Linear Algebra Autumn 2019 - Assignment 1

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Question 1

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & -4 & 4 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ -1 & 2 & -2 & 2 & 1 \end{bmatrix} \tag{1}$$

Row operation: $R1 \hookleftarrow (R1 + R4)$

$$\begin{bmatrix} 0 & 0 & -6 & 6 & 0 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ -1 & 2 & -2 & 2 & 1 \end{bmatrix} \tag{2}$$

Row operation: $R1 \leftrightarrow R4$

$$\begin{bmatrix} -1 & 2 & -2 & 2 & 1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -6 & 6 & 0 \end{bmatrix} \tag{3}$$

Row operation: $R1 \hookleftarrow (-1 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -6 & 6 & 0 \end{bmatrix} \tag{4}$$

Row operation: $R4 \hookleftarrow (\frac{1}{6} \times R4)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ -3 & 6 & 4 & 3 & 10 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \tag{5}$$

Row operation: $R2 \hookleftarrow (R2 + 3 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 10 & -3 & 7 \\ -2 & 4 & 2 & 2 & 6 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \tag{6}$$

Row operation: $R3 \hookleftarrow (R3 + 2 \times R1)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 10 & -3 & 7 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \tag{7}$$

Row operation: $R2 \leftrightarrow R4$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \tag{8}$$

Row operation: $R2 \hookleftarrow (-1 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 6 & -2 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \tag{9}$$

Row operation: $R3 \hookleftarrow (R3 - 6 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \tag{10}$$

Row operation: $R3 \hookleftarrow (\frac{1}{4} \times R3)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 10 & -3 & 7 \end{bmatrix} \tag{11}$$

Row operation: $R4 \hookleftarrow (R4 - 10 \times R2)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 & 7 \end{bmatrix} \tag{12}$$

Row operation: $R4 \hookleftarrow (\frac{1}{7} \times R4)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{13}$$

Row operation: $R4 \hookleftarrow (R4 - R3)$

$$\begin{bmatrix} 1 & -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Row operation: $R1 \hookleftarrow (R1 - 2 \times R2)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Row operation: $R2 \hookleftarrow (R2 + R3)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

Basic variables: x_1, x_3, x_4

Free variable: x_2

$$\mathbf{x} = \begin{bmatrix} x_1 = -1 + 2x_2 \\ x_3 = 1 \\ x_5 = 1 \end{bmatrix} \tag{17}$$

Question 2

Augmented matrix:

$$\begin{bmatrix} 3 & 6 & 1 & -2 & -4 & 6 \\ 1 & 2 & 0 & 1 & -1 & -1 \end{bmatrix} \tag{18}$$

Row operation: $R1 \leftrightarrow R2$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & -1 \\ 3 & 6 & 1 & -2 & -4 & 6 \end{bmatrix} \tag{19}$$

Row operation: $R2 \hookleftarrow (R2 - 3 \times R1)$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -5 & -1 & 9 \end{bmatrix} \tag{20}$$

Basic variables: x_1, x_3

Free variables: x_2, x_4, x_5

$$\mathbf{x} = \begin{bmatrix} x_1 = -1 - 2x_2 - x_4 + x_5 \\ x_3 = 9 + 5x_4 + x_5 \end{bmatrix} \tag{21}$$

Question 3

$$\det(\mathbf{A}) = 1\begin{vmatrix}9 & 4\\6 & 5\end{vmatrix} - 2\begin{vmatrix}8 & 4\\7 & 5\end{vmatrix} + 3\begin{vmatrix}8 & 9\\7 & 6\end{vmatrix} \tag{22}$$

$$= 1\,(45 - 24) - 2\,(40 - 28) - 4\,(48 - 63) \tag{23}$$

$$\det(\mathbf{A}) = -48 \tag{24}$$

$$\det(\mathbf{B}) = 0\begin{vmatrix}1 & 6 & 0\\1 & 2 & 0\\-2 & 0 & 3\end{vmatrix} - 1\begin{vmatrix}0 & 6 & 0\\1 & 2 & 0\\1 & 0 & 3\end{vmatrix} + 2\begin{vmatrix}0 & 1 & 0\\1 & 1 & 0\\1 & -2 & 3\end{vmatrix} - 3\begin{vmatrix}0 & 1 & 6\\1 & 1 & 2\\1 & -2 & 0\end{vmatrix} \tag{25}$$

