Mahidol University
Faculty of Information and Communication Technology

ITDS 122 Data Structures and Algorithm Analysis Lecture 14: Graph Algorithms

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2/2024

Teaching Lecture Hour : 30 Lab Hour : 30

- Recalling Python Coding
- Object-Oriented Programming with Python
- Algorithm Analysis
- Recursion
- Array-Based Sequences
- Stacks, Queues, Deques
- Linked Lists
- Review & checkpoint [Midterm]
- Trees and Search Trees *
- Maps and Hash Tables
- Graph Algorithms *
- Sorting
- Review & checkpoint [Final]

Graph Algorithms

- Transitive Closure
- Directed Acyclic Graphs (DAGs)
- Shortest Paths
- Minimum Spanning Tree (MST)

Transitive Closure

Introduction to Transitive Closure

- Goal: Determine reachability in a directed graph
- If vertex u can reach vertex v, then v is in the transitive closure of u
- Applications:
  - Routing & network analysis
  - Scheduling systems
  - Analyzing dependencies

Definition of Transitive Closure

- Transitive closure G* of a directed graph G:
  - Same vertex set as G
  - Edge (u, v) exists in G* iff there is a path from u to v in G
- Can include self-loops (path from u to u)

Motivation for Precomputing

- DFS/BFS answers reachability for a single source
- Inefficient for multiple queries
- Precompute transitive closure for constant-time reachability checks
- Efficient for dense graphs or matrix-based representations

Floyd-Warshall Algorithm

- Iteratively add paths by checking intermediate vertices
- Let vertices be v1, v2, …, vn
- For each k, add (i, j) if i → k and k → j exist
- Key idea: path i → k → j implies i → j

Pseudocode

Algorithm FloydWarshall($\vec{G}$):

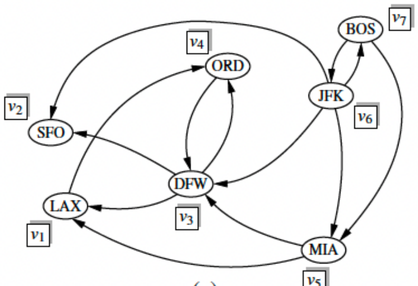
Input: A directed graph $\vec{G}$ with n vertices

Output: The transitive closure $\vec{G}^*$ of $\vec{G}$

let $v_1, v_2, \ldots, v_n$ be an arbitrary numbering of the vertices of $\vec{G}$
$\vec{G}_0 = \vec{G}$
for $k = 1$ to n **do**
    $\vec{G}_k = \vec{G}_{k-1}$
    for all i, j in $\{1, \ldots, n\}$ with $i \neq j$ and $i, j \neq k$ **do**
        if both edges (v_i, v_k) and (v_k, v_j) are in $\vec{G}_{k-1}$ **then**
            add edge (v_i, v_j) to $\vec{G}_k$ (if it is not already present)
return $\vec{G}_n$

Python Implementation

```python
class Graph:
    def __init__(self):
        self.adj = {}

    def add_vertex(self, v):
        if v not in self.adj:
            self.adj[v] = set()

    def add_edge(self, u, v):
        self.add_vertex(u)
        self.add_vertex(v)
        self.adj[u].add(v)

    def vertices(self):
        return list(self.adj.keys())

    def get_edge(self, u, v):
        return v if v in self.adj.get(u, set()) else None

    def insert_edge(self, u, v):
        self.adj[u].add(v)

    def __str__(self):
        return "\n".join(f"{u} -> {sorted(vs)}" for u, vs in self.adj.items())

```



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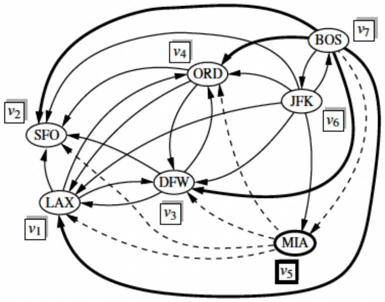

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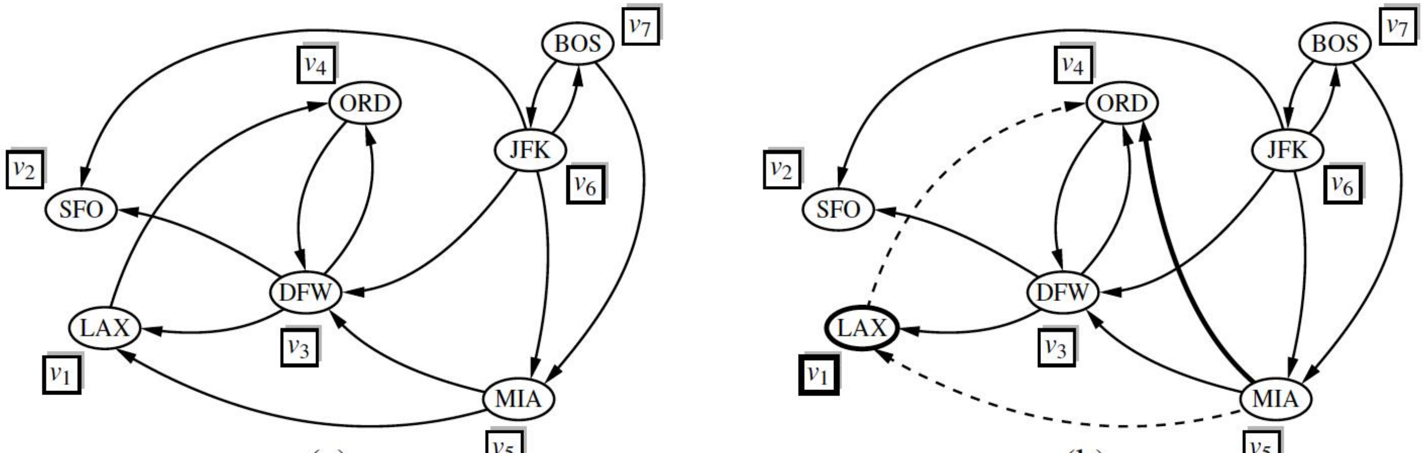
```python
def floyd_warshall(g):
    """Return a new graph that is the transitive closure of g."""
    closure = deepcopy(g)
    verts = list(closure.vertices())
    n = len(verts)

    for k in range(n):
        for i in range(n):
            if i != k and closure.get_edge(verts[i], verts[k]) is not None:
                for j in range(n):
                    if i != j != k and closure.get_edge(verts[k], verts[j]) is not None:
                        if closure.get_edge(verts[i], verts[j]) is None:
                            closure.insert_edge(verts[i], verts[j])

    return closure

```

```python
#
g = Graph()
g.add_edge("LAX", "ORD")
g.add_edge("DFW", "LAX")
g.add_edge("DFW", "SFO")
g.add_edge("DFW", "ORD")
g.add_edge("ORD", "DFW")
g.add_edge("MIA", "LAX")
g.add_edge("MIA", "DFW")
g.add_edge("JFK", "DFW")
g.add_edge("JFK", "MIA")
g.add_edge("JFK", "BOS")
g.add_edge("BOS", "JFK")



if __name__ == "__main__":
    print("Original Graph:")
    print(g)

    closure = floyd_warshall(g)
    print("\nTransitive Closure:")
    print(closure)

```

```
Original Graph:
LAX -> ['ORD']
ORD -> ['DFW']
DFW -> ['LAX', 'ORD', 'SFO']
SFO -> []
MIA -> ['DFW', 'LAX']
JFK -> ['BOS', 'DFW', 'MIA']
BOS -> ['JFK']

Transitive Closure:
LAX -> ['DFW', 'ORD', 'SFO']
ORD -> ['DFW', 'LAX', 'SFO']
DFW -> ['LAX', 'ORD', 'SFO']
SFO -> []
MIA -> ['DFW', 'LAX', 'ORD', 'SFO']
JFK -> ['BOS', 'DFW', 'LAX', 'MIA', 'ORD', 'SFO']
BOS -> ['DFW', 'JFK', 'LAX', 'MIA', 'ORD', 'SFO']
```

