

Question:

When $t = 0$ a switch closes connecting a voltage source to a series RL circuit. The source is characterised by: $v(t) = V_m \sin(\omega t + \phi)$. Find the time domain expression of the current, $i(t)$.

$$I(s) = \frac{V(s)}{Z(s)} \quad (1)$$

Finding $V(s)$

$$\mathcal{L}\{v(t)\} = \mathcal{L}\{V_m \sin(\omega t + \phi)\} = V_m \mathcal{L}\{\sin(\omega t + \phi)\} \quad (2)$$

$$V_m \mathcal{L}\{\sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi)\} \quad (3)$$

$$V_m \left(\mathcal{L}\{\sin(\omega t) \cos(\phi)\} + \mathcal{L}\{\cos(\omega t) \sin(\phi)\} \right) \quad (4)$$

Note that $\cos(\phi)$ and $\sin(\phi)$ are constants with respect to time can subsequently be removed from the argument of the Laplace transform.

$$V_m \left(\cos(\phi) \mathcal{L}\{\sin(\omega t)\} + \sin(\phi) \mathcal{L}\{\cos(\omega t)\} \right) \quad (5)$$

$$\therefore V(s) = V_m \left(\cos(\phi) \frac{\omega}{s^2 + \omega^2} + \sin(\phi) \frac{s}{s^2 + \omega^2} \right) \quad (6)$$

Finding $Z(s)$ by simple s-domain series impedance:

$$Z(s) = sL + R \Rightarrow \frac{1}{Z(s)} = \frac{1}{sL + R} = \frac{\frac{1}{L}}{s + \frac{R}{L}} \quad (7)$$

Then to current, $I(s)$

$$I(s) = \frac{V(s)}{Z(s)} = V_m \left(\cos(\phi) \frac{\omega}{s^2 + \omega^2} + \sin(\phi) \frac{s}{s^2 + \omega^2} \right) \frac{\frac{1}{L}}{s + \frac{R}{L}} \quad (8)$$

$$= V_m \left(\frac{\cos(\phi)\omega + \sin(\phi)s}{s^2 + \omega^2} \right) \frac{\frac{1}{L}}{s + \frac{R}{L}} \quad (9)$$

$$= \frac{V_m}{L} \left(\frac{\cos(\phi)\omega + \sin(\phi)s}{(s^2 + \omega^2)(s + \frac{R}{L})} \right) \quad (10)$$

We want to decompose the bottom into separate individual poles so we can use the residue theorem. It's helpful to decompose $s^2 + \omega^2$ into $(s + j\omega)(s - j\omega)$.

$$= \left(\frac{V_m}{L} \right) \frac{(\cos(\phi)\omega + \sin(\phi)s)}{(s + j\omega)(s - j\omega)(s + \frac{R}{L})} \quad (11)$$

Recall the residue theorem:

$$f(t) = 2\pi i \sum_{k=1}^m \frac{1}{(n-1)!} \lim_{s \rightarrow s_k} \frac{d^{n-1}}{ds^{n-1}} (s - s_k)^n F(s) \quad (12)$$

s.t.

m = number of poles

n = order of the pole

s_k = the k -th pole in $F(s)$

Using the residue theorem to solve inverse Laplace transforms:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \sum_{k=1}^m \text{Res}(e^{st} F(s), s_k) = \sum_{k=1}^m \frac{1}{(n-1)!} \lim_{s \rightarrow s_k} \frac{d^{n-1}}{ds^{n-1}} (s - s_k)^n e^{st} F(s) \quad (13)$$

Looking at the denominator of the frequency domain function of current, we can see the set of poles,

$$\{s\} = \left\{ -j\omega, j\omega, -\frac{R}{L} \right\}$$

Now we have to insert the poles and evaluate Eq.?? but we can first do some simplification. For our poles, they are all of order 1.

$$\sum_{k=1}^m \frac{1}{(n-1)!} \lim_{s \rightarrow s_k} \frac{d^{n-1}}{ds^{n-1}} (s - s_k)^n e^{st} F(s) \Big|_{n=1} = \sum_{k=1}^m \lim_{s \rightarrow s_k} (s - s_k)^n e^{st} F(s) \quad (14)$$

$$\sum_{k=1}^m \lim_{s \rightarrow s_k} (s - s_k) e^{st} F(s) = \lim_{s \rightarrow j\omega} (s - s_k) e^{st} F(s) + \lim_{s \rightarrow -j\omega} (s - s_k) e^{st} F(s) + \lim_{s \rightarrow \alpha} (s - s_k) e^{st} F(s) \quad (15)$$

$$\sum_{k=1}^m \lim_{s \rightarrow s_k} (s - s_k) e^{st} F(s) = (s - j\omega) e^{j\omega t} F(j\omega) + (s + j\omega) e^{-j\omega t} F(-j\omega) + (s - \alpha) e^{\alpha t} F(\alpha) \quad (16)$$

$$i++i$$