# NEWTON-COTES FORMULAE FOR NUMERICAL INTEGRATION

***BY***

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# Certification

This is to certify that this project was carried out by **AMAO, KAMALDEEN OLANREWAJU** of Matriculation Number 16/56EB046, for the award of Bachelor of Science B.Sc (Hons) degree in the Department of Mathematics, Faculty of Physical sciences, University of Ilorin, Ilorin, Nigeria.

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# Chapter 1

# GENERAL INTRODUCTION

## 1.1 INTRODUCTION

The general problem of numerical integration may be stated as follows. Given a set of data points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ of a function $y = f(x)$ is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y \text{ dx} \tag{1.1}$$

As in the case of numerical differentiation, one replaces $f(x)$ by an interpolating polynomial $\phi(x)$ and obtains, an integration, integral. Thus, different integration formulae can be obtained depending upon the type of the interpolation formulae used.

Let the interpolation points $x_i$, be equally spaced, that is let $x_i = x_0 + ih,\ i = 0, 1, 2, \cdots, n$ and let the end points of the interval of integration be placed such that

$$x_0 = a, \qquad x_n = b, \qquad h = \frac{b-a}{n}.$$

Then the definite integral

$$I = \int_a^b y \ \mathrm{dx} \tag{1.2}$$

is evaluated by an integration formula of the type

$$I_n = \sum_{i=0}^{n} Q_i y_i, \tag{1.3}$$

where the coefficient $Q_i$ are determined completely by the abscissa $x_i$. Integration formulae of the type **(1.3)** are called **Newton-Cotes Closed Integration Formulae**. They are **"closed"** since the end points $a$ and $b$ are the extreme abscissa in the formulae. It is easily seen that the integration formulae derived in **(1.2)** are the simplest Newton-Cotes Closed Formulae. On the other hand, formulae which do not employ the endpoints are called **Newton-Cotes Open Integration Formulae**. The Newton-Cotes Formulae are an extremely useful and straightforward family of numerical integration techniques.

## 1.2 STATEMENT OF THE PROBLEM

In order to obtain the solution of the numerical integration using Newton-Cotes Formulae, there must be three basic steps:

1. To integrate a function $f(x)$ over some interval $[a, b]$, we divide it into $n$ equal parts such that $f_n = f(x_n)$ and $h \equiv (b-a)/n$.

2. Then find the polynomials which approximate the tabulated function.

3. Integrate them to approximate the Area under the curve.

## 1.3 AIMS AND OBJECTIVES OF THE STUDY

The aim of this project is to study and obtain numerical solution of integration using Newton-Cotes Formulae. The objectives were to:

1. Obtain the method which are found to give sufficient accuracy in numerical integration;
2. Compare the numerical solution of the problem using the method stated in the aim with the exact value; and
3. recommend the method if it gives a sufficient accuracy for the solution of the problem.

## 1.4 SIGNIFICANCE OF STUDY

The method discussed in this project is extremely useful and straightforward family of numerical integration techniques. It is also used to find the fitting Polynomials which is used in Lagrange Interpolating Polynomials.

## 1.5 SCOPE OF THE STUDY

The scope of this project is meant to solve a numerical integration and provide a foundation knowledge for all subsequent steps in other project within the scope of work of this nature. With the knowledge of the subject, we tried to the project work explanatory by examples with concurrently understandable explanations, hereby creating a foreknowledge for people who are not familiar with the topic.

# Chapter 2

# LITERATURE REVIEW

## 2.1 REVIEW OF RELATED LITERATURE

Newton-Cotes Formulaemay be **"closed"** if the interval $[x_1, x_2]$ is included in the fit **"open"** if the points $[x_2, x_{n-1}]$ are used, or a variation of these two. If the formula uses $n$ points (closed or open), the coefficients, of terms sum to $n - 1$. In 1940, Daniell obtained remainders in interpolation and quadrature formulae. Hildebrand(1967), introduced some numerical solution in numerical analysis. Whittaker and Robinson (1967), derived some formulae of integration for the Newton-Cotes. Abrahmwitz and Stegun solved some problems of integration in handbook of Mathematical functions with formulae, graphs and mathematical tables. Press et al. (1992), obtained classical formulas for equally spaced abscissas.

Corbit (1996), obtained some solutions of numerical integration from Traphezoids to Root Mean Square (RMS). Ueberhuber (1997) computed some datas in numerical computations.

Fornberg(1998), calculated some weights in finite difference Formulas. Weerakoon and Fernando (2000), developed a variant of Newton's method with accelerated third-order convergence. Hasanov et al. (2002), modified Newton's method for numerical integration. Nedzhibov (2002), presented an iterative methods for solving nonlinear equations. Frontim and Sormom (2003), developed some variants of Newton's method with third order convergence.

In 2004, Ozban also developed some new variant of Newton's method. Lukic and Ralevic (2008), deduced Geometric Mean of Newton's method for simple and multiple roots.

Ababneh(2012), Solved problems on New Newton's method with third-order convergence for Solving nonlinear equations. Jain et al. (2012), established numerical methods for Scientific and engineering computation.

Jayakumar and Kalyanasundaran(2013), modified Newton's method using harmonic mean for solving nonlinear equations. More results an Newton-Cotescould be seen Layakumar(2013).

## 2.2 NUMERICAL INTEGRATION

The general problem of numerical integration may be stated as follows. Give a set of data point $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute the value of definite integral

$$I = \int_a^b y \text{ dx} \tag{2.1}$$

As in the case of numerical differentiation, one replaces $f(x)$ by an interpolating polynomial $phi(x)$ and obtains, on integration, an approximate value of the definite integral.

Thus, different integration formulae can be obtained depending upon the type of the interpolation formulae used. Using Newton's Forward Difference Formula to derive the general formula.

