Fundamentals of linear algebra

- Subspaces of a matrix:
  - Null space
  - Column space
  - Row space
- Basis for matrix subspaces
- Dimensions of matrix subspaces
- Rank of a matrix and the rank theorem

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Revision: subspace, spanning set, basis

Definition: A *subspace* H of a vector space V is a subset which (i) includes the zero vector of V; (ii) is closed under vector addition, and (iii) is closed under multiplication by scalars.

Theorem:
For $\mathbf{v}_1, \ldots \mathbf{v}_p \in V$, $\quad \mathrm{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is a subspace of V.

Definition: An indexed set $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_p\}$ is a *basis* in V if:
$\mathcal{B}$ is a linearly independent set, and $V = \mathrm{Span}\{\mathbf{b}_1, \ldots \mathbf{b}_p\}$

Spanning set theorem:
Let $\mathcal{S} = \{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ be a set in V, and $H = \mathrm{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$.

(a) If any $\mathbf{v}_i \in \mathcal{S}$ is a linear combination of the other vectors in $\mathcal{S}$, then the set formed from $\mathcal{S}$ by removing $\mathbf{v}_i$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of $\mathcal{S}$ is a basis for H.

Revision: dimension of a vector space

Definition: If V is spanned by a finite set, then V is called a *finite-dimensional* space, and the dimension of V, written as $\dim V = n$, is the number n of vectors in a basis for V.

Note: the dimension is a number of vectors in a basis, **not** a number of entries in each vector.

The dimension of the $\{\mathbf{0}\}$ vector space is defined to be zero.

If V is not spanned by a finite set then V is *infinite-dimensional*.

"The basis theorem":

Let V be a p-dimensional vector space, $p \geqslant 1$. Then:

- any linearly independent set of p elements in V is a basis for V;
- any set of p elements that spans V is a basis for V.

Subspaces related to a matrix

A specific set of vector subspaces of $\mathbb{R}^n$ arises in relation to linear systems of equations.

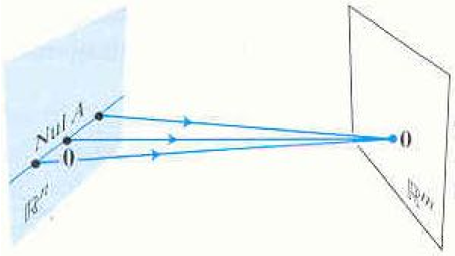
For a given matrix $\mathbf{A}$, the following spaces are defined:

- Null space
- Column space
- Row space

Null space

Definition: The **null space** of an $m \times n$ matrix $\mathbf{A}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$:

$$\mathrm{Nul}\,\mathbf{A} = \{\mathbf{x} : \ \mathbf{x} \in \mathbb{R}^n \ \text{ and } \ \mathbf{Ax} = \mathbf{0}\}$$



In terms of linear transformations, $\mathrm{Nul}\,\mathbf{A}$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped onto the zero vector of $\mathbb{R}^m$ via $\mathbf{x} \mapsto \mathbf{Ax}$.

Null space, example

Consider the following system of homogeneous equations:

$$\begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -5x_1 + 9x_2 + x_3 = 0 \end{cases} \quad \Leftrightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$$

Check if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ belong to $\operatorname{Nul} \mathbf{A}$:

$$\mathbf{Au} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Aw} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Thus $\mathbf{u} \in \operatorname{Nul} \mathbf{A}$ however $\mathbf{w} \notin \operatorname{Nul} \mathbf{A}$.

Null space

Theorem: For an $m \times n$ matrix $\mathbf{A}$, $\mathrm{Nul}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

Proof:

We must show that $\mathrm{Nul}\,\mathbf{A}$ satisfies the definition of a subspace.

1) Obviously, $\mathbf{0} \in \mathrm{Nul}\,\mathbf{A}$ because $\mathbf{A} \cdot \mathbf{0} = \mathbf{0}$.

