APPLICATION OF LAPLACE TRANSFORM IN SOLVING PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER DERIVATIVE

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CERTIFICATION

This is to certify that this project was carried out by **AKINWUNMI, Rilwan Babatunde** with Matriculation Number 17/56EB029 in the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria, for the award of Bachelor of Science (B.Sc.) degree in Mathematics.

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DEDICATION

I would like to dedicate the project to God, for the grace and faithfulness of God thus far. For His mercies, guidance and protection throughout my years of study.

ACKNOWLEDGMENTS

ABSTRACT

This project deals with

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Chapter 1

GENERAL

INTRODUCTION

1.1 INTRODUCTION

The Laplace transform is a globally used integral transform with many applications in Physics and Engineering. The Laplace transform method is applicable in the solution of differential equation to that of solving in algebraic problem. It is denoted by $\mathcal{L}\left[f(t)\right]$ or $\mathcal{L}\left[f\right]\left\{s\right\} = \int_0^\infty e^{-st} f(t)\mathrm{dt}$, it is a linear operator of a function $f(t)$ with a real argument $t(t \geq 0)$ that transform it to a function $F(s)$ with a complex argument s. This transformation is essentially bijective for the majority of practical uses, the respective pairs of $f(t)$ and $F(s)$ are matched in tables. The Laplace transform has the useful property that many relationships and operations over the original $f(t)$ correspond to simpler relationships and operation over the images $F(s)$.

It is named after Pierre-Simon Laplace, who introduced the transform

in his work in probability theory. The Laplace transform is related to the Fourier transform, but as the Fourier transform expresses a function or signal as a series of nodes of vibration (frequencies), the Laplace transform resolves a function into its moments. In physics and engineering, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillations, optical devices and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation from the time-domain, in which inputs and outputs are function of time to the frequency-domain, where the same inputs and outputs are function of complex angular frequency on radians per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analysing the behaviour of the system or in synthesizing a new system base on a set of specification.

On the other hand, PDE provide a quantitative description for many central models on physical, biological and social sciences. A Partial Differential Equation is an equation which involves derivatives of an unknown function with respect to two or more independent variables.

For example:

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} - \lambda v = 0 \qquad (1.1)$$

where λ is a constant.

The description is in terms of unknown functions of two or more independent variables, and the relation between partial derivatives with respect to those variables. The order of a PDE is the order of the highest derivative.

A PDE is said to be linear if the unknown function and all its partial derivatives occur to the first power only, and there are no product involving more than one of the terms, for example equation (1.1) is linear. And it is non-linear if the relations between the unknown functions and the partial derivatives involved are non-linear.

For example,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(w) \tag{1.2}$$

called a stationary heat equation with a non-linear source.

Despite the apparent simplicity of the underlying differential relations, non-linear PDE govern a vast array of complex phenomena of motion, reactions, diffusion, equilibrium, conservation and more. Due to their pivotal role in science and engineering and more recently, they spread into economic, financial, forecasting, image processing and other fields.

Indeed, these studies found their way into entries throughout the scientific literature. They reflect a rich development of mathematical theories and analytical techniques to solve PDEs and illustrate the phenomena they govern. Yet, analytical theories provide only a limited account for the array of complex phenomena governed by non-linear PDEs .

Over the past fifty years, scientific computation has emerged as the most versatile tool to complement theory and experiments. Modern numerical methods, in particular those for solving non-linear PDEs are the heart of many of these advanced scientific computation. Apparantly, numerical com-

putation have not only joined experiment and theory as one of the funda-
mental tools of investigation, they have also altered the kind of experiments
performed and have expanded the scope of theory.

1.2 STATEMENT OF THE PROBLEM

This study will try to apply Laplace transform in solving the following Par-
tial Differential Equations.

1. $\dfrac{\partial^2 V}{\partial x \partial y} = e^{-2y} \sin x$ with initial condition $Vx(x,0) = 0,\ \ Vy(0,y) = 0$

2. $\dfrac{\partial^2 V}{\partial y \partial x} = \cos x \cos y$ with initial condition $V(x,0) = 1 + \sin x,\ Vy(0,y) = -2\cos y$

1.3 DEFINITION OF RELEVANT TERMS

1.3.1 Linear Equation

A linear equation is an algebraic equation in which the highest power of the
unknown is one. For example;

$$ax + b = 0 \qquad (1.3)$$

1.3.2 Differential Equation

An equation involving derivatives or differential of one or more dependent
variable with respect to one or more independent variables is called differ-
ential equation. An example is;

$$\frac{d^2 y}{dx^2} + y = 0 \qquad (1.4)$$

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1.3.3 Ordinary Differential Equation

A differential equation which express a relationship between an independent variable, dependent variable and one or more differential coefficient of the dependent variable with respect to independent variable is called an ordinary differential equation. For example;

$$y'' + 5y = 0 \tag{1.5}$$

1.3.4 Partial Differential Equation

A partial differential equation (PDE) is an equation which involves derivatives of an unknown function with respect to two or more independent variables. An example is;

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} - \lambda u = 0 \tag{1.6}$$

where λ is a constant

1.3.5 Order of a Partial Differential Equation

The order of a partial differential equation is the order of the highest derivative involved. The most general *second*-order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.7}$$

1.3.6 Solution to a Partial Differential Equation

A *solution* (or a particular solution) to partial differential equation is a function $u(x, y, \ldots)$ that solves the equation or in other words, turns it into an identity when substituted into the equation. A solution is called *general* if it contain all particular solution of the equation involved. The term *exact*

solution is often used for second and higher-order non-linear PDEs to denote a *particular* solution.

1.3.7 Linear Partial Differential Equation

A partial differential equation is linear if it linear in the unknown function and all its derivatives with coefficient depending only on the independent variable. For example; equation (1.1)

1.3.8 Non-Linear Partial Differential Equation

A partial differential equation is said to be non-linear if its contains the product of the dependent variable or its derivatives. For example; equation (1.2)

1.3.9 Quasi-Linear Partial Differential Equation

A partial differential equation is quasi-linear if it is linear in the highest order derivatives with coefficients depending on the independent variable the unknown function and its derivatives of order lower than the order of the equation. For example;

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \tag{1.8}$$

where the function a, b and c can be involve u but not its derivatives

1.3.10　Homogeneous Partial Differential Equation

A PDE is said to be homogeneous if the equation does not contain a term independent of the unknown function and its derivatives. For example;

$$U_{xx} + U_{yy} = 0 \tag{1.9}$$

1.4　AIM AND OBJECTIVES OF STUDY

This project aim at the application of Laplace transform in solving partial differential equation of the second order derivative. The objectives of the study were to:

i. find the exact solution of the problem stated above;

ii. determine whether PDEs can be solved using Laplace substitution method; and

iii. investigate in the factors affecting the use of Laplace transform in solving differential equations.

1.5　SCOPE OF STUDY

The study on the application of Laplace transform in solving partial differential equations in the second derivative will be limited to second order PDEs . The study will cover on how to apply Laplace transform to PDEs of the second order derivative.

Chapter 2

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter gives an insight into various studies conducted by outstanding researchers, as well as explained terminologies in the regards to the application of Laplace transform in solving Partial Differential Equation in the second derivative.

2.2 LAPLACE TRANSFORM

The Laplace transform method is applicable in the solution of differential equation. The method reduces the problem of solving differential equation to that of solving an algebraic problem. Bendei being a different and efficient alternative to variation of parameters and undetermined coefficient, the Laplace transform method is particularly advantageous for input term that are piecewise-defined, periodic or impulse.

2.2.1 DEFINITION 1

A function $f(t)$ defined in the interval $[a, b]$ is said to have a **Jumping Discontinuity** at $t_0 \in [a, b]$ if $f(t)$ is discontinuous at t_0 and the one sided limits $f(t_0-)$ and $f(t_0+)$ exists in finite members.

If the discontinuity occurs at an endpoint $t_0 = a($ or $b)$, a jump discontinuity occurs if $f(a+)$ or $f(b-)$ exists in finite number.

For example, if
$$f(t) = \begin{cases} t^2 + 1, & t > 1 \\ 1 - t, & t < 1 \end{cases}$$

Then

$$f(1+) = \lim_{t \to 1}(t^2 + 1) = 2$$

$$f(1-) = \lim_{t \to 1}(1 - t) = 0$$

Since these two limits exist, the given function has a jump discontinuity at $t_0 = 1$ and its magnitude is $2 - 0 = 2$.

