Fundamentals of linear algebra

- Bases and coordinate systems
- Coordinate mapping
- Row space
- The rank theorem
- Change of basis

Friday, 11 May 2018

Revision: Basis

- An indexed set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a **basis** in V if:
  $\mathcal{B}$ is a linearly independent set, and $V = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$
- An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to impose a coordinate system on V.
- If $\mathcal{B}$ for V contains n vectors, then a coordinate system makes V behave like $\mathbb{R}^n$.

Theorem: (the *unique representation theorem*)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V.

Then $\forall\, \mathbf{x} \in V$ there exists a *unique* set of scalars $x_1, \ldots, x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n.$$

Revision: Spanning set theorem

- A basis is an "efficient" spanning set that contains only "necessary" vectors.
- A basis can be constructed from a spanning set by discarding unnecessary vectors.

Spanning set theorem:

Let $\mathcal{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and $H = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(a) If one of the vectors in $\mathcal{S}$, say $\mathbf{v}_i$, is a linear combination of the other vectors in $\mathcal{S}$, then the set formed from $\mathcal{S}$ by removing $\mathbf{v}_i$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of $\mathcal{S}$ is a basis for H.

Revision: Spanning set theorem (notes)

Notes:

- A basis is a smallest possible spanning set.
- A basis is also a largest possible linearly independent set.
- If $\mathcal{S}$ is a basis for V and is enlarged by one vector $\mathbf{w} \in V$ then the enlarged set cannot be linearly independent, because $\mathcal{S}$ spans V so $\mathbf{w}$ is a linear combination of the vectors of $\mathcal{S}$.
- If $\mathcal{S}$ is a basis for V and if $\mathcal{S}$ is made smaller by one vector $\mathbf{u} \in V$ then the reduced set cannot serve as a basis, because it will not span V anymore.

A linearly independent set can be enlarged to form a basis, but further enlargement destroys the linear independence.

A larger spanning set can be reduced to a basis, but further shrinking destroys the spanning property.

Coordinate systems (revision)

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V, and $\mathbf{x} \in V$. The *coordinates* of $\mathbf{x}$ *relative to basis* $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\mathbf{x}$) are the coefficients $x_1, \ldots, x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n.$$

If $x_1, \ldots, x_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is the *coordinate vector* of $\mathbf{x}$ (relative to $\mathcal{B}$),
or the $\mathcal{B}$*-coordinate vector* of $\mathbf{x}$.

Mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping defined by $\mathcal{B}$.

Coordinate systems

- **Example:** Standard basis $\{\mathbf{e}_1; \mathbf{e}_2\}$, and basis $\mathcal{B} = \{\mathbf{b}_1; \mathbf{b}_2\}$ with $\mathbf{b}_1 = \mathbf{e}_1$ and $\mathbf{b}_2 = [1; 2]$, for vector $\mathbf{x} = [1; 6]$.
- Coordinates $[1; 6]$ locate $\mathbf{x}$ relative to the standard basis.
- $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ locate $\mathbf{x}$ on the coordinate system of basis $\mathcal{B}$.

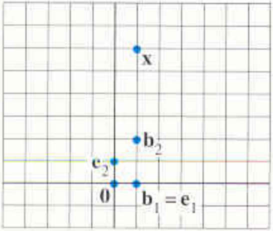


FIGURE 1 Standard graph paper.

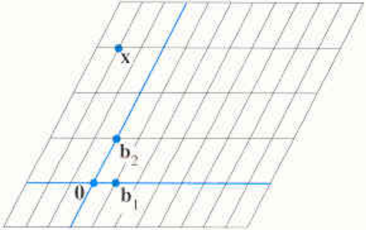


FIGURE 2 $\mathcal{B}$-graph paper.

Change of coordinates in $\mathbb{R}^n$

For a given basis $\mathcal{B}$ in $\mathbb{R}^n$, $\mathcal{B}$-coordinates are easily found.

Example:

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

To find the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = (c_1; c_2)$ of $\mathbf{x}$ with respect to basis $\mathcal{B}$, we need to solve the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

which is

$$\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

This equation can be solved by row-reducing the augmented matrix; the solution is $c_1 = 3$ and $c_2 = 2$.

Change of coordinates in $\mathbb{R}^n$

Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$

and we can write

$$[\mathbf{x}_{\mathcal{B}}] = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Note: The matrix $\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$

transforms $\mathcal{B}$-coordinates of vector $\mathbf{x}$ into standard coordinates for $\mathbf{x}$.

