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37233 Linear Algebra
Problem Set 5 – Solutions Part (b)

Note: you may use *Mathematica* to carry out any calculations you feel may be of use.

4. The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways of expressing $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -2 & -2 & 4 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right)$$
$$\implies \mathbf{c} = \begin{pmatrix} 5c_3 + 5 \\ -c_3 - 2 \\ c_3 \end{pmatrix} = c_3 \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}.$$

Choosing $c_3 = 0$ gives $c_1 = 5$, $c_2 = -2$, $c_3 = 0$ and

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -8 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 7 \end{pmatrix}.$$

Choosing $c_3 = 1$ gives $c_1 = 10$, $c_2 = -3$, $c_3 = 1$ and

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 10 \begin{pmatrix} 1 \\ -3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -8 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 7 \end{pmatrix}.$$

5. Let $\mathbb{S}$ be the set of doubly infinite sequences $\mathbb{S} = \{\dots, y_{-1}, y_0, y_1, \dots\}$. Prove that $\mathbb{S}$ is a vector space.

Solution: Verify the axioms. Let $u, v \in \mathbb{S}$. Then, for example:

(i) $u = (\dots, u_{-1}, u_0, u_1, \dots)$, $v = (\dots, v_{-1}, v_0, v_1, \dots) \implies u + v = (\dots, u_{-1} + v_{-1}, u_0 + v_0, u_1 + v_1, \dots) \in \mathbb{S}$, so $\mathbb{S}$ is closed under addition.

(ii) We have

$$\begin{aligned} u + v &= (\dots, u_{-1} + v_{-1}, u_0 + v_0, u_1 + v_1, \dots) \\ &= (\dots, v_{-1} + u_{-1}, v_0 + u_0, v_1 + u_1, \dots) \text{ (by commutativity of real numbers)} \\ &= v + u, \end{aligned}$$

so $\mathbb{S}$ is commutative under addition of sequences.

The other axioms are similarly verified by expressing each property in terms of the operations on its components, and using the properties of the component-level operations to verify the corresponding property of the sequence-level operation.

6. Prove that

(i) $\mathbb{P}_n$ is a vector space;

(ii) $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

Solution:

(i) As for question 2, with polynomials replacing sequences and checking that degrees of $(p+q)$ and (αp) (where α is a real constant) are at most n. (Why?)

(ii) We have to show that $0 \in \mathbb{P}_n$ and that $\mathbb{P}_n$ is closed under addition and scalar multiplication. Let $n \geq 0$. Then:

(*Zero*) Certainly $z(t) \equiv 0$ (that is, the polynomial $z(t) = 0 \forall\ t$) is a polynomial of degree not greater than n and for any $p \in \mathbb{P}_n$, $(z+p)(t) = p(t)\ \forall\ t$, so $z \in \mathbb{P}_n$ is the zero element of $\mathbb{P}_n$.

(*Closure under addition*) We need to show that the degree of a sum of polynomials is less than or equal to n. Let $p, q \in \mathbb{P}_n$, with

$$\begin{aligned} p(t) &= p_0 + p_1 t + \cdots + p_n t^n, \\ q(t) &= q_0 + q_1 t + \cdots + q_n t^n. \end{aligned}$$

Then

$$\begin{aligned} (p+q)(t) &= p(t) + q(t) \\ &= (p_0 + p_1 t + \cdots + p_n t^n) + (q_0 + q_1 t + \cdots + q_n t^n) \\ &= (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n \end{aligned}$$

which is a polynomial of degree at most n.

(*Closure under scalar multiplication*) is established similarly.

7. Hermite polynomials arise in the study of certain differential equations in mathematical physics. The first four of these polynomials are 1, $2t$, $4t^2 - 2$, and $8t^3 - 12t$. Show that the polynomials form a basis of $\mathbb{P}_3$.

