37233 Linear Algebra

Subject coordinator and lecturer:

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Classes

- One two-hour lecture (p.w.): Wednesdays 10:00 to 12:00
- One two-hour tutorial/computer laboratory (p.w.)

Subject Assessment

- Tutorial assignments (9 weekly; 40 marks each): 36%
- Written assignment (due in week 8: 8–10 May): 14%
- Final exam (in June; closed book): 50%
  (note: it is required to gain at least 40% of the exam marks)
- To pass the subject: at least 50% for the final combined mark

Subject contents

- Fundamentals of linear algebra
- Applications of linear algebra
- Computational methods

Software

- Wolfram Mathematica

Why do we need linear algebra?

Linear algebra is one of the fundamental areas of mathematics.

On a practical level, one of the aspects is solving linear equations.

- Linear models in science, engineering, economics, business ...
- Many systems in real world behave in an (approximately) linear manner over a significant range of parameters
- Genuinely nonlinear problems can be often linearised — that is, approximated by linear systems
- Natural phenomena are often described in terms of partial or ordinary differential equations. Solving these equations requires discretisation. This, in turn, leads to linear systems.

Applications of linear algebra

- Science
  - Mathematics
  - Astronomy
  - Physics
  - Chemistry
  - Biology
  - Statistics
  - ...
- Engineering (mechanical, electrical, ...)
- Economics and business
- Transport, logistics, ...
- “Big Data” analysis, IT, AI, machine learning, ...

Revision: Matrices. Determinants.

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For a 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is

$$\det \mathbf{A} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Properties (these properties hold for any $n \times n$ matrix)

$$\begin{vmatrix} a & b \\ c+e & d+f \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ e & f \end{vmatrix}$$

$$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinants

- Determinant of an identity (unitary) matrix equals to 1

$$\det \mathbf{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

- If two rows of $\mathbf{A}$ are same, then $\det \mathbf{A} = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

- An elementary row operation of addition of a multiple of one row to another row leaves the determinant unchanged

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c \pm ka & d \pm kb \end{vmatrix}$$

- Determinant of a matrix with a zero row equals to zero

$$\begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} = 0$$

Determinants

Definition: Let $\mathbf{A}$ be $n \times n$ matrix and a_{ij} an element of $\mathbf{A}$. The **cofactor** of a_{ij} is the $(n-1) \times (n-1)$ determinant A_{ij} obtained by:

1. deleting the i-th row and j-th column of $\mathbf{A}$ and
2. multiplying the resulting matrix determinant by $(-1)^{(i+j)}$.

The determinant of $\mathbf{A}$ can be found by *expansion* along row i:

$$\det \mathbf{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

For example, we can take an expansion along the first row:

$$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

Determinants

For example, for a 3×3 matrix, its determinant can be found as:

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Alternatively, expansion along the second row yields

$$\det \mathbf{A} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$= -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Linear algebra for solving linear equations

Linear equations

- Linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b, \tag{1}$$

where b and $a_1, a_2, \ldots a_n$ are real or complex numbers. (subscript n is an integer, a "counter" for the variables)

- For example, equations

$$4x_1 - 5x_2 + 2 = x_1, \qquad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are linear because they can be arranged in the form (1)

$$3x_1 - 5x_2 = -2, \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

- The following examples are not linear equations:

$$4x_1 - 5x_2 + 2 = x_1 \sin x_2, \quad x_2 = 2\sqrt{x_1} - 6$$

Systems of linear equations

- A system of linear equations (a linear system) is a collection of one or more linear equations involving the same variables

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8, \\ x_1 - x_3 = -7. \end{cases}$$

  In general, a system of m equations for n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \vdots \qquad\quad \vdots \qquad\quad \ldots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases}$$

- The solution of the system is a list of variables $x_1, x_2, \ldots, x_n$ that makes *each* equation a true statement.