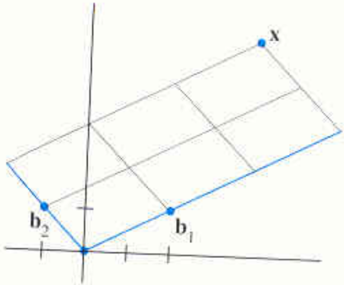


FIGURE 4

The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)$.

Change of coordinates in $\mathbb{R}^n$

Generally, let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis in $\mathbb{R}^n$.

Construct the following matrix: $\mathrm{P}_\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$.

Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ is equivalent to

$$\mathbf{x} = \mathrm{P}_\mathcal{B}\,[\mathbf{x}]_\mathcal{B}, \qquad \text{where} \qquad [\mathbf{x}]_\mathcal{B} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

$\mathrm{P}_\mathcal{B}$ is the *change of coordinates matrix* from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$: $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

The columns of $\mathrm{P}_\mathcal{B}$ form a basis for $\mathbb{R}^n$. $\mathrm{P}_\mathcal{B}$ is therefore invertible.

Left-multiplication by $\mathrm{P}_\mathcal{B}^{-1}$ converts $\mathbf{x}$ into $\mathcal{B}$-coordinate vector

$$\mathrm{P}_\mathcal{B}^{-1}\mathbf{x} = [\mathbf{x}]_\mathcal{B}.$$

This is a one-to-one linear transformation.

Coordinate mapping

By choosing a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ in space V we introduce a coordinate system in V.

Coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ links the space V to the space $\mathbb{R}^n$.

Points in V can be identified by equivalent points in $\mathbb{R}^n$.

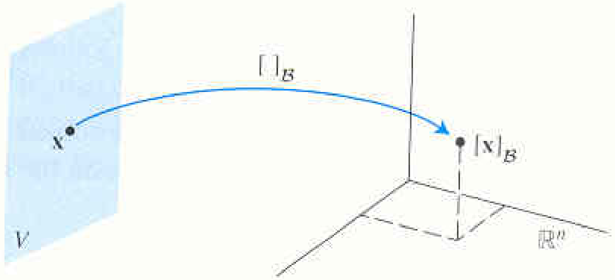


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Coordinate mapping

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof: Consider two arbitrary vectors in V,

$$\mathbf{u} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n,$$
$$\mathbf{v} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n.$$

Then $\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{b}_1 + \ldots + (c_n + d_n)\mathbf{b}_n$ and

$$[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}.$$

Thus the coordinate mapping preserves addition.

Coordinate mapping

Proof (continuation):

If r is an arbitrary scalar, then

$$\begin{aligned} r\mathbf{u} &= r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = \\ &= (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n, \end{aligned}$$

so

$$[r\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_{\mathcal{B}}.$$

Thus the coordinate mapping also preserves scalar multiplication.

Therefore, coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a linear transformation.

Coordinate mapping

- The linearity of the coordinate mapping extends to linear combinations: for $\mathbf{u}_1, \ldots, \mathbf{u}_n \in V$ and scalars $c_1, \ldots, c_n$

  $$[c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n]_\mathcal{B} = c_1[\mathbf{u}_1]_\mathcal{B} + \ldots + c_n[\mathbf{u}_n]_\mathcal{B}$$

  (the $\mathcal{B}$-coordinates of a linear combination of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, are the coefficients in the linear combination of their coordinate vectors).

- A one-to-one linear transformation from vector space V onto vector space W is called an *isomorphism* from V onto W.

  Notation and terminology may be different in V and W, however as vector spaces they have identical structures.

- Coordinate mapping is an isomorphism between V and $\mathbb{R}^n$.

Coordinate mapping

Example:

$\mathcal{P} = \{1, t, t^2, t^3\}$ is the standard basis in the space $\mathbb{P}_3$ (polynomials of a degree up to 3).

A typical element $\mathbf{p}$ of $\mathbb{P}_3$ has the form

$$\mathbf{p} = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

This is a linear combination of the standard basis vectors, so

$$[\mathbf{p}]_{\mathcal{P}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{P}}$ is an isomorphism of $\mathbb{P}_3$ onto $\mathbb{R}^4$.

All vector operations in $\mathbb{P}_n$ correspond to operations in $\mathbb{R}^{(n+1)}$.

Coordinate mapping

Example: Check if these polynomials are linearly dependent:

$$\mathbf{p}_1 = 1 + 2t^2, \quad \mathbf{p}_2 = 4 + t + 5t^2, \quad \mathbf{p}_3 = 3 + 2t$$

Solution: These $\mathbf{p}_i \in \mathbb{P}_2$ so their coordinate vectors are in $\mathbb{R}^3$:

$$[\mathbf{p}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad [\mathbf{p}_2]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \quad [\mathbf{p}_3]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

We then write these vectors as columns of a matrix, and check the linear dependence $x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 = \mathbf{0}$ by row reduction:

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot in the third column so the columns are linearly dependent.

