Linear Algebra, Assignment 5

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Question 1

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 7 & -6 \\ -2 & -1 & -9 & 7 \\ 1 & 13 & -8 & 9 \end{bmatrix}$$

Basis for a vector space spanned by the columns of **A** First, let's find the vector space spanned by **A**:

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 5 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}(\mathbf{A}) = 2 \quad \Rightarrow \quad \text{Span}\{\mathbf{A}\} = \mathbb{R}^2$$

A valid basis for this vector space is that given in the cartesian system:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} = \mathbb{R}^2$$

Question 2

With:

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

We can prove that B is a basis in $\mathbb{R}^4$ if it is linearly independent and if it spans $\mathbb{R}^4$.

(a) Linear Independence:

Forming an augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

Through row reduction we can reduce this to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix has 4 pivots, rank = 4 and hence must span $\mathbb{R}^4$.

(b) Find $\mathbf{x}$

If you have a basis s.t.

$$\mathbf{B} = \{\mathbf{b_1},\ \dots\ ,\mathbf{b_n}\}$$

We can redefine $\mathbf{B}$ to be:

$$\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \dots & \mathbf{b}_n\end{bmatrix}$$

Then $\mathbf{x}$ is expressed as $[\mathbf{x}]_B$ via the equation:

$$\mathbf{x} = \mathbf{B}\,[\mathbf{x}]_B$$

As a corrollary, it also then follows that:

$$[\mathbf{x}]_B = \mathbf{B}^{-1}\mathbf{x}$$

Solving $\mathbf{x} = \mathbf{B}\,[\mathbf{x}]_B$:

$$\mathbf{x} = \begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 2 & 2 & 0\\ 3 & 3 & 0 & 0\\ 4 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1\\ 0\\ 0\\ 2\end{bmatrix} = \begin{bmatrix}1\\ -2\\ -3\\ -4\end{bmatrix}$$

(c) Find $[\mathbf{y}]_B$

$$[\mathbf{y}]_B = \mathbf{B}^{-1}\mathbf{y}$$

Inverting $\mathbf{B}$, we get:

$$\mathbf{B}^{-1} = \begin{bmatrix}0 & 0 & 0 & \frac{1}{4}\\ 0 & 0 & \frac{1}{3} & -\frac{1}{4}\\ 0 & \frac{1}{2} & -\frac{1}{3} & 0\\ 1 & -\frac{1}{2} & 0 & 0\end{bmatrix}$$

Evaluating:

$$[\mathbf{y}]_B = \mathbf{B}^{-1}\mathbf{y} = \begin{bmatrix}0 & 0 & 0 & \frac{1}{4}\\ 0 & 0 & \frac{1}{3} & -\frac{1}{4}\\ 0 & \frac{1}{2} & -\frac{1}{3} & 0\\ 1 & -\frac{1}{2} & 0 & 0\end{bmatrix}\begin{bmatrix}10\\ 12\\ 9\\ 4\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}$$

Question 3

If the linear transformation $\mathbf{R}$ is:

$$\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}$$

Let's define the unit square as the set:

$$S = \{s_1, s_2, s_3, s_4\}$$

Where the elements of the set are:

$$s_1 = \begin{bmatrix}0\\ 0\end{bmatrix} \quad s_2 = \begin{bmatrix}1\\ 0\end{bmatrix} \quad s_3 = \begin{bmatrix}1\\ 1\end{bmatrix} \quad s_4 = \begin{bmatrix}0\\ 1\end{bmatrix}$$

Then mapping each element of S with $\mathbf{R}$, we get:

$$\mathbf{R}\ s_1 = \begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\mathbf{R}\ s_2 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\mathbf{R}\ s_3 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\mathbf{R}\ s_3 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Illustrated, it looks like so:

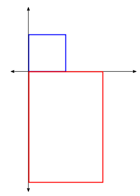

Figure 1: Blue is the unit square, red is the unit square transformed by $\mathbf{R}$

Question 4

(a) Domain and Codomain of $\mathcal{T}$

Because $\mathbf{A}$ is 4×3 with respect to its size:

$$\mathbf{A} : \mathbb{R}^4 \mapsto \mathbb{R}^3$$

Therefore

$$\text{Domain} = \mathbb{R}^4$$

$$\text{Codomain} = \mathbb{R}^3$$

(b) Range of $\mathcal{T}$

The issue is that the columns of $\mathbf{A}$ are linearly dependent. This means that the columns do not span $\mathbb{R}^3$, it is instead a plane in 3D space.

(c) Images of Vectors

$$\mathcal{T}(\mathbf{v}) = \begin{bmatrix} 1 & -2 & 7 & -6 \\ -2 & -1 & -9 & 7 \\ 1 & 13 & -8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$$

$$\mathcal{T}(\mathbf{u}) = \begin{bmatrix} 1 & -2 & 7 & -6 \\ -2 & -1 & -9 & 7 \\ 1 & 13 & -8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$$

$$\mathcal{T}(\mathbf{w}) = \begin{bmatrix} 1 & -2 & 7 & -6 \\ -2 & -1 & -9 & 7 \\ 1 & 13 & -8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 15 \end{bmatrix}$$

All the answers lie in a plane because the range of $\mathbf{A}$ is $\mathbb{R}^2$.

However, couldn't tell you why they are the same point in the plane ...