Visual Walkthrough



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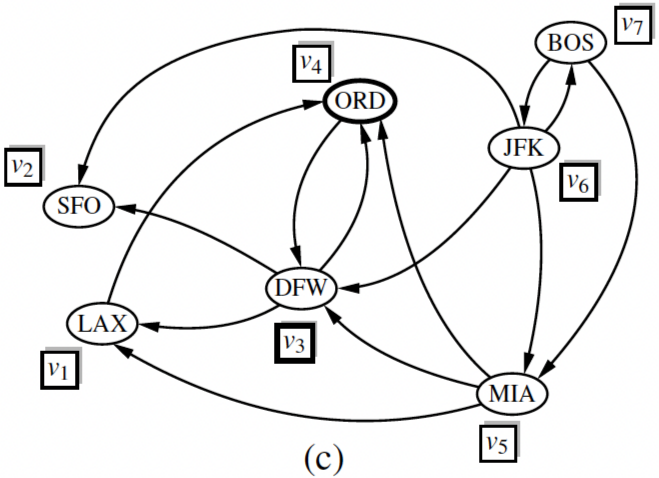
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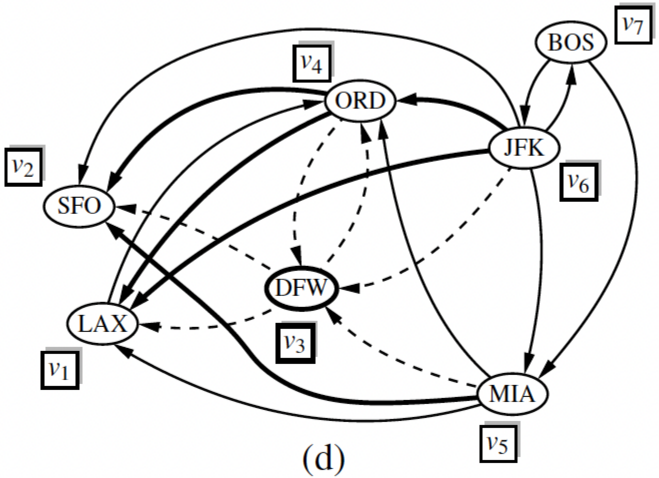
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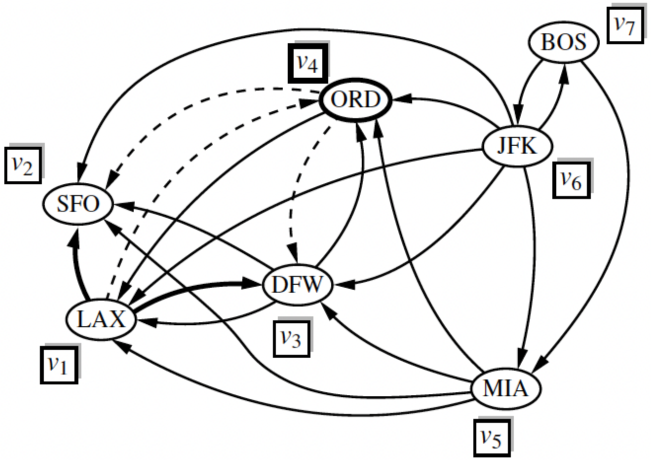
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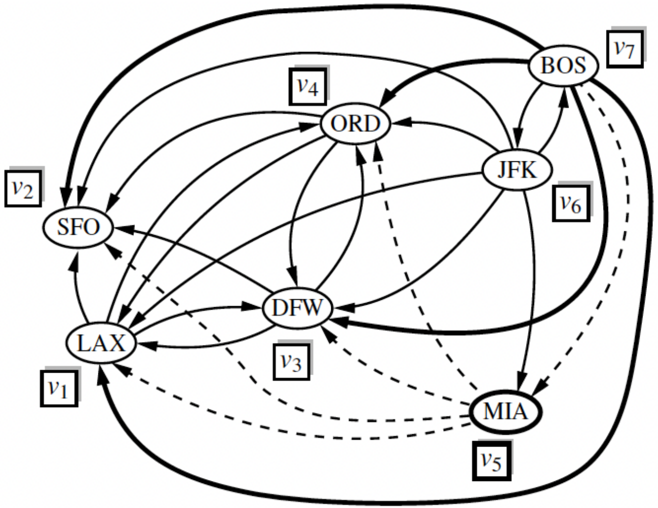
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Summary

- Transitive closure = precomputed reachability
- Floyd-Warshall is conceptually simple and easy to implement
- Use for dense graphs or matrix-based representations
- Alternatives: DFS/BFS for sparse graphs (lower time but repeated calls)

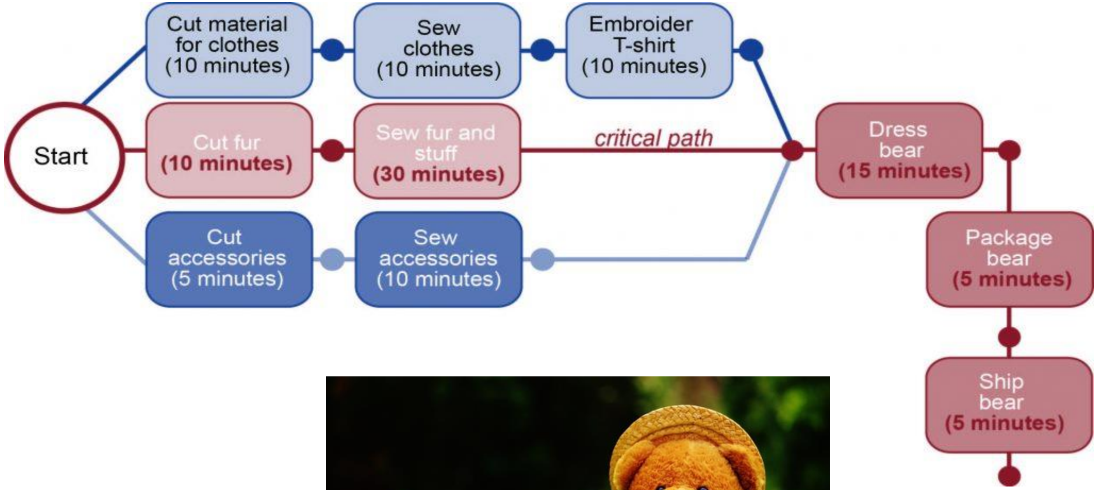
Directed Acyclic Graphs (DAGs)

Introduction to Directed Acyclic Graphs (DAGs)

- A Directed Acyclic Graph (DAG) is a directed graph with no cycles.
- Common applications:
  - Course prerequisites
  - Task scheduling
  - Inheritance hierarchies in OOP
- Important property: DAGs can be topologically ordered.

Motivation Example

- To manage a large project, break it into smaller tasks.
- Some tasks depend on others (e.g., foundation before walls).
- Represent tasks as vertices, dependencies as directed edges.
- If feasible, the graph must be a DAG.

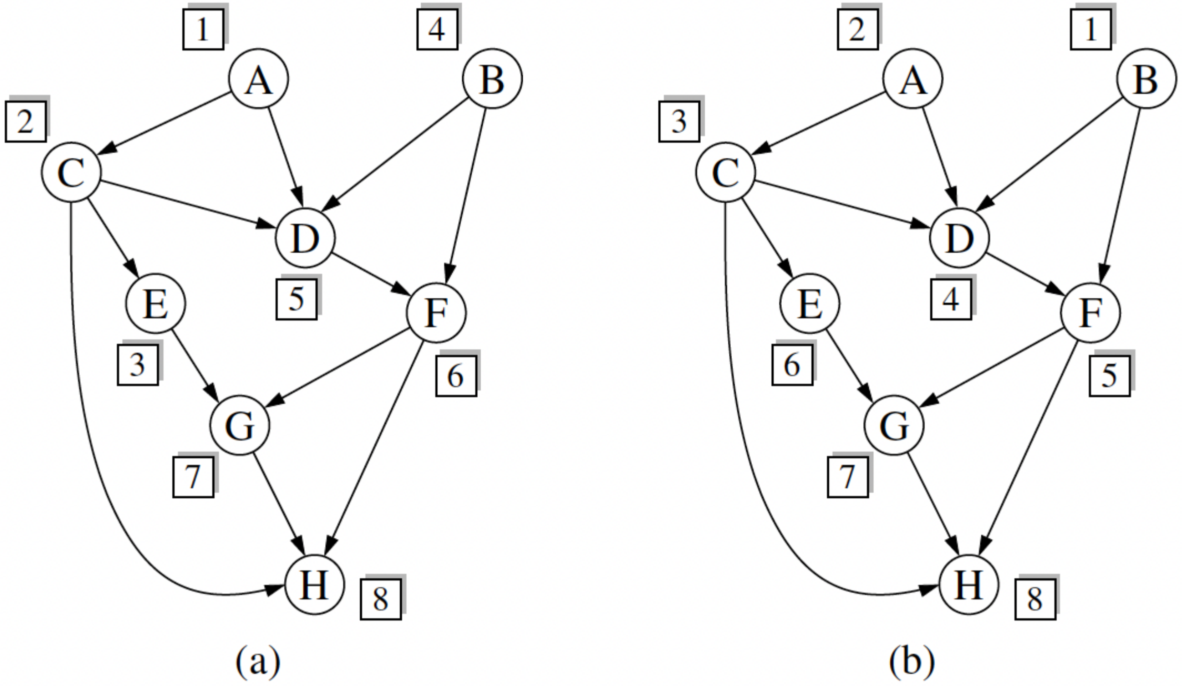


Topological Ordering

- A topological ordering of a DAG:
  - A sequence of vertices such that for every edge (u, v), u comes before v.
- Proposition: A graph has a topological order iff it is acyclic.

Visual Example

- Two possible topological orders for the same DAG:
  - Different valid sequences based on constraint satisfaction.
- Key point: multiple valid topological sorts may exist.



Topological Sorting Algorithm : Kahn's Algorithm

1. Initialize:
   - Count incoming edges (in-degree)
   - Add nodes with 0 in-degree to a ready list
2. While ready is not empty:
   - Remove node u from ready
   - Append u to topo
   - Decrease in-degree of u's neighbors
   - Add new nodes with 0 in-degree to ready

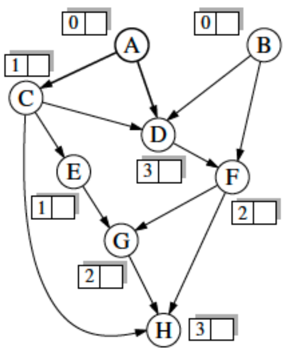
Python Implementation

```python
def topological_sort(g):
    """
    Return a list of vertices of directed acyclic graph g in topological order.
    If graph g has a cycle, the result will be incomplete.
    """
    topo = []                       # list of vertices placed in topological order
    ready = []                      # list of vertices that have no remaining constraints
    incount = {}                    # keep track of in-degree for each vertex

    for u in g:
        incount[u] = 0

    for u in g:
        for v in g[u]:
            incount[v] += 1

    for u in g:
        if incount[u] == 0:
            ready.append(u)

    while ready:
        u = ready.pop()
        topo.append(u)
        for v in g[u]:
            incount[v] -= 1
            if incount[v] == 0:
                ready.append(v)

    return topo
```

```python
graph = {
    'A': ['C','D'],
    'B': ['D','F'],
    'C': ['D', 'E','H'],
    'D': ['E', 'F'],
    'E': ['G'],
    'F': ['G', 'H'],
    'G': ['H'],
    'H': []
}

if __name__ == "__main__":
    ordering = topological_sort(graph)
    print("Topological Ordering:", ordering)
```

Example Walkthrough

- Highlight the step-by-step updates:
  - Track incount changes
  - Highlight which node is selected next
  - Show dashed edges as processed
- Visualize how nodes are added to final order.



