37233 Linear Algebra

Problem Sheet 9 Solutions

Question 1 (least squares - overdetermined system)

```
a = {{1, 3}, {4, 1}, {2, -1}, {3, 1}};
```

```
a // MatrixForm
```

$$\begin{pmatrix} 1 & 3 \\ 4 & 1 \\ 2 & -1 \\ 3 & 1 \end{pmatrix}$$

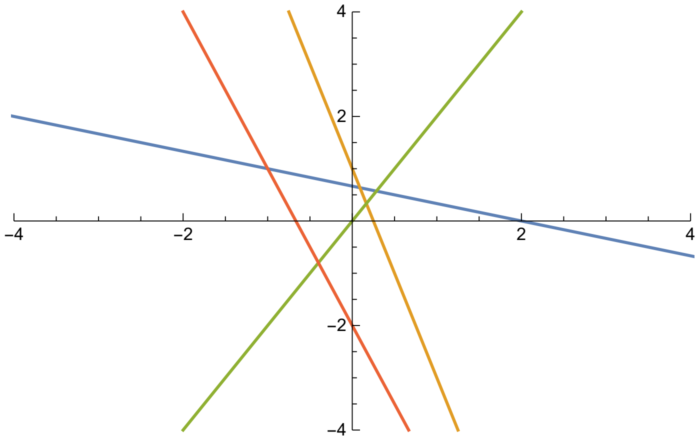
```
b = {2, 1, 0, -2};
```

```
b // MatrixForm
```

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

Just for interest, plot the lines defined by the inconsistent equations

```
p1 =
 Plot[{1/a[[1, 2]] * (-a[[1, 1]] * x + b[[1]]), 1/a[[2, 2]] * (-a[[2, 1]] * x + b[[2]]),
   1/a[[3, 2]] * (-a[[3, 1]] * x + b[[3]]), 1/a[[4, 2]] * (-a[[4, 1]] * x + b[[4]])},
  {x, -10, 10}, PlotRange → {{-4, 4}, {-4, 4}}]
```



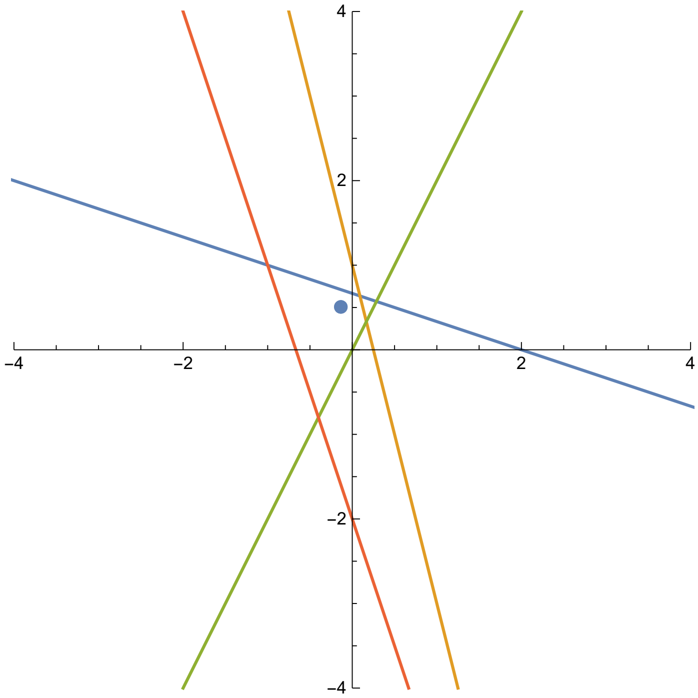
Find the least squares solution

```
sol = LinearSolve[Transpose[a].a, Transpose[a].b]
```

$$\left\{-\frac{5}{37}, \frac{75}{148}\right\}$$

Superimpose the point representing the solution on the previous diagram

```
Show[{p1, ListPlot[{sol}, {AspectRatio → 1, PlotStyle -> PointSize[0.02]}]},
 AspectRatio → 1]
```



Question 2 (least squares - data fitting)

```
data = {{-0.291996, 3.13651}, {0.664258, 9.06364}, {2.08586, 12.1178},
   {2.64251, 13.9248}, {4.08646, 18.0404}, {5.41087, 20.9689}}
```

