National Technical University of Athens
School of Electrical and Computer Engineering

Game Theory
Spring 2025 - Exercise Series 1

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Problem 1

We consider the following game:

	W	X	Y	Z
A	$(15, 42)$	$(13, 40)$	$(9, 23)$	$(0, 23)$
B	$(2, 19)$	$(2, 14)$	$(5, 13)$	$(1, 0)$
C	$(20, 7)$	$(20, 5)$	$(11, 3)$	$(1, 2)$
D	$(20, 45)$	$(3, 11)$	$(3, 5)$	$(1, 2)$

1. Is strategy B strictly dominated by A?

By definition, a strategy s_i of player 1 is *dominant* if $\mathbf{u_1}(s_i, t_j) \geq \mathbf{u_1}(s', t_j)$ for every strategy $s' \in S^1$ and every strategy $t_j \in S^2$.

Comparing outcomes of strategies A *&* B for player 1:

$$
\begin{aligned}
A(W) &= 15 > 2 = B(W) \\
A(X) &= 13 > 2 = B(X) \\
A(Y) &= 9 > 5 = B(Y) \\
A(Z) &= 0 < 1 = B(Z)
\end{aligned}
$$

In all columns except for Z, strategy A yields a better outcome than strategy B. However, since strategy B outperforms A in column Z, we can conclude that **B is neither strictly nor even weakly dominated by A**.

2. Does X strictly dominate Y?

For player 2, a strategy t_j is *strictly dominant* if $\mathbf{u_2}(s_i, t_j) > \mathbf{u_2}(s_i, t')$ for every strategy $t' \in S^2$ and for every strategy $s_i \in S^1$.

Evaluating the outcomes of strategies X *&* Y for player 2:

$$
\begin{aligned}
X(A) &= 40 > 23 = Y(A) \\
X(B) &= 14 > 13 = Y(B) \\
X(C) &= 5 > 3 = Y(C) \\
X(D) &= 11 > 5 = Y(D)
\end{aligned}
$$

Since the inequality holds in all cases, meaning that strategy X consistently yields better outcomes than Y in every row, we can safely conclude that **X strictly dominates Y**.

3. Does D weakly dominate B?

By definition, a strategy s_i of player 1 *weakly dominates* some other strategy s' if for every strategy t_j of player 2, it holds that $\mathbf{u_1}(s_i, t_j) \geq \mathbf{u_1}(s', t_j)$ and for at least one strategy t_j, we have $\mathbf{u_1}(s_i, t_j) > \mathbf{u_1}(s', t_j)$.

$$\begin{aligned} D(W) &= 20 > 2 = B(W) \\ D(X) &= 3 > 2 = B(X) \\ D(Y) &= 3 < 5 = B(Y) \\ D(Z) &= 1 = B(Z) \end{aligned}$$

Comparing the outcomes of strategies D *&* B for player 1, it is clear that **D does not dominate B**, not even weakly, since B yields a better outcome than D in column Y and both strategies produce the same outcome in column Z.

4. Is Z weakly dominated by W?

Similarly, a strategy t_j of player 2 *weakly dominates* some other strategy t' if for every strategy s_i of player 1, it holds that $\mathbf{u_2}(s_i, t_j) \geq \mathbf{u_2}(s_i, t')$ and for at least one strategy s_i, we have $\mathbf{u_2}(s_i, t_j) > \mathbf{u_2}(s_i, t')$.

$$\begin{aligned} W(A) &= 42 > 23 = Z(A) \\ W(B) &= 19 > 0 = Z(B) \\ W(C) &= 7 > 2 = Z(C) \\ W(D) &= 45 > 2 = Z(D) \end{aligned}$$

Since the inequality holds in all cases and strategy W consistently yields better outcomes than strategy Z in every row, we conclude that **Z is**, not only weakly, but **strictly dominated by W**.

5. Is there a dominant strategy for player 1?

$$\begin{aligned} C(W) &= D(W) > A(W), B(W) \\ C(X) &= 20 > A(X), B(X), D(X) \\ C(Y) &= 11 > A(Y), B(Y), D(Y) \\ C(Z) &= B(Z),\ D(Z) > A(Z) \end{aligned}$$

As we observe, strategy C weakly dominates over all other strategies of player 1. Thus, **C is a dominant strategy for player 1**.

6. Is C a best response to W?

A best response strategy for player 1 to strategy W of player 2 is defined as:

$$s^* \in \arg\max_{s \in S_1} u_1(s, W).$$

This means that the strategy s^* provides the maximum possible utility to player 1 when the opponent selects strategy W. Since the best outcome for player 1 under strategy W is 20, **both** strategies **C and D are best responses to W**.

