Fundamentals of linear algebra

- Lecture 4: final slides and revision

- Linear dependence or independence
- Linear combinations of vectors
- Vector spaces and subspaces
- Basis of a vector space
- Coordinate systems

Wednesday, 17 April 2018

Lecture 4: Homogeneous linear systems

A linear system of equations is *homogeneous* if it has form

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

This system always has at least one solution $\mathbf{x} = \mathbf{0}$.
This solution is called a *trivial solution*.

For a given homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ an important question is whether there is a nontrivial solution $\mathbf{x} \neq \mathbf{0}$.

Example:

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lecture 4: Homogeneous linear systems

$$\begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So x_1 and x_2 are basic variables, and x_3 is a free variable.

$$\begin{aligned} x_1 - (4/3)x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The solution set is:

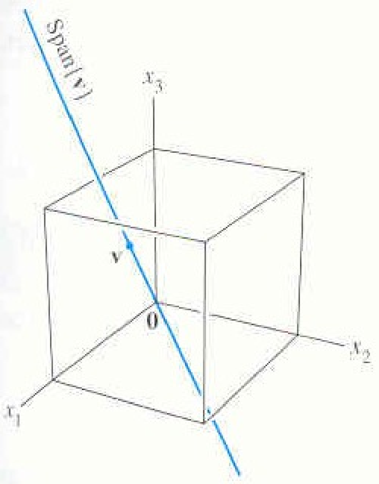
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

So every solution is scalar multiple $t\mathbf{v}$ of a vector $\mathbf{v}$, $\forall\, t \in \mathbb{R}$.

$$\mathbf{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \equiv t\mathbf{v}$$

Geometrically, this solution set represents a line though $\mathbf{0}$ in $\mathbb{R}^3$.

A homogeneous system has a non-trivial solution if and only if there is at least one free variable.



Example. Describe all the solutions of the homogeneous equation:

$$10x_1 - 3x_2 - 2x_3 = 0$$

A general solution is $x_1 = 0.3x_2 + 0.2x_3$. In a vector form, that is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

which is

$$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$$

where

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

Lecture 4: Homogeneous linear systems

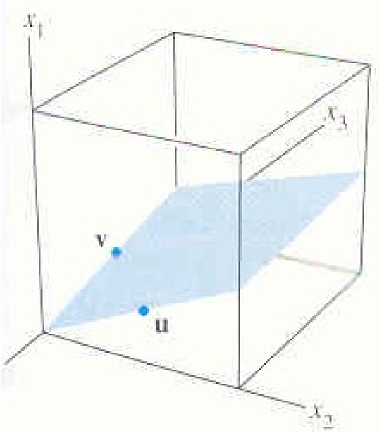
This solution $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$ is a *parametric equation* of a plane through the origin, defined by

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, so the solution set is $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

For any homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, solution set can be written as $\mathrm{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ (for some vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$).

If the only solution is zero vector, then the solution set is $\mathrm{Span}\{\mathbf{0}\}$.



Now consider $\mathbf{A}\mathbf{x} = \mathbf{b}$ (same $\mathbf{A}$) with the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system becomes

$$\begin{aligned} x_1 - (4/3)x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

The solution is $x_1 = (4/3)x_3 - 1$, $x_2 = 2$ and x_3 is a free variable.

In a vector form, the solution can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

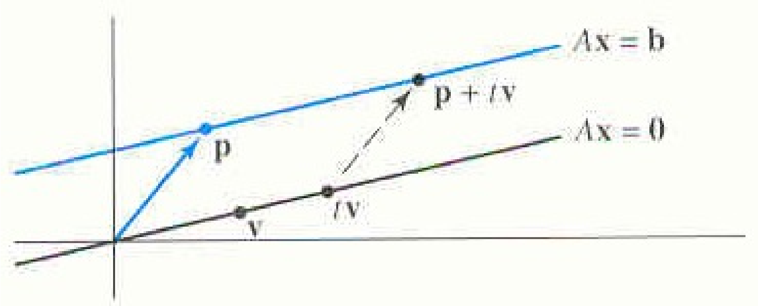
which is $\quad \mathbf{x} = \mathbf{p} + x_3\mathbf{v} \quad$ with $\quad \mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$

Recall the solution of the homogeneous system was $\mathbf{x} = x_3\mathbf{v}$.

