# Linear Algebra, Assignment 7

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# Question 1

## Question 1: Part 1 (a)

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & -6 & -1 & -3 \\ 1 & -8 & 9 & -3 & 10 \\ -2 & -9 & 7 & 1 & 0 \end{bmatrix}$$

$$\text{ref}(\mathbf{A}) = \begin{bmatrix} 2 & \frac{9}{2} & -\frac{7}{2} & -\frac{1}{2} & 0 \\ 0 & 5 & -5 & 1 & -4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

### Col(A)

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -8 \\ -9 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\} = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \Rightarrow \text{Col}(\mathbf{A}) = \{a \cdot \mathbf{u} + b \cdot \mathbf{v} + c \cdot \mathbf{w} = \mathbf{0} \mid \{a, b, c\} \in \mathbb{R}\}$$

### Nul(A)

Free variables:

$$\{x_3, x_5\}$$

Basic variables:

$$\{x_1, x_2, x_4\}$$

Expressing basic in terms of free:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - 5x_5 \\ x_3 + x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

So $\text{Nul}(\mathbf{A})$ is

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{Span}\{\mathbf{u_n}, \mathbf{v_n}\} \Rightarrow \text{Nul}(\mathbf{A}) = \{a \cdot \mathbf{u_n} + b \cdot \mathbf{v_n} = \mathbf{0} \mid \{a, b\} \in \mathbb{R}\}$$

**Row(A)**

$$\text{Span}\left\{\begin{bmatrix}2\\ \frac{9}{2}\\ -\frac{7}{2}\\ -\frac{1}{2}\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 5\\ -5\\ 1\\ -4\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 0\\ 1\\ 1\end{bmatrix}\right\} = \text{Span}\left\{\mathbf{u_r}, \mathbf{v_r}, \mathbf{w_r}, \right\} \Rightarrow \text{Row}(\mathbf{A}) = \{a \cdot \mathbf{u_r} + b \cdot \mathbf{v_r} + c \cdot \mathbf{w_r} = \mathbf{0} \mid \{a, b, c\} \in \mathbb{R}\}$$

# Question 1: Part 1 (b)

Basis for Col($\mathbf{A}$)

$$\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}7\\-8\\-9\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\end{bmatrix}\right\}$$

Basis for Nul($\mathbf{A}$)

$$\left\{\begin{bmatrix}-1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-5\\1\\0\\-1\\1\end{bmatrix}\right\}$$

Basis for Row($\mathbf{A}$)

$$\left\{\begin{bmatrix}2\\ \frac{9}{2}\\ -\frac{7}{2}\\ -\frac{1}{2}\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 5\\ -5\\ 1\\ -4\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 0\\ 1\\ 1\end{bmatrix}\right\}$$

# Question 1: Part 1 (c)

$\dim(\text{Col}(\mathbf{A})) = 3$
$\dim(\text{Nul}(\mathbf{A})) = 2$
$\dim(\text{Row}(\mathbf{A})) = \dim(\text{Col}(\mathbf{A})) = 3$

## Question 1: Part 1 (d)

$$x = \begin{bmatrix} 2 \\ -9 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$x \in \mathbf{Col(A)}$ **?**

Take the Col($\mathbf{A}$), form a matrix, invert and multiply by $x$.

$$\begin{bmatrix} 1 & 1 & 2 \\ 7 & -8 & -9 \\ -1 & -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -9 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{25}{13} \\ \frac{5}{13} \\ \frac{34}{13} \end{bmatrix}$$

$$\therefore x \in \mathrm{Col}(\mathbf{A})$$

$x \in \mathbf{Nul(A)}$ ?

The dimensions of the Null space do not match $x$ and so it **cannot** be in the Null space. i.e. the Null space is made of vectors that have 5 entries, where as $x$ only has 3.

$x \in \mathbf{Row(A)}$ ?

The dimensions of the Row space do not match $x$ and so it **cannot** be in the Row space. i.e. the Row space is made of vectors that have 5 entries, where as $x$ only has 3.

$y \in \mathbf{Col(A)}$ ?

The dimensions of the Column space does not match $y$ and so it **cannot** be in the Column space. i.e. the Column space is made of vectors that have 3 entries, where as $y$ has 5.

$y \in \mathbf{Nul(A)}$ ?

$$a\mathbf{u_n} + b\mathbf{v_n} = \mathbf{y}$$

$$a \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -5 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Form an augmented matrix:

$$\mathcal{N} = \begin{bmatrix} -1 & -5 & 4 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Get into reduced echelon form:

$$\text{rref}(\mathcal{N}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $a = 1$ and $b = -1$, so $y$ is a linear combination of the null vectors, meaning it lies in the space Null space.

$$\therefore y \in \text{Nul}(\mathbf{A})$$

$y \in \mathbf{Row}(\mathbf{A})$ ?

$$a\mathbf{u_r} + b\mathbf{v_r} + c\mathbf{w_r} = \mathbf{y}$$

$$a\begin{bmatrix} 2 \\ \frac{9}{2} \\ -\frac{7}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 5 \\ -5 \\ 1 \\ -4 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Form an augmented matrix:

$$\mathcal{C} = \begin{bmatrix} 2 & 0 & 0 & 4 \\ \frac{9}{2} & 5 & 0 & 0 \\ -\frac{7}{2} & -5 & 0 & 1 \\ -\frac{1}{2} & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \end{bmatrix}$$

Get into reduced echelon form:

$$\text{rref}(\mathcal{C}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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# Question 1: Part 2 (a)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 8 \\ 0 & -3 & 3 \\ -1 & 3 & -8 \end{bmatrix}$$

$$\text{ref}(\mathbf{A}) = \begin{bmatrix} 3 & 7 & 8 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Col(A)**