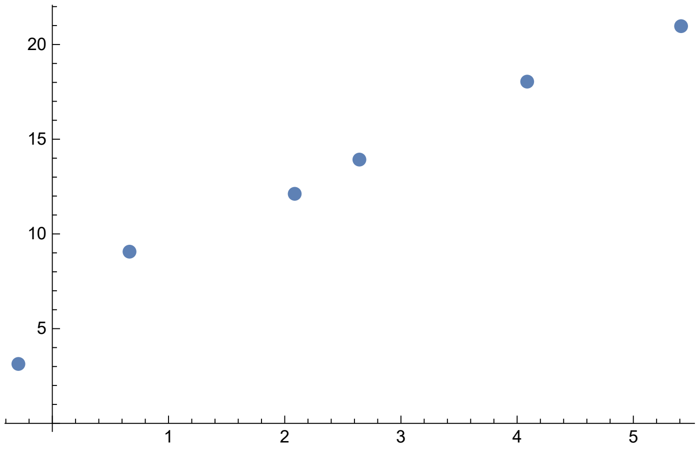
```
{{-0.291996, 3.13651}, {0.664258, 9.06364}, {2.08586, 12.1178},
 {2.64251, 13.9248}, {4.08646, 18.0404}, {5.41087, 20.9689}}
```

```
data // TableForm
```

-0.291996	3.13651
0.664258	9.06364
2.08586	12.1178
2.64251	13.9248
4.08646	18.0404
5.41087	20.9689

Plot the data points

```
points = ListPlot[data, {Joined → False, PlotStyle -> PointSize[0.02]}]
```



Set up the design matrix

```
xm = Transpose[{{1, 1, 1, 1, 1, 1}, Transpose[data][[1]]}]
```

```
{{1, -0.291996}, {1, 0.664258}, {1, 2.08586}, {1, 2.64251}, {1, 4.08646}, {1, 5.41087}}
```

```
xm // MatrixForm
```

$$\begin{pmatrix} 1 & -0.291996 \\ 1 & 0.664258 \\ 1 & 2.08586 \\ 1 & 2.64251 \\ 1 & 4.08646 \\ 1 & 5.41087 \end{pmatrix}$$

Set up the observed dependent values

```
ym = Transpose[data][[2]]
```

```
{3.13651, 9.06364, 12.1178, 13.9248, 18.0404, 20.9689}
```

Solve for the model parameters

```
sol = LinearSolve[Transpose[xm].xm, Transpose[xm].ym]
```

```
{5.63691, 2.97511}
```

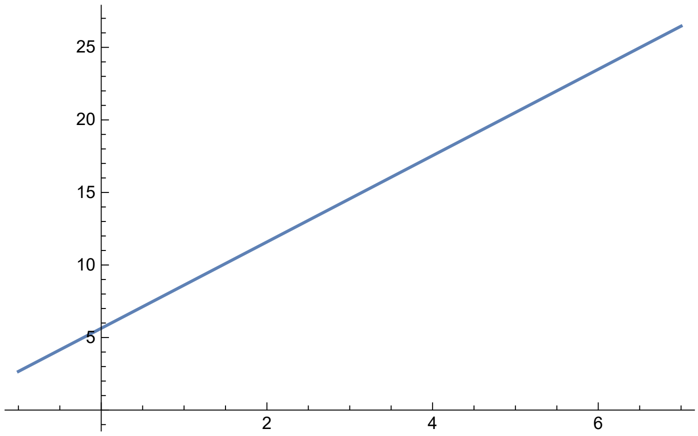
Plot the model equation

```
Clear[lbf];
lbf[x_] := sol[[1]] + sol[[2]] * x;
```

```
lbf[2]
```