Let the interval $[a, b]$ be divided into $n$ equal subintervals such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence the integral becomes

$$I = \int_{x_0}^{x_n} y \text{ dx}$$

Approximating $y$ by Newton's Forward Difference Formula, we obtain

$$I = \int_{x_0}^{x_n} \left[ y_0 + P\Delta y_0 + \frac{(P-1)}{2}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{6}\Delta^3 y_0 + \cdots \right] \text{dx}$$

Since $x = x_0 + Ph,\ \text{dx} = hdp$ and hence the above integral becomes

$$I = h\int_{x_0}^{x_n} \left[ y_0 + P\Delta y_0 + \frac{(P-1)}{2}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{6}\Delta^3 y_0 + \cdots \right] \text{dp}$$

Which gives on simplification

$$\int_{x_0}^{x_n} = nh\left[ y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta y_0 + \cdots \right] \qquad (2.2)$$

From this general formula, we can obtain different integration formulae by putting $n = 1, 2, 3, \cdots$ etc.

# Chapter 3

# THE NEWTON-COTES FORMULAE

## 3.1 TRAPEZOIDAL RULE

Setting $n = 1$ in the general formula in **(2.2)**, all difference higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1} y \text{ dx} = h(y_0 + \frac{1}{2}\Delta y_0) = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1) \qquad (3.1)$$

For the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_1}^{x_2} y \text{ dx} = \frac{h}{2}(y_1 + y_2) \qquad (3.2)$$

and so on. For the last interval $[x_{n-1}, x_n]$ we have

$$\int_{x_{n-1}}^{x_n} y \text{ dx} = \frac{h}{2}(y_{n-1} + y_n) \qquad (3.3)$$

Combining all these expressions, we obtain the

$$\int_{x_0}^{x_1} y \text{ dx} = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right] \qquad (3.4)$$

Which is known as the **Trapezoidal Rule**.

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by $n$ straight lines joining the point $(x_0, y0)$ and $(x_1, y_1), (x_1, y_1)$ and $(x_2, y_2), \cdots, (x_{n-1}, y_{n-1})$ and $(x_n, y_n)$. The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the $n$ trapeziums obtained. The error of the trapezoidal formula can be obtained in the following way.

Let $y = f(x)$ be continuous, well-behaved, and possess continuous derivatives in $[x_0, x_n]$. Expanding $y$ in a Taylor's series around $x = x_0$, we obtain

$$\begin{aligned}\int_{x_0}^{x_1} y \text{ dx} &= \int_{x_1}^{x_2} \left[ y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2} y_0'' + \cdots \right] dx \\ &= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0''' + \cdots \qquad (3.5)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{h}{2}(y_0 + y_1) &= \frac{h}{2}(y_0 + y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \cdots) \\ &= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \cdots \qquad (3.6)\end{aligned}$$

From **(3.5)** and **(3.6)**, we obtain

$$\int_{x_0}^{x_n} y \text{ dx} - \frac{h}{2}(y_0 + y_1) = -\frac{1}{12}h^3 y_0'' + \cdots, \qquad (3.7)$$

Which is the error in the interval $[x_0, x_1]$. Proceeding in a similar manner we obtain errors in the remaining subintervals, viz, $[x_1, x_2], [x_2, x_3], \cdots$ and $[x_{n-1}, x_n]$. We thus have

$$E = \frac{1}{12}h^3(y_0'' + y_1'' + \cdots + y_{n-1}'') \qquad (3.8)$$

Where $E$ is the total error. Assuming that $y''(\bar{x})$ is the largest value of the $n$ quantities on the right-hand of **(3.8)**, we obtain

$$E = \frac{1}{12}h^3 n y''(\bar{x}) = -\frac{b-a}{12}h^2 y''(\bar{x}) \tag{3.9}$$

Since $nh = b - a$

## 3.2 SIMPSON 1/3-RULE

This rule is obtained by putting $n = 2$ in **(2.2)**, that is by replacing the curve by $n/2$ arc of second-degree polynomials or parabolas. We have then

$$\int_{x_0}^{x_2} y \text{ dx} = 2h(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_2}^{x_4} y \text{ dx} = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\vdots$$

and finally

$$\int_{x_{n-2}}^{x_n} y \text{ dx} = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

summing up, we obtain

$$\begin{aligned}\int_{x_0}^{x_n} y \text{ dx} &= \tfrac{h}{3}\big[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) \\ &+ \ 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + y_n\big] \qquad (3.10)\end{aligned}$$

Which is known as **Simpson's** 1/3**-Rule**, or simply **Simpson's Rule**. It should be noted that this rule requires the division of the whole range into an even number of subintervals of width $h$.

Following the method, it can be shown that error in Simpson's rule is given by

$$\begin{aligned}
\int_a^b \text{ dx} &= \frac{h}{3}\big[y_0 + 4\big(y_1 + y_3 + y_5 + \cdots + y_{n-1}\big) \\
&+ 2\big(y_2 + y_4 + y_6 + \cdots + y_{n-1} + y_n\big) \\
&= \frac{b-a}{180}h^4 y^{iv}(\bar{x}),
\end{aligned} \tag{3.11}$$

Where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

## 3.3 SIMPSON'S 3/8-RULE

Setting $n = 3$ in **(2.2)**, we observe that all the differences higher than the third become zero and we obtain

$$\begin{aligned}
&= 3h\big(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\big) \\
&= 3h\big[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\big] \\
&= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)
\end{aligned}$$

Similarly,

$$\int_{x_3}^{x_6} y \text{ dx} = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

and so on. Summing up all these, we obtain

$$\begin{aligned}
\int_{x_0}^{x_n} y \text{ dx} &= \frac{3h}{8}\big[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots \\
&+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots \\
&+ \quad 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n) \qquad (3.12)
\end{aligned}
$$