Thus the solutions of inhomogeneous system $\mathbf{Ax} = \mathbf{b}$ are obtained by adding $\mathbf{p}$ to the solution of homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Vector $\mathbf{p}$ is a *particular solution* of $\mathbf{Ax} = \mathbf{b}$ for $x_3 = 0$.

Visualisation of $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ in $\mathbb{R}^2$:

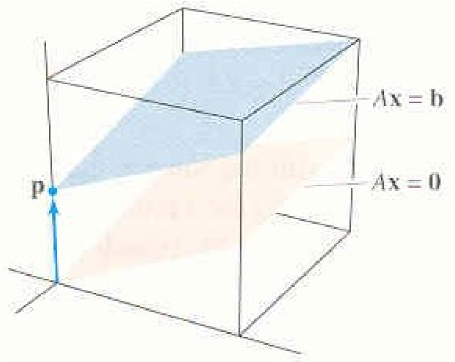


Solutions to a homogeneous system form a line through $\mathbf{0}$, and to the inhomogeneous system — a parallel line, shifted by $\mathbf{p}$.

Theorem:

Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ and let $\mathbf{p}$ be a solution.

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ is a set of all vectors in the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$, where $\mathbf{v}_0$ is any solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.



Proof: $\mathbf{Aw} = \mathbf{A}(\mathbf{p} + \mathbf{v}_0) = \mathbf{Ap} + \mathbf{Av}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Lecture 4: Inhomogeneous systems

Summary:

Step 1: Row-reduce the augmented matrix to REF

Step 2: Express each basic variable (variables of the pivot columns) in terms of any free variables

Step 3: Write a typical solution $\mathbf{x}$ in the vector form

Step 4: Decompose $\mathbf{x}$ into a linear combination of vectors using the free variables as scalars.

Solutions to a homogeneous system form a sub-space through $\mathbf{0}$.

Solutions to the corresponding inhomogeneous system form a parallel sub-space, shifted by vector $\mathbf{p}$ of a particular solution.

Lecture 4: Inhomogeneous systems

Example:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Basic variables are x_1, x_2, x_5 and free variables are x_3, x_4.

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 - 24 \\ x_2 &= 2x_3 - 2x_4 - 7 \\ x_5 &= 4 \end{aligned}$$

General solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix}$$

Lecture 4: Inhomogeneous systems

General solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix}$$

Therefore

$$\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v} + \mathbf{p}$$

where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Lecture 4: Inhomogeneous systems

$\mathbf{u}$ and $\mathbf{v}$ are solutions to the corresponding homogeneous system

$$\begin{array}{rcrcrcrcrl} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = 0 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 9x_5 & = 0 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = 0 \end{array}$$

Check by substituting $\mathbf{u} = (2;\ 2;\ 1;\ 0;\ 0)$:

$$\begin{array}{rcrcrcrcrl} & & 3 \cdot 2 & - & 6 \cdot 1 & + & 0 & + & 0 & = 0 \\ 3 \cdot 2 & - & 7 \cdot 2 & + & 8 \cdot 1 & - & 0 & + & 0 & = 0 \\ 3 \cdot 2 & - & 9 \cdot 2 & + & 12 \cdot 1 & - & 0 & + & 0 & = 0 \end{array}$$

Check by substituting $\mathbf{v} = (-3;\ -2;\ 0;\ 1;\ 0)$:

$$\begin{array}{rcrcrcrcrl} & & 3 \cdot (-2) & - & 0 & + & 6 \cdot 1 & + & 0 & = 0 \\ 3 \cdot (-3) & - & 7 \cdot (-2) & + & 0 & - & 5 \cdot 1 & + & 0 & = 0 \\ 3 \cdot (-3) & - & 9 \cdot (-2) & + & 0 & - & 9 \cdot 1 & + & 0 & = 0 \end{array}$$