Linear equations: graphical representation

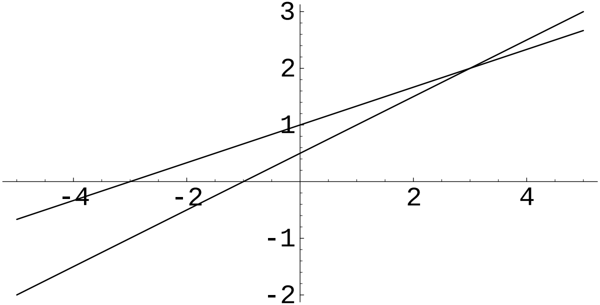
Consider a system of linear equations

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 3x_2 = 3. \end{cases}$$

It has a unique solution $x_1 = 3$ and $x_2 = 2$.

This may be represented graphically; in "Mathematica" type

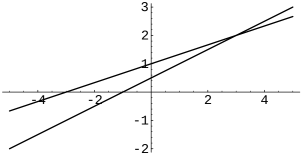
```
ContourPlot[{x1-2x2==-1,-x1+3x2==3},{x1,0,6},{x2,0,4},Axes->True]
```



Linear equations: graphical representation

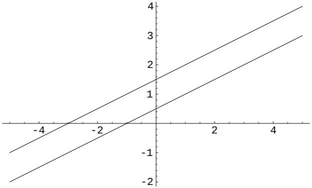
System 1: Unique solution

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 3x_2 = 3. \end{cases}$$



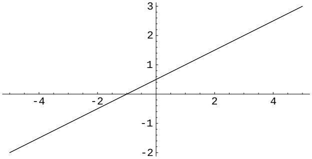
System 2: No solutions

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 2x_2 = 3. \end{cases}$$



System 3: Infinitely many solutions

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 2x_2 = 1. \end{cases}$$



Linear systems of equations

- A linear system may have

  - exactly one solution

  - no solutions

  - infinitely many solutions

- For 2 equations in 2 unknowns, easy to assess and visualise. Much harder or impossible in higher dimensions.

- We need a general tool to determine whether a system has a solution, and if so, whether the solution is unique.

Matrix representation of a linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

Set of coefficients a_{ij} forms an $m \times n$ matrix $\mathbf{A}$ of the system:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

The element a_{ij} located in i-th row and j-th column of $\mathbf{A}$.

Matrix representation of a linear system

The entire system of linear equations

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

can be compactly written using the notations

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

in a short form: $\qquad \mathbf{Ax} = \mathbf{b}.$

Augmented matrix of a linear system

For a linear system with m equations and n unknowns

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ldots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m. \end{cases}$$

an augmented matrix of the system is

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Gaussian reduction / elimination

In order to solve a linear system using matrix representation,
we need to reduce the augmented matrix to an *echelon form*.

This process is called Gauss – Jordan elimination,
achieved with a series of *row operations*.

Row operations that can be used:

- Swapping two rows
- Multiplying a row by a constant
- Adding a multiple of one row to another

Row reduction and echelon form

The first non-zero element in a row is called the **leading element**

The matrix in **echelon form** has the following properties:

- All non-zero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(where $\blacksquare$ is a non-zero number, and $*$ is any number)

Reduced echelon form (REF)

The next step is to obtain a reduced echelon form (REF):

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

REF matrix, in addition to **EF** form, has the properties that

- the leading entry in each non-zero row is 1
- the leading 1 is the only non-zero entry in its column

($\blacksquare$ is a non-zero number, $*$ is any number)

Matrices in EF and REF forms

Scheme of an EF form

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Scheme of a REF form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

($\blacksquare$ is a non-zero number, $*$ is any number)

Matrices in EF and REF forms

- The echelon form of a matrix (**EF**) is not unique, however reduced echelon form (**REF**) is unique.
- **Theorem:** Each matrix is row-equivalent to one and only one matrix in reduced echelon form (REF).

