Exercises week 7

Complexity 2011-2012

A. Silva, H. Barendregt, B. Westerbaan & B. Westerbaan

1. We will show SAT $\leq_P$ 3SAT in steps.

   Formulas in SAT might have $\rightarrow$s in them. Let SAT-ARROWLESS be the set of satisfiable boolean formulas that do not contain the implication connective $\rightarrow$.

   (a) Show SAT $\leq_P$ SAT-ARROWLESS. [Hint: $\neg a \vee b \Leftrightarrow a \rightarrow b$]

   The $\neg$s may be anywhere in formulas of SAT-ARROWLESS. Let SAT-ARROWLESS-MOVED$\neg$ be the satisfiable boolean formulas without $\rightarrow$s in which the $\neg$s are only found in front of a variable.

   (b) Show SAT-ARROWLESS $\leq_P$ SAT-ARROWLESS-MOVED$\neg$. [Hint: $\neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$ and $\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$]

   SAT-CNF are the satisfiable formulas in conjunctive normal form[1]. The distributive laws are:

$$(a \wedge b) \vee c \Leftrightarrow (a \vee c) \wedge (b \vee c)$$
$$(a \vee b) \wedge c \Leftrightarrow (a \wedge c) \vee (b \wedge c)$$

   For any formula in SAT-ARROWLESS-MOVED$\neg$ we can apply the distributive laws a few times such that it becomes an equivalent formula in conjunctive normal form.

   However, the conjunctive normal forms grow too big for this to be a polynomial reduction.

   (c) Show that in worst-case the length of a formula in conjunctive normal form is exponential in the original length.

   For any formula φ with variables $x_1, \ldots, x_n$ without $\rightarrow$s and all $\neg$s in front of variables, we can find a formula ψ with variables $x_1, \ldots, x_n, y_1, \ldots, y_k$ in conjunctive normal form that is equisatisfiable. That is: φ is satisfiable if and only if ψ is satisfiable. And furthermore the length of ψ is less than or equal $c \cdot n^2$ where c does not depend on ϕ.

[1] The formulas in conjuctive normal form are defined by the following grammar:

$$\begin{aligned}
\text{VARIABLE} &\Rightarrow x_1 | x_2 | \ldots \\
\text{LITERAL} &\Rightarrow \text{VARIABLE} | \neg\text{VARIABLE} \\
\text{DISJUNCTION} &\Rightarrow \text{LITERAL} | \text{DISJUNCTION} \vee \text{LITERAL} \\
\text{FORMULA-IN-CNF} &\Rightarrow (\text{DISJUNCTION}) | \text{FORMULA-IN-CNF} \wedge (\text{DISJUNCTION})
\end{aligned}$$

(d) Show that we can do this for literals.

(e) Show that if we can do this for φ_1 and φ_2, that we can do this for $\varphi_1 \wedge \varphi_2$.

(f) Show that if we can do this for φ_1 and φ_2, that we can do this for $\varphi_1 \vee \varphi_2$. [Hint: introduce a new variable y_i to denote whether φ_1 or φ_2 is true and rewrite φ_1 and φ_2.]

(g) Conclude $\textsc{sat-arrowless-moved}\neg \leq_P \textsc{sat-cnf}$.

3SAT are the satisfiable formulas in conjunctive normal form in which each disjunction contains exactly 3 'disjuncts'.

(h) Show that $\textsc{sat-cnf} \leq_P 3\textsc{sat}$.

(i) Show that 3SAT is NP-complete.

2SAT are the satisfiable formulas in conjunctive normal form in which each disjunction contains exactly 2 'disjuncts'.

(j) Show that 2SAT is in P.