Fundamentals of linear algebra

- Properties of vectors in $\mathbb{R}^n$

- Linear combinations of vectors

- Spanning sets

- Homogeneous and inhomogeneous linear systems

Friday, 13 April 2018

Vectors in $\mathbb{R}^n$

We generally have scalars, vectors, matrices, and higher tensors.

Scalar a is a single number. Vector $\mathbf{a}$ ($\equiv \vec{a}$) is a list of numbers.

We consider vectors in two-, three-, ... n-dimensional space:

$$
\left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \in \mathbb{R}^2, \qquad
\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] \in \mathbb{R}^3, \qquad
\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \in \mathbb{R}^n.
$$

For example,

$$
\mathbf{a} = \left[\begin{array}{c} -1 \\ 2 \end{array} \right], \qquad
\mathbf{b} = \left[\begin{array}{c} 3 \\ -4 \\ 5 \end{array} \right], \qquad
\mathbf{c} = \left[\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \right]
$$

Vectors in $\mathbb{R}^n$

Equality: all the corresponding components are equal

$$\mathbf{v} = \mathbf{u} \qquad \text{if} \qquad v_i = u_i \qquad \forall i = 1 \ldots n$$

Multiplication by a scalar: each component is multiplied

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

Addition: corresponding components are added

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

Vectors in $\mathbb{R}^2$

Equality examples

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Multiplication example

$$\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad c = 2, \quad \text{then} \quad c\mathbf{u} = 2 \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}.$$

Addition example

$$\mathbf{u} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \text{then} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}.$$

Vectors in $\mathbb{R}^2$

Example:

find $4\mathbf{u} - 3\mathbf{v}$ for vectors

$$\mathbf{u} = \left[\begin{array}{c} 1 \\ -2 \end{array} \right] \quad \text{and} \quad \mathbf{v} = \left[\begin{array}{c} -3 \\ -1 \end{array} \right].$$

Solution:

$$4\mathbf{u} - 3\mathbf{v} = 4 \left[\begin{array}{c} 1 \\ -2 \end{array} \right] - 3 \left[\begin{array}{c} -3 \\ -1 \end{array} \right] = \left[\begin{array}{c} 4 \\ -8 \end{array} \right] - \left[\begin{array}{c} -9 \\ -3 \end{array} \right] = \left[\begin{array}{c} 13 \\ -5 \end{array} \right].$$

Vectors in $\mathbb{R}^2$

Consider a 2D Cartesian coordinate system.

Each point is defined by an ordered pair of numbers (a, b).

We can identify a point (a, b) with the vector

$$\begin{bmatrix} a \\ b \end{bmatrix}.$$

FIGURE 1
Vectors as points.

Figure shows geometric representations of

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

For vectorial visualisation an arrow is drawn from the coordinate origin towards the point defined by the vector.
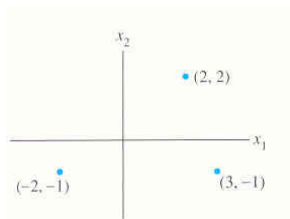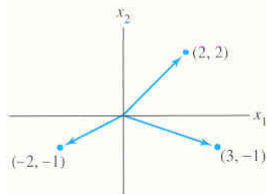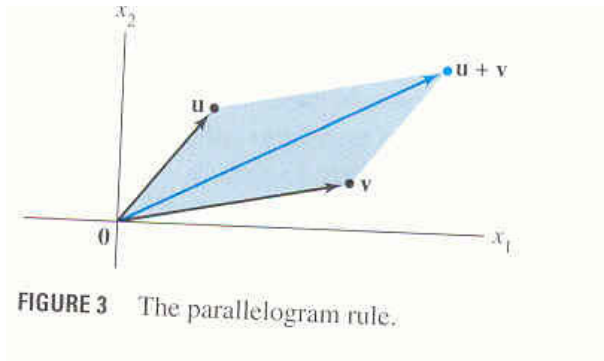

FIGURE 2
Vectors with arrows.

Vectors in $\mathbb{R}^2$

Parallelogram rule for vectorial addition

FIGURE 3 The parallelogram rule.