Coordinate mapping

So

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_3 is a free variable, and then $x_2 = -2x_3$ and $x_1 = 5x_3$.

Choosing $x_3 = 1$, we have $x_1 = 5$ and $x_2 = -2$:

$$5 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \mathbf{0}.$$

The corresponding relation $5\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$ then reads:

$$5(1 + 2t^2) - 2(4 + t + 5t^2) + (3 + 2t) = 0 \quad \forall\, t$$

Coordinate mapping

Example: Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\text{with} \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

Determine if $\mathbf{x} \in H$ and, if so, find $[\mathbf{x}]_{\mathcal{B}}$.

Solution: If $\mathbf{x} \in H$, then the following equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}.$$

The scalars c_1 and c_2, if they exist, are the $\mathcal{B}$ coordinates of $\mathbf{x}$.
Row reduction yields

$$\left[\begin{array}{rr|r} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \to \left[\begin{array}{rr|r} 3 & -1 & 3 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

Coordinate mapping

Thus the system is consistent; a unique solution is $c_1 = 2,\ c_2 = 3$.

The coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix},\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};$$

and

$$\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = 2\mathbf{v}_1 + 3\mathbf{v}_2$$

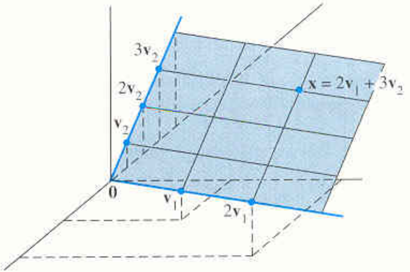


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

Dimension of a vector space

We have seen that a vector space V with the basis containing n vectors is isomorphic to $\mathbb{R}^n$. This number n is an intrinsic property of V (called the *dimension* of V) which does not depend on a particular choice of basis vectors.

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ with n vectors, then any set in V containing more than n vectors must be linearly dependent.

Note: This theorem implies that if a vector space V has a basis of n vectors, then any linearly independent set in V must have no more than n vectors.

Proof strategy:

- Create isomorphism to $\mathbb{R}^n$.
- Prove in $\mathbb{R}^n$ using matrix techniques.
- Apply a reverse isomorphism.

Dimension of a vector space

Proof: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with $p > n$ vectors. The coordinate vectors $[\mathbf{u}_1]_{\mathcal{B}}, \ldots [\mathbf{u}_p]_{\mathcal{B}}$ are linearly dependent in $\mathbb{R}^n$ because there are more vectors (p) than entries (n). Therefore $\exists\ c_1, \ldots, c_p$ not all zero such that

$$c_1[\mathbf{u}_1]_{\mathcal{B}} + \ldots + c_p[\mathbf{u}_p]_{\mathcal{B}} = \mathbf{0}.$$

The coordinate mapping is a linear transformation $V \mapsto \mathbb{R}^n$, so we therefore have

$$[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p]_{\mathcal{B}} = \mathbf{0} = [\mathbf{0}]_{\mathcal{B}} \qquad \text{in} \quad \mathbb{R}^n.$$

Since the transformation is one-to-one, we must have

$$c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p = \mathbf{0} \qquad \text{in} \quad V$$

Since the c_i are not all zero, $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly dependent.

Dimension of a vector space

Theorem: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof strategy: Double inequality: $a \leqslant b$ and $a \geqslant b \implies a = b$ (show that upper and lower bounds coincide)

Proof: Let $\mathcal{B}_1$ be a basis in V with n vectors, and $\mathcal{B}_2$ be any other basis in V with m vectors.

Since $\mathcal{B}_1$ is a basis and $\mathcal{B}_2$ is a linearly independent set, from the previous theorem $\mathcal{B}_2$ has no more than n vectors, so $m \leqslant n$.

Vice versa, since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is a linearly independent set, $\mathcal{B}_1$ has to have no more than m vectors, so $n \leqslant m$.

Thus $m = n$ and $\mathcal{B}_2$ consists of exactly n vectors.

Note: If a non-zero space V is spanned by a finite set S then there is a subset of S that is a basis in V by spanning set theorem.

Dimension of a vector space

Definition: If V is spanned by a finite set, then V is called a *finite-dimensional* space, and the dimension of V, written as $\dim V = n$, is the number n of vectors in a basis for V.

