The Vectorial Deffuant Model

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Content

The Standard Voter Model

- A spin system (or interacting particle system) is a Feller process on the set $\{0,1\}^V$, where V is a graph such as the $\mathbb{Z}^d$-lattice, and individual transitions involve one site only.

- In the standard voter model on $\mathbb{Z}^d$ we can think of each site being occupied by a person, who is either in favour or against a certain issue or candidate.

$$\eta_t : \mathbb{Z}^d \longrightarrow \{0,1\}, t \in [0,\infty) \tag{1}$$

- At an exponentially distributed time each agent randomly picks one of her neighbours and sets her opinion equal to that of the neighnour.

- The voter model is defined by the probability generator:

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \frac{1}{2d} \sum_{u \sim x} [f(\eta_{x,u}) - f(\eta)] \quad (2)$$

- $\eta_{x,u}$ denotes the configuration obtained from η by setting the configuration at $x \in \mathbb{Z}^d$ equal to the configuration at $u \in \mathbb{Z}^d$ and leaving all the other sites unchanged.

- Where f is continuous, real-valued and $|||f||| := \sum_{x,u} sup_\eta |f(\eta_{x,u}) - f(\eta)| < \infty$

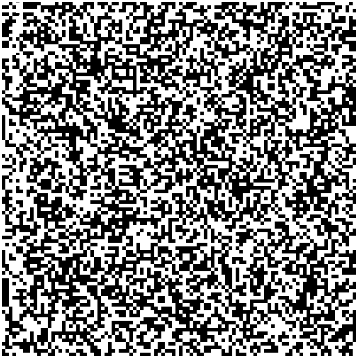


Figure 1: Voter Model, initial configuration

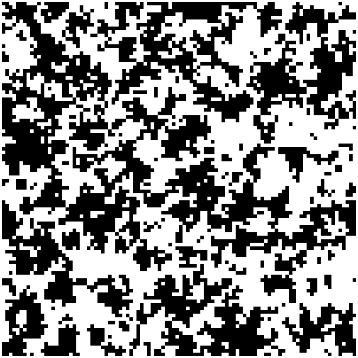


Figure 2: Voter model after 30K successful interactions

The vectorial Deffuant Model

- The vectorial Deffuant model extends in the voter model with a state space $\{0,1\}^F$,$F \in \mathbb{N}$ denotes the number of **features**, and incorporates homophily (with an interaction threshold).

$$\eta_t : \mathbb{Z}^d \longrightarrow \{0,1\}^F \tag{3}$$

- The state space is a hypercube equipped with the Hamming distance

$$H : \{0,1\}^F \times \{0,1\}^F \to \mathbb{N},\ H(u,v) := |\{i : u_i \neq v_i\}| \tag{4}$$

- $\theta \in \{0,1,...,F\}$ is called the **threshold**.
- Each individual picks a random neighbour at rate one and updates her configuration by moving one unit towards this neighbour along a random direction in the hypercube unless they differ in more than θ features or both neighbours already agree.

Vectorial Deffuant Model

- Formally the probability generator is given by:

$$Lf(\eta) = \sum_{x} |\{y : y \sim x\}|^{-1} \sum_{y \sim x} |\Omega(x,y,\eta)|^{-1} \sum_{u \in \Omega(x,y,\eta)} 1_{\{1 \leq H(\eta(x),\eta(y)) \leq \theta\}} [f(\eta_{x,u}) - f(\eta)] \tag{5}$$

where $\Omega(x, y, \eta) := \{u \in \{0, 1\}^F : H(u, \eta(y)) = H(\eta(x), \eta(y)) - 1\}$

- The system is started from a product measure where

$$P(\tilde{\eta}_0(x, i) = 1) = p, \ 1 \leq i \leq F \text{ and } \forall x \in \mathbb{Z}$$

