

Quantitative regular behaviours

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Calco-jnr, September 2009

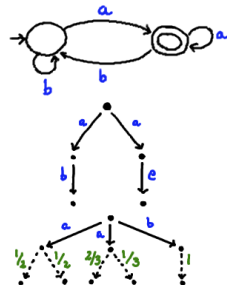
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e.g. DFA, LTS, PA,



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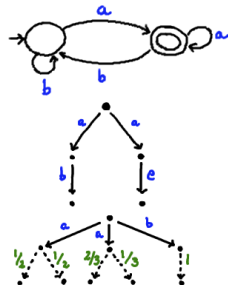
Syntax
RE, CCS, ...

$b^*a(b^*a)^*$

$a.b.0 + a.c.0$

$a(\frac{1}{2}.0 \oplus \frac{1}{2}.0) + \dots$

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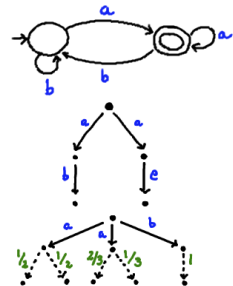
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$$P + 0 = P$$

$$p.P \oplus p'.P = (p+p').P$$

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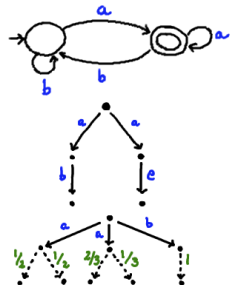
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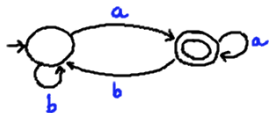
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Can we do all of this **uniformly** in a single framework?

What do these things have in common?



$$(S, t: S \rightarrow 2 \times S^A)$$

$$(S, t: S \rightarrow \mathcal{P}S^A)$$

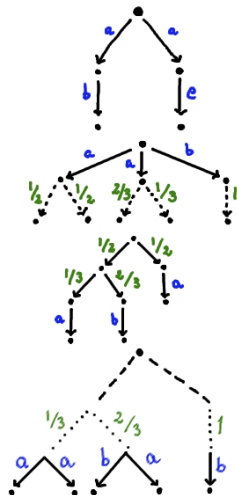
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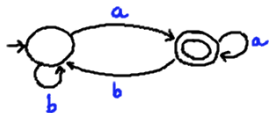
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$$(S, t: S \rightarrow GS)$$

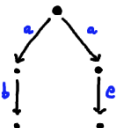
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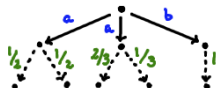
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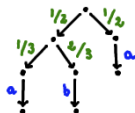
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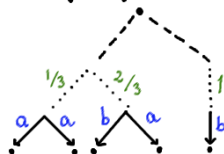
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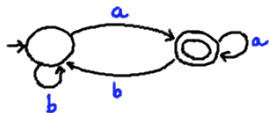


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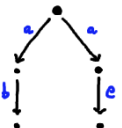
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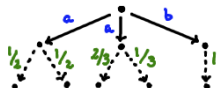
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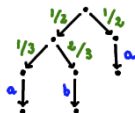
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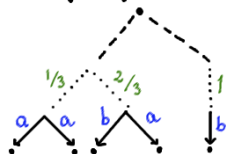
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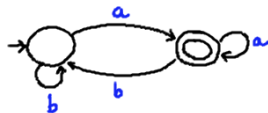


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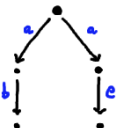
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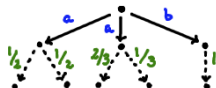
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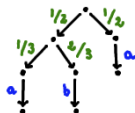
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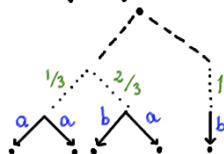
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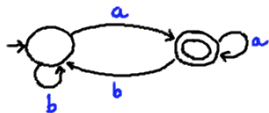


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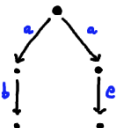
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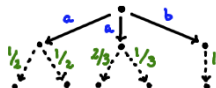
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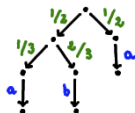
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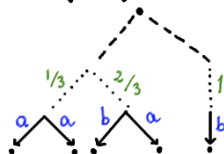
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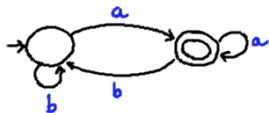


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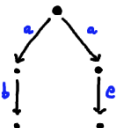
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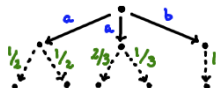
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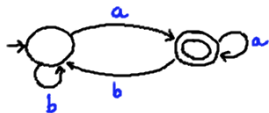


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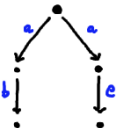
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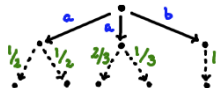
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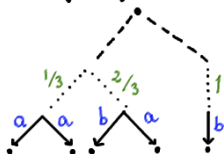
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The power of G

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The functor G determines:

- 1 notion of observational equivalence (coalg. bisimulation)
- 2 behaviour (final coalgebra)
- 3 set of expressions describing finite systems
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❶ + ❷ are classic coalgebra; ❸ + ❹ are LICS'09 and CONCUR'09

Quantitative coalgebras

- Generalizations of deterministic automata
- Quantitative coalgebras: set of states S and $t : S \rightarrow GS$

$$G ::= Id \mid B \mid G \times G \mid G + G \mid G^A \mid \mathbb{M}^G$$

\mathbb{M} is a monoid. $\mathcal{P} = 2^{Id}$ and $\mathcal{D}_\omega = \mathbb{R}^{Id}$

Examples

- $G = 2 \times Id^A$ Deterministic automata
- $G = (B \times Id)^A$ Mealy machines
- $G = (\mathcal{P}Id)^A$ LTS
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In this talk...