> **Note:** the dimension is a number of vectors in a basis, **not** a number of entries in each vector.

The dimension of the $\{\mathbf{0}\}$ vector space is defined to be zero.

If V is not spanned by a finite set then V is *infinite-dimensional*.

Example: The basis for $\mathbb{R}^n$ contains n vectors, so $\dim \mathbb{R}^n = n$.

Example: For standard polynomial basis $\{1, t, t^2\}$, $\dim \mathbb{P}_2 = 3$.

In general, $\dim \mathbb{P}_n = n + 1$.

The space $\mathbb{P}$ of all polynomials is infinite-dimensional.

Dimension of a vector space

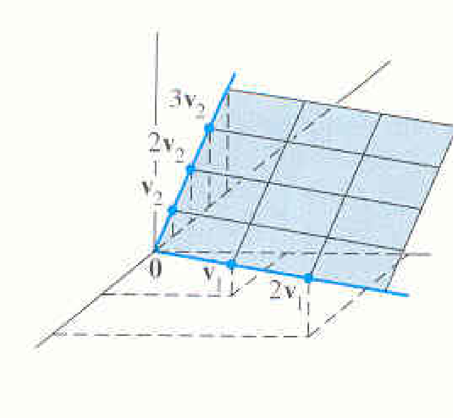
Example:

Let $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H, since vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other and hence are linearly independent.

So $\dim H = 2$.



Dimension of a vector space

Example: Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \right\} \qquad a, b, c, d \in \mathbb{R}.$$

Solution: Decomposing these vectors, we have

$$\begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} = a \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Therefore H is the set of all linear combination of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

Dimension of a vector space

By analysing these vectors, we notice the following

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

- $\mathbf{v}_1 \neq \mathbf{0}$
- $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.
- $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$ (since $\mathbf{v}_3 = -2\mathbf{v}_2$).
  Using the spanning set theorem, we may discard $\mathbf{v}_3$ and have the remaining set still spanning H.
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a linearly independent set and a basis for H.

Thus $\dim H = 3$.

Dimension of a subspace

Example: Various subspaces of $\mathbb{R}^3$:

(a) *0*-dimensional subspace: Only the zero subspace.
(b) *1*-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.
(c) *2*-dimensional subspaces: Any subspace spanned by two linearly independent vectors. Planes through the origin.
(d) *3*-dimensional subspace: Only $\mathbb{R}^3$ itself. Any three linearly independent vectors in $\mathbb{R}^3$ span the entire $\mathbb{R}^3$.

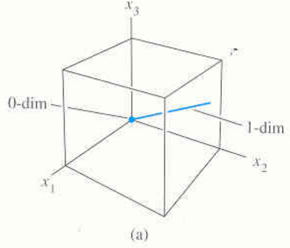


FIGURE 1 Sample subspaces of $\mathbb{R}^3$.

Dimension of a subspace

Theorem: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H, and $\dim H \leqslant \dim V$.

Proof: If $H = \{\mathbf{0}\}$, then certainly $\dim H = 0 \leqslant \dim V$.

Otherwise let $S = \{\mathbf{u}_1 \ldots \mathbf{u}_k\}$ be a linearly independent set in H.

If S spans H then S is a basis for H.

Otherwise there is a some $\mathbf{u}_{k+1}$ in H that is not in span of S.

Then the set $S' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is linearly independent.

As long as the new set S does not span subspace H we can continue to add linearly independent vectors to S, expanding it to a larger linearly independent set in H.

But the number of vectors can never exceed $\dim V$, which is a number of linearly independent vectors in the entire space V.

Eventually the expanded set S will span H and $\dim H \leqslant \dim V$.

Dimension of a subspace

The basis theorem:

Let V be a p-dimensional vector space, $p \geqslant 1$. Then:

- any linearly independent set of p elements in V is a basis for V;
- any set of p elements that spans V is a basis for V.

Proof:

Any linearly independent set S of p elements can be extended to a basis for V. But this basis should contain exactly p elements since $\dim V = p$. So S is a basis for V.

Suppose S has p elements and spans V. Since V is nonzero, the spanning set theorem implies that there exists a subset S' of S which is a basis of V. Since $\dim V = p$, S' must contain p vectors. Hence $S = S'$.

Dimensions of $\operatorname{Nul}\mathbf{A}$ and $\operatorname{Col}\mathbf{A}$

- Since the pivot columns of a matrix $\mathbf{A}$ form a basis of $\operatorname{Col}\mathbf{A}$, we know the dimension of $\operatorname{Col}\mathbf{A}$ once we know the pivots.
- Finding the dimension and basis vectors of $\operatorname{Nul}\mathbf{A}$ requires solving the equation $\mathbf{Ax} = \mathbf{0}$.
- However there is a shortcut to find the dimension of $\operatorname{Nul}\mathbf{A}$.