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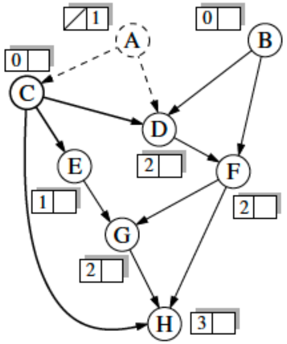
🛠 **Ready List**: [A, B]
📄 **Topological Order**: []



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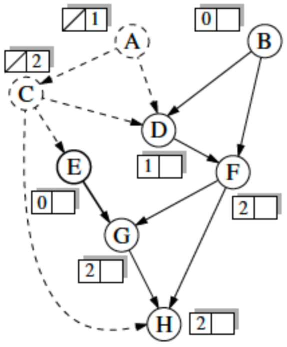
🛠 **Ready List**: [B, C]
📄 **Topological Order**: [A]

✅ **Selected Node:** A (first in the ready list)
📄 **Topological Order:** [A]
🛠 **Ready List:** [B]

📉 **Update in-degrees for neighbors of A:**
- A → C → incount[C] -= 1 → 1 → 0 ✅
- A → D → incount[D] -= 1 → 3 → 2

➕ **Add C to ready** (since incount[C] == 0)



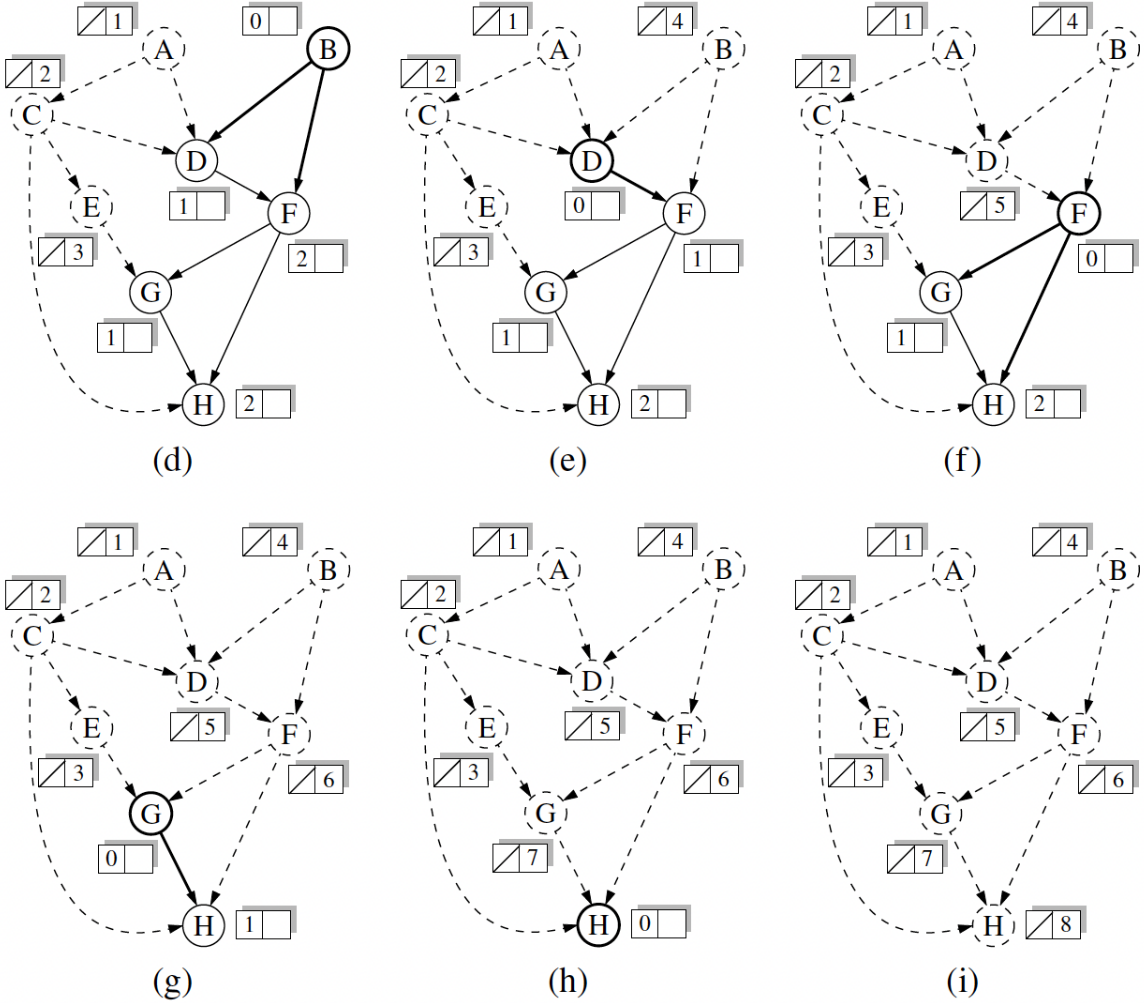
(c)

🛠 **Ready List**: [B, E]
📄 **Topological Order**: [A, C]

✅ **Selected Node:** C
🛠 **Ready List:** [B]

📉 **Update in-degrees for C's neighbors:**
- C → D → incount[D] -= 1 → 2 → 1
- C → E → incount[E] -= 1 → 1 → 0 ✅
- C → H → incount[H] -= 1 → 3 → 2

➕ **Add E to ready** (because incount[E] == 0)



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Performance Analysis

- Let n = number of vertices, m = number of edges
- Time complexity: $O(n + m)$
- Space complexity: $O(n)$
- Efficient for large sparse graphs

Detecting Cycles

- If topo does not include all nodes:
  - The graph contains a cycle
  - Some nodes remain with non-zero in-degree

Summary

- DAGs are fundamental in scheduling and dependency resolution
- Topological sorting is a key algorithm on DAGs
- Python implementation uses Kahn's algorithm efficiently
- Always verify acyclicity to ensure a valid ordering

Shortest Paths

Introduction to Shortest Paths

- Goal: Find the shortest (minimum-weight) path between vertices
- Applications:
  - Navigation systems
  - Routing in computer networks
  - Task scheduling
- Breadth-First Search (BFS) only works when all edge weights are equal

Weighted Graphs

- A weighted graph assigns a numeric weight $w(e)$ to each edge e
- For edge (u, v), use notation: $w(u, v) = w(e)$
- Weights can represent:
  - Distance
  - Cost
  - Time

Path and Distance Definitions

- A path from u to v is a sequence of edges
- The length of a path = sum of its edge weights
- The shortest path from u to v is a path with minimum total weight
- If no path exists: $d(u, v) = \infty$

Valid Weights for Shortest Path Algorithms

- Must use nonnegative edge weights for Dijkstra
- Negative-weight edges allow cycles that reduce cost arbitrarily
- Algorithms must avoid revisiting same nodes with lower cost in cycles

Dijkstra's Algorithm Overview

- Greedy algorithm for single-source shortest paths
- Always picks vertex with smallest tentative distance (D[v])
- Expands a "cloud" of known shortest paths from the source
- Relaxes edges from each newly added vertex

Edge Relaxation Explained

- For each edge (u, v):

```
if D[v] > D[u] + w(u, v):
    D[v] = D[u] + w(u, v)
```

- Ensures shortest known distance is improved
- D[v] values always shrink or stay the same

Dijkstra Pseudocode

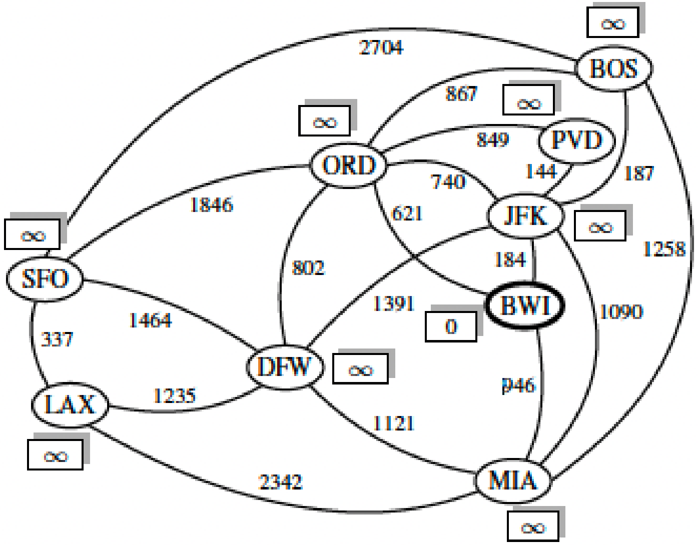
Algorithm ShortestPath(G, s):

Input: A weighted graph G with nonnegative edge weights, and a distinguished vertex s of G.