2.2.2 Definition 2

A function $f(t)$ is said to be **piecewise continuous** on a finite interval $I = [a, b]$ if it continuous at every point of $[a, b]$ except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

2.2.3 Definition 3

A function $f(t)$ is said to be **piecewise continuous (or sectionally continuous)** on a finite interval $[a, b]$ if $[a, b]$ can be divided into a finite number of subinterval such that

1. $f(t)$ is continuous in the interior of each subintervals

2. $f(t)$ approaches finite limits at t approaches either and points of each subintervals from its interiors.

2.2.4 Definition 4

A function $f(t)$ is said to be **piecewise continuous** on the infinite interval $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ $\forall$ $N > 0$.

A function which is piecewise continuous on a finite member of intervals as necessarily integrable over that interval.

2.2.5 Definition 5

A function $f(t)$ is said to be exponential order if a constant k exists such that $e^{-kt}|f(t)|$ is bounded for sufficiently large values of t, that is there exists a positive constant M such that

$$|f(t)| < Me^{-kt} \tag{2.1}$$

$\forall$ $t > T$ at which $f(t)$ is defined

2.2.6 Definition 6

The Laplace Transform of f is denoted by F and it gives an average value of f taken over all positive values of t such that the value $F(s)$ represents an average of f taken over possible time intervals of length s.

2.2.7 Definition 7

Let $f(t)$ be a real valued function on $[0, \infty]$. The Laplace transform of f is defined by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \mathrm{d}t \quad \text{for } s > 0 \tag{2.2}$$

and it is denoted by $\mathcal{L}\left[f(t)\right]$

This immediately raises the question of why to use such a procedure. The research is strongly motivated by real engineering problems. There, we encounter models for the dynamics of phenomena which depends on rate of change of functions e.g velocities and acceleration of particles or points in rigid bodies, ordinary calculus to solve ODE, provided that the function are nicely-behaved, that is, continuous and with continuous derivatives. But, there is much interest in engineering dynamical problems involving functions that input step change or spike impulses to system, a good example in a playing pool. Now, there is an easy way to smooth out discontinuities in functions of time: simply take an average value over all time. But an ordinary average will replace the function by a constant, so we use a kind of moving average which takes continuous averages over all possible intervals of t. This effectively deals with the discontinuities by encoding them as a smooth function of interval length s.

The amazing thing about using Laplace Transform is that we can convert a whole the algebra, find the transformed solution $F(s)$, then undo the transform to get back the required solution f as a function of t. Furthermore, it turns out that the transform of a derivatives of a function on a simple combination of the transform of the function and its initial value. So a calculus problem in converted into an algebraic problem involving polynomial functions, which is easier.

In addition, calculus operation of differentiation and integration are lin-

ear, so the Laplace Transform of a sum of functions in the sum of their Laplace transforms and multiplication of a function by a constant can be done before or after taking its transform.

2.3   FINDING LAPLACE TRANSFORM

In this section, we shall find some Laplace Transforms form first principle, that is, from the definition (2.2.3) describe some theorems that help find some transform, then use Laplace Transform to solve problems involving ODEs.

2.3.1   FROM THE DEFINITION

Here, we used (2.2.3) directly:

$$
\begin{aligned}
\text{For } f(t) = 1: \qquad \mathcal{L}\left[1\right] &= \int_0^\infty e^{-st} dt \\
&= -\frac{1}{s} \lim_{p \to \infty} \left[e^{-st} \right]_0^p \\
&= -\frac{1}{s} \lim_{p \to \infty} \left(\frac{1}{e^{sp}} - 1 \right) \\
&= \frac{1}{s}
\end{aligned}
$$

For $f(t) = t$:
$$\mathcal{L}[t] = \int_0^\infty t e^{-st} dt$$

$$= \frac{1}{s} \lim_{p \to \infty} \left\{ \left[-t e^{-st} \right]_0^p + \int_0^p e^{-st} dt \right\}$$

$$= \frac{1}{s} \lim_{p \to \infty} \left\{ -p e^{-sp} - \frac{1}{s} \left(e^{-sp} - 1 \right) \right\}$$

$$= \frac{1}{s^2}$$

For $f(t) = e^{at}$:
$$\mathcal{L}\left[e^{at} \right] = \int_0^\infty e^{at} e^{-st} dt$$

$$= \lim_{p \to \infty} \int_0^p e^{(a-s)t} dt$$

$$= \frac{1}{a-s} \lim_{p \to \infty} \left[e^{(a-s)t} \right]_0^p$$

$$= \frac{1}{a-s} \lim_{p \to \infty} \left\{ e^{(a-s)p} - 1 \right\}$$

$$= \frac{1}{s-a} , \quad s > a$$

For $f(t) = \cos kt$:
$$\mathcal{L}\left[\cos kt \right] = \int_0^\infty e^{-st} \cos kt \ dt$$

$$= \lim_{p \to \infty} \int_0^p e^{-at} \cos kt \ dt$$

$$= \frac{1}{k} \lim_{p \to \infty} \left\{ \left[e^{-st} \sin kt \right]_0^p + s \int_0^p e^{-st} \sin kt \ dt \right\}$$

$$= \frac{1}{k} \lim_{p \to \infty} \left\{ e^{-sp} \sin kp + \frac{s}{k} \left\{ \left[-e^{-st} \cos kt \right]_0^p \right. \right.$$

$$\left. \left. -s \int_0^p e^{-st} \cos kt \ dt \right\} \right\}$$

$$= \frac{1}{k} \lim_{p \to \infty} \left(e^{-sp} \sin kp \right) + \frac{s}{k^2} \lim_{p \to \infty} \left(-e^{-sp} \cos kt + 1 \right)$$

$$-\frac{s^2}{k^2} \lim_{p \to \infty} \int_0^p e^{-st} \cos kt$$

$$\mathcal{L}[\cos kt] = \frac{s}{k^2} - \frac{s^2}{k^2} \mathcal{L}[\cos kt]$$

$$\text{Hence,} \quad \mathcal{L}\left[\cos kt\right] = \frac{s}{s^2 + k^2} \quad, \quad s > 0$$

2.3.2 THEOREM

Suppose

1. $f(t)$ is piecewise continuous on $[0, \infty]$

2. $f(t)$ is of exponential order k.

Then $\mathcal{L}\left[f(t)\right]$ exists for $s > k$

Proof:

Since $f(t)$ is continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis.

Now assuming that $s > k$, then

$$\mathcal{L}\left[f(t)\right] = \left| \int_0^\infty e^{-st}f(t)dt \right| \leq \int_0^\infty |f(t)|e^{-st}dt$$

But by the given hypothesis $\exists M > 0$ such that $|f(t)| \leq Me^{-kt}$

So that,

$$\mathcal{L}\left[f(t)\right] \leq M \int_0^\infty e^{-st}f(t)dt = \frac{M}{s-k} \quad, \quad s > k$$

and hence the proof.

Example 2.1

Find the Laplace transform of the function

$$f(t) = \begin{cases} 4 \, , & 0 < t < 3 \\ 0 \, , & 3 < t < 5 \\ e^{2t} \, , & t > 5 \end{cases}$$

Solution

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^3 e^{-st} f(t)dt + \int_3^5 e^{-st} f(t)dt + \int_5^\infty e^{-st} f(t)dt$$

$$= 4\int_0^3 e^{-st}dt + \int_5^\infty e^{2t} e^{-st}dt$$

$$= -\frac{4}{5}\left(e^{-3s} - 1\right) - \lim_{p\to\infty}\int_5^p e^{-(s-2)t}dt \qquad \text{(where } p \text{ is dummy)}$$

$$= \frac{4}{5}\left(1 - e^{-3s}\right) - \frac{1}{s-2}\lim_{p\to\infty}\left[e^{-(s-2)p} - e^{-5(s-2)}\right]$$

$$= \frac{4}{5}\left(1 - e^{-3s}\right) + \frac{1}{s-2}e^{5(s-2)} \ , \quad s > 2$$

2.3.3   PROPERTIES OF LAPLACE TRANSFORMS

If f and g are two real valued functions of t and a, b, c are constants, then the following properties hold.

$$\mathcal{L}\left[af + bg\right] = a\mathcal{L}\left[f\right] + b\mathcal{L}\left[b\right] \tag{2.3}$$

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a) \tag{2.4}$$

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\mathcal{L}\left[f(t)\right] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots f^{n-1}(0) \tag{2.5}$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s) \tag{2.6}$$

$$\mathcal{L}\left[g(t)\right] = e^{-as}F(s) \tag{2.7}$$

Where

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

and

$$g(t) = \begin{cases} 0 & ,0 < t < a \\ f(t-a) & ,t > a \end{cases}$$

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Some of these results are established as follows

$$\mathcal{L}\left[af + bg\right] = \int_0^\infty e^{-st}\left[af(t) + bg(t)\right]dt$$

$$= \int_0^\infty \left[ae^{-st}f(t) + be^{-st}g(t)\right]dt$$

$$= a\int_0^\infty e^{-st}f(t) + b\int_0^\infty e^{-st}g(t)dt$$

$$\mathcal{L}\left[af + bg\right] = a\mathcal{L}\left[f\right] + b\mathcal{L}\left[g\right]$$

which is in equation (2.3)

Also,

$$\mathcal{L}\left[e^{at}f(t)\right] = \int_0^\infty e^{-st}e^{at}f(t)dt$$

$$\mathcal{L}\left[e^{at}f(t)\right] = \int_0^\infty e^{(-s-a)t}$$

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$$

which is equation (2.4). It is also called the First Shifting Theorem.