- A **pivot position** corresponds to the leading (non-zero) entry. A **pivot column** is a column that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduction to EF (example)

Finding solutions of a linear system using Gaussian reduction:

$$\left\{\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}\right.$$

Write the augmented matrix of this system:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Follow a step-wise procedure, using elementary row operations

- Swapping two rows
- Multiplying a row by a constant
- Adding a multiple of one row to another

Row reduction to EF (example)

Starting matrix: $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$

- **Step 1**: Begin with the first left non-zero column.
  Swap the rows to bring any zeros of the first column down.
  Select nonzero entry in the pivot column as a pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **Step 2**: Use row replacement operation to create zeros in all positions below the pivot (here, use $\mathsf{R}_2 \to \mathsf{R}_2 - \mathsf{R}_1$).

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Row reduction to EF (example)

- **Step 3**. Cover the row containing the pivot (and any rows above it). Apply steps 1–2 to the remaining sub-matrix. Repeat until there are no more non-zero rows to modify.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we divide the second row by 2)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we use $\mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2$)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction from EF to REF (example)

We have achieved an EF:

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

This is the end of the “forward” phase (down and to the right).

From here, we will work “backward” (to the left and up).

- **Step 4** : If a pivot is not 1, make it 1 by a scaling operation. Here, row operation $\mathrm{R}_1 \to \mathrm{R}_1/3$ leads to

$$\begin{bmatrix} {\color{red}1} & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row reduction from EF to REF (example)

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- **Step 5** : Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.

  Here, $\mathrm{R}_2 \to \mathrm{R}_2 - \mathrm{R}_3$ leads to

  $$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

  and $\mathrm{R}_1 \to \mathrm{R}_1 - 2\mathrm{R}_3$ leads to

  $$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The REF form ("backward phase")

- Done with the lowest-right pivot; address the next left-up

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

by making $\mathsf{R}_1 \to \mathsf{R}_1 + 3\mathsf{R}_2$:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This finally brings the matrix to the REF form.

The resulting matrix corresponds to an equivalent system

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

REF form and system solution

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{array}{rcl} x_1 - 2x_3 + 3x_4 & = & -24 \\ x_2 - 2x_3 + 2x_4 & = & -7 \\ x_5 & = & 4 \end{array}$$

- Note there are 3 equations for 5 variables.
- Variables with pivots are called **basic variables**: x_1, x_2, x_5.
- Variables without pivots are called **free variables**: x_3, x_4.
- In the final solution basic variables x_1, x_2, x_5 must be expressed in terms of free variables x_3, x_4.

The solution is:
$$\begin{array}{rcl} x_1 & = & -24 + 2x_3 - 3x_4 \\ x_2 & = & -7 + 2x_3 - 2x_4 \\ x_5 & = & 4 \end{array}$$

Various representations

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \Leftrightarrow \quad \begin{array}{rcl} x_1 & = & -24 + 2x_3 - 3x_4 \\ x_2 & = & -7 + 2x_3 - 2x_4 \\ x_5 & = & 4 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2x_3 - 3x_4 \\ -7 + 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Example 2

Second example to solve:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

To bring this to EF we eliminate x_1 in equation 3:

$$\text{Eq3} + 4 * \text{Eq1} \to \text{Eq3} \qquad \text{or} \qquad \text{R3} + 4 * \text{R1} \to \text{R3}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Example 2

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, we eliminate x_2 in equation 3. But first, factor R2.

$$\text{Eq2} \rightarrow \frac{1}{2} \times \text{Eq2} \qquad \text{or} \qquad \text{R2} \rightarrow \frac{1}{2} \times \text{R2}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, $\qquad \text{Eq3} + 3 * \text{Eq2} \rightarrow \text{Eq3} \qquad \text{or} \qquad \text{R3} + 3 * \text{R2} \rightarrow \text{R3}$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 2

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix of the system is now in an echelon form.

Now we continue towards REF, or we can also solve it directly:

Solving directly:

$$\begin{array}{lcl} \text{Eq3} & \Rightarrow & x_3 = 3 \\ \text{Eq2} & \Rightarrow & x_2 = 4x_3 + 4 = 4 \times 3 + 4 = 16 \\ \text{Eq1} & \Rightarrow & x_1 = 2x_2 - x_3 + 0 = 2 \times 16 - 3 = 29 \end{array}$$

This is called **backward substitution**.