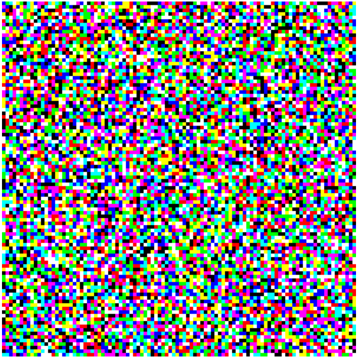


Figure 3: Initial configuration: F=3, $\theta = 1$, 100×100 torus

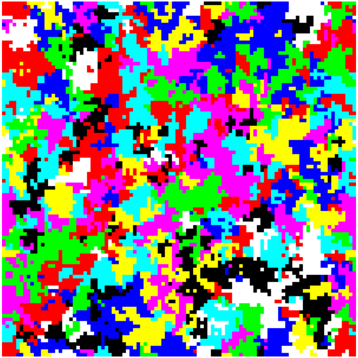


Figure 4: 100 million successful interactions later

Flux, Fixation and Clustering

- **Fluctuation** occurs whenever

$$P(\eta_t(x) \neq \eta_s(x) \text{ for some } t > s) = 1 \, \forall x \in V \, \forall s > 0 \tag{6}$$

- **Fixation** occurs if there exists a configuration η_∞ such that

$$P(\eta_t(x) = \eta_\infty \text{ eventually in } t) = 1 \, \forall x \in V \tag{7}$$

  i.e. the attributes of each individual is only updated a finite number of times, and a coexistence of different configurations persists.

- **Clustering** occurs if

$$\lim_{t \to \infty} P(\eta_t(x) = \eta_t(x)) = 1 \, \forall x \in V \tag{8}$$

  i.e. there is a convergence to global consensus.

Coupling with Annihilating Random Walks

- When $d = 1$ it is useful to visualise the vec. Deffuant model in terms of a collection of annihilating random walks.

Vectorial Deffuant Model

- We consider the edge between two nearest neighbours $x \sim y \in \mathbb{Z}$ as a space occupied by particles on F levels. A particle at level i on the edge $e = xy$ represents a disagreement between x and y on the Feature i.

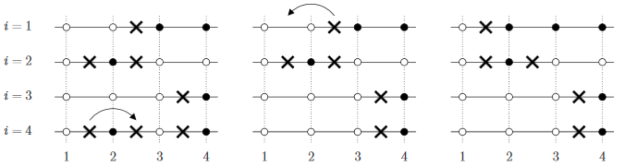


Figure 5: [8]

- Let $\tilde{\eta}_t : V \times \{1, 2, ..., F\} \to \{0, 1\}$ where $\tilde{\eta}_t(x, i) :=$ ith coordinate of $\eta_t(x)$. Then the process corresponding to disagreements is defined by

$$\xi_t(e, i) := 1_{\{\tilde{\eta}_t(x,i) \neq \tilde{\eta}_t(y,i)\}}$$

where $e = xy$ an edge. ξ_t corresponds to a system of F symmetric annihilating random walks (**non-independent**).

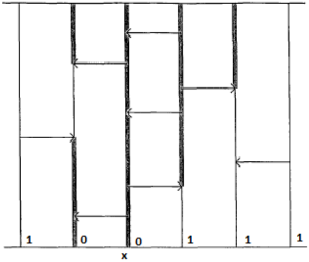
- Extinction of particles is equivalent to a clustering of the system.

$$\zeta_t(e) := \xi_t(e,1) + \xi_t(e,2) + ... + \xi_t(e,F) = H(\eta_t(x), \eta_t(y))$$

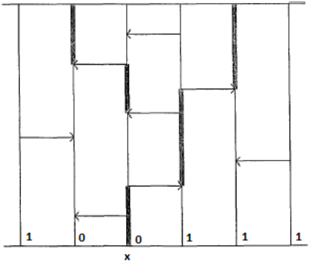
- An edge will be referred to as a blockade if more than θ particles are on the edge and as an active edge if at most θ particles are on the edge.

Graphical Representation

- We define a graphical representation for the vec. Deff. model which identifies the system with a percolation structure. The annihilating random walks and the dual process can be constructed from the graphical representation with a standard argument due to Harris (1972).



Voter Model



Dual of Voter Model

- We basically construct F voter models - a graphical representation for each feature with the difference that each path can be active or inactive according to threshold constraints.

