NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATION OF THIRD ORDER BY VARIATION ITERATION METHOD

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Certification

This is to certify that this project was carried out by **ADEOYE LATEEF ABIMBOLA** of Matriculation Number 16/56EB018, for the award of Bachelor of Science B.Sc (Hons) degree in the Department of Mathematics, Faculty of Physical sciences, University of Ilorin, Ilorin, Nigeria.

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Dedication

This Project is dedicated to Almighty God who in his infinite mercy saw me through to this very stage of my life and to my beloved parents Mr. and Mrs. Adeoye, who with their love and support have seen me through my education.

May God be with you and grant you long life.

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Abstract

In this project we discuss the numerical solution of Integro-Differential Equationof third order by Variational Iteration Method. It was also observed from the definition that Integro-Differential Equationcan be thought as an equation that involves both integrals and derivatives of an unknown function.

Table of Contents

Chapter 1

INTRODUCTION

Mathematics is the Science of numbers quantity and space. Under these we have numerical analysis, which brings about Integro-Differential Equationequation.

Solving Integro-Differential Equationis one of the major problems of numerical analysis. This is because such a wide variety of application leads to Integro-Differential Equationand so few can be solved analytically.
In Science, mathematics Methodology is used in analysis data set for solving system of equations but the main goal of numerical analysis is to simplify methods for obtaining solutions to mathematical problems, such mathematics has its basis in applied mathematics, such as engineering, physics and dynamics.

Numerical analysis according ot Gilat and Amos (American Heritage Dictionary) (2004) is defined as the study of approximation techniques for solving mathematical problems.

1.1 DEFINITION OF SOME RELEVANT TERMS

1.1.1 INTEGRO-DIFFERENTIAL EQUATION

An equation that involves both integral and derivatives of an unknown function is known as **Integro-Differential Equation**.

1.1.2 DIFFERENTIAL EQUATION

A mathematical equation that relates some unknown functions of one or more variables with derivatives is known as **differential equation**.

1.1.3 EQUATION

In Mathematics, an equation is a formulation of the form A and B, where A, B are expressions that may contains one or more variables called unknown and "=" denotes the equality of binary relation, equation say two thing are equal. Also an equation is a statement of an equality containing one or more variable.

Solving the equation consists of determining which values of the variables make the equality true. In this situation, variables are also known as unknown and the values which satisfy the equality are known as solutions.

1.1.4 SOLUTION

In mathematics, a value we can put in place of a variable (such as x) that makes the equation true, a solution is the set of values that satisfy a given set of equation or inequalities.

1.1.5 NUMERICAL SOLUTION

Numerical solution is the study of approximation techniques for solving mathematical problems, taking into account the extent of possible errors. It can also be defined as the branch of mathematics the studies of algorithm for approximating solutions to problems in infinitesimal Calculus. Numerical Methods for Integro-Differential Equations

1.1.6 VARIATIONAL

This is a method of calculating an upper bound on the lowest energy level of a quantum mechanical system and an approximation for the corresponding wave function. In the integral representing the expectation value of the hamiltonian operator, one substitutes a trial function for the true wave function and varies parameters in the trial function to minimize the integral.

1.1.7 VARIATIONAL ITERATION

This method is one of the well known semi-analytical methods for solving linear and non-linear ordinary as well as partial differential equation.

1.1.8 THIRD ORDER EQUATION

This is an equation at which the derivative of $f(x)$ is 3 e.g $f'''(x)$

Let $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

Therefore the third derivative of $f(x)$ is

$f'''(x) = 24x$

Or using Leibniz notation
$\frac{d^3}{dx^3}(x^4) = 24x$

Chapter 2

LITERATURE REVIEW

2.1 INTEGRO-DIFFERENTIAL EQUATION

Volterra, in the early 1900, studied the population growth where new type of equations have been developed and was termed as Integro-Differential Equations. In this type of equations, the unknown function $U(x)$ occurs in one side as an ordinary derivatives and appears in the other side under integral sign.