With the help of this property, we can have the following important results.

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \tag{2.8}$$

$$\mathcal{L}\left[e^{at}t^n\right] = \frac{n!}{(s-a)^{n+1}} \tag{2.9}$$

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$$\mathcal{L}\left[e^{at}\cosh bt\right] \;=\; \frac{s-a}{(s-a)^2-b^2} \tag{2.10}$$

$$\mathcal{L}\left[e^{at}\sinh bt\right] \;=\; \frac{b}{(s-a)^2-b^2} \tag{2.11}$$

$$\mathcal{L}\left[e^{at}\sin bt\right] \;=\; \frac{b}{(s-a)^2+b^2} \tag{2.12}$$

$$\mathcal{L}\left[e^{at}\cos bt\right] \;=\; \frac{s-a}{(s-a)^2+b^2} \tag{2.13}$$

2.3.4   THEOREM

Suppose that

(i) f is of exponential order

(ii) f is continuous on $[0,\infty)$

(iii) f' is also of exponential order

(iv) f' is piecewise continuous on $[0,\infty)$

Then

$$\mathcal{L}\left[f'\right] = s\mathcal{L}\left[f\right] - f(0) \tag{2.14}$$

Proof:

$$
\begin{aligned}
\mathcal{L}\left[f'\right] \;&=\; \int_0^{\infty} e^{-st} f'(t)\,dt \\[2mm]
&=\; \lim_{p\to\infty}\left[e^{-st}f(t)\right]_0^{p} + s\lim_{p\to\infty}\int_0^{p} e^{-st} f(t)\,dt \\[2mm]
&=\; \lim_{p\to\infty}\left(e^{-sp}f(p) - f(0) + s\int_0^{\infty} e^{-st} f(t)\,dt\right) \\[2mm]
&=\; \lim_{t\to\infty} e^{-st}f(t) - f(0) + s\mathcal{L}\left[f\right], \quad \text{(Since p is dummy)}
\end{aligned}
$$

By hypothesis (1) of theorem (2.3.2)

Now;

$$0 \leq \left| e^{-st} f(t) \right| \leq \lim_{t \to \infty} e^{-st} \left| f(t) \right|$$

$$\leq \lim_{t \to \infty} e^{-st} M e^{kt}$$

$$= \lim_{t \to \infty} M e^{(k-s)t} = 0$$

Provided that $s > k$

Hence

$$\lim_{t \to \infty} \left| e^{-st} f(t) \right| = 0$$

So that

$$\lim_{t \to \infty} e^{-st} f(t) = 0$$

consequently,

By using equation (2.14), we have;

$$\mathcal{L} \left[f'' \right] = s\mathcal{L} \left[f' \right] - f'(0)$$

$$= s \left\{ s\mathcal{L} \left[f \right] - f(0) \right\} - f'(0)$$

$$= s^2 \mathcal{L} \left[f \right] - s f(0) - f'(0)$$

Provided that;

1. f' is continuous on $[0, \infty)$

2. f'' is piecewise continuous on $[0, \infty)$

3. f, f', f'' are of exponential order

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Then by repeatedly applying (2.14), we get (2.5) provided that

1. $f^{(n-1)}$ is continuous on $[0, \infty)$

2. $f^{(n)}$ is piecewise continuous on $[0, \infty)$

3. $f,\ f',\ f''$ are of exponential order

Example 2.2

If $\mathcal{L}[\sin bt] = \dfrac{b}{s^2 + b^2}$,   Find $\mathcal{L}[\cos bt]$

Solution

$\mathcal{L}[f(t)]\, f' = s\mathcal{L}[f] - f(0) \equiv \mathcal{L}[\cos bt]$

$f'(t) = \cos bt;$

$\therefore\ \ f(t) = \dfrac{1}{b}\sin bt \quad f(0) = 0$

Hence;

$\mathcal{L}[\cos bt] = s\mathcal{L}\left[\dfrac{1}{b}\sin bt\right] - 0 = \dfrac{s}{b}\mathcal{L}[\sin bt] = \dfrac{s}{s^2 + b^2}$

Example 2.3

Find the Laplace transform of $e^{at}\cos bt$

Solution

$$\mathcal{L}\left[f(t)\right]e^{at}f(t) = F(s-a) \equiv \mathcal{L}\left[e^{at}\cos bt\right]$$

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[\cos bt\right] = \frac{s}{s^2+b^2} \equiv F(s)$$

Hence

$$\mathcal{L}\left[e^{at}\cos bt\right] = F(s-a) = \frac{s-a}{(s-a)^2+b^2}$$

Example 2.4

Determine $\mathcal{L}\left[t\cos bt\right]$

Solution

$$\mathcal{L}\left[tf(t)\right] = -F'(s) \equiv \mathcal{L}\left[t\cos bt\right]$$

Let $f(t) = \cos bt$

Since $\mathcal{L}\left[f(t)\right] = \dfrac{s}{s^2+b^2}$,   then

$$\mathcal{L}\left[tf(t)\right] = -\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$
$$= \frac{s^2-b^2}{(s^2+b^2)^2}$$

2.3.5   TABLE OF LAPLACE TRANSFORMS

Table 2.1: Some useful Laplace Transforms are listed below

S/N	$f(t)$	$\mathcal{L}[f]$	S/N	$f(t)$	$\mathcal{L}[f]$
1	1	$\frac{1}{s}$	7	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$

2	t^n	$\dfrac{n!}{s^{n+1}}$	8	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
3	$t^n e^{at},\ n\geq 0$	$\dfrac{n!}{(s-a)^{n+1}}$	9	$\cosh bt$	$\dfrac{s}{s^2-b^2}$
4	$t^a,\ a>0$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	10	$\sinh bt$	$\dfrac{b}{s^2-b^2}$
5	$\cos bt$	$\dfrac{s}{s^2+b^2}$	11	e^{at}	$\dfrac{1}{s-a}$
6	$\sin bt$	$\dfrac{b}{s^2+b^2}$	12	$\begin{cases} 1,\ 0\leq t\leq a \\[2mm] 0,\ t>a \end{cases}$	$\dfrac{1}{s}\left(1-e^{as}\right)$

2.4   THE INVERSE LAPLACE TRANSFORM

If

$$\mathcal{L}\left[f(t)\right]=F(s)$$

is the Laplace transform of the function $f(t)$, the function

$$f(t)=\mathcal{L}^{-1}\left[F(s)\right]$$

is called the **Inverse Laplace Transform of** $F(s)$

Some examples will be considered to determine the inverse transforms of some function. This will be aided by the use of the results in the Table 2.1

Example 2.5

Find the Inverse Laplace transform for the given functions

(i)$F(s)=\dfrac{6}{s^4}$ (ii)$F(s)=\dfrac{s}{s^2+9}$ (iii)$F(s)=\dfrac{s-3}{s^2-3s+25}$

Solution

From Table 2.1, we have

1. $\mathcal{L}^{-1}\left[\dfrac{6}{s^4}\right] = \mathcal{L}^{-1}\left[\dfrac{3!}{s^4}\right] = t^3$

2. $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+9}\right] = \mathcal{L}^{-1}\left[\dfrac{s}{s^2+3^2}\right] = \cos 3t$

3. $\mathcal{L}^{-1}\left[\dfrac{s-3}{s^2-3s+25}\right] = \mathcal{L}^{-1}\left[\dfrac{s-3}{(s-3)^2+16}\right] = e^{3t}\cos 4t$

2.4.1   LINEARITY PROPERTY

The Inverse transform also satisfies the linearity property. If the inverse transforms $\mathcal{L}^{-1}[F]$ and $\mathcal{L}^{-1}[G]$ of the two function f_1 and f_2 respectively exist, then;

$$\mathcal{L}^{-1}[aF + bG] = a\mathcal{L}^{-1}[F] + b\mathcal{L}^{-1}[G]$$

where a and b are constants.

