37233 Linear Algebra

Problem Sheet 10 Solutions

Question 1

Part (a)

$\boldsymbol{x}^T A \boldsymbol{x} = 4 x_1^2 + 6 x_1 x_2 + 2 x_2^2 + 2 x_2 x_3 + x_3^2$

Part (b)

When **x**=(2,-1,5), $\boldsymbol{x}^T A\boldsymbol{x} = 21$.

Question 2

Part (a)

```
a = MatrixForm[{{8, -3, 2}, {-3, 7, -1}, {2, -1, -3}}]
```

$$\begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

Part (b)

```
a = MatrixForm[{{0, 2, 3}, {2, 0, -4}, {3, -4, 0}}]
```

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{pmatrix}$$

Question 3

```
MatrixForm[a = {{3, -2}, {-2, 6}}]
```

$$\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$$

```
Eigenvalues[a]
```

{7, 2}

So, positive definite. Change of variables matrix:

```
p1 = Eigenvectors[a][[1]] / Norm[Eigenvectors[a][[1]]]
```

$\left\{-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\}$

```
p2 = Eigenvectors[a][[2]] / Norm[Eigenvectors[a][[2]]]
```

$\left\{\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\}$

```
MatrixForm[p = Transpose[{p1, p2}]]
```

$$\begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Check:

```
Clear[x, y];
{{x, y}}.a.{{x}, {y}} // Expand
```

$\{\{3 x^2 - 4 x y + 6 y^2\}\}$

```
Clear[u, v];
{u, v} = p.{x, y}
```

$\left\{-\frac{x}{\sqrt{5}} + \frac{2 y}{\sqrt{5}}, \frac{2 x}{\sqrt{5}} + \frac{y}{\sqrt{5}}\right\}$

```
{{u, v}}.a.{{u}, {v}} // Expand
```

$\{\{7 x^2 + 2 y^2\}\}$

Question 4

Part (a)

```
MatrixForm[a = {{5, -2}, {-2, 5}}]
```

$$\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$$

```
Eigensystem[a]
```

{{7, 3}, {{-1, 1}, {1, 1}}}

So maximum value subject to constraint is 7 (largest eigenvalue)

Part (b)

Maximum value is achieved when x is unit eigenvector associated with maximum eigenvalue, ie:

```
u = Eigensystem[a][[2, 1]] / Norm[Eigensystem[a][[2, 1]]]
```

$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

Part (c)

Maximum value subject to these two simultaneous constraints is the value of the second largest eigenvalue :

```
eval2 = Eigensystem[a][[1, 2]]
```

3

Question 5

There is, of course, the SIngularValueDecomposition command, but in the solution below we will do the calculations semi-manually, to illustrate the process.

```
MatrixForm[a = {{1, 1}, {0, 1}, {-1, 1}}]
```

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

```
MatrixForm[ata = Transpose[a].a]
```

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

So eigenvalues of $A^T A$ are 3 and 2. Hence singular values of A are $\sqrt{3}$ and $\sqrt{2}$ (in descending order of magnitude) and the matrix of singular values is

```
MatrixForm[s = {{Sqrt[3], 0}, {0, Sqrt[2]}, {0, 0}}]
```

$$\begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$$

The matrix of right singular vectors is matrix of corresponding eigenvectors of $A^T A$ (be careful to right these in the same order as the singular values):

```
Eigensystem[ata]
```

{{3, 2}, {{0, 1}, {1, 0}}}

So

```
MatrixForm[v = Transpose[{{0, 1}, {1, 0}}]]
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A unit vector **x** at which A**x** has maximum length is the normalised (*ie* unit length) right singular vector of A (*ie* eigenvector of $A^T A$) associated with the largest singular value: in this example,

```
Transpose[v][[1]]
```

{0, 1}

We could stop here, having answered the question - however, in the next question we will proceed to construct the full SVD.

Question 6

We still need to find the matrices V^T and U:

```
MatrixForm[vt = Transpose[v]]
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The orthogonal matrix U is 3x3. The first two columns are derived from A.V after normalisation:

```
{w1, w2} = {a.vt[[1]], a.vt[[2]]}
```

```
{{1, 1, 1}, {1, 0, -1}}
```

The third column needs to be found by applying Gram-Schmidt to a linearly independent vector in R^3.

```
LinearSolve[a, {0, 0, 1}]
```

```
LinearSolve[{{1, 1}, {0, 1}, {-1, 1}}, {0, 0, 1}]
```

so {0,0,1} will do as the linearly independent vector. To find w3:

```
w3 = {0, 0, 1} - ({0, 0, 1}.w1) / (w1.w1) w1 - ({0, 0, 1}.w2) / (w2.w2) w2
```

$$\left\{\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right\}$$

```
MatrixForm[u = Transpose[{w1 / Norm[w1], w2 / Norm[w2], w3 / Norm[w3]}]]
```

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

To check that U is orthogonal:

```
Transpose[u].u // FullSimplify
```

```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

Finally, check that $A = \mathrm{USV}^T$:

```
MatrixForm[u.s.vt]
```

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Question 7

Part (a)

The rank of A is given by the number of non-zero singular values - in this case 2.

Part (b)

An orthonormal basis for the column space of A is given by the columns of U that correspond to the non-zero singular values - in this case the first trwo columns

```
u1 = {0.4, 0.37, -0.84}
```

```
{0.4, 0.37, -0.84}
```

and

```
u2 = {-0.78, -0.33, -0.52}
```

```
{-0.78, -0.33, -0.52}
```

A basis for the null space of A is given by the right singular vectors (eigenvectors of $A^T A$) corresponding to the ‘zero’ singular values - in this case the third column of V (or third row of V^T):

```
v3 = {0.58, -0.58, 0.58}
```

```
{0.58, -0.58, 0.58}
```

To check: reconstruct A and find A.v3

```
a = {{0.4, -0.78, 0.43}, {0.37, -0.33, -0.87}, {-0.84, 0.52, -0.16}}.
  {{7.1, 0, 0}, {0, 3.1, 0}, {0, 0, 0}}.
  {{0.3, -0.51, -0.81}, {0.76, 0.64, -0.12}, {0.58, -0.58, 0.58}}
```

```
{{-0.98568, -2.99592, -2.01024},
 {0.01062, -1.99449, -2.00511}, {-0.56408, 4.07332, 4.6374}}
```

```
a.v3
```

$\{0., -2.22045\times 10^{-16}, 4.44089\times 10^{-16}\}$