In other to discuss the method, we consider the general nth order Integro-Differential Equationof the form:

$$U^n(x) = f(x) + \lambda \int_0^x K(x,t)U(t)\mathrm{d}t \tag{2.1}$$

where $U^n(x) = \dfrac{\mathrm{d}^n u}{dx^n}$, λ is a parameter and $K(x,t)$ is the kernel of the integral equation.

2.1.1 CLASSIFICATION OF INTEGRO-DIFFERENTIAL EQUATION

Integro-Differential Equationcan be classified into two namely;

i. Volterra Integro-Differential Equationand

ii. Fredholm Integro-Differential Equation

2.1.2 VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

This contains the unknown function $U(x)$ and one of it's derivatives $U^{(n)}(x),\ n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equation.

The Volterra Integro-Differential Equationis given in the form

$$U^n(x) = f(x) + \lambda \int_0^x K(x,t)U(t)\mathrm{d}t \tag{2.2}$$

where $U^{(n)}$ indicates the nth derivatives of $U(x)$. Other derivatives of less order may appear with $U^{(n)}$ at the left side

2.1.3 FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

This contains the unknown function $U(x)$ and one of it's derivatives $U^{(n)}(x),\ n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in Fredholm integration equations.

The Fredholm Integro-Differential Equationcan be given in the form

$$U^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)U(t)\mathrm{d}t \tag{2.3}$$

Examples of Integro-Differential Equationsare

$$\begin{aligned} \text{(i)}\ \ U'(x) &= -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tU(t)\mathrm{d}t && (2.4) \\ \text{(ii)}\ U'(x) &= 1 - \frac{1}{3}x + \int_0^1 tU(t)\mathrm{d}t,\ U(0) = 0 && (2.5) \\ \text{(iii)}\ U'(x) &= 1 + \int_0^x U(t)\mathrm{d}t,\ U(0) = 0 && (2.6) \end{aligned}$$

equation **(2.4)** and **(2.6)** are Volterra Integro-Differential Equationwhile **(2.5)** is Fredholm Integro-Differential Equation.

2.1.4 VARIATIONAL ITERATION METHOD

In this section, numerical method using in this project ot solve Integro-Differential Equationis Variational Iteration Method. In order to apply this method, we assume that the differential equation is of the form

$$LU + NU = g(t) \tag{2.7}$$

where L and N are linear and non-linear operators respectively and $g(t)$ is the source in homogeneous term.

The Variational iteration method presents a correction functional

$$U_{n+1}(x) = U_n(x) + \int_0^x \lambda(\xi)\left[LU_n(\xi) + N\bar{U}_n(\xi) - g(\xi)\right]\mathrm{d}\xi \tag{2.8}$$

where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function and $\bar{U}_n$ is a restricted value that mean it behaves as a constant, hence if $\delta \bar{U}_n = 0$ where δ is the Variational derivatives.

For a complete use of the Variational iteration method, we should follow tow steps

i. The determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally and

In other words, the correction functional **(2.8)** will give several approximation and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multiplier and its correction functional respectively.

$$\text{(i)} \begin{cases} U' + F(U(\xi), U'(\xi)) = 0, \ \lambda = -1 \\ U_{n+1} = U_n - \int_0^x \left[U_n' + F(U_n, U_n')\right] \mathrm{d}t \end{cases}$$

$$\text{(ii)} \begin{cases} U'' + F(U(\xi), U'(\xi), U''(\xi)) = 0, \ \lambda = (\xi - x) \\ U_{n+1} = U_n - \int_0^x (\xi - x) \left[U_n'' + F(U_n, U_n', U_n'')\right] \mathrm{d}\xi \end{cases}$$

$$\text{(iii)} \begin{cases} U''' + F(U(\xi), U'(\xi), U''(\xi), U'''(\xi)) = 0, \ \lambda = -\frac{1}{2}(\xi - x)^2 \\ U_{n+1} = U_n - \int_0^x -\frac{1}{2}(\xi - x)^2 \left[U_n'' + F(U(\xi), U'(\xi), U''(\xi), U'''(\xi))\right] \mathrm{d}\xi \end{cases}$$

For $n \geq 1$

Therefore for first order

$$\begin{aligned}
\lambda(\xi) &= -1 \\
\lambda(\xi) &= (\xi - x) \text{ For second order} \\
\lambda(\xi) &= -\frac{(\xi - x)^2}{2} \text{ for third order}
\end{aligned}$$

and so on.

ii. With $\lambda(\xi)$, we substitute the result into **(2.8)** where the restrictions should be determined.