Let $(x, y, i) \in V \times V \times \{1, 2, ..., F\}$ where $x \sim y$

- we let $(N_{x,y,i} : t \geq 0)$ be a collection of rate one Poisson process
- we let $T_{x,y,i}(n)$ denote the nth arrival time: $T_{x,y,i}(n) := \inf\{t : N_{x,y,i} = n\}$
- we let $(B_{x,y,i}(n) : n \geq 1)$ be a collection of independent Bernoulli random variables with

$$P(B_{x,y,i}(n) = +1) = P(B_{x,y,i}(n) = -1) = 1/2 \tag{9}$$

- we let $(U_{x,y,i}(n) : n \geq 1)$ be collection of independent Uniform(0,1) random variables.

At each time $t := T_{x,y,i}(n)$ an arrow is drawn from x to y with label i, $x \longrightarrow y$ if $B_{x,y,i}(n) = +1$ and from y to x if $B_{x,y,i}(n) = -1$.

The arrow $x \longrightarrow y$ is active if

$$\xi_{t-}(e,i) = 1 \text{ and } U_{x,y,i}(n) \leq r(\zeta_{t-}(e)) \text{ where } e = xy \tag{10}$$

where

$$r(m) := \begin{cases} m^{-1}, & \text{if } 0 < m \leq \theta \\ 0, & \text{if } \theta < m \leq F \end{cases}$$

We say there is an active-i-path from (z,s) to (x,t), denoted $(z,s) \rightsquigarrow^i (x,t)$, whenever there are sequences of times and vertices

$$s_0 = s < s_1 < ... < s_{n+1} = t \text{ and } x_0 = z, x_1, ..., x_n = x$$

such the following conditions hold:

- for $j = 1, 2, ..., n$, there is an active-i-arrow $x_{j-1} \longrightarrow x_j$ at time s_j, and
- for $j = 1, 2, ..., n$, there is no active-i-arrow pointing at $\{x_j\} \times (s_j, s_{j+1})$

and there is a *generalized active path* from (z,s) to (x,t), denoted $(z,s) \rightsquigarrow (x,t)$, if

- for $j = 1, 2, ..., n$, there is an active arrow $x_{j-1} \longrightarrow x_j$ at time s_j

Clustering of the one-dim. 2-Feature vec. Deffuant Model

1. We define a coupling of the 2-Feature vec. Deffuant model to the standard voter model by defining the process:

$$
\begin{aligned}
&\rho_t(x) := |\tilde{\eta}_t(x,1) - \tilde{\eta}_t(x,2)| \text{ for } x \in \mathbb{Z}^d \\
&A_1 := \{(0,1),(1,0)\} \longmapsto 1 \\
&A_0 := \{(1,1),(0,0)\} \longmapsto 0
\end{aligned}
\tag{11}
$$

2. We prove that the voter model fluctuates on $\mathbb{Z}$.

3. The expected number of particles per edge

$$u(t) := E(\zeta_t(e)) = P(\zeta_t(e) = 1) + 2P(\zeta_t(e) = 2)$$

   is independent of edge e due to translation invariance of the initial configuration and symmetric evolution rules.

4. $u(t)$ is non-increasing in time, $\lim_{t\to\infty} u(t)$ exists almost surely.

5. Let $e = (x)(x+1)$ and $\zeta_t(e) > 0$. Then the edge e is either a blockade or an active edge:
   - if e is a blockade it follows from the fluctuation of the voter model $\tau := \inf\{s > t : \zeta_s(e) \neq 2\} = \inf\{s > t : \rho_s(x) \neq \rho_s(x+1)\} < \infty$ a.s.. i.e. the blockade is broken in finite time.
   - if e is a live edge, it follows from the recurrence of 1 dimensional symmetric random walks that the active particle at e eventually hits another particle and annihilates or forms a blockade.

2 Features, Threshold 1

6. $u(t)$ is strictly decreasing. Particles go extinct almost surely.

Fixation in the one-dimensional vec. Deff. Model

We derive a sufficient condition for the fixation of the one-dimensional vec. Deff. Model starting from a product measure.
The following Lemma was first proven for cyclic particle systems in [2] and more recently extended to the vec. Deff. Model in [8].

