APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$\mathcal{L}\left[f(t)\right] = F(S) = \int_0^{-\infty} e^{-st} f(t)\mathrm{dt}$
$\mathcal{L}\left[f^n(t)\right] = s^n y - s^{n-1}y(0) - s^{n-2}y'(0) \cdots\cdots - sf^{n-2}(0) \cdots\cdots - y^{n-1}(0)$
$\mathcal{L}\left[u_x(x,t)\right] = \int_0^{\infty} e^{-st}u(x,t)\mathrm{dt} \equiv u(x,s)$
$\mathcal{L}\left[u_x(x,t)\right] = u_x(x,s)$
$\mathcal{L}\left[u_x(x,t)\right] = u_x(x,s)$
$\mathcal{L}\left[u_t(x,t)\right] = su(x,s) - u(x,0)$
$\mathcal{L}\left[u_t(x,t)\right] = s^2u(x,s) - su(x,0) - u_t(x,0)$
$F(s) = \mathcal{L}\left[f(t)\right]$ then,
$\mathcal{L}\left[u(t-a) \centerdot g(t-a)\right] = e^{-as}G(s)$

P.D.E of Order 2

Examples:

(1) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial x}(x,t), 0 < x < 2, t > 0 \quad u(0,t) = 0, u(2,t) = 0, u(x,0) = 3\sin(2\pi x)$

Solution:
$u_{xx}(x,t) = u_t(x,t)$

taking the Laplace transform

$\mathcal{L}\left[u_{xx}(x,t)\right] = \mathcal{L}\left[u_t(x,t)\right]$

$u_{xx}(x,s) = su(x,s) - u(x,0)$

Using the condition, $u(x,0) = 3\sin(2\pi x)$

$su(x,s) - 3\sin(2\pi x) = u_{xx}(x,s)$

$\Longrightarrow u_{xx}(x,s) - su(x,s) = -3\sin(2\pi x)$

$\dfrac{d^2u}{dx^2} - sU = -3\sin(2\pi x)$

Solving the Homogenous Problem

$\frac{d^2u}{dx^2} - su = 0$

The characteristic equation is given by

$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$

The homogenous solution is:

$$u_A(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

i.e $\frac{d^2u}{dx^2} - su = -3\sin(2\pi x) - - - - - - - - - - - - -(*)$

Let

$U = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x) - - - - - - - - - - (a)$

$U' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x) - - - - - - - - (b)$

$U'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - - - - - -(c)$

Substituting (a) and (c) in equation (*)

$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - s\Delta_1 \sin(2\pi x) - S\Delta_2 \cos(2\pi x) = -3\sin(2\pi x)$

$-4\pi^2\Delta_1 - S\Delta_1 = -3$ Also, $4\pi^2\Delta_2 - S\Delta_2 = 0$

$-\Delta_1\left[4\pi^2 + s\right] = -3$ $\Delta_2\left[s + 4\pi^2\right] = 0$

$\Delta_1 = \frac{3}{s + 4\pi^2}$ $\Delta_2 = 0$

The particular solution is:

$$u_p(x,s) = \frac{3}{s+4\pi^2}\sin(2\pi x)$$

The general solution is given by:

$$u(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x} + \frac{3\sin(2\pi x)}{s+4\pi^2}$$

Applying the boundary conditions $u(0,t) = 0,\ u(2,t) = 0$

$u(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$u(2,s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0 \quad$ [But $A_1 = -A_2$]

$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$

$A_2 \left[e^{-2\sqrt{s}} - e^{2\sqrt{s}} \right] = 0 \quad \Rightarrow A_2 = 0, A_1 = 0$

$$u(x,s) = \frac{3\sin(2\pi x)}{s + 4\pi^2}$$

Subtituing (a) and (c) in equation (*)

$$-c^2\pi^2 n_1 \sin(\pi x) - c^2\pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = \frac{-\sin(\pi x)}{s}$$

$$\Longrightarrow -c^2\pi^2 n_1 - s_2 n_1 = \frac{-1}{s} \Rightarrow -n_1 \left[s^2 + c^2\pi^2 \right] = \frac{-1}{s}$$

$$\Longrightarrow n_1 = \frac{1}{s\left[s^2 + c^2\pi^2\right]}$$

Also, $-c^2\pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$

$n_2 \left[s^2 + c^2\pi^2 \right] = 0$

$\Longrightarrow n_2 = 0$

Substituting 'n_1' and 'n_2' in (a)

$$u_p(x,s) = \frac{\sin(\pi x)}{s}$$

The general solution is given as:

