A coalgebraic view on bi-infinite streams

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Bi-infinite streams arise as a natural data structure in several contexts, such as signal processing [1], symbolic dynamics [2], (balanced) representation of real/rational numbers [3] or study of sets invariant under shift transformation [4].

In this paper, we will present a coalgebraic view of the set of bi-infinite streams which we shall denote by $A^{\mathbb{Z}}$ and is formally defined as

$$A^{\mathbb{Z}} = \{\sigma \mid \sigma : \mathbb{Z} \to A\}$$

We can easily prove that $A^{\mathbb{Z}} \cong (A \times A)^{\omega}$ and therefore the set $A^{\mathbb{Z}}$ is the final coalgebra for the functor $FX = (A \times A) \times X$.

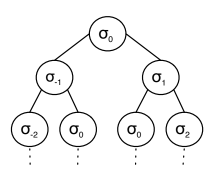
This reflects the fact that one can think about a bi-infinite stream denoted by $(\ldots, \sigma_{-2}, \sigma_{-1}, \underline{\sigma_0}, \sigma_1, \sigma_2, \ldots)$, as two infinite streams growing in parallel.

σ_0	σ_1	σ_2	$\ldots$
σ_{-1}	σ_{-2}	σ_{-3}	$\ldots$

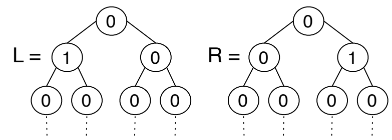
Using this observation, and defining a semiring structure on $A \times A$, we could reuse the calculus developed for streams [5] to deal with the bi-infinite case.

However, is this the only/best way to view bi-infinite streams coalgebraically? Will this reduction to the infinite case be too restrictive and not allow us to fully benefit from the structure of bi-infinite streams?

We shall now present another possible representation for bi-infinite streams. We can see $(\ldots, \sigma_{-2}, \sigma_{-1}, \underline{\sigma_0}, \sigma_1, \sigma_2, \ldots)$ as an infinite binary tree as follows:



The set T_A of infinite binary trees is the final coalgebra for the functor $GX = X \times A \times X$ and as showed in [6], by viewing trees as formal power series a very simple but surprisingly powerful coinductive calculus can be developed. In this framework, definitions are presented as behavioural differential equations and very compact closed formulae can be deduced for (rational) trees. For instance, in this framework the bi-infinite stream $(\ldots, 0, 1, 0, 1, \underline{0}, 1, 0, 1, 0, \ldots)$ would be represented by the formula $(L + R)(1 + (L - R)^2)^{-1}$, where L and R represent the following constant trees.



Note that not all $t \in T_A$ are representations of bi-infinite streams. However, we can prove that the subset of T_A containing valid representations of bi-infinite streams is a subcoalgebra of T_A and therefore, the existing calculus can be used to reason about bi-infinite streams.

Further questions still remain to be answered. We would like to classify the closed formulae that we have for trees in such a way that from its syntax could immediately be deduced if it is a valid representation of a bi-infinite stream.

We would also like to further exploit a specific class of bi-infinite streams, the ones which correspond to finite-tailed Laurent series and see if they give rise to a different type of coalgebra/calculus.

References

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