Taking the variation **(2.8)** with respect to the independent variables U_n we find

$$\frac{\delta U_{n+1}}{\delta U_n} = 1 + \frac{\delta}{\delta U_n}\left(\int_0^x \lambda(\xi)(LU_n(\xi) + N\bar{U}_n(\xi) - g(\xi))\mathrm{d}\xi\right) \tag{2.9}$$

or equivalently

$$\delta U_{n+1} = \delta U_n + \delta\left(\int_0^x \lambda(\xi)(LU_n(\xi))\mathrm{d}\xi\right) \tag{2.10}$$

Let $U_n'(\xi) = LU_n(\xi)$ in equation **(2.10)**

Using integration by parts for the determination of Lagrange Multiplier

$$\left.\begin{aligned}
&\textstyle\int_0^x \lambda(\xi)U_n'(\xi)\mathrm{d}\xi = \lambda(\xi)U_n(\xi) - \int_0^x \lambda'(\xi)U_n(\xi)\mathrm{d}\xi \\
&\textstyle\int_0^x \lambda(\xi)U_n''(\xi)\mathrm{d}\xi = \lambda(\xi)U_n'(\xi) - \lambda'(\xi)U_n(\xi) + \int_0^x \lambda''(\xi)U_n(\xi)\mathrm{d}\xi \\
&\textstyle\int_0^x \lambda(\xi)U_n'''(\xi)\mathrm{d}\xi = \lambda(\xi)U_n''(\xi) - \lambda'(\xi)U_n'(\xi) + \lambda''(\xi)U_n(\xi) \\
&\qquad\qquad\qquad\textstyle - \int_0^x \lambda'''(\xi)U_n(\xi)\mathrm{d}\xi
\end{aligned}\right\} \tag{2.11}$$

and so on. These identities are obtained by integrating by parts. Integrating the integral of **(2.10)** by parts using **(2.11)** we obtain

$$\delta U_{n+1} = \delta U_n(\xi)(1 + \lambda|_{\xi=x}) - \int_0^x \lambda' \; \delta U_n \; \mathrm{d}\xi \tag{2.12}$$

or equivalently

$$\delta U_{n+1} = \delta U_n(\xi)(1 + \lambda|_{\xi=x}) - \int_0^x \lambda' \; \delta U_n \; \delta\xi \tag{2.13}$$

The extremum condition of U_{n+1} require that $\delta_{n+1} = 0$. This means that the left hand side of **(2.13)** is zero and as a result the right hand side should be zero as well. This yield the stationary conditions:

$$1 + \lambda|_{\xi=x} = 0, \qquad \lambda'|_{\xi=x} = 0 \tag{2.14}$$

$$\lambda = -1 \tag{2.15}$$

As a second example, Let $LU_n(\xi) = U_n''(\xi)$ in **(2.10)**, integrating the integral of **(2.10)** by parts using **(2.11)** we obtain

$$\delta U_{n+1} = \delta U_n + \delta\lambda \left((U_n)'\right)_0^x - (\lambda'\delta U_n)_0^x + \int_0^x \lambda''\delta U_n \mathrm{d}\xi \tag{2.16}$$

or equivalent

$$\delta U_{n+1} = \delta U_n(\xi)(1 - \lambda'|_{\xi=x}) + \delta\lambda((U_n)'|_{\xi=x}) + \int_0^x \lambda'' U_n \mathrm{d}\xi \tag{2.17}$$