Output: The length of a shortest path from s to v for each vertex v of G.

```
Initialize D[s] = 0 and D[v] = ∞ for each vertex v ≠ s.
Let a priority queue Q contain all the vertices of G using the D labels as keys.
while Q is not empty do
   {pull a new vertex u into the cloud}
   u = value returned by Q.remove_min()
   for each vertex v adjacent to u such that v is in Q do
      {perform the relaxation procedure on edge (u,v)}
      if D[u] + w(u,v) < D[v] then
         D[v] = D[u] + w(u,v)
         Change to D[v] the key of vertex v in Q.
return the label D[v] of each vertex v
```

Example Execution



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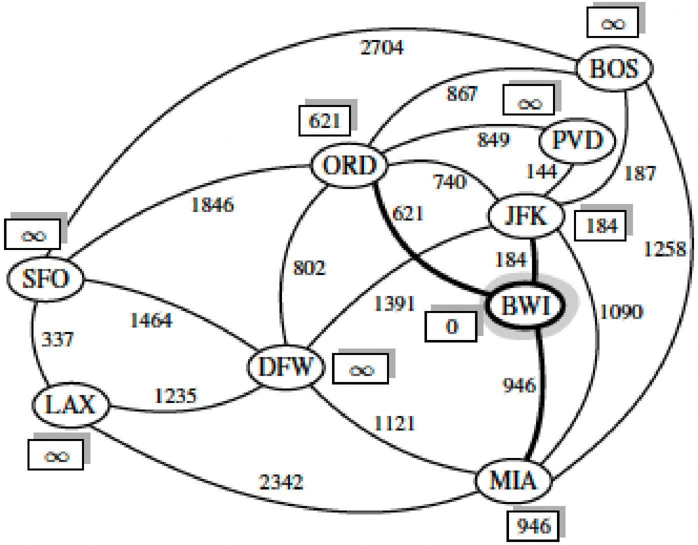

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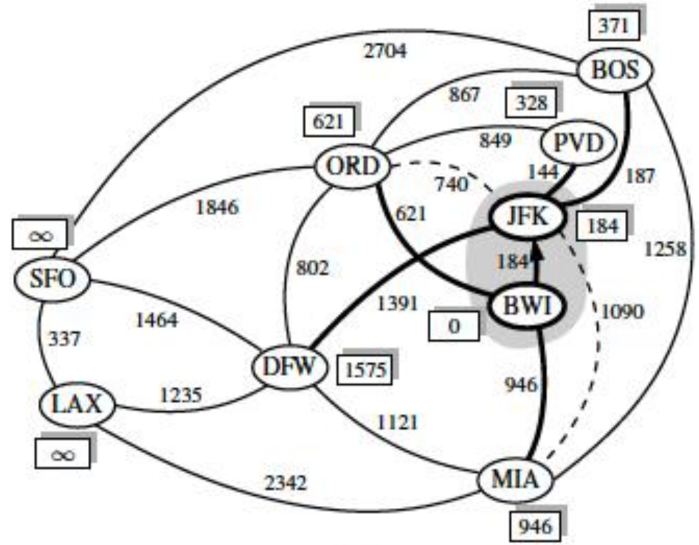
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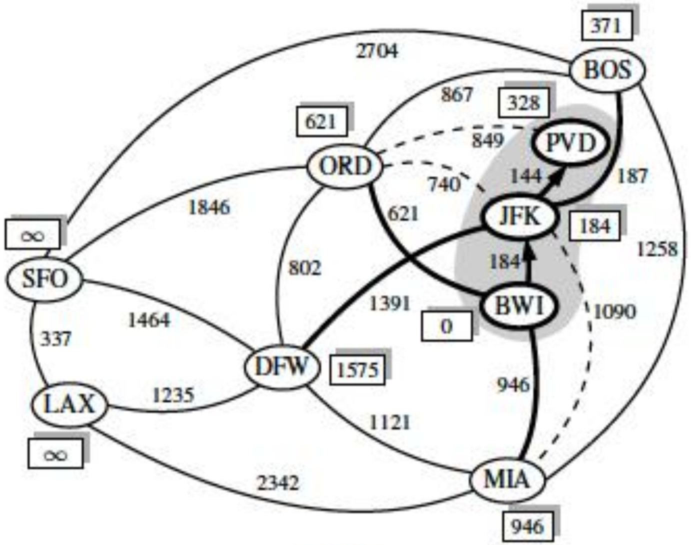
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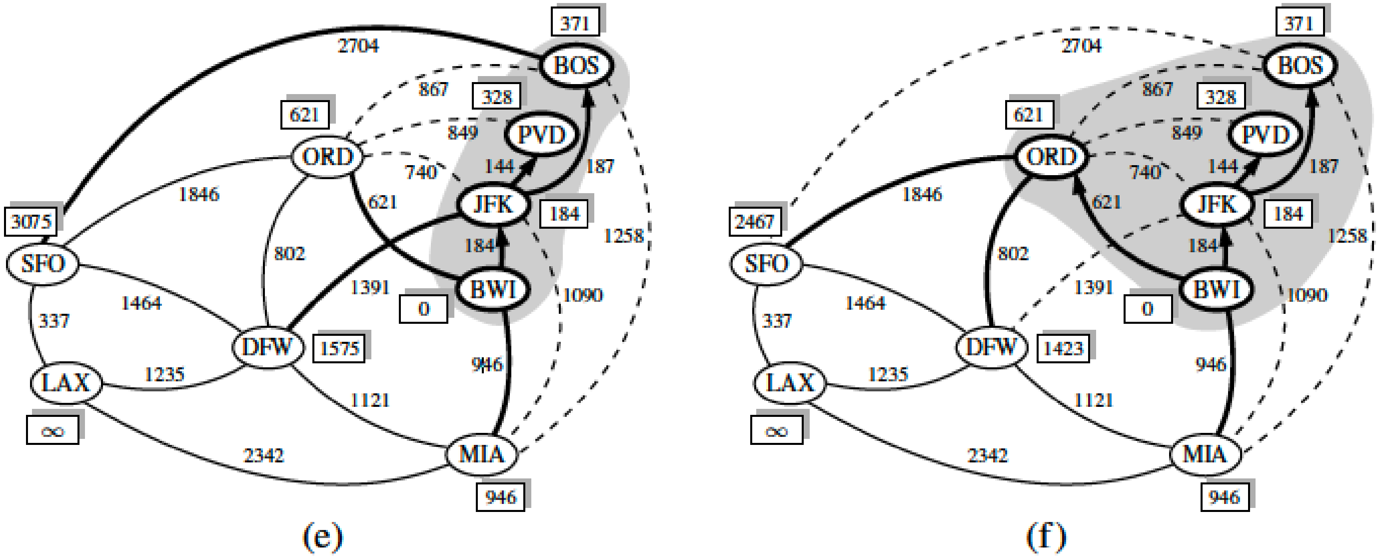

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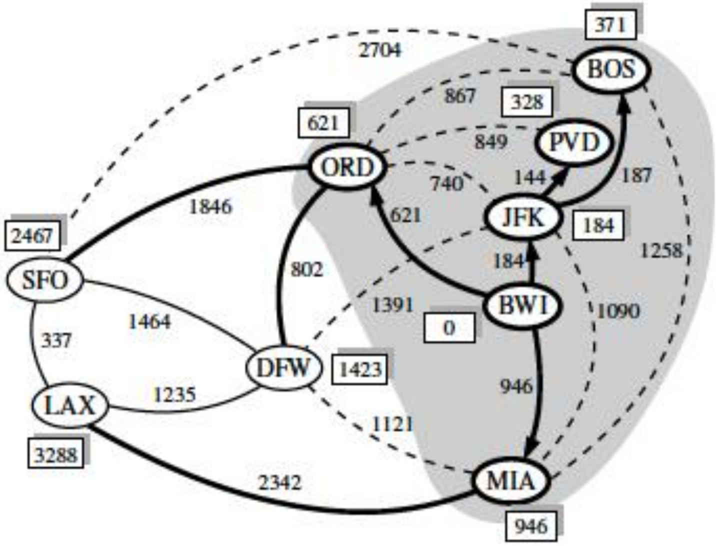

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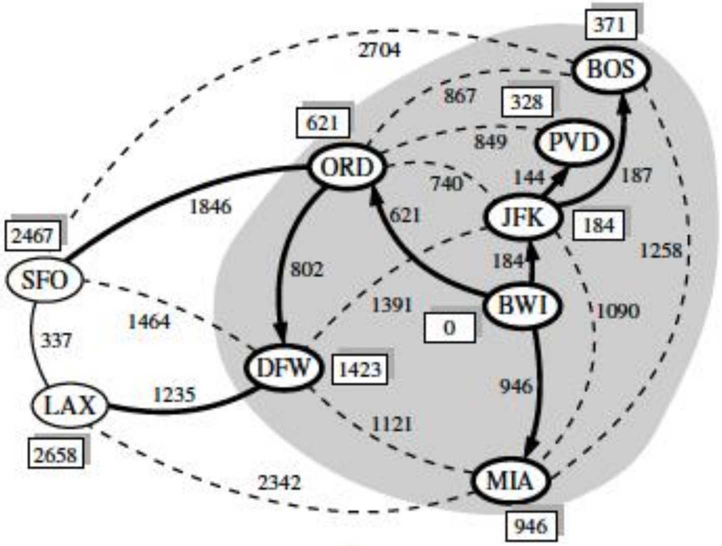

