Assignment 2 - Linear Algebra, Autumn 2019

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Question 1

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 9 & 1 & -3 & 4 \\ -6 & 0 & -2 & 2 \\ 0 & 2 & 4 & 8 \end{bmatrix} \tag{1}$$

For LU-decomposition with Doolittle's method:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \tag{2}$$

Hence:

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{9}{3} \tag{3}$$

$$u_{22} = a_{22} - l_{21}u_{12} = (1) - (3)(0) = 1 \tag{4}$$

$$u_{23} = a_{23} - l_{21}u_{13} = (-3) - (3)(-1) = 0 \tag{5}$$

$$u_{24} = a_{24} - l_{21}u_{14} = (4) - (3)(0) = 4 \tag{6}$$

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{-6}{3} = -2 \tag{7}$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{0 - (-2)(0)}{1} = 0 \tag{8}$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = (-2) - (-2)(-1) - (0)(0) = -4 \tag{9}$$

$$u_{34} = a_{34} - l_{31}u_{14} - l_{32}u_{24} = (2) - (-2)(0) - (0)(4) = 2 \tag{10}$$

$$l_{41} = \frac{a_{41}}{u_{11}} = \frac{0}{3} = 0 \tag{11}$$

$$l_{42} = \frac{a_{42} - l_{41}u_{12}}{u_{22}} = \frac{(2) - (0)(0)}{(1)} = 2 \tag{12}$$

$$l_{43} = \frac{a_{43} - l_{41}u_{13} - l_{42}u_{23}}{u_{33}} = \frac{(4) - (0)(-5) - (0)(0)}{-4} = -1 \tag{13}$$

$$a_{44} - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34} = (8) - (0)(0) - (2)(4) - (2)\left(\frac{1}{2}\right) = 8 - 8 + 2 = 2 \tag{14}$$

Therefore:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{15}$$

Question 2

Some matrix algebra indicates to us how to solve the system, given our matrices obtained during decomposition.

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{LUx} = \mathbf{b}$$

$$\mathbf{L^{-1}LUx} = \mathbf{L^{-1}b}$$

$$\mathbf{U^{-1}Ux} = \mathbf{U^{-1}L^{-1}b}$$

$$\mathbf{x} = \mathbf{U^{-1}L^{-1}b}$$

So, we must find $\mathbf{U^{-1}}$ and $\mathbf{L^{-1}}$, which is easily computable with python's sympy package.

```
>>> from sympy import *
>>> L = Matrix([[1,0,0,0],[3,1,0,0],[-2,0,1,0],[0,2,-1,1]])
>>> U = Matrix([[3,0,-1,0],[0,1,0,4],[0,0,-4,2],[0,0,0,2]])
>>> b = Matrix([0,18,-4,48])
>>> L.inv()
Matrix([
[ 1,  0, 0, 0],
[-3,  1, 0, 0],
[ 2,  0, 1, 0],
[ 8, -2, 1, 1]])
>>> U.inv()
Matrix([
[1/3, 0, -1/12, 1/12],
[  0, 1,     0,   -2],
[  0, 0,  -1/4,  1/4],
[  0, 0,     0,  1/2]])
>>> b = Matrix([0,18,-4,48])
>>> U.inv() * L.inv() * b
Matrix([
[1],
[2],
[3],
[4]])
```

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We get:

$$\mathbf{L^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 8 & -2 & 1 & 1 \end{bmatrix}$$

$$\mathbf{U^{-1}} = \begin{bmatrix} 1/3 & 0 & -1/12 & 1/12 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

And recalling that:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 18 \\ -4 \\ 48 \end{bmatrix}$$

Hence, as the code prior showed:

$$\mathbf{U^{-1}L^{-1}b} = \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Question 3

Part (a)

Using Crout's method:

$$l_{11} = a_{11} = -1$$

$$l_{21} = a_{21} = 3$$

$$l_{31} = a_{31} = 2$$

$$u_{12} = \frac{a_{12}}{l_{11}} = 1/(-1) = -1$$

$$u_{13} = \frac{a_{13}}{l_{11}} = 0/(-1) = 0$$

$$l_{22} = a_{22} - l_{21}u_{12} = (-1) - (3)(-1) = 2$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{(2) - (3)(0)}{2} = -1$$

$$l_{32} = a_{32} - l_{31}u_{12} = (-4) - (2)(1) = -2$$

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = (5) - (2)(0) - (-2)(-1) = 3$$

Hence we have:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (b) on next page.

Part (b)

Again, using python's sympy package:

```
>>> from sympy import *
>>> L = Matrix([[-1,0,0],[3,2,0],[2,-2,3]])
>>> U = Matrix([[1,-1,0],[0,1,-1],[0,0,1]])
>>> L*U
Matrix([
[-1,  1,  0],
[ 3, -1, -2],
[ 2, -4,  5]])
>>> b = Matrix([-2,10,3])
>>> U.inv() * L.inv() * b
Matrix([
[5],
[3],
[1]])
```

Hence:

$$\mathbf{x} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \tag{16}$$

Question 4

Part (a)

Because we have a symmetric matrix, we may be able to use Choleski's method.
Where:

$$\mathbf{A} = \mathbf{A}^T = \mathbf{LU} = (\mathbf{LU})^T \Rightarrow \mathbf{U}^T\mathbf{U}$$

$$u_{11} = \sqrt{a_{11}} = 2$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{6}{2} = 3$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{-2}{2} = -1$$

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{10 - 9} = 1$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{(-1) - (3)(-1)}{1} = 2$$

$$u_{33} = \sqrt{a_{33} - -u_{12}^2 - u_{23}^2} = \sqrt{(14) - (-2)^2 - (-1)^2} = \sqrt{14 - 5} = 3$$

Hence:

$$\mathbf{U} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{U}^T = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

Part (b) on next page.

Part (b)

$$\mathbf{A} = \mathbf{A} = \mathbf{A}^T = \mathbf{L}\mathbf{U} = (\mathbf{L}\mathbf{U})^T = \mathbf{U}^T\mathbf{L}^T \Rightarrow \mathbf{U}^T\mathbf{U}$$

And recalling that:

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix}$$

We can solve the system with python's sympy package:

```
>>> from sympy import *
>>> U = Matrix([[2,3,-1],[0,1,2],[0,0,3]])
>>> b = Matrix([2,3,-10])
>>> U.T * U       # Check that it is in fact equal to A
Matrix([
[ 4,  6, -2],
[ 6, 10, -1],
[-2, -1, 14]])
>>> U.inv() * U.T.inv() * b
Matrix([
[-3],
[ 2],
[-1]])
```

Hence:

$$\mathbf{x} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

Question 5

For the first two matrices (Q1 and Q3) you can test for the existence of the LU-decompositions with a sufficient condition of strict diagonal dominance.

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| \tag{17}$$

If the matrix is diagonally dominant, then it is non-singular and can also be decomposed into a lower-triangular $\mathbf{L}$ and an upper-triangular $\mathbf{U}$ that satisfy: $\mathbf{A} = \mathbf{LU}$

For question 4, because we are using the Cholesky method, for it to *definitely* work we need it to be a positive definite matrix. This can be satisfied with two steps:

1. If the matrix is diagonally dominant, and
2. If the diagonal elements are all greater than zero.

For our matrices, they all fail their respective tests, i.e. the matrices in Q1, Q3 and Q4 are all not strictly diagonally dominant.

However, because this is a sufficient but not necessary condition, we still succeeded in decomposition.