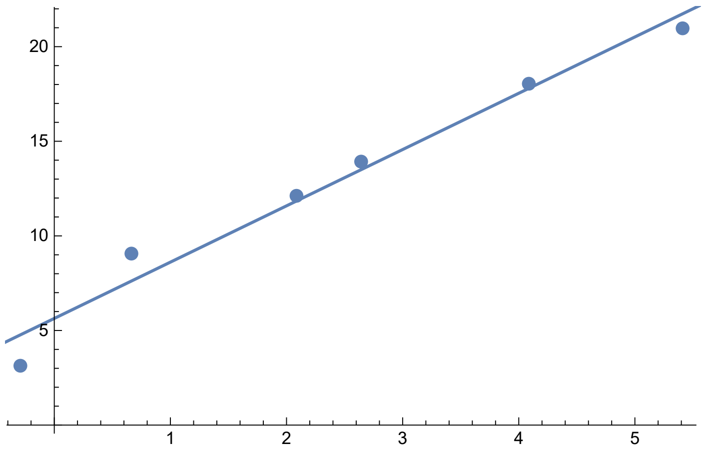
```
11.5871
```

```
plbf = Plot[lbf[x], {x, -1, 7}]
```



Superimpose the data on the model plot

```
Show[{points, plbf}]
```



Question 3

Part (a)

```
a = {{3, -1, -1}, {0, 2, 0}, {2, -2, 0}}
```

```
{{3, -1, -1}, {0, 2, 0}, {2, -2, 0}}
```

Mathematica has some useful built-in commands for finding eigenvalues (the Eigenvalues command) and eigenvectors (the Eigenvectors command), or both (the Eigensystem command):

```
e = Eigensystem[a]
```

```
{{2, 2, 1}, {{1, 0, 1}, {1, 1, 0}, {1, 0, 2}}}
```

The first element (a list of three numbers) is the list of eigenvalues that can be used to construct a diagonal matrix. The second element (a list of three vectors) is a list of eigenvectors, each associated with the corresponding eigenvalue in the first list. If these are linearly indpendent they can be used to costruct an invertible matrix P (the change of basis matrix from the standard basis to an eigenvector basis)

```
d = DiagonalMatrix[e[[1]]]
{{2, 0, 0}, {0, 2, 0}, {0, 0, 1}}

p = Transpose[e[[2]]]
{{1, 1, 1}, {0, 1, 0}, {1, 0, 2}}

a == p.d.Inverse[p]
True
```

To do the same calculations by hand is a bit more tedious, but straightforward. The outline is below:

```
eigenvals = Solve[Det[a - lam IdentityMatrix[3]] == 0, lam]
{{lam → 1}, {lam → 2}, {lam → 2}}

{lam1, lam2, lam3} = Table[eigenvals[[i, 1, 2]], {i, 1, Length[eigenvals]}]
{1, 2, 2}

d = DiagonalMatrix[{lam1, lam2, lam3}]
{{1, 0, 0}, {0, 2, 0}, {0, 0, 2}}

a - lam1 IdentityMatrix[3]
{{2, -1, -1}, {0, 1, 0}, {2, -2, -1}}

NullSpace[a - lam1 IdentityMatrix[3]]
{{1, 0, 2}}

p1 = NullSpace[a - lam1 IdentityMatrix[3]][[1]]
{1, 0, 2}

NullSpace[a - lam2 IdentityMatrix[3]]
{{1, 0, 1}, {1, 1, 0}}

p2 = NullSpace[a - lam2 IdentityMatrix[3]][[1]]
{1, 0, 1}

p3 = NullSpace[a - lam2 IdentityMatrix[3]][[2]]
{1, 1, 0}

p = Transpose[{p1, p2, p3}]
{{1, 1, 1}, {0, 0, 1}, {2, 1, 0}}

a == p.d.Inverse[p]
True
```

Part (b)

```
a = {{3, -1, -1}, {-2, 3, 1}, {4, -3, -1}}
{{3, -1, -1}, {-2, 3, 1}, {4, -3, -1}}
```

```
e = Eigensystem[a]
```

```
{{2, 2, 1}, {{0, -1, 1}, {0, 0, 0}, {1, 0, 2}}}
```

Not diagonalisable - A does not have three linearly independent eigenvectors (the second vector (which is not actually an eigenvector) associated with eigenvalue 2 is given as {0,0,0}. This is *Mathematica*'s way of indicating that it could not find (ie there does not exist) a second linearly independent eigenvector associated with this eigenvalue.