Example 2

Continue with the elimination into REF

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Elimination above the pivots (yes, may be done also in this order):

$$\text{Eq1} + 2 * \text{Eq2} \rightarrow \text{Eq1} \qquad \text{or} \qquad \text{R1} + 2 * \text{R2} \rightarrow \text{R1}$$

$$\begin{array}{rcrcrcr} x_1 & & & - & 7x_3 & = & 8 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 2

$$\begin{array}{ccccccc} x_1 & & - & 7x_3 & = & 8 \\ & x_2 & - & 4x_3 & = & 4 \\ & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\text{Eq1} + 7 * \text{Eq3} \to \text{Eq1} \qquad \text{or} \qquad \text{R1} + 7 * \text{R2} \to \text{R1}$$

$$\text{Eq2} + 4 * \text{Eq3} \to \text{Eq2} \qquad \text{or} \qquad \text{R2} + 4 * \text{R3} \to \text{R2}$$

$$\begin{array}{ccccc} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example with no solutions

Third example

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Upon doing $\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$ and $\mathsf{R}_3 \to (2\mathsf{R}_3 - 5\mathsf{R}_2) + \mathsf{R}_1$ we get

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & {\color{red}=} & 5 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The **inconsistency** ${\color{red}0 = 5}$ implies that
this system does not have any solutions.

Example with infinitely many solutions

Fourth example

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 6 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & -1/2 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix}$$

Doing again $\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$ and $\mathsf{R}_3 \to (2\mathsf{R}_3 - 5\mathsf{R}_2) + \mathsf{R}_1$ yields

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 6 \\ & & & & 0 & = & 0 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 equations in 3 unknowns $\longrightarrow$ 2 equations in 3 unknowns
$\Rightarrow$ only two independent equations

No contradiction, but no unique solution (infinitely many)

Summary

Case 1: Consistent system, unique solution:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Case 2: Inconsistent system, no solution:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Case 3: Consistent system with infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. System of equations is **consistent** if the solution is unique or there are infinitely many solutions.
2. System of equations is **inconsistent** if it has no solutions.

Applications of linear systems

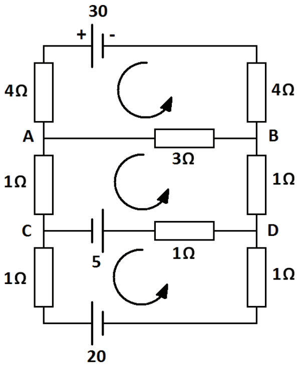
- Linear systems naturally arise in network analysis
- Network is a set of branches through which something "flows"
  - Electrical wires (electricity flow)
  - Economic linkages (money flow)
  - Pipes through which oil, gas or water flows
  - Fibres through which information flows (Internet)
- Branches meet at nodes or junctions
- A numerical measure is the rate of flow through a branch
- Analysis of networks is based on linear systems

Example: Electric circuits

Voltage drop across a resistor is given by Ohm's law $V = RI$

Kirchhoff's Voltage Law: *The algebraic sum of the IR voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.*

$$\sum_{i=1}^{N} R_i I_i = \sum_{i=1}^{M} V_i$$

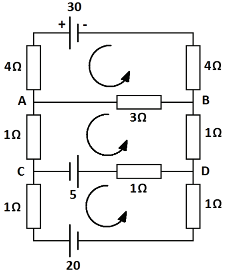


Example: Electric circuits

Loop 1: $4I_1 + 3I_1 - 3I_2 + 4I_1 = 30$

Loop 2: $-3I_1 + 3I_2 + I_2 + I_2 - I_3 + I_2 = 5$

Loop 3: $-I_2 + I_3 + I_3 + I_3 = -5 - 20$



$$\left\{\begin{array}{rcrcrcr} 11I_1 & - & 3I_2 & & & = & 30 \\ -3I_1 & + & 6I_2 & - & I_3 & = & 5 \\ & - & I_2 & + & 3I_3 & = & -25 \end{array}\right.$$