For vectors $\mathbf{u}$ and $\mathbf{v}$, the sum $\mathbf{u} + \mathbf{v}$ correspond to the fourth vertex of a parallelogram with vertices $\mathbf{u}$, $\mathbf{v}$, 0.

Vectors in $\mathbb{R}^2$

Example: geometric addition of these vectors $\mathbf{u}$, $\mathbf{v}$

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, \qquad \mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$
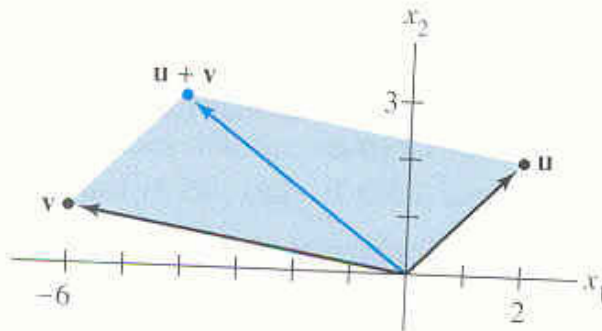

FIGURE 4

Vectors in $\mathbb{R}^2$

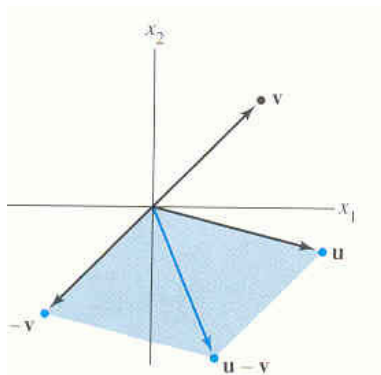Subtraction of vectors: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$

IGURE 7

/ector subtraction.

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ have 3 entries, e.g.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}.$$

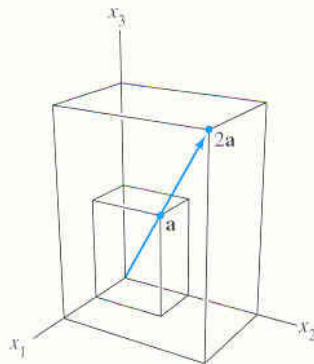Figure shows $\mathbf{a} = \mathbf{u}$ and $2\mathbf{a} = \mathbf{v}$.

FIGURE 6
Scalar multiples in $\mathbb{R}^3$.

Vectors in $\mathbb{R}^n$

Vectors in $\mathbb{R}^n$ space have n entries:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Vector with all entries equal to 0 is called a *zero vector*, $\mathbf{0}$:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(all components are zero)}$$

Equality of vectors, addition, scalar multiplication of vectors in $\mathbb{R}^n$ follow the same rules as for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
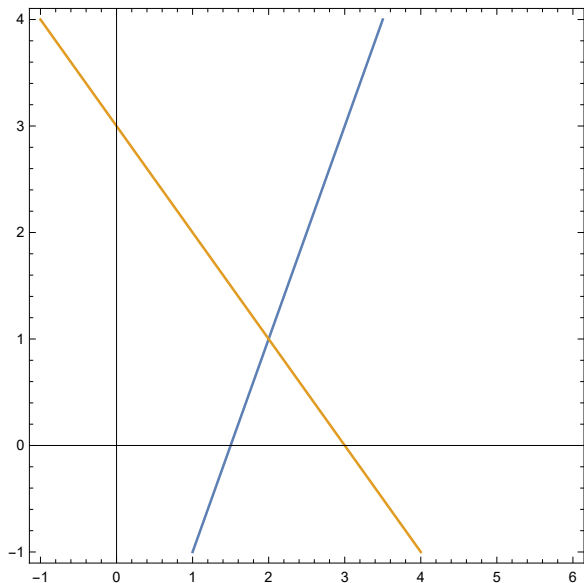
Geometric interpretation of linear systems

Consider an example

$$\begin{cases} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{cases}$$

By plotting the above equations in $\{x_1, x_2\}$ plane, we find the unique solution of this system as:

$x_1 = 2$ and $x_2 = 1$.