$u_g(x,s) = u_u(x,s) + u_p(x,s)$

$$u_g(x,s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]} \quad --------------- (**)$$

Apply the boundary conditions; $\quad u(0,t) = 0$ and $u(1,t) = 0$

$u(0,s) = A_1 + A_2 = 0 \quad \Longrightarrow A_1 = -A_2$

$u(1,s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$

Substituting $'A_1'$ and $'A_2'$ in eqution (**)

$$u(x,s) = \frac{\sin(\pi x)}{s\left[s^2 + c^2\pi^2\right]}$$

Applying Inverse Laplace Transform

$\mathcal{L}^{-1}\left[u(x,s)\right] = \sin(\pi x)\ \mathcal{L}^{-1}\left[\dfrac{1}{s\left[s^2+c^2\pi^2\right]}\right]$

Resolving $\dfrac{1}{s\left[s^2+c^2\pi^2\right]}$ into partial fractions

$\dfrac{1}{s\left[s^2+c^2\pi^2\right]} = \dfrac{A}{s} + \dfrac{Bs+D}{s^2+c^2\pi^2} = \dfrac{A\left[s^2+c^2\pi^2\right]+\left[Bs+D\right]s}{s\left[s^2+c^2\pi^2\right]}$

$1 = A\left[s^2+c^2\pi^2\right] + Bs^2 + Ds$

Taking the Inverse Laplace equation

$\mathcal{L}^{-1}\left[u(x,s)\right] = \mathcal{L}^{-1}\left[\dfrac{3\sin(2\pi x)}{s+4\pi^2}\right]$

$$\mathbf{u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x)}$$

(2) $\dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\dfrac{\partial^2 u}{\partial x^2}(x,t)\ ; 0 < x < 1, t > 0,$
$u(x,0) = 0, u_t(x,0) = 0, u(0,t) = 0, u(1,t) = 0$

Solution:

$u_{tt}(x,t) = c^2 u_{xx}(x,t) + \sin(\pi x)$

Taking the Laplace transform

$\mathcal{L}\left[u_{tt}(x,t)\right] = c^2\mathcal{L}\left[u_{xx}(x,t)\right] + \mathcal{L}\left[\sin(\pi x)\right]$

$s^2u(x,s) - su(x,0) - u_t(x,0) = c^2u_{xx}(x,s) + \dfrac{sin(\pi x)}{s}$

Applying the initial conditions; $u(x,0) = 0$ and $u_t(x,0) = 0$

$s^2u(x,s) - c^2u_{xx}(x,s) = \dfrac{\sin(\pi x)}{s}$

Re-arranging

$c^2u_{xx}(x,s) - s^2u(x,s) = -\dfrac{\sin(\pi x)}{s}$

Solving the Homogenous problem

$$c^2\frac{d^2u}{dx^2}(x,s) - s^2u(x,s) = 0$$

The Auxillary equation is given by:

$$c^2m^2 - s^2 = 0 \Longrightarrow m^2 - \left(\frac{s}{c}\right)^2 = 0 \Longrightarrow m = \pm\frac{s}{c}$$

The homogenous solution is: $u(x,s) = A_1e^{\frac{sx}{c}} + A_2e^{-\frac{sx}{c}}$

Solving the non-homogenous problem by method of Undetermined Coefficient:
i.e $c^2\frac{d^2u}{dx^2}(x,s) - s^2u(x,s) = -\frac{\sin(\pi x)}{s} - - - - - - - - - - -(*)$