Question 4

```
a = {{1, 5}, {5, 1}}
```

```
{{1, 5}, {5, 1}}
```

Mathematica has some useful built-in commands for finding eigenvalues (the Eigenvalues command) and eigenvectors (the Eigenvectors command), or both (the Eigensystem command):

```
e = Eigensystem[a]
```

```
{{6, -4}, {{1, 1}, {-1, 1}}}
```

The first element (a list of two numbers) is the list of eigenvalues that can be used to construct a diagonal matrix. The second element (a list of two vectors) is a list of eigenvectors, each associated with the corresponding eigenvalue in the first list. If these are linearly indpendent they can be used to costruct an invertible matrix P (the change of basis matrix from the standard basis to an eigenvector basis)

```
d = DiagonalMatrix[e[[1]]]
```

```
{{6, 0}, {0, -4}}
```

```
p = Transpose[e[[2]]]
```

```
{{1, -1}, {1, 1}}
```

```
a == p.d.Inverse[p]
```

```
True
```

To do the same calculations by hand is a bit more tedious, but straightforward. The outline is below:

```
eigenvals = Solve[Det[a - lam IdentityMatrix[2]] == 0, lam]
```

```
{{lam → -4}, {lam → 6}}
```

```
{lam1, lam2} = Table[eigenvals[[i, 1, 2]], {i, 1, Length[eigenvals]}]
```

```
{-4, 6}
```

```
d = DiagonalMatrix[{lam1, lam2}]
```

```
{{-4, 0}, {0, 6}}
```

```
a - lam1 IdentityMatrix[2]
```

```
{{5, 5}, {5, 5}}
```

```
NullSpace[a - lam1 IdentityMatrix[2]][[1]]
```

{-1, 1}

```
p1 = NullSpace[a - lam1 IdentityMatrix[2]][[1]]
```

{-1, 1}

```
NullSpace[a - lam2 IdentityMatrix[2]]
```

{{1, 1}}

```
p2 = NullSpace[a - lam2 IdentityMatrix[2]][[1]]
```

{1, 1}

```
p = Transpose[{p1, p2}]
```

{{-1, 1}, {1, 1}}

```
a == p.d.Inverse[p]
```

True

To construct the spectral decomposition: first represent p1 as a 2x1 matrix and make it unit norm:

```
MatrixForm[p1m = (Transpose[{p1}]) / Norm[p1]]
```

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now form $P_1 \, P_1{}^T$

```
MatrixForm[p1mat = p1m.Transpose[p1m]]
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Do the same for P_2:

```
MatrixForm[p2m = (Transpose[{p2}]) / Norm[p2]]
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
MatrixForm[p2mat = p2m.Transpose[p2m]]
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Finally, check that the decomposition is correct:

```
MatrixForm[decomposition = lam1 * p1mat + lam2 * p2mat]
```

$$\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$$

```
a == decomposition
```

True

Question 5

In each case we verify that the matrices are conformable for multiplication. Then, ...

Part (a)

$((B^\wedge T\,AB))^\wedge T = B^\wedge T\,A^\wedge T\,(B^\wedge T)^\wedge T = B^T A^T B = B^T$ AB (since A is symmetric). Hence B^T AB is symetric.

Part (b)

$((B^\wedge T\,B))^\wedge T = B^\wedge T\,(B^\wedge T)^\wedge T = B^T B$ (in effect, this follows from the previous part, replacing the matrix A by the identity matrix, which is also symmetric).

Part (c)

$((B\,B^\wedge T))^\wedge T = (B^\wedge T)^\wedge T\,B^\wedge T = B\,B^T$.

Question 6

$$
\begin{aligned}
A\,x.y &= \left(\sum_{j=1}^{n} x_j\, a_j\right).y \\
&= \sum_{j=1}^{n} x_j\, (a_j.y) \\
&= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} a_{ij}.y_i\right) \\
&= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} \left(A^T\right)_{ji}.y_i\right) \\
&= \sum_{j=1}^{n} x_j\, \left(A^T y\right)_j \quad \text{where } \left(A^T y\right)_j \text{ is the j ' th column of } A^T y \\
&= x.\left(A^T y\right) \\
&= x.(A\,y) \quad (\text{since A is symmetric})
\end{aligned}
$$