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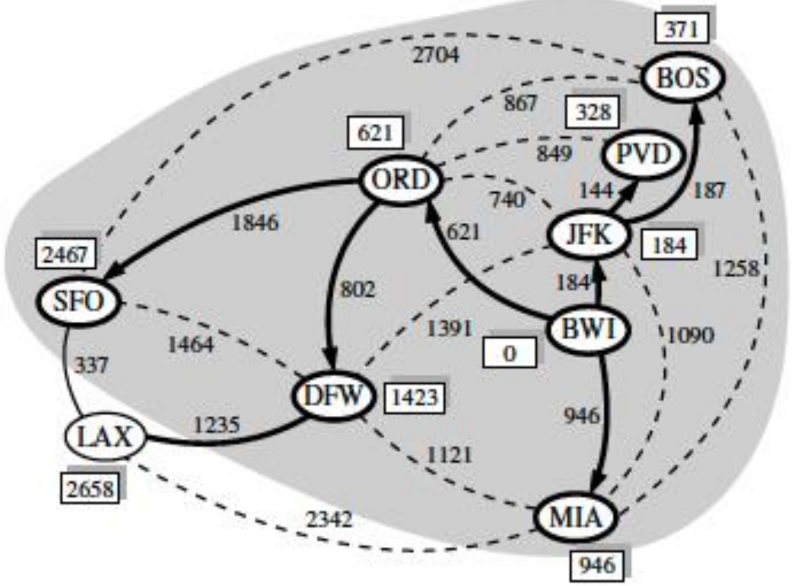
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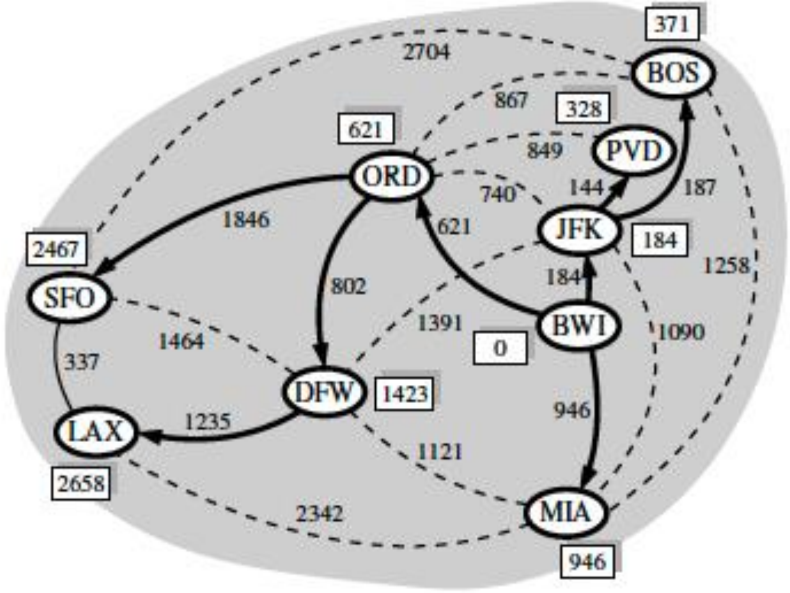
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Reconstructing Shortest Paths

- Dijkstra gives shortest distance, but not actual path
- To reconstruct:
  - For each $v \neq s$, find edge (u, v) such that $D[u] + w(u, v) = D[v]$
- Builds the shortest-path tree

Summary

- Dijkstra’s algorithm solves the single-source shortest path problem
- Assumes nonnegative weights
- Builds both distance map and shortest-path tree
- Efficient and correct with proper data structures

Minimum Spanning Tree (MST)

Minimum Spanning Tree (MST)

- Connect all vertices of a weighted, undirected graph
- Use minimum total edge weight
- Real-world: Lay out cable to connect offices with minimal cost

Problem Definition

- Given undirected graph G = (V, E) with weights w(e)
- Find tree T ⊆ E that spans all V with minimal:

```
w(T) = Σ w(u, v)   for all (u,v) in T
```

- T is a Minimum Spanning Tree (MST)

Prim-Jarník Algorithm - Concept

- Grow MST from an arbitrary start vertex
- Use a greedy strategy: always add lightest edge to expand MST
- Very similar structure to Dijkstra's Algorithm

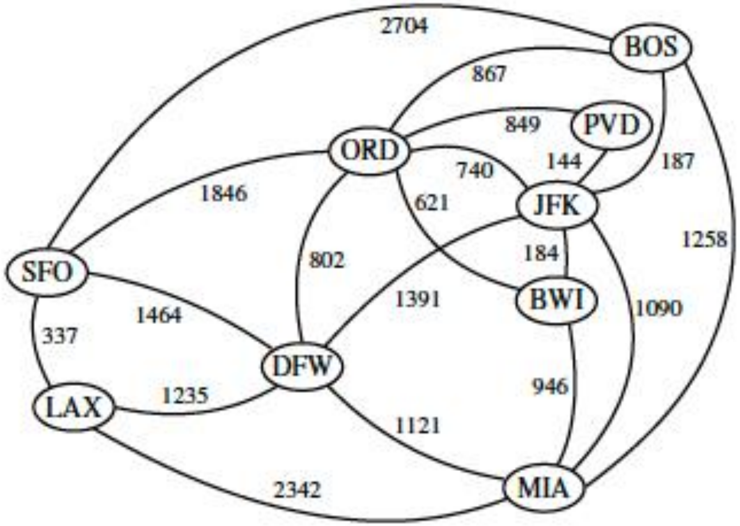
Prim-Jarník Pseudocode

Algorithm PrimJarnik(G):

  Input: An undirected, weighted, connected graph G with n vertices and m edges
  Output: A minimum spanning tree T for G

Pick any vertex s of G
$D[s] = 0$
for each vertex $v \neq s$ **do**
  $D[v] = \infty$
Initialize $T = \emptyset$.
Initialize a priority queue Q with an entry $(D[v], (v, \text{None}))$ for each vertex v, where $D[v]$ is the key in the priority queue, and (v, None) is the associated value.
while Q is not empty **do**
  (u, e) = value returned by Q.remove_min()
  Connect vertex u to T using edge e.
  for each edge $e' = (u, v)$ such that v is in Q **do**
    {check if edge (u, v) better connects v to T}
    if $w(u, v) < D[v]$ **then**
      $D[v] = w(u, v)$
      Change the key of vertex v in Q to $D[v]$.
      Change the value of vertex v in Q to (v, e').
return the tree T

MST Illustration



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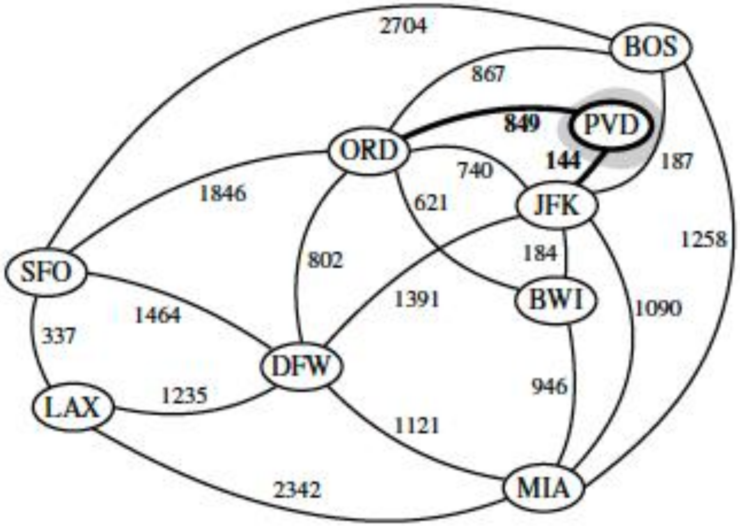

