$\boldsymbol{x}^{(k+1)} = L\boldsymbol{x}^{(k+1)} + U\boldsymbol{x}^{(k)} + \hat{\boldsymbol{c}}$

$\Rightarrow (I\text{-}L)\boldsymbol{x}^{(k+1)} = U\boldsymbol{x}^{(k)} + \hat{\boldsymbol{c}}$

$\Rightarrow \boldsymbol{x}^{(k+1)} = \Rightarrow (I\text{-}L)^{-1}U\boldsymbol{x}^{(k)} + (I\text{-}L)^{-1}\hat{\boldsymbol{c}}$,

and so we can choose $T = (I\text{-}L)^{-1}\, U$ and $\mathbf{c} = (I\text{-}L)^{-1}\hat{\boldsymbol{c}}$.

For the case of Q4 we have

$$x_1^{(k+1)} == 1/10\, x_2^{(k)} + 9/10;$$

$$x_2^{(k+1)} == 1/10 * x_1^{(k+1)} + 1/5 * x_3^{(k)} + 7/10;$$

$$x_3^{(k+1)} == 1/5\, x_2^{(k+1)} + 3/5;$$

We rewrite above in the matrix form

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1/10 & 0 & 0 \\ 0 & 1/5 & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{pmatrix} + \begin{pmatrix} 0 & 1/10 & 0 \\ 0 & 0 & 1/5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix} + \begin{pmatrix} 9/10 \\ 7/10 \\ 3/5 \end{pmatrix}$$

So

$$L = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

$$c = \begin{pmatrix} 9/10 \\ 7/10 \\ 3/5 \end{pmatrix}$$

and

$$x^{(k+1)} = Lx^{(k+1)} + Ux^{(k)} + c$$

The rest of the derrivation is as above

$T = (I - L)^{-1} U$ and $c1 = (I - L)^{-1} c$ and

$$x^{(k+1)} = Tx^{(k)} + c\,1$$

Question 8

Partial pivoting reduction steps are :

```
MatrixForm[aug0 = {{1/3, 1/2, 1/4, -1}, {1/4, 1/3, 1/5, 0}, {1/2, 1, 1/3, 2}}]
```

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & -1 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{2} & 1 & \frac{1}{3} & 2 \end{pmatrix}$$

```
MatrixForm[aug1 = {{1/2, 1, 1/3, 2}, {1/4, 1/3, 1/5, 0}, {1/3, 1/2, 1/4, -1}}]
```

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{3} & 2 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & -1 \end{pmatrix}$$

```
MatrixForm[aug2 = {{1/2, 1, 1/3, 2}, {0, -1/6, 1/30, -1}, {0, -1/6, 1/36, -7/3}}]
```

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{3} & 2 \\ 0 & -\frac{1}{6} & \frac{1}{30} & -1 \\ 0 & -\frac{1}{6} & \frac{1}{36} & -\frac{7}{3} \end{pmatrix}$$

```
MatrixForm[aug3 = {{1/2, 1, 1/3, 2}, {0, -1/6, 1/30, -1}, {0, 0, -1/180, -4/3}}]
```

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{3} & 2 \\ 0 & -\frac{1}{6} & \frac{1}{30} & -1 \\ 0 & 0 & -\frac{1}{180} & -\frac{4}{3} \end{pmatrix}$$

Back substitute:

$x_3 = -180 * (-4/3)$

240

$x_2 = (-6 * (-1 - 1/30 * x_3))$

54

$x_1 = 2\,(2 - x_2 - 1/3 * x_3)$

-264