# Linear Algebra, Assignment 8

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# Question 1

## Part (a)

$$\mathbf{y} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$

Finding $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}(\mathbf{y}) = \sum_{i=1}^{p} \frac{\mathbf{y} \cdot \mathbf{u_i}}{\mathbf{u_i} \cdot \mathbf{u_i}} \cdot \mathbf{u_i} \quad \Rightarrow \quad p = 1 \quad \Rightarrow \quad \hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \cdot \mathbf{u_1}$$

$$\mathbf{y} \cdot \mathbf{u_1} = (7)(8) + (1)(-6) = 56 - 6 = 50$$

$$\mathbf{u_1} \cdot \mathbf{u_1} = (8)(8) + (-6)(-6) = 64 + 36 = 100$$

$$\hat{\mathbf{y}} = \frac{50}{100} \begin{bmatrix} 8 \\ -6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$

$$\therefore \ \hat{\mathbf{y}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$$

$$\therefore \mathbf{z} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Hence the orthogonal decomposition is:

$$\mathbf{y} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

## Part (b)

This would just be the modulus of $\mathbf{z}$.

$$\|\mathbf{z}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

# Question 2

$$\mathcal{W} = \text{Span}\left\{\mathbf{u_1}, \mathbf{u_2}\right\}$$

$$\mathbf{y_1} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} \qquad \mathbf{y_2} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{u_1} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{u_2} = \begin{bmatrix} -15 \\ 2 \\ 4 \end{bmatrix}$$

## Part (a)

First vector:

$$\mathbf{y_1} = \hat{\mathbf{y_1}} + \mathbf{z_1}$$

$$\hat{\mathbf{y_1}} = \text{proj}_{\mathcal{W}}\left(\mathbf{y_1}\right) = \sum_{i=1}^{p=2} \frac{\mathbf{y_1} \cdot \mathbf{u_i}}{\mathbf{u_i} \cdot \mathbf{u_i}} \cdot \mathbf{u_i}$$

$$\hat{\mathbf{y_1}} = \frac{\mathbf{y_1} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \cdot \mathbf{u_1} + \frac{\mathbf{y_1} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \cdot \mathbf{u_2} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{z_1} = \mathbf{y_1} - \hat{\mathbf{y_1}} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

$$\therefore \mathbf{y_1} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

Second vector:

$$\mathbf{y_2} = \hat{\mathbf{y_2}} + \mathbf{z_2}$$

$$\hat{\mathbf{y_2}} = \frac{\mathbf{y_2} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \cdot \mathbf{u_1} + \frac{\mathbf{y_2} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \cdot \mathbf{u_2} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{z_2} = \mathbf{y_2} - \hat{\mathbf{y_2}} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = 0$$

$$\therefore \mathbf{y_2} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## Part (b)

Distance from $\mathbf{y_1}$ to $\mathcal{W}$ is equal to the modulus of the $\mathbf{z}$ component of the orthogonal decomposition.

$$\|\mathbf{z_1}\| = \sqrt{2^2 + 3^2 + 6^2} = 7$$

$$\|\mathbf{z_2}\| = 0$$

Therefore, $\mathbf{y_1}$ is 7 units from $\mathcal{W}$ and $\mathbf{y_2}$ is 0 units from $\mathcal{W}$.

$$\therefore \mathbf{y_2} \in \mathcal{W}$$

# Question 3

## Part (a)

The Gram-Schmidt (a method used to convert a basis to an orthogonal basis) can concisely be written as:

$$\mathbf{v_k} = \mathbf{x_k} - \sum_{j=1}^{k-1} \text{proj}_{\mathbf{v_j}}\left(\mathbf{x_k}\right)$$

Our vectors are:

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{a_2} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{a_3} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Finding our orthogonal set ...

$$\mathbf{v_1} = \mathbf{x_1}$$

$$\mathbf{v_2} = \mathbf{x_2} - \text{proj}_{\mathbf{v_1}}\left(\mathbf{x_2}\right) = \mathbf{v_1} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{x_3} - \text{proj}_{\mathbf{v_1}}\left(\mathbf{x_3}\right) - \text{proj}_{\mathbf{v_2}}\left(\mathbf{x_3}\right) = \mathbf{v_1} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{x_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2}$$

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} -1/10 \\ -1/10 \\ 3/10 \\ 3/10 \end{bmatrix}$$

## Part (b)

Finding our orthonormal set:

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} \qquad \mathbf{u_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} \qquad \mathbf{u_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|}$$

$$\mathbf{u_1} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad \mathbf{u_2} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix} \qquad \mathbf{u_3} = \begin{bmatrix} -\sqrt{5}/10 \\ -\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix}$$

# Question 4

Ways to check whether a vector belongs to certain spaces:

$$\mathbf{v} \in \text{Col}(\mathbf{A}) \text{ if } \exists\ \mathbf{A}, \mathbf{x} \ \text{ s.t. } \mathbf{Ax} = \mathbf{v}$$

$$\mathbf{v} \in \text{Row}(\mathbf{A}) \text{ if } \exists\ \mathbf{A}, \mathbf{x} \ \text{ s.t. } \mathbf{A}^T\mathbf{x} = \mathbf{v}$$

$$\mathbf{v} \in \text{Nul}(\mathbf{A}) \text{ if } \exists\ \mathbf{A} \ \text{ s.t. } \mathbf{Av} = \mathbf{0}$$

## Part (a)

Can a vector $\mathbf{v}$ exist s.t. $\mathbf{v} \in \text{Col}(\mathbf{A})$ and $\mathbf{v} \in \text{Row}(\mathbf{A})$?
If $\mathbf{v} \in \text{Col}(\mathbf{A})$, then:

$$\mathbf{Ax} = \mathbf{v}$$

But if $\mathbf{v} \in \text{Row}(\mathbf{A})$, then:

$$\mathbf{A}^T\mathbf{x} = \mathbf{v}$$

Then:

$$\mathbf{A}^{-1}\mathbf{A}^T\mathbf{x} = \mathbf{x}$$

Meaning the following must hold for $\mathbf{v}$ to be contained in both vector spaces.

$$\mathbf{A}^{-1}\mathbf{A}^T = \mathbf{I}$$

Corollary:
This must also mean that $\mathbf{A}$ is a square, non-singular matrix.
**Note:** This might mean (I'm not sure) that $\mathbf{A}$ has to be the identity?

## Part (b)

Can a vector $\mathbf{v}$ exist s.t. $\mathbf{v} \in \text{Row}(\mathbf{A})$ and $\mathbf{v} \in \text{Nul}(\mathbf{A})$?
The matrix $\mathbf{A}$ is a linear map s.t. for an $m \times n$ matrix:

$$\mathbf{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

It should also be noted that:

$$\text{Row}(\mathbf{A}) \subset \mathbb{R}^m$$

$$\text{Nul}(\mathbf{A}) \subset \mathbb{R}^m$$

Then if $\mathbf{A}$ is a square, non-singular matrix, then:

$$\mathbf{A}^T\mathbf{x} = \mathbf{v} \qquad \mathbf{A}\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

$$\therefore\ \mathbf{A}^T\mathbf{x} = \mathbf{0}$$

By invertible-matrix-theorem:

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \Rightarrow \exists\ (\mathbf{A}^T)^{-1}\ \forall\ \mathbf{A}^{-1}$$

Meaning that $\mathbf{x} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$ to lie in both Row($\mathbf{A}$) and Nul($\mathbf{A}$).

## Part (c)

Can a vector $\mathbf{v}$ exist s.t. $\mathbf{v} \in \text{Col}(\mathbf{A})$ and $\mathbf{v} \in \text{Nul}(\mathbf{A})$?
The matrix $\mathbf{A}$ is a linear map s.t. for an $m \times n$ matrix:

$$\mathbf{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

It should also be noted that:

$$\text{Col}(\mathbf{A}) \subset \mathbb{R}^m$$

$$\text{Nul}(\mathbf{A}) \subset \mathbb{R}^n$$

Meaning that $m = n$ for there to be a vector, $\mathbf{v}$, that lies in both spaces.
What we are now really asking is that:

$$\exists\ \mathbf{v} \mid \mathbf{v} \in \big(\text{Col}(\mathbf{A})\ \cap\ \text{Nul}(\mathbf{A})\big)$$

A solution is the trivial solution, $\mathbf{v} = \mathbf{0}$.
Constraint from the column space:

$$\mathbf{A}\mathbf{x} = \mathbf{v} = \mathbf{0}$$

Constraint from the null space:

$$\mathbf{A}\mathbf{v} = \mathbf{0}$$

And again, if $\mathbf{A}$ is a square, invertible matrix, then:

$$\mathbf{v} = \mathbf{0}$$

Meaning:

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

And hence:

$$\mathbf{x} = \mathbf{0}$$