This rule, called **Simpson's** $3/8$**-Rule**, is not so accurate as Simpson's Rule, the formula being $(3/80)h^5 y^{iv}(\bar{x})$

## 3.4 BOOLE'S AND WEDDLE'S RULES

If we wish to retain differences up to those of the fourth order, we should integrate between $x_0$ and $x_4$ and obtain **Boole's Formula**

$$
\int_{x_0}^{x_n} y \, \mathrm{dx} = \frac{2h}{45}\big(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\big) \qquad (3.13)
$$

The leading team in the error of this formula can shown to be

$$
\frac{-8h^7}{945} y^{vi}(\bar{x})
$$

If, one the other hand we integrate between $x_0$ and $x_6$ retaining difference up to those of sixth order, we obtain **Weddle's Rule**

$$
\int_{x_0}^{x_6} y \, \mathrm{dx} = \frac{3h}{10}\big(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\big) \qquad (3.14)
$$

the error in which is given by $-(h^7/140)y^{vi}(\bar{x})$.

These two formulae can also be generalized as in the previous cases. It should, however, be noted that the number of strips will have to be a multiple of four in the case of Boole's Rule and a multiple of six for Weddle's Rule.

## 3.5 USE OF CUBIC SPLINES

If $S(x)$ is the **Cubic Spline** in the interval $(x_{i-1}, x_i)$, then we have

$$
\begin{aligned}
I &= \int_{x_0}^{x_n} y \text{ dx} \approx \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S(x) \text{ dx} \\
&= \sum_{i=x_{i-1}}^{n} \int_{x_{i-1}}^{x_i} \left\{ \frac{1}{6h} \left[ (x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i \right] \right. \\
&+ \left. \frac{1}{h}(x_i - x) \left[ y_{i-1} - \frac{h^2}{6} M_{i-1} \right] + \frac{1}{h}(x - x_{i-1}) \left( y_i - \frac{h^2}{6} M_i \right) \right\} \text{dx}
\end{aligned}
$$

Using

$$
\begin{aligned}
S_i(x) &= \frac{1}{h_i} \left[ \frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i + \left( y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) (x_i - x) \right. \\
&+ \left. \left( y_i - \frac{h_i^2}{6} M_i (x - x_{i-1}) \right) \right]
\end{aligned}
$$

On carrying out the integration and simplifying, we obtain

$$
I = \sum_{i=1}^{n} \left[ \frac{h}{2}(y_{i-1} + y_i) - \frac{h^3}{24}(M_{i-1} + M_i) \right] \tag{3.15}
$$

Where $M_i$, the **Spline Second-derivatives**, are calculated from recurrence relation

$$
M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \; i = 1, 2, \cdots, n-1
$$

## 3.6 ROMBERG INTEGRATION

This method can often be used to improve the approximate results obtained by the finite-difference methods. It application to the numerical evaluation of definite integrals, for example in the use of Trapezoidal Rule, can be described, as follows.

We consider the integral

$$I = \int_a^b y \text{ dx}$$

and evaluate it by the Trapezoidal Rule **(3.3)** with two different subintervals of widths $h_1$ and $h_2$ to obtain the approximate values $I_1$ and $I_2$, respectively. Then **(3.9)** gives errors $E_1$ and $E_2$ as

$$E_1 = -\frac{1}{12}(b-a)h^2,\ y''(\bar{x}) \tag{3.16}$$

and

$$E_2 = -\frac{1}{12}(b-a)h^2,\ y''(\bar{\bar{x}}) \tag{3.17}$$

Since the term $y''(\bar{\bar{x}})$ in **(3.17)** is also the largest value of $y''$, it is reasonable to assume that the quantities $y''(\bar{x})$ and $y''(\bar{\bar{x}})$ are very nearly the same. We therefore have

$$\frac{E_1}{E_2} = \frac{h_1^2}{h_2^2}$$

and hence

$$\frac{E_2}{E_2 - E_1} = \frac{h_2^2}{h_2^2 - h_1^2}$$

Since $E_2 - E_1 = I_2 - I_1$, this gives

$$E_2 = \frac{h_2^2}{h_2^2 - h_1^2}(I_2 - I_1) \tag{3.18}$$

We therefore obtain a new approximation $I_3$ defined by

$$I_3 = I_2 - E_2 = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \tag{3.19}$$

Which, in general, would be closer to the actual value provided that the errors decrease monotonically and are the same sign.

If we now set

$$h_2 = \frac{1}{2}h_1 = \frac{1}{2}h$$

eqn **(3.19)** can be written in the more convenient form

$$I\left(h, \frac{1}{2}h\right) = \frac{1}{3}\left[4I\left(\frac{1}{2}h\right) - I\left(h\right)\right] \tag{3.20}$$

Where $I(h) = I_1,\ I(\frac{1}{2}h) = I_2$ and $I(h, \frac{1}{2}h) = I_3$

with this notation, the following table can be formed

$$\begin{array}{ccccccc}
I(h) & & & & & & \\
 & I(h, \frac{1}{2}h) & & & & & \\
I(\frac{1}{2}h) & & I(h, \frac{1}{2}h, \frac{1}{4}h) & & & & \\
 & I(\frac{1}{2}h, \frac{1}{4}h) & & I(h, \frac{1}{2}h, \frac{1}{4}h, \frac{1}{8}h) & & & \\
I(\frac{1}{4}h) & & I(\frac{1}{2}h, \frac{1}{4}h, \frac{1}{8}h) & & & & \\
 & I(\frac{1}{4}h, \frac{1}{8}h) & & & & & \\
I(\frac{1}{8}h) & & & & & &
\end{array}$$

The computations can be stopped when two successive values are sufficiently close to each other. This method, due to L.F. Richardson, is called the deferred approach to the limit and Systematic tabulation of this is called **Romberg Integration**.