Geometric interpretation of linear systems

Let us analyse this linear system

$$\left\{ \begin{array}{rrrcl} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{rr} 2 & -1 \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[\begin{array}{c} 3 \\ 3 \end{array} \right]$$

Multiplication of the matrix by the vector $\mathbf{x}$ yields

$$\left[\begin{array}{rrr} 2x_1 & - & x_2 \\ x_1 & + & x_2 \end{array} \right] = \left[\begin{array}{c} 3 \\ 3 \end{array} \right]$$

Splitting the variables on left-hand side we get

$$\left[\begin{array}{c} 2x_1 \\ x_1 \end{array} \right] + \left[\begin{array}{c} -x_2 \\ x_2 \end{array} \right] = \left[\begin{array}{c} 3 \\ 3 \end{array} \right]$$

So the system can be rewritten in the form

$$x_1 \left[\begin{array}{c} 2 \\ 1 \end{array} \right] + x_2 \left[\begin{array}{c} -1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 3 \\ 3 \end{array} \right]$$

Note the first vector is the first column, and the second vector is the second column of the matrix of the linear system.

Geometric interpretation of linear systems

So, the original system was rewritten in the form

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which can be represented as $x_1\mathbf{v_1} + x_2\mathbf{v_2} = \mathbf{b}$ where

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

- Multiplication of a matrix by a vector from the right can be regarded as a linear combination of its columns.
- Solving a linear system is therefore equivalent to finding weights x_1 and x_2 for vectors $\mathbf{v_1}$, $\mathbf{v_2}$ to produce vector $\mathbf{b}$.
- In other words, solving $\mathbf{Ax} = \mathbf{b}$ is finding a linear combination of column vectors from $\mathbf{A}$ to produce vector $\mathbf{b}$.

Geometric interpretation of linear systems

Graphical visualisation for the unique solution $x_1 = 2$ and $x_2 = 1$.

$$\begin{cases} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{cases}$$

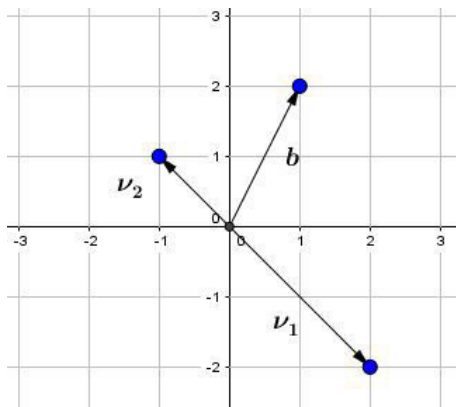$$x_1 \mathbf{v_1} + x_2 \mathbf{v_2} = \mathbf{b}$$
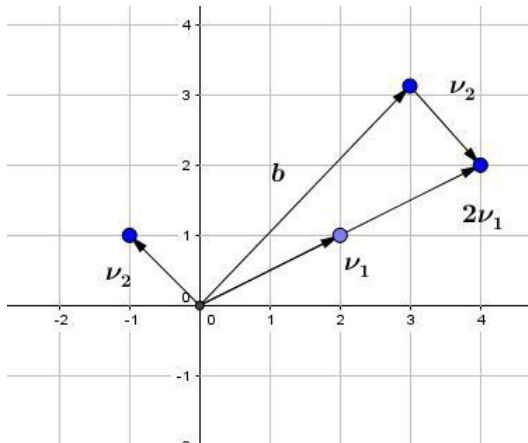
$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Geometric interpretation of linear systems

Another example:

$$\begin{cases} 2x_1 & - & x_2 & = & 1, \\ -2x_1 & + & x_2 & = & 2. \end{cases}$$

Here, no solutions.

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} = x_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}$$

Geometric interpretation of linear systems

Another example:

$$\left\{ \begin{array}{rrrrr} 2x_1 & - & x_2 & = & -3 \\ -2x_1 & + & x_2 & = & 3 \end{array} \right.$$

Infinite number of solutions

$(x_2 = 2x_1 + 3 \quad \forall x_1)$

$$x_1 \mathbf{v_1} + x_2 \mathbf{v_2} = x_1 \left[\begin{array}{c} 2 \\ -2 \end{array} \right] + (2x_1 + 3) \left[\begin{array}{c} -1 \\ 1 \end{array} \right] = \left[\begin{array}{c} -3 \\ 3 \end{array} \right] = \mathbf{b}$$

