37233 Linear Algebra

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Classes

- One two-hour lecture (p.w.)
- One two-hour tutorial/computer laboratory (p.w.)

Subject Assessments

- Weekly Tutorial Assignments (starting week 2): weight 15%
- Written Assignment: weight 25 %
- Final exam: weight 60%
  It is required to gain at least 40% at the final exam
- To pass the subject it is necessary to obtain at least 50% for the final combined mark

Teaching linear algebra

- Purely theoretical approach (abstract)
- Application-oriented approach (hands on approach)

Subject contents

- Fundamentals of Linear Algebra
- Applications of Linear Algebra
- Computational Methods

Software

- Wolfram Mathematica

Contents of Lecture 1

- What is the subject of Linear Algebra?
- Why do we need Liner Algebra?
- Applications of Linear Algebra

- A brief review
  - Linear systems of equations
  - Row reduction / elimination (Gauss–Jordan)
  - Determinants $\det \mathbf{A}$
  - Inverse of a matrix $\mathbf{A}^{-1}$

The subject of linear algebra

- Linear algebra is one of the essential parts of mathematics. In short, it is the study of linear equations.
- Linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation that can be written as

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b, \tag{1}$$

  where b and $a_1, a_2, \ldots a_n$ are real or complex numbers. The subscript n is any integer number.
- Equations

$$4x_1 - 5x_2 + 2 = x_1, \qquad x_2 = 2(\sqrt{6} - x_1) + x_3$$

  are linear because they can be arranged as in (1)

$$3x_1 - 5x_2 = -2, \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

- The equations

$$4x_1 - 5x_2 + 2 = x_1 \sin x_2, \quad x_2 = 2\sqrt{x_1} - 6$$

  are not linear.

Why do we need linear algebra?

- Linear models in science, engineering, economics, statistics . . .
- Many systems in real world behave in a linear manner over a significant parameter ranges, even though they are nonlinear
- Genuinely nonlinear problems can be often linearised — approximated by linear systems
- Natural phenomena are often described in terms of partial or ordinary differential equations. Solving these equations requires discretisation. This, in turn, leads to linear systems.

Applications of linear algebra

- Science
  - Physics
  - Chemistry
  - Biology
  - ...
- Engineering (mechanical, electrical, ...)
- Economics
- Statistics
- Big Data Analysis

Systems of linear equations

- A system of linear equations (a linear system) is a collection of one or more linear equations involving the same variables

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 1.5x_3 & = & 8, \\ x_1 & & & - & x_3 & = & -7. \end{array}\right.$$

In general

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

- The solution of the system is a list of numbers $x_1, x_2, \dots, x_n$ that makes each equation a true statement.

System of linear equations

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

Set of coefficients a_{ij} is the matrix $\mathbf{A}[m \times n]$ of linear system

$$\mathbf{A} = \left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right],$$

The element a_{ij} located in i-th row and j-th column of $\mathbf{A}$.

Matrix representation of a linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In short, $\quad \mathbf{A}\mathbf{x} = \mathbf{b}.$

Augmented matrix of a linear system

- For a linear system with m equations and n unknowns

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

  an **augmented matrix** of the system is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Systems of Linear Equations

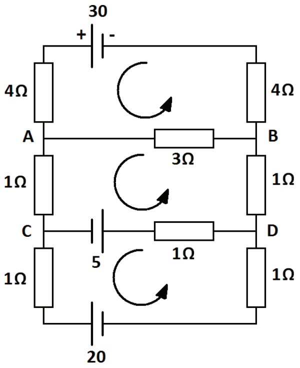
- Linear system naturally arise in network analysis
- Network is a set of branches through which something "flows"
  - Electrical wires (electricity flow)
  - Economic linkages (money flow)
  - Pipes through which oil, gas or water flows
  - Fibres through which information flows (Internet)
- Branches meet at nodes or junctions
- A numerical measure is the rate of flow through a branch
- Analysis of networks is based on linear systems