For all $(z,i) \in \mathbb{Z} \times \{1,2,...,F\}$, let

$$T(z,i) := \inf\{t : (z,0) \rightsquigarrow^{i} (0,t)\} \tag{12}$$

Then the vec. Deff. Model fixates whenever

$$\lim_{N\to\infty} P(T(z,i) < \infty \text{ for some } z < -N \text{ and } i = 1,2,...,F) = 0 \tag{13}$$

The contribution variable

"[...] construct a random interval such that all the blockades initially in the interval must have been destroyed by either active particles initially in in this interval or active particles that result from the the destruction of these blockades." [8, p. 552]

$$H_N := \{T(z,i) < \infty : \text{ for some } z < -N \text{ and some } i = 1,2,...,F\} \tag{14}$$

For $T_N := \inf\{T(z,i) : z < -N \text{ and } i = 1,2,...,F\}$

$$H_N = \{T_N < \infty\} \tag{15}$$

Conditional on the event H_N let z^* be s.t. $T(z^*,i) = T_N$ and let

$$\begin{aligned} z_- &:= \min\{z \in \mathbb{Z} : (z,0) \rightsquigarrow (0,T_N)\} \leq z^* < -N \\ z_+ &:= \max\{z \in \mathbb{Z} : (z,0) \rightsquigarrow (0,S) \text{ for some } S < T_N\} \geq 0 \end{aligned} \tag{16}$$

where $\rightsquigarrow$ now denotes a generalized active path. And let $I_N := (z_-, z_+)$. The contribution variable is defined

$$cont : E(\mathbb{Z}) \to \mathbb{Z}$$

e **blockade**

$$\begin{aligned} cont(e) := \#\{&\text{active particles that either annihilate or become frozen as a result} \\ &\text{of a jump onto } e \text{ before } e \text{ becomes a live edge }\} - \\ &\#\{\text{particles initially at } e \text{ that ever become active}\} \end{aligned} \tag{17}$$

e **active**

$$\begin{aligned} cont(e) := \#\{&\text{active particles that either annihilate or become frozen as a result} \\ &\text{of a jump onto } e \text{ before the first jump of an active particles initially at } e\} \\ &- \#\{\text{particles initially at } e\} \end{aligned}$$

It follows by definition of I_N

$$
\begin{aligned}
&H_N = \{T_N < \infty\} \subseteq \{\sum_{e \in I_N} cont(e) \leq 0\} \\
&\Rightarrow \{\sum_{e \in I_N} cont(e) > 0\} \subseteq \{T_N = \infty\}
\end{aligned}
\tag{18}
$$

The weight function

The weight function, as it is named in [8] is an explicit random function $\phi : E(\mathbb{Z}) \to \mathbb{Z}^{\Omega}$ defined to be stochastically smaller that the contribution. So that $\{E(\phi(e)) > 0\} \subseteq \{T_N = \infty\}$ holds for any $e \in E(\mathbb{Z})$.
The first jump of an active particle onto a blockade is equally likely to occur at each level and results in

- an annihilating event with probability j/F
- a blockade increase - extra particle on the edge - with probability $1 - j/F$

We define the weight function:

$$\begin{aligned} &\phi : E(\mathbb{Z}) \to \mathbb{Z}^{\Omega} \\ &\phi(e) := \begin{cases} -j, & \text{if } \zeta_0(e) = j \leq \theta \\ j + 2(X_{e,j} - \theta), & \text{if } \zeta_0(e) = j > \theta \end{cases} \end{aligned} \tag{19}$$

where $X_{e,j} := Bernoulli(1 - j/F)$ and $(X_{e,j})_{e \in E(\mathbb{Z}), 1 \leq j \leq F}$ independent.

Lemma
For all $\epsilon > 0$, there exists constants $c_6 > 0$,$C_0 > 0$ (depending on F) such that

$$P((\sum_{e\in(-N,0)} \phi(e) - NE\phi(e)) \notin (-\epsilon N, \epsilon N)) \leq C_0 \exp(-c_6 N)$$

for large N and any $e \in E(\mathbb{Z})$.

Theorem
The system fixates whenever $E\phi(e) > 0$.