Lecture 4: Inhomogeneous systems

Once again: $\mathbf{u}$ and $\mathbf{v}$ are solutions to the homogeneous system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= 0 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 &= 0 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 0 \end{aligned}$$

The general solution is: $\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v}$, so: $\mathbf{x} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Then, $\mathbf{p}$ is a particular solution to the inhomogeneous system, which is obtained by specifying $x_3 = x_4 = 0$

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 \quad &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 \quad &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 \quad &= 15 \end{aligned}$$

The general solution here is: $\mathbf{x} = \mathbf{p} + x_3\mathbf{u} + x_4\mathbf{v}$.

Lecture 4: Summary

- Properties of vectors in $\mathbb{R}^n$, and linear combinations.

- Equivalence between $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mathbf{x} = \mathbf{b} \qquad \Leftrightarrow \qquad x_1\mathbf{a}_1 + \ldots x_n\mathbf{a}_n = \mathbf{b}$$

- Homogeneous $\mathbf{Ax} = \mathbf{0}$ and inhomogeneous $\mathbf{Ax} = \mathbf{b}$ systems:

  Relation $\mathbf{w} = \mathbf{v}_0 + \mathbf{p}$ between the solutions.

Lecture 5

- Linear dependence or independence
- Linear combinations of vectors
- Vector spaces and subspaces
- Basis of a vector space
- Coordinate systems

Linear independence

Definitions:

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

  has only the trivial solution (with all $c_i = 0$).

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is *linearly dependent* if there are weights $c_1, c_2, \ldots c_m$, not all equal to zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

  The above relation is a linear dependence relation.

Quite obviously, a set of vectors is linearly dependent if and only if it is not linearly independent (and vice versa).

Linear independence: Example

Example: Determine if the set $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is linearly dependent

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To do so, we need to solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ and check if there is a nontrivial solution.

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to REF as

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear independence: Example

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has x_1, x_2 as basic, and x_3 as a free variable.

Now we can obtain the linear dependence equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ in an explicit form, by solving

$$\left\{ \begin{array}{rcrcl} x_1 & - & 2x_3 & = & 0 \\ x_2 & + & x_3 & = & 0 \\ & & 0 & = & 0 \end{array} \right. \qquad \left| \qquad \begin{array}{rcl} x_1 & = & 2x_3 \\ x_2 & = & -x_3 \\ x_3 & \in & \mathbb{R} \end{array} \right.$$

Any non-zero value for x_3 yields a nontrivial solution; e.g. $x_3 = 1$:

$$2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

Thus, vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

Any other coefficients satisfying the above relations are suitable.

Linear independence: Remarks

- A matrix equation $\mathbf{Ax} = \mathbf{0}$ is equivalent to the vector form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

  Any linear dependence relation between the columns of $\mathbf{A}$ corresponds to a nontrivial solution of $\mathbf{Ax} = \mathbf{0}$.

  So, the columns of matrix $\mathbf{A}$ are linearly independent if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

- Single vector $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$ (because $c\,\mathbf{v} = \mathbf{0}$ has only the trivial solution for $\mathbf{v} \neq \mathbf{0}$).

- Zero vector is linearly dependent as the equation $c \cdot \mathbf{0} = \mathbf{0}$ has infinite number of non-trivial solutions.

Linear independence: Example

Example:

Check if the columns of this matrix are linearly independent:

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$$

The EF indicates that there are no free variables

So the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$ has only a trivial solution, implying that vectors $\{\mathbf{a}_i\}$ are linearly independent.

In other words, each column has a pivot, so the $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution and columns of $\mathbf{A}$ are linearly independent.

Linear independence for two vectors

Example 1: $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

We can see that $\mathbf{v}_2 = 2\mathbf{v}_1$, so $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ which implies that the set of $\mathbf{v}_1$ and $\mathbf{v}_2$ is linearly dependent.