The extremum condition of U_{n+1} requires that $\delta U_{n+1} = 0$. This means that the left hand side of **(2.17)** is zero and as a result the right hand side should be zero as well. This yields the stationary condition

$$\lambda = \xi - x$$

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximation $U_{n+1},\ n \geq 0$ of the solution $U(x)$ will be readily obtained upon using selective function $U_0(x)$. However, for fast convergence, the function $U_0(x)$ should be selected by suing the initial condition as follows:

$$\begin{aligned} &U_0(x) = U(0) \text{ for first order } U_n' \\ &U_0(x) = U(0) + xU'(0) \text{ for second order } U_n'' \\ &U_0(x) = U(0) + xU'(0) + \tfrac{1}{2!}x^2U''(0) \text{ for third order } U_n''' \end{aligned} \tag{2.18}$$

and so on consequently, the solution

$$U(x) = \lim_{n\to\infty} U_n(x) \tag{2.19}$$

Chapter 3

METHODOLOGY

NUMERICAL EXAMPLES ON INTEGRO-DIFFERENTIAL EQUATION

3.1 EXAMPLES

Here, we consider the third order Integro-Differential Equations

$$\text{i) } U'''(x) = -1 + \int_0^x U(t)\mathrm{d}t \tag{3.1}$$

with the initial condition $U(0) = 1,\ U'(0) = 1$ and $U''(0) = -1$

$$\text{ii) } U'''(x) = e^x - 1 + \int_0^1 tU(t)\mathrm{d}t \tag{3.2}$$

with the initial condition $U(0) = 1,\ U'(0) = 1$ and $U''(0) = 1$

Equation **(3.1)** is Volterra while **(3.2)** is Fredholm.

3.1.1 EXAMPLE 1

Consider the third order Integro-Differential Equation

$$U'''(x) = 1+x+\frac{1}{3!}x^3+\int_0^x (x-t)U(t)\mathrm{d}t$$

with the initial conditions $U(01,\ U'(0) = 0$ and $U''(0) = 1$

$$U'''(x)-1-x-\frac{1}{3!}x^3-\int_0^x (x-t)U(t)\mathrm{d}t = 0 \tag{3.3}$$

The correction functional for this equation is given by

$$U_{n+1}(x) = U_n(x)-\frac{1}{2}\int_0^x (\xi-x)^2 \left(U_n'''(x) - 1 - x - \frac{1}{3!}x^3 - \int_0^x (x-t)U_n(t)\mathrm{d}t\right) \mathrm{d}\xi \tag{3.4}$$

where $\lambda = -\frac{1}{2}(\xi - x)^2$ for third order Integro-Differential Equation

The zeroth approximation $U_0(x)$ can be selected by using the initial conditions, hence we set

$$U_0(x) = U(0) + xU'(0) + \frac{1}{2!}x^2U''(0) = 1 + \frac{1}{2!}x^2 \tag{3.5}$$

using this selection into the correction function gives the following successive approximations

$$U_0(x) = 1 + \frac{1}{2!}x^2$$

For $n = 0$

$$U_1(x) = U_0(x)-\frac{1}{2}\int_0^x (\xi-x)^2 \left(U_0'''(x) - 1 - x - \frac{1}{3!}x^3 - \int_0^x (x-t)U_0(t)\mathrm{d}t\right) \mathrm{d}\xi$$

Hence $U_0(x) = 1 + \frac{1}{2!}x^2$

$$U_0'(x) = x, \quad U_0''(x) = 1 \quad \text{and} \quad U_0'''(x) = 0$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[0 - 1 - x - \frac{1}{3!}x^3\right. \\
&- \left.\int_0^x (x-t)\left(1 + \frac{1}{2!}t^2\right)\mathrm{d}t\right]\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-1 - x - \frac{1}{3!}x^3\right. \\
&- \left.\int_0^x \left(x + \frac{1}{2!}xt^2 - t - \frac{1}{2!}t^3\right)\mathrm{d}t\right]\mathrm{d}\xi
\end{aligned}
$$

$$
U_1(x) = U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-1 - x - \frac{1}{3!}x^3 - \left[xt + \frac{xt^3}{6} - \frac{t^2}{2} - \frac{t^4}{8}\right]_0^x\ \right]\mathrm{d}\xi
$$

