APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$\mathcal{L}[f(t)] = F(S) = \int_0^{-\infty} e^{-st} f(t) \mathrm{dt}$
$\mathcal{L}[f^n(t)] = S^n Y - S^{n-1} Y(0) - S^{n-2} Y'(0) \cdots\cdots - S f^{n-2}(0) \cdots\cdots - y^{n-1}(0)$
$\mathcal{L}[U_x(x,t)] = \int_0^{\infty} e^{-st} U(x,t) \mathrm{dt} \equiv U(x,s)$
$\mathcal{L}[U_x(x,t)] = U_x(x,s)$
$\mathcal{L}[U_x(x,t)] = U_x(x,s)$
$\mathcal{L}[U_t(x,t)] = SU(x,s) - U(x,0)$
$\mathcal{L}[U_t(x,t)] = S^2 U(x,s) - SU(x,0) - U_t(x,0)$
$F(s) = \mathcal{L}[f(t)]$ then,
$\mathcal{L}[U(t-a) \boldsymbol{.} g(t-a)] = e^{-as} G(s)$

P.D.E of Order 2

Examples:

(1) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial x}(x,t), 0 < x < 2, t > 0 \quad U(0,t) = 0, U(2,t) = 0, U(x,0) = 3\sin(2\pi x)$

Solution:
$U_{xx}(x,t) = U_t(x,t)$

taking the Laplace transform

$\mathcal{L}[U_{xx}(x,t)] = \mathcal{L}[U_t(x,t)]$

$U_{xx}(x,s) = sU(x,s) - U(x,0)$

Using the condition, $U(x,0) = 3\sin(2\pi x)$

$SU(x,s) - 3\sin(2\pi x) = U_{xx}(x,s)$

$\Longrightarrow U_{xx}(x,s) - sU(x,s) = -3\sin(2\pi x)$

$\dfrac{d^2 u}{dx^2} - sU = -3\sin(2\pi x)$

Solving the Homogenous Problem

$\frac{d^2u}{dx^2} - sU = 0$

The characteristic equation is given by

$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$

The homogenous solution is:

$$U_A(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

i.e $\frac{d^2u}{dx^2} - sU = -3\sin(2\pi x) - - - - - - - - - - - - - (*)$

Let

$U = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x) - - - - - - - - - - (a)$

$U' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x) - - - - - - - - (b)$

$U'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - - - - - - (c)$

Substituting (a) and (c) in equation (*)

$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - S\Delta_1 \sin(2\pi x) - S\Delta_2 \cos(2\pi x) = -3\sin(2\pi x)$

$-4\pi^2\Delta_1 - S\Delta_1 = -3$ Also, $4\pi^2\Delta_2 - S\Delta_2 = 0$

$-\Delta_1\left[4\pi^2 + s\right] = -3$ $\Delta_2\left[s + 4\pi^2\right] = 0$

$\Delta_1 = \frac{3}{s+4\pi^2}$ $\Delta_2 = 0$

The particular solution is:

$$U_p(x,s) = \frac{3}{s+4\pi^2}\sin(2\pi x)$$

The general solution is given by:

$$U(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x} + \frac{3\sin(2\pi x)}{s+4\pi^2}$$

Applying the boundary conditions $U(0,t) = 0,\ U(2,t) = 0$

$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$U(2, s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0 \quad$ [But $A_1 = -A_2$]

$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$

$A_2 \left[e^{-2\sqrt{s}} - e^{2\sqrt{s}}\right] = 0 \quad \Rightarrow A_2 = 0, A_1 = 0$

$$U(x, s) = \frac{3\sin(2\pi x)}{s + 4\pi^2}$$

Subtituing (a) and (c) in equation (*)

$-c^2\pi^2 n_1 \sin(\pi x) - c^2\pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = \dfrac{-\sin(\pi x)}{s}$

$\Longrightarrow -c^2\pi^2 n_1 - s_2 n_1 = \dfrac{-1}{s} \Rightarrow -n_1 \left[s^2 + c^2\pi^2\right] = \dfrac{-1}{s}$

$\Longrightarrow n_1 = \dfrac{1}{s\left[s^2 + c^2\pi^2\right]}$

Also, $-c^2\pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$

$n_2 \left[s^2 + c^2\pi^2\right] = 0$

$\Longrightarrow n_2 = 0$

Substituting 'n_1' and 'n_2' in (a)

$$U_p(x, s) = \frac{\sin(\pi x)}{s}$$

The general solution is given as:

$U_g(x, s) = U_u(x, s) + U_p(x, s)$

$U_g(x, s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \dfrac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]}$ ---------------(**)