$$\det(\mathbf{B}) = (0)B_{11} + B_{12} + (2)B_{13} + (3)B_{14} \tag{26}$$

Calculating the cofactors:

$$B_{12} = (-1)^{1+2}\begin{vmatrix}0 & 6 & 0\\1 & 2 & 0\\1 & 0 & 3\end{vmatrix} = (-1)(-6)(3-0) = 18 \tag{27}$$

$$B_{13} = (-1)^{1+3}\begin{vmatrix}0 & 1 & 0\\1 & 1 & 0\\1 & -2 & 3\end{vmatrix} = (1)(-1)(3-0) = -3 \tag{28}$$

$$B_{14} = (-1)^{1+4}\begin{vmatrix}0 & 1 & 6\\1 & 1 & 2\\1 & -2 & 0\end{vmatrix} = (-1)\big((-1)(0-2) + 6(-3)\big) = (-1)(2-18) = 16 \tag{29}$$

Then calculating the determinant:

$$\det(\mathbf{B}) = (0) + 18 + (2)(-3) + (3)(16) = 12 + 48 \tag{30}$$

$$\det(\mathbf{B}) = 60 \tag{31}$$

Question 4

$$(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 7 \\ -1 & 2 & 0 \\ -2 & 5 & 3 \end{bmatrix} \tag{32}$$

If: $\det\mathbf{A} = 0$ then it is a singular matrix.

$$\det(\mathbf{A}) = 1\Big((2)(3) - (5)(0)\Big) + 7\Big((-1)(5) - (2)(-2)\Big) \tag{33}$$

$$\det(\mathbf{A}) = 6 - 7 = -1 \tag{34}$$

$$\therefore \det(\mathbf{A}) \neq 0 \Rightarrow \mathbf{A} \text{ is non-singular} \Rightarrow \mathbf{A} \text{ has an inverse} \tag{35}$$

Append identity matrix, forming augmented matrix for Gaussian elimination.

$$\mathbf{A}|\mathbf{I} = \begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ -2 & 5 & 3 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

Row operation: $R3 \hookleftarrow (R3 + 2 \times R1)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \end{bmatrix} \tag{37}$$

Row operation: $R2 \leftrightarrow R3$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ -1 & 2 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{38}$$

Row operation: $R3 \hookleftarrow (R3 + R1)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 2 & 7 & 1 & 1 & 0 \end{bmatrix} \tag{39}$$

Row operation: $R3 \hookleftarrow \left(R3 - \frac{2}{5}R2\right)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 1 & \frac{-2}{5} \end{bmatrix} \tag{40}$$

Row operation: $R3 \hookleftarrow (5 \times R3)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 17 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \tag{41}$$

Row operation: $R2 \hookleftarrow (R1 - 17 \times R3)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 5 & 0 & -15 & -85 & 35 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \tag{42}$$

Row operation: $R2 \leftarrow \left(\frac{1}{5} \times R2\right)$

$$\begin{bmatrix} 1 & 0 & 7 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -17 & 7 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \tag{43}$$

Row operation: $R1 \leftarrow (R1 - 7 \times R3)$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & -35 & 14 \\ 0 & 1 & 0 & -3 & -17 & 7 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} = \mathbf{I}|\mathbf{A}^{-1} \tag{44}$$

$$\therefore \mathbf{A}^{-1} = \begin{bmatrix} -6 & -35 & 14 \\ -3 & -17 & 7 \\ 1 & 1 & -2 \end{bmatrix} \tag{45}$$

Question 5

$$\mathbf{A} = \begin{bmatrix} a_{11} & 2 & 1 \\ 5 & -1 & 2 \\ -3 & 1 & -1 \end{bmatrix} \tag{46}$$

If $\det(\mathbf{A}) = 0 \Rightarrow \mathbf{A}$ is singular

$$\det(\mathbf{A}) = a_{11}\Big((-1)(-1) - (2)(1)\Big) - 2\Big((5)(-1) - (-3)(2)\Big) + 1\Big((5)(1) - (-1)(-3)\Big) = 0 \tag{47}$$

$$0 = a_{11}\,(1-2) - 2\,(-5+6) + 1\,(5-3) \tag{48}$$

$$a_{11} = 0 \tag{49}$$

Therefore, if you want $\mathbf{A}$ to be singular a_{11} must be set to zero.