Example 2: $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Suppose there are non-zero scalars c, d such that $c\mathbf{v}_3 + d\mathbf{v}_4 = \mathbf{0}$. Then $\mathbf{v}_3 = (-d/c)\mathbf{v}_4$, implying them to be multiples of each other. That is not the case, so $\{\mathbf{v}_3, \mathbf{v}_4\}$ is an independent set.

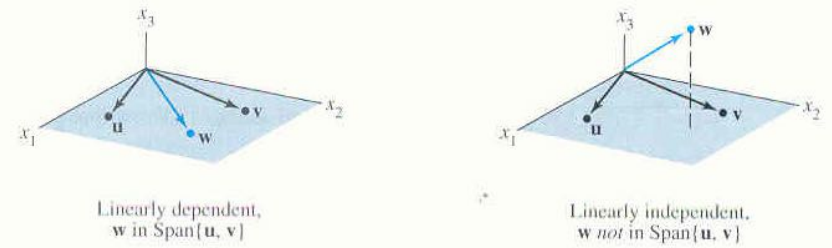
Example 3: $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}.$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other. They span the (x_1, x_2) plane in $\mathbb{R}^3$.

Linear independence for many vectors

It is straightforward to check if two vectors are linearly dependent:
It is sufficient to find out whether they are multiples of each other.

For many vectors, a more formal consideration is required.



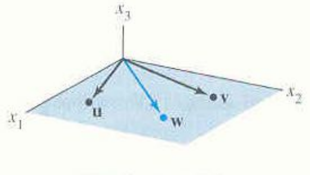
Linear independence (three vectors)

Theorem:

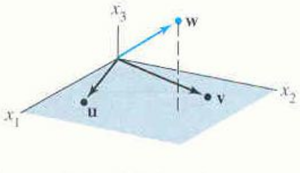
$\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Proof: If $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$, which can be rewritten as $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ with $c_3 = -1$ and non-trivial pair c_1, c_2. Therefore, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Conversely, if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set, then $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ for some non-trivial c_1, c_2, c_3, therefore $\mathbf{w} = -(c_1/c_3)\mathbf{u} - (c_2/c_3)\mathbf{v}$, implying $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.



Linearly dependent,
$\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$



Linearly independent,
$\mathbf{w}$ *not* in Span$\{\mathbf{u}, \mathbf{v}\}$

Linear independence: Theorems

Theorem:

- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ of two or more vectors in $\mathbb{R}^n$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.
- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.

Theorem:

- Any set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$. That is, if a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

Compose $\mathbf{A} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$, an $n \times p$ matrix. Then $\mathbf{Ax} = \mathbf{0}$ corresponds to n equations with p unknowns. If $p > n$, there are more variables than equations so there must be free variables and non-trivial solutions, so the columns of $\mathbf{A}$ are linearly dependent.

Linear independence: Example

Consider a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_6\}$ given by the columns of

$$\mathbf{v} = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Upon row reduction, we find which $\mathbf{v}_i$ are linearly independent.

$$\mathbf{v} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are pivots in columns 1, 2, 4, 6.

Therefore $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_6$ are linearly independent.

Linear independence: Example

Let us determine the linear dependence explicitly:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 & - & 2x_5 \\ 3x_3 & + & 2x_5 \\ x_3 & & \\ & & x_5 \\ & & x_5 \\ & 0 & \end{bmatrix}$$

Linear independence: Example

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 & - & 2x_5 \\ 3x_3 & + & 2x_5 \\ x_3 & & \\ & & x_5 \\ & & x_5 \\ & 0 & \end{bmatrix}$$

Using this solution in the form

$$\begin{cases} x_1 = -2x_3 - 2x_5 \\ x_2 = 3x_3 + 2x_5 \end{cases} \quad \text{and} \quad \begin{cases} x_4 = x_5 \\ x_6 = 0 \end{cases}$$

we can rewrite the vector equation as follows:

$$\begin{aligned} x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 + x_5\mathbf{v}_5 + x_6\mathbf{v}_6 &= \mathbf{0} \\ (-2x_3 - 2x_5)\mathbf{v}_1 + (3x_3 + 2x_5)\mathbf{v}_2 + x_3\mathbf{v}_3 + x_5\mathbf{v}_4 + x_5\mathbf{v}_5 + 0 \cdot \mathbf{v}_6 &= \mathbf{0} \\ x_3(-2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3) + x_5(-2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5) &= \mathbf{0} \end{aligned}$$

Linear independence: Example

So: $x_3(-2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3) + x_5(-2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5) = \mathbf{0}$

This relation must be true for any x_3 and x_5, therefore

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v}_5 = 2\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4$$

This can be also seen directly from the reduced matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dependence coefficients appear in the non-pivot columns.