Geometric interpretation of linear systems

Next example:

$$\left\{ \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \right.$$

$$x_1 \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + x_2 \left[\begin{array}{c} -2 \\ 3 \end{array} \right] = \left[\begin{array}{c} -1 \\ 3 \end{array} \right]$$

Solution is $x_1 = 3$, $x_2 = 2$
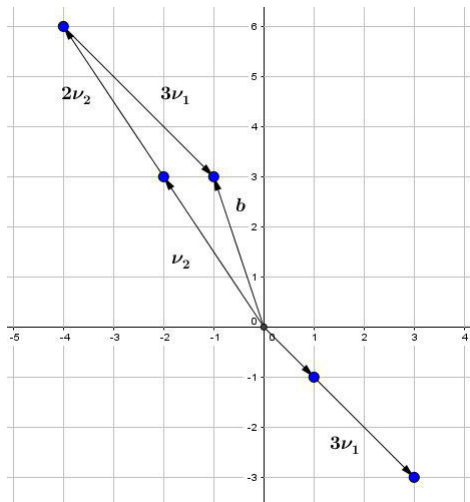
Vector space in $\mathbb{R}^n$

- We can add vectors and multiply vectors by a scalars in $\mathbb{R}^n$. This results in a vector within the same space $\mathbb{R}^n$.

- Vector operations in $\mathbb{R}^n$ satisfiy the following conditions. For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ and any scalars c, d:

  (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  (iv) $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
  (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
  (vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$
  (viii) $1\mathbf{u} = \mathbf{u}, \quad (-\mathbf{u}) = (-1)\mathbf{u}$

- The satisfying set of vectors is called a **vector space**

Linear combinations

Definition: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n \in \mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_n$, ($c_i \in \mathbb{R}$ are any real numbers including zero)

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n$$

is called a **linear combination** of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$
with coefficients $\{c_1, c_2 \ldots, c_n\}$.

Example:

$$\mathbf{v}_1 = \left[\begin{array}{c} 1 \\ -1 \end{array}\right] \quad \mathbf{v}_2 = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

$$\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = 2\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + 3\left[\begin{array}{c} 2 \\ 1 \end{array}\right] = \left[\begin{array}{c} 8 \\ 1 \end{array}\right]$$

Linear combinations can be used to describe general solutions of systems of linear equations.

Linear combinations

Example: integer linear combinations of

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

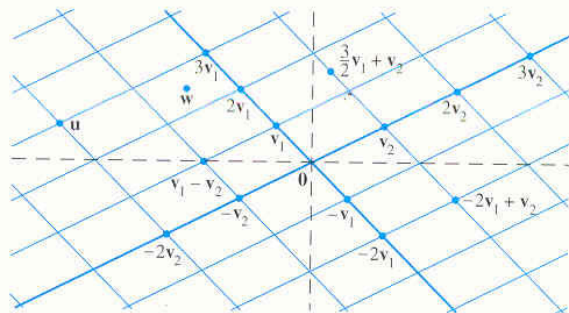have form $\mathbf{u} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, where x_1 and x_2 are any integers.

FIGURE 8   Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Linear combinations

Example: Given

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

can $\mathbf{b}$ be represented as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$?
That is, are there scalars x_1, x_2 such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 = \mathbf{b}$$

Solution: Expand the vector equation.

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad \Leftrightarrow$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{rrrrr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & + & 5x_2 & = & 4 \\ -5x_1 & + & 6x_2 & = & -3 \end{array}$$

Linear combinations

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

Reduce to REF with $R_2 \to R_2 + 2R_1$ and then $R_3 \to R_3 + 5R_1$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is $x_1 = 3$, $x_2 = 2$.

Note: the original vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \mid \mathbf{b} \end{bmatrix}$$

Linear combinations

So, the solution was $x_1 = 3$, $x_2 = 2$, which means

$$\mathbf{b} = 3\mathbf{a}_1 + 2\mathbf{a}_2,$$

and thus $\mathbf{b}$ can be presented as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$; vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$ form the the augmented matrix $\begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, | \, \mathbf{b} \end{bmatrix}$.