7. Is there a Nash equilibrium with social welfare below 30?

By definition, a strategy profile (s, t) is a Nash equilibrium, if no player has a unilateral incentive to deviate, given the other player's choice. Thus, (s, t) is a Nash equilibrium, if $u_1(s, t) \geq u_1(s', t)$ for every strategy $s' \in S^1$ and $u_2(s, t) \geq u_2(s, t')$ for every strategy $t' \in S^2$. So in order to answer the question we need to find a Nash equilibrium that satisfies the condition $SW(s, t) < 30$.

$\boldsymbol{SW}$	W	X	Y	Z
A	57	53	32	23
B	21	16	18	1
C	27	25	14	3
D	65	14	8	3

Player 1 (row player) chooses between A, B, C, D and player 2 (column player) chooses between W, X, Y, Z to maximize their payoff.

If player 2 plays:
- column W, then best response from player 1 is row C or D.
- column X, then best response from player 1 is row C.
- column Y, then best response from player 1 is row C.
- column Z, then best response from player 1 is either row B, C or D.

If player 1 plays:
- row A, then best response from player 2 is column W.
- row B, then best response from player 2 is column W.
- row C, then best response from player 2 is column W.
- row D, then best response from player 2 is column W.

A Nash equilibrium occurs where both players are playing their best responses simultaneously. Since W is the dominant strategy for player 2, player 1's optimal response to W can be either row C, or D, as both yield the same outcome for him. Thus, there exist two Nash equilibria: $(D, W) = (20, 45)$ and $(C, W) = (20, 7)$. Among these, only the second equilibrium has a social welfare below 30, specifically yielding a total of 27.

I hereby include Problem 2 from the 2024 assignment, as I mistakenly assumed it was part of our assignment upon encountering the wrong document in the "Written Assignments" section on Helios (see: `https://helios.ntua.gr/mod/assign/view.php?id=21547`*). By the time I realised my mistake, I had already completed the task and considered it worthwhile to include it in my submission nonetheless.*

Problem 2 ***from last year***

We represent this as a normal-form game with the following payoff matrix:

$$\begin{array}{ccc} & \textbf{C1} & \textbf{C2} \\ \textbf{C1} & \left(\frac{w_1}{2}, \frac{w_1}{2}\right) & (w_1, w_2) \\ \textbf{C2} & (w_2, w_1) & \left(\frac{w_2}{2}, \frac{w_2}{2}\right) \end{array}$$

A Nash equilibrium occurs when no player benefits from deviating. For pure strategies, we examine each case:

- **(C1, C1)**: both players have incentive to deviate because $w_2 > w_1/2$.
- **(C2, C2)**: both players have incentive to deviate because $w_2 > w_2/2$.
- **(C1, C2)**: no player benefits from deviating.
- **(C2, C1)**: no player benefits from deviating.

Hence: The profiles (C1, C2) *&* (C2, C1) are Nash equilibria of this game.

Let's now examine the case of mixed strategies. By definition, a mixed strategy profile ($\mathbf{p}$,$\mathbf{q}$) constitutes a Nash equilibrium if:

- $u_1(\mathbf{p}, \mathbf{q}) \geq u_1(\mathbf{e}^i, \mathbf{q})$ for every pure strategy $\mathbf{e}^i$ of player 1
- $u_2(\mathbf{p}, \mathbf{q}) \geq u_2(\mathbf{p}, \mathbf{e}^j)$ for every pure strategy $\mathbf{e}^j$ of player 2

		q	***1−q***
		C1	**C2**
p	**C1**	$\left(\frac{w_1}{2}, \frac{w_1}{2}\right)$	(w_1, w_2)
1−p	**C2**	(w_2, w_1)	$\left(\frac{w_2}{2}, \frac{w_2}{2}\right)$

Let p be the probability that player 1 chooses C1, so the probability they choose C2 is $1-p$. player's 1 choice of p should make player 2 indifferent between C1 *&* C2.

$$u_2(\text{C1}) = p \cdot \frac{w_1}{2} + (1-p) \cdot w_1 = \frac{pw_1}{2} + w_1 - pw_1 = w_1 - \frac{pw_1}{2}$$
$$u_2(\text{C2}) = p \cdot w_2 + (1-p) \cdot \frac{w_2}{2} = pw_2 + \frac{w_2}{2} - \frac{pw_2}{2} = \frac{w_2}{2} + \frac{pw_2}{2}$$

To find the value of p that makes player 2 indifferent between choosing C1 and C2, we equate their expected utilities:

$$w_1 - \frac{pw_1}{2} = \frac{w_2}{2} + \frac{pw_2}{2} \Rightarrow w_1 - \frac{w_2}{2} = \frac{p}{2}(w_1 + w_2) \Rightarrow \boldsymbol{p = \frac{2w_1 - w_2}{w_1 + w_2}}$$

Now, let q be the probability that player 2 chooses C1, so the probability they choose C2 is $1 - q$. Similarly, player's 2 choice of q should make player 1 indifferent between choosing C1 and C2.