Solution: Let

$$\begin{aligned} H_0(t) &= 1, \\ H_1(t) &= 2t, \\ H_2(t) &= -2 + 4t^2, \\ H_3(t) &= -12t + 8t^3. \end{aligned}$$

To establish linear independence we must show that

$$c_0 H_0(t) + c_1 H_1(t) + c_2 H_2(t) + c_3 H_3(t) \equiv 0$$

has the unique solution (for c_0, c_1, c_2, c_3): $c_0 = c_1 = c_2 = c_3 = 0$. Now, denoting the linear combination by $Z(t)$:

$$\begin{aligned} Z(t) &= c_0 H_0(t) + c_1 H_1(t) + c_2 H_2(t) + c_3 H_3(t) \\ &= c_0(1) + c_1(2t) + c_2(-2 + 4t^2) + c_3(-12t + 8t^3) \\ &= (c_0 - 2c_2) + (2c_1 - 12c_3)t + (4c_2)t^2 + (8c_3)t^3 \\ &= 0\ \forall t \end{aligned}$$

if and only if

$$\begin{aligned} 8c_3 &= 0, \\ 4c_2 &= 0, \\ 2c_1 - 12c_3 &= 0, \\ c_0 - 2c_2 &= 0. \end{aligned}$$

Clearly these equations have the unique solution $c_0 = c_2 = c_2 = c_3 = 0$, so the polynomials are linearly independent.

To show that they span $\mathbb{P}_3$, let p be an arbitrary element of $\mathbb{P}_3$:

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 \tag{1}$$

for some $p_0, p_1, p_2, p_3 \in \mathbb{R}$. We must show that there exist constants $d_0, d_1, d_2, d_3 \in \mathbb{R}$ such that

$$\begin{aligned} p(t) &= d_0 H_0(t) + d_1 H_1(t) + d_2 H_2(t) + d_3 H_3(t) \\ &= d_0(1) + d_1(2t) + d_2(-2 + 4t^2) + d_3(-12t + 8t^3) \qquad (2) \\ &= (d_0 - 2d_2) + (2d_1 - 12d_3)t + (4d_2)t^2 + (8d_3)t^3. \end{aligned}$$

Such constants will exist if and only if (equating the coefficients of like power functions in (1) and (2)) the following system of equations is consistent:

$$\begin{array}{ccccccc} d_0 & & -2d_2 & & = & p_0 \\ & 2d_1 & & -12d_3 & = & p_1 \\ & & 4d_2 & & = & p_2 \\ & & & 8d_3 & = & p_3 \end{array}$$

or, in matrix form,

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \mathbf{d} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Clearly the matrix is invertible (it is already in row echelon form with no non-pivot columns, and square) and so the system is consistent for all p_0, p_1, p_2, p_3. Thus $\{H_0, H_1, H_2, H_3\}$ spans $\mathbb{P}_3$ and, since it is linearly independent, is a basis for $\mathbb{P}_3$.

8. Given the basis, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \right\}$, find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{pmatrix} 8 \\ -9 \\ 6 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

so $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$.

9. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

Solution: Let B be the matrix $(\mathbf{b}_1 \ \ldots \ \mathbf{b}_n)$. Then the coordinate vector $[\mathbf{b}_k]_{\mathcal{B}}$ of $\mathbf{b}_k$ (relative to the basis $\mathcal{B}$) is the vector satisfying

$$B[\mathbf{b}_k]_{\mathcal{B}} = \mathbf{b}_k.$$

Given that the left hand side is the matrix-vector product which effectively selects the kth column of the matrix (*ie* $\mathbf{b}_k$), the vector $[\mathbf{b}_k]_{\mathcal{B}}$ must have a 0 for every entry other than the kth, and a 1 for the kth entry. That is,

$$[\mathbf{b}_k]_{\mathcal{B}} = \mathbf{e}_k.$$

10. Let V be a vector space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. Prove that Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Solution: We must show that $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ contains $\mathbf{0}$ ad is closed under addition and scalar multiplication.

(i) (*Zero*): Certainly $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n \ \in \text{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$.

(ii) (*Closure under addition*): Let $\mathbf{u}, \mathbf{w} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then there exist $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ in $\mathbb{R}$ such that

$$\begin{aligned} \mathbf{u} &= c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \\ \mathbf{w} &= d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) + (d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n) \\ &= (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_n + d_n)\mathbf{v}_n \\ &\in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}. \end{aligned}$$

(iii) (*Closure under scalar multipication*): Let $\mathbf{u}$ be as above and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \alpha\mathbf{u} &= \alpha(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) \\ &= (\alpha c_1)\mathbf{v}_1 + \cdots + (\alpha c_n)\mathbf{v}_n \\ &\in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}. \end{aligned}$$

This completes the proof.