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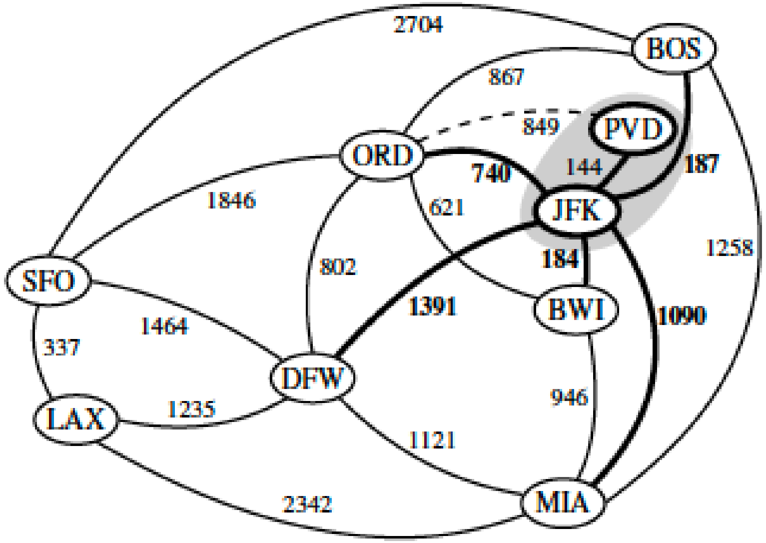
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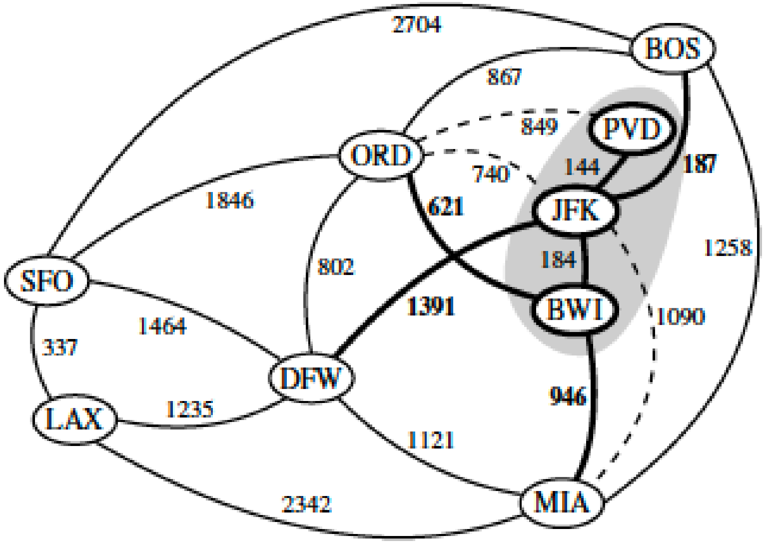
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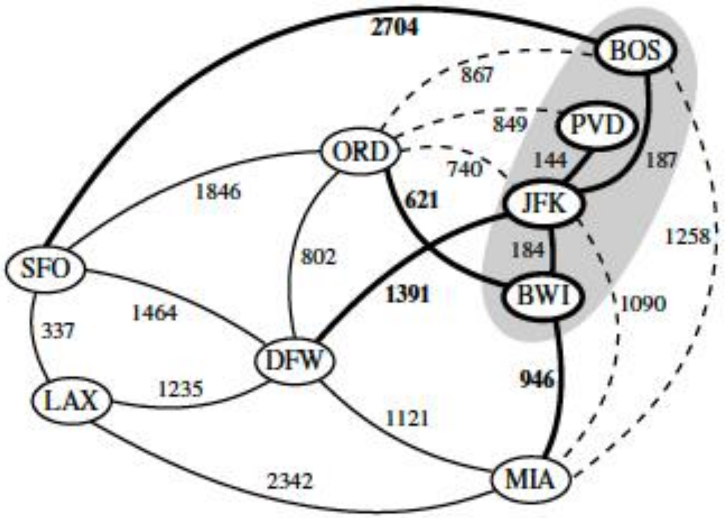
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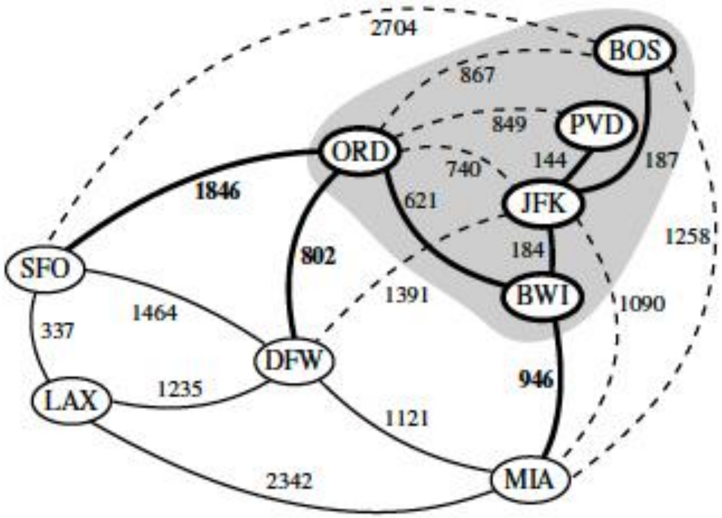
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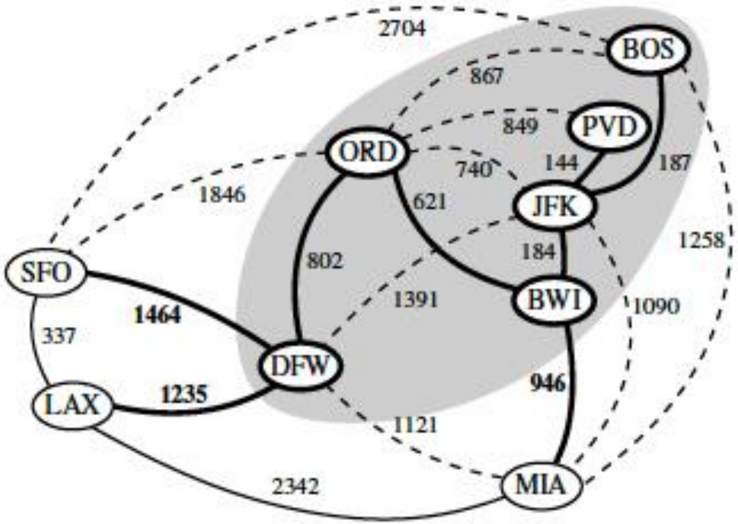

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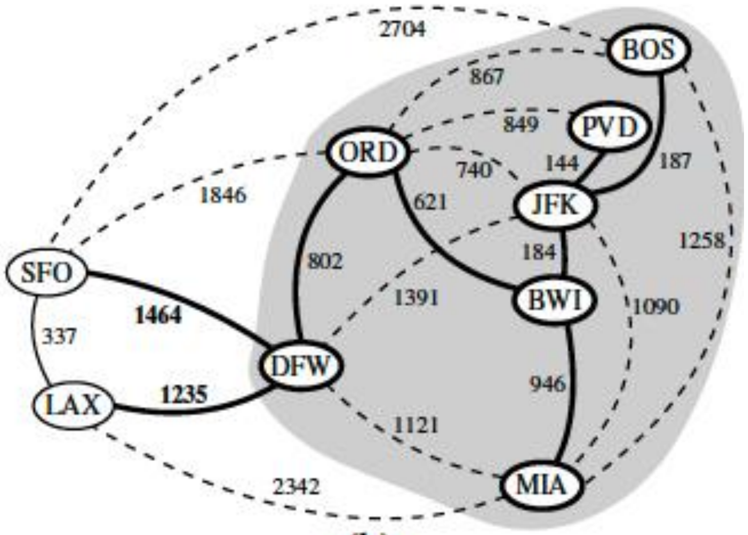

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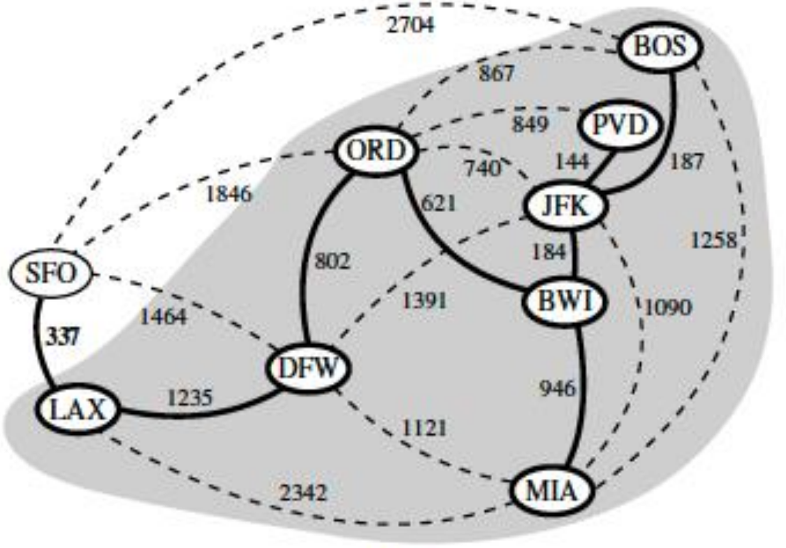
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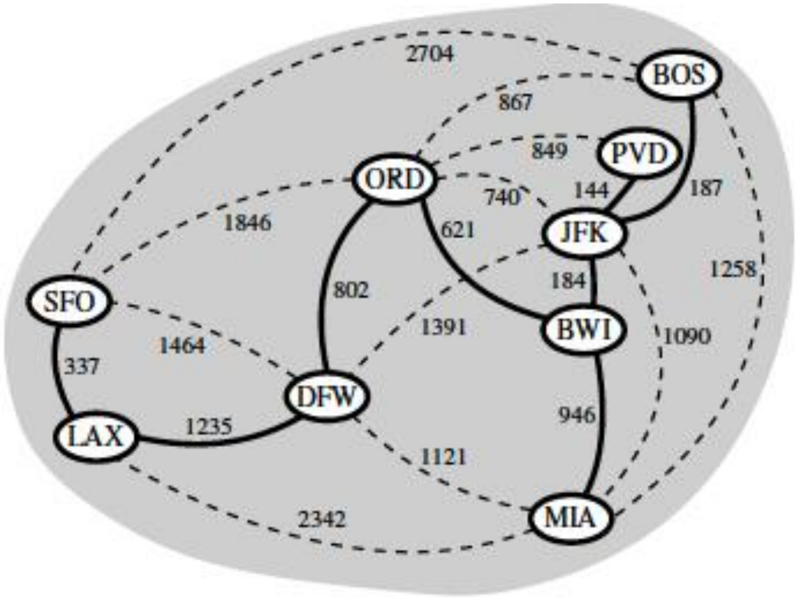
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Kruskal's Algorithm for MST

- Greedy algorithm for Minimum Spanning Tree (MST)
- Builds MST by adding lightest edges one-by-one
- Keeps growing a forest (set of trees), until connected

Kruskal's Algorithm Concept

- Start: All vertices are individual clusters (singleton sets)
- Sort edges by weight (min → max)
- For each edge (u, v):
  - If u and v are in different clusters → add edge to MST
  - Merge their clusters
  - If u and v are already connected → skip (to avoid cycle)