$$u_1(\text{C1}) = q \cdot \frac{w_1}{2} + (1-q) \cdot w_1 = \frac{qw_1}{2} + w_1 - qw_1 = w_1 - \frac{qw_1}{2}$$
$$u_1(\text{C2}) = q \cdot w_2 + (1-q) \cdot \frac{w_2}{2} = qw_2 + \frac{w_2}{2} - \frac{qw_2}{2} = \frac{w_2}{2} + \frac{qw_2}{2}$$

Setting the expected utilities equal to solve for q:

$$w_1 - \frac{qw_1}{2} = \frac{w_2}{2} + \frac{qw_2}{2} \Rightarrow w_1 - \frac{w_2}{2} = \frac{q}{2}(w_1 + w_2) \Rightarrow \boldsymbol{q = \frac{2w_1 - w_2}{w_1 + w_2}}$$

Hence: Both players randomize between their actions, choosing C1 with probability p and C2 with probability $1 - p$, where $p = q$. The profile

$$\boxed{\text{pl. 1: } (p, 1-p), \quad \text{pl. 2: } (p, 1-p)}$$

is a symmetric mixed-strategy Nash equilibrium of the game. It exists as long as $0 \leq p \leq 1$, which holds true for all $w_1, w_2 > 0$ with $w_2 \leq 2w_1$ *&* $w_1 \leq 2w_2$. This condition is satisfied since it is given that $\frac{1}{2} < \frac{w_1}{w_2} < 2$.

Problem 2 ***this year***

In this problem we are asked to modify the classical Prisoner's Dilemma game, which involves a pure strategy set $\{C, D\}$ for both players and is defined by the following payoff matrix:

	C	**D**
C	$(2, 2)$	$(0, 3)$
D	$(3, 0)$	$(1, 1)$

where C denotes cooperation with the other player and D denotes defection, meaning betrayal of the other player.

With $u_1(s, t)$ and $u_2(s, t)$ as the original utilities of players 1 and 2, we will denote their modified utility functions as follows, to capture the influence of one player's payoff on the other's utility, thus introducing a degree of altruism:

$$u_1'(s, t) = u_1(s, t) + \alpha u_2(s, t) \quad \textit{\&} \quad u_2'(s, t) = u_2(s, t) + \alpha u_1(s, t)$$

where $\alpha \in [0, 1]$ quantifies each player's concern for the other player's payoff.

(i) Modified Prisoner's Dilemma for $\alpha = \frac{2}{3}$

Having set $\alpha = \frac{2}{3}$ we compute the new payoffs for each strategy profile:

$$\begin{aligned}
(C, C) &\to \left(2 + \tfrac{2}{3} \cdot 2,\ 2 + \tfrac{2}{3} \cdot 2\right) = \left(\tfrac{10}{3}, \tfrac{10}{3}\right) \\
(C, D) &\to \left(0 + \tfrac{2}{3} \cdot 3,\ 3 + \tfrac{2}{3} \cdot 0\right) = (2,\ 3) \\
(D, C) &\to \left(3 + \tfrac{2}{3} \cdot 0,\ 0 + \tfrac{2}{3} \cdot 3\right) = (3,\ 2) \\
(D, D) &\to \left(1 + \tfrac{2}{3} \cdot 1,\ 1 + \tfrac{2}{3} \cdot 1\right) = \left(\tfrac{5}{3}, \tfrac{5}{3}\right)
\end{aligned}$$

which is giving us the following modified payoff matrix:

	C	**D**
C	$\left(\frac{10}{3}, \frac{10}{3}\right)$	$(2, 3)$
D	$(3, 2)$	$\left(\frac{5}{3}, \frac{5}{3}\right)$

This game does no longer represent the classical Prisoner's Dilemma, where both players have a strict incentive to defect, regardless of the other player's choice. The original game consists of a unique Nash equilibrium at (D, D), which is Pareto dominated by (C, C), as both players would be strictly better off cooperating, yet mutual defection prevails due to dominant strategies.

In the modified game with $\alpha = \frac{2}{3}$, the incentive structure changes fundamentally. More specifically, if player 2 plays C, player 1 strictly prefers C over D, since $\frac{10}{3} > 3$. If player 2 plays D, player 1 still prefers C, as $2 > \frac{5}{3}$. By symmetry, the same is true for player 2. As a result, both players now strictly prefer to cooperate, making C a strictly dominant strategy. Hence, the modified game with $\alpha = \frac{2}{3}$ no longer captures the classical Prisoner's Dilemma, as the original core incentive to defect has been replaced by a clear incentive to cooperate.

(ii) α values to retain the Prisoner's Dilemma core

The game retains the classical Prisoner's Dilemma structure when defection is a strictly dominant strategy for both players, yielding a unique Nash equilibrium at (D, D), while (C, C) is Pareto superior to mutual defection (D, D). A strategy profile is Pareto dominant over another if it improves at least one player's payoff without decreasing any other player's payoff.

