National Technical University of Athens
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Game Theory
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Problems

Problem 1 $\mathcal{VCG}$

We consider a Vickrey–Clarke–Groves (VCG) auction involving three bidders, denoted by 1, 2 and 3 and a set of three items $M = \{a, b, c\}$. Each bidder i has a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$. The valuations for each bidder are given:

S	$v_1(S)$	$v_2(S)$	$v_3(S)$
$\emptyset$	0	0	0
$\{a\}$	1	0	0.5
$\{b\}$	0	0	0.5
$\{c\}$	0	0	1.5
$\{a, b\}$	1	2	1
$\{a, c\}$	1	1	2
$\{b, c\}$	0	1	2
$\{a, b, c\}$	2	3	3

Table 1: Valuation functions v_1, v_2, v_3 over all subsets of $M = \{a, b, c\}$

1.1 Socially Optimal Allocation

An allocation $\mathcal{A} = (S_1, S_2, S_3)$ is a distribution of the items in M among the bidders such that $S_i \subseteq M$ for each $i \in \{1, 2, 3\}$, the sets S_1, S_2, S_3 are pairwise disjoint thus $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{3} S_i \subseteq M$. A socially optimal allocation is one that satisfies these conditions while maximizing the total valuation:

$$\mathrm{SW}(\mathcal{A}) = \sum_{i=1}^{3} v_i(S_i).$$

That is, $\mathcal{A}$ defines a partition of the item set M into three disjoint bundles, one for each agent. Note that we do not impose any constraint on the cardinality of S_i as agents may receive arbitrary-sized bundles, including the empty set. So 2^M denotes the power set of M, thus the set of all subsets of M:

$$2^M = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

and $(2^M)^3$ is the set of all ordered triples (S_1, S_2, S_3) with $S_i \subseteq M$, representing arbitrary assignments of subsets to the three bidders. The subset $\mathcal{F} \subseteq (2^M)^3$ consists of all feasible disjoint partitions of M, thus the ones that satisfy $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{3} S_i = M$. In other words, each item is allocated to exactly one bidder, meaning that no item is left unassigned and no item is simultaneously assigned to more than one bidder. The objective is to determine an allocation $\mathcal{A}^* \in \mathcal{F}$ such that:

$$\mathcal{A}^* \in \arg\max_{\mathcal{A} \in \mathcal{F}} \mathrm{SW}(\mathcal{A}).$$

where $\mathcal{A}^*$ is a feasible allocation from the set $\mathcal{F}$ that maximizes the social welfare among all possible allocations. Since $|M| = 3$, the set of all allocations that assign each item to exactly one bidder corresponds to the set of functions $f : M \to N$. Each such function induces an allocation (S_1, S_2, S_3), where $S_i = f^{-1}(i)$, forming a disjoint and exhaustive partition of M

across the bidders. Thus, the total number of feasible allocations $|\mathcal{F}|$ is $3^3 = 27$.

We evaluate $\mathrm{SW}(\mathcal{A})$ for each feasible allocation $\mathcal{A} = (S_1, S_2, S_3) \in \mathcal{F}$, where $S_i = f^{-1}(i)$ for some function $f : M \to N$ as previously described. The total welfare is computed as $\sum_{i=1}^{3} v_i(S_i)$. We now examine a selection of allocations that are promising candidates for maximizing social welfare, selected based on high-value bundles in the valuation table.

Promising Allocations

Allocation	Bundle Assignment	Valuations	SW
$\mathcal{A}_1$	$(\{a\}, \{b\}, \{c\})$	$v_1 = 1,\ v_2 = 0,\ v_3 = 1.5$	2.5
$\mathcal{A}_2$	$(\emptyset, \{a, b\}, \{c\})$	$v_1 = 0,\ v_2 = 2,\ v_3 = 1.5$	3.5
$\mathcal{A}_3$	$(\emptyset, \{a, c\}, \{b\})$	$v_1 = 0,\ v_2 = 1,\ v_3 = 0.5$	1.5
$\mathcal{A}_4$	$(\emptyset, \{a, b, c\}, \emptyset)$	$v_2 = 3$	3.0
$\mathcal{A}_5$	$(\{a, b\}, \emptyset, \{c\})$	$v_1 = 1,\ v_2 = 0,\ v_3 = 1.5$	2.5
$\mathcal{A}_6$	$(\emptyset, \{b, c\}, \{a\})$	$v_1 = 0,\ v_2 = 1,\ v_3 = 0.5$	1.5
$\mathcal{A}_7$	$(\emptyset, \{a\}, \{b, c\})$	$v_2 = 0,\ v_3 = 2$	2.0
$\mathcal{A}_8$	$(\{a, c\}, \{b\}, \emptyset)$	$v_1 = 1,\ v_2 = 0$	1.0

Among all evaluated allocations, $\mathcal{A}_2 = (\emptyset, \{a, b\}, \{c\})$ yields the highest social welfare of 3.5. This outcome exploits bidder 2's complementarity over $\{a, b\}$ and bidder 3's strong valuation for $\{c\}$, while excluding bidder 1, whose marginal contribution is strictly lower. We therefore conclude that the socially optimal allocation $\mathcal{A}^*$ is $\mathcal{A}_2$ with $\mathrm{SW}(\mathcal{A}_2) = 3.5$.

In contrast, the highest social welfare achievable under bijective allocations, where each bidder receives exactly one item, is achieved by $\mathcal{A}_1$, with $\mathrm{SW}(\mathcal{A}_1) = 2.5$. Therefore, the general allocation $\mathcal{A}^*$ yields strictly higher welfare and thus dominates all bijective allocations in terms of efficiency. This outcome reflects bidder 2's complementarity over the bundle $\{a, b\}$, which cannot be realized under bijective allocations.

1.2 $\mathcal{VCG}$ Payments

Under the $\mathcal{VCG}$ mechanism, each bidder pays an amount equal to their externality imposed on the others. More specifically, the difference between the maximum welfare achievable by the other players when the bidder is excluded and the welfare actually achieved by them under the optimal allocation. Let $\mathcal{A}^*$ denote the optimal allocation and S_j^* the bundle allocated to player j in $\mathcal{A}^*$. Then the VCG payment for player i is given by:

$$p_i = \max_{\mathcal{A}_{-i}} \sum_{j \neq i} v_j(S_j) - \sum_{j \neq i} v_j(S_j^*), \tag{1}$$

where the maximum is taken over all feasible allocations of the items to players $\{1, 2, 3\} \setminus \{i\}$.

1.2.1 Player 1

Excluding player 1, the remaining players are $\{2, 3\}$ and the set of items remains $M = \{a, b, c\}$. We compute the VCG payment for player 1 by first determining the maximum welfare attainable by players 2 and 3 in the absence of player 1. We evaluate all 8 feasible partitions of $M = \{a, b, c\}$ between bidders 2 and 3 such that their bundles are disjoint.

Feasible Allocations Excluding Player 1

Label	Bundle Assignment (2, 3)	Valuations	SW
$\mathcal{A}_1$	$(\{a\}, \{b, c\})$	$v_2 = 0,\ v_3 = 2$	2.0
$\mathcal{A}_2$	$(\{b\}, \{a, c\})$	$v_2 = 0,\ v_3 = 2$	2.0
$\mathcal{A}_3$	$(\{c\}, \{a, b\})$	$v_2 = 0,\ v_3 = 1$	1.0
$\mathcal{A}_4$	$(\{a, b\}, \{c\})$	$v_2 = 2,\ v_3 = 1.5$	3.5
$\mathcal{A}_5$	$(\{a, c\}, \{b\})$	$v_2 = 1,\ v_3 = 0.5$	1.5
$\mathcal{A}_6$	$(\{b, c\}, \{a\})$	$v_2 = 1,\ v_3 = 0.5$	1.5
$\mathcal{A}_7$	$(\{a, b, c\}, \emptyset)$	$v_2 = 3,\ v_3 = 0$	3.0
$\mathcal{A}_8$	$(\emptyset, \{a, b, c\})$	$v_2 = 0,\ v_3 = 3$	3.0

Under the optimal allocation $\mathcal{A}_4$, players 2 and 3 receive $\{a, b\}$ and $\{c\}$ respectively, yielding:

$$\max_{\mathcal{A}_{-1}} \sum_{j \neq 1} v_j(S_j) = \sum_{j \neq 1} v_j(S_j^*) = v_2(\{a, b\}) + v_3(\{c\}) = 2 + 1.5 = 3.5$$

Player 1 neither receives any items in the optimal allocation nor alters the welfare attainable by the remaining players. Consequently, his presence does not impose any externality on the system and by equation (1), his VCG payment is $p_1 = 3.5 - 3.5 = 0$. Thus, he receives no goods and incurs no payment.

1.2.2 Player 2

Excluding player 2, the remaining players are $\{1, 3\}$ and the item set remains $M = \{a, b, c\}$. We compute the VCG payment for player 2 by evaluating all 8 feasible partitions of the items between players 1 and 3, ensuring disjoint bundles.

Feasible Allocations Excluding Player 2

Label	Bundle Assignment (1, 3)	Valuations	SW
$\mathcal{A}_1$	$(\{a\}, \{b, c\})$	$v_1 = 1,\ v_3 = 2$	3.0
$\mathcal{A}_2$	$(\{b\}, \{a, c\})$	$v_1 = 0,\ v_3 = 2$	2.0
$\mathcal{A}_3$	$(\{c\}, \{a, b\})$	$v_1 = 0,\ v_3 = 1$	1.0
$\mathcal{A}_4$	$(\{a, b\}, \{c\})$	$v_1 = 1,\ v_3 = 1.5$	2.5
$\mathcal{A}_5$	$(\{a, c\}, \{b\})$	$v_1 = 1,\ v_3 = 0.5$	1.5
$\mathcal{A}_6$	$(\{b, c\}, \{a\})$	$v_1 = 0,\ v_3 = 0.5$	0.5
$\mathcal{A}_7$	$(\{a, b, c\}, \emptyset)$	$v_1 = 2,\ v_3 = 0$	2.0
$\mathcal{A}_8$	$(\emptyset, \{a, b, c\})$	$v_1 = 0,\ v_3 = 3$	3.0

The maximum social welfare achievable in the absence of player 2 is attained under both allocations $\mathcal{A}_1$ and $\mathcal{A}_8$, each yielding:

$$\max_{\mathcal{A}_{-2}} \sum_{j \neq 2} v_j(S_j) = \max\{v_1(\{a\}) + v_3(\{c\}),\ v_1(\emptyset) + v_3(\{a, b, c\})\} = \max\{1 + 2,\ 0 + 3\} = 3.0.$$

In the original optimal allocation $\mathcal{A}^* = (\emptyset, \{a, b\}, \{c\})$, players 1 and 3 are allocated $\emptyset$ and $\{c\}$, respectively, resulting in a realized welfare contribution of:

$$\sum_{j \neq 2} v_j(S_j^*) = v_1(\emptyset) + v_3(\{c\}) = 0 + 1.5 = 1.5.$$

Consequently, the externality imposed by player 2 on the remaining participants is 1.5. By equation (1), the corresponding VCG payment is:

$$p_2 = 3.0 - 1.5 = 1.5.$$

1.2.3 Player 3

Excluding player 3, the remaining players are $\{1,2\}$ and the item set remains $M = \{a,b,c\}$. We compute the VCG payment for player 3 by evaluating all 8 feasible partitions of the items between players 1 and 2, ensuring disjoint bundles.