Generally, a vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the system of linear equations represented by an augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, \ldots \, \mathbf{a}_n \, | \, \mathbf{b} \end{bmatrix}$$

One of the key concepts in linear algebra is a set of all vectors that are generated as a linear combination of a fixed set of vectors.

Definition: For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, the set of all their linear combinations is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ *spanned* (generated) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Therefore $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

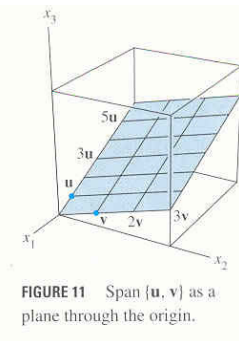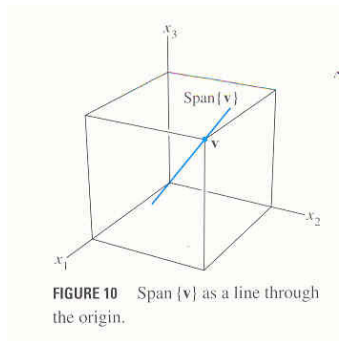$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with arbitrary scalars $c_1, c_2, \ldots, c_p$.

Geometric interpretation of span

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$.
Then $\mathrm{Span}\{\mathbf{v}\} = c\mathbf{v}$ is the set of all scalar multiples of $\mathbf{v}$.
This can be visualised as the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.

FIGURE 10 Span $\{\mathbf{v}\}$ as a line through the origin.

FIGURE 11 Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

If $\mathbf{u}$, $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{u}$ not a multiple of $\mathbf{v}$, then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = c_1\mathbf{u} + c_2\mathbf{v}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}, \mathbf{v}, \mathbf{0}$.

Linear combinations and span

Consider vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b} \in \mathbb{R}^3$.

$\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin and vectors $\mathbf{a}_1$, $\mathbf{a}_2$.

Three equivalent statements:

- $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$
- $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$
- Equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution.

Linear combinations and span

Example:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

We can check whether $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ by solving the system.

Row reduce the augmented matrix $[\mathbf{a}_1\, \mathbf{a}_2\,|\,\mathbf{b}]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The bottom row in the REF shows an inconsistency $0 \stackrel{!}{=} -2$, so there are no solutions and therefore $\mathbf{b} \notin \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$, that is, vector $\mathbf{b}$ is not a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Linear combinations and span

If $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{a}_1 \ldots \mathbf{a}_n$ and if $\mathbf{x} \in \mathbb{R}^n$ then the product $\mathbf{Ax}$ is the linear combination of the columns of $\mathbf{A}$:

$$
\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n.
$$

Example:

$$
\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4 \cdot 1 & + & 3 \cdot 2 & + & 7 \cdot (-1) \\ 4 \cdot 0 & + & 3 \cdot (-5) & + & 7 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.
$$

Linear combinations and span

Example: Express $-3\mathbf{v}_1 + 2\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix product:

$$-3\mathbf{v}_1 + 2\mathbf{v}_2 + 7\mathbf{v}_3 = \left[\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}\right] \left[\begin{array}{c} -3 \\ 2 \\ 7 \end{array}\right]$$

In general, $\mathbf{A}\mathbf{x} = \mathbf{b}$ denotes the linear combination

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n = \mathbf{b},$$

which can be regarded as a system of linear equations with unknowns $x_1, x_2, \ldots, x_n$.

This system can be solved using the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{array}\right]$$

using standard Gaussian elimination.

Comments:

$\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.

We have already considered the equivalence of this to
$\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n\}$

An equivalent question is whether the system $A\mathbf{x} = \mathbf{b}$ is consistent

A more tricky question is whether $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$.

Linear combinations and span

Example: Check if the system $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$, given

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \qquad \begin{array}{l} \mathsf{R_2 \to R_2 + 4R_1} \\ \mathsf{R_3 \to R_3 + 3R_1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right] \qquad \mathsf{R_3 \to R_3 - \dfrac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{1}{2}(-4b_1 - b_2) + b_3 \end{array}\right]$$

which tells that the system is not consistent for every $\mathbf{b}$.

Linear combinations and span

So, the EF of the augmented matrix upon the reduction is:

$$A_4 = \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{bmatrix}$$

The system is consistent if $b_1 - b_2/2 + b_3 = 0$.

