37233 Linear Algebra

Problem Sheet 6 Solutions - Part A

Function definitions

Question 1

In[1]:=
```
p = Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}}];
```

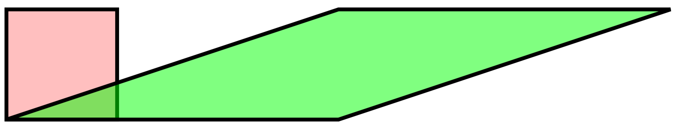
In[6]:=
```
Clear[t];
t[x_] := {{3, 3}, {0, 1}}.x
```

In[8]:=
```
tp = Polygon[{t[{0, 0}], t[{1, 0}], t[{1, 1}], t[{0, 1}]}]
```

Out[8]=
```
Polygon[{{0, 0}, {3, 0}, {6, 1}, {3, 1}}]
```

In[9]:=
```
Show[{Graphics[{Opacity[0.5], EdgeForm[Thick], Pink, p}],
  Graphics[{Opacity[0.5], EdgeForm[Thick], Green, tp}]}]
```

Out[9]=



This is a shearing transformation. A square transforms into a parallelogram.

Question 2

In[10]:=
```
p = Polygon[{{0, 0}, {1, 0}, {1, 1}}}];
```

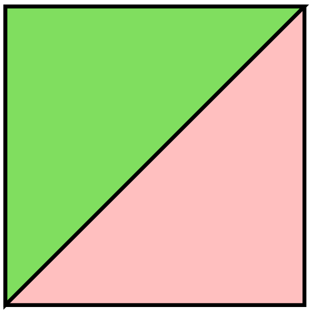
In[10]:=
```
Clear[t];
t[x_] := {{0, 1}, {1, 0}}.x
```

In[12]:=
```
tp = Polygon[{t[{0, 0}], t[{1, 0}], t[{1, 1}] }]
```

Out[12]=
```
Polygon[{{0, 0}, {0, 1}, {1, 1}}]
```

In[13]:=
```
Show[{Graphics[{Opacity[0.5], EdgeForm[Thick], Pink, p}],
  Graphics[{Opacity[0.5], EdgeForm[Thick], Green, tp}]}]
```

Out[13]=



This is a reflection transformation along the line y=x.

A triangle reflects into a triangle along y=x.

Question 3

In[14]:=
```
e1 = {1, 0, 0};
e2 = {0, 1, 0};
e3 = {0, 0, 1};
```

In[17]:=
```
y1 = {3, 5, -7};
y2 = {2, 0, 3};
y3 = {-1, 3, 5};
```

Part (a)

In[20]:=
```
image = -1 y1 + 2 y2 + 1 y3
```

Out[20]= {0, -2, 18}

Part (b)

In[21]:=
```
MatrixForm[tmatrix = Transpose[{y1, y2, y3}]]
```

Out[21]//MatrixForm=

$$\begin{pmatrix} 3 & 2 & -1 \\ 5 & 0 & 3 \\ -7 & 3 & 5 \end{pmatrix}$$

Check:

In[22]:=
```
tmatrix.{-1, 2, 1}
```

Out[22]= {0, -2, 18}

Question 4

Part (a)

Clearly $H = \text{Span}(\{v_1, v_2\}) = \text{Span}(B)$ so we need only show that B is linearly independent:
Vectors are not multiples of each other therefore they are linearly independent.

In[27]:=
```
a = {{3, -1}, {6, 0}, {2, 1}};
```

Part (b)

Solve B**y**=**x** for **y**=[**x**]$_B$:

```
GaussianReduce[a, {3, 12, 7}];
```

Number of rows = 3, number of columns = 2, coefficient matrix = $\begin{pmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{pmatrix}$, RHS = $\begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 2 & 6 \\ 2 & 1 & 7 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 2 & 6 \\ 0 & \frac{5}{3} & 5 \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Pivot columns are: {1, 2}

Row echelon form is $\begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Back-substitute for solution:

$x_2 = 3$

$x_1 = 2$

Reduced row echelon form is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

General solution is $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

A particular solution is $\mathbf{x}_p = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So the system is consistent, hence x∈H, and the coordinate vector $[x]_B$ is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Question 5