Feasible Allocations Excluding Player 3

Label	Bundle Assignment (1, 2)	Valuations	SW
$\mathcal{A}_1$	$(\{a\},\{b,c\})$	$v_1 = 1,\ v_2 = 1$	2.0
$\mathcal{A}_2$	$(\{b\},\{a,c\})$	$v_1 = 0,\ v_2 = 1$	1.0
$\mathcal{A}_3$	$(\{c\},\{a,b\})$	$v_1 = 0,\ v_2 = 2$	2.0
$\mathcal{A}_4$	$(\{a,b\},\{c\})$	$v_1 = 1,\ v_2 = 0$	1.0
$\mathcal{A}_5$	$(\{a,c\},\{b\})$	$v_1 = 1,\ v_2 = 0$	1.0
$\mathcal{A}_6$	$(\{b,c\},\{a\})$	$v_1 = 0,\ v_2 = 0$	0.0
$\mathcal{A}_7$	$(\{a,b,c\},\emptyset)$	$v_1 = 2,\ v_2 = 0$	2.0
$\mathcal{A}_8$	$(\emptyset,\{a,b,c\})$	$v_1 = 0,\ v_2 = 3$	3.0

The maximum social welfare achievable in the absence of player 3 is obtained under allocation $\mathcal{A}_8$, where player 1 receives nothing and player 2 receives the entire set:

$$\max_{\mathcal{A}_{-3}} \sum_{j\neq 3} v_j(S_j) = v_1(\emptyset) + v_2(\{a,b,c\}) = 0 + 3 = 3.0.$$

In the original optimal allocation $\mathcal{A}^* = (\emptyset, \{a,b\}, \{c\})$, players 1 and 2 are allocated $\emptyset$ and $\{a,b\}$, respectively, yielding:

$$\sum_{j\neq 3} v_j(S_j^*) = v_1(\emptyset) + v_2(\{a,b\}) = 0 + 2 = 2.0.$$

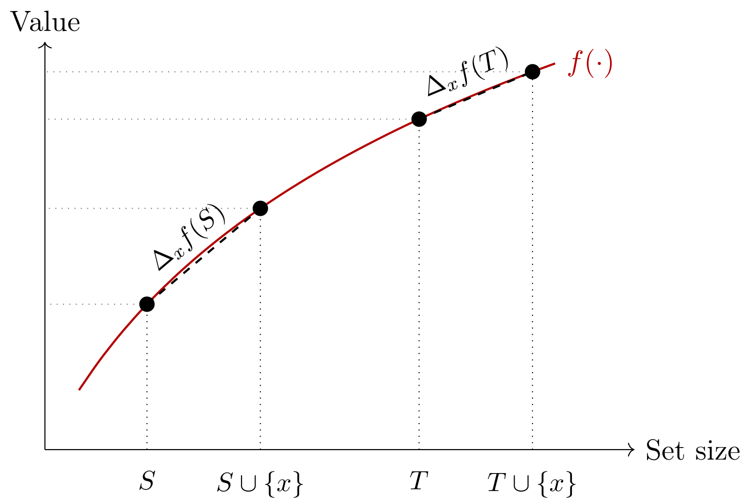
Therefore, the externality imposed by player 3 on the remaining participants is 1.0. By equation (1), the corresponding VCG payment is:

$$p_3 = 3.0 - 2.0 = 1.0.$$

1.3 Submodularity of Valuation Functions

We investigate whether the valuation functions $v_1, v_2, v_3 : 2^M \to \mathbb{R}_{\geq 0}$ exhibit submodularity. A set function $f : 2^M \to \mathbb{R}$, where 2^M is the power set of a finite ground set M, is called *submodular* if it satisfies a natural condition of decreasing incremental returns. Formally, f is submodular if for all subsets $S \subseteq T \subseteq M$ and for every element $x \in M \setminus T$, the marginal increase in value from adding x to the smaller set S is at least as large as the increase from adding it to the larger set T. That is

$$f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T) \tag{2}$$



Submodularity Illustration

This inequality expresses the property of diminishing marginal returns, whereby the marginal contribution of an element x weakly decreases as the reference set grows. In our context of valuation functions, submodularity holds if the marginal value of acquiring an additional item does not increase as more items are already held by the bidder. This condition excludes the presence of complementarities, ensuring that the valuation grows in a concave manner over the lattice of subsets. We proceed to verify whether the functions v_1, v_2, v_3 satisfy this property based on the provided valuations.

1.3.1 Valuation Function v_1

We verify the submodularity condition by identifying explicit counterexamples that violate the inequality (2) for some sets $A \subseteq B \subseteq M$ and element $x \in M \setminus B$. First, let $A = \{a\}$, $B = \{a, b\}$ and $x = c$. Clearly, $A \subseteq B \subseteq M$ and $x \notin B$. From the valuation table 1 we know that:

$$v_1(A \cup \{x\}) - v_1(A) = v_1(\{a, c\}) - v_1(\{a\}) = 1 - 1 = 0,$$
$$v_1(B \cup \{x\}) - v_1(B) = v_1(\{a, b, c\}) - v_1(\{a, b\}) = 2 - 1 = 1.$$

This yields $v_1(A \cup \{x\}) - v_1(A) = 0 \not\geq 1 = v_1(B \cup \{x\}) - v_1(B)$ which directly contradicts the submodularity inequality.

A second violation arises by selecting $A = \{a\}$, $B = \{a, c\}$ and $x = b$, which satisfy the conditions $A \subseteq B \subseteq M$ and $x \in M \setminus B$. From the valuation table 1, we have:

$$v_1(A \cup \{x\}) - v_1(A) = v_1(\{a, b\}) - v_1(\{a\}) = 1 - 1 = 0,$$
$$v_1(B \cup \{x\}) - v_1(B) = v_1(\{a, b, c\}) - v_1(\{a, c\}) = 2 - 1 = 1.$$

This yields a strict violation of the submodularity equation (2), since $0 \not\geq 1$. In both examples we proposed, the marginal gain from adding x to the smaller set A is actually less than that from adding it to the larger set B. Consequently, the valuation function $\boldsymbol{v_1}$ **is not submodular**.

1.3.2 Valuation Function v_2

We assess whether $v_2 : 2^M \to \mathbb{R}_{\geq 0}$ satisfies the submodularity condition (2) by searching for explicit violations.

Consider the sets $A = \{c\}$, $B = \{b, c\}$ and $x = a$, where clearly $A \subseteq B \subseteq M$ and $x \notin B$. Using the valuation table 1, we compute:

$$v_2(A \cup \{x\}) - v_2(A) = v_2(\{a, c\}) - v_2(\{c\}) = 1 - 0 = 1,$$
$$v_2(B \cup \{x\}) - v_2(B) = v_2(\{a, b, c\}) - v_2(\{b, c\}) = 3 - 1 = 2.$$

Since $1 \ngeq 2$, the inequality (2) is violated and we can safely conclude that $\boldsymbol{v_2}$ **is not submodular**.

1.3.3 Valuation Function v_3

We now examine whether the valuation function $v_3 : 2^M \to \mathbb{R}_{\geq 0}$ satisfies the submodularity condition (2). Consider the sets $A = \{a\}$, $B = \{a, b\}$ and $x = c$. From the valuation table 1, we compute:

$$v_3(A \cup \{x\}) - v_3(A) = v_3(\{a, c\}) - v_3(\{a\}) = 2 - 0.5 = 1.5,$$
$$v_3(B \cup \{x\}) - v_3(B) = v_3(\{a, b, c\}) - v_3(\{a, b\}) = 3 - 1 = 2.$$

Since $1.5 \ngeq 2$, the condition (2) is violated and we can safely conclude that $\boldsymbol{v_3}$ **is not submodular**.

Conclusion

Based on our analysis, we conclude that none of the valuation functions v_1, v_2, or v_3 are submodular. This conclusion follows explicit counterexamples that violate the submodularity condition (2), which requires that the marginal value of adding an item to a set does not increase as the set grows. Each valuation function exhibits *complementarity*, whereby the value assigned to a bundle of items exceeds the sum of the values of its individual elements. This characteristic directly contradicts the *diminishing marginal returns* property required for submodularity. For instance, in the case of v_2, we observe that

$$v_2(\{a, b\}) = 2 \quad > \quad v_2(\{a\}) + v_2(\{b\}) = 0 + 0 = 0,$$

indicating that the combined acquisition of items a and b yields strictly greater utility than acquiring them separately. Such behavior indicates that items are valued more together than separately, a defining trait of complementary valuations. This reflects superadditivity, where the value of a union of disjoint sets exceeds the sum of their individual values.

Problem 2 Truthful Payments under Greedy Allocation

In mechanism design, allocating m items to n players with valuation functions $v_j : 2^{[m]} \to \mathbb{R}_{\geq 0}$ typically aims to maximize social welfare. Our approach is the greedy allocation algorithm, which assigns items sequentially to the player with the highest marginal value at each step. We investigate whether there exist payment rules that render this algorithm truthful under the assumption that valuations are submodular. However, since the greedy allocation rule is not monotone in general, we are led to suspect that no payment rule can render the mechanism truthful. This observation suggests that the greedy algorithm may inherently violate the necessary condition for dominant-strategy incentive compatibility.