Let:

$$u(x,s) = n_1\sin(\pi x) + n_2cos(\pi x) - - - - - - - - - - - - - - (a)$$
$$u_x(x,s) = \pi n_1\cos(\pi x) - \pi n_2\sin(\pi x) - - - - - - - - - - - - -(b)$$
$$u_{xx}(x,s) = -\pi^2n_1\sin(\pi x) - \pi^2n_2\cos(\pi x) - - - - - - - - - - -(c)$$

Comparing or Equating the Coefficients of boths sides:

$s: \quad D = 0$
$s^2: \ A + B = 0 \Longrightarrow A = -B$
constant : $1 = Ac^2\pi^2 \Longrightarrow A = \frac{1}{c^2\pi^2} \Longrightarrow B = \frac{-1}{c^2\pi^2}$

$$\therefore \frac{1}{s\left[s^2+c^2\pi^2\right]} = \frac{1}{sc^2\pi^2} - \frac{s}{(c^2\pi^2)(s^2+c^2\pi^2)} = \frac{1}{c^2\pi^2}\left[\frac{1}{s} - \frac{s}{s^2+c^2\pi^2}\right]$$

$$u(x,t) = \frac{\sin(\pi x)}{c^2\pi^2}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+(c\pi)^2}\right]$$

$$\mathbf{u(x,t) = \frac{\sin(\pi x)}{c^2\pi^2}\left[1 - \cos(c\pi t)\right]}$$

NEXT:
Solution of non-linear PDE, by the combined Laplace transform and the new Modified Variational Iteration Method.

SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATION BY THE COMBINED LAPLACE TRANSFORM METHOD AND THE NEW MODIFIED VARIATIONAL ITERATION METHOD

Presenting a reliable combined Laplace transform and the new modified variational Iteration method to solve some non-linear Partial Differential Equations. This method is more efficient and easy to handle non-linear PDEs.

Recall,

$$\mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0)$$

$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

Where $F(x,s)$ is the Laplace transform of (x,t) [x is considered as a dummy variable and t, a parameter].

Illustrating the basic concept of **<u>He's Variational Iteration Method</u>**, we consider the following general differential equations:

$$Lu(x,t) + Nu(x,t) = g(x,t) - - - - - - - - -(i)$$

$$\text{with the initial condition, } u(x,0) = G(x) - - - -(ii)$$

Where L is a linear operator of the first Order, N is a non-linear operator and $g(x,t)$ is non-homogenous term. According to Variational Iteration Method, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda\left[LU(x,s) + N(x,s) - g(x,s)\right]ds - - - - - \, (iii)$$

where λ is a Lagrange Multiplier ($\lambda = -1$), the subscripts $'n'$ denotes the nth approximation, $\bar{U}_n$ is considered as a restricted variation, i.e $\partial\bar{U}_n = 0$.
Equation (iii) is called a **Correction Functional**
Obtaining the Lagrange Multiplier $'\lambda'$ by using Integration by Part of Equation (i), but the Lagrange Multiplier is of the form $\lambda = \lambda(x,t)$.
Then, Laplace transform of equation (iii), then the correction functional will be in the form:

$$\mathcal{L}\left[u_{n+1}(x,t)\right] = \mathcal{L}\left[u_n(x,t)\right] + \mathcal{L}\left[\int_0^t \lambda(x,t)\left(Lu_n(x,s) + Nu_n(x,s) - g(x,s)\right)ds\right],\ n \geq 0 - - - - - -(iv)$$

Therefore,

$$\mathcal{L}\left[u_{n+1}(x,t)\right] = \mathcal{L}\left[u_n(x,t)\right] + \lambda\left[Lu_n(x,t) + Nu_n(x,t) - g(x,t)\right] - - - - - - - - - -(v)$$

To find the optimal value of $\lambda(x,t)$, we first take the Variation with respect to $u_n(x,t)$ and in such a case, the integration is basically the single convolution to t, and hence, Laplace transform is appropriate to use.