Apply the boundary conditions; $\quad U(0, t) = 0$ and $U(1, t) = 0$

$U(0, s) = A_1 + A_2 = 0 \quad \Longrightarrow A_1 = -A_2$

$U(1, s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$

Substituting $'A_1'$ and $'A_2'$ in eqution (**)

$$U(x, s) = \frac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]}$$

Applying Inverse Laplace Transform

$$\mathcal{L}^{-1}\left[U(x,s)\right] = \sin(\pi x)\ \mathcal{L}^{-1}\left[\frac{1}{s\left[s^2+c^2\pi^2\right]}\right]$$

Resolving $\dfrac{1}{s\left[s^2+c^2\pi^2\right]}$ into partial fractions

$$\frac{1}{s\left[s^2+c^2\pi^2\right]} = \frac{A}{s} + \frac{Bs+D}{s^2+c^2\pi^2} = \frac{A\left[s^2+c^2\pi^2\right]+\left[Bs+D\right]s}{s\left[s^2+c^2\pi^2\right]}$$

$$1 = A\left[s^2+c^2\pi^2\right] + Bs^2 + Ds$$

Taking the Inverse Laplace equation

$$\mathcal{L}^{-1}\left[U(x,s)\right] = \mathcal{L}^{-1}\left[\frac{3\sin(2\pi x)}{s+4\pi^2}\right]$$

$$\mathbf{U(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x)}$$

(2) $\dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\dfrac{\partial^2 u}{\partial x^2}(x,t)\ ; 0<x<1, t>0,$
$U(x,0)=0, U_t(x,0)=0, U(0,t)=0, U(1,t)=0$

Solution:

$$U_{tt}(x,t) = c^2U_{xx}(x,t) + \sin(\pi x)$$

Taking the Laplace transform

$$\mathcal{L}\left[U_{tt}(x,t)\right] = c^2\mathcal{L}\left[U_{xx}(x,t)\right] + \mathcal{L}\left[\sin(\pi x)\right]$$

$$s^2U(x,s) - sU(x,0) - U_t(x,0) = c^2U_{xx}(x,s) + \frac{sin(\pi x)}{s}$$

Applying the initial conditions; $U(x,0)=0$ and $U_t(x,0)=0$

$$s^2U(x,s) - c^2U_{xx}(x,s) = \frac{\sin(\pi x)}{s}$$

Re-arranging

$$c^2U_{xx}(x,s) - s^2U(x,s) = -\frac{\sin(\pi x)}{s}$$

Solving the Homogenous problem

$$c^2\frac{d^2u}{dx^2}(x,s) - s^2U(x,s) = 0$$

The Auxillary equation is given by:

$$c^2m^2 - s^2 = 0 \Longrightarrow m^2 - \left(\frac{s}{c}\right)^2 = 0 \Longrightarrow m = \pm\frac{s}{c}$$

The homogenous solution is: $U(x,s) = A_1e^{\frac{sx}{c}} + A_2e^{-\frac{sx}{c}}$

Solving the non-homogenous problem by method of Undetermined Coefficient:

i.e $c^2\frac{d^2u}{dx^2}(x,s) - s^2U(x,s) = -\frac{\sin(\pi x)}{s}$ −−−−−−−−−−−(∗)

Let:

$U(x,s) = n_1\sin(\pi x) + n_2 cos(\pi x)$ −−−−−−−−−−−−−− (a)
$U_x(x,s) = \pi n_1\cos(\pi x) - \pi n_2\sin(\pi x)$ −−−−−−−−−−−−−(b)
$U_{xx}(x,s) = -\pi^2 n_1\sin(\pi x) - \pi^2 n_2\cos(\pi x)$ −−−−−−−−−−−(c)

Comparing or Equating the Coefficients of boths sides:

$s: \quad D = 0$
$s^2: \ A + B = 0 \Longrightarrow A = -B$
constant : $1 = Ac^2\pi^2 \Longrightarrow A = \frac{1}{c^2\pi^2} \Longrightarrow B = \frac{-1}{c^2\pi^2}$

$$\therefore \frac{1}{s\left[s^2 + c^2\pi^2\right]} = \frac{1}{sc^2\pi^2} - \frac{s}{(c^2\pi^2)(s^2 + c^2\pi^2)} = \frac{1}{c^2\pi^2}\left[\frac{1}{s} - \frac{s}{s^2 + c^2\pi^2}\right]$$

$$U(x,t) = \frac{\sin(\pi x)}{c^2\pi^2}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2 + (c\pi)^2}\right]$$

$$\mathbf{U(x,t)} = \frac{\sin(\pi\mathbf{x})}{\mathbf{c^2}\pi^2}\left[\mathbf{1} - \cos(\mathbf{c}\pi\mathbf{t})\right]$$

NEXT:

Solution of non-linear PDE, by the combined Laplace transform and the new Modified Variational Iteration Method.

SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATION BY THE COMBINED LAPLACE TRANSFORM METHOD AND THE NEW MODIFIED VARIATIONAL ITERATION METHOD

Presenting a reliable combined Laplace transform and the new modified variational Iteration method to solve some non-linear Partial Differential Equations. This method is more efficient and easy to handle non-linear PDEs.