Example: Electric circuits

Voltage drop across a resistor is given by Ohm's law $V = RI$

Kirchhoff's Voltage Law: *The algebraic sum of the IR voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.*

$$\sum_{i=1}^{N} R_i I_i = \sum_{i=1}^{M} V_i$$

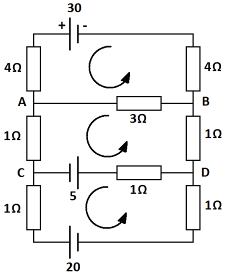


Example: Electric circuits

Loop 1: $4I_1 + 3I_1 - 3I_2 + 4I_1 = 30$

Loop 2: $-3I_1 + 3I_2 + I_2 + I_2 - I_3 + I_2 = 5$

Loop 3: $-I_2 + I_3 + I_3 + I_3 = -5 - 20$



$$\begin{cases} 11I_1 - 3I_2 = 30 \\ -3I_1 + 6I_2 - I_3 = 5 \\ -I_2 + 3I_3 = -25 \end{cases}$$

The loop currents are $I_1 = 3\,\mathrm{A}$, $I_2 = 1\,\mathrm{A}$ and $I_3 = -8\,\mathrm{A}$

- The total current in the branch AB is $I_1 - I_2 = 3 - 1 = 2\,\mathrm{A}$
- The current in branch CD is $I_2 - I_3 = 9\,\mathrm{A}$

Example: Balancing chemical equations

- Chemical equations describe the quantities of substances consumed and produced by chemical reactions:

$$(x_1)\,\mathrm{C_3H_8} + (x_2)\,\mathrm{O_2} \quad \rightarrow \quad (x_3)\,\mathrm{CO_2} + (x_4)\,\mathrm{H_2O}$$

  Balancing requires finding amounts x_1, x_2, x_3, x_4 such that the total amounts of carbon C, hydrogen H, and oxygen O atoms on the left match the corresponding numbers on the right.

$$\mathrm{C_3H_8} : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} \quad \mathrm{O_2} : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{CO_2} : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{H_2O} : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  The solution of this system is $x_1 = 1$, $x_2 = 5$, $x_3 = 3$, $x_4 = 4$.

Linear equations: graphical representation

Consider a system of linear equations

$$\begin{cases} x_1 & - & 2x_2 & = & -1, \\ -x_1 & + & 3x_2 & = & 3. \end{cases}$$

It has a unique solution $x_1 = 3$ and $x_2 = 2$.

