APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$L[y^{n}(t)] = s^{n}Y - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - s\,f^{n-2}(0) - \cdots - y^{n-1}(0)$$

$$L[u(x,t)] = \int_0^{\infty} e^{-st} u(x,t)\,dt \equiv U(x,s)$$

$$L[u_x(x,t)] = U_x(x,s)$$

$$L[u_t(x,t)] = sU(x,s) - U(x,0)$$

$$L[u_{tt}(x,t)] = s^2U(x,s) - sU(x,0) - u_t(x,0)$$

If $F(s) = L\{f(t)\}$ then,

$$L\{u(t-a)\cdot g(t-a)\} = e^{-as}G(s).$$

Linear P.D.E Of Order 2

Examples: (1) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t)$, $0<x<2,\ t>0$

$u(0,t)=0$, $u(2,t)=0$; $u(x,0) = 3\sin(2\pi x)$.

Solution

$$u_{xx}(x,t) = u_t(x,t)$$

taking the Laplace transform

$$L[u_{xx}(x,t)] = L[u_t(x,t)]$$

$$U_{xx}(x,s) = sU(x,s) - u(x,0)$$

Using the condition, $u(x,0) = 3\sin(2\pi x)$ we have;

$$sU(x,s) - 3\sin(2\pi x) = U_{xx}(x,s)$$

$$\Rightarrow U_{xx}(x,s) - sU(x,s) = -3\sin(2\pi x)$$

$$\frac{d^2U}{dx^2} - sU = -3\sin(2\pi x)$$

Solving the Homogenous Problem

$$\frac{d^2U}{dx^2} - sU = 0$$

The characteristic equation is given by

$$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$$

the homogenous Solution is :

$$U_h(x,s) = A_1 e^{\sqrt{5}x} + A_2 e^{-\sqrt{5}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient i.e

$$\frac{d^2u}{dx^2} - 5u = -3\sin(2\pi x) \quad ---- (*)$$

Let $u = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x)$ ---- (a)

$u' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x)$ ---- (b)

$u'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x)$ ---- (c)

Substituting (a) and (c) in equation (*)

$$-4\pi^2\Delta_1\sin(2\pi x) - 4\pi^2\Delta_2\cos(2\pi x) - 5\Delta_1\sin(2\pi x) - 5\Delta_2\cos(2\pi x) = -3\sin(2\pi x)$$

$-4\pi^2\Delta_1 - 5\Delta_1 = -3$ Also, $-4\pi^2\Delta_2 - 5\Delta_2 = 0$

$-\Delta_1[4\pi^2 + 5] = -3$ $\qquad \Delta_2[5 + 4\pi^2] = 0$

$\Delta_1 = \frac{3}{5+4\pi^2}$ $\qquad \Delta_2 = 0$

∴ the particular solution is :

$$U_p(x,s) = \frac{3}{5+4\pi^2}\sin(2\pi x)$$

the general Solution is given by : $U(x,s) = U_h(x,s) + U_p(x,s)$

$$U(x,s) = A_1 e^{\sqrt{5}x} + A_2 e^{-\sqrt{5}x} + \frac{3\sin(2\pi x)}{5+4\pi^2}$$

Applying the boundary Conditions

$u(0,t) = 0, \quad u(2,t) = 0$

$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$U(2,s) = A_1 e^{2\sqrt{5}} + A_2 e^{-2\sqrt{5}} = 0$ [But $A_1 = -A_2$]