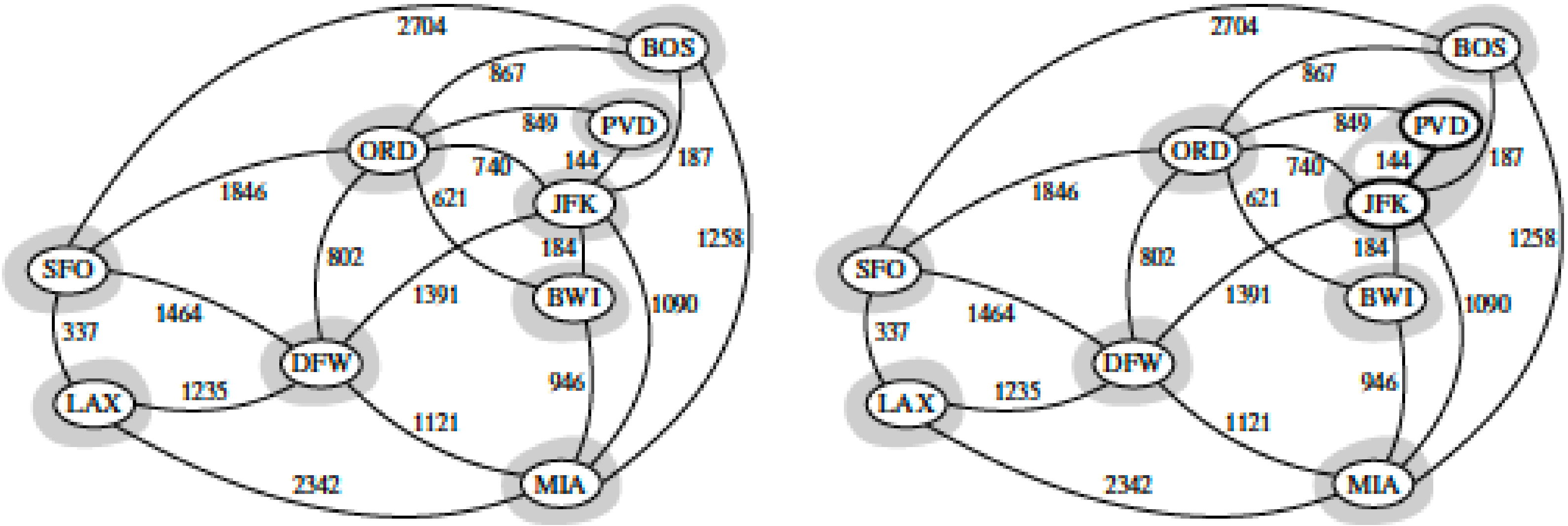
Kruskal's Pseudocode

Algorithm Kruskal(G):

  Input: A simple connected weighted graph G with n vertices and m edges
  Output: A minimum spanning tree T for G

for each vertex v in G **do**
  Define an elementary cluster $C(v) = \{v\}$.
Initialize a priority queue Q to contain all edges in G, using the weights as keys.
$T = \emptyset$ $\{T$ will ultimately contain the edges of the MST$\}$
while T has fewer than $n-1$ edges **do**
  (u,v) = value returned by Q.remove_min()
  Let $C(u)$ be the cluster containing u, and let $C(v)$ be the cluster containing v.
  if $C(u) \neq C(v)$ **then**
    Add edge (u,v) to T.
    Merge $C(u)$ and $C(v)$ into one cluster.
return tree T

Kruskal’s MST algorithm Illustration



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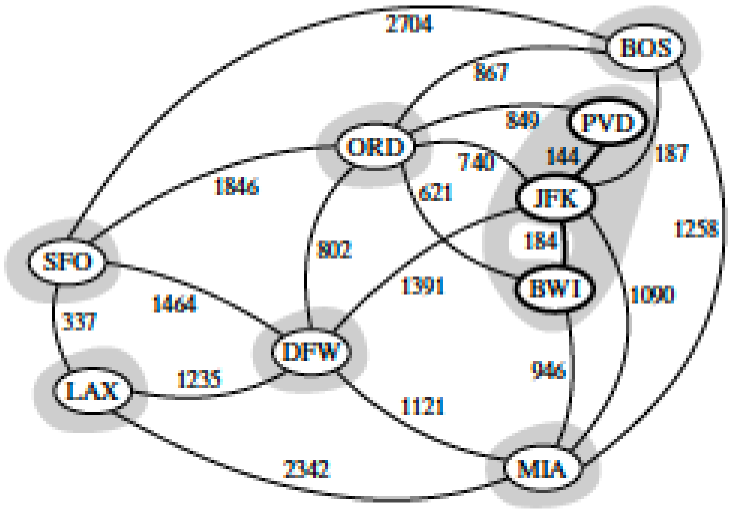
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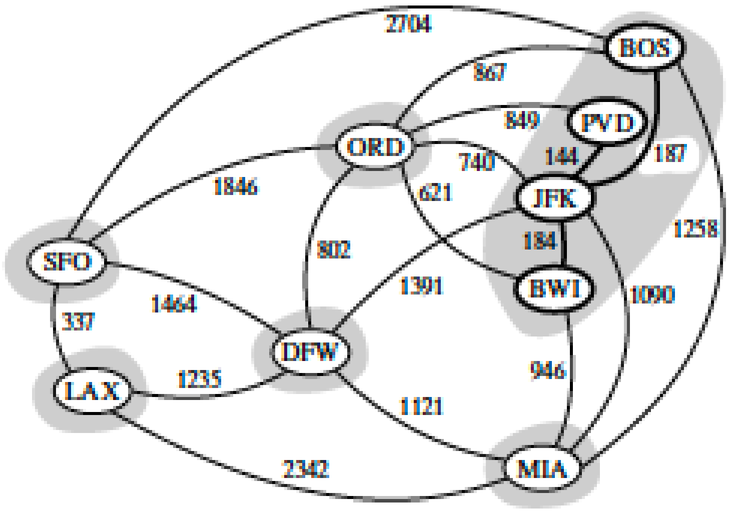
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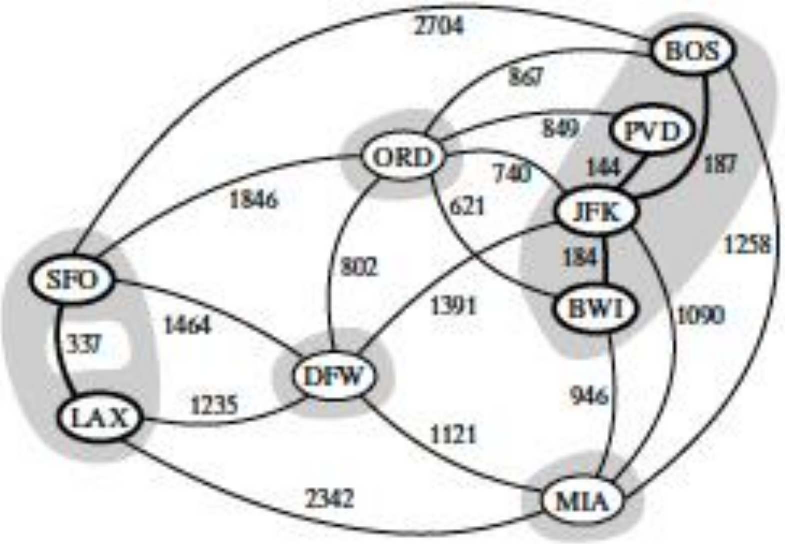
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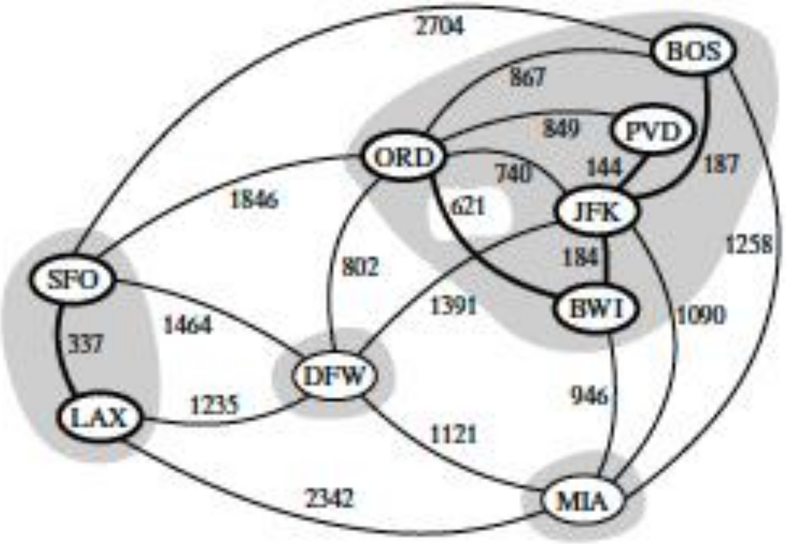
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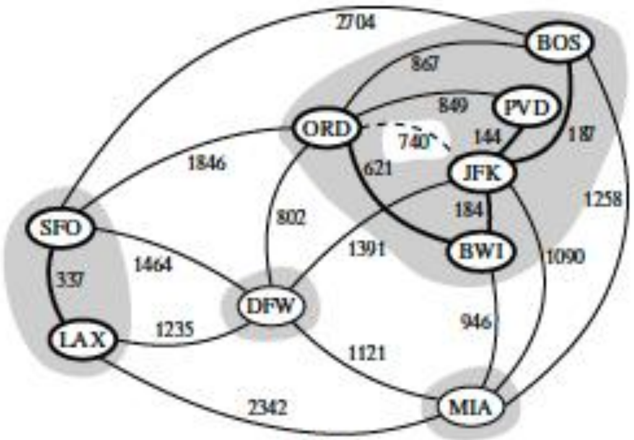