To examine whether defection strategy remains dominant, we will analyze player's 1 incentives in each scenario. So, if player 2 plays C, then player's 1 modified payoffs are

$$u_1'(C, C) = 2 + 2\alpha \quad \& \quad u_1'(D, C) = 3.$$

In this case defection strategy would be strictly preferred if and only if $3 > 2 + 2\alpha$, which is equivalent to $\alpha < \frac{1}{2}$. On the other hand, if player 2 plays D, then player's 1 payoffs become

$$u_1'(C, D) = 3\alpha \quad \& \quad u_1'(D, D) = 1 + \alpha,$$

in which case again defection is strictly preferred if and only if $1 + \alpha > 3\alpha \Rightarrow \alpha < \frac{1}{2}$. Due to the symmetry of the game, the exact same reasoning applies to player 2, resulting in the same threshold condition $\alpha < \frac{1}{2}$ for the defection strategy to be strictly dominant.
The final essential part of the classic Prisoner's Dilemma game is that the cooperative outcome (C, C) Pareto dominates (D, D). Under the modified payoffs, we have:

$$u_1'(C,C) = 2 + 2\alpha > 1 + \alpha = u_1'(D,D) \quad \& \quad u_2'(C,C) = 2 + 2\alpha > 1 + \alpha = u_2'(D,D),$$

which is clearly true for all $\alpha \in [0, 1]$, ensuring that (C, C) remains strictly better for both players than (D, D). Concluding, the modified game retains the defining characteristics of the Prisoner's Dilemma for all $\alpha \in [0,\ \frac{1}{2})$.

(iii) α values that make (C, C) a Nash equilibrium

So, let's express the modified game in terms of α, using the new utility function of each player:

$$u_1'(s,t) = u_1(s,t) + \alpha u_2(s,t), \quad u_2'(s,t) = u_2(s,t) + \alpha u_1(s,t).$$

which yields the following payoff matrix:

	C	**D**
C	$(2 + 2\alpha, 2 + 2\alpha)$	$(3\alpha, 3)$
D	$(3, 3\alpha)$	$(1 + \alpha, 1 + \alpha)$

In order for (C, C) to constitute a Nash equilibrium, no player should have an incentive to deviate unilaterally. This requires that each player's utility at the profile (C, C) is at least as great as their utility from deviating to another strategy, assuming the other player chooses C.
For player 1, the condition we discussed above is expressed as

$$u_1'(C,C) \geq u_1'(D,C) \quad \Leftrightarrow \quad 2 + 2\alpha \geq 3 \quad \Leftrightarrow \quad \alpha \geq \tfrac{1}{2}.$$

Similarly, for player 2:

$$u_2'(C,C) \geq u_2'(C,D) \quad \Leftrightarrow \quad 2 + 2\alpha \geq 3 \quad \Leftrightarrow \quad \alpha \geq \tfrac{1}{2}.$$

Therefore, the profile (C, C) is a Nash equilibrium in the modified game if and only if $\alpha \geq \frac{1}{2}$.

Problem 3

	2	3	4	$\cdots$	99	100
2	$(2,2)$	$(4,0)$	$(4,0)$	$\cdots$	$(4,0)$	$(4,0)$
3	$(0,4)$	$(3,3)$	$(5,1)$	$\cdots$	$(5,1)$	$(5,1)$
4	$(0,4)$	$(1,5)$	$(4,4)$	$\cdots$	$(6,2)$	$(6,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
99	$(0,4)$	$(1,5)$	$(2,6)$	$\cdots$	$(99,99)$	$(101,97)$
100	$(0,4)$	$(1,5)$	$(2,6)$	$\cdots$	$(97,101)$	$(100,100)$

(i) Iterated elimination of weakly dominated strategies

We examine which strategies survive the process of iterated elimination of **weakly dominated strategies**. We are going to fix player's 1 strategy to $k \in \{2, \ldots, 100\}$. Their payoff depends on player 2's choice y:

- If $y = k$, then $u_1(k,k) = k$.
- If $y < k$, then $u_1(k,y) = y - 2$ (penalty for choosing the higher value).
- If $y > k$, then $u_1(k,y) = k + 2$ (bonus for choosing the lower value).

We are going to prove our intuition, which says that choosing $k-1$ always gives us at least equal or bigger payoff than when choosing k. More specifically, if the opponent says k, then:

$$u_1(k,k) = k, \quad \text{but} \quad u_1(k-1,k) = (k-1) + 2 = k+1 > k.$$

Hence, strategy k is weakly dominated by $k-1$ and should thus get eliminated. We may repeat this process iteratively. Since $100 \prec 99 \prec 98 \prec \cdots \prec 3$, strategy 2 is the only one that cannot be undercut by the opponent and therefore survives the process of iterated elimination of weakly dominated strategies. Any strategy involving $k > 2$ can be undercut by the other player simply by the last choosing a lower number. When both players select strategy 2, they each get a payoff of 2 euros. In this case, neither of the two can improve his outcome by deviating unilaterally. For instance, if one player deviates to 3 while the other sticks with 2, the deviator receives $k - 2 = 0$, whereas the other player receives 4. Thus, it is now clear to us that neither player has an incentive to deviate from this profile, confirming $(2,2)$ as the only pure strategy Nash equilibrium of our game.