As we have already stated, a valuation function v_j is submodular if for all $S \subseteq T \subseteq [m]$ and item $i \notin T$, the marginal gain satisfies $v_j(S \cup \{i\}) - v_j(S) \geq v_j(T \cup \{i\}) - v_j(T)$. The greedy algorithm operates by sequentially allocating items to players based on marginal valuations. It begins by initializing empty bundles $S_1, \ldots, S_n$ for each player, such that $S_j = \emptyset$ for all $j \in [n]$. Then, for each item $i \in [m]$, the algorithm selects the player $j \in [n]$ who experiences the highest marginal increase in value from receiving item i. Thus, player j is chosen according to the rule:

$$j = \arg\max_{\ell \in [n]} \{v_\ell(S_\ell \cup \{i\}) - v_\ell(S_\ell)\}.$$

The selected player's bundle is then updated by including item i, that is, $S_j \leftarrow S_j \cup \{i\}$. The process carries on until all items have been allocated. A mechanism is truthful if reporting true valuations maximizes each player's utility $u_j = v_j(S_j) - p_j$, where p_j is the payment.

2.1 Counterexample

We consider $n = 2$ players and $m = 3$ items $\{a, b, c\}$ and define submodular valuations making sure we have decreasing marginal gains.

Set	$v_1(S)$	$v_2(S)$
$\emptyset$	0	0
$\{a\}$	2	1
$\{b\}$	2	1
$\{c\}$	1	1
$\{a, b\}$	3	1.5
$\{a, c\}$	2.5	1.5
$\{b, c\}$	2.5	1.5
$\{a, b, c\}$	3.5	1.8

Table 2: Valuation functions v_1 and v_2 for all subsets of $\{a, b, c\}$

Algorithm 1 Greedy Allocation Algorithm

Input: Items $[m] = \{1, 2, \ldots, m\}$, players $[n] = \{1, 2, \ldots, n\}$, valuations $\{v_j : 2^{[m]} \to \mathbb{R}_{\geq 0}\}_{j \in [n]}$, item order $\pi : [m] \to [m]$

Output: Allocation $\{S_1, S_2, \ldots, S_n\}$, where $S_j \subseteq [m]$ for each player j

```
Initialize S_j ← ∅ for all j ∈ [n]
for each item i in order π(1), π(2), ..., π(m) do
    Compute marginal gains: Δ_j ← v_j(S_j ∪ {i}) − v_j(S_j) for all j ∈ [n]
    j* ← argmax_{j∈[n]} Δ_j    ▷ Break ties deterministically, e.g., favor lower-indexed player
    S_{j*} ← S_{j*} ∪ {i}
end for
return {S_1, S_2, ..., S_n}
```

We now simulate the execution of the greedy allocation algorithm under the valuation functions v_1 and v_2 as defined in Table 2. The algorithm proceeds by iteratively assigning each item to the player for whom the item yields the highest marginal increase in value, defined as:

$$\Delta_j(i) = v_j(S_j \cup \{i\}) - v_j(S_j),$$

where S_j denotes the current bundle allocated to player j and $i \in M$ is the item under consideration. Let us fix the item order as $\pi = (a, b, c)$ and initialize both bundles as empty: $S_1 = S_2 = \emptyset$. The allocation shall proceed as follows:

Item a. At the first iteration, both players begin with empty bundles. The marginal values of assigning item a are computed as:

$$\Delta_1(a) = v_1(\{a\}) - v_1(\emptyset) = 2 - 0 = 2 \quad \& \quad \Delta_2(a) = v_2(\{a\}) - v_2(\emptyset) = 1 - 0 = 1.$$

Since player 1 has the highest marginal valuation for item a, the item is assigned to him. As a result, the bundles are updated to $S_1 = \{a, b\}$ and $S_2 = \emptyset$.

Item b. In the second step, item b is considered and current allocation is $S_1 = \{a\}$, $S_2 = \emptyset$. The marginal valuations are:

$$\Delta_1(b) = v_1(\{a, b\}) - v_1(\{a\}) = 3 - 2 = 1 \quad \& \quad \Delta_2(b) = v_2(\{b\}) - v_2(\emptyset) = 1 - 0 = 1.$$

Here, both players exhibit equal marginal gain. We assume the tie-breaking rule favors the player with the smaller index. Thus, player 1 receives item b. Consequently, the state of the allocation is updated to $S_1 = \{a, b\}, \quad S_2 = \emptyset$.

Item c. Finally, we consider item c and the bundles are currently $S_1 = \{a, b\}$, $S_2 = \emptyset$. We compute:

$$\Delta_1(c) = v_1(\{a, b, c\}) - v_1(\{a, b\}) = 3.5 - 3 = 0.5 \quad \& \quad \Delta_2(c) = v_2(\{c\}) - v_2(\emptyset) = 1 - 0 = 1,$$

where player 2 achieves a strictly higher marginal benefit from item c and is therefore assigned this item. Ultimately, the algorithm allocates $S_1 = \{a, b\}$ to Player 1, with value $v_1(\{a, b\}) = 3$ & utility $u_1 = 3 - p_1$ and $S_2 = \{c\}$ to Player 2, with value $v_2(\{c\}) = 1$ & utility $u_2 = 1 - p_2$.

2.1.1 Misreporting by Player 1

Suppose Player 1 deviates from their true valuation v_1 and instead reports a submodular valuation function v_1'.

Set	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\boldsymbol{v_1(S)}$	0	2	2	1	3	2.5	2.5	3.5
$\boldsymbol{v_1'(S)}$	0	1.5	1.5	0.8	2.5	2	2	2.7

Table 3: Misreported valuation v_1' compared to true valuation v_1

To verify that the reported valuation function v_1' is submodular, we test the submodularity condition directly. Recall that a function $f : 2^M \to \mathbb{R}$ is submodular if for all $A \subseteq B \subseteq M$ and $x \in M \setminus B$, it holds that $f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$. We will exhaustively verify the submodularity condition over the ground set $M = \{a, b, c\}$ for all $A \subseteq B \subseteq M$ and $x \in M \setminus B$, using the reported valuation function v_1'.

We first consider the sets $A = \emptyset$, $B = \{a\}$ and element $x = b$. The marginal gain from adding b to A is $v_1'(\{b\}) - v_1'(\emptyset) = 1.5$, while the marginal gain from adding b to B is $v_1'(\{a,b\}) - v_1'(\{a\}) = 2.5 - 1.5 = 1.0$. Since $1.5 \geq 1.0$, the inequality (2) holds. As another example, for $A = \emptyset$, $B = \{a\}$ and $x = c$, we compute $v_1'(\{c\}) - v_1'(\emptyset) = 0.8$ and $v_1'(\{a,c\}) - v_1'(\{a\}) = 2 - 1.5 = 0.5$, again satisfying the condition (2). Given that the ground set contains three items, we have exactly twelve valid combinations of sets $A \subseteq B \subseteq M$ and elements $x \in M \setminus B$ to test. To maintain brevity, we will summarize our findings in the matrix below.

Set A	**Set** B	**Item** x	$\Delta_x v_1'(A)$	$\Delta_x v_1'(B)$
$\emptyset$	$\{a\}$	b	$v_1'(\{b\}) - v_1'(\emptyset) = 1.5$	$v_1'(\{a,b\}) - v_1'(\{a\}) = 2.5 - 1.5 = 1$
$\emptyset$	$\{a\}$	c	$v_1'(\{c\}) - v_1'(\emptyset) = 0.8$	$v_1'(\{a,c\}) - v_1'(\{a\}) = 2 - 1.5 = 0.5$
$\emptyset$	$\{b\}$	a	$v_1'(\{a\}) - v_1'(\emptyset) = 1.5$	$v_1'(\{a,b\}) - v_1'(\{b\}) = 2.5 - 1.5 = 1$
$\emptyset$	$\{b\}$	c	0.8	$v_1'(\{b,c\}) - v_1'(\{b\}) = 2 - 1.5 = 0.5$
$\emptyset$	$\{c\}$	a	1.5	$v_1'(\{a,c\}) - v_1'(\{c\}) = 2 - 0.8 = 1.2$
$\emptyset$	$\{c\}$	b	1.5	$v_1'(\{b,c\}) - v_1'(\{c\}) = 2 - 0.8 = 1.2$
$\{a\}$	$\{a,b\}$	c	$v_1'(\{a,c\}) - v_1'(\{a\}) = 0.5$	$v_1'(\{a,b,c\}) - v_1'(\{a,b\}) = 2.7 - 2.5 = 0.2$
$\{a\}$	$\{a,c\}$	b	$v_1'(\{a,b\}) - v_1'(\{a\}) = 1$	$v_1'(\{a,b,c\}) - v_1'(\{a,c\}) = 2.7 - 2 = 0.7$
$\{b\}$	$\{a,b\}$	c	$v_1'(\{b,c\}) - v_1'(\{b\}) = 0.5$	$v_1'(\{a,b,c\}) - v_1'(\{a,b\}) = 0.2$
$\{c\}$	$\{a,c\}$	b	$v_1'(\{b,c\}) - v_1'(\{c\}) = 1.2$	$v_1'(\{a,b,c\}) - v_1'(\{a,c\}) = 0.7$
$\{b\}$	$\{b,c\}$	a	$v_1'(\{a,b\}) - v_1'(\{b\}) = 1$	$v_1'(\{a,b,c\}) - v_1'(\{b,c\}) = 2.7 - 2 = 0.7$
$\{c\}$	$\{b,c\}$	a	$v_1'(\{a,c\}) - v_1'(\{c\}) = 1.2$	$v_1'(\{a,b,c\}) - v_1'(\{b,c\}) = 0.7$

Table 4: Exhaustive verification of submodularity condition for v_1'

In every case listed above, the marginal gain of adding x to a smaller set A is actually equal to the marginal gain of adding x to a larger set B. Therefore v_1' is submodular over M.

We now simulate the execution of the greedy allocation algorithm using the misreported valuation v_1' from Table 3 and the true valuation v_2 from Table 2. As before, the algorithm proceeds by sequentially assigning items to the player with the highest marginal value, we assume items arrive in the fixed order $\pi = (a, b, c)$ and both bundles are initially empty: $S_1 = S_2 = \emptyset$. The allocation unfolds as follows:

Item a. At the initial step, both players have empty allocations and the marginal gains are:

$$\Delta_1(a) = v_1'(\{a\}) - v_1'(\emptyset) = 1.5 - 0 = 1.5 \quad \& \quad \Delta_2(a) = v_2(\{a\}) - v_2(\emptyset) = 1 - 0 = 1.$$

Player 1 has the higher marginal gain and is therefore allocated item a. As a result, the bundles are updated to $S_1 = \{a\}, \quad S_2 = \emptyset$.

