University of Technology Sydney
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37233 Linear Algebra Problem Set 8 – Solutions–Part A

Note: you may use *Mathematica* to carry out any calculations you feel may be of use.

Question 1.

Let

$$\mathbf{u} = \begin{pmatrix} 3 \\ 2 \\ -5 \\ 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} -4 \\ -1 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ 7 \\ 4 \\ 0 \end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} 1 \\ -8 \\ 15 \\ -7 \end{pmatrix}.$$

Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}\}$ orthogonal? What are the lengths of vectors $\mathbf{u}$ and $\mathbf{v}$?

Solution: The set is not orthogonal. For example, $\mathbf{u} \cdot \mathbf{v} = -12 - 2 + 10 = -4 \neq 0$. Also,

$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{9+4+25} = \sqrt{38}, \\ \|\mathbf{v}\| &= \sqrt{16+1+4+36} = \sqrt{57}. \end{aligned}$$

Question 2.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ and express $\mathbf{x}$ as a linear combination of $\mathbf{u}$'s, where

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Solution: $\mathbf{u}^T\mathbf{u} = \begin{pmatrix} 3 & -3 & 0 \\ 2 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{pmatrix}$ is diagonal, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal. Also,

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{15+9+0}{9+9+0}\mathbf{u}_1 + \frac{10-6-1}{4+4+1}\mathbf{u}_2 + \frac{5-3+4}{1+1+16}\mathbf{u}_3 \\ &= \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3 \end{aligned}$$

Question 3.

Construct a matrix $\mathbf{U}$ with orthonormal vectors from the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

and calculate $U^T U$ and UU^T.

Solution:

$$\|\mathbf{u}_1\| = \sqrt{1/9 + 1/9 + 1/9} = 1/\sqrt{3} \implies \hat{\mathbf{u}}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix},$$

$$\|\mathbf{u}_2\| = \sqrt{1/4 + 0 + 1/4} = 1/\sqrt{2} \implies \hat{\mathbf{u}}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

and

$$U^T U = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$UU^T = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}.$$

Question 4.

Let W be the subspace spanned by the $\mathbf{u}$'s. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}, \ \ \mathbf{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$$

Solution: $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1 + 3 - 2 = 0$ so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthogonal. Also,

$$\begin{aligned} \text{Proj}_W \mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\ &= \frac{-1+4+3}{1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1+12-6}{1+9+4} \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 3/2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3/2 \\ 7/2 \\ 1 \end{pmatrix} \in W, \end{aligned}$$

and

$$\mathbf{y} - \text{Proj}_W\mathbf{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 7/2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -5/2 \\ 1/2 \\ 2 \end{pmatrix} \in W^{\perp},$$

so

$$\mathbf{y} = \begin{pmatrix} 3/2 \\ 7/2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5/2 \\ 1/2 \\ 2 \end{pmatrix}$$

as required.

Question 5.

Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix}.$$

Solution: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 - 1 - 1 - 1 = 0$ so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is orthogonal. Also,

$$\begin{aligned}
\text{Proj}_W\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= \frac{9+1-5+1}{9+1+1+1}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \frac{3-1+5-1}{1+1+1+1}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 3/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 3/2 \\ -3/2 \\ 3/2 \\ -3/2 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix},
\end{aligned}$$

which is the closest point to $\mathbf{y}$ in W.

Question 6.

Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where

$$\mathbf{z} = \begin{pmatrix} 3 \\ -7 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Solution: Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We have $\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 - 1 + 0 - 1 = 0$ so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is orthogonal. Also,

$$
\begin{aligned}
\text{Proj}_W \mathbf{z} &= \frac{\mathbf{z} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{z} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= \frac{6+7-6+3}{4+1+9+1}\begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix} + \frac{3-7+0-3}{1+1+0+1}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 4/5 \\ -2/3 \\ -2 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -7/3 \\ -7/3 \\ 0 \\ 7/3 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -3 \\ -2 \\ 3 \end{pmatrix}
\end{aligned}
$$

is the best approximation.

Question 7.

Use the Gram-Schmidt process to produce an orthogonal basis for the space W spanned by vectors

$$\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}], \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}.$$

Construct an orthonormal basis for W.

Solution: See separate *Mathematica* output (Solutions part (b))

Question 8.

Let $\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(a) Apply the Gram–Schmidt process to $\mathcal{V}$ to construct an orthonormal basis for $\mathbb{R}^3$.

(b) Let V be the matrix whose columns are the elements of the basis $\mathcal{V}$. Construct a QR factorisation of V.

Solution: See separate *Mathematica* output (Solutions part (b))

Question 9.

Find:

(a) an orthogonal basis for the column space;

(b) a QR factorization

of the matrix A:

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}.$$

Solution: See separate *Mathematica* output (Solutions part (b))