(ii) Existence of other pure strategy equilibria

Suppose there exists a pure strategy Nash equilibrium (x,y) with $x \neq y$. Without loss of generality, we assume that $x < y$, so player 1 receives $x+2$ and player 2 receives $x-2$. But player 2 could instead choose strategy x, matching player 1, and receive $x > x-2$. Hence, player 2 has an incentive to unilaterally deviate from his strategy, so (x,y) with $x \neq y$ cannot constitute a Nash equilibrium. Let's now study strategies with $x = y$, like (k, k) for $k > 2$. If both players declare k, they each receive k, but one could deviate to $k - 1$ and receive $k + 1$, while the other would get $k - 3$. Meaning one player could profit by undercutting his opponent. The profile $(2, 2)$ is the only pure strategy Nash equilibrium, as the value 2 cannot be undercut (being the minimum allowed) and any deviation to a higher value strategy results in a strictly lower payoff for the deviator.

Problem 4

Consider the utility functions of the two players:

$$u_1(x, y) = 2\sqrt{y} - (x + 2)^2 + 5xy$$
$$u_2(x, y) = x - \frac{y^2}{2} + \alpha xy$$

The second player, the employee, selects optimal effort level y to maximize $u_2(x, y)$, taking salary x as fixed:

$$\frac{\partial u_2}{\partial y} = -y + \alpha x = 0 \Rightarrow y = \alpha x$$

The first player, the employer, selects optimal salary x to maximize $u_1(x, y)$, taking employee's effort level y as fixed:

$$\frac{\partial u_1}{\partial x} = -2(x + 2) + 5y = 0 \Rightarrow y = \frac{2(x + 2)}{5}$$

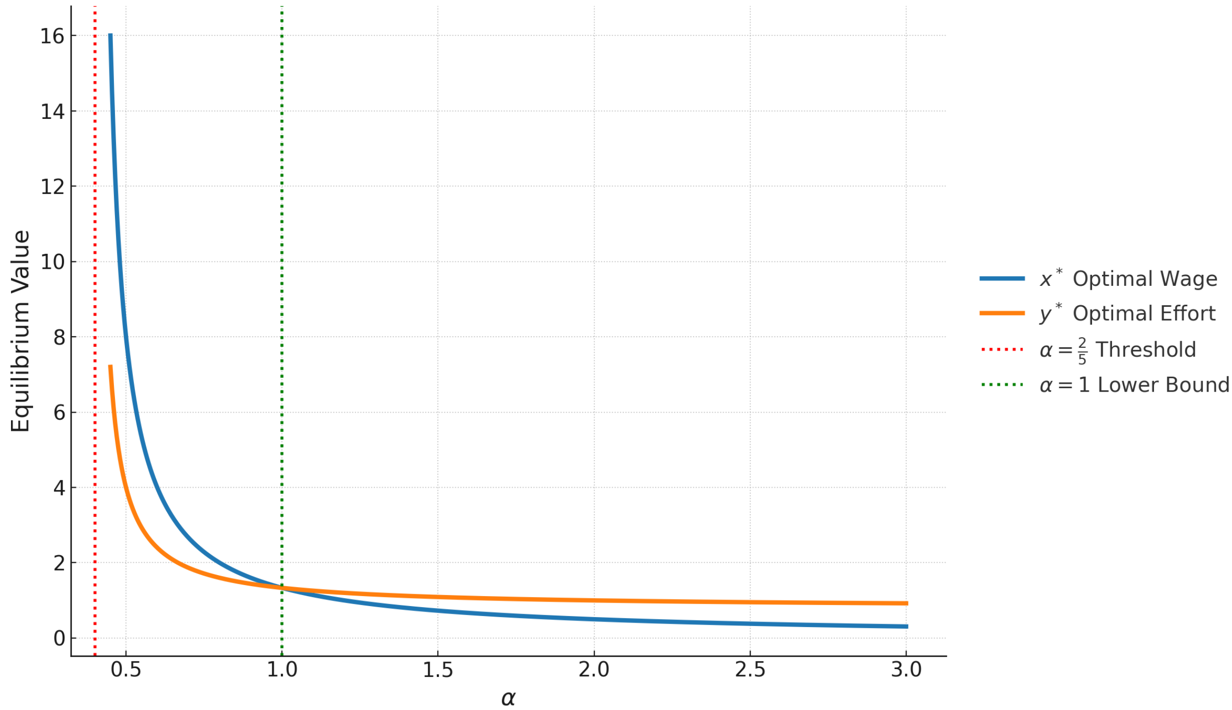
A Nash equilibrium occurs when both players best respond to one another, i.e., when the two expressions for y coincide:

$$\alpha x = \frac{2(x + 2)}{5} \Rightarrow 5\alpha x = 2x + 4 \Rightarrow x(5\alpha - 2) = 4 \Rightarrow x^* = \frac{4}{5\alpha - 2}$$