So the right-hand side must have the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + (1/2)b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

i.e.

$$\mathbf{b} \in \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

Linear combinations and span

Geometrically:
the columns of $\mathbf{A}$ span a plane
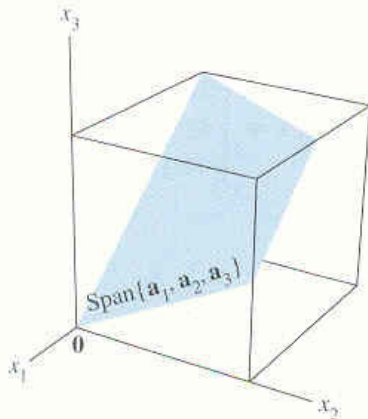
$$b_1 - (1/2)b_2 + b_3 = 0.$$

FIGURE 1

The columns of $A = [\, \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \,]$ span a plane through $\mathbf{0}$.

Linear combinations and span

The statement that columns of $\mathbf{A}$ span $\mathbb{R}^m$ means that every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of $\mathbf{A}$.

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ spans $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^m$$

Theorem. For $m \times n$ matrix $\mathbf{A}$, the following statements are logically equivalent (either all true or all false for a given matrix):

- For each $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution,
- The columns of $\mathbf{A}$ span $\mathbb{R}^m$
- $\mathbf{A}$ has a pivot position in every row.

Note: the latter statement refers to the coefficients matrix $\mathbf{A}$, not the augmented matrix $[\mathbf{A} \,|\, \mathbf{b}]$.

Linear combinations and span

Returning to the previous example

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right]$$

note that the matrix only has two pivots and the system is consistent only if $b_1 - (1/2)b_2 + b_3 = 0$. Any vector which is spanned by the columns of $\mathbf{A}$ must have the form

$$\left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array}\right] \qquad \text{where} \qquad b_1 - \frac{1}{2}b_2 + b_3 = 0$$

Linear combinations and span

In particular, each column of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$$

satisfies the same constraint as $b_1 - (1/2)b_2 + b_3 = 0$, as follows:

First column: $\quad 1 - (-4)/2 - 3 = 0$

Second column: $\quad 3 - 2/2 - 2 = 0$

Third column: $\quad 4 - (-6)/2 - 7 = 0$

Linear combinations and span

Furthermore, the vector space spanned by the columns of $\mathbf{A}$ is generated by only two vectors (since there are only two pivots).

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + (1/2)b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

So any vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is consistent belongs to the span of these vectors:

$$\mathbf{b} \in \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

Linear combinations and span

This was an example where all the statements of the previous theorem are simultaneously false:

- " For each $\mathbf{b} \in \mathbb{R}^3$, $\mathbf{Ax} = \mathbf{b}$ has a solution "
  False: System only has a solution if $b_1 - b_2/2 + b_3 = 0$.

- " The columns of $\mathbf{A}$ span $\mathbb{R}^3$ "
  False: The columns of $\mathbf{A}$ span a plane (an $\mathbb{R}^2$ area within $\mathbb{R}^3$)

- " $\mathbf{A}$ has a pivot position in every row "
  False: $\mathbf{A}$ has only 2 pivots, not 3.

The two pivots correspond to the plane spanned by the two vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Linear combinations and span

Another example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}.$$

Do these vectors span $\mathbb{R}^3$? Applying the theorem, we will perform a reduction on $\begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \end{bmatrix}$ and see if we get a pivot in every row.

$$R_3 \to R_3 + R_1 \qquad\qquad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ -1 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}.$$

So there are three pivots and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

Homogeneous linear systems

A linear system of equations is *homogeneous* if it has form

$$\mathbf{Ax} = \mathbf{0}$$

This system always has at least one solution $\mathbf{x} = \mathbf{0}$. This solution is called a *trivial solution*.

For a given equation $\mathbf{Ax} = \mathbf{0}$ the important question whether there is a nontrivial solution $\mathbf{x} \neq \mathbf{0}$.

