Question:
When $t = 0$ a switch closes connecting a voltage source to a series RL circuit. The source is characterised by: $v(t) = V_m \sin(\omega t + \phi)$. Find the time domain expression of the current, $i(t)$.

$$I(s) = \frac{V(s)}{Z(s)} \tag{1}$$

Finding $V(s)$

$$\mathcal{L}\Big\{v(t)\Big\} = \mathcal{L}\Big\{V_m \sin(\omega t + \phi)\Big\} \; = V_m\mathcal{L}\Big\{\sin(\omega t + \phi)\Big\} \tag{2}$$

$$V_m \; \mathcal{L}\Big\{\sin(\omega t)\cos(\phi) + \cos(wt)\sin(\phi)\Big\} \tag{3}$$

$$V_m\Big(\mathcal{L}\Big\{\sin(\omega t)\cos(\phi)\Big\} + \mathcal{L}\Big\{\cos(wt)\sin(\phi)\Big\}\Big) \tag{4}$$

Note that $\cos(\phi)$ and $\sin(\phi)$ are constants with respect to time can subsequently be removed from the argument of the Laplace transform.

$$V_m\Big(\cos(\phi)\; \mathcal{L}\Big\{\sin(\omega t)\Big\} + \sin(\phi)\; \mathcal{L}\Big\{\cos(wt)\Big\}\Big) \tag{5}$$

$$\therefore\; V(s) = V_m\left(\cos(\phi)\;\frac{\omega}{s^2+\omega^2} + \sin(\phi)\;\frac{s}{s^2+\omega^2}\right) \tag{6}$$

Finding Z(s) by simple s-domain series impedance:

$$Z(s) = sL + R \quad \Rightarrow \quad \frac{1}{Z(s)} = \frac{1}{sL+R} = \frac{\frac{1}{L}}{s+\frac{R}{L}} \tag{7}$$

Then to current, $I(s)$

$$I(s) = \frac{V(s)}{Z(s)} = V_m\left(\cos(\phi)\;\frac{\omega}{s^2+\omega^2} + \sin(\phi)\;\frac{s}{s^2+\omega^2}\right)\frac{\frac{1}{L}}{s+\frac{R}{L}} \tag{8}$$

$$= V_m\left(\frac{\cos(\phi)\omega + \sin(\phi)s}{s^2+\omega^2}\right)\frac{\frac{1}{L}}{s+\frac{R}{L}} \tag{9}$$

$$= \frac{V_m}{L}\left(\frac{\cos(\phi)\omega + \sin(\phi)s}{(s^2+\omega^2)(s+\frac{R}{L})}\right) \tag{10}$$

We want to decompose the bottom into separate individual poles so we can use the residue theorem. It's helpful to decompose $s^2 + \omega^2$ into $(s + j\omega)(s - j\omega)$.

$$= \left(\frac{V_m}{L}\right)\frac{(\;\cos(\phi)\omega + \sin(\phi)s\;)}{(s+j\omega)(s-jw)(s+\frac{R}{L})} \tag{11}$$

Recall the residue theorem:

$$f(t) = 2\pi i \sum_{k=1}^{m} \frac{1}{(n-1)!} \lim_{s\to s_k} \frac{d^{n-1}}{ds^{n-1}} (s-s_k)^n F(s) \tag{12}$$

s.t.
m = number of poles
n = order of the pole
s_k = the k-th pole in $F(s)$

Using the residue theorem to solve inverse Laplace transforms:

$$\mathcal{L}^{-1}\Big\{F(s)\Big\} = f(t) = \sum_{k=1}^{m} \mathrm{Res}(e^{st}F(s), s_k) = \sum_{k=1}^{m} \frac{1}{(n-1)!} \lim_{s\to s_k} \frac{d^{n-1}}{ds^{n-1}} (s-s_k)^n \; e^{st} F(s) \tag{13}$$

Looking at the denominator of the frequency domain function of current, we can see the set of poles,

$$\{s\} = \Big\{-j\omega,\; j\omega,\; -\frac{R}{L}\Big\}$$

Now we have to insert the poles and evaluate Eq.13 but we can first do some simplification. For our poles, they are all of order 1.

$$\sum_{k=1}^{m} \frac{1}{(n-1)!} \lim_{s\to s_k} \frac{d^{n-1}}{ds^{n-1}} (s-s_k)^n \; e^{st} F(s)\Bigg|_{n=1} = \sum_{k=1}^{m} \lim_{s\to s_k} (s-s_k)^n \; e^{st} F(s) \tag{14}$$

$$\sum_{k=1}^{m} \lim_{s\to s_k} (s-s_k)\, e^{st} F(s) = \lim_{s\to j\omega} (s-s_k)\, e^{st} F(s) + \lim_{s\to -j\omega} (s-s_k)\, e^{st} F(s) + \lim_{s\to \alpha} (s-s_k)\, e^{st} F(s) \tag{15}$$

$$\sum_{k=1}^{m} \lim_{s\to s_k} (s-s_k)\, e^{st} F(s) = (s-j\omega) e^{j\omega t}\; F(j\omega) + (s+j\omega)\; e^{-j\omega t}\; F(-j\omega) + (s-\alpha)\; e^{\alpha t}\; F(\alpha) \tag{16}$$

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