Thus, for example

$$\mathcal{L}^{-1}\left[\frac{2}{s-2} + \frac{3s}{s^2+25} - \frac{7(s-3)}{s^2-3s+25}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right] - 7\mathcal{L}^{-1}\left[\frac{s-3}{s^2-3s+25}\right]$$

$$= 2e^{2t} + 3\cos 5t - 7e^{3t}\cos 4t$$

2.5 APPLICATION OF LAPLACE TRANSFORM TO SOLUTION OF INITIAL VALUE PROBLEMS (IVPs)

Initial Value Problem(IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain.

A good example is given;

$$\frac{d^2y}{dx^2} = 2 - 11x \qquad y'(0) = 4, \quad y(0) = 1$$

It is possible to solve certain initial value problem in ODE by the use of Laplace transform.

Let us consider the second order ordinary differential equation(ODE)

$$y'' + a_1 y' + a_0 y = g(t) \tag{2.15}$$

with the associated initial conditions

$$y(0) = k_1 \qquad\qquad y'(0) = k_2 \tag{2.16}$$

Taking the transform of (2.15)

$$\mathcal{L}\left[y''\right] + a_1 \mathcal{L}\left[y'\right] + a_0 \mathcal{L}\left[y\right] = \mathcal{L}\left[g(t)\right] \tag{2.17}$$

But from (2.5)

$$\mathcal{L}\left[y'\right] = s\mathcal{L}\left[y\right] - y(0) = s\mathcal{L}\left[y\right] - k_1 \tag{2.18}$$

$$\mathcal{L}\left[y''\right] = s^2 \mathcal{L}\left[y\right] - sy(0) - y'(0) = s^2 \mathcal{L}\left[y\right] - sk_1 - k_2 \tag{2.19}$$

Substitute equation (2.18) and equation (2.19) into equation (2.17), to have

$$s^2 \mathcal{L}\left[y\right] - sk_1 - k_2 + a_1(s\mathcal{L}\left[y\right] - k_1) + a_0 \mathcal{L}\left[y\right] = \mathcal{L}\left[g\right]$$

That is,

$$(s^2 + a_1 s + a_0)\mathcal{L}[y] - k_1(s + a_1) - k_2 = \mathcal{L}[g] \qquad (2.20)$$

This is

$$\mathcal{L}[y] = \frac{k_1(s + a_1) + k_2 + \mathcal{L}[g]}{s^2 + a_1 s + a_0} \qquad (2.21)$$

The solution of equation (2.15) is then obtained as

$$y(t) = \mathcal{L}^{-1}[\mathcal{L}[y]] \qquad (2.22)$$

with the aid of the table of transform.

Example 2.6

Solve the IVP

$$y'' + 2y' + y = t^2 + 1 \qquad \text{(i)}$$

$$y(0) = 0 ; \quad y'(0) = 1 \qquad \text{(ii)}$$

Solution

We take the transform of (i) to have

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[t^2] + \mathcal{L}[1]$$

Now insert equation (2.18) and equation (2.19) with $k_1 = 0$ and $k_2 = 1$, to get

$$(s^2\mathcal{L}[y] - 1) + 2s\mathcal{L}[y] + \mathcal{L}[y] = \mathcal{L}[t^2] + \mathcal{L}[1]$$

since $\mathcal{L}[y] = \dfrac{-s^2 - 2s}{(s + 1)(s^2 + 2s + 2)}$

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By expressing the RHS in term of partial fraction, we have

$$\mathcal{L}\left[y\right] = \frac{1}{s+1} - \frac{2s+2}{s^2+2s+2}$$

so that

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{s+1}{s^2+2s+2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]
\end{aligned}
$$

Using the results in the Table 2.1, we have

$$y = e^{-t} - 2e^{-t}\cos t$$

as the desired solution.

2.6   CONVOLUTION

We shall consider a method for finding the inverse Laplace transform of a function which is the product of two Laplace transforms.

The **convolution** of two functions f and g is defined by

$$f \times g(t) = \int_0^t f(x)g(t-x)dx \tag{2.23}$$

where f and g are piecewise continuous functions on $[0, \infty)$ and are of exponential order.

Clearly,

$$f \times g = g \times f$$

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Since by letting $u = t - x$ in equation (2.23), we have

$$\begin{aligned} f \times g(t) &= \int_0^t f(x)g(t-x)dx \\ &= -\int_0^t f(t-u)g(u)du \\ &= \int_0^t g(u)f(t-u) = g \times f(t) \end{aligned}$$

The transform of the convolution of f and g is given by

$$\mathcal{L}\left[f \times g\right] = \mathcal{L}\left[f\right]\mathcal{L}\left[g\right] \tag{2.24}$$

or equivalently

$$f \times g = \mathcal{L}^{-1}\left[\mathcal{L}\left[f\right]\mathcal{L}\left[g\right]\right] \tag{2.25}$$

Example 2.8

If $f(t) = 2t$ and $g(t) = e^t$, find the convolution of f and g

Solution

$$f \times g = \int_0^t f(x)g(t-x)dx$$

$$= \int_0^t 2xe^{t-x}dx$$

$$= 2\left[\left[-xe^{t-x}\right]_0^t + \int_0^t e^{t-x}dx\right]$$

$$= 2\left[-t - \left[e^{t-x}\right]_0^t\right]$$

$$= 2\left[-t - \left(1 - e^t\right)\right]$$

$$= 2\left(e^t - t - 1\right)$$

Example 2.9

Determine the inverse transform of

$$\frac{s+1}{s(s^2+4)}$$

Solution

$$\frac{s+1}{s(s^2+4)} = \frac{s}{s(s^2+4)} + \frac{1}{s(s^2+4)} \equiv F(s) + G(s)$$

For

$$F(s) = \frac{s}{s(s^2+4)}$$

Let

$$F_1(s) = \frac{1}{s}, \quad F_2(s) = \frac{s}{s^2+4}$$

so that

$$f_1(t) = 1 \qquad f_2(t) = \cos 2t$$

$$\mathcal{L}^{-1}\left[F(s)\right] = g * f(t)$$

$$= \int_0^t \cos 2x \, dx = \frac{1}{2}\sin 2t$$

Also for

$$G(s) = \frac{1}{s(s^2+4)}$$

Let

$$G_1(s) = \frac{1}{s}, \quad G_2(s) = \frac{1}{s^2+4} = \frac{1}{2}\left(\frac{2}{s^2+4}\right)$$

Then

$$g_1(t) = 1 \qquad g_2(t) = \frac{1}{2}\sin 2t$$

Hence;

$$\mathcal{L}^{-1}\left[G(s)\right] = g \times f(t)$$

$$= \frac{1}{2}\int_0^t \sin 2x \, dx$$

$$\mathcal{L}^{-1}\left[G(s)\right] = -\frac{1}{4}\left[\cos 2x\right]_0^t = \frac{1}{4}\left(1 - \cos 2t\right)$$

Therefore;

$$\mathcal{L}^{-1}\left[F(s) + G(s)\right] = \mathcal{L}^{-1}\left[F(s)\right] + \mathcal{L}^{-1}\left[G(s)\right]$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4}\left(1 - \cos 2t\right)$$

Example 2.10

Solve by the property of convolution the IVP

$$y'' + y = \cos 2t \tag{i}$$

$$y(0) = y'(0) = 0 \tag{ii}$$

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Solution

We take the transform of equation (i) to have

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[\cos 2t\right]$$

Next, we apply equation (2.15) to have

$$\left\{s^2\mathcal{L}\left[y\right] - sy(0) - y'(0)\right\} + \mathcal{L}\left[y\right] = \frac{s}{s^2 + 4}$$

This gives

$$(s^2 + 1)\mathcal{L}\left[y\right] = \frac{s}{s^2 + 4}, \quad \text{by use of (ii)}$$

Thus,

$$\mathcal{L}\left[y\right] = \frac{s}{(s^2 + 1)(s^2 + 4)}$$

So then,

$$y = \mathcal{L}^{-1}\left[\frac{s}{(s^2 + 1)(s^2 + 4)}\right]$$

Let

$$F(s) = \frac{s}{s^2 + 1}, \qquad G(s) = \frac{1}{s^2 + 4} \equiv \frac{1}{2}\left(\frac{2}{s^2 + 4}\right)$$

