University of Technology Sydney
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37233 Linear Algebra
Problem Set 8

Note: you may use *Mathematica* to carry out any calculations you feel may be of use.

Question 1.

Let

$$\mathbf{u} = \begin{pmatrix} 3 \\ 2 \\ -5 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -4 \\ -1 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ 7 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 1 \\ -8 \\ 15 \\ -7 \end{pmatrix}.$$

Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}\}$ orthogonal? What are the lengths of vectors $\mathbf{u}$ and $\mathbf{v}$?

Question 2.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ and express $\mathbf{x}$ as a linear combination of $\mathbf{u}$'s, where

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Question 3.

Construct a matrix $\mathbf{U}$ with orthonormal vectors from the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

and calculate $U^T U$ and UU^T.

Question 4.

Let W be the subspace spanned by the $\mathbf{u}$'s. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$$

Question 5.

Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix}.$$

Question 6.

Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where

$$\mathbf{z} = \begin{pmatrix} 3 \\ -7 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Question 7.

Use the Gram-Schmidt process to produce an orthogonal basis for the space W spanned by vectors

$$\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}.$$

Construct an orthonormal basis for W.

Question 8.

Let $\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(a) Apply the Gram–Schmidt process to $\mathcal{V}$ to construct an orthonormal basis for $\mathbb{R}^3$.

(b) Let V be the matrix whose columns are the elements of the basis $\mathcal{V}$. Construct a $\mathbf{QR}$ factorisation of V.

Question 9.

Find:

(a) an orthogonal basis for the column space;

(b) a QR factorization

of the matrix A:

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}.$$