And, thus:

$$y^* = \alpha x^* = \frac{4\alpha}{5\alpha - 2}$$

This equilibrium exists as long as the denominator is not equal to zero. Moreover, due to the *No Positive Transfers (NPT)* property, the denominator must also be strictly positive, since negative values of x or y would imply the existence of a negative wage or negative effort. Therefore, we have $5\alpha - 2 > 0 \Rightarrow \alpha > \frac{2}{5}$, which is true since we have $\alpha \geq 1$ as a given. Thus, **for all $\alpha \geq 1$, the pair $(x^*, y^*) = \left(\frac{4}{5\alpha-2}, \frac{4\alpha}{5\alpha-2}\right)$ is a Nash equilibrium with pure strategies**.



Problem 5

(i) Pure strategy equilibria with α parameter

We seek the values of α for which the following zero-sum game possesses at least one pure strategy Nash equilibrium.

$$\begin{bmatrix} \alpha & 5 & 6 \\ 3 & 6 & 1 \\ 2 & 3 & 7 \end{bmatrix}$$

For a 2-player zero-sum game, we are always interested in the best among the worst-case scenarios:

- For player 1, this translates to: $v_1 = \max_i \min_j A_{ij}$, which means we take the minimum of each row and select the best among them.
- Similarly, for player 2, we compute: $v_2 = \min_j \max_i A_{ij}$. That is, we find the maximum value in each column and then select the smallest among them.

A pure strategy Nash equilibrium, also known as a saddle point, exists if there is a cell in the matrix that is simultaneously the minimum in its row and the maximum in its column. We examine the profile α in position (1,1). For this cell to satisfy the saddle point condition, it must be the smallest in its row and the largest in its column, which translates to $\alpha \leq 5$ and $\alpha \geq 3$. Therefore, **the game has at least one pure strategy Nash equilibrium if and only if** $3 \leq \alpha \leq 5$.

(ii) Pure & Mixed Strategy Nash Equilibria

We are given the following 2-player game:

	C1	**C2**	**C3**	**C4**
R1	$(0, 0)$	$(5, 2)$	$(3, 4)$	$(6, 5)$
R2	$(2, 6)$	$(3, 5)$	$(5, 3)$	$(1, 0)$

Pure Strategy Nash Equilibria

Player 1 (row player) chooses between R_1 and R_2 and player 2 (column player) chooses between C_1, C_2, C_3, C_4 to maximize their payoff.

If player 2 plays:

- column C1, then best response from player 1 is row R2, with $u_1 = 2$.
- column C2, then best response from player 1 is row R1, with $u_1 = 5$.
- column C3, then best response from player 1 is row R2, with $u_1 = 5$.
- column C4, then best response from player 1 is row R1, with $u_1 = 6$.

If player 1 plays:

- row R1, then best response from player 2 is column C4, with $u_2 = 5$.
- row R2, then best response from player 2 is column C1, with $u_2 = 6$.

A Nash equilibrium occurs where both players are playing their best responses simultaneously. In our game there no weakly or strictly dominant strategies for either of the players, so we will look for mutual best responses. It's obvious that:

- At $(R1, C4)$, R1 is the best response to C4 & C4 is the best response to R1.
- At $(R2, C1)$, R2 is the best response to C1 & C1 is the best response to R2.

These are the only profiles where both players are simultaneously playing best responses. Therefore, the game possesses exactly two pure strategy Nash equilibria:

$$(R1, C4) \quad \& \quad (R2, C1).$$

Mixed Strategy Nash Equilibria

		q_1	q_2	q_3	q_4
		C1	**C2**	**C3**	**C4**
p	**R1**	$(0, 0)$	$(5, 2)$	$(3, 4)$	$(6, 5)$
$1-p$	**R2**	$(2, 6)$	$(3, 5)$	$(5, 3)$	$(1, 0)$

We begin by analyzing a restricted support that involves only the columns C_1 and C_4, since they appear to be player's 2 best responses. Let player 2 assign probability q to C_1 and $1-q$ to C_4, since $q_2 = q_3 = 0$, and player 1 play R_1 with probability p and R_2 with $1-p$.