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -3 \\ 3 \end{bmatrix}\right\} = \text{Span}\{\mathbf{u}, \mathbf{v}\} \Rightarrow \text{Col}(\mathbf{A}) = \{a \cdot \mathbf{u} + b \cdot \mathbf{v} = \mathbf{0} \mid \{a, b\} \in \mathbb{R}\}$$

**Nul(A)**

Free variables

$$\{x_3\}$$

Basic variables:

$$\{x_1, x_2\}$$

Expressing basic in terms of free:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\} = \text{Span}\{\mathbf{u_n}\} \Rightarrow \text{Nul}(\mathbf{A}) = \{a \cdot \mathbf{u_n} = \mathbf{0} \mid \{a\} \in \mathbb{R}\}$$

**Row(A)**

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\} = \text{Span}\{\mathbf{u_r}, \mathbf{v_r}\} \Rightarrow \text{Row}(\mathbf{A}) = \{a \cdot \mathbf{u_r} + b \cdot \mathbf{v_r} = \mathbf{0} \mid \{a, b\} \in \mathbb{R}\}$$

## Question 1: Part 2 (b)

Basis for Col($\mathbf{A}$)

$$\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -3 \\ 3 \end{bmatrix}\right\}$$

Basis for Nul($\mathbf{A}$)

$$\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Basis for Row($\mathbf{A}$)

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

## Question 1: Part 2 (c)

$\dim(\text{Col}(\mathbf{A})) = 2$

$\dim(\text{Nul}(\mathbf{A})) = 1$

$\dim(\text{Row}(\mathbf{A})) = \dim(\text{Col}(\mathbf{A})) = 2$

## Question 1: Part 2 (d)

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 2 \\ -6 \\ 10 \end{bmatrix}$$

$x \in \textbf{Col}(\mathbf{A})$ ?

The dimensions do not match up.

$x$ has three entries, $\text{Col}(\mathbf{A})$ has four entries, there is no way that $x$ can be contained in the space spanned by $\text{Col}(\mathbf{A})$.

$$\therefore x \notin \text{Col}(\mathbf{A})$$

$x \in \textbf{Nul}(\mathbf{A})$ ?

If $x$ was in $\text{Nul}(\mathbf{A})$, it would have to be a linear combination of its single basis vector.

$$a \cdot \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5a \\ a \\ a \end{bmatrix}$$

This shows us that $a$ would have to simultaneously be equal to 2 and 3, meaning that:

$$x \notin \text{Nul}(\mathbf{A})$$

$x \in \textbf{Row}(\mathbf{A})$ ?

If $x$ was a linear combination of the basis vectors for the row space, we would have:

$$a \cdot \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Forming an augmented matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 5 & -1 & 3 \end{bmatrix} \Rightarrow \text{rref}\left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 5 & -1 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $a = 2$ and $b = 1$, meaning that:

$$x \in \text{Row}(\mathbf{A})$$

$y \in \mathbf{Col}(\mathbf{A})$ ?

If $y$ is contained $\text{Col}(\mathbf{A})$ then:

$$a \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 7 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -6 \\ 10 \end{bmatrix}$$

Where $\{a, b\} \in \mathbb{R}$.
Forming an augmented matrix with the basis matrix of $\text{Col}(\mathbf{A})$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 2 \\ 0 & -3 & -6 \\ -1 & 3 & 10 \end{bmatrix}$$

$$\text{rref}\left(\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 2 \\ 0 & -3 & -6 \\ -1 & 3 & 10 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence:

$$y \in \text{Col}(\mathbf{A}) \quad \because \quad y = -4 \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 7 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -6 \\ 10 \end{bmatrix}$$

$y \in \mathbf{Nul}(\mathbf{A})$ ?

Dimensions not appropriate.

$$y \notin \text{Nul}(\mathbf{A})$$

$y \in \mathbf{Row}(\mathbf{A})$ ?

Dimensions not appropriate.

$$y \notin \text{Row}(\mathbf{A})$$

# Question 2

## Part (a)

Matrix is 7 equations (rows) by 8 variables (columns), meaning the size is $(7 \times 8)$.

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

If two of the rows are linearly dependent, then there must be rank 6.

We can also prove this with the rank theorem:

$$\text{Rank}(\mathbf{A}) + \dim(\text{Nul}(\mathbf{A})) = n$$

Where $n :=$ number of columns

$$\therefore n = 8$$

and with there being two dependent equations:

$$\dim(\text{Nul}(\mathbf{A})) = 2$$

so

$$\text{Rank}(\mathbf{A}) = n - \dim(\text{Nul}(\mathbf{A})) = 8 - 2$$

$$\therefore \text{Rank}(\mathbf{A}) = 6$$

## Part (b)

$$\dim(\text{Nul}(\mathbf{A})) = 2$$

## Part (c)

No, it does not have a solution for any $\mathbf{b}$. $\mathbf{b}$ would be of size $(7 \times 1)$. However, the column space of $\mathbf{A}$ only spans up to $\mathbb{R}^6$. Because of this, $\mathbf{Ax} = \mathbf{b}$ cannot express any $\mathbf{b}$. For that, $\text{Col}(\mathbf{A})$ would have to span $\mathbb{R}^7$.

## Part (d)

By the dimensions of the matrix.

Domain $= \mathbb{R}^8$

Codomain $= \mathbb{R}^7$

## Part (e)

Becuase of the two dependent rows, the range can only be spanning up to $\mathbb{R}^6$.

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