This may be represented graphically; in "Mathematica" type

```
ContourPlot[{x1-2x2==-1,-x1+3x2==3},{x1,0,6},{x2,0,4},Axes->True]
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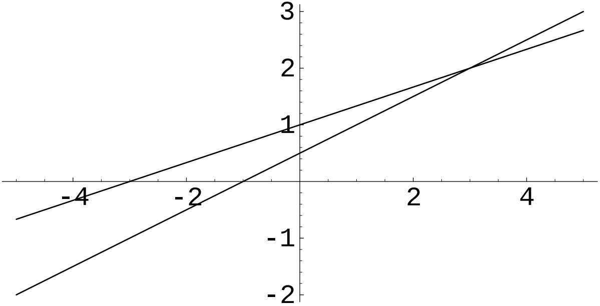
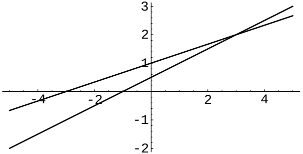


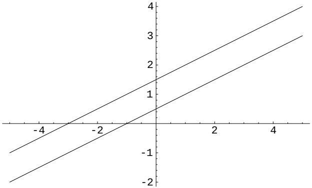
System 1: Unique solution

$$\left\{\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1, \\ -x_1 & + & 3x_2 & = & 3. \end{array}\right.$$



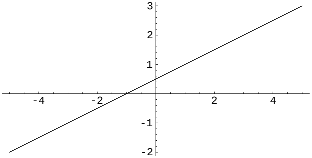
System 2: No solutions

$$\left\{\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1, \\ -x_1 & + & 2x_2 & = & 3. \end{array}\right.$$



System 3: Infinitely many solutions

$$\left\{\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1, \\ -x_1 & + & 2x_2 & = & 1. \end{array}\right.$$



Linear systems of equations

- A linear system may have
  - exactly one solution
  - no solutions
  - infinitely many solutions
- For 2 equations in 2 unknowns, easy to assess and visualise. Much harder or impossible in higher dimensions.
- We need a general tool to understand whether a system has a solution, and if so, whether the solution is unique.

Gaussian reduction / elimination

In the process of Gauss–Jordan elimination,
an augmented matrix is reduced to a diagonal form.

Row operations that can be used:

- Adding a multiple of one row to another
- Multiplying a row by a constant
- Swapping two rows

Row reduction and echelon form

- The first non-zero element in a row is called the **leading element** of the row
- The reduction of a matrix to **echelon form (EF)** occurs via a sequence of row operations
- The matrix in **echelon form** has the following properties:
  - All non-zero rows are above any rows of all zeroes.
  - Each **leading entry** of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zeroes.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(where $\blacksquare$ is a non-zero number, and $*$ is any number)

Reduced echelon form (REF)

- The next step is to obtain a reduced echelon form (REF):

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- **REF** matrix, in addition to **EF** form, has the properties
  - The **leading entry** in each non-zero row is 1
  - The leading 1 is the only non-zero entry in its column

($\blacksquare$ is a non-zero number, $*$ is any number)

Matrices in EF and REF forms

- Scheme of EF form ($\blacksquare$ is non-zero number, $*$ is any number)

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

- Scheme of REF form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Matrices in EF and REF forms

- The echelon form of a matrix (**EF**) is not unique, however reduced echelon form **REF** is unique.
- **Theorem**: Each matrix is row-equivalent to one and only one matrix in **reduced echelon form REF**.
- A **pivot position** corresponds to leading 1 in REF. A **pivot column** is a column that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduction to EF and REF forms

- Finding solutions of a linear system using Gaussian reduction:

$$\begin{cases} \phantom{3x_1 -{}} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

- Write the augmented matrix of this system:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- Follow a step-wise procedure, using elementary row operations
  - Adding a multiple of one row to another
  - Multiplying a row by a constant
  - Swapping two rows

Row reduction to EF and REF forms

- **Step 1**: Begin with the leftmost non-zero column.

$$\begin{bmatrix} \color{red}{0} & 3 & -6 & 6 & 4 & -5 \\ \color{red}{3} & -7 & 8 & -5 & 8 & 9 \\ \color{red}{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Step 2**: Select nonzero entry in the pivot column as a pivot. If necessary, interchange rows.

$$\begin{bmatrix} \color{red}{3} & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **Step 3**: Use row replacement operation to create zeros in all positions below the pivot (here, use $R_2 \to R_2 - R_1$).

$$\begin{bmatrix} \color{red}{3} & -9 & 12 & -9 & 6 & 15 \\ \color{blue}{0} & 2 & -4 & 4 & 2 & -6 \\ \color{blue}{0} & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The EF form ("forward phase")

- **Step 4**. Cover the row containing the pivot (and any rows above it). Apply all steps 1-3 to the remaining sub-matrix. Repeat until there are no more non-zero rows to modify.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we divide the second row by 2)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we use $R_3 \to R_3 - 3R_2$)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The REF form (“backward phase”)

- $$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- **Step 5**. Beginning with the rightmost pivot and working upward and to the left, create zeroes above each pivot. If pivot is not 1, make it 1 by a scaling operations.
  Row operation $R_1 \to R_1/3$ leads to

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

then, $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - 2R_3$ lead to

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The REF form ("backward phase")

- $$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

  now, $R_1 \to R_1 + 3R_2$. This finally brings us to the REF form

  $$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- Note there are 3 equations for 5 variables.
- Variables with pivots are called **basic variables**: x_1, x_2, x_5.
- Variables without pivots are called **free variables**: x_3, x_4.
- In the final solution basic variables x_1, x_2, x_5 must be expressed in terms of free variables x_2, x_4.

- $$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
- The solution is

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_5 &= 4 \end{aligned}$$