$-A_2 e^{2\sqrt{5}} + A_2 e^{-2\sqrt{5}} = 0$

$A_2[e^{-2\sqrt{5}} - e^{2\sqrt{5}}] = 0 \Rightarrow A_2 = 0$

$\Rightarrow A_1 = 0$

$$U(x,s) = \frac{3\sin(2\pi x)}{5+4\pi^2}$$

Substituting (a) and (c) in equation (*)

$$-c^2\pi^2 n_1 \sin(\pi x) - c^2\pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = -\frac{\sin \pi x}{s}$$

$$\Rightarrow \quad -c^2\pi^2 n_1 - s^2 n_1 = -\frac{1}{s} \Rightarrow -n_1\left[s^2 + c^2\pi^2\right] = -\frac{1}{s}$$

$$n_1 = \frac{1}{s\left[s^2+c^2\pi^2\right]}$$

Also, $-c^2\pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$

$$n_2\left[s^2 + c^2\pi^2\right] = 0 \quad \Rightarrow n_2 = 0$$

Substituting 'n_1' and 'n_2' in (e)

$$U_p(x,s) = \frac{\sin(\pi x)}{s\left[s^2+c^2\pi^2\right]}$$

The general solution is given as:

$$U_g(x,s) = U_h(x,s) + U_p(x,s)$$

$$U_g(x,s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{\sin(\pi x)}{s\left[s^2+c^2\pi^2\right]} \quad \text{———} (**)$$

Applying the boundary Conditions

$u(0,t) = 0$ and $u(1,t) = 0$

$$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

$$U(1,s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$$

Substituting 'A_1' and 'A_2' in eqn (**)

$$U(x,s) = \frac{\sin(\pi x)}{s\left[s^2+c^2\pi^2\right]}$$

Applying Inverse Laplace Transform

$$L^{-1}\left[U(x,s)\right] = \sin(\pi x)\, L^{-1}\left[\frac{1}{s\left[s^2+c^2\pi^2\right]}\right]$$

Resolving $\frac{1}{s\left[s^2+c^2\pi^2\right]}$ into partial fractions

$$\frac{1}{s\left[s^2+c^2\pi^2\right]} = \frac{A}{s} + \frac{Bs+D}{s^2+c^2\pi^2} = \frac{A\left[s^2+c^2\pi^2\right] + \left[Bs+D\right]s}{s\left[s^2+c^2\pi^2\right]}$$

$$1 = A\left[s^2+c^2\pi^2\right] + Bs^2 + Ds$$

SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE COMBINED LAPLACE TRANSFORM METHOD AND THE NEW MODIFIED VARIATIONAL ITERATION METHOD

Presenting a reliable combined Laplace transform and the new modified variational iteration method to solve some non-linear partial differential equations. This method is more efficient and easy to handle non-linear PDEs.

Recall,

$$\mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0)$$

$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

Where $f(x,s)$ is the Laplace transform of (x,t) [x is considered as a dummy variable and t, a parameter]

Illustrating the basic concept of <u>He's Variational Iteration Method</u>, we consider the following general differential equations:

$$LU(x,t) + NU(x,t) = g(x,t) \quad \text{------ (i)}$$

with the initial condition, $U(x,0) = h(x)$ ------ (ii)

Where L is a linear operator of the first order, N is a non-linear operator and $g(x,t)$ is non-homogenous term. According to Variational Iteration Method we can construct a correction functional as follows:

$$U_{n+1} = U_n + \int_0^t \lambda\left[LU(x,s) + N\tilde{U}(x,s) - g(x,s)\right]ds \quad n \geq 0 \quad \text{(iii)}$$

where λ is a Lagrange Multiplier ($\lambda = -1$), the subscript 'n' denotes the nth approximation, $\tilde{U}_n$ is considered as a restricted variation, i.e $\delta\tilde{U}_n = 0$.

Equation (iii) is called a Correction Functional

Obtaining the Lagrange Multiplier 'λ' by using integration by part of Equation (i), but the Lagrange Multiplier is of the form $\lambda = \lambda(x,t)$

then ⟹ ~~taking~~ Laplace transform of Equation (iii), then the correction functional will be in the form:

$$\mathcal{L}\left[U_{n+1}(x,t)\right] = \mathcal{L}\left[U_n(x,t)\right] + \mathcal{L}\left[\int_0^t \lambda(x,t)\left(LU_n(x,s) + N\tilde{U}_n(x,s) - g(x,s)\right)ds\right], \quad n \geq 0 \quad \text{------ (iv)}$$

