University of Technology Sydney
Department of Mathematical and Physical Sciences

37233 Linear Algebra Problem Set 5

Note: you may use *Mathematica* to carry out any calculations you feel may be of use.

Question 1.

Are the columns of $\begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 3 & 3 & 6 & 1 & 14 \\ 0 & -1 & 0 & -2 & -9 \end{pmatrix}$ linearly independent? If not, find all possible linear combinations that sum to $\mathbf{0}$, *ie* the linear dependencies.

Question 2.

Prove: an indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. If S is linearly dependent then some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots \mathbf{v}_{j-1}$.

Question 3.

Are the columns of $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \\ 0 & 12 & -2 \\ 3 & 3 & 4 \end{pmatrix}$ linearly independent? What about the columns of A^T?

Question 4.

The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways of expressing $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Question 5.

Let $\mathbb{S}$ be the set of doubly infinite sequences $\mathbb{S} = \{, \ldots, y_{-1}, y_0, y_1, \ldots\}$. Prove that $\mathbb{S}$ is a vector space.

Question 6.

Prove that

(i) $\mathbb{P}_n$ is a vector space;

(ii) $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

.../Over

Question 7.

Hermite polynomials arise in the study of certain differential equations in mathematical physics. The first four of these polynomials are 1, $2t$, $4t^2 - 2$, and $8t^3 - 12t$. Show that the polynomials form a basis of $\mathbb{P}_3$.

Question 8.

Given the basis, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \right\}$, find $[\mathbf{x}]_B$ for $\mathbf{x} = \begin{pmatrix} 8 \\ -9 \\ 6 \end{pmatrix}$.

Question 9.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

Question 10.

Let V be a vector space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$. Prove that Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.