2) Consider $\{\mathbf{u},\, \mathbf{v}\} \in \mathbb{R}^n$ such that $\{\mathbf{u},\, \mathbf{v}\} \in \mathrm{Nul}\,\mathbf{A}$.
This implies that $\mathbf{Au} = \mathbf{0}$ and $\mathbf{Av} = \mathbf{0}$.
Then $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so $(\mathbf{u} + \mathbf{v}) \in \mathrm{Nul}\,\mathbf{A}$.
Also $\mathbf{A}(c\,\mathbf{u}) = c(\mathbf{Au}) = c \cdot \mathbf{0} = \mathbf{0}$, so $c\,\mathbf{u} \in \mathrm{Nul}\,\mathbf{A}$.

Therefore, $\mathrm{Nul}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

Null space, example

Example: Let $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$ be the set of all vectors $\mathbf{h}$ in $\mathbb{R}^4$

with the coordinates satisfying equations: $\begin{cases} b = c - a \\ d = a - 2b + 5c \end{cases}$

By rearranging the governing equations we obtain

$$\begin{cases} -a - b + c = 0 \\ a - 2b + 5c - d = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & -2 & 5 & -1 \end{bmatrix} \mathbf{h} = \mathbf{0}$$

With the previous theorem we see that H is a subspace of $\mathbb{R}^4$.

Note: For a relation between the coordinates such that the corresponding system were *inhomogeneous*, the set would *not* be a subspace (as $\mathbf{0}$ is not a solution of an inhomogeneous system).

Explicit description of $\mathrm{Nul}\,\mathbf{A}$

There is no immediate connection between $\mathrm{Nul}\,\mathbf{A}$ and a_{ij}.

Solving $\mathbf{Ax} = \mathbf{0}$ provides an explicit description of $\mathrm{Nul}\,\mathbf{A}$.

Example: Find a spanning set for the null space of

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(1): find a general solution of $\mathbf{Ax} = \mathbf{0}$ in terms of free variables:

$$\to \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, x_1, x_3 are basic, and x_2, x_4, x_5 are free variables:

$x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$.

Explicit description of $\operatorname{Nul} \mathbf{A}$

(2): Using $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$
we decompose the general solution as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

where we can introduce

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and thereby $\operatorname{Nul} \mathbf{A} = \operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Explicit description of $\mathrm{Nul}\,\mathbf{A}$

Notes:

- The spanning set for $\mathrm{Nul}\,\mathbf{A}$ obtained by solving $\mathbf{Ax} = \mathbf{0}$ is linearly independent because the relation

$$x_2\mathbf{u}_2 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5 = \mathbf{0}$$

  is only true if $x_2 = x_4 = x_5 = 0$.

- When $\mathrm{Nul}\,\mathbf{A}$ contains non-zero vectors, the number of vectors in the spanning set for $\mathrm{Nul}\,\mathbf{A}$ equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Column space

Definition: The **column space** of an $m \times n$ matrix $\mathbf{A}$ is the set of all linear combinations of the columns of $\mathbf{A}$:

$$\mathrm{Col}\,\mathbf{A} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\}$$

- Each $\mathbf{a}_i \in \mathbb{R}^m$, therefore $\mathrm{Col}\,\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

  (a spanning set is a subspace of the corresponding vector space [L5])

- In terms of solving linear systems:

  $\mathrm{Col}\,\mathbf{A} = \{\mathbf{b} \,:\, \mathbf{b} = \mathbf{A}\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n\}$

  (since $\mathbf{A}\mathbf{x}$ is a linear combination of the columns of $\mathbf{A}$)

- In terms of linear transformations:

  $\mathrm{Col}\,\mathbf{A}$ is the range of the transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.

- $\mathrm{Col}\,\mathbf{A} = \mathbb{R}^m$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution $\forall\,\mathbf{b} \in \mathbb{R}^m$.

Column space, example

Example: Find a matrix $\mathbf{A}$ such that $\mathbf{W} = \operatorname{Col}\mathbf{A}$ if

$$\mathbf{W} = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \right\}, \quad a,\, b \in \mathbb{R}$$

Solution: Write $\mathbf{W}$ as a set of linear combinations

$$\mathbf{W} = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The vectors in the above spanning set are the columns of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{W} = \operatorname{Col}\mathbf{A}$$

Comparison between $\operatorname{Nul}\mathbf{A}$ and $\operatorname{Col}\mathbf{A}$

Example: Check if $\mathbf{u}$ and $\mathbf{v}$ belong to $\operatorname{Nul}\mathbf{A}$, $\operatorname{Col}\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(a) Clearly, $\mathbf{u} \notin \operatorname{Col}\mathbf{A}$, because $\operatorname{Col}\mathbf{A}$ is a subspace of $\mathbb{R}^3$.