$$
U_1(x) = U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-1 - x - \frac{1}{3!}x^3 - \left(x^2 + \frac{x^4}{6} - \frac{x^2}{2} - \frac{x^4}{8}\right)\right]\mathrm{d}\xi
$$

$$
U_1(x) = U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left(-1 - x - \frac{1}{3!}x^3 - x^2 - \frac{x^4}{6} + \frac{x^2}{2} + \frac{x^4}{8}\right)\mathrm{d}\xi
$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x \left[-\xi^2 - x\xi^2 - \frac{x^3\xi^2}{6} - x^2\xi^2 - \frac{x^4\xi^2}{6} + \frac{x^2\xi^2}{2} + \frac{x^4\xi^2}{8} + 2x\xi\right. \\
&+ 2x^2\xi + \frac{2x^4\xi}{6} + 2x^3\xi + \frac{2x^5\xi}{6} - \frac{2x^3\xi}{2} - \frac{2x^5\xi}{8} - x^2 - x^3 - \frac{x^5}{6} - x^4 - \frac{x^6}{6} \\
&+ \left.\frac{x^4}{2} + \frac{x^6}{8}\right]\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\left[-\frac{\xi^3}{3} - \frac{x\xi^3}{3} - \frac{x^3\xi^3}{18} - \frac{x^2\xi^3}{3} - \frac{x^4\xi^3}{18} + \frac{x^2\xi^3}{6} + \frac{x^4\xi^3}{24} + \frac{2x\xi^2}{2}\right. \\
&+ \frac{2x^2\xi^2}{2} + \frac{2x^4\xi^2}{12} + \frac{2x^3\xi^2}{2} + \frac{2x^5\xi^2}{12} - \frac{2x^3\xi^2}{4} - \frac{2x^5\xi^2}{16} - x^2\xi - x^3\xi - \frac{x^5\xi}{6} \\
&- \left.x^4\xi - \frac{x^6\xi}{6} + \frac{x^4}{2} + \frac{x^6}{8}\right]_0^x
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
U_1(x) &= U_0(x)+\frac{x^3}{6}+\frac{x^4}{6}+\frac{x^6}{36}+\frac{x^5}{6}+\frac{x^7}{36}-\frac{x^5}{12}-\frac{x^7}{48}-\frac{x^3}{2}-\frac{x^4}{2}-\frac{x^6}{12}-\frac{x^5}{4}\\
&- \frac{x^7}{12}+\frac{x^5}{4}+\frac{x^7}{16}+x^3+x^4+\frac{x^6}{6}+x^5+\frac{x^7}{6}-\frac{x^7}{6}-\frac{x^5}{4}-\frac{x^7}{8}
\end{aligned}
$$

$$
U_1(x) = 1+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7
$$

and so on. The Variational Integration Method admits he use of

$$
U(x)=\lim_{n\to\infty}U_n(x)
$$

that gives the exact solution

$$
U(x)=e^x-x \tag{3.7}
$$

3.1.2 EXAMPLE 2

$$
U'''(x)=e^x-1+\int_0^1 tU(t)\mathrm{d}t
$$

with initial conditions $U(0)=1,\quad U'(0)=1,\quad U''(0)=1$

$$
U'''(x)-e^x+1-\int_0^1 tU(t)\mathrm{d}t \tag{3.8}
$$

The correction functional for this equation is given by

$$
U_{n+1}(x)=U_n(x)+\int_0^x \lambda(\xi)\left[U_n'''(x)-e^x+1-\int_0^1 tU_n(t)\mathrm{d}t\right]\mathrm{d}\xi
$$

where $\lambda=-\frac{1}{2}(\xi-x)^2$ for third order Integro-Differential Equation.