## 3.7 NEWTON-COTES INTEGRATION FORMULAE

Let the interpolation points, $x_i$, be equally spaced, that is let $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$, and let the endpoints of the interval of integration be placed such that

$$x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n}$$

The definite integral

$$I = \int_a^b y \text{ dx} \tag{3.21}$$

is evaluated by an integration formula of the type

$$I_n = \sum_{i=0}^{n} c_i y_i \tag{3.22}$$

Where the coefficients $c_i$ are determined completely by the abscissa $x_i$. Integration Formulae of the type **(3.22)** are called **Newton-Cotes Closed Integration Formulae**.

They are **"closed"** since the endpoints $a$ and $b$ are extreme abscissa in the formulae. It is easily seen that the integration formulae derived eqns **(3.18)** - **(3.21)** are simplest Newton-Cotes Closed Formulae.

On the other hand, formulae which do not employ the endpoints are called **Newton-Cotes, Open Integration Formulae**.

We give below the five simplest Newton-Cotes Open Integration Formulae

$$\text{(i)} \int_{x_0}^{x_2} y \text{ dx} = 2hy_1 + \frac{h^3}{3}y''(\bar{x}), (x_0 \leq \bar{x} < x_2) \tag{3.23}$$

$$\text{(ii)} \int_{x_0}^{x_3} y \text{ dx} = \frac{3h}{2}(y_1 + y_2) + \frac{3h^3}{4}y''(\bar{x}), (x_0 < \bar{x} < x_3) \tag{3.24}$$

$$\text{(iii)} \int_{x_0}^{x_4} y \text{ dx} = \frac{4h}{3}(2y_1 - y_2 + 2y_3) + \frac{14}{45}h^5 y^{iv}(\bar{x}), \quad (x_0 \leq \bar{x} < x_4) \tag{3.25}$$

$$(iv) \int_{x_0}^{x_5} y \text{ dx} = \frac{5h}{24}(11y_1 + y_2 + y_3 + 11y_4) + \frac{95}{144}h^5 y^{iv}(\bar{x}), (x_0 < \bar{x} < x_5) \tag{3.26}$$

$$(v) \int_{x_0}^{x_6} y \text{ dx} = \frac{6h}{20}(11y_1 - 14y_2 + 26y_3 - 14y_4 + 11y_5) + \frac{41}{140}h^7 y^{vi}(\bar{x}), (x_0 < \bar{x} < x_6) \tag{3.27}$$

A convenient method for determining the coefficients in the Newton-Cotes Formulae is the method of **Undetermined Coefficients**.

## 3.8 PRINCIPLE OF NEWTON-COTES INTEGRATION

This subsection only cover the Newton-Cotes closed formulas.

The interval of integration $[a, b]$ is partitioned by the points.

$$\left\{a, a + \frac{b-a}{n}, a + 2 \cdot \frac{b-a}{n}, \cdots, b\right\}$$

We estimate the integral of $f(x)$ on this interval by using the Lagrange Interpolating Polynomial through the following points.

$$\left\{(a, f(a)), \left(a + \frac{b-a}{n}, f\left(a + \frac{b-a}{n}\right)\right), \cdots, (b, f(b))\right\}$$

The formula for the integral of this Lagrange Polynomial simplifies to a linear combination of the values of $f(x)$ at the points

$$\left\{x_i = a + i \cdot \frac{b-a}{n} \mid i = 0, 1, 2, 3, \cdots, n\right\}$$

## 3.9 ERROR ANALYSIS OF NEWTON-COTES

One would expect that the error would grow smaller as we use larger $n$ in the Newton-Cotes method. It turns out that this not correct. The reason is that using equally spaced points, the Lagrange interpolating Polynomial may give a very bad approximation of the function away from the interpolating points. It can be so bad that the integrals of these polynomials do not converge to the integral of the function $f(x)$ as $n \to \infty$.

The classic example is the following function over the interval $[-4, 4]$

$$f(x) = \frac{1}{1 + x^2}$$

# Chapter 4

# COMPUTATIONAL PROBLEMS

In this chapter, three problems was solved by Closed Newton-Cotes Formulae and the results obtained were compared with the exact solution.

## 4.1 NUMERICAL EXAMPLES

**Example 4.1**

Consider the integral:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\ \mathrm{dx}, \qquad h = \frac{\pi}{24}$$

Solving for the exact value

$$(1)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\ \mathrm{dx}, \quad h = \frac{\pi}{24}$$

By trigonometry identity

$\sin x = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$

Let $I = \displaystyle\int \frac{\sin x}{x}\ \mathrm{dx}$

$$I = \int \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})\ \mathrm{dx}}{x}$$

$$I = \int \frac{\sin(\frac{x}{2})\cos(\frac{x}{2})}{\frac{x}{2}}\ \mathrm{dx}$$

Put $\sin\dfrac{x}{2} = t \implies \dfrac{x}{2} = \sin^{-1}(t)$

$$\implies \frac{\mathrm{dt}}{\mathrm{dx}} = \frac{1}{2}\cos(\frac{x}{2})$$

$$\implies \cos(\frac{x}{2})\mathrm{dx} = 2\mathrm{dt}$$

Therefore,

$$I = 2\int \frac{t\ \mathrm{dt}}{\sin^{-1}(t)}$$

$$2\int t\ \mathrm{cosec}(t)\ \mathrm{dt}$$

Integrating by parts

$$
\begin{aligned}
I &= 2\left[\ \operatorname{cosec}^{-1} t \cdot \frac{t^2}{2} - \int \frac{-1}{t\sqrt{t^2-1}} \cdot \frac{t^2}{2}\ \mathrm{dt}\right] \\
&= 2\left[\frac{t^2\ \operatorname{cosec}^{-1}(t)}{2} + \frac{1}{2}\int \frac{t}{\sqrt{t^2-1}}\ \mathrm{dt}\right]
\end{aligned}
$$