(b) To find out if $\mathbf{u} \in \operatorname{Nul}\mathbf{A}$, there is no need for an explicit description of $\operatorname{Nul}\mathbf{A}$; we just check if $\mathbf{A}\mathbf{u} = \mathbf{0}$:

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \mathbf{0} \qquad \Rightarrow \qquad \mathbf{u} \notin \operatorname{Nul}\mathbf{A}$$

Comparison between $\mathrm{Nul}\,\mathbf{A}$ and $\mathrm{Col}\,\mathbf{A}$

Example: Check if $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathrm{Nul}\,\mathbf{A}$, $\mathrm{Col}\,\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Clearly, $\mathbf{v} \notin \mathrm{Nul}\,\mathbf{A}$, because $\mathrm{Nul}\,\mathbf{A}$ is a subspace of $\mathbb{R}^4$.

(d) If $\mathbf{v} \in \mathrm{Col}\,\mathbf{A}$, it must be a solution to $\mathbf{Ax} = \mathbf{v}$, so we form the required augmented matrix and check with row reduction:

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{array}\right]$$

We see that $\mathbf{Ax} = \mathbf{v}$ is consistent, so $\mathbf{v} \in \mathrm{Col}\,\mathbf{A}$.

For a given $m \times n$ matrix $\mathbf{A}$:

- $\operatorname{Nul}\mathbf{A}$ is a subspace of $\mathbb{R}^n$;
  $\operatorname{Col}\mathbf{A}$ is a subspace of $\mathbb{R}^m$.
- $\operatorname{Nul}\mathbf{A}$ is implicitly defined by $\mathbf{Ax} = \mathbf{0}$;
  $\operatorname{Col}\mathbf{A}$ is explicitly defined as $\operatorname{Span}\{\mathbf{a}_i\}$.
- There is no direct relation between $\operatorname{Nul}\mathbf{A}$ and a_{ij};
  There is a direct relation between $\operatorname{Col}\mathbf{A}$ and columns of $\mathbf{A}$;
- A vector of $\operatorname{Nul}\mathbf{A}$ is obtained by solving $\mathbf{Ax} = \mathbf{0}$;
  A vector of $\operatorname{Col}\mathbf{A}$ is obtained as a linear combination of $\{\mathbf{a}_i\}$.
- Checking if $\mathbf{v} \in \operatorname{Nul}\mathbf{A}$ is done by computing if $\mathbf{Av} = \mathbf{0}$;
  Checking if $\mathbf{v} \in \operatorname{Col}\mathbf{A}$ requires solving $\mathbf{Ax} = \mathbf{v}$.
- $\operatorname{Nul}\mathbf{A} = \{\mathbf{0}\}$ if and only if $\mathbf{Ax} = \mathbf{0}$ only for $\mathbf{x} = \mathbf{0}$;
  $\operatorname{Col}\mathbf{A} = \mathbb{R}^m$ if and only if $\mathbf{Ax} = \mathbf{b}$ has a solution $\forall\, \mathbf{b} \in \mathbb{R}^m$.
- If $\mathcal{T} : \mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one, then $\operatorname{Nul}\mathbf{A} = \{\mathbf{0}\}$;
  If $\mathcal{T} : \mathbf{x} \mapsto \mathbf{Ax}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then $\operatorname{Col}\mathbf{A} = \mathbb{R}^m$.

Basis for $\mathrm{Nul}\,\mathbf{A}$

Solution of a homogeneous system produces a linearly independent set given that it is expressed via free variables.

Thus the resulting spanning vectors form a basis for $\mathrm{Nul}\,\mathbf{A}$.

Example: Consider one the earlier examples with $\mathbf{Ax} = \mathbf{0}$ for

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

which we have denoted as $\mathbf{x} = x_2\mathbf{u}_2 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5$.

Vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ span $\mathrm{Nul}\,\mathbf{A}$ and are linearly independent, therefore $\mathcal{U} = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\}$ is a basis for $\mathrm{Nul}\,\mathbf{A}$.