Then

$$f(t) = \cos t \qquad g(t) = \frac{1}{2}\sin 2t$$

Thus,

$$
\begin{aligned}
y &= g \times f(t) \\
&= \frac{1}{2}\int_0^t \sin 2t \cos(t - x)dx \\
&= \frac{1}{4}\int_0^t \left[\sin(x + t) + \sin(3x - t)dx\right]
\end{aligned}
$$

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That is, apply trigonometric property

$$\sin A + \sin B = 2\sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

Then

$$
\begin{aligned}
y &= -\frac{1}{4}\left\{\Big[\cos(x+t)\Big]_0^t + \frac{1}{3}\Big[\cos(3x-t)\Big]_0^t\right\} \\
&= -\frac{1}{4}\left\{\cos 2t - \cos t + \frac{1}{3}\Big(\cos 2t - \cos t\Big)\right\} \\
&= -\frac{1}{4}\left\{\frac{4}{3}\cos 2t - \frac{4}{3}\cos t\right\} \\
y &= \frac{1}{3}\Big(\cos t - \cos 2t\Big)
\end{aligned}
$$

This is the required solution

Chapter 3

PARTIAL DIFFERENTIATION EQUATION

3.1   INTRODUCTION

The key defining property of partial differential equation (PDE) in that there is more than one independent variables $x, y, \ldots$. There is a dependent variable that is an unknown function of these variable, $u(x, y, \ldots)$. We will often denotes its derivatives by subscripts; this $\dfrac{\partial u}{\partial x} = u_x$, and so on.

A Partial Differential Equation (or briefly a PDE) is mathematical equation that involves two or more independent variables, an unknown function(dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0 \qquad (3.1)$$

This is the most general PDE in two independent variables of *first* order.

3.2 FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

3.2.1 General Form of First-Order Partial Differential Equation

A first order partial differential equation with independent variables has the general form

$$F\left(x_1, x_2, \cdots, x_n, w, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \cdots, \frac{\partial w}{\partial x_n}\right) = 0 \tag{3.2}$$

where $w = w(x_1, x_2, \cdots, x_n)$ in the unknown function and $F(\cdot)$ is a given function.

3.2.2 Quasilinear Equations, Characteristic System, General Solution

A general form of first-order quasilinear partial differential equation (PDE) with two independent variables has the general form

$$f(x, y, w)\frac{\partial w}{\partial x} + g(x, y, w)\frac{\partial w}{\partial y} = h(x, y, w) \tag{3.3}$$

Such equations are encountered in various applications(continuum mechanics, gas dynamics, hydrodynamics, heat and mars transfer, wave theory, acoustics, multiphere flows, chemical engineering, e.t.c.). If the function f, g and h are independent of the unknown w, then equation (3.3) is called *linear*.

Characteristics System, General Solution

The System of ordinary differential equations

$$\frac{dx}{f(x,y,w)} = \frac{dy}{g(x,y,w)} = \frac{dw}{h(x,y,w)} \tag{3.4}$$

is known as the *characteristics system* of equation (3.3)

Suppose that two independent particular solutions of this system have been found in the form

$$u_i(x,y,w) = c_1 \qquad u_2(x,y,w) = c_2 \tag{3.5}$$

where c_1 and c_2 are arbitrary constants, such particular solutions are known as integral of system (3.4). Then the general solution to (3.3) can be written as

$$\Phi(u_1, u_2) = 0 \tag{3.6}$$

where Φ is an arbitrary function of two variables. With equation (3.6) solved for u_2, one often specifies the general solution in the form $u_2 = \varphi(u_1)$ where $\varphi(u)$ is an arbitrary function of one variable.

Remark: If $h(x,y,w) \equiv 0$, then $w = c_2$ can be used as the second integral in equation (3.5)

Example 3.1

Consider the linear equation $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = b$

The associated characteristics system of ordinary differential equations.

$$\frac{dx}{1} = \frac{dy}{a} = \frac{dw}{b}$$

has two integrals

$$y - ax = c_1 \quad , \quad w - bx = c_2$$

Therefore, the general solution to this PDE can be written as

$$w - bx = \varphi(y - ax) \quad , \quad \text{or} \quad w = bx + \varphi(y - ax)$$

where $\varphi(z)$ is an arbitrary function.

3.2.3 Generalized Cauchy Problem

Generalized Cauchy Problem: Find a solution $w = w(x, y)$ to equation (3.3) satisfying the initial conditions

$$x = \varphi_1(\xi), \quad y = \varphi_2(\xi), \quad w = \varphi_3(\xi) \tag{3.7}$$

where ξ is a parameter $(\alpha \leq \xi \leq \beta)$ and the $\varphi_k(\xi)$ are given functions.

Geometric interpretation: Find an integral surface of equation (3.3) passing through the line defined parametrically by equation (3.7).

3.2.4 Classical Cauchy Problem

Classical Cauchy Problem: Find a solution $w = w(x, y)$ of equation (3.3) satisfying the initial condition

$$w = \varphi(y) \text{ at } x = 0, \tag{3.8}$$

where $\varphi(y)$ is a given function.

It is often convenient to represent the classical Cauchy Problem as a generalized Cauchy Problem by rewriting condition (3.8) in the parametric form

$$x = 0, \quad y = \xi \quad w = \varphi(\xi)$$

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Existence and Uniqueness Theorem

If the coefficients f, g and h of equation (3.3) and the functions φ_k in (3.7) are continuously differentiable with respect to each of their arguments and if the inequalities $f(\varphi_2) - g(\varphi_1) \neq 0$ and $(\varphi_1)^2 + (\varphi_2)^2 \neq 0$ hold along the curve (3.7), then there is a unique solution to the Cauchy Problem (in a neighbourhood of the curve (3.7))

Procedure of solving the Cauchy Problem

The procedure for solving the Cauchy Problem (3.3), (3.7) involves several steps. First, two independent integrals (3.5) are determined. Then, to find the constants of integration c_1 and c_2, the initial data (3.7) must be substituted into the integrals (3.5) to obtain

$$u_1(\varphi_i(\xi), \varphi_2(\xi), \varphi_3(\xi)) = c1, \quad u_2(\varphi_i(\xi), \varphi_2(\xi), \varphi_3(\xi)) = c_2 \qquad (3.9)$$

Eliminating c_1 and c_2 from (3.5) and (3.9) yields

$$
\begin{aligned}
u_1(x, y, w) &= u_1(\varphi_i(\xi), \varphi_2(\xi), \varphi_3(\xi)) \\
u_2(x, y, w) &= u_2(\varphi_i(\xi), \varphi_2(\xi), \varphi_3(\xi))
\end{aligned}
\qquad (3.10)
$$

Formulas (3.10) are a parametric form of the solution to the Cauchy Problem (3.3), (3.7). In some cases, one may succeed in eliminating the parameter ξ from relations (3.10), thus obtaining the solution in an explicit form.

In the cases where first integral (3.5) of the characteristics system (3.4) cannot be found using analytical methods, one should employ numerical methods to solve the Cauchy Problem (3.3), (3.7) (or (3.3), (3.8)).

3.3 SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

3.3.1 Linear, Semilinear and Non-linear Second-Order PDEs

Linear Second-Order PDEs and their properties

A Second-Order linear partial differential equation with two independent variables has the form

$$
\begin{aligned}
a(x,y)\frac{\partial^2 w}{\partial x^2} + 2b(x,y)\frac{\partial^2 w}{\partial x \partial y} + c(x,y)\frac{\partial^2 w}{\partial \partial y^2} &= \alpha(x,y)\frac{\partial w}{\partial x} \\
&\quad + \beta(x,y)\frac{\partial w}{\partial y} + \gamma(x,y)w + \delta(x,y)
\end{aligned}
\tag{3.11}
$$

If $\delta \equiv 0$, equation (3.11) is a *homogeneous linear equation* and if $\delta \neq 0$, it is a non-homogeneous linear equation. The functions $a(x,y), b(x,y), \ldots, \gamma(x,y), \delta(x,y)$ are called coefficients of equation (3.11).

Some properties of a homogenous linear equation (with $\delta \equiv 0$)

1. A homogenous linear equation has a particular solution $w = 0$

2. The principle of linear superposition holds; namely, if $u_1(x,y)$, $u_2(x,y)$, $\ldots, u_k(x,y)$ are particular solutions to homogeneous linear solution equation, then the function $A_1 u_1(x,y) + A_2 u_2(x,y) + \cdots + A_n u_n(x,y)$, $A_1, A_2, \ldots, A_n$ are arbitrary numbers is also an exact solution to that equation.