thus;

$$\frac{\partial}{\partial u_n}\mathcal{L}\left[u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\frac{\partial}{\partial u_n}\mathcal{L}\left[Lu_n(x,t) + Nu_n(x,t) - g(x,t)\right] - - - (vi)$$

Equation (vi) becomes,

$$\mathcal{L}\left[\partial u_{n+1}(x,t)\right] = \mathcal{L}\left[\partial u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\partial\mathcal{L}\left[Lu_n(x,t)\right] - - - - - - - (vii)$$

$\{i.e \ \ \partial N\bar{u}_n(x,t) = 0$ and $\partial g(x,t) = 0\}$

We assume that L is a linear first-order Partial Differential Operator in this chapter given by $\frac{\partial}{\partial t}$ then, equation (vii) can be written in the form

$$\mathcal{L}\left[\partial u_{n+1}(x,t)\right] = \mathcal{L}\left[\partial u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\left[s\mathcal{L}\partial u_n(x,t)\right]$$

the extreme condition of $u_{n+1}(x,t)$ requires that $\partial u_{n+1}(x,t) = 0$

$$\Longrightarrow 0 = \mathcal{L}\left[\partial u_n(x,t)\right]\left[1 + s\mathcal{L}\left[\lambda(x,t)\right]\right]$$

$$\Longrightarrow 1 + s\mathcal{L}\left[\lambda(x,t)\right] = 0$$

$$\Longrightarrow s\mathcal{L}\left[\lambda(x,t)\right] = -1$$

$$\Longrightarrow \mathcal{L}\left[\lambda(x,t)\right] = \frac{-1}{s}$$

Taking the Laplace Inverse of both sides

$$\lambda(x,t) = \mathcal{L}^{-1}\left[\frac{-1}{s}\right]$$

$$\lambda(x,t) = -1$$

This implies $\lambda = -1$

Substituting $(\lambda = -1)$ in equation (iii)

$$u_{n+1} = u_n - \int_0^t \left[Lu_n(x,s) + N\bar{u}_n(x,s) - g(x,s)\right]ds - - - - - - - - - (viii)$$

The successive approximation 'u_{n+1}' of the solution 'u' will be readily obtained by using the determined Lagrange Multiplier and any selective function u_n consequently, the solution is given by:

$$u(x,t) = \lim_{x\to\infty} u_n(x,t)$$

Also, from equation (i)

$$i.e \;\; LU(x,t) + Nu(x,t) = g(x,t) - - - - - - - (*)$$

taking the Laplace transform of both sides, we have

$$\mathcal{L}\left[LU(x,t)\right] + \mathcal{L}\left[Nu(x,t)\right] = \mathcal{L}\left[g(x,t)\right]$$

Using the differential property of Laplace transform and initial condition (ii), we have:

$$s\mathcal{L}\left[u_{x,t}\right] - h(x) = \mathcal{L}\left[g(x,t)\right] - \mathcal{L}\left[Nu(x,t)\right] - - - - - - - - - - - - (**)$$

Applying the inverse Laplace transform on both side of the equation (**), we find:

$$u(x,t) = G(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[(Nu(x,t))\right]\right] - - - - - - (***)$$

Where $G(x,t)$ represents the terms arising from the source term and the prescribed initial condition $\left[i.e\; G(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left[g(xt,)\right] + h(x)\right]\right\}\right]$

Taking the first Parital derivates with respect to 't' of equation (***) to obtain:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}G(x,t) - \frac{\partial}{\partial t}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[Nu(xt,)\right]\right\} - - - - - - (****)$$

By the correction functional of the Variation iteration method:

$$u_{n+1}(x,t) = u_n - \int_0^t \left\{(u_n)_1(x,s) - \frac{\partial}{\partial s}G(x,s) + \frac{\partial}{\partial s}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[Nu(x,s)\right]\right\}\right\} ds$$

or

$$u_{n+1}(x,t) = G(x,t) - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[Nu_n(x,t)\right]\right\} - - - - - - - - - (****)$$