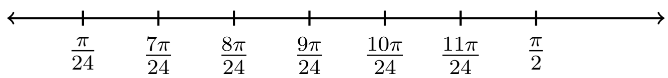
Let $u = t^2 - 1$ du$= 2t$dt $\frac{\mathrm{du}}{2} = t$ dt

$$
\begin{aligned}
&= 2\left[\frac{t^2\ \operatorname{cosec}^{-1}(t)}{2} + \frac{1}{2}\int \frac{1}{\sqrt{2}}\frac{\mathrm{du}}{2}\right] \\
&= t^2\ \operatorname{cosec}^{-1}(t) + \frac{1}{2}\frac{u^{\frac{1}{2}}}{-\frac{1}{2}+1} + C \\
&= t^2\ \operatorname{cosec}^{-1}(t) + \sqrt{u} \\
&= \frac{\sin^2(\frac{x}{2})}{\sin^{-1}(\frac{x}{2})} + \sqrt{\sin^2\frac{x}{2} - 1} + C \\
I &= \frac{2\sin^2(\frac{x}{2})}{x} + \sqrt{\sin^2\left(\frac{x}{2}\right) - 1} + C
\end{aligned}
$$

So, for $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}$ dx we have

$$
\begin{aligned}
&= \left[\frac{2\sin^2(\frac{x}{2})}{x} + \sqrt{\sin^2(\frac{x}{2}) - 1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left[\frac{2\sin^2(\frac{\pi}{4})}{\frac{\pi}{2}} + \sqrt{\sin^2(\frac{\pi}{4}) - 1}\quad\right] - \left[\frac{2\sin^2(\frac{\pi}{8})}{\frac{\pi}{4}} + \sqrt{\sin^2(\frac{\pi}{8}) - 1}\quad\right] \\
&= \left[\frac{2(\frac{1}{2})}{\frac{\pi}{2}} + \sqrt{\frac{1}{2} - 1}\quad\right] - \left[\frac{2(0.146)}{\frac{\pi}{4}} + \sqrt{(0.146) - 1}\quad\right] \\
&= 0.611786 \\
&\approxeq \mathbf{0.6118}
\end{aligned}
$$

$$f(x) = \frac{\sin x}{x}, \qquad \text{Let } h\frac{\pi}{24}$$




$$x_0 = \frac{\pi}{24} \implies f(\frac{\pi}{24}) = \frac{\sin\frac{\pi}{24}}{\frac{\pi}{24}} = \frac{0.7071}{45} = 0.0157$$

$$x_1 = \frac{7\pi}{24} \implies f(\frac{7\pi}{24}) = \frac{\sin(\frac{7\pi}{24})}{\frac{7\pi}{24}} = \frac{0.7934}{52.5} = 0.0151$$

$$x_2 = \frac{8\pi}{24} \implies f(\frac{8\pi}{24}) = \frac{\sin(\frac{8\pi}{24})}{\frac{8\pi}{24}} = \frac{0.8660}{60} = 0.0144$$

$$x_3 = \frac{9\pi}{24} \implies f(\frac{9\pi}{24}) = \frac{\sin(\frac{9\pi}{24})}{\frac{9\pi}{24}} = \frac{0.9239}{67.5} = 0.0137$$

$$x_4 = \frac{10\pi}{24} \implies f(\frac{10\pi}{24}) = \frac{\sin(\frac{10\pi}{24})}{\frac{10\pi}{24}} = \frac{0.9659}{75} = 0.0129$$

$$x_5 = \frac{11\pi}{24} \implies f(\frac{11\pi}{24}) = \frac{\sin(\frac{11\pi}{24})}{\frac{11\pi}{24}} = \frac{0.9914}{82.5} = 0.0120$$

$$x_6 = \frac{\pi}{2} \implies f(\frac{\pi}{2}) = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{1}{90} = 0.011$$

**Using Trapezoidal**

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_{n-1}) + \cdots + f(x_n)\right]$$

$$
\begin{aligned}
T_6 &= \frac{7.5}{2}\left[0.0157 + 2(0.0151) + 2(0.014) + 2(0.0137) + 2(0.0129) + 2(0.0120) + 0.0111\right] \\
&= \frac{7.5}{2}\left[0.0157 + 0.0302 + 0.0288 + 0.0274 + 0.0258 + 0.024 + 0.0111\right] \\
&= 3.75[0.163] \\
&= 0.61125 \\
&\approxeq 0.6113
\end{aligned}
$$

**Using Simpson 1/3-Rule**

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_{n-1}) + \cdots f(x_n)\right]$$

$$
\begin{aligned}
S_6 &= \frac{7.5}{3}\left[0.0157 + 4(0.0151) + 2(0.0144) + 4(0.0137) + 2(0.0129) + 4(0.0120) + 0.0111\right] \\
&= \frac{7.5}{3}[0.0157 + 0.0604 + 0.0288 + 0.0548 + 0.0258 + 0.048 + 0.011] \\
&= 2.5[0.2446] \\
&\approx 0.6115
\end{aligned}
$$