- ... we present a **systematic** way to derive from the functor **G**: languages of (generalized) regular expressions and
- ... sound and complete axiomatizations thereof for **quantitative systems**;
- ... we show the correspondence between language and systems (generalizing **Kleene's theorem**);
- ... we apply the framework to several types of probabilistic automata **recovering old results and deriving new ones**.

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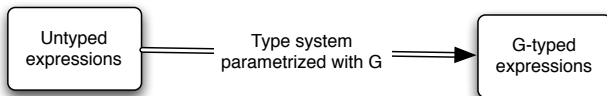
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How do we define E_G ?



G-expressions

$$\begin{array}{lcl} \text{Exp} \ni \varepsilon & ::= & \emptyset \mid \varepsilon \oplus \varepsilon \mid \mu x. \gamma \\ & & \mid b \quad B \\ & & \mid l\langle \varepsilon \rangle \mid r\langle \varepsilon \rangle \quad G_1 \times G_2 \\ & & \mid l[\varepsilon] \mid r[\varepsilon] \quad G_1 + G_2 \\ & & \mid a(\varepsilon) \quad G^A \\ & & \mid m \cdot \varepsilon \quad M^G \end{array}$$

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$$\varepsilon ::= \mu x. \varepsilon \mid \bigoplus_{i \in 1 \dots n} p_i \cdot \varepsilon \quad \text{for } p_i \in (0, 1] \text{ such that } \sum_{i \in 1 \dots n} p_i = 1$$

Kleene's Theorem

The goal is:

G – expressions **correspond to** Finite G – coalgebras and vice-versa.
What does it mean **correspond**?

Final coalgebras exist for quantitative coalgebras.

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 GS & \xrightarrow{Gh} & G\Omega_G & \xleftarrow{G[\![\cdot]\!] } & G\text{Exp}_G
 \end{array}$$

correspond \equiv mapped to the same element of the final coalgebra
 \equiv **bisimilar**

A generalized Kleene Theorem

Theorem

- 1 *Let (S, g) be a G -coalgebra. If S is finite then there exists for any $s \in S$ a G -expression ε_s such that $\varepsilon_s \sim s$.*
- 2 *For all G -expressions ε , there exists a finite G -coalgebra (S, g) such that $\exists_{s \in S} s \sim \varepsilon$.*

The proof provides algorithms to construct an expression from a system and vice-versa.

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Axiomatization

$$\left. \begin{array}{ll} \varepsilon_1 \oplus \varepsilon_2 & \equiv \varepsilon_2 \oplus \varepsilon_1 \\ \varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & \equiv (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\ \varepsilon_1 \oplus \varepsilon_1 & \equiv \varepsilon_1, \text{ } \textcolor{red}{G} \text{ polynomial} \\ \varepsilon \oplus \emptyset & \equiv \varepsilon \end{array} \right\} \textcolor{blue}{G}$$

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Sound and complete w.r.t \sim

Similar for $G_1 + G_2$ and G^A

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Results I : Segala systems – $\mathcal{P}(D_\omega(Id))^A$

$$\varepsilon:: = \emptyset \mid \varepsilon \boxplus \varepsilon \mid \mu X. \varepsilon \mid X \mid a(\{\varepsilon'\})$$

$$\varepsilon':: = \bigoplus_{i \in 1 \dots n} p_i \cdot \varepsilon_i$$

where $a \in A$, $p_i \in (0, 1]$ and $\sum_{i \in 1 \dots n} p_i = 1$

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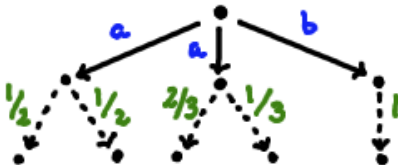
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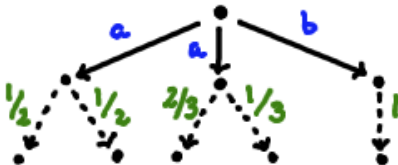
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↑↑
Kleene's Theorem

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Results II : Stratified systems – $D_\omega(Id) + (B \times Id) + 1$

$$\varepsilon ::= \mu x. \varepsilon \mid x \mid \langle b, \varepsilon \rangle \mid \bigoplus_{i \in 1 \dots n} p_i \cdot \varepsilon_i \mid \downarrow$$

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New syntax and axiomatization.

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New syntax and axiomatization.

Conclusions and future work

Conclusions

- Framework to **uniformly** derive language and axioms for quantitative coalgebras (weighted automata, probabilistic automata, etc)
- Examples show the effectiveness of the framework: known syntaxes recovered, new ones derived.

Future work

- Apply the framework to other systems, *e.g.* alternating systems.
- Automation: `Circ` — Coinductive prover