

Linear Algebra Autumn 2019 - Assignment 1

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Question 1

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & -4 & 4 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ -1 & 2 & -2 & 2 & 1 \end{bmatrix} \quad (1)$$

Row operation: $R1 \leftarrow (R1 + R4)$

$$\begin{bmatrix} 0 & 0 & -6 & 6 & 0 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ -1 & 2 & -2 & 2 & 1 \end{bmatrix} \quad (2)$$

Row operation: $R1 \leftrightarrow R4$

$$\begin{bmatrix} -1 & 2 & -2 & 2 & 1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -6 & 6 & 0 \end{bmatrix} \quad (3)$$

Row operation: $R1 \leftarrow (-1 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -6 & 6 & 0 \end{bmatrix} \quad (4)$$

Row operation: $R4 \leftarrow (\frac{1}{6} \times R4)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (5)$$

Row operation: $R2 \leftarrow (R2 + 3 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 10 & -3 & 7 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (6)$$

Row operation: $R3 \leftarrow (R3 + 2 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 10 & -3 & 7 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (7)$$

Row operation: $R2 \leftrightarrow R4$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \quad (8)$$

Row operation: $R2 \leftarrow (-1 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \quad (9)$$

Row operation: $R3 \leftarrow (R3 - 6 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \quad (10)$$

Row operation: $R3 \leftarrow (\frac{1}{4} \times R3)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \quad (11)$$

Row operation: $R4 \leftarrow (R4 - 10 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 & 7 \end{bmatrix} \quad (12)$$

Row operation: $R4 \leftarrow (\frac{1}{7} \times R4)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (13)$$

Row operation: $R4 \leftarrow (R4 - R3)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

Row operation: $R1 \leftarrow (R1 - 2 \times R2)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

Row operation: $R2 \leftarrow (R2 + R3)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

Basic variables: x_1, x_3, x_4

Free variable: x_2

$$\mathbf{x} = \begin{bmatrix} x_1 = -1 + 2x_2 \\ x_3 = 1 \\ x_5 = 1 \end{bmatrix} \quad (17)$$

Question 2

Augmented matrix:

$$\begin{bmatrix} 3 & 6 & 1 & -2 & -4 & 6 \\ 1 & 2 & 0 & 1 & -1 & -1 \end{bmatrix} \quad (18)$$

Row operation: $R1 \leftrightarrow R2$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & -1 \\ 3 & 6 & 1 & -2 & -4 & 6 \end{bmatrix} \quad (19)$$

Row operation: $R2 \leftarrow (R2 - 3 \times R1)$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -5 & -1 & 9 \end{bmatrix} \quad (20)$$

Basic variables: x_1, x_3

Free variables: x_2, x_4, x_5

$$\mathbf{x} = \begin{bmatrix} x_1 = -1 - 2x_2 - x_4 + x_5 \\ x_3 = 9 + 5x_4 + x_5 \end{bmatrix} \quad (21)$$

Question 3

$$\det(\mathbf{A}) = 1 \begin{vmatrix} 9 & 4 \\ 6 & 5 \end{vmatrix} - 2 \begin{vmatrix} 8 & 4 \\ 7 & 5 \end{vmatrix} + 3 \begin{vmatrix} 8 & 9 \\ 7 & 6 \end{vmatrix} \quad (22)$$

$$= 1(45 - 24) - 2(40 - 28) - 4(48 - 63) \quad (23)$$

$$\det \mathbf{A} = -48 \quad (24)$$

$$\det(\mathbf{B}) = 0 \begin{vmatrix} 1 & 6 & 0 \\ 1 & 2 & 0 \\ -2 & 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 0 & 6 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 & 6 \\ 1 & 1 & 2 \\ 1 & -2 & 0 \end{vmatrix} \quad (25)$$

$$\det(\mathbf{B}) = (0)B_{11} + B_{12} + (2)B_{13} + (3)B_{14} \quad (26)$$

Calculating the cofactors:

$$B_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 6 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (-1)(-6)(3 - 0) = 18 \quad (27)$$

$$B_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 3 \end{vmatrix} = (1)(-1)(3 - 0) = -3 \quad (28)$$

$$B_{14} = (-1)^{1+4} \begin{vmatrix} 0 & 1 & 6 \\ 1 & 1 & 2 \\ 1 & -2 & 0 \end{vmatrix} = (-1)((-1)(0 - 2) + 6(-3)) = (-1)(2 - 18) = 16 \quad (29)$$

Then calculating the determinant:

$$\det(\mathbf{B}) = (0) + 18 + (2)(-3) + (3)(16) = 12 + 48 \quad (30)$$

$$\det \mathbf{B} = 60 \quad (31)$$

Question 4

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 7 \\ -1 & 2 & 0 \\ -2 & 5 & 3 \end{bmatrix} \quad (32)$$

If: $\det \mathbf{A} = 0$ then it is a singular matrix.

$$\det \mathbf{A} = 1(2 \times 3 - 5 \times 0) + 7(-1 \times 5 - 2 \times (-2)) \quad (33)$$

$$\det \mathbf{A} = 6 - 7 = -1 \quad (34)$$

$$\therefore \det \mathbf{A} \neq 0 \Rightarrow \mathbf{A} \text{ is not singular} \Rightarrow \mathbf{A} \text{ has an inverse} \quad (35)$$

Append identity matrix, forming augmented matrix for Gaussian elimination.

$$\mathbf{A}|\mathbf{I} = \begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ -2 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \quad (36)$$

Row operation: $R3 \leftarrow (R3 + 2 \times R1) 5$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \end{bmatrix} \quad (37)$$

Row operation: $R2 \leftrightarrow R3$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ -1 & 2 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (38)$$

Row operation: $R3 \leftarrow (R3 + R1)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 2 & 7 & 1 & 1 & 0 \end{bmatrix} \quad (39)$$

Row operation: $R3 \leftarrow (R3 - \frac{2}{5}R2)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 1 & \frac{-2}{5} \end{bmatrix} \quad (40)$$

Row operation: $R3 \leftarrow (5 \times R3)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 5 & -2 \end{bmatrix} \quad (41)$$

Row operation: $R2 \leftarrow (R1 - 17 \times R3)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 0 & -15 & -85 & 35 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \quad (42)$$

Row operation: $R2 \leftarrow (\frac{1}{5} \times R2)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -17 & 7 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \quad (43)$$

Row operation: $R1 \leftarrow (R1 - 7 \times R2)$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & -35 & 14 \\ 0 & 1 & 0 & -3 & -17 & 7 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} = \mathbf{I} | \mathbf{A}^{-1} \quad (44)$$

$$\therefore \mathbf{A}^{-1} = \begin{bmatrix} -6 & -35 & 14 \\ -3 & -17 & 7 \\ 1 & 1 & -2 \end{bmatrix} \quad (45)$$

Question 5

$$\mathbf{A} = \begin{bmatrix} a_{11} & 2 & 1 \\ 5 & -1 & 2 \\ -3 & 1 & -1 \end{bmatrix} \quad (46)$$

If $\det \mathbf{A} = 0 \Rightarrow \mathbf{A}$ is singular

$$\det \mathbf{A} = a_{11} ((-1) \times (-1) - (2 \times 1)) - 2 (5 \times (-1) - ((-3) \times 2)) + 1 (5 \times 1 - (-1) \times (-3)) = 0 \quad (47)$$

$$0 = a_{11} (1 - 2) - 2 (-5 + 6) + 1 (5 - 3) \quad (48)$$

$$a_{11} = 0 \quad (49)$$

Therefore, if you want \mathbf{A} to be singular a_{11} must be set to zero.