Lecture 6

- *Linear Transformations*
- *Fundamental Spaces of Linear Algebra*
  - *Null Space*
  - *Column Space*
  - *Basis for Null Space*
  - *Basis for Column Space*

Linear Transformations

- A matrix equation $A\mathbf{x} = \mathbf{b}$ can be considered as a vector equation
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}.$$
- Matrix equation $\mathbf{Ax} = \mathbf{b}$ can be viewed as a "function" or mapping.
- We can see this if we think of the matrix $\mathbf{A}$ as an object that acts on the variable $\mathbf{x}$, in this case a vector, to produce the new vector $\mathbf{b}$; $\mathbf{b} = \mathbf{Ax}$.
- This is similar to the action of a function $y = f(x)$ (generalization of functions).
- The correspondence from $\mathbf{x}$ into $A\mathbf{x}$ is a function from one set of vectors to another.
- Pre-multiplication by $\mathbf{A}$ transforms $\mathbf{x}$ into $\mathbf{b}$.

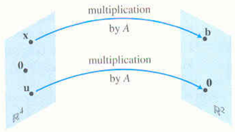

FIGURE 1 Transforming vectors via matrix

Linear Transformations

Example

$$\begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

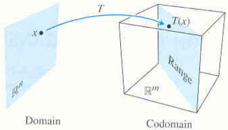

FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^2$.

Linear Transformations

Definition: A transformation (function or mapping) T from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

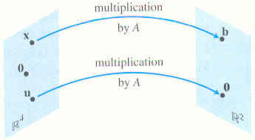

FIGURE 1 Transforming vectors via matrix multiplication.

- The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.
- The notation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For a given $\mathbf{x}$ in $\mathbb{R}^n$ the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$.
- The set of all images $T(\mathbf{x})$ is called the **range** of T.

Linear Transformations

- For each $\mathbf{x}$ in $\mathbb{R}^n$, a **linear** transformation $T(\mathbf{x})$ can be computed as $\mathbf{Ax}$ where $\mathbf{A}$ is an $m \times n$ matrix: $T(\mathbf{x}) = \mathbf{Ax}$.
- Note, the domain of T is $\mathbb{R}^n$ when $\mathbf{A}$ has n columns, and the codomain of T is $\mathbb{R}^m$ when each column of $\mathbf{A}$ has m entries.
- The range of T is the set of all linear combinations of the columns of $\mathbf{A}$ given each image is in the form $\mathbf{Ax}$.
- We denote the matrix transformation as $\mathbf{x} \mapsto \mathbf{Ax}$.

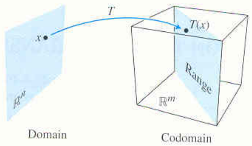

FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Linear Transformations

Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Find the image of $\mathbf{u}$.
The transformation is $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ so that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

$$T(\mathbf{u}) = \mathbf{A}\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

Linear Transformations

Definition: The transformation T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all scalars c.

If T is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$
Indeed

$$T(\mathbf{0}) = T(0 \times \mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$$

Also

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

Linear Transformation

Example: Show that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is a linear transformation where $\mathbf{A}$ is an $m \times n$ matrix and

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n.$$

So

$$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = (x_1+y_1)\mathbf{a}_1 + \ldots + (x_n+y_n)\mathbf{a}_n$$
$$= x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n + y_1\mathbf{a}_1 + \ldots + y_n\mathbf{a}_n = T(\mathbf{x}) + T(\mathbf{y}).$$
$$T(c\mathbf{x}) = (cx_1)\mathbf{a}_1 + \ldots + (cx_n)\mathbf{a}_n =$$
$$= c(x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n) = cT(\mathbf{x}).$$

Therefore

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}),$$
$$T(c\mathbf{x}) = cT(\mathbf{x})$$

and T is a linear transformation.

Linear Transformations

Example: Let
$T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$.
Show that T is **not** a linear transformation.
In matrix form:

$$T(x_1, x_2) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

So $T(\mathbf{x}) = A\mathbf{x} + \mathbf{q}$, where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}, \text{ and } \mathbf{q} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

If T is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$. So $T(\mathbf{0}) = \mathbf{q} \neq \mathbf{0}$.