x_1, x_2, x_5 (basic) are expressed in terms of x_3, x_4 (free).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2x_3 - 3x_4 \\ -7 + 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Gaussian reduction / elimination

Further examples

First example to solve:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

To bring this to EF we eliminate x_1 in equation 3:

$$\text{Eq3} + 4 * \text{Eq1} \to \text{Eq3} \qquad \text{or} \qquad \text{R3} + 4 * \text{R1} \to \text{R3}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, we eliminate x_2 in equation 3. But first, factor R2.

$$\text{Eq2} \to \frac{1}{2} \times \text{Eq2} \qquad \text{or} \qquad \text{R2} \to \frac{1}{2} \times \text{R2}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, $\quad \text{Eq3} + 3 * \text{Eq2} \to \text{Eq3} \qquad \text{or} \qquad \text{R3} + 3 * \text{R2} \to \text{R3}$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix of the system is now in an echelon form.

Now we continue towards REF, or we can also solve it directly:

Solving directly:

$$\begin{array}{lcl} \text{Eq3} & \Rightarrow & x_3 = 3 \\ \text{Eq2} & \Rightarrow & x_2 = 4x_3 + 4 = 4 \times 3 + 4 = 16 \\ \text{Eq1} & \Rightarrow & x_1 = 2x_2 - x_3 + 0 = 2 \times 16 - 3 = 29 \end{array}$$

This is called **backward substitution**.

- Continue with the elimination into REF

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Elimination above the pivots:

$$\mathrm{Eq1} + 2 * \mathrm{Eq2} \to \mathrm{Eq1} \qquad \text{or} \qquad \mathrm{R1} + 2 * \mathrm{R2} \to \mathrm{R1}$$

$$\begin{array}{rcrcrcr} x_1 & & & - & 7x_3 & = & 8 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- $$\begin{array}{rcrcl} x_1 & & - & 7x_3 & = & 8 \\ & x_2 & - & 4x_3 & = & 4 \\ & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- $$\text{Eq1} + 7 * \text{Eq3} \to \text{Eq1} \qquad \text{or} \qquad \text{R1} + 7 * \text{R2} \to \text{R1}$$

  $$\text{Eq2} + 4 * \text{Eq3} \to \text{Eq2} \qquad \text{or} \qquad \text{R2} + 4 * \text{R3} \to \text{R2}$$

- $$\begin{array}{ccccc} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Gaussian reduction / elimination

- Second example

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Upon doing $R_1 \leftrightarrow R_2$ and $R_3 \to (2R_3 - 5R_2) + R_1$ we get

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The **inconsistency** $0 = 5$ implies that this system does not have any solutions.

Gaussian reduction / elimination

- Third example

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 6 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & -1/2 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix}$$

Doing again $R_1 \leftrightarrow R_2$ and $R_3 \to (2R_3 - 5R_2) + R_1$ yields

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 6 \\ & & & & 0 & = & 0 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 equations in 3 unknowns $\longrightarrow$ 2 equations in 3 unknowns
$\Rightarrow$ only two independent equations

No contradiction, but no unique solution (infinitely many)

- Case 1: Consistent system, unique solution:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

  Case 2: Inconsistent system, no solution:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

  Case 3: Consistent system with infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- 1. System of equations is **consistent** if the solution is unique or there are infinitely many solutions.
  2. System of equations is **inconsistent** if it has no solutions.

Revision: Matrices. Determinants.

Determinants

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $\det \mathbf{A} = ad - bc$ is the determinant of the 2×2 matrix.

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties (these properties hold for any $n \times n$) matrix

- $$\begin{vmatrix} a & b \\ c+e & d+f \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ e & f \end{vmatrix}$$
- $$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
- $$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinants

- Determinant of an identity (unitary) matrix is 1

$$\det \mathbf{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

- If two rows of $\mathbf{A}$ are same, then $\det \mathbf{A} = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

- The elementary row operation of subtraction a multiple of one row from another row leaves the determinant unchanged

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c \pm ka & d \pm kb \end{vmatrix}$$

- Determinant with a zero row is zero

$$\begin{vmatrix} 0 & 0 \\ d & c \end{vmatrix} = 0$$

Determinants

- **Definition**: Let $\mathbf{A}$ be $n \times n$ matrix and a_{ij} an element of $\mathbf{A}$. The **cofactor** of a_{ij} denoted by A_{ij}, is the $(n-1) \times (n-1)$ determinant obtained by
  1. deleting the i-th row and j-th column of $\mathbf{A}$ and
  2. multiplying the resulting matrix determinant by $(-1)^{(i+j)}$.
- The determinant obtained by deleting the i-th row and j-th column of $\mathbf{A}$ is called a minor of a_{ij}.
- For a 3×3 matrix: $\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants

- Let $\mathbf{A}$ be $n \times n$ matrix. Then the determinant of $\mathbf{A}$ is the number $a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$.
- Expansion along i-th row is $\det \mathbf{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$
- For a 3×3 matrix expansion along the second row reads $\det \mathbf{A} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.
- So

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$-a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Inverse of a matrix