  Let $\mathbf{A}$ be an $m \times n$ matrix and suppose the equation $\mathbf{Ax} = \mathbf{0}$ has k free variables.

  A spanning set for $\operatorname{Nul}\mathbf{A}$ has exactly k linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, one for each free variable.

  So $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for $\operatorname{Nul}\mathbf{A}$, and the number of free variables is the size of the basis.

Therefore, given the equation $\mathbf{Ax} = \mathbf{0}$:

- The dimension of $\operatorname{Nul}\mathbf{A}$ is the number of free variables.
- The dimension of $\operatorname{Col}\mathbf{A}$ is the number of pivot columns.

Dimensions of $\mathrm{Nul}\,\mathbf{A}$ and $\mathrm{Col}\,\mathbf{A}$

Example: To find the dimensions of the null space and column space of $\mathbf{A}$, we reduce the augmented matrix to echelon form.

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Row reduce the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Pivots in columns 1 and 3. Free variables: x_2, x_4 and x_5.

So $\dim(\mathrm{Nul}\,\mathbf{A}) = 3$ and $\dim(\mathrm{Col}\,\mathbf{A}) = 2$.

Row space and rank

Definition:

If $\mathbf{A}$ is an $m \times n$ matrix each row of $\mathbf{A}$ has n entries and thus can be identified with a vector in $\mathbb{R}^n$.

The set of all linear combinations of these row vectors is called the *row space* of $\mathbf{A}$ and is denoted by $\mathrm{Row}\,\mathbf{A}$.

Notes:

- Each row has n entries, so $\mathrm{Row}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$.
- Since the rows of $\mathbf{A}$ are identical with columns of $\mathbf{A}^{\mathsf{T}}$,

$$\mathrm{Col}\,\mathbf{A}^{\mathsf{T}} = \mathrm{Row}\,\mathbf{A}$$

Row space and rank

Theorem: If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row-equivalent, then their row spaces are equal. If $\mathbf{B}$ is in echelon form, the non-zero rows of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$ and $\mathbf{B}$.

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by row operations, the rows of $\mathbf{B}$ are the linear combinations of the rows of $\mathbf{A}$.
- Furthermore, any linear combination of the rows of $\mathbf{B}$ is automatically a linear combination of rows of $\mathbf{A}$. Thus the row space of $\mathbf{B}$ is contained in the row space of $\mathbf{A}$.
- Row operations are reversible, so the same argument implies that the row space of $\mathbf{A}$ is contained in the row space of $\mathbf{B}$.
- Therefore, it must be that the two row spaces are the same.
- Finally, if $\mathbf{B}$ is in echelon form, its non-zero rows are linearly independent. Thus the non-zero rows of $\mathbf{B}$ form a basis of the common row space of $\mathbf{B}$ and $\mathbf{A}$.

Row space and rank

Example: Find bases and dimensions for $\operatorname{Col}\mathbf{A}$, $\operatorname{Nul}\mathbf{A}$, $\operatorname{Row}\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$\to \mathbf{B} = \begin{bmatrix} 1 & -3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-zero rows of $\mathbf{B}$ or $\mathbf{C}$ are the first three rows.

Pivot columns in $\mathbf{B}$ or $\mathbf{C}$ are columns 1, 2 and 4.

Number of free variables in $\mathbf{B}$ or $\mathbf{C}$ is 2.

Thus: $\dim(\operatorname{Row}\mathbf{A}) = 3$; $\dim(\operatorname{Col}\mathbf{A}) = 3$; $\dim(\operatorname{Nul}\mathbf{A}) = 2$.

Row space and rank

A basis for $\mathrm{Row}\,\mathbf{A}$ (and $\mathrm{Row}\,\mathbf{B}$) is given by non-zero rows of $\mathbf{B}$:

$$\mathrm{Row}\,\mathbf{A} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 20 \end{bmatrix} \right\}$$

A basis for $\mathrm{Col}\,\mathbf{A}$ (but not $\mathrm{Col}\,\mathbf{B}$) is given by pivot columns in $\mathbf{A}$:

$$\mathrm{Col}\,\mathbf{A} = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Row space and rank

Now we use REF and solve $\mathbf{Cx} = \mathbf{0}$; free variables are x_3 and x_5:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 & -x_5 \\ 2x_3 & -3x_5 \\ x_3 & \\ & 5x_5 \\ & x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Therefore $$\operatorname{Nul} \mathbf{A} = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

The rank theorem

Definition: $\operatorname{rank} \mathbf{A} = \dim(\operatorname{Col} \mathbf{A})$

The *rank* of matrix $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$.