Basis for $\operatorname{Col} \mathbf{A}$

Recall: finding pivot columns of a matrix through row-reduction is equivalent to finding linearly independent columns.

Theorem: Pivot columns of a $\mathbf{A}$ form a basis for $\operatorname{Col} \mathbf{A}$.

Proof: Let $\mathbf{B}$ be the REF form of $\mathbf{A}$. The pivot columns of $\mathbf{B}$ are linearly independent. Then the corresponding columns of $\mathbf{A}$ are also linearly independent (any linear relation between $\mathbf{a}_i$ implies the same linear relation for $\mathbf{b}_i$, via row-equivalence).

Thus, any non-pivot column of $\mathbf{A}$ is a linear combination of the pivot columns of $\mathbf{A}$. Therefore the non-pivot columns can be discarded from the spanning set for $\operatorname{Col} \mathbf{A}$, recall the spanning set theorem. This leaves the pivot columns of $\mathbf{A}$ as a basis for $\operatorname{Col} \mathbf{A}$.

Note: It is important that the pivot columns of $\mathbf{A}$ itself, and not those of the REF form, are a basis for $\operatorname{Col} \mathbf{A}$. The column space of $\mathbf{B}$ is not necessarily the same as the column space of $\mathbf{A}$.

Basis for $\mathrm{Col}\,\mathbf{A}$

Example: Find a basis for $\mathrm{Col}\,\mathbf{A}$ of $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$.

We obtain with row reduction

$$\rightarrow \quad \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 2 & -2 & 13 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{B}.$$

The REF form shows that $\mathbf{b}_1$, $\mathbf{b}_3$ and $\mathbf{b}_5$ are linearly independent, whereas $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.

Therefore $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are linearly independent, so $\mathbf{a}_2 = 4\mathbf{a}_1$ and $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ can be discarded from the spanning set.

Thus $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for $\mathrm{Col}\,\mathbf{A}$.

Basis for $\mathrm{Col}\,\mathbf{A}$

More explicitly, with that example we found the bases as:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{for} \quad \mathrm{Col}\left(\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\} \quad \text{for} \quad \mathrm{Col}\left(\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \right)$$

However even though matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, neither $\mathcal{A}$ is a basis for $\mathrm{Col}\,\mathbf{B}$, nor $\mathcal{B}$ is for $\mathrm{Col}\,\mathbf{A}$.

Only the indices of the columns which make the basis are the same.

Dimensions of $\mathrm{Nul}\,\mathbf{A}$ and $\mathrm{Col}\,\mathbf{A}$

Recall that the dimension of a vector (sub)space equals to the number of basis vectors required for that (sub)space.

Since the pivot columns of a matrix $\mathbf{A}$ form a basis of $\mathrm{Col}\,\mathbf{A}$, we know the dimension of $\mathrm{Col}\,\mathbf{A}$ once we know the pivots.

We have seen that a basis of $\mathrm{Nul}\,\mathbf{A}$ is formed by the vectors arising with free variables; there are as many vectors as free variables.

Therefore, by proceeding with solving $\mathbf{Ax} = \mathbf{0}$ we can establish:

- $\dim(\mathrm{Nul}\,\mathbf{A})$ is the number of free variables.
- $\dim(\mathrm{Col}\,\mathbf{A})$ is the number of pivot columns.

Dimensions of $\mathrm{Nul}\,\mathbf{A}$ and $\mathrm{Col}\,\mathbf{A}$

Examples: Return to the earlier examples:

$$\mathbf{A}_1 = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1 and 3, and free variables are x_2, x_4 and x_5.

Therefore $\dim(\mathrm{Nul}\,\mathbf{A}_1) = 3$ and $\dim(\mathrm{Col}\,\mathbf{A}_1) = 2$.

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1, 3 and 5, and free variables are x_2 and x_4.

Therefore $\dim(\mathrm{Nul}\,\mathbf{A}_2) = 2$ and $\dim(\mathrm{Col}\,\mathbf{A}_2) = 3$.

Row space of matrix

Definition:

If $\mathbf{A}$ is an $m \times n$ matrix, each row of $\mathbf{A}$ has n entries and thus can be identified with a vector in $\mathbb{R}^n$.