Thus, $\mathbf{v}_3$ depends on $\{\mathbf{v}_1, \mathbf{v}_2\}$, and $\mathbf{v}_5$ depends on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$.

Linear independence: Example

Example: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$
must be linearly dependent because there are only two entries in each vector ($n = 2$) but there are 3 vectors ($p = 3$).

Indeed, if we compose the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, upon REF reduction the corresponding matrix is

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The basic variables are x_1, x_2 and the free variable is x_3.

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \left| \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \end{array} \right.$$

So the vector equation becomes $-x_3\mathbf{v}_1 + x_3\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ and so the explicit form of the linear dependence is: $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear independence: Summary

- A single vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.
- A set of two non-zero vectors $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one is a multiple of the other.
- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.
- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.
- Any set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$.
- If a set $S = \{\mathbf{v}_1, \ldots \mathbf{v}_m\} \in \mathbb{R}^n$ contains $\mathbf{0}$, then the set is linearly dependent.
  (suppose $\mathbf{v}_i = \mathbf{0}$, then $0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 1\mathbf{v}_i + \ldots + 0\mathbf{v}_m = \mathbf{0}$ which demonstrates a linear dependence)

Vector space: Definition

A *vector space* in $\mathbb{R}^n$ is a non-empty set V of vectors, on which two operations — addition and multiplication by real scalars — are defined, subject to these axioms:

(i) $(\mathbf{u}+\mathbf{v}) \in V$

(ii) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(iii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iv) $\exists\, \mathbf{0}$ such that $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$

(v) $\forall\, \mathbf{u}\ \exists\, (-\mathbf{u})$ such that $\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}$

(vi) $c\,\mathbf{u} \in V$

(vii) $c\,(\mathbf{u}+\mathbf{v}) = c\,\mathbf{u}+c\,\mathbf{v}$

(viii) $(c+d)\mathbf{u} = c\,\mathbf{u}+d\,\mathbf{u}$

(ix) $c\,(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$

(x) $1\mathbf{u} = \mathbf{u}, \quad (-\mathbf{u}) = (-1)\mathbf{u}$

These rules must hold for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and for any c, $d \in \mathbb{R}$.

Vector spaces: Examples

1. An $\mathbb{R}^n$ space itself is a vector space.

2. A set of all arrows (directed line segments) in 2D, with

- multiplication $c\mathbf{v}$ defined to produce an arrow with the length $|c|$ times the length of $\mathbf{v}$ and pointing in the same direction as $\mathbf{v}$ for $c > 0$ or in the opposite direction for $c < 0$ (Fig. 1);
- addition defined by parallelogram rule shown in Fig. 2; e.g. axiom $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ is verified in Fig. 3.

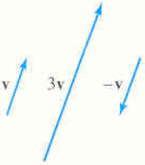


FIGURE 1

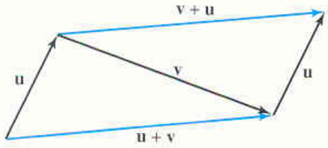


FIGURE 2 $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.

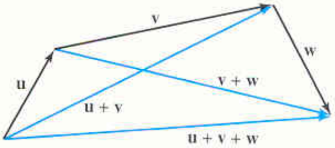


FIGURE 3 $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Vector spaces: Examples

3. Let $\mathcal{S}$ be a space of "double-infinite" sequences of numbers:

$$\mathfrak{Y} = \langle y_k \rangle = \{\ldots\ y_{-2},\ y_{-1},\ y_0,\ y_1,\ y_2, \ldots\}$$

If $\mathfrak{Z} = \langle z_k \rangle$ is another element of $\mathcal{S}$, the sum is $\mathfrak{Y} + \mathfrak{Z} = \langle y_k + z_k \rangle$.