Notes

- Since $\operatorname{Row} \mathbf{A} = \operatorname{Col} \mathbf{A}^\mathsf{T}$, then $\dim(\operatorname{Row} \mathbf{A}) = \operatorname{rank} \mathbf{A}^\mathsf{T}$.
- The dimension of $\operatorname{Nul} \mathbf{A}$ is sometimes called the *nullity* of $\mathbf{A}$.

The rank theorem: For an $m \times n$ matrix $\mathbf{A}$,

$$\dim(\operatorname{Col} \mathbf{A}) = \dim(\operatorname{Row} \mathbf{A}) = \operatorname{rank} \mathbf{A}$$

This common dimension, the rank of matrix $\mathbf{A}$, also equals to the number of pivot positions in $\mathbf{A}$ and also satisfies the equation

$$\operatorname{rank} \mathbf{A} + \dim(\operatorname{Nul} \mathbf{A}) = n$$

The rank theorem

Proof: By definition, $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A})$, which equals to the number of basis vectors for $\operatorname{Col}\mathbf{A}$, which is the number of pivot columns in $\mathbf{A}$. Equivalently, $\operatorname{rank}\mathbf{A}$ is then the number of pivot columns in an echelon form $\mathbf{B}$ of $\mathbf{A}$.

Because $\mathbf{B}$ has a non-zero row for each pivot, and these rows form a basis for $\operatorname{Row}\mathbf{A}$, $\operatorname{rank}\mathbf{A}$ is also the dimension of $\operatorname{Row}\mathbf{A}$,

$$\dim(\operatorname{Row}\mathbf{A}) = \dim(\operatorname{Col}\mathbf{A})$$

The dimension of $\operatorname{Nul}\mathbf{A}$ is the number of columns of $\mathbf{A}$ which correspond to free variables, so which are *not* the pivot columns.

Obviously, the total number of columns n is the sum of the number of columns with pivots, and the number of columns without pivots, and therefore

$$\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = n$$

The rank theorem

Examples:

(a) If $\mathbf{A}$ is a 7×9 matrix with a two-dimensional null space, what is the rank of $\mathbf{A}$?

— As $\mathbf{A}$ has 9 columns, $\operatorname{rank} \mathbf{A} + 2 = 9$ and hence $\operatorname{rank} \mathbf{A} = 7$.

(b) Could a 6×9 matrix $\mathbf{B}$ have a two-dimensional null space?

— If a 6×9 matrix had a two-dimensional null space it would have to have rank 7 by the rank theorem.

However the columns of $\mathbf{B}$ are vectors in $\mathbb{R}^6$, and so $\dim(\operatorname{Col} \mathbf{B}) \leqslant 6$; therefore $\operatorname{rank} \mathbf{B} \leqslant 6$.

Thus $\mathbf{B}$ cannot have a two-dimensional null space.

The rank theorem

Example: Suppose that two solutions of a homogeneous system of 40 equations in 42 variables are found.

These solutions are not multiples, and all other solutions can be constructed by adding appropriate multiples of these two solutions.

What can be said about the solution of an associated inhomogeneous system?

Solution: Let $\mathbf{A}$ be the 40×42 coefficient matrix of the system.

The two solutions are linearly independent and span $\mathrm{Nul}\,\mathbf{A}$.

So $\dim(\mathrm{Nul}\,\mathbf{A}) = 2$.

By the rank theorem, $\dim(\mathrm{Col}\,\mathbf{A}) = 42 - 2 = 40$.

Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ whose dimension is 40, $\mathrm{Col}\,\mathbf{A}$ must be all of $\mathbb{R}^{40}$.

Then that every inhomogeneous equation $\mathbf{Ax} = \mathbf{b}$ has a solution.

The invertible matrix theorem (summary of results)

Let $\mathbf{A}$ be $n \times n$ matrix. Then each of the following statements is equivalent to the statement that $\mathbf{A}$ is an invertible matrix:

- $\mathbf{A}$ has n pivot positions
- $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution
- The columns of $\mathbf{A}$ form a linearly independent set
- The transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one
- The columns of $\mathbf{A}$ span $\mathbb{R}^n$
- The columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$
- The transformation $\mathbf{x} \mapsto \mathbf{Ax}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$
- There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}$
- $\mathbf{A}^{\mathsf{T}}$ is invertible
- $\mathrm{Col}\,\mathbf{A} = \mathbb{R}^n$
- $\dim(Col\,\mathbf{A}) = n$
- $\mathrm{rank}\,\mathbf{A} = n$
- $\mathrm{Nul}\,\mathbf{A} = \{\mathbf{0}\}$
- $\dim(\mathrm{Nul}\,\mathbf{A}) = 0$

Change of basis

A basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for an n-dimensional space V, imposes a coordinate system for V via coordinate mapping onto $\mathbb{R}^n$.

