Using the Gaussian Elimination method,

$$\left(\begin{array}{cccc|c} 16 & 16 & -4 & -24 & 0 \\ 16 & 32 & -12 & -24 & 16 \\ -4 & -12 & 30 & 1 & -63 \\ -24 & -24 & 1 & 46 & 20 \end{array}\right)$$

$$\begin{array}{lll} R_1' = & \frac{R_1}{16} & \Longrightarrow \\ R_2' = & R_2 - R_1 & \Longrightarrow \\ R_3' = & 4R_3 + R_1 & \Longrightarrow \\ R_4' = & 2R_4 + 3R_1 & \Longrightarrow \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -\frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 16 & -8 & 0 & 16 \\ 0 & -32 & 116 & -20 & -252 \\ 0 & 0 & -10 & 20 & 40 \end{array}\right)$$

$$\begin{array}{ll} R_2' = & \frac{R_2}{8} \\ R_3' = & \frac{R_3}{4} \\ R_4' = & \frac{R_4}{10} \end{array} \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -\frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & -8 & 29 & -5 & -63 \\ 0 & 0 & -1 & 2 & 4 \end{array}\right)$$

$$R_3' = \quad R_3 + R_4 \quad \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 25 & -5 & -55 \\ 0 & 0 & -1 & 2 & 4 \end{array}\right)$$

$$R_3' = \quad \frac{R_3}{5} \quad \Longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 5 & -1 & -11 \\ 0 & 0 & -1 & 2 & 4 \end{array}\right)$$

$$R_4' = \quad 5R_4 + R_3 \quad \implies \left(\begin{array}{cccc|c} 1 & 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 5 & -1 & -11 \\ 0 & 0 & 0 & 9 & 9 \end{array}\right)$$

This implies:

(i) $9x_4 = 9 \implies x_4 = 1$

(ii) $5x_3 - x_4 = -11$
$5x_3 - 1 = -11$
$5x_3 = -11 + 1$
$x_3 = -\dfrac{10}{5} = -2$

(iii) $2x_2 - x_3 = 2$
$2x_2 - (-2) = 2$
$2x_2 + 2 = 2$
$2x_2 = 2 - 2 = 0$
$x_2 = \dfrac{0}{2} = 0$

(iv) $x_1 + x_2 - \dfrac{x_3}{4} - \dfrac{3x_4}{2} = 0$

$x_1 + \dfrac{2}{4} - \dfrac{3}{2} = 0$

$x_1 + \dfrac{2-6}{4} = 0$

$x_1 + \left(-\dfrac{4}{4}\right) = 0$

$x_1 - 1 = 0 \implies x_1 = 1$

Hence, $x_1 = 1, x_2 = 0, x_3 = -2$, and $x_4 = 1$

$$x = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Example 2: Solve the Linear System of equations below

$6x + 15y + 55z = 76$
$15x + 55y + 225z = 295$
$55x + 255y + 979z = 1259$

Solution

(i) Using Cholesky decomposition method

$$\begin{pmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 76 \\ 295 \\ 1259 \end{pmatrix}$$

Cholesky decomposition: $A = L \cdot L^T$, Every symmetric positive definite matrix A can be decomposed into a product of a unique lower triangular matrix L and its transpose. Here, the matrix is symmetric positive definite.

Formula $l_{ki} = \dfrac{a_{ki} - \sum\limits_{j=1}^{i-1} l_{ij} - l_{kj}}{l_{ii}}$ and $l_{kk} = \sqrt{a_{kk} - \sum\limits_{j=1}^{k-1} l_{kj}^2}$