The loop currents are $I_1 = 3$ A, $I_2 = 1$ A and $I_3 = -8$ A

- The total current in the branch AB is $I_1 - I_2 = 3 - 1 = 2$ A
- The current in branch CD is $I_2 - I_3 = 9$ A

Example: Balancing chemical equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions:

$$(x_1)\,\mathrm{C_3H_8} + (x_2)\,\mathrm{O_2} \quad\rightarrow\quad (x_3)\,\mathrm{CO_2} + (x_4)\,\mathrm{H_2O}$$

Balancing requires finding amounts x_1, x_2, x_3, x_4 such that the total amounts of carbon C, hydrogen H, and oxygen O atoms on the left match the corresponding numbers on the right.

$$\mathrm{C_3H_8} : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} \quad \mathrm{O_2} : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{CO_2} : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{H_2O} : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of this system is $x_1 = 1,\ x_2 = 5,\ x_3 = 3,\ x_4 = 4.$

Revision: Inverse of a matrix

Definition: An $n \times n$ matrix $\mathbf{A}$ is said to be invertible
if there is an $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{CA} = \mathbf{I} \quad \text{and} \quad \mathbf{AC} = \mathbf{I},$$

where $\mathbf{I}$ is a unitary $n \times n$ matrix. Then $\mathbf{C}$ is the inverse of $\mathbf{A}$.

$\mathbf{C}$ is uniquely determined by $\mathbf{A}$:
Indeed, suppose $\mathbf{B}$ is another inverse of $\mathbf{A}$. Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

The inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \text{and} \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

A non-invertible matrix is called a **singular** matrix.
An invertible matrix is called a **non-singular** matrix.

Inverse of a matrix

Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Check

$$\mathbf{AC} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{CA} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem:

If $\quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad$ and $\quad \det \mathbf{A} = ad - bc \neq 0$

then $\mathbf{A}$ is invertible and $\quad \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of a matrix

Theorem: A matrix is invertible if and only if $\det \mathbf{A} \neq 0$.

So if $\det \mathbf{A} = 0$, then $\mathbf{A}$ is not invertible (singular).

Theorem:

a) If $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

b) If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Theorem:

If $\mathbf{A}$ is an invertible $n \times n$ matrix then $\forall\, \mathbf{b} \in \mathbb{R}^n$,
the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

$\mathbf{A}^{-1}$ by Gaussian elimination

Theorem: An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if it is row-equivalent to identity matrix $\mathbf{I}$, and a sequence of elementary row operations that reduces $\mathbf{A}$ to $\mathbf{I}$, transforms $\mathbf{I}$ into $\mathbf{A}^{-1}$.

This gives an easy algorithm for calculating $\mathbf{A}^{-1}$:

Row reduce the augmented matrix $[\mathbf{A} \,|\, \mathbf{I}]$.
If $\mathbf{A}$ is row-reduced to $\mathbf{I}$, then $[\mathbf{A} \,|\, \mathbf{I}]$ is row-reduced to $[\mathbf{I} \,|\, \mathbf{A}^{-1}]$.
Otherwise $\mathbf{A}$ does not have an inverse.

In practice $\mathbf{A}^{-1}$ is seldom computed directly ($2n^3$ operations).
Row reduction is faster and often more accurate.

$\mathbf{A}^{-1}$ by Gaussian elimination

Example: Find inverse of a matrix, if it exists

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

We form the extended augmented matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$\mathsf{R}_3 \rightarrow \mathsf{R}_3 - 4\mathsf{R}_1$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$\mathsf{R}_3 \rightarrow \mathsf{R}_3 + 3\mathsf{R}_2$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$\mathsf{R}_3 \rightarrow \mathsf{R}_3/2$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$\mathsf{R}_2 \to \mathsf{R}_2 - 2\mathsf{R}_3$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$\mathsf{R}_1 \to \mathsf{R}_1 - 3\mathsf{R}_3$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

Thus $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \left[\begin{array}{ccc} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{array}\right]$

See you next Wednesday

C B 0 5 C . 0 1 . 0 3 1