Item b. At this point, the current allocation is $S_1 = \{a\}$, $S_2 = \emptyset$ and the marginal gains are computed as:

$$\Delta_1(b) = v_1'(\{a, b\}) - v_1'(\{a\}) = 2.5 - 1.5 = 1.0 \quad \& \quad \Delta_2(b) = v_2(\{b\}) - v_2(\emptyset) = 1 - 0 = 1.$$

Both players have equal marginal gain, but assume the tie-breaking rule favors Player 1. Thus, item b is assigned to them and the bundles become $S_1 = \{a, b\}$, $S_2 = \emptyset$.

Item c. Now, the bundles are $S_1 = \{a, b\}$, $S_2 = \emptyset$ and we evaluate the marginal valuations:

$$\Delta_1(c) = v_1'(\{a, b, c\}) - v_1'(\{a, b\}) = 2.7 - 2.5 = 0.2 \quad \& \quad \Delta_2(c) = v_2(\{c\}) - v_2(\emptyset) = 1 - 0 = 1.$$

Since player 2 has the greater marginal value, item c is assigned to them. The final allocation is $S_1 = \{a, b\}$, $S_2 = \{c\}$.

Eventually, Player 1, under the misreport v_1', receives items $\{a, b\}$. Evaluated using their true valuation v_1, this gives a value of $v_1(\{a, b\}) = 3$. Hence, Player 1's utility from misreporting is $u_1' = 3 - p_1'$, where p_1' is the payment required under the mechanism. Player 2 receives bundle $\{c\}$, with value $v_2(\{c\}) = 1$ and utility $u_2' = 1 - p_2'$. Hereby we demonstrated how the greedy algorithm may admit profitable deviations through misreports, even when the reported valuations remain submodular.

Conclusion

When Player 1 misreports their valuation as v_1', they still receive the bundle $\{a, b\}$ and their true valuation remains $v_1(\{a, b\}) = 3$. However, the mechanism determines payments based on the reported (lower) marginal values. To ensure truthfulness, meaning, to prevent Player 1 from benefiting by misreporting, it must hold that their utility from truthful reporting is at least as high as from misreporting: $u_1 = 3 - p_1 \geq u_1' = 3 - p_1'$, which implies $p_1' \geq p_1$.

Problem 3 Truthful Mechanism Design for Public Project

We will present a truthful mechanism for deciding on the construction of a public park, based on the Clarke pivot rule, a special case of the $\mathcal{V}$ickrey-$\mathcal{C}$larke-$\mathcal{G}$roves mechanism. Our goal is to ensure that all agents - citizens have an incentive to report their true valuations.

3.1 Mechanism Design

We consider a setting with n agents , indexed by $i = 1, \ldots, n$. Each agent i has a private valuation $v_i \in \mathbb{R}$ representing the benefit they derive from the construction of a public park. The total cost of constructing the park is a known constant $C > 0$. Each agent submits a reported valuation $\hat{v}_i$, which may differ from their true valuation v_i.

The mechanism proceeds to compute the sum of all reported valuations as $\sum_{i=1}^{n} \hat{v}_i$ and the park is constructed if and only if this sum meets or exceeds the cost C. If the park is not built, no agent is charged and their utility is zero. If the park is constructed, each agent i is charged an amount p_i determined according to the Clarke pivot rule. Specifically, the payment is given by

$$p_i = \max\left\{C - \sum_{j \neq i} \hat{v}_j, 0\right\}.$$

This rule ensures that no agent pays more than the marginal value of their presence in achieving the total required for construction and that no agent is assigned a negative payment. If the park is not constructed, then the utility that agent i obtains under this mechanism is given by $u_i = 0$, whereas if the park is constructed, the agent's utility is given by $u_i = v_i - p_i$.

The Clarke mechanism is incentive compatible in that, under standard assumptions, each agent maximizes their utility by reporting their true valuation, irrespective of the reports of others. This property arises because each agent's payment depends only on the reported valuations of the other agents, and the agent's own report affects only whether the park is constructed, not how much they are charged. More precisely, for agent i, let $S_{-i} = \sum_{j \neq i} \hat{v}_j$. The decision to construct the park depends on whether $\hat{v}_i + S_{-i} \geq C$. The agent's payment, however, is given by $p_i = \max\{C - S_{-i}, 0\}$, independent of $\hat{v}_i$. Thus, the agent's report influences only the decision, not the payment, except through the aggregate outcome. Given this structure, an agent maximizes utility by reporting truthfully. If their true valuation satisfies $v_i \geq C - S_{-i}$, then they benefit from the park being constructed, and reporting truthfully helps ensure that outcome. Conversely, if $v_i < C - S_{-i}$, then they prefer the park not to be constructed and reporting truthfully again leads to a better outcome. Hence, truth-telling is a dominant strategy for each agent.

3.2 Application of the Clarke Mechanism

Suppose we have three agents - citizens, thus $n = 3$ and the cost of the park is $C = 100$. Let the true valuations be $v_1 = 50$, $v_2 = 40$ and $v_3 = 30$. We will assume all agents report truthfully, so that $\hat{v}_i = v_i \; \forall \; i$. The total reported valuation is $50 + 40 + 30 = 120$, which exceeds the cost $C = 100$, so the park does get constructed and the payments will be computed as follows:

$$\begin{aligned}
p_1 &= \max\{100 - (\hat{v}_2 + \hat{v}_3),\, 0\} = \max\{100 - (40 + 30),\, 0\} = 30,\\
p_2 &= \max\{100 - (\hat{v}_1 + \hat{v}_3),\, 0\} = \max\{100 - (50 + 30),\, 0\} = 20,\\
p_3 &= \max\{100 - (\hat{v}_1 + \hat{v}_2),\, 0\} = \max\{100 - (50 + 40),\, 0\} = 10.
\end{aligned}$$

with the utilities resulting to:

$$\begin{aligned}
u_1 &= v_1 - p_1 = 50 - 30 = 20,\\
u_2 &= v_2 - p_2 = 40 - 20 = 20,\\
u_3 &= v_3 - p_3 = 30 - 10 = 20.
\end{aligned}$$

Let us now suppose that agent 1 chooses to unilaterally deviate and report $\hat{v}_1 = 0$, while the others report truthfully. Then the total reported valuation is $0 + 40 + 30 = 70 < 100$ and the park does not get constructed. Well in that case, all payments and utilities along with u_1, the utility of the agent who chose to deviate, drop to zero, which is strictly worse than the utility under truthful reporting. Thus, agent 1 can safely conclude that deviating was not beneficial for him.

Observation

A notable observation though, is that the total payment collected when all agents report truthfully, $\sum_i p_i = 60$, falls short of the total cost $C = 100$. In this setting, the purpose of the individual payments p_i is not to cover the total cost C, as explicitly stated in the assignment, but rather to incentivize agents to truthfully report their valuations v_i. Under the Clarke mechanism, each agent's payment $p_i = C - \sum_{j \neq i} \hat{v}_j$ (or zero if this expression is negative) reflects the externality that agent i imposes on the rest of the group by influencing the collective decision to construct the project. As a consequence, the total sum of payments $\sum_i p_i$ may be strictly less than the project cost C and the remaining amount must be financed through alternative means.

Problem 4 Identical Item Auction

We examine an auction setting in which a seller offers k identical indivisible items (e.g., laptops) to $n \geq k + 1$ competing bidders. Each bidder i has a private valuation v_i for obtaining a single item, and the bidder derives no additional value from acquiring more than one unit, thus has no interest in this endeavour. The utility of bidder i is $u_i = v_i - p_i$ if allocated an item at price p_i or zero otherwise.

4.1 Truthful Mechanism for Social Welfare Maximization

In the first part, the allocation rule is fixed: the auctioneer awards the k identical items to the k highest bidders according to the submitted bids $b_1, b_2, \ldots, b_n$. The goal is to determine a payment scheme that ensures truthfulness—that is, a dominant-strategy incentive compatible mechanism.

The canonical solution in single-parameter environments of this form is to use threshold (or critical value) pricing, a concept rooted in the characterization of truthful mechanisms for single-parameter domains. For each winning bidder, the price they pay should correspond to the minimum bid they could have submitted and still remained among the top k. Since bids are sorted in decreasing order, the k-th winning bid is $b_{(k)}$, and the first losing bid is $b_{(k+1)}$. Thus, a natural and effective pricing rule is to charge every winner the $(k+1)$-*st highest bid*. Under this rule, each of the top k bidders receives one unit and pays the same uniform price $p_i = b_{(k+1)}$, independently of their own bid. The utility of bidder i is then:

$$u_i = \begin{cases} v_i - b_{(k+1)}, & \text{if } b_i \geq b_{(k)}, \\ 0, & \text{otherwise.} \end{cases}$$

This payment rule guarantees truthfulness because it removes any incentive to misreport. Overstating one's bid cannot lower the price and may lead to a loss in utility if the price exceeds their value. Understating one's bid increases the risk of not receiving an item, forfeiting a positive utility. Hence, truthful bidding becomes a dominant strategy. This mechanism is individually rational as long as each player's true value exceeds the uniform price they must pay. Also, the allocation is efficient since items are allocated to those who value them the most.

4.2 Revenue-Maximizing Truthful Mechanism

We consider the setting of a single-parameter multi-unit auction where k identical items are to be allocated among $n \geq k+1$ bidders, each of whom demands at most one unit. Each bidder i has a private valuation v_i, drawn independently from a regular distribution D_i, which may actually differ across bidders. Our goal is to design a truthful mechanism that maximizes the expected revenue of the seller.

4.2.1 Proposed Mechanism: Allocation and Payments

The mechanism follows the structure of Myerson's optimal auction for unit-demand settings with multiple identical items. Each bidder i submits a bid b_i, which the mechanism interprets as their reported valuation. Using the known distribution D_i, the mechanism computes the *virtual valuation* of bidder i as

$$\phi_i(b_i) = b_i - \frac{1 - F_i(b_i)}{f_i(b_i)},$$

where F_i and f_i are the cumulative distribution function and probability density function of D_i, respectively. The mechanism allocates the k items to the k bidders with the highest non-negative virtual valuations. To ensure incentive compatibility, each winning bidder pays their *critical payment*, defined as the lowest bid they could have submitted and still remained among the winners. This payment corresponds to the threshold at which their virtual value would still qualify them for allocation, given the other bids.

4.2.2 Computation of Reserve Prices

To ensure that only bidders who contribute positively to the expected revenue are eligible to receive items, the mechanism enforces *individual reserve prices*. The reserve price r_i for bidder i is defined as the smallest value such that

$$\phi_i(r_i) = 0.$$

This condition guarantees that the expected virtual surplus from allocating an item to bidder i is non-negative. Bidders who submit bids below their reserve price are excluded from consideration, regardless of the competitiveness of their bid relative to others.