- **Definition**: An $n \times n$ matrix $\mathbf{A}$ is said to be invertible if there is an $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{CA} = \mathbf{I} \quad \text{and} \quad \mathbf{AC} = \mathbf{I},$$

where $\mathbf{I}$ is a unitary $n \times n$ matrix. Then $\mathbf{C}$ is an inverse of $\mathbf{A}$.

- $\mathbf{C}$ is uniquely determined by $\mathbf{A}$.
Indeed, suppose $\mathbf{B}$ is another inverse of $\mathbf{C}$. Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

This unique inverse $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$, so that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \text{and} \quad \mathbf{AA}^{-1} = \mathbf{I}.$$

- A non-invertible matrix is called a **singular** matrix.
An invertible matrix is called a **non-singular** matrix.

- Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Then

$$\mathbf{AC} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{CA} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Theorem:**

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \det \mathbf{A} = ad - bc \neq 0$$

$$\text{then } \mathbf{A} \text{ is invertible and} \quad \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a matrix

- If $\det \mathbf{A} = ad - bc = 0$, then $\mathbf{A}$ is not invertible.
- The matrix is invertible if and only if $\det \mathbf{A} \neq 0$.
- **Theorem:**
  If $\mathbf{A}$ is an invertible $n \times n$ matrix then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
- **Theorem**:

  a) If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible, and $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.

  b) If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices, then so is $\mathbf{AB}$, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Inverse of a matrix

- $$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n. \end{cases}$$

  The system can be written in the matrix form

  $$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

  In short $\mathbf{Ax} = \mathbf{b}$

  $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Inverse of a matrix

- So the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is (given $\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x}$)

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}, \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

- One of the way to find $\mathbf{A}^{-1}$ is using the adjoint matrix $\mathbf{A}^{adj}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}(\mathbf{A}^{c})^{T} = \frac{1}{\det \mathbf{A}}\mathbf{A}^{adj}.$$

$$\mathbf{A}^{c} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

$\mathbf{A}^{-1}$ by Gaussian elimination

- **Theorem:** A $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ is row-equivalent to identity matrix $\mathbf{I}$, and in this case any sequence of elementary row operations that reduces $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ into $\mathbf{A}^{-1}$.
- This gives an algorithm of finding $\mathbf{A}^{-1}$.
- Row reduce the augmented matrix $[\mathbf{A}\ \mathbf{I}]$. If $\mathbf{A}$ is row equivalent to $\mathbf{I}$, then $[\mathbf{A}\ \mathbf{I}]$ is row equivalent to $[\mathbf{I}\ \mathbf{A}^{-1}]$. Otherwise $\mathbf{A}$ does not have an inverse.
- In practice $\mathbf{A}^{-1}$ is seldom computed directly ($2N^3$ operations). Row reduction is faster and often more accurate.
- Example: Find inverse of a matrix, if it exists

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

- We form the extended augmented matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 3R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

- $R_3 \to R_3/2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

- So $\mathbf{A} \sim \mathbf{I}$, and $\mathbf{A}$ is invertible, $\mathbf{A}^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

. . .

Homogeneous versus inhomogeneous linear systems

- A linear system can be written as $\mathbf{Ax} = \mathbf{b}$.
- If $\mathbf{b} \neq \mathbf{0}$, the system is called **inhomogeneous**
- If $\mathbf{b} = \mathbf{0}$, the system is called **homogeneous** $(\mathbf{Ax} = 0)$
- For the inhomogeneous system from the previous example the explicit linear system is

$$\left\{ \begin{array}{cccccccc} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array} \right.$$

- The corresponding homogeneous system reads

$$\left\{ \begin{array}{cccccccc} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & 0 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 0 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 0 \end{array} \right.$$

Homogeneous linear systems

- For the homogeneous linear system

$$\begin{cases} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = 0 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 0 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 0 \end{cases}$$

the augmented matrix and its REF are as follows

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 0 \\ 3 & -7 & 8 & -5 & 8 & 0 \\ 3 & -9 & 12 & -9 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- So the solution is

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_5 &= 0 \end{aligned}$$

Basic variables x_1, x_2, x_5 are expressed in terms of free x_3, x_4.

Inhomogeneous linear systems

- **Theorem**: Suppose the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set is, all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}$ where $\mathbf{v}$ is any solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.
- In our examples, a particular solution of the inhomogeneous system $\mathbf{p}$ and the solution set of the homogeneous system $\mathbf{v}$ were

$$\mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

so the solution set of the inhomogeneous system is indeed

$$\mathbf{w} = \mathbf{p} + \mathbf{v}$$

See you next Friday

C B 06.03.022