In each case:
(1) to determine if **b** is in the Nul(A) we simply test whether A**b**=**0**,
(2) to determine if **b** is in Col(A) we need to reduce the augmented matrix (A|**b**) and determine whether the sysem is consistent.

```
MatrixForm[a = {{30, 20, 18}, {50, 0, -20}, {70, 30, 17}}]
```

$$\begin{pmatrix} 30 & 20 & 18 \\ 50 & 0 & -20 \\ 70 & 30 & 17 \end{pmatrix}$$

Part (i)

For vector **b**:

```
MatrixForm[b = {5, 15, 15}]
```

$$\begin{pmatrix} 5 \\ 15 \\ 15 \end{pmatrix}$$

```
a.b
```

{720, -50, 1055}

This is not **0**, so **b** is not in Nul(A). To determine if it is in Col(A):

```
GaussianReduce[a, b];
```

Number of rows = 3, number of columns = 3, coefficient matrix = $\begin{pmatrix} 30 & 20 & 18 \\ 50 & 0 & -20 \\ 70 & 30 & 17 \end{pmatrix}$, RHS = $\begin{pmatrix} 5 \\ 15 \\ 15 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 30 & 20 & 18 & 5 \\ 50 & 0 & -20 & 15 \\ 70 & 30 & 17 & 15 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{6} \\ 0 & -\frac{100}{3} & -50 & \frac{20}{3} \\ 70 & 30 & 17 & 15 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{6} \\ 0 & -\frac{100}{3} & -50 & \frac{20}{3} \\ 0 & -\frac{50}{3} & -25 & \frac{10}{3} \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{6} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot columns are: {1, 2}

Row echelon form is $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{6} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Back-substitute for solution:

x_3 is a free variable

$x_2 = -\frac{1}{5} - \frac{3 x_3}{2}$

$x_1 = \frac{1}{10} (3 + 4 x_3)$

Reduced row echelon form is $\begin{pmatrix} 1 & 0 & -\frac{2}{5} & \frac{3}{10} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

General solution is $\mathbf{x} = \begin{pmatrix} \frac{1}{10} (3 + 4 x_3) \\ -\frac{1}{5} - \frac{3 x_3}{2} \\ x_3 \end{pmatrix}$

A particular solution is $\mathbf{x}_p = \begin{pmatrix} \frac{3}{10} \\ -\frac{1}{5} \\ 0 \end{pmatrix}$

A basis for the null space is $\left\{ \begin{pmatrix} \frac{2}{5} \\ -\frac{3}{2} \\ 1 \end{pmatrix} \right\}$

The system is consistent, so **b** is in Col(A).

Part (ii)

For vector **c**:

```
MatrixForm[c = {15, 5, -10}]
```

$\begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix}$

```
a.c
```

{370, 950, 1030}

This is not **0**, so **c** is not in Nul(A). To determine if it is in Col(A):

```
GaussianReduce[a, c];
```

Number of rows = 3, number of columns = 3, coefficient matrix = $\begin{pmatrix} 30 & 20 & 18 \\ 50 & 0 & -20 \\ 70 & 30 & 17 \end{pmatrix}$, RHS = $\begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 30 & 20 & 18 & 15 \\ 50 & 0 & -20 & 5 \\ 70 & 30 & 17 & -10 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{2} \\ 0 & -\frac{100}{3} & -50 & -20 \\ 70 & 30 & 17 & -10 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{2} \\ 0 & -\frac{100}{3} & -50 & -20 \\ 0 & -\frac{50}{3} & -25 & -45 \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{3}{5} \\ 0 & 0 & 0 & -35 \end{pmatrix}$

System is inconsistent: no solutions

The system is inconsistent, so **c** is not in Col(A).