4.2.3 Justification of Truthfulness

The mechanism is truthful because it satisfies the two canonical conditions for dominant-strategy incentive compatibility in single-parameter environments. First, the allocation rule is monotone in a bidder's report: bidding a higher value, hence increasing one's virtual value, can only improve the chance of being allocated an item. Secondly, the payment rule is based on critical values. Each winner pays the minimum amount they could have bid and still won, which aligns incentives and ensures that truthful bidding maximizes utility. Thus, no bidder can benefit from misreporting their valuation.

4.2.4 Justification of Revenue Optimality

The mechanism achieves revenue optimality in expectation by maximizing the expected virtual surplus, which coincides with the expected revenue, under Myerson's method. The use of bidder-specific virtual valuations accounts for the heterogeneity in value distributions and the application of reserve prices ensures that items are not allocated to bidders whose participation would reduce expected revenue. The assumption of regularity ensures that virtual valuation functions are monotone, allowing the mechanism to maintain both truthfulness and optimal allocation. Ultimately, these elements combined guarantee that the mechanism yields the maximum possible expected revenue among all truthful mechanisms under the stated conditions.

Problem 5 Truthful Mechanism for Job Scheduling

Hereby we explore a scheduling auction with 5 players, each with a unit-length job and one machine to serve them all. Player i's value for position j, $j \in \{1, \ldots, 5\}$ is $v_i \cdot \alpha^{j-1}$, where $\alpha \in (0, 1)$ is publicly known and v_i is private. Players bid b_i and jobs are scheduled in decreasing bid order.

5.1 Utility and Truthfulness

The mechanism assigns player with bid $b_{(i)}$ (the i-th highest) to position i, paying $p_i = b_{i+1} \cdot \alpha^{i-1}$, with $b_6 = 0$. If player i's bid b_i is the k-th highest, they get:

- position k,
- value $v_i \cdot \alpha^{k-1}$,
- pay $p_i = b_{k+1} \cdot \alpha^{k-1}$, where b_{k+1} is the next highest bid and
- utility $u_i = v_i \cdot \alpha^{k-1} - b_{k+1} \cdot \alpha^{k-1} = \alpha^{k-1}(v_i - b_{k+1})$.

A mechanism is truthful if bidding $b_i = v_i$ maximizes utility for all v_i, b_i, and others' bids $\mathbf{b}_{-i}$. From Myerson's theorem for single-parameter domains, we know that truthfulness requires a monotone allocation and specific payments. The allocation is monotone since increasing b_i can only improve or maintain player i's position. Let us now test truthfulness with a counterexample.

Counterexample: Non-truthfulness of the Mechanism

We consider $\alpha = 0.8$, valuations $v_1 = 10$, $v_2 = 9$, $v_3 = 8$, $v_4 = 7$, $v_5 = 6$. If all players bid truthfully ($b_i = v_i$), the order is $10, 8, 6, 4, 2$.

For player 2 in position 2, his payment and utility are computed as follows:

$$p_2 = b_3 \cdot \alpha = 8 \cdot 0.8 = 6.4 \quad \& \quad u_2 = 9 \cdot 0.8 - 6.4 = 7.2 - 6.4 = 0.8.$$

If player 2 decides to lie and bid $b_2' = 7.5 < v_2 = 9$, the order becomes $10, 8, 7.5, 7, 6$, placing him in position 3 and he gets:

$$p_2' = b_4 \cdot \alpha^2 = 7 \cdot 0.64 = 4.48 \quad \& \quad u_2' = 9 \cdot 0.8^2 - 4.48 = 9 \cdot 0.64 - 4.48 = 5.76 - 4.48 = 1.28.$$

Since $u_2' = 1.28 > u_2 = 0.8$, bidding $b_2 = 7.5 < v_2$ increases utility, so the mechanism is not truthful. The payment rule doesn't align with the critical values needed for incentive compatibility, as it depends on the next bid without accounting for the full externality.

5.2 Payments

5.2.1 Payments that ensure truthfulness and individual rationality

To make the mechanism truthful, we use Myerson's characterization for single-parameter domains. Player i's value is $v_i \cdot x_i(\mathbf{b})$, where $x_i = \alpha^{k_i-1}$ if assigned position k_i. The allocation rule is monotone (higher bids yield better positions), so the payment is computed as:

$$p_i(v_i, \mathbf{b}_{-i}) = v_i x_i(v_i, \mathbf{b}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{b}_{-i}) dz \tag{3}$$

This ensures truthfulness, because the player maximizes their utility by reporting their true valuation v_i. We now argue that the mechanism is also individually rational, i.e., each player's utility is non-negative when bidding truthfully.

Player i's utility is

$$u_i = v_i x_i(v_i, \mathbf{b}_{-i}) - p_i = \int_0^{v_i} x_i(z, \mathbf{b}_{-i}) dz.$$

Since $x_i(z, \mathbf{b}_{-i})$ always takes the form α^{k-1} for some position $k \in \{1, 2, 3, 4, 5\}$ and $\alpha \in (0, 1)$, it follows that $x_i(z, \mathbf{b}_{-i}) \geq 0$ for all $z \in [0, v_i]$. Therefore, the integral is non-negative:

$$u_i = \int_0^{v_i} x_i(z, \mathbf{b}_{-i}) dz \geq 0.$$

Ultimately, the player never receives negative utility when reporting truthfully, which satisfies individual rationality.

5.2.2 Justification of truthfulness via Myerson's formula

Let $b_{-i,(1)} \geq \cdots \geq b_{-i,(4)}$ be the sorted bids excluding i, with $b_{-i,(0)} = \infty$, $b_{-i,(5)} = 0$. If player i bids z:

$$x_i(z, \mathbf{b}_{-i}) = \begin{cases} \alpha^0 = 1 & \text{if } z > b_{-i,(1)}, \\ \alpha^1 & \text{if } b_{-i,(1)} > z > b_{-i,(2)}, \\ \alpha^2 & \text{if } b_{-i,(2)} > z > b_{-i,(3)}, \\ \alpha^3 & \text{if } b_{-i,(3)} > z > b_{-i,(4)}, \\ \alpha^4 & \text{if } z < b_{-i,(4)}. \end{cases}$$

If v_i places player i in position k_i (i.e., $b_{-i,(k_i-1)} > v_i > b_{-i,(k_i)}$), then the integral becomes:

$$\int_0^{v_i} x_i(z, \mathbf{b}_{-i}) dz = \int_0^{b_{-i,(4)}} \alpha^4 dz + \int_{b_{-i,(4)}}^{b_{-i,(3)}} \alpha^3 dz + \int_{b_{-i,(3)}}^{b_{-i,(2)}} \alpha^2 dz + \int_{b_{-i,(2)}}^{b_{-i,(1)}} \alpha^1 dz + \int_{b_{-i,(1)}}^{v_i} \alpha^0 dz$$

$$= \alpha^4 b_{-i,(4)} + \alpha^3 (b_{-i,(3)} - b_{-i,(4)}) + \alpha^2 (b_{-i,(2)} - b_{-i,(3)}) + \alpha^1 (b_{-i,(1)} - b_{-i,(2)}) + \alpha^{k_i-1}(v_i - b_{-i,(k_i)}).$$

Substituting this into Myerson's payment formula (3):

$$p_i = v_i\alpha^{k_i-1} - \big[\alpha^4 b_{-i,(4)} + \alpha^3(b_{-i,(3)} - b_{-i,(4)}) + \alpha^2(b_{-i,(2)} - b_{-i,(3)}) + \alpha^1(b_{-i,(1)} - b_{-i,(2)}) + \alpha^{k_i-1}(v_i - b_{-i,(k_i)})\big]$$

$$= \alpha^{k_i-1} b_{-i,(k_i)} + \sum_{m=1}^{k_i-1} \alpha^{m-1}(b_{-i,(m)} - b_{-i,(m+1)}).$$

We now express this in terms of the full bid vector $\mathbf{b}$. Since $b_{-i,(m)} = b_{(m)}$ for $m < k_i$ and $b_{-i,(k_i)} = b_{(k_i+1)}$, the payment simplifies to:

$$p_i = \alpha^{k_i-1} b_{(k_i+1)} + \sum_{m=1}^{k_i-1} \alpha^{m-1}(b_{(m)} - b_{(m+1)}).$$

This payment is independent of v_i and depends only on the bids of other players, meaning that bidding truthfully is a dominant strategy. Thus, our mechanism is actually truthful.

5.2.3 Equivalence with VCG payments

The VCG mechanism charges each player the loss in social welfare their participation causes to the others. Let:

$$W = \sum_{j=1}^{5} b_{(j)}\alpha^{j-1} \tag{4}$$

$$W_{-i} = \sum_{m=1}^{4} b_{-i,(m)}\alpha^{m-1} \tag{5}$$

Then, player i's VCG payment is:

$$p_i = W_{-i} - (W - b_i\alpha^{k_i-1}) \xRightarrow{(4),\,(5)} p_i = \left[\sum_{m=1}^{4} b_{-i,(m)}\alpha^{m-1}\right] - \left[\sum_{j=1}^{5} b_{(j)}\alpha^{j-1} - b_i\alpha^{k_i-1}\right]$$

After simplifying and aligning terms, noting how positions shift after removing player i, we arrive at the same formula as in Myerson's payments:

$$p_i = \alpha^{k_i-1} b_{(k_i+1)} + \sum_{m=1}^{k_i-1} \alpha^{m-1}(b_{(m)} - b_{(m+1)}).$$

Thus, we conclude that the VCG and Myerson payment rules yield exactly the same payments in this setting.

5.3 Reserve Prices

Valuations v_i are assumed to be i.i.d. draws from a regular distribution D, with CDF $F(v)$, PDF $f(v)$, and increasing virtual valuation function:

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}.$$

5.3.1 Ensuring truthfulness with reserve prices

To preserve truthfulness, we modify the allocation rule to exclude players whose valuation is below a reserve price r^*. The modified mechanism works as follows:

- Players with $b_i < r^*$ are excluded from the allocation.
- Remaining players are ordered by bid and assigned positions as before.
- Payments are computed using Myerson's formula, where each threshold bid is replaced with $\max(b_{(m)}, r^*)$.