Therefore,

$$\mathcal{L}\left[U_{n+1}(x,t)\right] = \mathcal{L}\left[U_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t) * \left[LU_n(x,t) + N\tilde{U}_n(x,t) - g(x,t)\right]\right] \quad \text{------ (v)}$$

To find the optimal value of $\lambda(x,t)$, we first take the variation with respect to $U_n(x,t)$

and in such a case, the integration is basically the single convolution with respect to t, and hence, Laplace transform is appropriate to use.

$$\Rightarrow = \mathcal{L}\left[U_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\mathcal{L}\left[LU_n(x,t) + N\tilde{U}_n(x,t) - g(x,t)\right]$$

where $*$ is a single convolution w.r.t t.

Using the differentiation property of Laplace transform and initial condition (ii), we have:

$$s\mathcal{L}[u(x,t)] - h(x) = \mathcal{L}[g(x,t)] - \mathcal{L}[Nu(x,t)] \quad \text{------} \quad (*\,*)$$

Applying the inverse Laplace transform on both sides of Equation (* *), we find:

$$u(x,t) = G(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[Nu(x,t)]\right] \quad \text{------} \quad (*\,*\,*_a)$$

where $G(x,t)$ represents the terms arising from the source term and the prescribed initial condition $\left[\text{i.e } G(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}[g(x,t)] + h(x)\right]\right\}\right]$

Taking the first partial derivative with respect to 't' of Equation (* * *) to obtain:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}G(x,t) - \frac{\partial}{\partial t}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu(x,t)]\right\} \quad \text{---} \quad (*\,*\,*_b)$$

By the correction functional of the Variation Iteration Method

$$u_{n+1} = u_n - \int_0^t \left\{(u_n)_s(x,s) - \frac{\partial}{\partial s}G(x,s) + \frac{\partial}{\partial s}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu(x,s)]\right\}\right\} ds$$

or

$$u_{n+1} = G(x,t) - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu_n(x,t)]\right\} \quad \text{----} \quad [*\,*\,*\,*]$$

Equation [* * * *] is the new modified correction functional of Laplace transform and the Variational Iteration method, and the solution is given by:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

<u>Next:</u> Solving some non-linear PDEs by using the new modified Variational Iteration Laplace transform method:

<u>Examples:</u> [1] $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}$; $u(x,0) = x^2$

<u>Solution</u>

Given $u_t = u_x^2 + uu_{xx}$; $u(x,0) = x^2$

Taking Laplace transform, subject to the initial condition, we have:

$$\mathcal{L}[u_t] = \mathcal{L}[u_x^2] + \mathcal{L}[uu_{xx}]$$

$$s\mathcal{L}[u(x,t)] - u(x,0) = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$s\mathcal{L}[u(x,t)] - x^2 = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$\mathcal{L}[u(x,t)] = \frac{x^2}{s} + \frac{1}{s}\mathcal{L}[u_x^2 + uu_{xx}]$$

Taking the Inverse Laplace transform to obtain:

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{x^2}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(u_x^2 + uu_{xx}\right)\right]$$

Thus; $\frac{\delta}{\delta u_n}\mathcal{L}\left[u_{n+1}(x,t)\right]=\frac{\delta}{\delta u_n}\mathcal{L}\left[u_n(x,t)\right]+\mathcal{L}\left[\lambda(x,t)\right]\frac{\delta}{\delta u_n}\mathcal{L}\left[Lu_n(x,t)+N\tilde{u}_n(x,t)-g(x,t)\right]$ ——— (vi)

Equation (vi) becomes,

$$\mathcal{L}\left[\delta u_{n+1}(x,t)\right]=\mathcal{L}\left[\delta u_n(x,t)\right]+\mathcal{L}\left[\lambda(x,t)\right]\delta\mathcal{L}\left[Lu_n(x,t)\right] \quad \text{——— (vii)}$$

$$\left\{\text{i.e } \delta N\tilde{u}_n(x,t)=0 \text{ and } \delta g(x,t)=0\right\}$$

