APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$\mathcal{L}\left[f(t)\right] = F(S) = \int_0^{-\infty} e^{-st} f(t)\mathrm{dt}$
$\mathcal{L}\left[f^n(t)\right] = S^n Y - S^{n-1}Y(0) - S^{n-2}Y'(0) \cdots\cdots - Sf^{n-2}(0) \cdots\cdots - y^{n-1}(0)$
$\mathcal{L}\left[U_x(x,t)\right] = \int_0^{\infty} e^{-st} U(x,t)\mathrm{dt} \equiv U(x,s)$
$\mathcal{L}\left[U_x(x,t)\right] = U_x(x,s)$
$\mathcal{L}\left[U_x(x,t)\right] = U_x(x,s)$
$\mathcal{L}\left[U_t(x,t)\right] = SU(x,s) - U(x,0)$
$\mathcal{L}\left[U_t(x,t)\right] = S^2U(x,s) - SU(x,0) - U_t(x,0)$
$F(s) = \mathcal{L}\left[f(t)\right]$ then,
$\mathcal{L}\left[U(t-a) \centerdot g(t-a)\right] = e^{-as}G(s)$

P.D.E of Order 2

Examples:

(1) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial x}(x,t), 0 < x < 2, t > 0 \quad U(0,t) = 0, U(2,t) = 0, U(x,0) = 3\sin(2\pi x)$

Solution:
$U_{xx}(x,t) = U_t(x,t)$

taking the Laplace transform

$\mathcal{L}\left[U_{xx}(x,t)\right] = \mathcal{L}\left[U_t(x,t)\right]$

$U_{xx}(x,s) = sU(x,s) - U(x,0)$

Using the condition, $U(x,0) = 3\sin(2\pi x)$

$SU(x,s) - 3\sin(2\pi x) = U_{xx}(x,s)$

$\Longrightarrow U_{xx}(x,s) - sU(x,s) = -3\sin(2\pi x)$

$\dfrac{d^2u}{dx^2} - sU = -3\sin(2\pi x)$

Solving the Homogenous Problem

$\frac{d^2u}{dx^2} - sU = 0$

The characteristic equation is given by

$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$

The homogenous solution is:

$$U_A(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

i.e $\frac{d^2u}{dx^2} - sU = -3\sin(2\pi x) - - - - - - - - - - - - - (*)$

Let

$U = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x) - - - - - - - - - - (a)$

$U' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x) - - - - - - - - (b)$

$U'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - - - - - - (c)$

Substituting (a) and (c) in equation (*)

$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - S\Delta_1 \sin(2\pi x) - S\Delta_2 \cos(2\pi x) = -3\sin(2\pi x)$

$-4\pi^2\Delta_1 - S\Delta_1 = -3$ Also, $4\pi^2\Delta_2 - S\Delta_2 = 0$

$-\Delta_1\left[4\pi^2 + s\right] = -3$ $\Delta_2\left[s + 4\pi^2\right] = 0$

$\Delta_1 = \frac{3}{s + 4\pi^2}$ $\Delta_2 = 0$

The particular solution is:

$$U_p(x,s) = \frac{3}{s + 4\pi^2}\sin(2\pi x)$$

The general solution is given by:

$$U(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x} + \frac{3\sin(2\pi x)}{s + 4\pi^2}$$

Applying the boundary conditions $U(0,t) = 0,\ U(2,t) = 0$

$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$U(2,s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0 \quad [\text{But } A_1 = -A_2]$
$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$
$A_2 \left[e^{-2\sqrt{s}} - e^{2\sqrt{s}}\right] = 0 \quad \Rightarrow A_2 = 0, A_1 = 0$

$$U(x,s) = \frac{3\sin(2\pi x)}{s + 4\pi^2}$$

Subtituing (a) and (c) in equation (*)
$$-c^2\pi^2 n_1 \sin(\pi x) - c^2\pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = \frac{-\sin(\pi x)}{s}$$
$$\Longrightarrow -c^2\pi^2 n_1 - s_2 n_1 = \frac{-1}{s} \Rightarrow -n_1\left[s^2 + c^2\pi^2\right] = \frac{-1}{s}$$
$$\Longrightarrow n_1 = \frac{1}{s\left[s^2 + c^2\pi^2\right]}$$

Also, $-c^2\pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$
$n_2\left[s^2 + c^2\pi^2\right] = 0$
$\Longrightarrow n_2 = 0$

Substituting 'n_1' and 'n_2' in (a)
$$U_p(x,s) = \frac{\sin(\pi x)}{s}$$

The general solution is given as:

$U_g(x,s) = U_u(x,s) + U_p(x,s)$
$$U_g(x,s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]} \quad --------------(**)$$

Apply the boundary conditions; $U(0,t) = 0$ and $U(1,t) = 0$
$U(0,s) = A_1 + A_2 = 0 \quad \Longrightarrow A_1 = -A_2$
$U(1,s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$

Substituting 'A_1' and 'A_2' in eqution (**)
$$U(x,s) = \frac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]}$$

Applying Inverse Laplace Transform

$$\mathcal{L}^{-1}\left[U(x,s)\right] = \sin(\pi x)\ \mathcal{L}^{-1}\left[\frac{1}{s\left[s^2+c^2\pi^2\right]}\right]$$

Resolving $\dfrac{1}{s\left[s^2+c^2\pi^2\right]}$ into partial fractions

$$\frac{1}{s\left[s^2+c^2\pi^2\right]} = \frac{A}{s} + \frac{Bs+D}{s^2+c^2\pi^2} = \frac{A\left[s^2+c^2\pi^2\right]+\left[Bs+D\right]s}{s\left[s^2+c^2\pi^2\right]}$$

$$1 = A\left[s^2+c^2\pi^2\right] + Bs^2 + Ds$$

Taking the Inverse Laplace equation

$$\mathcal{L}^{-1}\left[U(x,s)\right] = \mathcal{L}^{-1}\left[\frac{3\sin(2\pi x)}{s+4\pi^2}\right]$$

$$\mathbf{U(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x)}$$

(2) $\dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\dfrac{\partial^2 u}{\partial x^2}(x,t)\ ; 0 < x < 1, t > 0,$
$U(x,0) = 0, U_t(x,0) = 0, U(0,t) = 0, U(1,t) = 0$

<u>Solution:</u>

$$U_{tt}(x,t) = c^2 U_{xx}(x,t) + \sin(\pi x)$$

Taking the Laplace transform

$$\mathcal{L}\left[U_{tt}(x,t)\right] = c^2\mathcal{L}\left[U_{xx}(x,t)\right] + \mathcal{L}\left[\sin(\pi x)\right]$$

$$s^2U(x,s) - sU(x,0) - U_t(x,0) = c^2U_{xx}(x,s) + \frac{sin(\pi x)}{s}$$

Applying the initial conditions; $U(x,0) = 0$ and $U_t(x,0) = 0$

$$s^2U(x,s) - c^2U_{xx}(x,s) = \frac{\sin(\pi x)}{s}$$

Re-arranging

$$c^2U_{xx}(x,s) - s^2U(x,s) = -\frac{\sin(\pi x)}{s}$$