This preserves truthfulness because the allocation remains monotone (higher bids still yield better positions) and the payments still follow Myerson's structure with an adjusted lower bound. It also maintains individual rationality, since players are never charged more than their value and those with a value lower than acceptable, are simply excluded.

5.3.2 Computing the reserve price

The reserve price r^* is defined as the point where the virtual value becomes zero:

$$\phi(r^*) = 0 \quad \Rightarrow \quad r^* = \frac{1 - F(r^*)}{f(r^*)}.$$

Consider the case where $D \sim \text{Uniform}[0, 1]$. Then:

$$F(v) = v, \quad f(v) = 1 \quad \& \quad \phi(v) = v - (1 - v) = 2v - 1.$$

Solving $\phi(r^*) = 0$ yields $2r^* - 1 = 0 \Rightarrow \boldsymbol{r^* = \frac{1}{2}}$. This threshold allows only players with non-negative virtual value, aligning with revenue maximization and truthfulness.

Problem 6 Revenue-Optimal Auction with Two Bidders

We consider a single-item auction with two bidders. Bidder 1's value x is drawn from the uniform distribution on $[0, 4]$, and bidder 2's value y is drawn from the uniform distribution on $[1, 3]$. The values x and y are independent.

6.1 Virtual Valuations and Reserve Prices

To compute the reserve prices for the two bidders, we apply the virtual valuation function $\phi(v) = v - \frac{1-F(v)}{f(v)}$, using each bidder's value distribution. These reserves correspond to the lowest values for which a bidder's virtual value becomes non-negative, and thus determine whether they are eligible to win under the revenue-maximizing mechanism.

For bidder 1, the value x follows a uniform distribution on the interval $[0, 4]$. Using the standard expressions for the CDF and PDF of a uniform distribution on $[a, b]$, namely $F(v) = \frac{v-a}{b-a}$ and $f(v) = \frac{1}{b-a}$, we substitute $a = 0$ and $b = 4$ to obtain $F_1(x) = \frac{x}{4}$ and $f_1(x) = \frac{1}{4}$. Hence, the virtual valuation is

$$\boldsymbol{\phi_1(x)} = x - \frac{1 - F_1(x)}{f_1(x)} = x - \frac{1 - \frac{x}{4}}{\frac{1}{4}} = x - (4 - x) = \mathbf{2x - 4}.$$

Setting $\phi_1(r_1) = 0$ yields the reserve price $\boldsymbol{r_1 = 2}$.

Similarly, for bidder 2 whose value $y \sim \text{Uniform}[1, 3]$, the CDF and PDF are $F_2(y) = \frac{y-1}{2}$ and $f_2(y) = \frac{1}{2}$, respectively. The virtual valuation becomes

$$\boldsymbol{\phi_2(y)} = y - \frac{1 - \frac{y-1}{2}}{\frac{1}{2}} = y - (3 - y) = \mathbf{2y - 3}.$$

Solving $\phi_2(r_2) = 0$, we find $\boldsymbol{r_2 = 1.5}$.

6.2 Optimal and Second-Price Auctions

6.2.1 Optimal Auction

According to Myerson's optimal auction theory, the seller should allocate the item to the bidder with the highest non-negative virtual value.

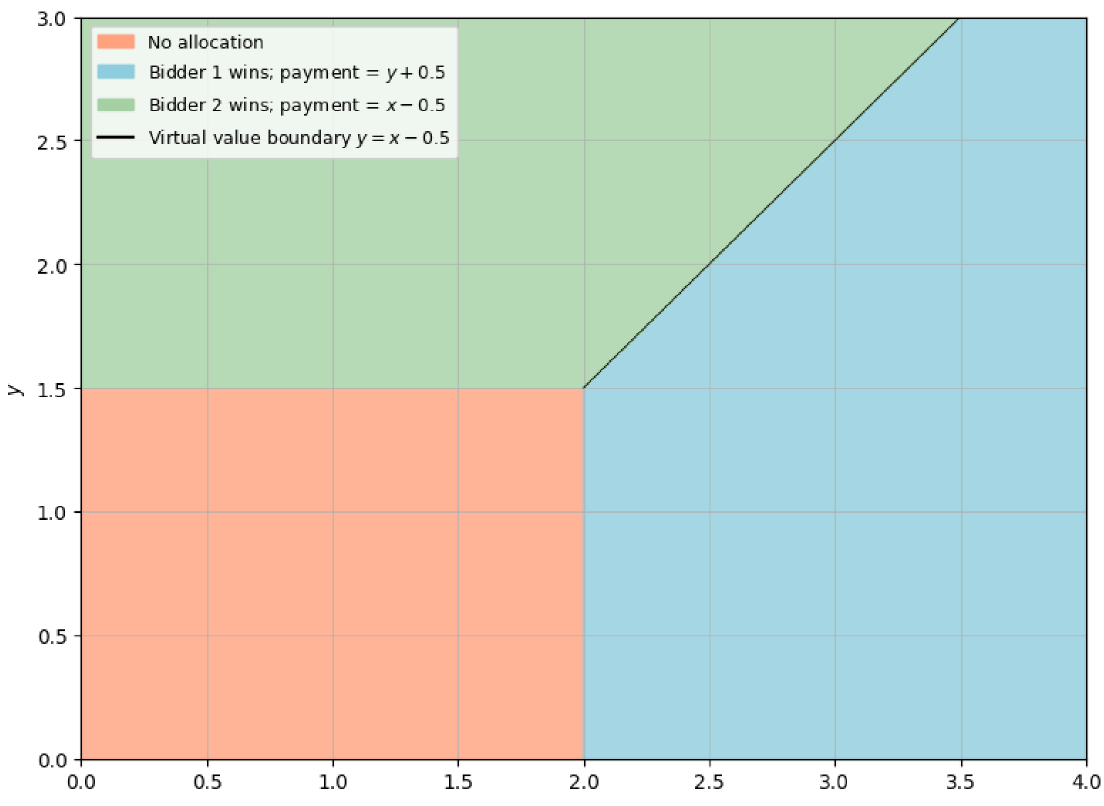


Figure 1: Allocation regions and virtual value boundary in Myerson's optimal auction

The allocation rule is determined by the following cases:

- Bidder 1 wins if $\phi_1(x) > \phi_2(y)$ and $\phi_1(x) \geq 0$, which is equivalent to $x > y + 0.5$ and $x \geq 2$.
- Bidder 2 wins if $\phi_2(y) > \phi_1(x)$ and $\phi_2(y) \geq 0$, which simplifies to $y > x - 0.5$ and $y \geq 1.5$.
- No allocation occurs if both virtual values are negative, i.e., $x < 2$ and $y < 1.5$.

The payment rule in the optimal auction corresponds to critical value pricing. If bidder 1 wins, their payment equals the lowest value they could have reported and still won. That value x' satisfies $\phi_1(x') = \phi_2(y)$, which gives $2x' - 4 = 2y - 3 \Rightarrow x' = y + 0.5$. Similarly, if bidder 2 wins, their payment is determined by the threshold y' for which $\phi_2(y') = \phi_1(x)$, leading to $y' = x - 0.5$.

6.2.2 Second-Price Auction with Common Reserve $r = 2$

In a second-price auction with a reserve price of 2, bidders are eligible to win only if their bids are at least 2. If both bidders satisfy this condition, the higher bidder wins and pays the maximum between the other bid and the reserve price. If only one bidder meets the reserve, that bidder wins and pays exactly the reserve. If neither bidder meets the reserve price, the item remains unsold and the seller earns zero revenue.

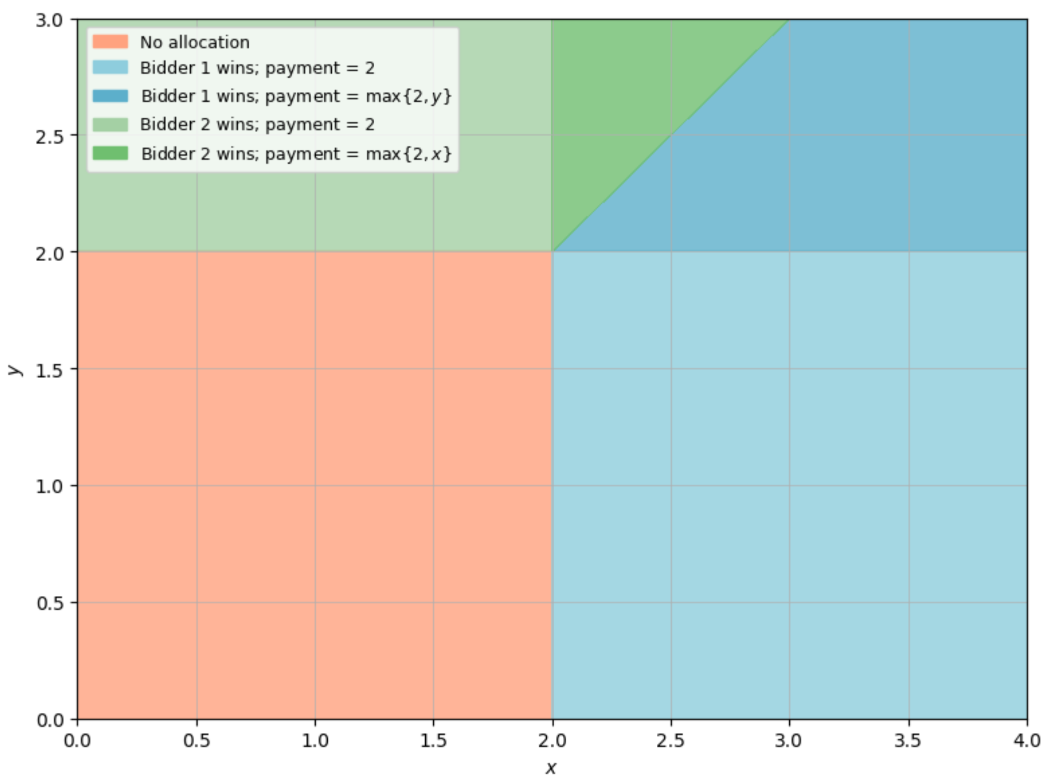


Figure 2: Allocation and payment regions under second-price auction with reserve price 2.

6.2.3 Standard Second-Price Auction (No Reserve)

In the absence of any reserve price, the standard second-price auction always allocates the item. The bidder with the higher bid wins and pays the other bidder's bid.