The zeroth approximation $U_0(x)$ can be selected by

$$
\begin{aligned}
U_0(x) &= U(0)+xU'(0)+\frac{1}{2!}x^2U''(0)\\
U_0(x) &= 1+x+\frac{1}{2!}x^2 \qquad (3.9)
\end{aligned}
$$

Using the selection into the correction functional gives the following successive approximations

$U_0(x) = 1 + x + \frac{1}{2!}x^2$

for $n = 0$

$$U_1(x) = U_0(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left[U_0'''(x) - e^x + x - \int_0^1 tU_0(x)\mathrm{d}t \right] \mathrm{d}\xi$$

Since $U_0(x) = 1 + x + \frac{1}{2!}x^2$

$$U_0'(x) = 1 + x, \quad U_0''(x) = 1 \text{ and } U_0'''(x) = 0$$

Therefore

$$\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[0 - e^x + x - \int_0^1 t\left(1 + t + \frac{1}{2!}t^2\right)\mathrm{d}t\right]\mathrm{d}\xi \\
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-e^x + x - \int_0^1 \left(t + t^2 + \frac{t^3}{2!}\right)\mathrm{d}t\right]\mathrm{d}\xi \\
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-e^x + x - \left[\frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}\right]_0^1\right]\mathrm{d}\xi \\
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x (\xi^2 - 2x\xi + x^2)\left[-e^x + x - \frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right]\mathrm{d}\xi \\
U_1(x) &= U_0(x) - \frac{1}{2}\int_0^x \left[-\xi^2 e^x + x\xi - \frac{\xi^2}{2} - \frac{\xi^2}{3} - \frac{\xi^2}{8} + 2x\xi e^x - 2x^2\xi + \frac{2x\xi}{2} + \frac{2x\xi}{2}\right. \\
&+ \left.\frac{2x\xi}{8} - x^2 e^x + x^3 - \frac{x^2}{2} - \frac{x^2}{3} - \frac{x^2}{8}\right]\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\left[-\frac{\xi^3 e^x}{3} + \frac{x\xi^2}{2} - \frac{\xi^3}{6} - \frac{\xi^3}{9} - \frac{\xi^3}{24} + x\xi^2 e^x - x^2\xi + \frac{x\xi^2}{2} + \frac{x\xi^2}{3}\right.\\
&+ \left.\frac{x\xi^2}{8} - x^2\xi e^x + x^3\xi - \frac{x^2\xi}{2} - \frac{x^2\xi}{3} - \frac{x^2\xi}{8}\right]_0^x
\end{aligned}
$$

$$
\begin{aligned}
U_1(x) &= U_0(x) - \frac{1}{2}\left[-\frac{x^3 e^x}{3} + \frac{x^3}{2} - \frac{x^3}{6} - \frac{x^3}{9} - \frac{x^3}{24} + x^3 e^x - x^3 + \frac{x^3}{2} + \frac{x^3}{3}\right.\\
&+ \left.\frac{x^3}{8} - x^3 e^x + x^3 - \frac{x^3}{2} - \frac{x^3}{3} - \frac{x^3}{8}\right]
\end{aligned}
$$

$$
U_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{x^3 e^x}{6} - \frac{x^4}{144}
$$

and so on. Canceling the noise terms give the exact solution

$$
U(x) = e^x \tag{3.10}
$$

3.1.3 EXAMPLE 3

$$
U'''(x) = -1 + \int_0^x U(t)\mathrm{d}t
$$

with initial conditions $U(0) = 1, \quad U'(0) = 1, \quad U''(0) = -1$

$$
U'''(x) + 1 - \int_0^x U(t)\mathrm{d}t = 0 \tag{3.11}
$$

The correction functional for this equation is given by

$$
U_{n+1}(x) = U_n(x) + \int_0^x \lambda(\xi)\left[U'''(x) + 1 - \int_0^x U(t)\mathrm{d}t\right] \tag{3.12}
$$

where $\lambda = -\frac{1}{2}(\xi - x)^2$ for third order Integro-Differential Equation.