**Using Simpson 3/8-Rule**

$$S_n = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_{n-1}) + f(x_n)\right]$$

$$\begin{aligned}
S_6 &= \frac{3(7.5)}{8}\left[0.0157 + 3(0.0151) + 3(0.0144) + 2(0.0137) + 3(0.0129) + 3(0.0120) + 0.0111\right] \\
&= 2.8125[0.0157 + 0.0453 + 0.0432 + 0.0274 + 0.0387 + 0.011] \\
&= 2.8125[0.2174] \\
&= 0.61144375 \\
&\approx 0.6114
\end{aligned}$$

**Using Boole's Rule**

$$B_n = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

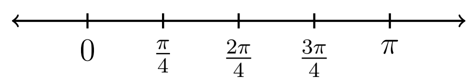
$$\begin{aligned}
B_6 &= \frac{2(7.5)}{45}[7(0.0157) + 32(0.0151) + 12(0.0144) + 32(0.0137) + 12(0.0129) + 32(0.0120) + 7(0. \\
&= 0.33[0.1099 + 0.4832 + 0.1728 + 0.4384 + 0.1548 + 0.384 + 0.0777] \\
&= 0.33[1.8208] \\
&= 0.600864 \\
&\approx 0.6009
\end{aligned}$$

**Example 4.2**

$$\int_0^{\pi} \frac{\mathrm{dx}}{x+\cos x}, \quad h = \frac{\pi}{4}, \quad h = \frac{\pi}{8}$$

**Solution**

$$f(x) = \frac{1}{x+\cos x}$$




$$\begin{aligned}
x_0 &= 0 \implies f(0) = \frac{1}{0+\cos 0} = \frac{1}{1} = 1 \\
x_1 &= \frac{\pi}{4} \implies f\left(\frac{\pi}{4}\right) = \frac{1}{\frac{\pi}{4}+\cos\left(\frac{\pi}{4}\right)} = \frac{1}{45+0.7071} = \frac{1}{45.7071} = 0.0219 \\
x_2 &= \frac{2\pi}{4} \implies f\left(\frac{2\pi}{4}\right) = \frac{1}{\frac{2\pi}{4}+\cos\left(\frac{2\pi}{4}\right)} = \frac{1}{90+0} = 0.0111 \\
x_3 &= \frac{3\pi}{4} \implies f\left(3\frac{\pi}{4}\right) = \frac{1}{\frac{3\pi}{4}+\cos\left(\frac{3\pi}{4}\right)} = \frac{1}{135+(-0.7071)} = \frac{1}{134.2929} = 0.0074 \\
x_4 &= \pi \implies f(\pi) = \frac{1}{\pi+\cos(\pi)} = \frac{1}{180+(-1)} = \frac{1}{179} = 0.0056
\end{aligned}$$

**Using Trapezoidal**

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_{n-1}) + \cdots + f(x_n)\right]$$

$$
\begin{aligned}
T_4 &= \frac{45}{2}\left[1 + 2(0.0219) + 2(0.0111) + 2(0.074)_0.0056\right] \\
&= 22.5\left[1 + 0.0438 + 0.0222 + 0.0148 + 0.0056\right] \\
&= 22.5[1.2196] \\
&= 27.441
\end{aligned}
$$

**Using Simpson 1/3-Rule**

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_{n-1}) + \cdots f(x_n)\right]$$

$$
\begin{aligned}
S_4 &= \frac{45}{3}\left[1 + 4(0.219) + 2(0.0111) + 4(0.0074) + 0.0056\right] \\
&= 15\left[1 + +0.876 + 0.0222 + 0.0296 + 0.0056\right] \\
&= 15[1.9574] \\
&= 29.361
\end{aligned}
$$

**Using Simpson 3/8-Rule**

$$S_n = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_{n-1}) + f(x_n)\right]$$

$$\begin{aligned}
S_4 &= \frac{3(45)}{8}\left[1 + 3(0.0219) + 3(0.0111) + 2(0.0074) + 0.0056\right] \\
&= 16.875[1 + 0.0657 + 0.0333 + 0.0148 + 0.0056] \\
&= 16.875[1.1194] \\
&= 18.889875 \\
&\approx 18.89
\end{aligned}$$

**Using Boole's Rule**

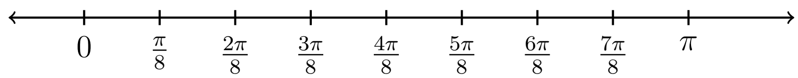
$$B_n = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

$$\begin{aligned}
B_4 &= \frac{2(45)}{45}\left[7(1) + 32(0.0219) + 12(0.0111) + 32(0.0074) + 7(0.0056)\right] \\
&= 2[7 + 0.7008 + 0.1332 + 0.2368 + 0.0392] \\
&= 2[8.11] \\
&= 16.22
\end{aligned}$$

