Linear Algebra, Assignment 4

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Question 1

$$\mathbf{a_1} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{a_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{b_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \mathbf{b_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{b_3} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Calculations:

$$\mathbf{a_1} - 4\mathbf{a_2} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 0 \end{bmatrix}$$

$$\mathbf{b_1} + 2\mathbf{b_3} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$(\mathbf{a_2} + 3\mathbf{b_2})$ is undefined.

$$\mathbf{b_1} + \mathbf{b_2} - 3\mathbf{b_3} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Question 2

$$\begin{bmatrix} 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Set up an augmented matrix:

$$\begin{bmatrix} 2 & -2 & b_1 \\ 3 & 3 & b_2 \\ 4 & -4 & b_3 \end{bmatrix}$$

Reduce to row echelon form:

$$R_3 \hookleftarrow (R_3 - 2R_1) \Rightarrow \begin{bmatrix} 2 & -2 & b_1 \\ 3 & 3 & b_2 \\ 0 & 0 & b_3 - 2b_1 \end{bmatrix}.$$

$$R_2 \hookleftarrow (R_2 - \frac{3}{2}R_1) \Rightarrow \begin{bmatrix} 2 & -2 & b_1 \\ 0 & 6 & b_2 - \frac{3}{2}b_1 \\ 0 & 0 & b_3 - 2b_1 \end{bmatrix}.$$

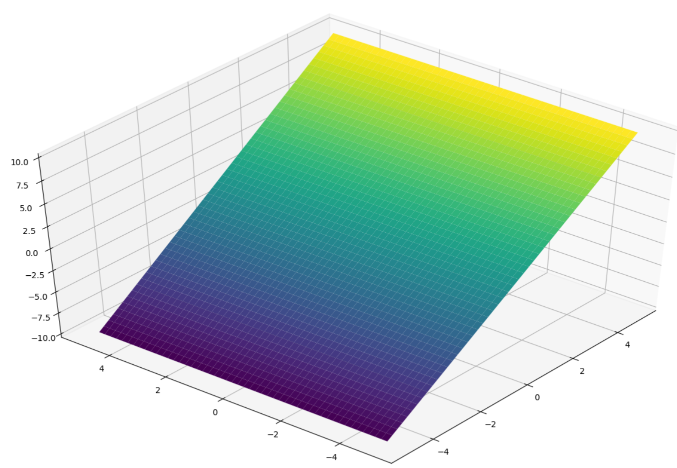
This means that to have a consistent $\mathbf{b}$, we must have: $b_3 - 2b_1 = 0$

Leading us to:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = b_1\mathbf{u} + b_2\mathbf{v}.$$

Meaning that a set of solutions that are consistent with the system would lie in a plane made of linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

Visual, geometric interpretation of linear combinations of $\mathbf{u}$ and $\mathbf{v}$:



Question 3

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v_4} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

First, we form an augmented matrix of the vectors:

$$\mathbf{A} = \left[\begin{array}{cccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} \end{array} \right] = \begin{bmatrix} 3 & 2 & 3 & 2 \\ 2 & -1 & 3 & 0 \\ 2 & -1 & 3 & 1 \end{bmatrix}.$$

Getting into Reduced-Row-Echelon Form:

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & \frac{9}{7} & 0 \\ 0 & 1 & \frac{-3}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Before continuing with our investigation we must define the term 'rank'.

The '*rank*' of a matrix is defined as: *The dimension of vector space spanned by its columns.*

We can see that the matrix has three pivots, (which, as a side-note, also means that it is of full-rank).

Hence if the matrix is of rank 3, then this means that the matrix spans 3-dimensional vector space, $\mathbb{R}^3$.

Therefore:

$$\text{Span}\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\} \in \mathbb{R}^3.$$

Question 4

$$\begin{bmatrix} 1 & 1 & 4 & 1 \\ 1 & 2 & 9 & 0 \\ 2 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Forming an augmented matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 & 1 & 0 \\ 1 & 2 & 9 & 0 & 0 \\ 2 & 1 & 3 & 3 & 0 \end{bmatrix}.$$

Getting $\mathbf{A}$ into Reduced-Row-Echelon-Form:

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Basic variables} = \{x_1, x_2\}.$$

$$\text{Free variables} = \{x_3, x_4\}.$$

So, to satisfy our system, we must have a certain set of values for $\mathbf{x}$ that satisfy the system and equal $\mathbf{0}$. That set of x-values is given by our row-reduction.

Expressing the basic variables in terms of the free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ -5x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence the solution set is:

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Question 5

$$\begin{bmatrix}1 & 1 & -1 & -2\\ 2 & -1 & -3 & -5\\ 1 & 2 & -6 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}1\\ 7\\ -6\end{bmatrix}.$$

Forming an augmented matrix:

$$\mathbf{A} = \begin{bmatrix}1 & 1 & -1 & -2 & 1\\ 2 & -1 & -3 & -5 & 7\\ 1 & 2 & -6 & -7 & -6\end{bmatrix}.$$

Transforming to Reduced-Row-Echelon form:

$$\mathrm{rref}(\mathbf{A}) = \begin{bmatrix}1 & 0 & 0 & -1 & 4\\ 0 & 1 & 0 & 0 & -2\\ 0 & 0 & 1 & 1 & 1\end{bmatrix}.$$

Therefore:

$$\text{Basic variables} = \{x_1, x_2, x_3\}.$$

$$\text{Free variable} = \{x_4\}$$

$$\begin{bmatrix}x1\\ x2\\ x3\\ x4\end{bmatrix} = \begin{bmatrix}x_4+4\\ -2\\ -x_4+1\\ x_4\end{bmatrix} = x_4\begin{bmatrix}1\\ 0\\ -1\\ 1\end{bmatrix} + \begin{bmatrix}4\\ -2\\ 1\\ 0\end{bmatrix}$$

Hence the solution set is:

$$\left\{ n\begin{bmatrix}1\\ 0\\ -1\\ 1\end{bmatrix} + \begin{bmatrix}4\\ -2\\ 1\\ 0\end{bmatrix},\ \forall\ n \in \mathbb{R} \right\}$$