University of Technology Sydney
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37233 Linear Algebra
Problem Set 7

Note: you may use *Mathematica* to carry out any calculations you feel may be of use.

Question 1.

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{pmatrix}.$$

Find the bases and their dimensions for:

(a) $\text{Row}\,(A)$

(b) $\text{Col}\,(A)$

(c) $\text{Nul}\,(A)$

(d) $\text{Nul}\,(A^T)$

Question 2.

Let

$$A = \begin{pmatrix} 3 & 0 & -1 & 1 \\ 3 & 0 & -1 & 2 \\ 3 & 0 & 5 & 3 \end{pmatrix}.$$

Find the bases and their dimensions for:

(a) $\text{Row}\,(A)$

(b) $\text{Col}\,(A)$

(c) $\text{Nul}\,(A)$

(d) $\text{Nul}\,(A^T)$

Question 3.

Suppose a system of nine equations in ten unknowns has a solution for all possible right hand sides of the equations. Is it possible to find two non-zero solutions of the associated homogeneous system that are not multiples of one another?

Question 4.

A homogeneous system of twelve equations in eight unknowns has two fixed solutions that are not multiples of one another, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than 12 homogeneous equations? If so, how many?

Question 5.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

(a) Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

Question 6.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space $\mathbb{R}^2$. Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

$$\mathbf{b}_1 = \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{c}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \ \mathbf{c}_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Question 7.

The Legendre polynomials are a family of polynomial functions that have many applications. The first five Legendre polynomials are given by

$$\begin{aligned}
p_0(x) &= 1, \\
p_1(x) &= x, \\
p_2(x) &= \frac{1}{2}(-1 + 3x^2), \\
p_3(x) &= \frac{1}{2}(-3x + 5x^3), \\
p_4(x) &= \frac{1}{8}(3 - 30x^2 + 35x^4).
\end{aligned}$$

(a) Write down the coordinate vectors of each of these Legendre polynomials with respect to the standard basis in $\mathbb{P}_4$ (that is, the basis of power functions $\{1, x, x^2, x^3, x^4\}$).

(b) Use the coordinate mapping to show that the set $\mathcal{L} = \{p_0, \ldots, p_4\}$ is a basis for $\mathbb{P}_4$

(c) Let f be the Maclaurin polynomial of degree 4 for e^x. Find $[f]_{\mathcal{E}}$. Find $[f]_{\mathcal{L}}$. Hint: find the change of coordinates matrix from $\mathcal{E}$ to $\mathcal{L}$.