Linear Transformations

The transformation approach in linear systems has a dynamical view of matrix-vector multiplication.

Example: Let $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

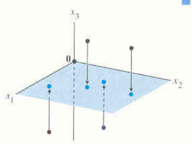

FIGURE 3
A projection transformation.

This transformation projects points in $\mathbb{R}^3$ into $\mathbb{R}^2$. This is a **projection** transformation.

Linear Transformations

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = r\mathbf{x}$, where $r \in \mathbb{R}$. Show that T is a linear transformation.

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= r(c\mathbf{u} + d\mathbf{v}) = c(r\mathbf{u}) + d(r\mathbf{v}) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}). \end{aligned}$$

T is called a *dilation*.

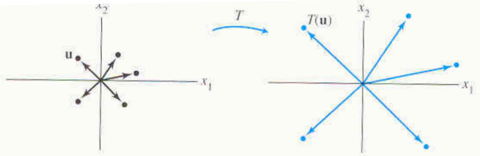

FIGURE 5 A dilation transformation.

Linear Transformations

Example 13: Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is called a *shear* transformation.

It can be shown that the image of a square is a parallelogram.

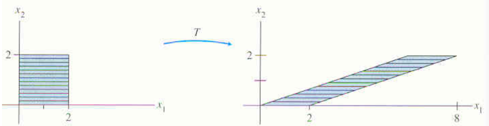

FIGURE 4 A shear transformation.

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

Standard basis, standard matrix

In $\mathbb{R}^2$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called the *standard unit vectors in* $\mathbb{R}^2$ and form the *standard basis* for $\mathbb{R}^2$. So

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2.$$

For any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2).$$

In matrix-vector form we can write

$$T(\mathbf{x}) = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\mathbf{x}$$

The matrix $\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix}$ is called the *standard matrix* of the linear transformation T. (Extends to $T : \mathbb{R}^n \to \mathbb{R}^m$)

Linear Transformations

Example: Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and}$$

$$\mathbf{y}_1 = \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}, \ \mathbf{y}_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \ \mathbf{y}_3 = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}.$$

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a linear transformation that maps $\mathbf{e}_k$ to $\mathbf{y}_k$ for $k = 1, 2, 3$.

a) Write down the standard matrix representation of T;

b) Find the image of $\mathbf{u} = (3 \ \ 2 \ \ -1)^T$.

Linear Transformations

Solution: (a) The standard matrix $\mathbf{A}$ of T has columns given by the $\mathbf{y}_k$:

$$\mathbf{A} = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)] = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]$$

$$= \begin{bmatrix} 3 & 2 & -1 \\ 5 & 0 & 3 \\ -7 & 3 & 5 \end{bmatrix}.$$

(b) The vector $\mathbf{u}$ is

$$\mathbf{u} = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}.$$

The image of $\mathbf{u}$ is

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 3 & 2 & -1 \\ 5 & 0 & 3 \\ -7 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ 12 \\ -20 \end{bmatrix}.$$

Linear Transformations

Example: Find the standard matrix for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$.

Solution:

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

$$T(\mathbf{e}_2) = 3\mathbf{e}_2 = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Therefore

$$\begin{aligned} \mathbf{A} &= (T(\mathbf{e}_1)\ T(\mathbf{e}_2)) \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}. \end{aligned}$$

Linear Transformations

Example: Define the matrix of a transformation that rotates a vector around the origin in $\mathbb{R}^2$ by φ radians.

$$T(\mathbf{e}_1) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}.$$

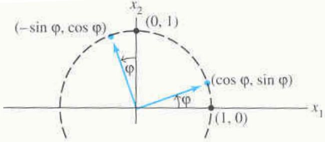

FIGURE 1 A rotation transformation.

Therefore

$$\mathbf{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Linear Transformations

Example (I) Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$ and $T(\mathbf{x}) = A\mathbf{x}$. Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$.