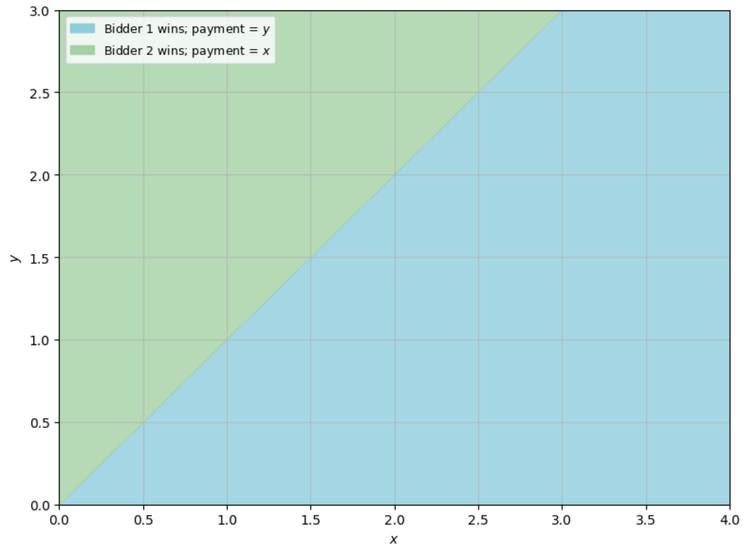


Figure 3: Allocation and payments in a standard second-price auction without reserve prices.

6.3 Expected Revenue as a Function of x

We study how the seller's expected revenue changes depending on player 1's value $x \in [2, 4]$, under both the optimal auction and the second-price auction with reserve prices.

6.3.1 Revenue for $x \in [2, 3]$

6.3.1.1 Optimal Auction

Fix a value $x \in [2, 3]$ for bidder 1. As established earlier, he is allocated the item whenever $\phi_1(x) > \phi_2(y)$ and $\phi_1(x) \geq 0$, which holds precisely when $y < x - 0.5$ and $y \geq 1.5$. Since $y \sim$ Uniform$[1, 3]$, the relevant interval for integration is $y \in [1.5, \min(x - 0.5, 3)]$. In this region, the payment made by bidder 1 corresponds to the critical value x' such that $\phi_1(x') = \phi_2(y)$, yielding a payment of $x' = y + 0.5$. Hence, the expected revenue contributed by cases where bidder 1 wins is

$$R_{\text{opt},1}(x) = \int_{1.5}^{\min(x-0.5,3)} (y + 0.5) \cdot \frac{1}{2}\, dy = \frac{1}{2}\left[\frac{1}{2}y^2 + 0.5y\right]_{1.5}^{x-0.5} = \frac{1}{4}x^2 - 1.$$

In the complementary region, where $y > x - 0.5$ and $y \geq 1.5$, bidder 2 wins and pays the threshold value y' for which $\phi_2(y') = \phi_1(x)$, which simplifies to $y' = x - 0.5$. Since $x \in [2, 3]$, the integration domain is $y \in [\max(1.5, x - 0.5), 3]$ and the payment is constant. Thus, the expected revenue from cases where bidder 2 wins is

$$R_{\text{opt},2}(x) = \int_{\max(1.5,x-0.5)}^{3} (x - 0.5) \cdot \frac{1}{2}\, dy = \frac{1}{2}(x - 0.5)(3.5 - x) = -\frac{1}{2}x^2 + 2x - \frac{7}{8}.$$

Adding the two contributions, the total expected revenue for the seller under the optimal mechanism when $x \in [2, 3]$ is given by

$$R_{\text{opt}}(x) = R_{\text{opt},1}(x) + R_{\text{opt},2}(x) = \left(\frac{1}{4}x^2 - 1\right) + \left(-\frac{1}{2}x^2 + 2x - \frac{7}{8}\right) = -\frac{1}{4}x^2 + 2x - \frac{15}{8}$$

6.3.1.2 Second-Price Auction with Reserve Price 2

Under the second-price auction with a reserve price of 2, only bids greater than or equal to 2 are eligible to receive the item. For values $x \in [2, 3]$, bidder 1 always meets the reserve and thus qualifies. Bidder 2, whose value $y \sim \text{Uniform}[1, 3]$, qualifies only if $y \geq 2$. We distinguish two mutually exclusive cases based on the value of y.

First, when $y \in [2, \min(x, 3)]$, both bidders qualify and bidder 1 has the higher bid. In this case, bidder 1 wins and pays the second-highest bid, which is y. The expected revenue from this region is given by the integral

$$R_{\text{2nd},1}(x) = \int_2^{\min(x,3)} y \cdot \frac{1}{2}\,dy = \frac{1}{2}\left[\frac{1}{2}y^2\right]_2^x = \frac{1}{4}(x^2 - 4),$$

where the integrand y reflects the payment and $\frac{1}{2}$ is the probability density of the uniform distribution over $[1, 3]$.

Second, when $y \in [1, 2)$, only bidder 1 satisfies the reserve condition. In this case, bidder 1 wins by default and pays exactly the reserve price of 2. This contribution to expected revenue is computed as

$$R_{\text{2nd},2}(x) = \int_1^2 2 \cdot \frac{1}{2}\,dy = 1.$$

Summing both components, the expected revenue for the seller under the second-price auction with reserve 2 when $x \in [2, 3]$ is

$$R_{\text{2nd}}(x) = R_{\text{2nd},1}(x) + R_{\text{2nd},2}(x) = \frac{1}{4}(x^2 - 4) + 1 = \frac{1}{4}x^2 - 1 + 1 = \frac{1}{4}x^2.$$

6.3.2 Revenue for $x \in [3, 4]$

6.3.2.1 Optimal Auction

For values $x \in [3, 4]$, we continue our analysis under the optimal mechanism using the same virtual valuation-based allocation rules. Recall that bidder 1 wins when $\phi_1(x) = 2x - 4 > \phi_2(y) = 2y - 3$, which is equivalent to $x > y + 0.5$ and is valid only if $x \geq 2$, which is already satisfied in the $[3, 4]$ interval. Bidder 1 thus receives the item when $y \in [1.5, \min(x - 0.5, 3)]$. In this region, the payment is computed using critical value pricing and equals $y + 0.5$. The expected revenue generated in this region is:

$$R_{\text{opt},1}(x) = \int_{1.5}^{\min(x-0.5,3)} (y + 0.5) \cdot \frac{1}{2}\,dy = \begin{cases} \frac{x^2}{4} - 1 & \text{if } 3 \leq x \leq 3.5, \\ 2.0625 & \text{if } 3.5 < x \leq 4. \end{cases}.$$

Complementarily, when $y \in [\max(1.5, x-0.5), 3]$, bidder 2 wins. In this region, the virtual valuation of bidder 2 exceeds that of bidder 1, and the payment equals the lowest value y could take for which this is true, i.e., $x - 0.5$. The expected revenue from this region is thus:

$$R_{\text{opt},2}(x) = \int_{\max(1.5,x-0.5)}^{3} (x-0.5)\cdot\frac{1}{2}\,dy = \frac{1}{2}(x-0.5)(3.5-x) = -\frac{1}{2}x^2 + 2x - \frac{7}{8}.$$

Summing both expressions yields the total expected revenue under the optimal auction for $x \in [3,4]$:

$$R_{\text{opt}}(x) = R_{\text{opt},1}(x) + R_{\text{opt},2}(x) = \begin{cases} -\dfrac{x^2}{4} + 2x - \dfrac{15}{8} & \text{if } 3 \le x \le 3.5 \\ -\dfrac{x^2}{2} + 2x + \dfrac{19}{16} & \text{if } 3.5 < x \le 4 \end{cases}.$$

6.3.2.2 Second-Price Auction with Reserve Price 2

In the second-price auction with a reserve price of 2, both bidders are eligible to win only if their values meet or exceed the reserve. Since $x \in [3,4]$, bidder 1 always qualifies. Bidder 2, drawn from Uniform$[1,3]$, qualifies if $y \ge 2$. We distinguish two regions depending on whether bidder 2 qualifies.

When $y \in [2,3]$, both bidders qualify. Since $x > y$ always in this range (as $x \ge 3$), bidder 1 wins and pays the maximum of y and the reserve, which is y. The expected revenue over this region is

$$R_{\text{2nd},1}(x) = \int_2^3 y\cdot\frac{1}{2}\,dy = \frac{1}{2}\int_2^3 y\,dy = \frac{1}{2}\left[\frac{y^2}{2}\right]_2^3 = \frac{1}{2}\left(\frac{9}{2} - \frac{4}{2}\right) = \frac{1}{2}\cdot\frac{5}{2} = \frac{5}{4}$$

When $y \in [1,2)$, bidder 2 does not meet the reserve and is excluded. In this case, bidder 1 wins by default and pays the reserve price of 2. The corresponding expected revenue contribution from this region is

$$R_{\text{2nd},2}(x) = \int_1^2 2\cdot\frac{1}{2}\,dy = 1.$$

Therefore, the total expected revenue for the seller under the second-price auction with reserve when $x \in [3,4]$ is:

$$R_{\text{2nd}}(x) = R_{\text{2nd},1}(x) + R_{\text{2nd},2}(x) = \frac{5}{4} + 1 = \frac{9}{4}.$$

6.3.3 Visual comparison of Expected Revenue across Auctions for $x \in [2,4]$

To visualize how the seller's expected revenue depends on bidder 1's value $x \in [2,4]$, we computed and plotted revenue functions for both the optimal auction and the second-price auction with a reserve price of 2. Bidder 1's value x is treated as a parameter, while bidder 2's value $y \sim$ Uniform$[1,3]$ is random. For each value of x, we calculate expected revenue by integrating over all possible values of y, weighted by the uniform distribution.