Equation (****) is the new modified correction functional of Laplace transform and the Variational Iteration Method, and the solution is given by:

$$u(x,t) = \lim_{u\to\infty} u_n(x,t)$$

<u>**Next:**</u> Solving some non-linear PDE, by using the new modified variational iteration Laplace transformation method:

Examples:

(1) $\dfrac{\partial u}{\partial t} = \left(\dfrac{\partial u}{\partial x}\right)^2 + u\dfrac{\partial^2 u}{\partial x^2}$; $u(x,0) = x^2$

Solution:

Given $u_t = u_x^2 + uu_{xx}$; $u(x,0) = x^2$

Taking Laplace transform, subject to the initial condition, we have:

$\mathcal{L}[u_t] = \mathcal{L}\left[u_x^2\right] + \mathcal{L}[uu_{xx}]$

$s\mathcal{L}[u(x,t)] - u(x,0) = \mathcal{L}\left[u_x^2 + uu_{xx}\right]$

$s\mathcal{L}[u(x,t)] - x^2 = \mathcal{L}\left[u_x^2 + uu_{xx}\right]$

$\mathcal{L}[u(x,t)] = \dfrac{x^2}{s} + \dfrac{1}{s}\mathcal{L}\left[u_x^2 + uu_{xx}\right]$

Taking the Inverse Laplace transform to obtain:

$u(x,t) = \mathcal{L}^{-1}\left[\dfrac{x^2}{s}\right] + \mathcal{L}^{-1}\left[\dfrac{1}{s}\mathcal{L}\left[u_x^2 + uu_{xx}\right]\right]$

$u(x,t) = x^2 + \mathcal{L}^{-1}\left\{\dfrac{1}{s}\mathcal{L}\left[u_n^2 + uu_{xx}\right]\right\}$

The new correction functional is given as:

$$u_{n+1}(x,t) = x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[(u_n)_x + u_n(u_n)_{xx}\right]\right\} \quad n \geq 0$$

The solution in series form is given by:

$$u_0(x,t) = x^2 \ \left[\text{ or } u_0(x,t) = u(x,0)\right]$$

$$\begin{aligned}
u_1(x,t) &= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[(u_0)_x^2 + (u_0)(u_0)_{xx}\right]\right\} \\
u_1(x,t) &= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[4x^2 + 2x^2\right]\right\} \\
&= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\frac{4x^2}{s} + \frac{2x^2}{s}\right]\right\} \\
&= x^2 + \mathcal{L}^{-1}\left\{\frac{6x^2}{s^2}\right\} \\
u_1(x,t) &= x^2 + 6x^2 t
\end{aligned}$$

$$
\begin{aligned}
u_2(x,t) &= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[(u_1)_x^2 + (u_1)\,(u_1)_{xx}\right]\right\} \\
u_2(x,t) &= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[(2x+12xt)^2 + (x^2+6x^2t)(2+12t)\right]\right\} \\
&= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[6x^2 + 72x^2t + 216x^2t^2\right]\right\} \\
&= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\frac{6x^2}{s} + \frac{72x^2}{s^2} + \frac{216x^2\cdot 2}{s^3}\right]\right\} \\
&= x^2 + \mathcal{L}^{-1}\left[\frac{6x^2}{s^2} + \frac{72x^2}{s^3} + \frac{432x^2}{s^4}\right] \\
u_1(x,t) &= x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3
\end{aligned}
$$

The series solution is given by:

$$
\begin{aligned}
u(x,t) &= x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3 + -----------\\
u(x,t) &= x^2\left[1 + 6t + 36t^2 + 72t^3 + -------------\right] \\
u(x,t) &= \frac{x^2}{1-6t}
\end{aligned}
$$

(2) $u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2t^2$, $u(x,0) = 0, \dfrac{\partial u(x,0)}{\partial u} = x$

$\mathcal{L}\left[u_{tt}\right] = \mathcal{L}\left[x^2t^2\right] + \mathcal{L}\left[u_{xx} - u^2\right]$