Part (iii)

For vector **d**:

```
MatrixForm[d = {-8, 30, -20}]
```

$\begin{pmatrix} -8 \\ 30 \\ -20 \end{pmatrix}$

```
a.d
```

{0, 0, 0}

So **d** is in Nul(A). To determine if it is in Col(A):

```
GaussianReduce[a, d];
```

Number of rows = 3, number of columns = 3, coefficient matrix = $\begin{pmatrix} 30 & 20 & 18 \\ 50 & 0 & -20 \\ 70 & 30 & 17 \end{pmatrix}$, RHS = $\begin{pmatrix} -8 \\ 30 \\ -20 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 30 & 20 & 18 & -8 \\ 50 & 0 & -20 & 30 \\ 70 & 30 & 17 & -20 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & -\frac{4}{15} \\ 0 & -\frac{100}{3} & -50 & \frac{130}{3} \\ 70 & 30 & 17 & -20 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & -\frac{4}{15} \\ 0 & -\frac{100}{3} & -50 & \frac{130}{3} \\ 0 & -\frac{50}{3} & -25 & -\frac{4}{3} \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & \frac{2}{3} & \frac{3}{5} & -\frac{4}{15} \\ 0 & 1 & \frac{3}{2} & -\frac{13}{10} \\ 0 & 0 & 0 & -23 \end{pmatrix}$

System is inconsistent: no solutions

The system is inconsistent, so **d** is not in Col(A).

Question 6

A spanning set for Nul(A) is given by solving Ax = 0:

```
MatrixForm[a = {{1, -6, 9, 0, -2}, {0, 1, 2, -4, 5}, {0, 0, 0, 5, 1}, {0, 0, 0, 0, 0}}]
```

$$\begin{pmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
GaussianReduce[a, {0, 0, 0, 0}];
```

Number of rows = 4, number of columns = 5, coefficient matrix = $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, RHS = $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix = $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot position in row 1, column 1

Reducing augmented matrix row 2 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot position in row 2, column 2

Reducing augmented matrix row 3 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot position in row 3, column 4

Reducing augmented matrix row 4 ... $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot columns are: {1, 2, 4}

Row echelon form is $\begin{pmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Back-substitute for solution:

x_5 is a free variable

$x_4 = -\frac{x_5}{5}$

x_3 is a free variable

$x_2 = -2 x_3 - \frac{29 x_5}{5}$

$x_1 = -21 x_3 - \frac{164 x_5}{5}$

Reduced row echelon form is $\begin{pmatrix} 1 & 0 & 21 & 0 & \frac{164}{5} & 0 \\ 0 & 1 & 2 & 0 & \frac{29}{5} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

General solution is $\mathbf{x} = \begin{pmatrix} -21 x_3 - \frac{164 x_5}{5} \\ -2 x_3 - \frac{29 x_5}{5} \\ x_3 \\ -\frac{x_5}{5} \\ x_5 \end{pmatrix}$

A particular solution is $\mathbf{x}_p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

A basis for the null space is $\left\{ \begin{pmatrix} -21 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{164}{5} \\ -\frac{29}{5} \\ 0 \\ -\frac{1}{5} \\ 1 \end{pmatrix} \right\}$

A spanning set for Col(A) is given by the pivot columns of A:

```
{MatrixForm[Transpose[a][[1]]],
 MatrixForm[Transpose[a][[2]]], MatrixForm[Transpose[a][[4]]]}
```

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 5 \\ 0 \end{pmatrix} \right\}$

Question 7

Solution : There are 3 pivot columns so the number of vectors in basis of Col (A) = 3.
There are 2 free variables so number of vectors in basis of Nul (A) = 2.

Question 8

In[32]:=
```
A = {{1, 2, 0, 2, 1}, {-1, -2, 1, 1, 0}, {1, 2, -3, -7, -2}};
```

In[33]:=
```
A // MatrixForm
```

Out[33]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{pmatrix}$$

In[34]:=
```
RowReduce[A] // MatrixForm
```

Out[34]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivots: in the first and in the third columns. So the first and the third columns of matrix A are the basis vectors for the column space:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

Basis for the Null space is given by the spanning set of vectors such that Ax=0. The solution of this homogeneous equation is given by
x1= -2x2 - 2x4 - x5, x3=-3x4 - x5, where x2, x4, x5 are free variables. In the vector form we have

$$x=\begin{pmatrix} -2\,x2 - 2\,x4 - x5 \\ x2 \\ -3\,x4 - x5 \\ x4 \\ x5 \end{pmatrix} = x2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x4\begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x5\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

So the basis for the Nul A is given by the set

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$