**2nd Solution**

$$\int_0^{\pi} \frac{\mathrm{dx}}{x + \cos x}, \quad h = \frac{\pi}{8}$$

$$f(x) = \frac{1}{x + \cos x}$$




$$\begin{aligned}
x_0 &= 0 \implies f(0) = \frac{1}{0 + \cos 0} = \frac{1}{1} = 1 \\
x_1 &= \frac{\pi}{8} \implies f\left(\frac{\pi}{8}\right) = \frac{1}{\frac{\pi}{8} + \cos\left(\frac{\pi}{8}\right)} = \frac{1}{22.5 + 0.9239} = \frac{1}{23.4239} = 0.0427 \\
x_2 &= \frac{2\pi}{8} \implies f\left(\frac{2\pi}{8}\right) = \frac{1}{\frac{2\pi}{8} + \cos\left(\frac{2\pi}{8}\right)} = \frac{1}{45 + \cos 45} = 0.0219 \\
x_3 &= \frac{3\pi}{8} \implies f\left(\frac{3\pi}{8}\right) = \frac{1}{\frac{3\pi}{8} + \cos\left(\frac{3\pi}{8}\right)} = \frac{1}{135 + (-0.7071)} = \frac{1}{134.2929} = 0.0074 \\
x_4 &= \frac{4\pi}{8} \implies f\left(\frac{4\pi}{8}\right) = \frac{1}{\frac{4\pi}{8} + \cos\left(\frac{4\pi}{8}\right)} = \frac{1}{90 + \cos 90} = 0.0111 \\
x_5 &= \frac{5\pi}{8} \implies f\left(\frac{5\pi}{8}\right) = \frac{1}{\frac{5\pi}{8} + \cos\left(\frac{5\pi}{8}\right)} = \frac{1}{112.5 + (-0.3827)} = \frac{1}{112.1173} = 0.0089 \\
x_6 &= \frac{6\pi}{8} \implies f\left(\frac{6\pi}{8}\right) = \frac{1}{\frac{6\pi}{8} + \cos\left(\frac{6\pi}{8}\right)} = \frac{1}{135 + \cos 135} = 0.0074
\end{aligned}$$

$$x_7 = \frac{7\pi}{8} \implies f\left(\frac{7\pi}{8}\right) = \frac{1}{\frac{7\pi}{8} + \cos\left(\frac{7\pi}{8}\right)} = \frac{1}{157.5 + (-0.9239)} = \frac{1}{156.5761} = 0.0064$$

$$x_8 = \pi \implies f(\pi) = \frac{1}{\pi + \cos(\pi)} = \frac{1}{180 + (-1)} = 0.0056$$

**Using Trapezoidal**

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_{n-1}) + \cdots + f(x_n)\right]$$

$$\begin{aligned} T_8 &= \frac{22.5}{2}[1 + 2(0.0427) + 2(0.0219) + 2(0.0147) + 2(0.0111) + 2(0.0089) \\ &+ 2(0.0074) + 2(0.0064) + 0.0056] \\ &= 11.25[1 + 0.0854 + 0.0438 + 0.0294 + 0.0222 + 0.0178 + 0.0148 \\ &+ 0.0128 + 0.0056] \\ &= 11.25[1.2318] \\ &= 13.85775 \\ &\approx 13.86 \end{aligned}$$

**Using Simpson 1/3-Rule**

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_{n-1}) + \cdots f(x_n)\right]$$

$$
\begin{aligned}
S_8 &= \frac{22.5}{3}\,[1+4(0.0427)+2(0.0219)+4(0.0147)+2(0.0111)+4(0.0089) \\
&+ 2(0.0074)+4(0.0064)+0.0056] \\
&= 7.5\,[1+0.1708+0.0438+0.0588+0.0222+0.0356+0.0148 \\
&+ 0.0256+0.0056] \\
&= 7.5[1.3772] \\
&= 10.329
\end{aligned}
$$

**Using Simpson 3/8-Rule**

$$S_n = \frac{3h}{8}\,[f(x_0)+3f(x_1)+3f(x_2)+2f(x_3)+3f(x_{n-1})+f(x_n)]$$

$$
\begin{aligned}
S_8 &= \frac{3(22.5)}{8}\,[1+3(0.0427)+3(0.0219)+2(0.0147)+3(0.0111)+3(0.0089) \\
&+ 2(0.0074)+3(0.0064)+0.0056] \\
&= 8.4375\,[1+0.1281+0.0657+0.0294+0.0333+0.0267+0.0148 \\
&+ 0.0192+0.0056] \\
&= 8.4375[1.3228] \\
&= 11.161125 \\
&\approx 11.16
\end{aligned}
$$

**Using Boole's Rule**

$$B_n = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

$$\begin{aligned}
B_8 &= \frac{2(22.5)}{45}\,[7(1) + 32(0.0427) + 12(0.0219) + 32(0.0147) + 12(0.0111) + 32(0.0089) \\
&+ 12(0.0074) + 32(0.0064) + 7(0.0056)] \\
&= \frac{45}{45}\,[7 + 1.3664 + 0.2628 + 0.4704 + 0.1332 + 0.2848 + 0.0888 \\
&+ 0.2048 + 0.00392] \\
&= 9.8504 \\
&\approx 9.850
\end{aligned}$$

**Example 4.3**

$$\int_0^{\pi} \frac{\mathrm{dx}}{1+x^2}, \quad h = \frac{\pi}{4},$$

**Solution**

$$\int_0^{\pi} \frac{1}{1+x^2} \mathrm{d}x$$

Let $x = \tan u$

$$\mathrm{d}x = \sec^2 u \mathrm{d}u$$

$$\int_0^{\pi} \frac{1}{1+\tan^2 u} \cdot \sec^2 u \mathrm{d}u$$

$$\int_0^{\pi} \frac{1}{\sec^2 u} \cdot \sec^2 u \mathrm{d}u$$

$$\int_0^{\pi} \mathrm{d}u$$

$$\implies u\Big|_0^{\pi}$$

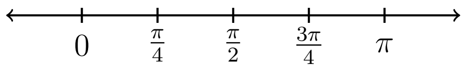
$$= \tan^{-1}(x)\Big|_0^{\pi}$$

$$= \tan^{-1}(\pi)$$

$$= 1.26262725$$

$$f(x) = \frac{\mathrm{d}x}{1+x^2}, \qquad \text{let } h = \frac{\pi}{4}$$

$$f(x) = \frac{\mathrm{d}x}{1 + x^2}$$




$$\begin{aligned}
x_0 &= 0 \implies f(0) = \frac{1}{1 + (2)^2} = \frac{1}{1} = 1 \\
x_1 &= \frac{\pi}{4} \implies f(\frac{\pi}{4}) = \frac{1}{1 + \left(\frac{\pi}{4}\right)^2} = \frac{1}{1.61685} = 0.061848 \\
x_2 &= \frac{\pi}{2} \implies f(\frac{\pi}{2}) = \frac{1}{1 + \left(\frac{\pi}{2}\right)^2} = \frac{1}{3.46740} = 0.28868 \\
x_3 &= \frac{3\pi}{4} \implies f(\frac{3\pi}{4}) = \frac{1}{1 + \left(\frac{3\pi}{4}\right)^2} = \frac{1}{6.55165} = 0.15263 \\
x_4 &= \pi \implies f(\pi) = \frac{1}{1 + (\pi)^2} = \frac{1}{4.14169} = 0.24145
\end{aligned}$$