We assume that L is a linear first-Order Partial Differential Operator in this chapter given by $\frac{\partial}{\partial t}$ then, equation (vii) can be written in the form

$$\mathcal{L}\left[\delta u_{n+1}(x,t)\right]=\mathcal{L}\left[\delta u_n(x,t)\right]+\mathcal{L}\left[\lambda(x,t)\right]\left[s\mathcal{L}\left[\delta u_n(x,t)\right]\right]$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$

$$\Rightarrow \quad 0=\mathcal{L}\left[\delta u_n(x,t)\right]\left[1+s\mathcal{L}\left[\lambda(x,t)\right]\right]$$

$$\Rightarrow \quad 1+s\mathcal{L}\left[\lambda(x,t)\right]=0$$

$$s\mathcal{L}\left[\lambda(x,t)\right]=-1$$

$$\mathcal{L}\left[\lambda(x,t)\right]=\frac{-1}{s}$$

Taking the Laplace Inverse of both sides

$$\lambda(x,t)=\mathcal{L}^{-1}\left[\frac{-1}{s}\right]$$

$$\lambda(x,t)=-1$$

~~this~~ implies $\lambda=-1$

Substituting $(\lambda=-1)$ in equation (iii)

$$u_{n+1}=u_n-\int_0^t\left[Lu_n(x,s)+N\tilde{u}_n(x,s)-g(x,s)\right]ds \quad \text{——— (viii)}$$

The successive approximation 'u_{n+1}' of the solution 'u' will be readily obtained by using the determined Lagrange multiplier and any selective function u_0 consequently, the solution is given by:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$

Also, from equation (i)

$$\text{i.e } Lu(x,t)+Nu(x,t)=g(x,t) \quad \text{—— (*)}$$

Taking the Laplace transform of both sides, we have

$$\mathcal{L}\left[Lu(x,t)\right]+\mathcal{L}\left[Nu(x,t)\right]=\mathcal{L}\left[g(x,t)\right]$$

$$u(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[u_x^2 + u u_{xx}\right]\right\}$$

The new correction functional is given as:

$$u_{n+1}(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_n)_x^2 + u_n (u_n)_{xx}\right]\right\} \qquad n \geq 0$$

The solution in series form is given by:

$$u_0(x,t) = x^2 \quad \left[\text{or } u_0(x,t) = u(x,0)\right]$$

$$u_1(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_0)_x^2 + (u_0)(u_0)_{xx}\right]\right\}$$

$$u_1(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[4x^2 + 2x^2\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s}\left[\frac{4x^2}{s} + \frac{2x^2}{s}\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{6x^2}{s^2}\right\}$$

$$u_1(x,t) = x^2 + 6x^2 t$$

$$u_2(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_1)_x^2 + (u_1)(u_1)_{xx}\right]\right\}$$

$$u_2(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(2x + 12xt)^2 + (x^2 + 6x^2 t)(2 + 12t)\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s} L\left[6x^2 + 72x^2 t + 216x^2 t^2\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s}\left[\frac{6x^2}{s} + \frac{72x^2}{s^2} + \frac{216x^2 \cdot 2}{s^3}\right]\right\}$$

$$= x^2 + L^{-1}\left[\frac{6x^2}{s^2} + \frac{72x^2}{s^3} + \frac{432x^2}{s^4}\right]$$

$$u_2(x,t) = x^2 + 6x^2 t + 36x^2 t^2 + 72x^2 t^3$$

The series solution is given by:

$$u(x,t) = x^2 + 6x^2 t + 36x^2 t^2 + 72x^2 t^3 + \cdots$$

$$= x^2\left[1 + 6t + 36t^2 + 72t^3 + \cdots\right]$$

$$u(x,t) = \frac{x^2}{1 - 6t}$$

(2) — Non linear

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2t^2 \; , \; u(x,0)=0 \; , \; \frac{\partial u(x,0)}{\partial t} = x$$

$$L[u_{tt}] = L[x^2t^2] + L[u_{xx} - u^2]$$

$$s^2 L[u(x,t)] - s\,u(x,0) - \frac{\partial u(x,0)}{\partial t} = \frac{2x^2}{s^3} + L[u_{xx} - u^2]$$