Proof idea: Let

$$H_N = \{T(z,i) < \infty : \text{ for some } z < -N \text{ and some } i = 1, 2, ..., F\}$$

and it holds that

$$H_N \subseteq \{\sum_{e\in I_N} cont(e) \le 0\} \overset{\phi(e)\preceq cont(e)}{\subseteq} \{\sum_{e\in I_N} \phi(e) \le 0\}$$
$$\overset{Def.\ I_N}{\subseteq} \{\sum_{e\in[l,r]} \phi(e) \le 0 \text{ for some } l < -N \text{ and some } r \ge 0\} \tag{20}$$

Let $\epsilon := E\phi(e) > 0$, then there exist constants $c_6 > 0$, C_0 s.t.

$$P(\sum_{e\in(-N,0)} \phi(e) \le 0) = P(\sum_{e\in(-N,0)} \phi(e) \le N(E\phi(e) - \epsilon)) \le C_0 \exp(-c_6 N) \tag{21}$$

Parameters leading to fixation

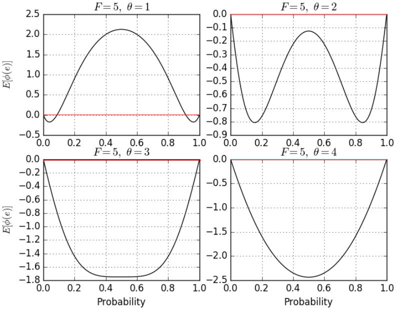
We now try to determine for which parameters the function $E\phi(e) : [0,1] \to \mathbb{R}$ takes positive values.

$$E\phi(e)(p) = \sum_{j \leq \theta} (-j) p_j(p) + \sum_{j > \theta} (j + 2(1 - \frac{j}{F} - \theta)) p_j(p)$$

and

$$p_j(p) = \binom{F}{j} (2p(1-p))^j (p^2 + (1-p)^2)^{F-j}$$

Fixation in one-dimension



Some basic properties of $E\phi(e) : [0,1] \to \mathbb{R}$:

- $E\phi(e)$is continuous differentiable in p.
- $E\phi(e)(p) = 0$ for $p \in \{0,1\}$.
- $\frac{dE\phi(e)}{dp} = \sum_{j \leq \theta}(-j)\frac{d}{dp}p_j + \sum_{j>\theta}(j + 2(1 - \frac{j}{F}))\frac{d}{dp}p_j$ where

$$\begin{aligned} \frac{d}{dp}p_j &= \frac{d}{dp}\binom{F}{j}(2p(1-p))^j(p^2 + (1-p)^2)^{F-j} \\ &= \binom{F}{j}j(2p(1-p))^{j-1}(2-4p)(p^2+(1-p)^2)^{F-j} \\ &+ \binom{F}{j}(2p(1-p))^j(F-j)(4p-2)(p^2+(1-p)^2)^{F-j-1} \end{aligned} \tag{22}$$

- $\frac{dE\phi(e)}{dp}(0) = -2F$ and $\frac{dE\phi(e)}{dp}(1) = 2F$

Fixation in one-dimension

- Symmetry: $E\phi(e)(p) = E\phi(e)(1-p)$. This follows from the symmetry of $p_j = \binom{F}{j}(2p(1-p))^j(p^2+(1-p)^2)^{F-j}$

- $E\phi(e) \leq 0$ for $F \leq 2\theta - 1$.

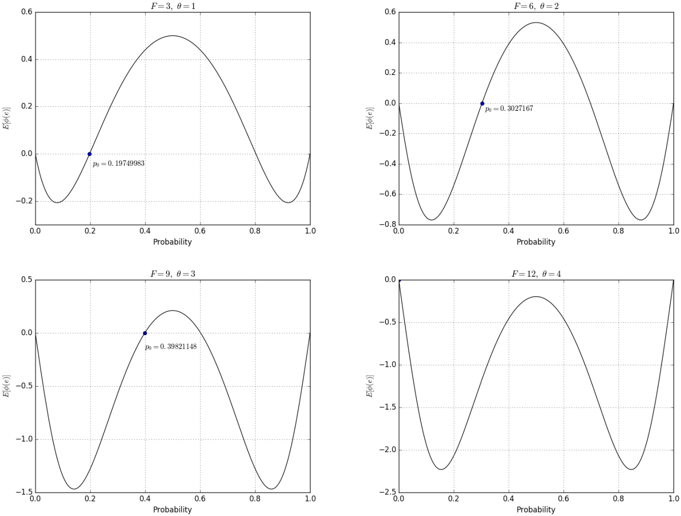
Results:

(1) There exists an open interval $(a, 1-a) \subset [0,1]$ such that the system with $F \geq 4\theta$ fixates when started from a product measure with density $p \in (a, 1-a)$.