3. Suppose equation (3.11) has a particular solution $\widetilde{w} = \widetilde{w}(x,y,\mu)$ that depends on a parameter μ(but can depend on x and y). Then, by differentiating $\widetilde{w}$ in the respect to μ, one obtains other solutions to the equation $\dfrac{\partial \widetilde{w}}{\partial \mu}, \dfrac{\partial^2 \widetilde{w}}{\partial \mu^2}, \ldots, \dfrac{\partial^k \widetilde{w}}{\partial \mu^k}, \ldots$

4. Let $\widetilde{w} = \widetilde{w}(x, y, \mu)$ be a particular solution as described in property 3. Multiplying $\widetilde{w}$ by an arbitrary function $\varphi(\mu)$ and integrating the resulting expression with respect to μ over some interval $[\mu_1, \mu_2]$, one obtain a new function $\int_{\mu_1}^{\mu_2} (x, y, \mu)\varphi(\mu)d\mu$, which is also a solution to the original homogeneous linear equation.

5. Suppose the coefficients of the homogeneous linear equation (3.11) are independent of x. Then: (i)there is a particular solution of the form $w = e^{\lambda x}u(y)$ where λ arbitrary number and $u(y)$ is determined by a linear second-order ordinary differential equation, and (ii) differentiating any particular solution with respect to x also results in a particular solution to equation (3.11)

Properties 2-5 are widely used for constructing solutions to problems governed by linear PDEs.

Examples of particular solution to linear PDEs can be found in the subsections Heat equation and Laplace equation below.

Semi-linear and Non-linear Second-order PDEs

A second-order partial differential equation with two independent variables has the form

$$a(x, y)\frac{\partial^2 w}{\partial x^2} + 2b(x, y)\frac{\partial^2 w}{\partial x \partial y} + c(x, y)\frac{\partial^2 w}{\partial y^2} = F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) \quad (3.12)$$

In the general case, a second-order non-linear partial differential equation with two independent variables has the form

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \quad (3.13)$$

The classification and the procedure for reducing linear and semi-linear equations of the form (3.11) and (3.12) to a canonical form are only determined by the left-hand side of the equation.

3.3.2 Some Linear Equation Encountered in Applications

Three basic types of linear partial differential equation are distinguished - *parabolic, hyperbolic* and *elliptic.* The solution of the equation pertaining to each of the types have their own characteristic qualitative differences.

Heat Equation (a parabolic equation)

1. The simplest of a *parabolic equation* is the *heat equation*

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0 \qquad (3.14)$$

   where the variables t and x play the role of the time and a spatial coordinate, respectively. Note that the equation (3.14) contains only one highest derivative term.
   Equation (3.14) is often encountered in the theory of heat and mass transfer. It describes one-dimensional unsteady mass-exchange processes with constant diffusivity.

2. The heat equation (3.14) has infinitely many particular solutions(this fact is commonly to many PDE).

Wave Equation (a hyperbolic equation)

1. The simplest example of a *hyperbolic equation* is the *wave equation*

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \qquad (3.15)$$

where the variables t and x play the role of time and the spatial coordinate, respectively. Note that the highest derivative term in equation (3.15) differ in sign.

This equation is also known as the *equation of vibration of a string*. It is often encountered in elasticity, aerodynamics, acoustics, electrodynamics

2. The general solution of the wave equation (3.15) is

$$w = \varphi(x + t) + \psi(x - t) \tag{3.16}$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary twice continuous differentiable function. This solution has the physical interpretation of two travelling waves of arbitrary shape that propagate to the right and to the left along the axis with a constant speed equal to 1.

Laplace Equation (an elliptic equation)

1. The simplest example of an *elliptic equation* is the *Laplace equation*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{3.17}$$

where x and y play the the role of the spatial coordinates. Note that the highest derivatives terms in equation (3.17) have like signs. The Laplace equation is often written briefly as $\Delta w = 0$, where Δ is the Laplace operator.

The Laplace equation is often considered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, in heat and mass transfer theory,

this equation describes steady-state temperature distribution in the absence of the heat sources and sinks in the domain under study. A solution to the Laplace equation (3.17) is called a *harmonic function.*

2. Note some particular solutions of the Laplace equation (3.17)

$$y \;=\; Ax + By + c$$

$$y \;=\; A(x^2 - y^2) + Bxy$$

$$y \;=\; \frac{Ax}{x^2 + y^2} + By + c$$

$$y \;=\; (A\sinh\mu x + B\cosh\mu x)(C\cos\mu y + D\sin\mu y)$$

$$y \;=\; (A\cos\mu x + B\sin\mu x)(C\sinh\mu y + D\cosh\mu y)$$

where A, B, C, D and μ are arbitrary constants.

A fairly general method for constructing solutions to the Laplace equation (3.17) involves the following.

Let $f(z) = u(x, y) + iv(x, y)$ be any analytic function of the complex variables ($z = x + uiy$ and v are real functions of the real variables x and y; $i^2 = -1$). Then the real and imaginary parts of f both satisfy the Laplace equations;

$$\Delta u = 0, \quad \Delta = 0$$

Thus, by specifying analytic functions $f(z)$ and looking their real and imaginary parts, one obtain various solutions of the Laplace equation (3.17)

3.3.3 Classification of Second-Order Partial Differential Equations

Types of Equations

Any semi-linear partial differential equation of the second order with two independent variables (3.12) can be reduced, by appropriate manipulations, to a simple equation that has one of the three highest derivative combinations specified above in examples (3.14), (3.15) and (3.17).

Given a point (x, y) equation (3.12) is said to be

$$\text{parabolic if } b^2 - ac = 0,$$
$$\text{hyperbolic if } b^2 - ac > 0,$$
$$\text{elliptic if } b^2 - ac < 0$$

a this point.

Characteristic Equations

In order to reduce equation (3.12) to a canonical form, we should first write out the characteristic equation

$$a(dy)^2 - 2b \; dx \; dy + c(dx)^2 = 0$$

which with $a \neq 0$ splits into two equations

$$ady - (b + \sqrt{b^2 - ac})dx = 0 \qquad\qquad (3.18)$$

and

$$ady - (b - \sqrt{b^2 - ac})dx = 0; \qquad\qquad (3.19)$$

Remark. If $a \equiv 0$, the simpler equation

$$dx = 0$$

should be used instead of (3.18) and (3.19). The first equation has the

$$x = c$$

Canonical form of parabolic equations (case $\mathbf{b^2 - ac = 0}$).

In this case, equations (3.18) and (3.19) coincide and have a common general integral,

$$w(x, y) = c$$

By passing from x, y to new independent variables, ξ, η in accordance with the relations

$$\xi = u(x, y), \quad \eta = \eta(x, y),$$

where $\eta = \eta(x, y)$ is any twice differentiable function that satisfies the condition of non-degeneracy of the Jacobian $\dfrac{D(\xi, \eta)}{D(x, y)}$ in the given domain, one reduces equation (3.12) to the canonical form

$$\frac{\partial^2 w}{\partial \eta^2} = F_1\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right) \tag{3.20}$$

As η, we can take $\eta = x$ or $\eta = y$.

It is apparent that the transformed equation (3.20) has only one highest derivative term, just as the heat equation (3.14).

Reduction of Second Order Linear Equations to Canonical Form

Let the general second order linear PDE of two variables be

$$\mathcal{L}[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{3.21}$$

or

$$au_{xx} + 2bu_{xy} + cu_{yy} = \phi(x, y, u, u_x, u_y) \tag{3.22}$$

and let $(\xi, \eta) = (\xi(x, y), \eta(x, y))$ be a non-singular transformation
we write,

$$w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \tag{3.23}$$

where w is a solution of a second-order equation of the same type. Using
the chain rule one finds that

$$u_x = w_\xi \xi_x + w_\eta \eta_x,$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y,$$

$$u_{xx} = w_{\xi\xi} \xi_x^2 + 2w_{\xi\eta} \xi_x \eta_x + w_{\eta\eta} \eta_2^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx} \tag{3.24}$$

$$u_{xy} = w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

$$u_{yy} = w_{\xi\xi} \xi_y^2 + 2w_{\xi\eta} \xi_y \eta_y + w_{\eta\eta} \eta_2^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy}$$

Substituting these formulas into (3.21), we see that as satisfies the following
linear PDE after transformation:

$$\mathcal{L}[w] : Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G \tag{3.25}$$

or

$$\mathcal{L}_0[w] : Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} = \varphi(\xi, \eta, w, w_\xi, w_\eta) \tag{3.26}$$

Where ϕ becomes φ and the new coefficients of the principal part of the

linear operator $\mathcal{L}$ are given by

$$
\begin{aligned}
A(\xi, \eta) &= a\xi_x^2 + 2b\xi_x\eta_x + c\eta_y^2, \\
B(\xi, \eta) &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\
C(\xi, \eta) &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2
\end{aligned}
$$

Example 3.2

Prove that the equation

$$
x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0 \tag{1}
$$

is parabolic and find its canonical form; find the general solution on the half-plane $x > 0$.