Augmented matrix is

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array}\right)$$

Therefore $x_1 = 1.5$, and $x_2 = -0.5$ and

$$\mathbf{x} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$$

The image of $\mathbf{x}$ is $\mathbf{b}$.

Example (I): Are there any other $\mathbf{x}$ with the image $\mathbf{b}$?

Any $\mathbf{x}$ whose image under T is $\mathbf{b}$ must satisfy the system above. The system has a unique solution so there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$.

Linear Transformations

Example (II): Determine if $\mathbf{c}$ is in the range of the transformation T.

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

Vector $\mathbf{c}$ is in the range of $T(\mathbf{x})$ if $\mathbf{c}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$.
So we need to solve the system $A\mathbf{x} = \mathbf{c}$.
The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right)$$

From the third equation $0 = -35$. So the system is inconsistent and $\mathbf{c}$ is **not** in the range of T.
Example (I) is a uniqueness problem for a system of linear equations $A\mathbf{x} = \mathbf{b}$: Is $\mathbf{b}$ the image of a unique $\mathbf{x}$ in $\mathbb{R}^m$?
Example (II) is an existence problem: Is there any $\mathbf{x}$ whose image is $\mathbf{c}$?

Linear Transformations

The concept of linear transformations provides a new way to understand existence and uniqueness questions.

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

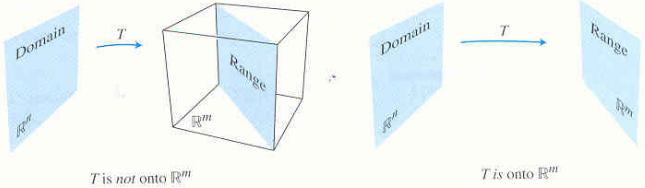

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

Linear Transformations

Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

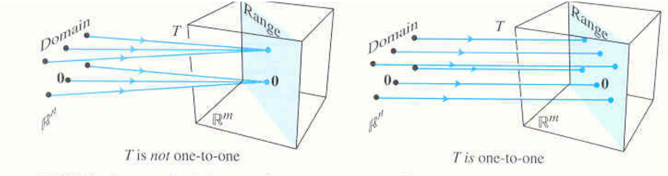

FIGURE 4 Is every **b** the image of at most one vector?

- Equivalently, T is one-to-one if, for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all.
- "Is T one-to-one?" is a uniqueness question.

Example : Let T be the linear transformation whose standard matrix is

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

(a) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?
(b) Is T one-to-one?

Solution: It is clear that the augmented matrix $[\mathbf{A}\ \mathbf{b}]$ always has a solution for any $\mathbf{b}$. Therefore this is an onto mapping. However there are free parameters and each $\mathbf{b}$ is the image of more then one $\mathbf{x}$, so it is **not** one-to-one.

Linear Transformations

Theorem: Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
Proof: Since T is linear then, $T(\mathbf{0}) = \mathbf{0}$.
If T is one-to-one, then the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution: the trivial solution $\mathbf{x} = \mathbf{0}$.
We prove the converse by contradiction. Let $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, and assume that T is *not* one-to-one.
Then there is a $\mathbf{b}$ that is the image of at least two different vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$: $T(\mathbf{u}) = \mathbf{b}$, $T(\mathbf{v}) = \mathbf{b}$.
But since T is linear,
$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$
The vector $\mathbf{u} - \mathbf{v}$ is not zero since $\mathbf{u} \neq \mathbf{v}$ and so the equation $T(\mathbf{x}) = \mathbf{0}$ has more than one solution: $\mathbf{0}$ and $\mathbf{u} - \mathbf{v}$. This contradicts the fact that $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, from which it follows that the assumption the T is not one-to-one must be false, that is, T is one-to-one.

Linear Transformations

Theorem: Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\mathbf{A}$ be the standard matrix for T. Then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{A}$ span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of $\mathbf{A}$ are linearly independent.

Proof: a) The columns of $\mathbf{A}$ span $\mathbb{R}^m$ if and only if for each $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ the equation $\mathbf{Ax} = \mathbf{b}$ is consistent, which means the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$. This is true only when T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b) The equation $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ has only the trivial solution if T is one-to-one. This can happen only if the columns of $\mathbf{A}$ are independent. Otherwise there will be nontrivial solutions, which implies linear dependence.