Therefore,

The zeroth approximation $U_0(x)$ can be selected by using initial condition, hence we set

$$\begin{aligned} U_0(x) &= U(0) + xU'(0) + \frac{1}{2!}x^2U''(0) \\ &= 1 + x(1) + \frac{1}{2!}x^2(-1) \\ U_0(x) &= 1 + x - \frac{1}{2!}x^2 \end{aligned}$$

for $n = 0$

$$U_1(x) = U_0(x) - \frac{1}{2}\int_0^x (\xi - x)^2 \left[U_0'''(x) + 1 - \int_0^x U_0(t)\mathrm{d}t\right] \mathrm{d}\xi$$

Since $U_0(x) = 1 + x - \frac{1}{2!}x^2$

$$U_0'(x) = 1 - x, \quad U_0''(x) = -1 \text{ and } U_0'''(x) = 0$$

$$U_1(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{2}\int_0^x \left(\xi^2 - 2x\xi + x^2\right)\left[0 + 1 - \int_0^x \left(1 + t - \frac{1}{2}t^2\right)\mathrm{d}t\right]\mathrm{d}\xi$$

$$U_1(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\int_0^x \left(\xi^2 - 2x\xi + x^2\right)\left[1 - \int_0^x \left(1 + t - \frac{t^2}{2}\right)\mathrm{d}t\right]\mathrm{d}\xi$$

$$U_1(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\int_0^x \left(\xi^2 - 2x\xi + x^2\right)\left[1 - \left[t + \frac{t}{2} - \frac{t^3}{6}\right]_0^x \mathrm{d}t\right]\mathrm{d}\xi$$

$$U_1(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\int_0^x \left(\xi^2 - 2x\xi + x^2\right)\left[1 - x - \frac{x^2}{2} + \frac{x^3}{6}\right]\mathrm{d}\xi$$

$$\begin{aligned} U_1(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\int_0^x \left[\xi^2 - x\xi^2 - \frac{x^2\xi^2}{2} + \frac{x^2\xi^2}{6} - 2x\xi + 2x^2\xi + \frac{2x^3\xi}{2}\right. \\ &- \left.\frac{2x^3\xi}{6} - x^2 - x^3 - \frac{x^4}{2} + \frac{x5}{6}\right]\mathrm{d}\xi \end{aligned}$$

$$\begin{aligned} U_1(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\left[\frac{\xi^3}{3} - \frac{x\xi^3}{3} - \frac{x^2\xi^3}{6} + \frac{x^2\xi^4}{24} - \frac{2x\xi^2}{2} + \frac{2x^2\xi^2}{2} + \frac{2x^3\xi^2}{2}\right. \\ &- \left.\frac{2x^3\xi^2}{6} - x^2\xi - x^3\xi - \frac{x^4\xi}{2} + \frac{x^5\xi}{6}\right]_0^x \end{aligned}$$

$$\begin{aligned} U_1(x) &= 1 + x - \frac{1}{2}x^2 - \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{3} - \frac{x^5}{6} + \frac{x^6}{24} - \frac{2x^3}{2} + \frac{2x^4}{2} + \frac{2x^5}{6}\right. \\ &- \left.\frac{2x^6}{12} - x^3 - x^4 - \frac{x^5}{2} + \frac{x^6}{6}\right] \end{aligned}$$

$$U_1(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{6!}x^6$$

$$U_1(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

and so on. The Variational Iteration Method admits the use of

$$U(x) = \lim_{n \to \infty} U_n(x)$$

That gives the exact solution

$$U(x) = \cos x + \sin x \tag{3.13}$$

Chapter 4

4.1 CONVERTING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS TO INITIAL VALUE PROBLEMS

The conversion process is obtained by differentiating both sides of the Volterra Integro-Differential Equationas many times until we get rid of the integral sign. To perform the differentiation, the integral at the right side, the Leibnitz rule should be used. The initial conditions should be determined by using a variety of integral equation that will obtain in the process of differentiation.

To give the clear overview of this method, we discuss the following illustrative examples.

1. Solve the Volterra Integro-Differential Equationby converting it to an initial value problem

$$U''(x) = -1 - x + \int_0^x (x-t)U(t)\mathrm{d}t, \quad U(0) = 1, \quad U'(0) = 1 \tag{4.1}$$

4.1.1 THE NOISE TERMS PHENOMENON

The noise terms is defined as the identical terms with opposite sign that arise in the components $U_0(x)$ and $U_1(x)$. Other noise terms may appear between other components.