The set of all linear combinations of these row vectors is called the *row space* of $\mathbf{A}$, denoted as $\mathrm{Row}\,\mathbf{A}$.

Notes:

- Each row has n entries, so $\mathrm{Row}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$.
- Since the rows of $\mathbf{A}$ are identical with columns of $\mathbf{A}^\mathsf{T}$,

$$\mathrm{Row}\,\mathbf{A} = \mathrm{Col}\,\mathbf{A}^\mathsf{T}$$

Row space of matrix

Theorem: If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row-equivalent, then their row spaces are equal. If $\mathbf{B}$ is in echelon form, the non-zero rows of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$ and $\mathbf{B}$.

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by row operations, the rows of $\mathbf{B}$ are the linear combinations of the rows of $\mathbf{A}$.
- Furthermore, any linear combination of the rows of $\mathbf{B}$ is automatically a linear combination of rows of $\mathbf{A}$. Thus the row space of $\mathbf{B}$ is contained in the row space of $\mathbf{A}$.
- Row operations are reversible, so the same argument implies that the row space of $\mathbf{A}$ is contained in the row space of $\mathbf{B}$.
- Therefore, it must be that the two row spaces are the same.
- Finally, if $\mathbf{B}$ is in echelon form, its non-zero rows are linearly independent. Thus the non-zero rows of $\mathbf{B}$ form a basis of the common row space of $\mathbf{B}$ and $\mathbf{A}$.

Row space of matrix (example)

Example: Find bases and dimensions for $\mathrm{Col}\,\mathbf{A}$, $\mathrm{Nul}\,\mathbf{A}$, $\mathrm{Row}\,\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$\to \mathbf{B} = \begin{bmatrix} 1 & -3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-zero rows of $\mathbf{B}$ or $\mathbf{C}$ are the first three rows.

Pivot columns in $\mathbf{B}$ or $\mathbf{C}$ are columns 1, 2 and 4.

Number of free variables in $\mathbf{B}$ or $\mathbf{C}$ is 2.

Thus: $\dim(\mathrm{Row}\,\mathbf{A}) = 3$; $\dim(\mathrm{Col}\,\mathbf{A}) = 3$; $\dim(\mathrm{Nul}\,\mathbf{A}) = 2$.

Row space of matrix (example)

A basis for $\operatorname{Row}\mathbf{A}$ (and $\operatorname{Row}\mathbf{b}$) is given by non-zero rows of $\mathbf{b}$:

$$\operatorname{Row}\mathbf{A} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 20 \end{bmatrix} \right\}$$

A basis for $\operatorname{Col}\mathbf{A}$ (but not $\operatorname{Col}\mathbf{b}$) is given by pivot columns in $\mathbf{A}$:

$$\operatorname{Col}\mathbf{A} = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Row space of matrix (example)

Now we use REF and solve $\mathbf{C}\mathbf{x} = \mathbf{0}$; free variables are x_3 and x_5:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 & -x_5 \\ 2x_3 & -3x_5 \\ x_3 & \\ & 5x_5 \\ & x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Therefore $$\operatorname{Nul} \mathbf{A} = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

Rank of a matrix

Definition: $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A})$

The *rank* of matrix $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$.

Notes

- Since $\operatorname{Row}\mathbf{A} = \operatorname{Col}\mathbf{A}^{\mathsf{T}}$, then $\dim(\operatorname{Row}\mathbf{A}) = \operatorname{rank}\mathbf{A}^{\mathsf{T}}$.
- The dimension of $\operatorname{Nul}\mathbf{A}$ is sometimes called the *nullity* of $\mathbf{A}$.

The rank theorem: For an $m \times n$ matrix $\mathbf{A}$,

$$\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = \operatorname{rank}\mathbf{A}$$

This common dimension, the rank of matrix $\mathbf{A}$, also equals to the number of pivot positions in $\mathbf{A}$ and also satisfies the equation

$$\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = n$$

Rank of a matrix

Proof: By definition, $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A})$, which equals to the number of basis vectors for $\operatorname{Col}\mathbf{A}$, which is the number of pivot columns in $\mathbf{A}$. Equivalently, $\operatorname{rank}\mathbf{A}$ is then the number of pivot columns in an echelon form $\mathbf{B}$ of $\mathbf{A}$.