The scalar multiple is defined by $c \cdot \mathfrak{Y} = \langle c y_k \rangle$.

For $\mathcal{S}$, all the axioms can be verified, so this is a vector space.

Such vector spaces arise in engineering when a signal (such as electrical, optical or mechanical) is measured at discrete times.

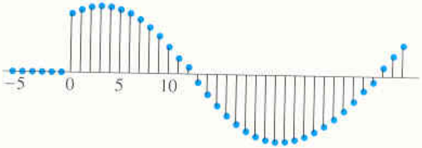


FIGURE 4 A discrete-time signal.

Vector spaces: Examples

4. For $n \geqslant 0$ let $\mathcal{P}_n$ be a set of polynomials of a degree up to n

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where variable t and coefficients $a_0 \ldots a_n$ are real numbers.

The degree of $\mathbf{p}$ is the highest power of t with non-zero coefficient (for $\mathbf{p}(t) = a_0 \neq 0$ it is zero). The *zero polynomial* has all $a_i = 0$.

Given $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n$, the sum is defined as

$$\{\mathbf{p}+\mathbf{q}\}(t) = \mathbf{p}(t)+\mathbf{q}(t) = (a_0+b_0)+(a_1+b_1)t+\ldots+(a_n+b_n)t^n$$

The scalar multiple $c\,\mathbf{p}$ is the polynomial defined by

$$c\,\mathbf{p}(t) = ca_0 + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n$$

The axioms of a vector space are satisfied so $\mathcal{P}_n$ is a vector space.

Vector spaces: Examples

5. Let $\mathcal{F}$ be a space of all real-valued functions defined on a number space $\mathcal{D}$.

- Addition is defined as the function $\{\mathbf{f}+\mathbf{g}\}$ with the value equal to $\mathbf{f}(t)+\mathbf{g}(t)\ \forall\, t \in \mathcal{D}$
- Scalar multiplication by c is defined as the function $c\,\mathbf{f}$ with the value $c\cdot\mathbf{f}(t)\ \forall\, t \in \mathcal{D}$
- Two functions are equal if their values are equal $\forall\, t \in \mathcal{D}$
- The zero vector in $\mathcal{F}$ is $\mathbf{f}_0(t) \equiv 0\ \forall\, t \in \mathcal{D}$
- The negative of $\mathbf{f}$ is $\bar{\mathbf{f}}$ such that $\bar{\mathbf{f}}(t) = -\mathbf{f}(t)$

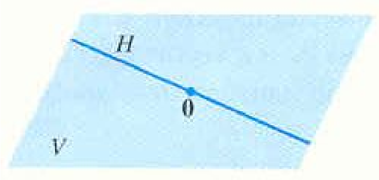
All the axioms are valid for $\mathcal{F}$ so it is a vector space.

Subspaces

Definition: A **subspace** H of a vector space V is a subset of vectors with the following properties:

- H includes the zero vector of V
- H is closed under vector addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\;\; \mathbf{u}+\mathbf{v} \in H$
- H is closed under multiplication by scalars:
  $\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R}$, $c\mathbf{v} \in H$

Every subspace is a vector space and satisfies the ten axioms.



Subspaces: Examples

1. A set consisting of only the zero vector of a vector space V is a subspace of V and is called *zero subspace* $\{\mathbf{0}\}$.
2. Consider space $\mathcal{P}$ of all polynomials with real coefficients, with operations in $\mathcal{P}$ defined as for real-valued functions. Then $\mathcal{P}$ is a subspace of the space $\mathcal{F}$ of all real-valued functions operating on $\mathbb{R}$, and $\mathcal{P}_n$ is the subspace of $\mathcal{P}$.
3. A line within $\mathbb{R}^2$, not passing through the origin, is not a subspace of $\mathbb{R}^2$, as it does not contain the $\mathbf{0}$ vector of $\mathbb{R}^2$.
4. A plane within $\mathbb{R}^3$, not including the origin, is not a subspace of $\mathbb{R}^3$ because this plane does not contain the $\mathbf{0}$ vector of $\mathbb{R}^3$.

