The Spanning Set Theorem

The Spanning Set Theorem

A basis is an "efficient" spanning set that contains no unnecessary vectors.
A basis can be constructed from a spanning set by discarding unnecessary vectors.
Example 3: Let

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
Show that

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$$

then find a basis for the subspace H. (Note: H is a subspace because it is given by $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.)

Solution:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

Every vector in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3.$$

Now let $\mathbf{x}$ be any vector in H
$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.
Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, therefore

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2. \end{aligned}$$

Thus $\mathbf{x}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ and H is identical to $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.
Note that $\mathbf{v}_1, \mathbf{v}_2$ are independent because they are not multiples:
$\mathbf{v}_1 \neq c\mathbf{v}_2$.

The Spanning Set Theorem

Spanning Set Theorem:
Let $\mathbb{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(a) If one of the vectors in $\mathbb{S}$, say $\mathbf{v}_i$, is a linear combination of the remaining vectors in $\mathbb{S}$, then the set formed from $\mathbb{S}$ by removing $\mathbf{v}_i$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of $\mathbb{S}$ is a basis for H.

Proof:
a) We may suppose that $\mathbf{v}_p$ is a linear combination of
$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}$, so
$\mathbf{v}_p = a_1\mathbf{v}_1 + \ldots + a_{p-1}\mathbf{v}_{p-1}$.
Given any $\mathbf{x}$ in H we may write

$$\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p.$$

By substituting the expression for $\mathbf{v}_p$ into this relation we can express $\mathbf{x}$ as a linear combination of only $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans H, because $\mathbf{x}$ was an arbitrary element of H.

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Proof (cont.):

- b) If the original spanning set $\mathbb{S}$ is linearly independent, then it is already a basis for H.
- If this is not the case then one of the vectors can be dropped from the set given part (a).
- We can continue this "dropping" process until the remaining set is linearly independent and hence is a basis for H.
- If the spanning set is eventually reduced to one vector, that vector will be nonzero because $H \neq \{\mathbf{0}\}$. Since a single nonzero vector $\mathbf{v}$ is independent, it will be a basis for H.

Two Views of a Basis

- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent, since a smaller set will no longer span V.
- Thus a basis is a spanning set which is as small as possible.
- A basis is also a linearly independent set that is as large as possible.
- If S is a basis for V and if S is enlarged by one vector $\mathbf{w}$ from V then the new set cannot be linearly independent, because S spans V and $\mathbf{w}$ is a linear combination of vectors from S.

Two Views of a Basis

Example 4: A linearly independent set can be enlarged to form a basis — but further enlargement destroys the linear independence. A spanning set can be shrunk to a basis — further shrinking destroys the spanning property.

A linearly independent set which does **not** span $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\},$$

A basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

Spans $\mathbb{R}^3$ but is linearly dependent:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}.$$