Because $\mathbf{B}$ has a non-zero row for each pivot, and these rows form a basis for $\operatorname{Row}\mathbf{A}$, $\operatorname{rank}\mathbf{A}$ is also the dimension of $\operatorname{Row}\mathbf{A}$,

$$\dim(\operatorname{Row}\mathbf{A}) = \dim(\operatorname{Col}\mathbf{A})$$

The dimension of $\operatorname{Nul}\mathbf{A}$ is the number of columns of $\mathbf{A}$ which correspond to free variables, so which are *not* the pivot columns.

The total number of columns n is the sum of the number of pivot columns and the number of columns without pivots, and therefore

$$\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = n$$

Rank of a matrix

Examples:

(a) If $\mathbf{A}$ is a 7×9 matrix with a two-dimensional null space, what is the rank of $\mathbf{A}$?

— As $\mathbf{A}$ has 9 columns, $\operatorname{rank} \mathbf{A} + 2 = 9$ and hence $\operatorname{rank} \mathbf{A} = 7$.

(b) Could a 6×9 matrix $\mathbf{B}$ have a two-dimensional null space?

— If a 6×9 matrix had a two-dimensional null space it would have to have rank 7 by the rank theorem.

However the columns of $\mathbf{B}$ are vectors in $\mathbb{R}^6$, and so $\dim(\operatorname{Col} \mathbf{B}) \leqslant 6$; therefore $\operatorname{rank} \mathbf{B} \leqslant 6$.

Thus $\mathbf{B}$ cannot have a two-dimensional null space.

Rank of a matrix

Example: Suppose that two solutions of a homogeneous system of 40 equations in 42 variables are found.

These solutions are not multiples, and all other solutions can be constructed by adding appropriate multiples of these two solutions.

What can be said about the solution of an associated inhomogeneous system?

Solution: Let $\mathbf{A}$ be the 40×42 coefficient matrix of the system.

The two solutions are linearly independent and span $\operatorname{Nul} \mathbf{A}$.

So $\dim(\operatorname{Nul} \mathbf{A}) = 2$.

By the rank theorem, $\dim(\operatorname{Col} \mathbf{A}) = 42 - 2 = 40$.

Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ with dimension of 40, $\operatorname{Col} \mathbf{A}$ must be all of $\mathbb{R}^{40}$.

Then that every inhomogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution.

The invertible matrix theorem (summary of results)

Statements equivalent to $\mathbf{A}$ being an $n \times n$ invertible matrix:

- There is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $\mathbf{A}^{\mathsf{T}}$ is invertible
- $\mathcal{T} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one
- $\mathcal{T} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- $\mathbf{A}$ has n pivot positions in the REF form
- The columns (rows) of $\mathbf{A}$ form a linearly independent set
- The columns (rows) of $\mathbf{A}$ span $\mathbb{R}^n$
- $\mathrm{Col}\,\mathbf{A} = \mathbb{R}^n$
- The columns (rows) of $\mathbf{A}$ form a basis of $\mathbb{R}^n$
- $\dim(\mathrm{Col}\,\mathbf{A}) = \dim(\mathrm{Row}\,\mathbf{A}) = n$
- $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution
- $\mathrm{Nul}\,\mathbf{A} = \{\mathbf{0}\}$
- $\dim(\mathrm{Nul}\,\mathbf{A}) = 0$
- $\mathrm{rank}\,\mathbf{A} = n$

Summary

- Null space, column space and row space of a matrix
- Bases for null, column and row spaces
- Dimensions of null, column and row spaces
- Rank of a matrix: $\operatorname{rank} \mathbf{A} = \dim(\operatorname{Col} \mathbf{A})$
- Rank theorem: $\operatorname{rank} \mathbf{A} + \dim(\operatorname{Nul} \mathbf{A}) = n$
  (for an $m \times n$ matrix $\mathbf{A}$)

Next lecture

See you next Wednesday

22 May 2019

Written assignment (essay) is due this week (on 15–17 May)

Assignment 6 is due this week (on 15–17 May)

Assignment 7 is due next week (on 22–24 May)