Any $\mathbf{x} \in V$ is identified uniquely by its $\mathcal{B}$-coordinates $[\mathbf{x}]_\mathcal{B}$ in $\mathbb{R}^n$.

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

$$[\mathbf{x}]_\mathcal{B} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

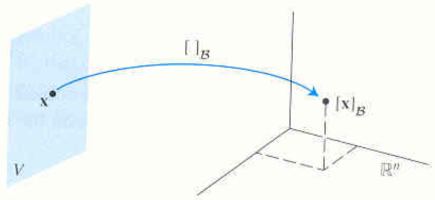


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

In a sense, $[\mathbf{x}]_\mathcal{B}$ lists the coefficients to build $\mathbf{x}$ as a linear combination of the basis vectors in $\mathcal{B}$.

Change of basis

Example: Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$, and some $\mathbf{x}$:

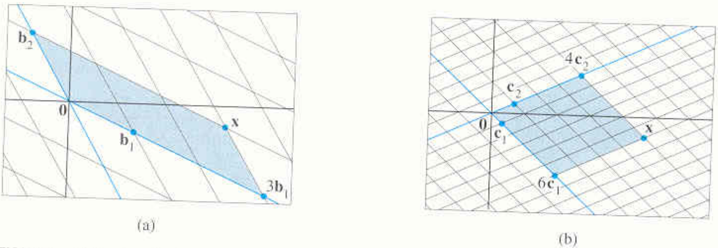


FIGURE 1 Two coordinate systems for the same vector space.

On the left, $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. On the right, the same $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$.

That is, $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and $[\mathbf{x}]_{\mathcal{C}} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

A connection between these coordinate systems can be established as we know how $\mathbf{b}_1$ and $\mathbf{b}_2$ are connected to $\mathbf{c}_1$ and $\mathbf{c}_2$.

Change of basis

Apply $\mathcal{C}$ coordinates to $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. Since the mapping is linear,

$$[\mathbf{x}]_{\mathcal{C}} = [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} = \left[\begin{array}{cc} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{array}\right] \left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

Given that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$, the columns are:

$$[\mathbf{b}_1]_{\mathcal{C}} = \left(\begin{array}{r} 4 \\ 1 \end{array}\right), \quad [\mathbf{b}_2]_{\mathcal{C}} = \left(\begin{array}{r} -6 \\ 1 \end{array}\right).$$

Therefore the connection between coordinates is

$$[\mathbf{x}]_{\mathcal{C}} = \left[\begin{array}{rr} 4 & -6 \\ 1 & 1 \end{array}\right] \left(\begin{array}{c} 3 \\ 1 \end{array}\right) = \left(\begin{array}{c} 6 \\ 4 \end{array}\right)$$

Change of basis

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of space V. Then there is a unique $n \times n$ matrix $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}.$$

The columns of $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in basis $\mathcal{B}$. That is, $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$.

$\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ is called the *change of coordinate matrix* from $\mathcal{B}$ to $\mathcal{C}$.

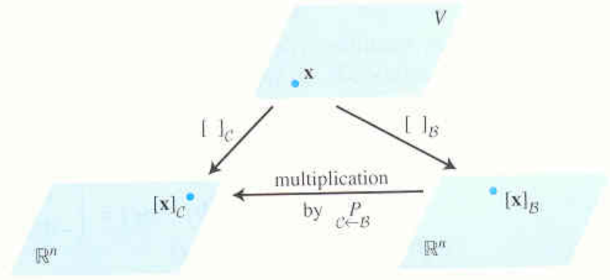


FIGURE 2 Two coordinate systems for V.