$4\times3\times2$

$$s^2 L[u(x,t)] = \frac{2x^2}{s^3} + x + L[u_{xx} - u^2]$$

$\frac{2x^2+}{s^3}$ $\quad 4!$

$$L[u(x,t)] = \frac{2x^2}{s^5} + \frac{x}{s^2} + \frac{1}{s^2} L[u_{xx} - u^2]$$

$$u(x,t) = \frac{x^2t^4}{12} + xt + L^{-1}\left\{\frac{1}{s^2} L[u_{xx} - u^2]\right\}$$

$$u_{n+1}(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[(u_n)_{xx} - (u_n)^2]\right\}$$

$$u_0(x,t) = u(x,0) + t\frac{\partial u(x,0)}{\partial t} = \underline{xt}.$$

$$u_1(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[-x^2t^2]\right\}$$

$\frac{2x^2}{s^3} = \frac{2x^2}{s^5}$

$$u_1(x,t) = xt + \frac{x^2t^4}{12} - \frac{x^2t^4}{12} = xt$$

$$u_2(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[-x^2t^2]\right\}$$

$$= \underline{\underline{xt}}.$$

$$u(x,t) = \underline{\underline{xt}}.$$

[2] $\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u - u^2 = te^{-x}$; $u(x,0) = 0$, $\frac{\partial u}{\partial t} = e^{-x} = u_t$

Solution

Given $u_{tt} + (u_x)^2 + u - u^2 = te^{-x}$

- taking the Laplace transform on both sides subject to the initial conditions, we have:

$$\mathcal{L}[u_{tt}] + \mathcal{L}[u_x^2] + \mathcal{L}[u] - \mathcal{L}[u^2] = \mathcal{L}[te^{-x}]$$

$$s^2\mathcal{L}[u(x,t)] - su(x,0) - u_t(x,t) = \mathcal{L}[te^{-x}] - \mathcal{L}[-u^2 + u + u_x^2]$$

$$s^2\mathcal{L}[u(x,t)] - e^{-x} = \mathcal{L}[te^{-x}] - \mathcal{L}[u^2 - u - u_x^2] = \mathcal{L}[te^{-x} + u^2 - u - u_x^2]$$

$$s^2\mathcal{L}[u(x,t)] = e^{-x} + \mathcal{L}[te^{-x} + u^2 - u - u_x^2]$$

~~taking the inverse Laplace transform to obtain:~~

$$\mathcal{L}[u(x,t)] = \frac{e^{-x}}{s^2} + \frac{1}{s^2}\mathcal{L}[te^{-x} + u^2 - u - u_x^2]$$

- taking the inverse Laplace transform to obtain:

$$u(x,t) = te^{-x} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[te^{-x} + u^2 - u - u_x^2]\right\}$$

- the new correction functional is given as:

$$u_{n+1}(x,t) = te^{-x} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[te^{-x} + u_n^2 - u_n - (u_n)_x^2]\right\} \quad n \geq 0$$

$$u_0(x,t) = te^{-x} \quad \left[\text{or } u_0(x,t) = u(x,0) + t\frac{\partial u}{\partial t}(x,0)\right]$$

$$u_1(x,t) = te^{-x} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[te^{-x} + u_0^2 - u_0 - (u_0)_x^2]\right\}$$

$$= te^{-x} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[te^{-x} + t^2e^{-2x} - te^{-x} - t^2e^{-2x}]\right\}$$

$$= te^{-x} + 0$$

$$u_1(x,t) = te^{-x}$$

Also, $u_2(x,t) = te^{-x} + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[te^{-x} + u_1^2 - u_1 - (u_1)_x^2]\right\}$

$$u_2(x,t) = te^{-x}$$

- the series solution is given by:

$$u(x,t) = te^{-x}$$