Example:

$$
\begin{array}{rcrcrcl}
3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\
-3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\
6x_1 & + & x_2 & - & 8x_3 & = & 0
\end{array}
$$

$$
\begin{bmatrix}
3 & 5 & -4 & 0 \\
-3 & -2 & 4 & 0 \\
6 & 1 & -8 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
3 & 5 & -4 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & -4/3 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Homogeneous linear systems

$$\begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So x_1 and x_2 are basic variables, and x_3 is a free variable.

$$\begin{aligned} x_1 - 4x_3/3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The solution set is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$
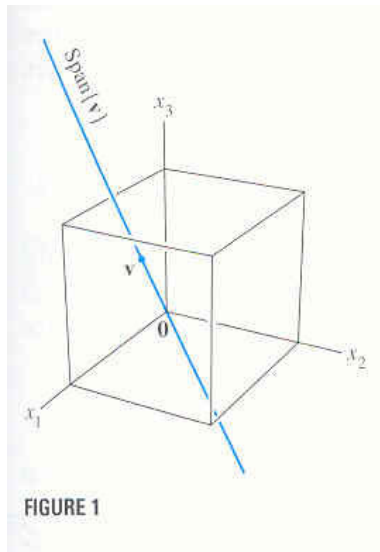
Homogeneous linear systems

So every solution is scalar multiple $t\mathbf{v}$ of vector $\mathbf{v}$ for all $t \in \mathbb{R}$.

$$\mathbf{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{v}$$

Geometrically, this solution set represents a line though $\mathbf{0}$ in $\mathbb{R}^3$.

A homogeneous system has a nontrivial solution if and only if there is at least one free variable.

FIGURE 1

Homogeneous linear systems

Example. Describe all the solutions of the homogeneous equation:

$$10x_1 - 3x_2 - 2x_3 = 0$$

The general solution is $x_1 = 0.3x_2 + 0.2x_3$. In a vector form, that is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

which is

$$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v},$$

where

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

Homogeneous linear systems

This solution $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$ is a *parametric equation* of a plane through the origin.

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, so the solution set is $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Conclusion: For any homogeneous equation $\mathbf{Ax} = \mathbf{0}$, solution set can be written as $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

If the only solution is zero vector, then the solution set is $\mathrm{Span}\{\mathbf{0}\}$.
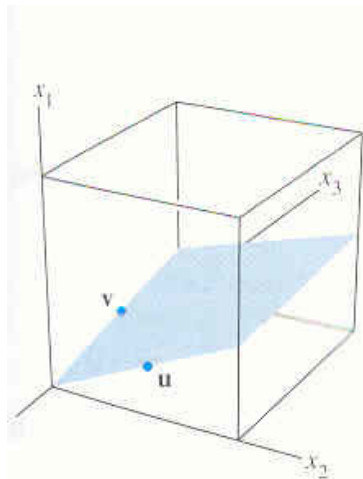

FIGURE 2

Inhomogeneous systems

Now consider solutions of $\mathbf{Ax} = \mathbf{b}$ with augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

The system becomes

$$\begin{aligned} x_1 - 4x_3/3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

The solution is $x_1 = 4/3x_3 - 1$, $x_2 = 2$ and x_3 is a free variable.

Inhomogeneous systems

In vector form the solution can be written as

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}
$$

which is

$$
\mathbf{x} = \mathbf{p} + x_3\mathbf{v} = \mathbf{p} + t\mathbf{v}
$$

with

$$
\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.
$$

Inhomogeneous systems

Recall the solution of the homogeneous system was $\mathbf{x} = x_3\mathbf{v}$.

Thus the solutions of inhomogeneous system $A\mathbf{x} = \mathbf{b}$ are obtained by adding vector $\mathbf{p}$ to the solution of homogeneous system $A\mathbf{x} = \mathbf{0}$.

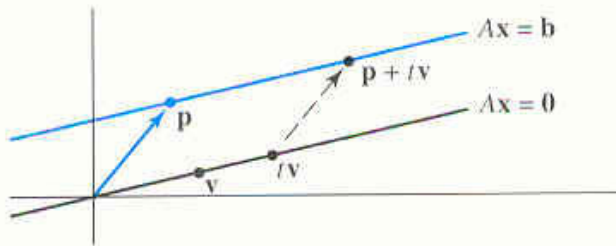Vector $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ for $x_3 = 0$.

FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Theorem:

Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ and let $\mathbf{p}$ be a solution.

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ is a set of all vectors in the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$, where $\mathbf{v}_0$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
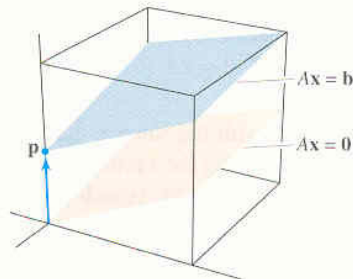

FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Indeed: $\mathbf{Aw} = \mathbf{A}(\mathbf{p} + \mathbf{v}_0) = \mathbf{Ap} + \mathbf{Av}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Inhomogeneous systems

Summary:

Step 1: Row-reduce the augmented matrix to REF

Step 2: Express each basic variable (variables of the pivot columns) in terms of any free variables

Step 3: Write a typical solution $\mathbf{x}$ in the vector form

Step 4: Decompose $\mathbf{x}$ into a linear combination of vectors using the free variables as scalars.

Solutions to a homogeneous system form a plane through $\mathbf{0}$.

Solutions to the corresponding inhomogeneous system form a parallel plane, shifted by $\mathbf{p}$.

Inhomogeneous systems

Example:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Basic variables are x_1, x_2, x_5 and free variables are x_3, x_4.

$$\begin{array}{rcl} x_1 & = & 2x_3 - 3x_4 - 24 \\ x_2 & = & 2x_3 - 2x_4 - 7 \\ x_5 & = & 4 \end{array}$$

General solution in vector form:

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{ccccccc} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{array}\right]$$

Inhomogeneous systems

General solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix}$$

Therefore

$$\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v} + \mathbf{p},$$

where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Inhomogeneous systems

Points to note:

- There are infinitely many possible solutions since x_3 and x_4 are free variables and may take any values.

- Vector $\mathbf{p}$ is a solution which is obtained by specifying $x_3 = x_4 = 0$. This solution is called a *particular solution*.

Check: Substitute $x_3 = x_4 = 0$ back into the original system:

$$3x_2 + 4x_5 = -5$$
$$3x_1 - 7x_2 + 9x_5 = 9$$
$$3x_1 - 9x_2 + 6x_5 = 15$$

Inhomogeneous systems

u and **v** are solutions to the corresponding homogeneous system

$$
\begin{array}{rcrcrcrcrcl}
 & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = 0 \\
3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 9x_5 & = 0 \\
3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = 0
\end{array}
$$

Substitute $\mathbf{u} = (2;\ 2;\ 1;\ 0;\ 0)$:

$$
\begin{array}{rcrcrcrcl}
 & & 3 \cdot 2 & - & 6 \cdot 1 & + & 0 & + & 0 & = 0 \\
3 \cdot 2 & - & 7 \cdot 2 & + & 8 \cdot 1 & - & 0 & + & 0 & = 0 \\
3 \cdot 2 & - & 9 \cdot 2 & + & 12 \cdot 1 & - & 0 & + & 0 & = 0
\end{array}
$$

Substitute $\mathbf{v} = (-3;\ -2;\ 0;\ 1;\ 0)$:

$$
\begin{array}{rcrcrcrcl}
 & & 3 \cdot (-2) & - & 0 & + & 6 \cdot 1 & + & 0 & = 0 \\
3 \cdot (-3) & - & 7 \cdot (-2) & + & 0 & - & 5 \cdot 1 & + & 0 & = 0 \\
3 \cdot (-3) & - & 9 \cdot (-2) & + & 0 & - & 9 \cdot 1 & + & 0 & = 0
\end{array}
$$

Inhomogeneous systems

In summary, $\mathbf{u}, \mathbf{v}$ are solutions to the homogeneous system

$$3x_2 - 6x_3 + 6x_4 + 4x_5 = 0$$
$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 = 0$$
$$3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 0$$

Its general solution is: $\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v}$, which is: $\mathbf{x} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Then, $\mathbf{p}$ is a particular solution to the inhomogeneous system

$$3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$$
$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 = 9$$
$$3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15$$

Its general solution is: $\mathbf{x} = \mathbf{p} + x_3\mathbf{u} + x_4\mathbf{v}$.