Change of basis

The columns of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$.
$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is a square matrix, and coordinate mapping is one-to-one, therefore $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. By multiplying both the sides of

$$[\mathbf{x}_{\mathcal{C}}] = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

by $(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ we obtain

$$(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\,[\mathbf{x}_{\mathcal{C}}] = [\mathbf{x}]_{\mathcal{B}}.$$

Therefore the matrix $(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ converts $\mathcal{C}$ coordinates into $\mathcal{B}$ coordinates:

$$(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}.$$

Note: For the standard basis $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, each $[\mathbf{b}_i]_{\mathcal{E}} = \mathbf{b}_i$. Then $\mathbf{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ (sometimes denoted as $\mathbf{P}_{\mathcal{B}}$) is:

$$\mathbf{P}_{\mathcal{E}\leftarrow\mathcal{B}} \equiv \mathbf{P}_{\mathcal{B}} = \left[\mathbf{b}_1,\, \ldots, \mathbf{b}_n\right] \equiv \Big[[\mathbf{b}_1]_{\mathcal{E}},\, \ldots, [\mathbf{b}_n]_{\mathcal{E}}\Big].$$

Change of basis

Example: Find $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Solution: The columns of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\mathcal{C}$-coordinates of $\mathbf{b}_1$ and $\mathbf{b}_2$.

Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $[\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then $\begin{cases} \mathbf{b}_1 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 \\ \mathbf{b}_2 = y_1\mathbf{c}_1 + y_2\mathbf{c}_2 \end{cases}$

or, in matrix-vector form,

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{b}_1, \qquad \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{b}_2.$$

To find x_1, x_2, y_1, y_2 at once, we form the doubly augmented matrix: $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & | & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.

Change of basis

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

$$\left[\mathbf{c}_1 \ \mathbf{c}_2 \mid \mathbf{b}_1 \ \mathbf{b}_2\right] = \left[\begin{array}{rr|rr} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{rr|rr} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

so the change of coordinate matrix is

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

Change of basis

Note:

$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ appeared in the right-hand side upon row reduction:

$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

The first column of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ results from row reducing $[\,\mathbf{c}_1\ \mathbf{c}_2 \mid \mathbf{b}_1\,]$ to $\left[\,\mathbf{I} \mid [\mathbf{b}_1]_{\mathcal{C}}\right]$ and likewise for the second column of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$, so

$$\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] \to \left[\begin{array}{c|c} \mathbf{I} & \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

This procedure is used for changing between any two bases in $\mathbb{R}^n$.

Change of basis

Another way to change coordinates from $\mathcal{B}$ to $\mathcal{C}$ is to combine the transitions into standard coordinates, realised by $\mathbf{P}_{\mathcal{B}}$ and $\mathbf{P}_{\mathcal{C}}$.

$$\forall\, \mathbf{x} \in \mathbb{R}^n \qquad \mathbf{P}_{\mathcal{B}}\, [\mathbf{x}]_{\mathcal{B}} = \mathbf{x} \qquad \mathbf{P}_{\mathcal{C}}\, [\mathbf{x}]_{\mathcal{C}} = \mathbf{x}$$

From the last relation we obtain

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\, \mathbf{x}$$

and subsequently using the first relation we obtain

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\, \mathbf{x} = \mathbf{P}_{\mathcal{C}}^{-1}\, \mathbf{P}_{\mathcal{B}}\; [\mathbf{x}]_{\mathcal{B}}$$

Therefore

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \mathbf{P}_{\mathcal{C}}^{-1}\, \mathbf{P}_{\mathcal{B}} \qquad \text{and} \qquad \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \mathbf{P}_{\mathcal{B}}^{-1}\, \mathbf{P}_{\mathcal{C}}$$

This approach is however slower than the direct transformation.

Change of basis

Example: Find $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ by using $\mathbf{P}_{\mathcal{C}}$ and $\mathbf{P}_{\mathcal{B}}$ for the two bases:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix};\quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

We first construct $\mathbf{P}_{\mathcal{C}}$ and $\mathbf{P}_{\mathcal{B}}$ and then use $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = (\mathbf{P}_{\mathcal{C}})^{-1}\mathbf{P}_{\mathcal{B}}$.

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \qquad \mathbf{P}_{\mathcal{C}} = \begin{bmatrix} -7 & -5 \\ 9 & 7 \end{bmatrix}.$$

Then

$$\mathbf{P}_{\mathcal{C}}^{-1} = \begin{bmatrix} -7/4 & -5/4 \\ 9/4 & 7/4 \end{bmatrix}.$$

and so

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = (\mathbf{P}_{\mathcal{C}})^{-1}\,\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

Summary

- Coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$
- Dimensions of vector spaces
- $\operatorname{rank} \mathbf{A} = \dim(\operatorname{Col} \mathbf{A})$
- $\operatorname{rank} \mathbf{A} + \dim(\operatorname{Nul} \mathbf{A}) = n$
- Change of basis $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}} = (\mathbf{P}_{\mathcal{C}})^{-1} \mathbf{P}_{\mathcal{B}}$