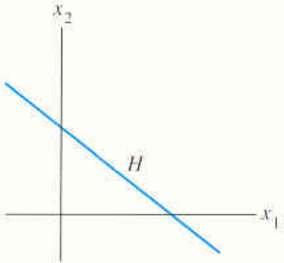


FIGURE 8
A line that is not a vector space.

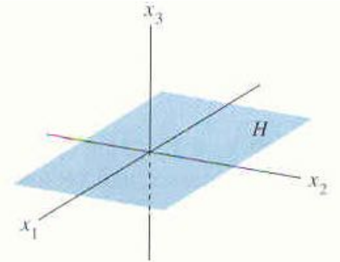
Subspaces: Examples

5. The entire vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. Vectors in $\mathbb{R}^3$ have three entries whereas vectors in $\mathbb{R}^2$ have two. However, a set like

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}, \quad (s, t) \in \mathbb{R}$$

is a subset of $\mathbb{R}^3$ that looks exactly like $\mathbb{R}^2$.

Indeed, this subset includes the zero vector of $\mathbb{R}^3$, and the set is closed: Any multiplication by a scalar or any addition of two vectors, produces a vector from this subset (because the third component is always zero).

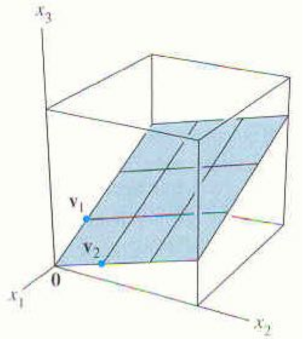


Subspaces: Examples

6. Consider $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where vectors $\mathbf{v}_1,\ \mathbf{v}_2 \in V$.

(a) The zero vector is in H because

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$



(b) Any two vectors in H can be written as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$$

therefore their sum $\mathbf{u} + \mathbf{w} \in H$ because

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \\ &= (c_1 + s_1)\mathbf{v}_1 + (c_2 + s_2)\mathbf{v}_2 \end{aligned}$$

(c) For any $c \in \mathbb{R}$ vector $c\,\mathbf{u} \in H$ because

$$c\,\mathbf{u} = c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2$$

Thus H is a subspace of V.

Subspaces spanned by a set

Theorem:

For $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in V$, $\quad \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ is a subspace of V.

This is called a subspace spanned (generated) by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$.

Example 1: Let H be a set of all vectors of the form $\big[(a-3b);\ (b-a);\ a;\ b\big]$ where a, b are arbitrary scalars:

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \equiv a\mathbf{v}_1 + b\mathbf{v}_2.$$

This rearrangement demonstrates that $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Therefore, H is a subspace of $\mathbb{R}^4$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Subspaces spanned by a set: Example

Example 2: Find h such that $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$, if

$$\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: vector $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if $\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

This vector equation corresponds to the augmented matrix

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

which is only consistent if $h = 5$, so then $\mathbf{y} = [-4; 3; 5]$.

Subspaces spanned by a set: Example

Continue reduction towards REF, taking into account $h = 5$:

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is now equivalent to the system

$$\begin{cases} x_1 + 7x_3 = 1 \\ x_2 - 2x_3 = -1 \end{cases} \quad \Bigg| \quad \begin{aligned} x_1 &= 1 - 7x_3 \\ x_2 &= -1 + 2x_3 \end{aligned}$$

Setting the free variable $x_3 = 0$, we get $x_1 = 1$ and $x_2 = -1$.

Thus $\mathbf{y} = 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$, which is easily checked:

$$\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

Next lecture

See you two weeks later

1 May 2019

Assignment 4 is due Week 7 (on 1–3 May)

Written assignment (essay) is due Week 9 (on 15–17 May)

No tutorial classes this week (17–19 April)