**<u>Using Trapezoidal</u>**

$$\begin{aligned}
T_n &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_{n-1}) + \cdots + f(x_n))\right] \\
T_4 &= \frac{\pi}{8}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\
&= \frac{\pi}{8}\left[1 + 2(0.61848) + 2(0.28868) + 2(0.15263) + 0.24145\right] \\
&= 0.39269(1 + 1.36368 + 0.57736 + 0.30526 + 0.24145) \\
&= 0.39269(3.48775) \\
&= 1.36960
\end{aligned}$$

**Using Simpson 1/3 Rule**

$$
\begin{aligned}
S_n &= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_{n-1}) + \cdots + f(x_n)\right] \\
S_4 &= \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\
S_4 &= \frac{\pi}{12}(1 + 4(0.61848) + 2(0.28868) + 4(0.15263) + 0.24145) \\
&= 0.26179(1 + 2.47392 + 0.57736 + 0.61052 + 0.24145) \\
&= 0.26179(4.90325) \\
&= 1.28362
\end{aligned}
$$

**Using Simpson 3/8 Rule**

$$
\begin{aligned}
S_n &= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_{n-1}) + f(x_n)\right] \\
S_4 &= \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + f(x_4)) \\
&= \frac{3\pi}{32}(1 + 3(0.61848) + 3(0.28868) + 2(0.15263) + 0.24145) \\
&= 0.29452(1 + 1.85544 + 0.86604 + 0.30526 + 0.24145) \\
&= 0.29452(4.26819) \\
&= 1.25706
\end{aligned}
$$

**Using Boole's Rule**

$$\begin{aligned}
B_n &= \frac{2h}{45}(7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)7f(x_4)) \\
B_4 &= \frac{2h}{45}(7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)7f(x_4)) \\
B_4 &= \frac{\pi}{90}(7(1)+32(0.61848)+12(0.28868)+32(0.15263)+7(0.24145)) \\
&= 0.03490(7+19.79136+3.46416+4.88416+1.69015) \\
&= 0.03490(36.82983) \\
&= 1.28536
\end{aligned}$$

| Methods | | Error |
|---|---|---|
| Exact | 0.6118 | |
| Trapezoidal | 0.6113 | 0.0005 |
| Simpson 1/3-rule | 0.6115 | 0.0003 |
| Simpson 3/8-rule | 0.6114 | 0.0004 |
| Boole's rule | 0.6009 | 0.0109 |

Table 4.1: (Result for Example 1)

| Methods | | Error |
|---|---|---|
| Exact | 29.889 | |
| Trapezoidal | 27.441 | 2.448 |
| Simpson 1/3-rule | 29.361 | 0.528 |
| Simpson 3/8-rule | 18.890 | 10.999 |
| Boole's rule | 16.220 | 13.669 |

Table 4.2: (Result for Example 2, $h = \frac{\pi}{4}$)

| Methods | | Error |
|---|---|---|
| Exact | 29.889 | |
| Trapezoidal | 13.860 | 16.029 |
| Simpson 1/3-rule | 10.329 | 19.56 |
| Simpson 3/8-rule | 11.160 | 18.729 |
| Boole's rule | 9.850 | 20.039 |

Table 4.3: (Result for Example 2, $h = \frac{\pi}{8}$)

| Methods | | Error |
|---|---|---|
| Exact | 29.889 | |
| Trapezoidal | 13.860 | 0.10698 |
| Simpson 1/3-rule | 10.329 | 0.021 |
| Simpson 3/8-rule | 11.160 | 0.0056 |
| Boole's rule | 9.850 | 0.02274 |

Table 4.4: (Result for Example 3, $h = \dfrac{\pi}{4}$)

## 4.2 DISCUSSION OF RESULTS

In Table **(4.1)**,it shown that Simpson 1/3-rule is the most accurate of the Closed Newton-Cotes Formulaeand its also shows that Boole's rule is the least accurate of the Closed Newton-Cotes Formulae.

Table **(4.2)** also shows that Simpson 1/3-rule is the most accurate, while Boole's rule is also the least accurate among the Closed Newton-Cotes Formulae.

# Chapter 5

# SUMMARY, CONCLUSION AND RECOMMENDATION

## 5.1 SUMMARY

The numerical solution of definite integration by some Newton-Cotes formulae has been presented. For this projects, the Trapezoidal, Simpson's $\frac{1}{3}$, Simpson's $\frac{3}{8}$ and Boole quadrature formulae were adopted. The derivative of these formulae was also documented. The numerical solutions were applied to three selected problems and the numerical results were compared with the exact solution. These results confirm the accuracy of the schemes.

## 5.2 CONCLUSION

In the course of this study, it was numerical integration examples considered that Simpson 1/3-rule is the most accurate of the closed Newton-Cotes

Formulae.

## 5.3 RECOMMENDATION

Based on what we have considered in this study, it was shown closed Newton-Cotes Formulae for Numerical Integration produced near accurate value to the exact.

It is recommended that a formulae be done so that it value could be compare with the most accurate of the closed Newton-Cotes Formulae.