The expected revenue under the optimal auction is given by the piecewise function below

$$R_{\text{opt}}(x) = \begin{cases} -\dfrac{1}{4}x^2 + 2x - \dfrac{15}{8}, & \text{for } x \in [2, 3.5], \\ -\dfrac{1}{2}x^2 + 2x + \dfrac{19}{16}, & \text{for } x \in (3.5, 4]. \end{cases}$$

The expected revenue under the second-price auction with reserve price $r = 2$ is

$$R_{\text{2nd}}(x) = \begin{cases} \dfrac{1}{4}x^2, & \text{for } x \in [2, 3], \\ \dfrac{9}{4}, & \text{for } x \in [3, 4]. \end{cases}$$

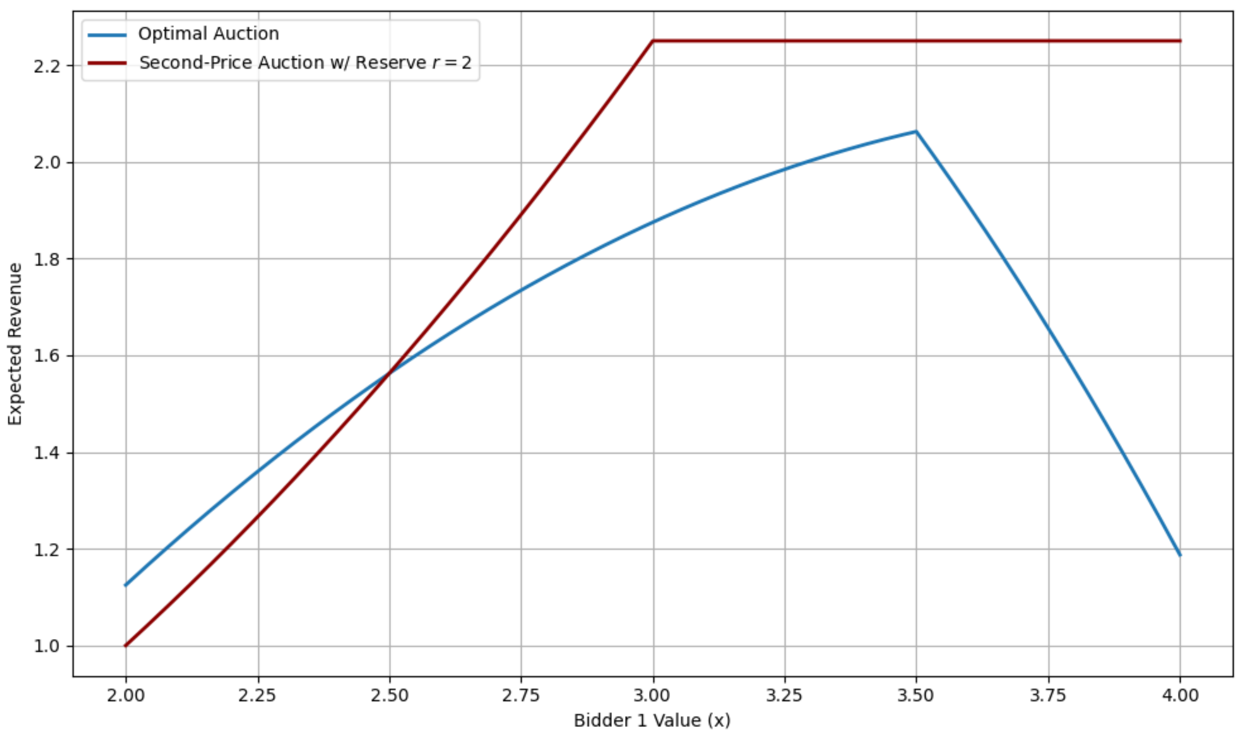


Figure 4: Expected revenue as a function of the valuation of bidder 1.

The graph illustrates how the seller's expected revenue varies with bidder 1's value $x \in [2, 4]$, under both the optimal auction and the second-price auction with a reserve price $r = 2$. In the optimal auction, the revenue function changes structure at $x = 3.5$, reflecting a shift in allocation behavior due to virtual valuations. For the second-price auction, revenue increases quadratically on $[2, 3]$ and becomes constant at $\frac{9}{4}$ for $x \in [3, 4]$, as bidder 1 always wins and pays either the reserve or bidder 2's bid. Notably, the second-price auction yields higher expected revenue than the optimal auction for much of the interval, specifically for $x > 2.5$.

6.3.4 Visual comparison of Expected Revenue across Auctions for $y \in [1, 3]$

To analyze how the seller's expected revenue depends on bidder 2's value $y \in [1, 3]$, we fix y and treat bidder 1's value $x \sim \text{Uniform}[2, 4]$ as the random variable. For each fixed y, we compute expected revenue by integrating over x, weighted by the uniform density.

In the optimal auction, allocation depends on virtual valuations: bidder 2 wins when $\phi_2(y) > \phi_1(x)$, which reduces to $y > x - 0.5$. If bidder 2 wins, he pays $x - 0.5$ and if bidder 1 wins, he pays $y + 0.5$. This leads to two revenue regions: when $y < 1.5$, bidder 1 always wins, and revenue is simply $y + 0.5$. When $y \geq 1.5$, we integrate over the corresponding winning regions for both bidders, resulting in a quadratic revenue expression.

$$R_{\text{opt}}(y) = \begin{cases} y + 0.5, & \text{for } y \in [1, 1.5], \\ -\dfrac{1}{4}y^2 + \dfrac{3}{2}y + \dfrac{5}{16}, & \text{for } y \in [1.5, 3]. \end{cases}$$

In the second-price auction with a reserve price of 2, bidder 2 is excluded unless $y \geq 2$. For $y < 2$, bidder 1 always wins and pays the reserve, yielding constant revenue of 2. For $y \geq 2$, if $x < y$, bidder 2 wins and pays x; if $x \geq y$, bidder 1 wins and pays y. Integrating over these intervals gives a quadratic function for the revenue in this range.

$$R_{\text{2nd}}(y) = \begin{cases} 2, & \text{for } y \in [1, 2), \\ -\dfrac{1}{4}y^2 + 2y - 1, & \text{for } y \in [2, 3]. \end{cases}$$

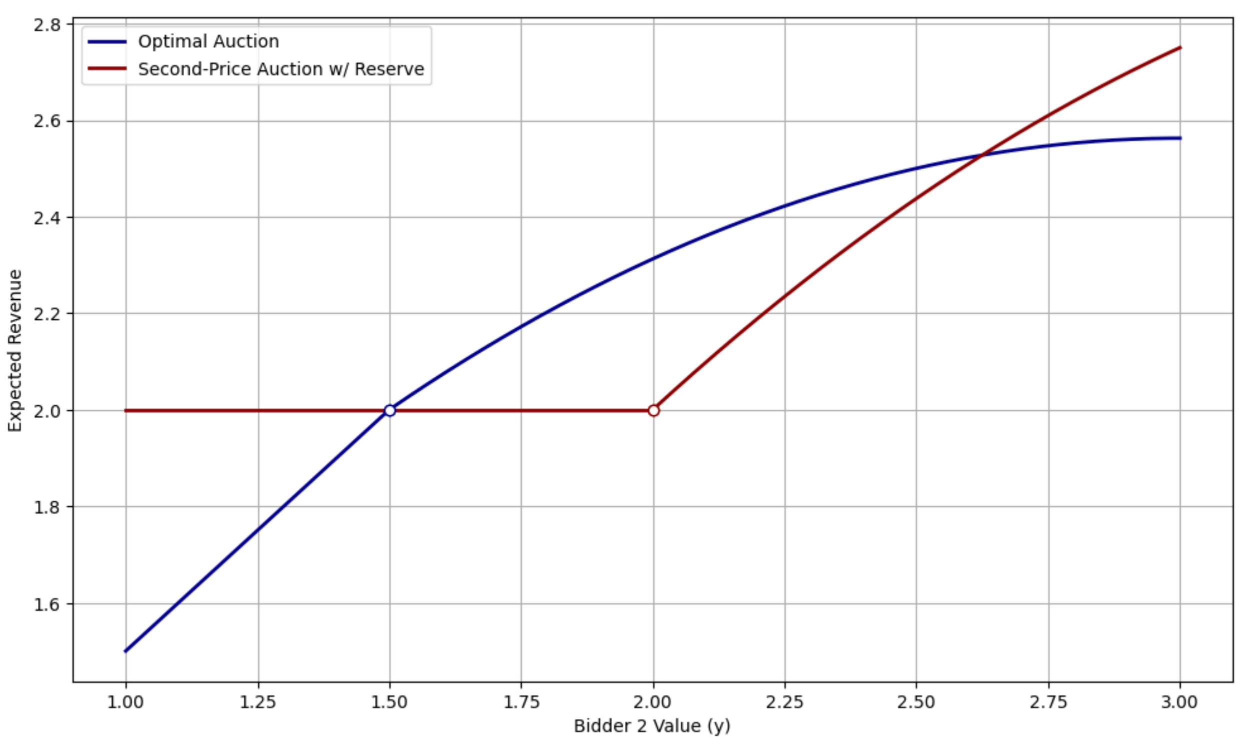


Figure 5: Expected revenue as a function of bidder 2's valuation.

The graph shows how the seller's expected revenue varies with bidder 2's value $y \in [1, 3]$, under both the optimal auction and the second-price auction with a reserve price $r = 2$. In the optimal auction, revenue increases smoothly and changes structure at $y = 1.5$. In the second-price auction, revenue is constant for $y < 2$, where bidder 2 does not qualify due to the reserve and increases quadratically for $y \geq 2$. The comparison shows that the optimal auction yields higher expected revenue than the second-price auction for bidder 2 values in the interval $y \in [1.5, 2.625]$. However, the second-price auction with reserve outperforms the optimal mechanism both when $y < 1.5$, due to its fixed reserve pricing, and when $y > 2.625$, as bidder 2 becomes sufficiently strong to dominate the bidding and drive up expected revenue.

6.3.5 Impact of Asymmetric Treatment in Optimal Auction

In the optimal auction, the seller treats the two bidders differently through the use of virtual valuations, a concept introduced in Myerson's optimal auction theory. Each bidder's true valuation is transformed into a virtual value, which captures not only the bidder's willingness to pay but also the marginal revenue the seller can extract from allocating the item to that bidder.

This transformation allows the seller to allocate the item not to the bidder with the highest raw value, but to the one whose virtual valuation is highest. The key idea is that virtual values account for the distributional characteristics of each bidder's type, such as their density or support. In particular, if a bidder tends to have higher valuations, the mechanism will be more inclined to allocate the item to them, even when their reported value is not strictly the highest.

In our setting, bidder 1's values lie in the range $[2, 4]$, while bidder 2's values lie in $[1, 3]$, making bidder 1 generally more valuable. The optimal mechanism reflects this by setting a higher bar for bidder 2 to win. As a result, bidder 2 receives the item less frequently, and bidder 1 is often allocated the item even when the values are close. Furthermore, critical value pricing ensures that the winning bidder pays the minimum value they would have needed to report to still win, which maximizes the expected payment the seller can extract under incentive compatibility.

By differentiating between bidders in this way, allocating based on virtual values and tailoring thresholds to each bidder's distribution, the optimal auction is able to extract more surplus from stronger bidders and limit revenue loss from weaker ones. This targeted strategy is what enables the mechanism to maximize expected revenue, especially in asymmetric environments like the one we study.