$$\begin{aligned}
l_{11} &= \sqrt{a_{11}} = \sqrt{6} = 2.44495 \\
l_{21} &= \frac{a_{21}}{l_{11}} = \frac{15}{2.4495} = 6.1237 \\
l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{55 - (6.1237)^2} = \sqrt{55 - 37.5} = 4.1833 \\
l_{31} &= \frac{a_{31}}{l_{11}} = \frac{55}{2.4495} = 22.4537 \\
l_{32} &= \frac{a_{32} - l_{31} \times l_{21}}{l_{22}} = \frac{225 - (22.4537) \times (6.1237)}{4.1833} = \frac{225 - 137.5}{4.1833} = 20.9165 \\
l_{33} &= \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{979 - (22.4537)^2 - (20.9165)^2} = \sqrt{979 - 941.6667} = 6.1101
\end{aligned} \tag{1}$$

$$\text{So,}\quad L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \quad = \quad \begin{pmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{pmatrix}$$

$$L \times L^T \quad = \quad \begin{pmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{pmatrix} \begin{pmatrix} 2.4495 & 6.1237 & 22.4537 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.117 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{pmatrix}$$

Now, $Ax = B$, and $A = LL^T \implies LL^T x = B$

Let $L^T x = y$, then $Ly = B$

$$\implies \quad \begin{bmatrix} 2.4495 & 0 & 0 \\ 6.1237 & 4.1833 & 0 \\ 22.4537 & 20.9165 & 6.1101 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 76 \\ 295 \\ 1259 \end{bmatrix}$$

$2.4495 y_1 = 76 \implies y_1 = 31.0269$

$6.1237 y_1 + 4.1833 y_2 = 295$

$6.1237(31.0269) + 4.1833 y_2 = 295$

$4.1833 y_2 = 295 - 190 = 105$

$y_2 = 25.0998$

$22.4537 y_1 = 20.9165 y_2 + 6.1101 y_3 = 1259$

$22.4537(31.0269) + 20.9165(25.0998) + 6.1101 y_3 = 1259$

$y_3 = \frac{37.3333}{6.1101} = 6.1101$

Now, $L^T x = y$

$$\begin{bmatrix} 2.4495 & 6.1237 & 22.4537 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.1101 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 31.0269 \\ 25.0998 \\ 6.1101 \end{bmatrix}$$

Using back substitution method

$$6.1101z = 6.1101 \implies z = 1$$

$4.1833y + 20.9165x = 25.0998$

$4.1833y + 20.9165(1) = 25.0998$

$4.1833y = 25.0998 - 20.9165 = 4.1833$

$y = 1$

$2.4495x + 6.1237y + 22.4537z = 31.0269$

$2.4495x + 6.1237(1) + 22.4537(1) = 31.0269$

$2.4495x = 31.0269 - 28.5774 = 2.4495$

$x = \dfrac{2.4495}{2.4495} = 1$

Hence, $x = 1, y = 1$ and $z = 1$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii)Using Gaussian Elimination Method

$$\left[\begin{array}{ccc|c} 6 & 15 & 55 & 76 \\ 15 & 55 & 225 & 295 \\ 55 & 225 & 979 & 1259 \end{array}\right]$$

$$\begin{aligned} R_1' &= \tfrac{R_1}{6} \\ R_2' &= \tfrac{R_2}{5} \\ R_3' &= \tfrac{R_3}{5} \end{aligned} \implies \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{55}{6} & \frac{38}{3} \\ 3 & 11 & 45 & 59 \\ 11 & 45 & \frac{979}{5} & \frac{1259}{5} \end{array}\right)$$

$$\begin{aligned} R_2' &= R_2 - 3R_1 \\ R_3' &= R_3 - 11R_1 \end{aligned} \implies \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{55}{6} & \frac{38}{3} \\ 0 & \frac{7}{2} & \frac{35}{2} & 21 \\ 0 & \frac{35}{2} & \frac{2849}{30} & \frac{1687}{15} \end{array}\right)$$

$$R_3' = \quad 7R_3 - 35R_2 \quad \Longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{55}{6} & \frac{38}{3} \\ 0 & \frac{7}{2} & \frac{35}{2} & 21 \\ 0 & 0 & \frac{784}{15} & \frac{784}{15} \end{array}\right)$$