(2) There exists an open interval $(a, 1-a) \subset [0,1]$ such that the system with $F = 4\theta - 1$ fixates when started from a product measure with density $p \in (a, 1-a)$.

(3) There exists an open interval $(a, 1-a) \subset [0,1]$ such that the system with $F = 4\theta - n$. where $n \in \mathbb{N}$ and $n \leq \min\{\theta - 1, 2\theta\}$, fixates when started from a product measure with density $p \in (a, 1-a)$.

Fixation in one-dimension



Simulation on Torus

- **Regions:** contiguous sites with the same configuration
- **Zones:** contiguous sites with compatible configuration, i.e. no blockades
- **Disagreements:** blocked edges

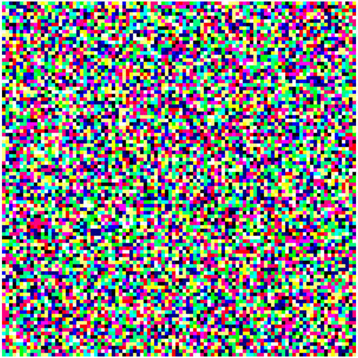


Figure 6: Initial configuration: F=4, $\theta = 2$, 100×100 torus

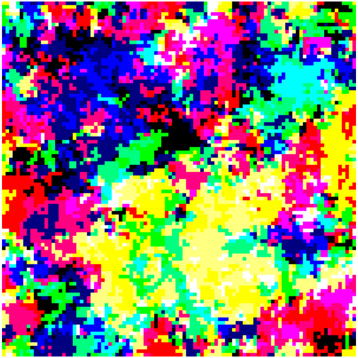
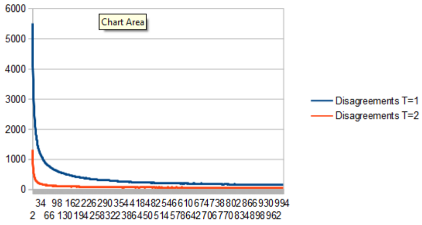


Figure 7: 10 million successful interactions later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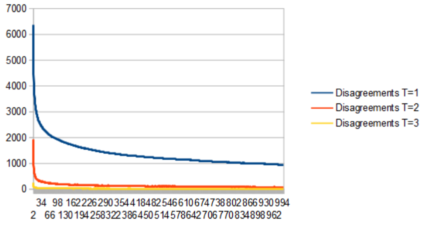
Disagreements T=1
Disagreements T=2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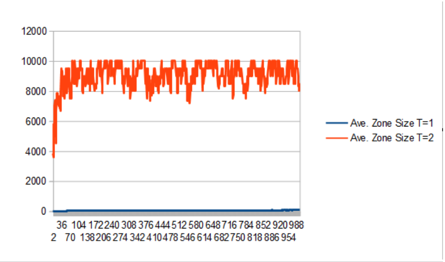
Simulation



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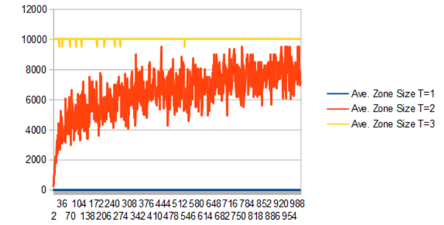

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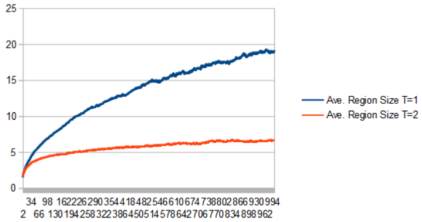
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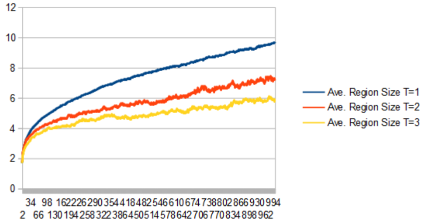
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Ave. Region Size T=1
Ave. Region Size T=2
Ave. Region Size T=3

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Questions?