To make player 1 indifferent between R_1 and R_2, we compute:

$$U_1(R_1) = 0 \cdot q + 6(1-q) = 6(1-q)$$
$$U_1(R_2) = 2q + 1(1-q) = 2q + 1 - q = q + 1$$

Equating $U_1(R_1) = U_1(R_2)$ gives us:

$$6(1-q) = q + 1 \Rightarrow 6 - 6q = q + 1 \Rightarrow 7q = 5 \Rightarrow q = \frac{5}{7}.$$

To ensure player 2 is also indifferent between strategies C_1 and C_4, we equate his expected payoffs:

$$U_2(C_1) = 0 \cdot p + 6(1-p) = 6(1-p)$$
$$U_2(C_4) = 5p + 0 \cdot (1-p) = 5p$$

Where equating $U_2(C_1) = U_2(C_4)$, gives us:

$$6(1-p) = 5p \Rightarrow 6 - 6p = 5p \Rightarrow 11p = 6 \Rightarrow p = \frac{6}{11}.$$

Therefore, **we obtain a mixed strategy Nash equilibrium supported on the subset of columns $\mathbf{C_1}$ *&* $\mathbf{C_4}$**, where both players are indifferent between their own choices and thus have no incentive to unilaterally deviate: $\boldsymbol{(p = (\frac{6}{11}, \frac{5}{11}),\ q = (\frac{5}{7}, 0, 0, \frac{2}{7}))}$

To find a second mixed strategy Nash equilibrium, we will now consider a restricted support that involves only the columns C_2 and C_3. Let player 2 assign probability q_2 to C_2 and $q_3 = 1 - q_2$ to C_3, and player 1 play R_1 with probability p and R_2 with probability $1-p$. In order, to make player 1 indifferent between R_1 and R_2, we compute:

$$U_1(R_1) = 5q_2 + 3q_3$$
$$U_1(R_2) = 3q_2 + 5q_3$$

Equating $U_1(R_1) = U_1(R_2)$, we get:

$$5q_2 + 3q_3 = 3q_2 + 5q_3 \Rightarrow 2q_2 = 2q_3 \Rightarrow q_2 = q_3 = \frac{1}{2}.$$

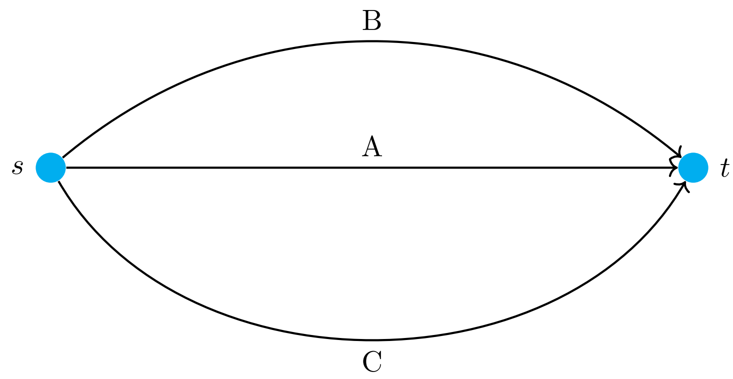
To ensure player 2 is also indifferent between C_2 and C_3, we do the same:

$$U_2(C_2) = 2p + 5(1-p) = -3p + 5 \quad \& \quad U_2(C_3) = 4p + 3(1-p) = p + 3$$

$$U_2(C_2) = U_2(C_3) \Rightarrow -3p + 5 = p + 3 \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}.$$

Therefore, **we have obtained another mixed strategy Nash equilibrium supported on the subset of columns $\mathbf{C_2}$ *&* $\mathbf{C_3}$ where: $\boldsymbol{(p = (\frac{1}{2}, \frac{1}{2}),\ q = (0, \frac{1}{2}, \frac{1}{2}, 0))}$**.

Problem 6



To find a pure strategy Nash equilibrium in this congestion game, we want a profile where no player has an incentive to deviate from their chosen route. Let:

- a be the number of players choosing route A,
- b be the number choosing route B,
- c be the number choosing route C.

Since each route's delay depends linearly on the number of users, the delays experienced by players are:

$$d_A = 5a, \quad d_B = 7.5b, \quad d_C = 10c$$

We will set the utility function of each player $u = -delay$. Each player wants to maximize their utility which translates to minimizing their own delay. As previously discussed, in order to reach a Nash equilibrium, no player should have an incentive to switch routes. This implies that the delays, or the utilities, must be equal across all routes that are used. Consequently, we require that: $5a = 7.5b = 10c$. Also, since the total number of players is fixed, we have an additional constraint which says: $a + b + c = 17$. Therefore, we end up with the following system of equations:

$$\begin{cases} a + b + c = 17 \\ 5a = 7.5b \Rightarrow a = 1.5b \\ 5a = 10c \Rightarrow a = 2c \end{cases}$$

By solving the subsystem $a = 1.5b$ *&* $a = 2c$, we identify the first three integer solutions, ordered by the total number of players $a + b + c$:

$$(a, b, c) = (0,\ 0,\ 0), where\ a + b + c = 0$$
$$(a, b, c) = (6,\ 4,\ 3), where\ a + b + c = 13$$
$$(a, b, c) = (12,\ 8,\ 6), where\ a + b + c = 26$$

Each of these corresponds to a distribution of players across the three routes such that the delay experienced - or utility achieved - on each route is identical.