As stated above, these identical terms with opposite sign may exist for some equations and it may not appear for other equation.

By canceling the noise terms between $U_0(x)$ and $U_1(x)$, even through $U_1(x)$ contains further terms, the remaining non-canceled terms of $U_0(x)$ may gives the exact solution of the integral equation. The appearance of the noise terms between $U_0(x)$ and $U_1(x)$ is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it is necessary to show that the non-canceling terms of $U_0(x)$ satisfy the given integral equation. On the other hand, if the non-canceled terms of $U_0(x)$ did not satisfy the given integral equation, or the noise term did not appear between $U_0(x)$ and $U_1(x)$, then it is necessary to determine more components of $U(x)$ to determine the solution in a series form.

Let assume

$$\begin{aligned} U_0(x) &= \sin x + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 \\ U_1(x) &= -\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \end{aligned}$$

Canceling the noise terms $\frac{1}{3!}x^3 - \frac{1}{4!}x^4$ from $U_0(x)$ gives the exact solution $U(x) = \sin x$

4.2 SOLUTION OF AN INTEGRAL EQUATION

A solution of a differential or an integral equation arises in any of the following two types:

1. **Exact Solution**

   The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions.

   Examples of exact solutions are as follows:

$$
\begin{aligned}
U(x) &= x + e^x \\
U(x) &= \sin x + e^{2x} \\
U(x) &= 1 + \cosh x + \tan x
\end{aligned}
$$

   and many others.

2. **Series Solution**

   For concrete problems, sometimes we cannot obtain exact solutions. In this case, we determine the solution in a series form that may converge to exact solution. If such a solution exists. Other series can be used for numerical purposes. The more terms that we determine, the higher accuracy level that we can achieve.

A solution of an integral or Integro-Differential Equationof a function $U(x)$ that satisfies the given equation. In other words, the obtained solution $U(x)$ must satisfy both sides of the examined equation. The following examples will be examined to explain the meaning of a solution.

Show that $U(x) = \sin x$ is a solution of Volterra Integro-Differential Equation

$$U'(x) = 1 - \int_0^x U(t)\mathrm{d}t \tag{4.2}$$

Using $U(x) = \sin x$ into both side of **(4.2)**, we find

$$\begin{aligned} \text{LHS} &= U'(x) = \cos x \\ \text{RHS} &= 1 - \int_0^x \sin x\mathrm{d}t = 1 - (-\cos t)\Big|_0^x = \cos x \end{aligned}$$

Example 2

Show that $U(x) = x + e^x$ is a solution of the Fredholm Integro-Differential Equation

$$U''(x) = e^x - \frac{4}{3}x + \int_0^1 xtU(t)\mathrm{d}t \tag{4.3}$$

Using $U(x) = x + e^x$ into both sides of **(4.3)**, we find

$$\begin{aligned} \text{LHS} &= U''(x) = e^x \\ \text{RHS} &= e^x - \frac{4}{3}x + x\int_0^1 t(t+e^t)\mathrm{d}t \\ &= e^x - \frac{4}{3}x + x\left(\frac{1}{3}t^3 + te^t - e^t\right)\Big|_0^1 = e^x \end{aligned}$$

Chapter 5

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 SUMMARY

In this Project, we discuss the numerical solution of Integro-Differential Equationof third order by Variational Iteration Method

- The types
- Method for solving
- Conversion of Volterra Integro-Differential Equationto Initial Value Problem
- The noise term phenomenon

It was observed from the definition that Integro-Differential Equationcan be thought as an equation that involves both integrals and derivatives of an unknown function.

5.2 CONCLUSION

It can be shown that Integro-Differential Equationof a function $U(x)$ that satisfies the given equation. It can be proved using the the obtained solution $U(x)$ which must satisfy both sides of the examined equation.

5.3 RECOMMENDATION

The study of numerical is very important and it's applicable in various fields. It can be used in data analysis sets for solving system equation but the main goal of numerical analysis is to simplify method for obtaining solution to mathematical problems.

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