Example: Let

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2).$$

Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$? (and what **are** m, n?)
Solution: The matrix of the linear transformation is

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

The columns of $\mathbf{A}$ are linearly independent (they are not multiples). Therefore T is one-to-one. The columns of $\mathbf{A}$ span $\mathbb{R}^3$ if and only if A has three pivots, **but** $\mathbf{A}$ has only two columns. So the columns of $\mathbf{A}$ do not span $\mathbb{R}^3$ and T is **not** onto.

Null Spaces, Column Spaces

The Null Space of a Matrix

- Consider the following system of homogeneous equations:

$$x_1 - 3x_2 - 2x_3 = 0$$
$$-5x_1 + 9x_2 + x_3 = 0$$

- In matrix-vector form this system is written as $\mathbf{Ax} = \mathbf{0}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}.$$

- The set of all $\mathbf{x}$ that satisfy the linear system is called the *solution set* of the system.
- We call the set of all $\mathbf{x}$ that satisfy $\mathbf{Ax} = \mathbf{0}$ the *null space* of the matrix $\mathbf{A}$.
- **Definition:** The **null space** of an $m \times n$ matrix $\mathbf{A}$, written as *Nul* $\mathbf{A}$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation, *Nul* $\mathbf{A} = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$.

The Null Space of a Matrix

The dynamic description of *Nul* $\mathbf{A}$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ that are mapped onto the zero vector of $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ (Fig.1).

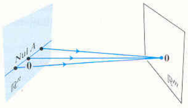

FIGURE 1

Example: Let $\mathbf{A}$ and $\mathbf{u}$ be given by

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

Does $\mathbf{u} \in$ *Nul* $\mathbf{A}$? **Solution:** $\mathbf{Au} = \mathbf{0}$?

$$A\mathbf{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $\mathbf{u}$ is in *Nul* A.

The Null Space of a Matrix

Theorem: The null space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $\mathbf{Ax} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$. *Nul A* is a subset of $\mathbb{R}^n$ given $\mathbf{A}$ has n columns and $\mathbf{x}$ has n components.

Proof: We must show that *Nul A* satisfies the three properties of a subspace. 1) $\mathbf{0}$ is in *Nul A* of course. 2) Next let $\mathbf{u}, \mathbf{v} \in \; NulA$. Then $\mathbf{Au} = \mathbf{0}$ and $\mathbf{Av} = \mathbf{0}$. To show that $\mathbf{u} + \mathbf{v}$ is in *Nul A* we must show that $A(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{u} + \mathbf{v}$ is in *Nul A*.

3) $\mathbf{A}(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$, which shows that $c\mathbf{u}$ is in *Nul A*. Thus *Nul A* is a subspace of $\mathbb{R}^n$.

An Explicit Description of *Nul A*

There is no clear connection between $Nul\ A$ and the elements of **A**. However, solving the system $A\mathbf{x} = \mathbf{0}$ amounts to producing an explicit description of $Nul\ A$.

Example: Find a spanning set for the null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Solution: *Step 1*: find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of free variables.

Row reduce the augmented matrix $[A\ \ \mathbf{0}]$ to reduced row echelon form in order to write the basic variables in terms of the free variables.

An Explicit Description of *Nul A*

Solution (cont) The augmented matrix and the RREF form are

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basic variables are x_1, x_3, while x_2, x_4, x_5 are free.

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We obtain

$x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$.