Solution

We identify: $a = x^2 \quad 2b = -2xy \quad c = y^2$

therefore $b^2 - ac = x^2 y^2 - x^2 y^2 = 0$

and the equation is parabolic

The equation for the characteristics is

$$
\frac{dy}{dx} = -\frac{y}{x} \tag{2}
$$

and the solution is

$$
xy = \text{ constant}
$$

We define

$$
\eta(x, y) = xy \tag{3}
$$

The second variable can be chosen in

$$\xi(x, y) = x \tag{4}$$

Let $v(\xi, \eta) = u(x, y)$. From (3) and (4), we have;

$$\eta_x = y \qquad \xi_x = 1$$
$$\eta_y = x \qquad \xi = 0$$

By chain rule, we have

$$u_x = yv_\eta + v_\xi \qquad u_y = u_\eta$$
$$u_{xx} = y^2 v_{\eta\eta} + 2yv_{\xi\eta} + v_{\eta\eta} \qquad u_{yy} = x^2 v_{\eta\eta}$$
$$u_{xy} = v_\eta + x_y v_{\eta\eta} + xv_{\xi\eta}$$

Substituting the new coordinates ξ and η into (1), we obtain:

$$x^2(y^2 v_{\eta\eta} + 2yv_{\xi\eta} + v_{\eta\eta}) - 2xy(v_\eta + x_y v_{\eta\eta} + xv_{\xi\eta}) + x^2 y^2 v_{\eta\eta} + xyv_{\eta\eta} + xv_\xi + xyv_\eta \tag{5}$$

Thus

$$\xi^2 v_{\xi\xi} + \xi v_\xi = 0 \tag{6}$$

or

$$v_{\xi\xi} + \left(\frac{1}{\xi}\right) v_\xi = 0 \tag{7}$$

and this is the desired canonical form

Setting

$$w = v_\xi \tag{8}$$

we arrive at the first-order ODE

$$w_\xi + \left(\frac{1}{\xi}\right) w = 0 \tag{9}$$

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The solution is

$$\ln w = \ln \xi + \widetilde{f}(\eta) \tag{10}$$

or

$$w = f(\eta)/\xi \tag{11}$$

Hence v satisfies

$$v = \int v_\xi d\xi = \int w d\xi = \int \frac{f(\eta)}{\xi} d\xi = f(\eta) \ln \xi + g(\eta) \tag{12}$$

Therefore, the general solution $u(x, y)$ of (1) is

$$u(x, y) = f(x, y) \ln x + g(x, y) \tag{13}$$

where $f, g \in C^2(R)$ are arbitrary real functions

Two Canonical forms of hyperbolic equation (case $b^2 - ac > 0$)

1. The general integral

$$u_1(x, y) = c_1 \quad , \quad u_2(x, y) = c_2$$

of equation (3.18) and (3.19) are real and different. These integrals determines two different families of real characteristics.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = u_1(x, y) , \quad \eta = u_2(x, y),$$

one reduces equation (3.12) to

$$\frac{\partial^2 w}{\partial \xi \partial y} = F_2 \left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta} \right)$$

This is the *first canonical form of a hyperbolic equation.*

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2. The transformation

$$\xi = t + z; \quad \eta = t - z$$

brings the above equation to another canonical form

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = F_3\left(t, z, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial z}\right),$$

where $F_3 = 4F_2$. This is called the *second canonical form of a hyperbolic equation*. Apart from notation, the left-hand side of the last equation coincides with that of the wave equation (3.15)

Example 3.3

Consider the Triconi equation

$$u_{xx} + xu_{yy} = 0 \qquad x < 0 \tag{1}$$

Find a mapping $q = q(x, y), r = r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system

Solution

It follows that the Triconi equation is variable coefficient equation with

$$a = 1, \qquad b = 0; \qquad c = x; \qquad (x < 0) \tag{2}$$

We calculate the discriminant, $\delta(\mathcal{L}) = -x > 0$, and therefore the PDE is hyperbolic.

The zeroes of the characteristic polynomial are given by

$$\lambda = \frac{b + \sqrt{\delta}}{a} = \sqrt{-x},$$

and $\qquad\qquad$ (3)

$$\lambda_2 = \frac{b - \sqrt{\delta}}{a} = -\sqrt{-x}$$

Therefore, from the characteristics equation (3.18) and (3.19)

$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} = \lambda_1 = \sqrt{-x},$$

and $\qquad\qquad$ (4)

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} = \lambda_2 = -\sqrt{-x}$$

Integrating the above two ODEs, we obtain the characteristics of the wave equation as

$$y = -\frac{2}{3}\left(-x\right)^{\frac{3}{2}} + K_1 \quad \text{and} \quad y = \frac{2}{3}\left(-x\right)^{\frac{3}{2}} + K_2 \qquad (5)$$

where K_1 and K_2 are the constants of integration.

We see that the two families of characteristics for the wave equation are given by $\frac{3}{2}y + (-x)^{\frac{3}{2}} = \text{constant}$ and $\frac{3}{2}y - (-x)^{\frac{3}{2}} = \text{constant}$.
It follows, that the transformation

$$q = \frac{3}{2}y + (-x)^{\frac{3}{2}}, \qquad r = \frac{3}{2}y - (-x)^{\frac{3}{2}} \qquad (6)$$

reduces the Triconi equation to canonical form.
Thus, the new independent variables are

$$q(x,y) = \frac{3}{2}y + (-x)^{\frac{3}{2}}, \qquad r(x,y) = \frac{3}{2}y - (-x)^{\frac{3}{2}} \qquad (7)$$

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Clearly,

$$q_x = -r_x = -\frac{3}{2}\left(-x\right)^{\frac{1}{2}} \qquad\qquad q_y = r_y = \frac{3}{2} \tag{8}$$

Define $(q, r) = u(x, y)$. By the chain rule

$$u_x = -\frac{3}{2}\left(-x\right)^{\frac{1}{2}}v_q + \frac{3}{2}\left(-x\right)^{\frac{1}{2}}v_r, \qquad\qquad u_y = \frac{3}{2}v_q + \frac{3}{2}v_r,$$

$$u_{xx} = -\frac{9}{4}xv_{qq} - \frac{9}{4}xv_{rr} + 2\frac{9}{4}xv_{qr} + \frac{3}{4}\left(-x\right)^{-\frac{1}{2}}(v_q - v_r)$$

$$u_{yy} = \frac{9}{4}(v_{qq} + v_{rr} + 2v_{qr})$$

Substituting these expression into the Triconi equation we obtain

$$-\frac{9}{4}xv_{qq} - \frac{9}{4}xv_{rr} + 2\frac{9}{4}xv_{qr} + \frac{3}{4}\left(-x\right)^{-\frac{1}{2}}(v_q - v_r) + \frac{9}{4}xv_{qq} + \frac{9}{4}xv_{rr} + 2\frac{9}{4}xv_{qr} = 0$$

which implies that

$$-9\left(q - r\right)^{\frac{2}{3}}\left[v_{qr} - \frac{v_q - v_r}{6(q - r)}\right] = 0$$

Hence,

$$v_{qr} = \frac{v_q - v_r}{6(q - r)} = \psi(q, r, v, v_q, v_r) \qquad (\because q \neq r) \tag{9}$$

Canonical form of elliptic equations (case $b^2 - ac < 0$)

In this case, the general integral of equations (3.18) and (3.19) are complex conjugate; these determines two families of complex characteristics.

Let the general integral of equation (3.18) have the form

$$u_1(x, y) + iu_2(x, y) = c, \qquad i^2 = -1,$$

where $u_{(}x, y)$ and $u_2(x, y)$ are real-valued functions.

By passing from x, y to new independent variables, ξ, η in accordance with the relations

$$\xi = u_1(x, y), \quad \eta = u_2(x, y),$$

one reduces equation (3.12) to the canonical form

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = F_1 \left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta} \right)$$

Apart from notation, the left-hand side of the last coincides with that of the Laplace equation (3.17).