Recall,

$$\mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0)$$

$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

Where $F(x,s)$ is the Laplace transform of (x,t) [x is considered as a dummy variable and t, a parameter].

Illustrating the basic concept of **<u>He's Variational Iteration Method</u>**, we consider the following general differential equations:

$$LU(x,t) + NU(x,t) = g(x,t) --------(i)$$

$$\text{with the initial condition, } U(x,0) = G(x) ----(ii)$$

Where L is a linear operator of the first Order, N is a non-linear operator and $g(x,t)$ is non-homogenous term. According to Variational Iteration Method, we can construct a correction functional as follows:

$$U_{n+1} = U_n + \int_0^t \lambda\left[LU(x,s) + N(x,s) - g(x,s)\right] ds ----- (iii)$$

where λ is a Lagrange Multiplier ($\lambda = -1$), the subscripts $'n'$ denotes the nth approximation, $\bar{U}_n$ is considered as a restricted variation, i.e $\partial\bar{U}_n = 0$.
Equation (iii) is called a **Correction Functional**
Obtaining the Lagrange Multiplier $'\lambda'$ by using Integration by Part of Equation (i), but the Lagrange Multiplier is of the form $\lambda = \lambda(x,t)$.
Then, Laplace transform of equation (iii), then the correction functional will be in the form:

$$\mathcal{L}\left[U_{n+1}(x,t)\right] = \mathcal{L}\left[U_n(x,t)\right] + \mathcal{L}\left[\int_0^t \lambda(x,t)\left(LU_n(x,s) + NU_n(x,s) - g(x,s)\right) ds\right],\ n \geq 0 -----(iv)$$

Therefore,

$$\mathcal{L}\left[U_{n+1}(x,t)\right] = \mathcal{L}\left[U_n(x,t)\right] + \lambda\left[LU_n(x,t) + NU_n(x,t) - g(x,t)\right] ---------(v)$$

To find the optimal value of $\lambda(x,t)$, we first take the Variation with respect to $U_n(x,t)$ and in such a case, the integration is basically the single convolution to t, and hence, Laplace transform is appropriate to use.

thus;

$$\frac{\partial}{\partial U_n}\mathcal{L}\left[U_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\frac{\partial}{\partial U_n}\mathcal{L}\left[LU_n(x,t) + NU_n(x,t) - g(x,t)\right] - - - (vi)$$

Equation (vi) becomes,

$$\mathcal{L}\left[\partial U_{n+1}(x,t)\right] = \mathcal{L}\left[\partial U_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\partial\mathcal{L}\left[LU_n(x,t)\right] - - - - - - - (vii)$$

$\{i.e\ \ \partial N\bar{U}_n(x,t) = 0$ and $\partial g(x,t) = 0\}$

We assume that L is a linear first-order Partial Differential Operator in this chapter given by $\frac{\partial}{\partial t}$ then, equation (vii) can be written in the form

$\mathcal{L}\left[\partial U_{n+1}(x,t)\right] = \mathcal{L}\left[\partial U_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\left[s\mathcal{L}\partial U_n(x,t)\right]$

the extreme condition of $U_{n+1}(x,t)$ requires that $\partial U_{n+1}(x,t) = 0$

$\Longrightarrow 0 = \mathcal{L}\left[\partial U_n(x,t)\right]\left[1 + s\mathcal{L}\left[\lambda(x,t)\right]\right]$

$\Longrightarrow 1 + s\mathcal{L}\left[\lambda(x,t)\right] = 0$

$\Longrightarrow s\mathcal{L}\left[\lambda(x,t)\right] = -1$

$\Longrightarrow \mathcal{L}\left[\lambda(x,t)\right] = \dfrac{-1}{s}$

Taking the Laplace Inverse of both sides

$\lambda(x,t) = \mathcal{L}^{-1}\left[\dfrac{-1}{s}\right]$

$\lambda(x,t) = -1$

This implies $\lambda = -1$

Substituting ($\lambda = -1$) in equation (iii)

$$U_{n+1} = U_n - \int_0^t \left[LU_n(x,s) + N\bar{U}_n(x,s) - g(x,s)\right] ds - - - - - - - - - (viii)$$

The successive approximation 'U_{n+1}' of the solution 'u' will be readily obtained by using the determined Lagrange Multiplier and any selective function U_n consequently, the solution is given by:

$$U(x,t) = \lim_{x\to\infty} U_n(x,t)$$

Also, from equation (i)

$$i.e \;\; LU(x,t) + NU(x,t) = g(x,t) - - - - - - - - (*)$$

taking the Laplace transform of both sides, we have

$$\mathcal{L}\left[LU(x,t)\right] + \mathcal{L}\left[NU(x,t)\right] = \mathcal{L}\left[g(x,t)\right]$$