Step 2: Decompose the vector giving the general solution as a linear combination of vectors where the coefficients are free variables

An Explicit Description of *Nul A*

Solution (cont)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

$Nul\ \mathbf{A} = Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$

An Explicit Description of *Nul A*

Notes:

1. The spanning set produced by the method is automatically linearly independent because the free variables are the coefficients of the spanning vectors. The relation
$$x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} = \mathbf{0}$$
is true only if $x_2 = x_4 = x_5 = 0$.
2. When *Nul A* contains nonzero vectors, the number of vectors in the spanning set for *Nul A* equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

The Column Space of a Matrix

- Another subspace associated with a matrix is its column space. Unlike the null space, the column space is defined explicitly via linear combinations.
- **Definition:** The *column space* of an $m \times n$ matrix $\mathbf{A}$, written as *Col A*, is the set of all linear combinations of the columns of A. If $\mathbf{A} = [\mathbf{a}_1 \ldots \mathbf{a}_n]$, then $Col\ A = Span\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

  Note: Since $Span\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a subspace by Theorem, the next theorem follows from the definition of *Col A* and the fact that the columns of A are in $\mathbb{R}^m$.
- **Theorem:** The column space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

- $Col\ A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$ since $A\mathbf{x}$ is a linear combination of the columns of A.
- The notation $A\mathbf{x}$ for vectors in $Col\ A$ also shows that $Col\ A$ is the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

The Column Space of a Matrix

Recall that the columns of $\mathbf{A}$ span $\mathbb{R}^m$ if only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$. We can restate this as:
The column space of an $m \times n$ matrix $\mathbf{A}$ is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$.

Contrast Between *Nul A* and *Col A*

Example: Find a vector in *Col A* and a nonzero vector in *Nul A*,

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

Solution: Any column of **A** is in *Col A* from the definition of the column space as $Span\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. So we can choose the first column as an answer:

$$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \in Col\ A.$$

To find a vector in null space of **A** we need to find a vector that satisfies $A\mathbf{x} = \mathbf{0}$. We row reduce the augmented matrix $[A\ \ \mathbf{0}]$

$$[A\ \mathbf{0}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We obtain $x_1 = -9x_3$, $x_2 = 5x_3$ and $x_4 = 0$ and x_3 is free variable. By choosing $x_3 = 1$ we find a vector in *Nul A* as $\mathbf{x} = (-9, 5, 1, 0)$.

Contrast Between *Nul A* and *Col A*

Example: Let

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \text{ and } A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

(a) Determine if $\mathbf{u}$ is in $Nul\ A$. Can $\mathbf{u}$ be in $Col\ A$?

(b) Determine if $\mathbf{v}$ is in $Col\ A$. Can $\mathbf{v}$ be in $Nul\ A$?

Solution:

(a) An explicit description of $Nul\ A$ is not needed here. We just need to check if $A\mathbf{u} = \mathbf{0}$ or not.

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq 0,$$

so $\mathbf{u}$ is not in $Nul\ A$. With four entries $\mathbf{u}$ can not be part of the $Col\ A$ since $Col\ A$ is a subspace of $\mathbb{R}^3$.

Solution (cont):
(b) Determine if $\mathbf{v}$ is in *Col A*. Can $\mathbf{v}$ be in *Nul A*?
To answer to this question we need to form the augmented matrix $[A \;\; \mathbf{v}]$ and use row reduction:

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -1 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

It is clear that the system $A\mathbf{x} = \mathbf{v}$ is consistent so $\mathbf{v}$ is in *Col A*. With only three entries $\mathbf{v}$ can not be in *Nul A* since this is a subspace of $\mathbb{R}^4$.

Contrast Between *Nul A* and *Col A*

Consider a matrix $\mathbf{A}$ of order $m \times n$. Then:

- *Nul A* is a subspace of $\mathbb{R}^n$
  Col A is a subspace of $\mathbb{R}^m$
- *Nul A* is implicitly defined by $A\mathbf{x} = \mathbf{0}$.
  Col A is explicitly defined; given by the span of the columns of A.
- To find vectors in *Nul A*, row operations in $[A\ \mathbf{0}]$ are required.
  Easy to find vectors in *Col A* (any linear combination of columns of $\mathbf{A}$ will be in the column space).
- There is no obvious relation between *Nul A* and the entries in $\mathbf{A}$.
- There is an obvious relation between *Col A* and entries in A: each column of $\mathbf{A}$ is in *Col A* .