Using backward substitution method,

$\dfrac{784}{15}z = \dfrac{784}{15} \implies z = 1$

$\dfrac{7}{2}y + \dfrac{35}{2}z = 21$

$\dfrac{7}{2}y + \dfrac{35}{2} = 21 \implies \dfrac{7}{2}y = 21 - \dfrac{35}{2}$

$\dfrac{7}{2}y = \dfrac{7}{2} \implies y = 1$

$x + \dfrac{5}{2}y + \dfrac{55}{6}z = \dfrac{38}{3}$

$x + \dfrac{5}{2} + \dfrac{55}{6} = \dfrac{38}{3}$

$x + \dfrac{35}{3} = \dfrac{38}{3}$

$x = \dfrac{38}{3} - \dfrac{35}{3} = \dfrac{3}{3} = 1$

Hence, $x = 1,\ y = 1$, and $z = 1$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(3) Solve;

$$\begin{aligned} 7x + 10y + 5z = 42 \\ 11x + 6y + 2z = 31 \\ 11x + 14y + 8z = 63 \end{aligned}$$

Solution

(1)Using the Gaussian Elimination Method.

The linear equation given can be written in matrix form:

$$\begin{bmatrix} 7 & 10 & 5 \\ 11 & 6 & 2 \\ 11 & 14 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \\ 63 \end{bmatrix}$$

The Augmented matrix is given by:

$$\left[\begin{array}{ccc|c} 7 & 10 & 5 & 42 \\ 11 & 6 & 2 & 31 \\ 11 & 14 & 8 & 63 \end{array}\right] \begin{array}{c} R_2' \to R_2 - \frac{11}{7}R_1 \\ R_3' \to R_3 - \frac{11}{7}R_1 \end{array} \left[\begin{array}{ccc|c} 7 & 10 & 15 & 42 \\ 0 & -\frac{88}{7} & -\frac{51}{7} & -47 \\ 0 & -\frac{12}{7} & \frac{1}{7} & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 7 & 10 & 15 & 42 \\ 0 & -\frac{88}{7} & -\frac{51}{7} & -47 \\ 0 & -\frac{12}{7} & \frac{1}{7} & -3 \end{array}\right] R_3' \to R_3 - \frac{12}{88}R_2 \left[\begin{array}{ccc|c} 7 & 10 & 15 & 42 \\ 0 & -\frac{88}{7} & -\frac{51}{7} & -47 \\ 0 & 0 & \frac{25}{22} & \frac{75}{22} \end{array}\right]$$

Using back substitution method

$$\frac{25}{22}z = \frac{75}{22} \implies z = \frac{75}{25} = 3$$

$$-\frac{88}{7}y - \frac{51}{7}z = -47$$

$$-\frac{88}{7}y - \frac{153}{7} = -47 \implies \frac{88}{7}y = 47 - \frac{153}{7}$$

$$y = \frac{176}{7} \times \frac{7}{88} \implies y = 2$$

$$7x + 10y + 5z = 42$$

$$7x + 10(2) + 5(3) = 42 \implies 7x = 42 - 35 = 7$$

Hence, $x = 1,\ y = 2$, and $z = 3$.

(2)Using Cholesky decomposition method

<u>Note:</u> The matrix is not symmetric; thereby not a symmetric positive definite matrix.

Using $l_{ki} = \dfrac{a_{ki} - \sum\limits_{j=1}^{i-1} l_{ij} - l_{kj}}{l_{ii}}$ and $l_{kk} = \sqrt{a_{kk} - \sum\limits_{j=1}^{k-1} l_{kj}^2}$

$l_{11} = \sqrt{a_{11}} = \sqrt{7} = 2.6458$

$l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{11}{2.6458} = 4.1575$

$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{6 - (4.1575)^2} = \sqrt{6 - 17.2848} = \left[\text{No real answer}\right]$

This implies, Cholesky decomposition method fails for a non-symmetric positive definite matrix.