Example 3.4

Show that the equation

$$u_{xx} + x^2 u_{yy} = 0 \tag{1}$$

is elliptic everywhere except on the coordinate axis $x = 0$, find the characteristics variables and hence write the equation in canonical form

Solution

The given equation in variable coefficient equation where

$$a = 1 \quad , \quad b = 0 \ , \qquad c = x^2 \tag{2}$$

The discriminant, $\delta = b^2 - ac = -x^2 < 0$ for $x \neq 0$, and therefore the PDE is elliptic.

The roots of the characteristics polynomial are given by

$$\lambda_1 = \frac{b - i\sqrt{ac - b^2}}{a} = -ix \quad \text{and} \quad \lambda_2 = \frac{b + i\sqrt{ac - b^2}}{a} = ix \tag{3}$$

Therefore, from the characteristic equation (3.18) and (3.19), we have

$$\frac{dy}{dx} = -ix, \qquad \frac{dy}{dx} = ix \qquad (4)$$

Integrating the above two ODEs, we obtain the characteristics in the complex plain as

$$y = -i\frac{x^2}{2} + C_1 \quad , \qquad y = i\frac{x^2}{2} + C_2 \qquad (5)$$

where C_1 and C_2 are the complex constants. We see that the two families of complex characteristics for the elliptic equation are given by

$$y + i\frac{x^2}{2} = \text{constant} \quad \text{and} \quad y - i\frac{x^2}{2} = \text{constant} \qquad (6)$$

It follows, then, that the transformation

$$\alpha = y + i\frac{x^2}{2}, \qquad \beta = y - i\frac{x^2}{2} \qquad (7)$$

The real and imaginary parts of α and β give the required transformation variables ξ and η.

Thus, we have

$$\xi = \frac{\alpha + \beta}{2} = y \qquad \eta = \frac{\alpha - \beta}{2i} = \frac{x^2}{2} \qquad (8)$$

With these choice of coordinates variable, equation (3.25) reduces to following canonical form

$$u_{xx} = w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi\xi_{xx} + w_\eta\eta_{xx} = x^2 w_{\eta\eta} + w_\eta$$
$$\qquad (9)$$
$$u_{yy} = w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi\xi_{yy} + w_\eta\eta_{yy} = w_{\xi\xi}$$

Substituting these relations in the given PDE and noting that $x^2 = 2\eta$, we obtain

$$w_{\xi\xi} + w_{\eta\eta} = -\frac{1}{2\eta}w_\eta \qquad (10)$$

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This is the canonical form of the given elliptic PDE.

Therefore, the PDE

$$u_{xx} + x^2 u_{yy} = 0$$

in rectangular coordinate system (x, y) has been transformed to PDE

$$w_{\xi\xi} + w_{\eta\eta} = -\frac{1}{2\eta} w_\eta$$

in curvilinear coordinate system (ξ, η). Here ξ =constant, these represent a family of straight lines parallel to x-axis and η =constant lines presents family of parabolas.

Table 3.1: Summary

$b^2 - ac$	> 0	$= 0$	< 0
Canonical Form	$\dfrac{\partial^2 w}{\partial\xi\partial\eta} + \cdots = 0$	$\dfrac{\partial^2 w}{\partial\eta^2} + \cdots = 0$	$\dfrac{\partial^2 w}{\partial\xi^2} + \dfrac{\partial^2 w}{\partial\eta^2} + \cdots = 0$
Type	Hyperbolic	Parabolic	Elliptic

Chapter 4

SOLUTION TO PDEs USING LAPLACE TRANSFORM

This chapter deals with the solution of partial differential equation using Laplace transform.

Problem 1:

Consider the partial differential equation

$$\frac{\partial^2 v}{\partial x \partial y} = e^{-2y} \sin x \tag{1}$$

with initial conditions

$$V_x(x,0) = 0, \ V_y(0,y) = 0 \tag{2}$$

Solution

In the above initial value problem (2) ;

$$\mathcal{L}v(x, y) = \frac{\partial^2 v}{\partial x \partial y} \quad ; \qquad \mathcal{L}(x, y) = e^{-2y} \sin x$$

and the general term $Rv(x, y)$ is zero.

Equation (1) can be rewritten in the form;

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = e^{-2y} \sin x \tag{3}$$

Substituting $\dfrac{\partial v}{\partial y} = V$ in equation (3), we get

$$\frac{\partial V}{\partial x} = e^{-2y} \sin x \tag{4}$$

This is non-homogeneous partial differential equation of first order. Taking Laplace transform on both sides of the equation (4) with respect to x, we have

$$SV(s, y) - V(0, y) = \mathcal{L}_x\left[e^{-2y} \sin x\right]$$

$$SV(s, y) - 0 = e^{-2y}\left(\frac{1}{s^2 + 1}\right) \tag{5}$$

$$V(s, y) = e^{-2y}\frac{1}{s}\left(\frac{1}{s^2 + 1}\right)$$

Taking the inverse Laplace transform of equation (5) with respect to x, we get

$$V(x, y) = e^{-2y}(1 - \cos x)$$

$$\frac{\partial V(x, y)}{\partial x} = e^{-2y}(1 - \cos x) \tag{6}$$

This is the partial differential equation of first order in the variables x and y.

Taking Laplace transform of equation (6) with respect to y, we obtain

$$SV(x, s) - V(x, 0) = \mathcal{L}_y \left[e^{-2y}(1 - \cos x) \right]$$

$$SV(x, s) - 0 = (1 - \cos x)\frac{1}{s + 2} \tag{7}$$

$$V(x, s) = (1 - \cos x)\frac{1}{s}\left(\frac{1}{s^2 + 1} \right)$$

Taking inverse Laplace transform of equation (7) with resect to y, we obtain

$$V(x, y) = \frac{1}{2}(1 - \cos x)(1 - e^{-2y}) \tag{8}$$

This is the required exact solution of the given partial differential equation.

Problem 2

Consider the partial differential equation

$$\frac{\partial^2 v}{\partial y \partial x} = \cos x \cos y \tag{1}$$

with initial conditions

$$v(x, 0) = 1 + \sin x \quad ; \quad v_y(0, y) = -2 \cos y \tag{2}$$

Solution

Assume both $v_x(x, y)$ and $v_y(x, y)$ are differentiable in the domain of definition of function $U(x, y)$ [Young's theorem].

This implies that,

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$$

Given initial value problems (2) ;

$$\mathcal{L}v(x, y) = \frac{\partial^2 v}{\partial x \partial y} \quad , \quad h(x, y) = \cos x \cos y$$

and the general term $RV(x, y)$ is zero.

Equation (1) can be rewritten in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \cos x \cos y \tag{3}$$

Substituting $\dfrac{\partial v}{\partial y} = V$ in equation (3), we get

$$\frac{\partial V}{\partial x} = \cos x \cos y \tag{4}$$

Taking Laplace transform of equation (4) with respect to x, we get

$$SV(s, y) - V(0, y) = \mathcal{L}_x\left[\cos x \cos y\right]$$

$$SV(s, y) + 2\cos y = \cos y\left(\frac{s}{s^2 + 1}\right) \tag{5}$$

$$V(s, y) = -\frac{2\cos y}{s} + \cos y\frac{1}{s}\left(\frac{s}{s^2 + 1}\right)$$

Taking inverse Laplace transform of equation (5) with respect to x, we get

$$V(x, y) = -2\cos y + \cos y \sin x$$

$$\frac{\partial v(x, y)}{\partial x} = \cos y(-2 + \sin x) \tag{6}$$

This is the partial differential equation of first order in the variables x and y.

Taking Laplace transform of equation (6) with respect to y, we obtain

$$SV(x, s) - V(x, 0) = \mathcal{L}_y\left[\cos y(-2 + \sin x)\right]$$

$$SV(x, s) - (1 + \sin x) = \frac{s}{s^2 + 1}(-2 + \sin x) \tag{7}$$

$$V(x, s) = \frac{1 + \sin x}{s} + \frac{1}{s}\left(\frac{s}{s^2 + 1}\right)(-2 + \sin x)$$

Taking the inverse Laplace transform of equation (7) with respect to y, we get

$$V(x, y) = 1 + \sin x + \sin y(-2 + \sin x)$$

$$V(x, y) = \sin x(1 + \sin y) - 2 \sin y + 1$$

This is the required exact solution of the given partial differential equation.

Chapter 5