Contrast Between *Nul A* and *Col A*

- A typical vector $\mathbf{v}$ in *Nul A* has the property $A\mathbf{v} = \mathbf{0}$
  A typical vector $\mathbf{v}$ in *Col A* has the property that $A\mathbf{x} = \mathbf{v}$ is consistent.
- Given a specific vector $\mathbf{v}$ it is easy to tell if $\mathbf{v}$ is in *Nul A*. Just compute $A\mathbf{v}$.
  Given a specific vector $\mathbf{v}$, row operations on $[A\ \mathbf{v}]$ are required to tell if $\mathbf{v}$ is in *Col A*.
- $Nul\ A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  $Col\ A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
- $Nul\ A = \{\mathbf{0}\}$ if the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
  $Col\ A = \mathbb{R}^m$ if the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Bases for *Nul A* and *Col A*

Solution of a homogeneous system always produces a set that is linearly independent given that it is expressed via free variables.

Example: Find a basis for the null space of the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Solution: The general solution of $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $\mathbf{x} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$, where

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Every linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is an element of *Nul A*. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for *Nul A* and, since the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are independent, it is a basis for *Nul A*.

Bases for *Nul A* and *Col A*

Example Find a basis for *Col B*, where

$$B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_5) = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: Note, each nonpivot column of B is a linear combination of the pivot columns. Indeed,

$$\mathbf{b}_2 = 4\mathbf{b}_1 \text{ and } \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3.$$

By using the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$ and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span *Col B*.

Bases for *Nul A* and *Col A*

Let

$$S = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since $\mathbf{b_1} \neq \mathbf{0}$ and no vector in S is a linear combination of the vectors that preceded it, S is a linearly independent set. Thus S is a basis for *Col B*.

Bases for *Nul A* and *Col A*

Question: What about matrices which are **not** in RREF form?

Principle: Row reduction transforms a matrix $\mathbf{A}$ to a row-equivalent matrix $\mathbf{B}$ in RREF. Although the entries are different, the equations $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$ have exactly the same set of solutions.
The columns of $\mathbf{A}$ have exactly the same linear dependence relationships as the columns of $\mathbf{B}$, since row operations do not affect the linear dependence relations among the columns of a matrix.

Bases for *Nul A* and *Col A*

Example: It can be shown that the matrix

$$\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5) = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}$$

is row-equivalent to

$$\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_5) = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(*i.e.* RREF of $\mathbf{A}$ is $\mathbf{B}$.) Find a basis for *Col A*.

Bases for *Nul A* and *Col A*

Solution:

We have established (previous example) that

$$\mathbf{b}_2 = 4\mathbf{b}_1 \text{ and } \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3.$$

So we can expect

$$\mathbf{a}_2 = 4\mathbf{a}_1 \text{ and } \mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3.$$

Therefore we may discard $\mathbf{a}_2, \mathbf{a}_4$ when selecting the minimal spanning set for *Col A*.

The set $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is linearly independent because any linear relations on $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ would imply the same relations on $\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$.

Thus $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for *Col A*.

Bases for *Nul A* and *Col A*

Theorem: The pivot columns of a matrix A form a basis for *Col* **A**.
Proof: the proof uses the arguments described above.

Let **B** be the RREF form of **A**. The set of pivot columns of **B** is linearly independent. Since **A** is row equivalent to **B**, the pivot columns of **A** are also linearly independent, because any linear relation between the columns of A would imply the same linear relations among the columns of **B**.

Any nonpivot column of **A** is a linear combination of the pivot columns of **A**. Therefore the nonpivot columns can be discarded from the spanning set for *Col A* by the Spanning Set Theorem. This leaves the pivot columns of **A** as a basis for *Col* **A**.

Bases for *Nul* **A** and *Col* **A**

Note: it is important to use the pivot columns of **A** itself as a basis for *Col A*. The pivot columns of **B** cannot, in general, be used as a basis for *Col A*, as the column space of **B** is not necessarily the same as the column space of **A**.
In our last example the last entries of the pivot columns of **B** are all zeroes:

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Clearly they cannot span the columns of **A**:

$$A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5) = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}.$$