However, as shown above, there is no integer solution to the subsystem where $a+b+c=17$. It follows that, in the case of 17 players, there is no possible allocation among the three routes that yields equal delays for all players. The closest profile that achieves equal delays across all routes is when $a=6, b=4, c=3$ where $a+b+c=13$, leading to delays of 30 on each path. However, this requires only 13 players. Since 17 is not divisible in a way that balances delays equally across all routes, any integer partition of 17 players results in at least one route having strictly lower delay, motivating players to deviate. Thus, there is no pure strategy Nash equilibrium for the case of 17 players where all players simultaneously have no incentive to deviate.

Problem 7

(i) Optimal ϵ for given strategy profile

		$q_1=1/2$ C1	$q_2=1/2$ C2
$p_1=1/2$	R1	$\left(\frac{1}{2}-\delta, \delta\right)$	$\left(0, \frac{1}{2}-\delta\right)$
$p_2=1/2$	R2	$\left(\delta, \frac{1}{2}-\delta\right)$	$\left(\frac{1}{2}-\delta, \delta\right)$

A strategy profile (p, q) is an ϵ-Nash equilibrium if no player can improve their payoff by more than ϵ through unilaterally deviating. Therefore, ϵ is given by the maximum gain achievable by either player through such a deviation:

$$\epsilon = \max(\text{Player 1 best response gain, Player 2 best response gain})$$

Player 1 has the following payoff matrix:

$$A = \begin{bmatrix} \frac{1}{2}-\delta & 0 \\ \delta & \frac{1}{2}-\delta \end{bmatrix}$$

We want to find the smallest $\epsilon \geq 0$ such that this profile is an ϵ-Nash equilibrium. The expected payoff for Player 1 under the strategy profile (p, q) is:

$$u_1(p,q) = p^\top A q = \frac{1}{2}\left(\frac{1}{4} - \frac{\delta}{2} + \frac{1}{4}\right) = \frac{1-\delta}{4}$$

Evaluating player's 1 utility if they unilaterally deviate to a pure strategy, we get that his expected utility is $u_1 = \frac{1}{2}(\frac{1}{2}-\delta)+\frac{1}{2}\cdot 0 = \frac{1}{4}-\frac{\delta}{2}$ if he chooses $R1$ and $u_1 = \frac{1}{2}\cdot\delta+\frac{1}{2}(\frac{1}{2}-\delta) = \frac{1}{4}$ if he choose $R2$. The best response is row 2, with utility $\frac{1}{4}$. So the regret - or gain in outcome from deviating - for player 1 is: $\epsilon_1 = \frac{1}{4} - \frac{1-\delta}{4} = \frac{\delta}{4}$

Since our game is symmetric, player 2 also has the same regret. Thus, **the smallest ϵ such that $(p,q) = \left(\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\right)$ is an ϵ-Nash equilibrium is:** $\epsilon(\delta) = \frac{\delta}{4}$

(ii) Uniform Strategy in k-Sparse Games

We have two $n \times n$ payoff matrices A *&* B with values in $[0, 1]$ and each row and column has at most k non-zero entries, so our game is k-sparse. We aim to show that the profile (p, p), where p is the uniform mixed strategy $p = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, constitutes an ϵ-Nash equilibrium for all $\epsilon \geq \frac{k}{n}$. In other words, no player can improve their expected payoff by more than ϵ through unilateral deviation.

If player 1 deviates from p to some pure strategy $i \in \{1, \dots, n\}$, then their expected utility shall become:

$$u_1(i, p) = \sum_{j=1}^{n} A_{i,j} \cdot \frac{1}{n}$$

Since at most k entries in row i are non-zero and all payoffs lie in the interval $[0, 1]$, the maximum possible value of each non-zero entry is 1 and the total expected utility from pure strategy i is bounded above by:

$$u_1(i, p) \leq \frac{k}{n}$$

The utility player 1 gets from playing the uniform strategy p against p is:

$$u_1(p, p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n^2} A_{i,j}$$

As one can easily observe, the above is simply the average of all entries in A, which are non-negative, so: $u_1(p, p) \geq 0$. Thus, the gain from deviating - regret - for player 1 is:

$$u_1(i, p) - u_1(p, p) \leq \frac{k}{n}$$

Symmetrically, the same is true for player 2, so no player can gain more than $\frac{k}{n}$ by deviating and **for all $\epsilon \geq \frac{\mathbf{k}}{\mathbf{n}}$, the uniform strategy profile (p, p) is an ϵ-Nash equilibrium.**