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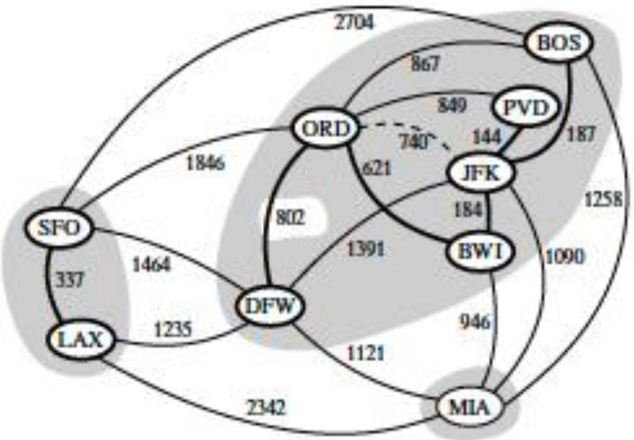

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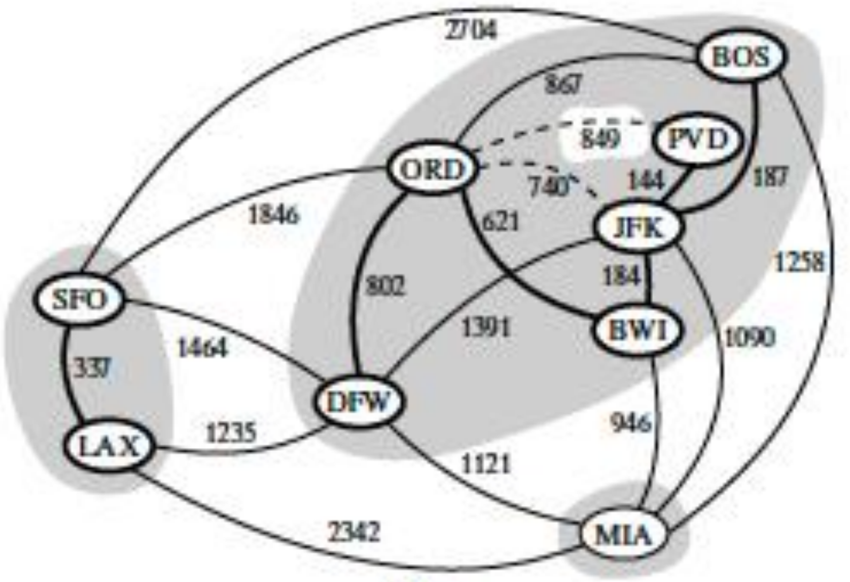
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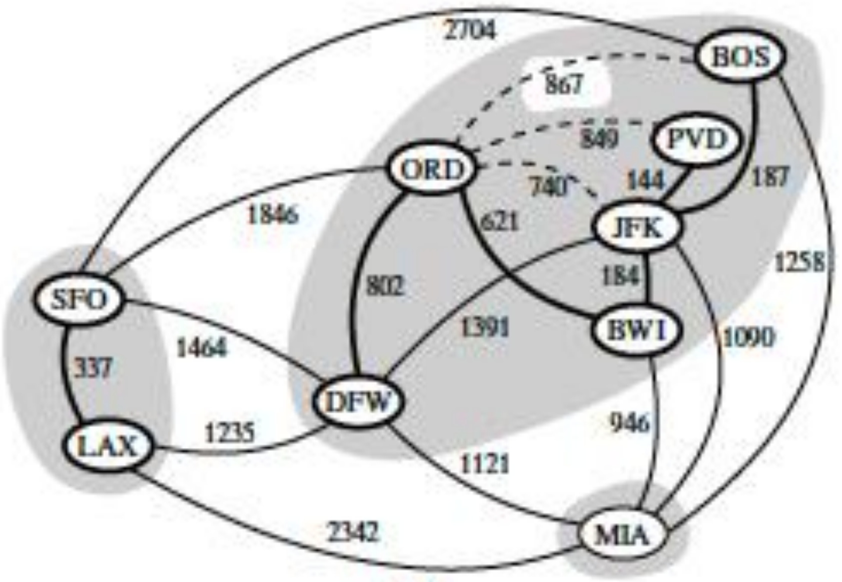
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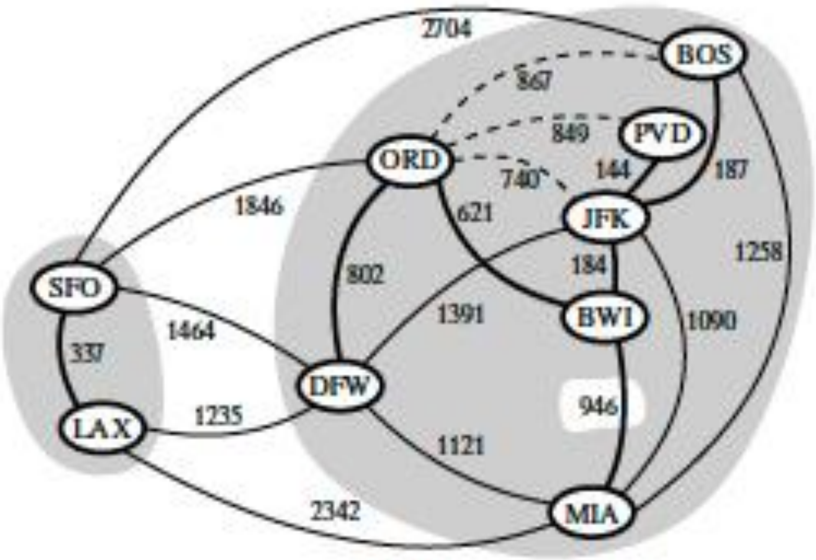
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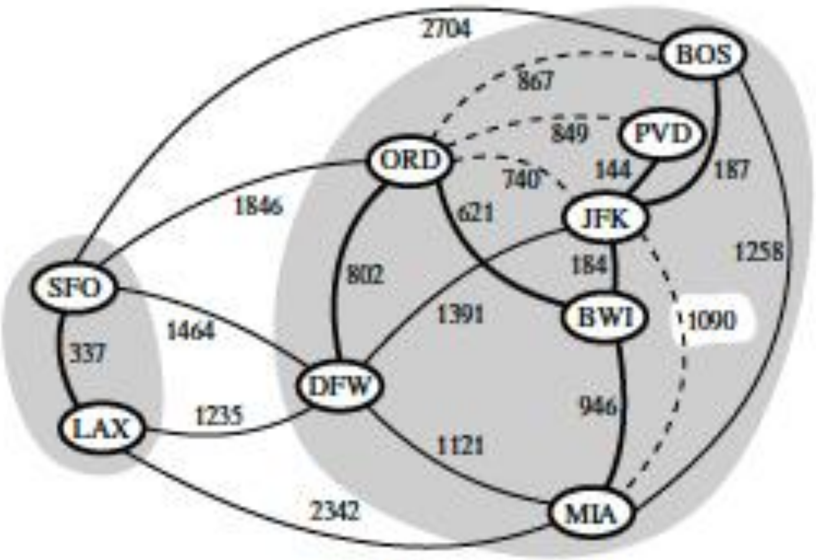
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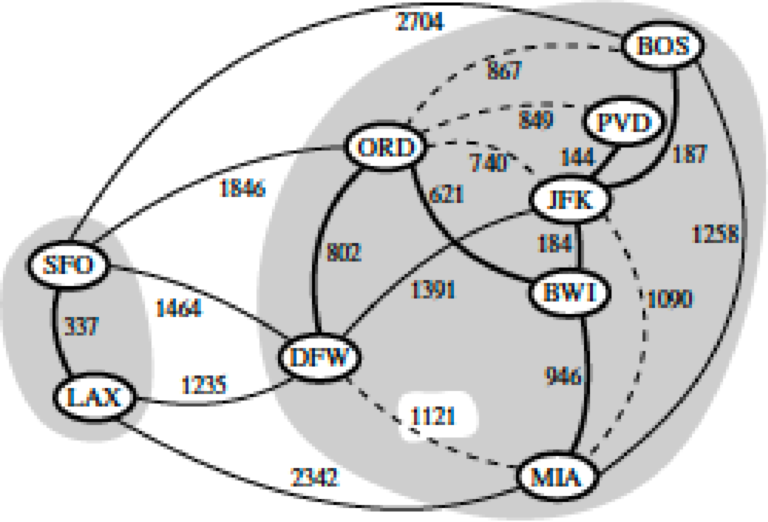
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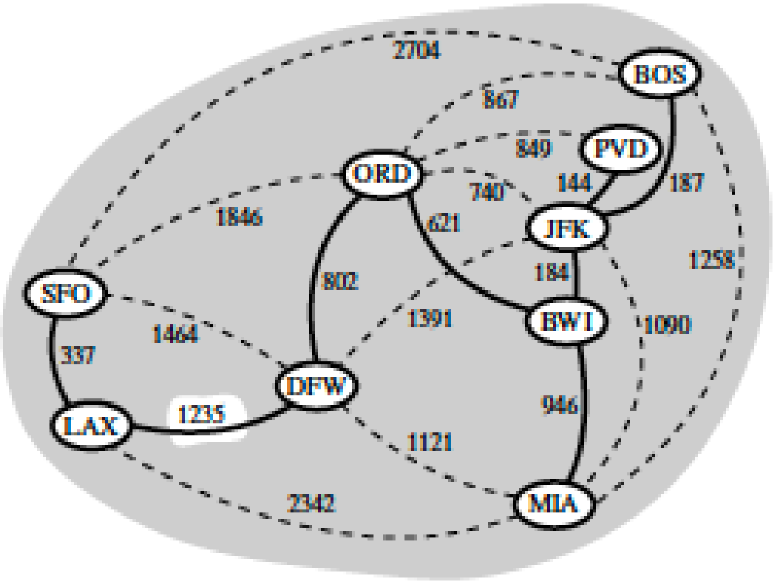
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