SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATION USING ADOMIAN DECOMPOSITION

BY

ELUTADE, ISAIAH ABIOLA
17/56EB050

A PROJECT SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF ILORIN, ILORIN, KWARA STATE, NIGERIA.

IN PARTIAL FULFILMENT OF REQUIREMENTS FOR THE AWARD OF BACHELOR OF SCIENCE (B. Sc.) DEGREE IN MATHEMATICS.

February, 2022

CERTIFICATION

This is to certify that this project was carried out by **ELUTADE, Isaiah Abiola** with Matriculation Number 17/56EB050 in the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria, for the award of Bachelor of Science (B.Sc.) degree in Mathematics.

....

Prof. A.O. Taiwo Date

Supervisor

....

Prof. K. Rauf Date

Head of Department

....

Prof.o Date

External Examiner

ACKNOWLEDGMENTS

All praises

i

DEDICATION

I would like to dedicate the project to God, for the grace and faithfulness of God thus far. For His mercies, guidance and protection throughout my years of study.

ABSTRACT

In this project,

Table of Contents

Chapter 1

INTRODUCTION TO ADOMIAN DECOMPOSITION METHOD (ADM)

1.1 General Introduction

The Adomian Decomposition Method (ADM) was first established in the 1980's by George Adomian, chairman of Center for Applied Mathematics at the University of Georgia. In the recent years, this method of Adomian Decomposition has be paid attention to in the field of applied mathematics and series solution. In addition, the ADM is widely used to obtain the solution to many types of linear or non-linear ordinary differential equation and integral equations.

The ADM gives the accurate and efficient solution to problem in a direct and simply way without the use of linearization and partubation which can change the physical behaviour of the method.

For the purpose of this study, we consider the solution of second order differential equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x)$$

or

$$y''P(x) + y'Q(x) + R(x)y = F(x)$$

with initial conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = y_1$$

Where $P(x), Q(x), R(x)$ and $F(x)$ are continuous functions of x; and y_0 and y_1 are given constant.

1.2 Definition of Relative Terms

1.2.1 Differential Equation

Differential equation is an equation that contains at least one derivative of an unknown function, either ordinary derivative $\left(\frac{d}{dy}\right)$, or a partial derivative $\left(\frac{\partial}{\partial x}\right)$.

1.2.2 General Second Order Ordinary Differential Equation

The general Second Order Ordinary Differential Equation of an independent variable x and dependent variable y is given by

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = f(x) \qquad \text{Or}$$

$$y''P(x) + y'Q(x) + R(x)y = f(x)$$

where $P(x), Q(x), R(x), f(x)$ are continuous functions of x.

1.2.3 Initial Value Problem

An initial-value problem for the second order equation

$$y''P(x) + y'Q(x) + R(x)y = f(x)$$

consists of finding a solution y of the differential equation that also satisfies initial condition of the form

$$y(x_0) = y_0 \qquad\qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants and $P(x), Q(x), R(x), f(x)$ are continuous functions of x and $P(x) \neq 0$.

1.2.4 Operator

An operator is a function that takes a function as an argument instead of number. Examples of operators are:

$$L = \frac{d}{dx} \ ; \ L = \frac{\partial}{\partial x} \ ; \ \ L = \int dx \ ; \ L = \int_a^b dx$$

1.2.5 Adomian Polynomials ($\mathbf{A_n}$)

The Adomian Polynomials of a non-linear differential equation is obtained using the formula

$$A_n = \frac{1}{n!}\ \frac{d^n}{d\lambda^2}\left[N\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

1.3 Aims and Objectives

1.3.1 Aim

The aim of this project work is to adopt the method of Adomian decomposition to solve Linear and Non-linear Second Order Differential Equation of the form

$$y''P(x) + y'Q(x) + R(x)y = f(x)$$

where $P(x), Q(x), R(x), f(x)$ are continuous functions of x.

1.3.2 Objectives

The objectives of this study are to

1. Describe Adomian decomposition method
2. Use ADM to solve linear ordinary differential equation
3. determine the solution of second order ordinary differential equation by Adomian decomposition method

1.4 Significance of the Project

The importance or significance of this project is to apply Adomian Decomposition Method to solve linear and non-linear ordinary differential equation of second order will converges the problem to its exact solution.

1.5 Outline of the Project

The project is divided into five chapters. Chapter one consists of the introduction, definition of relevant terms, aims and objectives, significance of the project and project outline.

Chapter two consist of literature review, Chapter three consists of methodology; method of solution of ADM; Numerical examples on Adomian Polynomial; Numerical examples of linear ordinary differential equation.

Chapter four consist of methodology, Numerical examples on Non-linear ordinary differential equation. Chapter five consist of discussion of result; conclusion and recommendation for further study.

Chapter 2

Literature Review

2.1 Introduction

The study of differential equation began in 1675 when Leibniz wrote the equation

$$\int x dx = \frac{1}{2}x^2 \tag{2.1}$$

Leibniz inaugurated the differential sign $\left(\frac{dy}{dx}\right)$ and integral sign $\left(\int\right)$ in (1675) a hundred years before the period of initial discovery of general method of integrating ordinary differential equation ended. (Ince 1956).

According to Sasser (2005) in Orapine, Tiza and Amon(2016) the search for general methods of integrating began when Newton classified the first order differential equation into three classes:

$$\frac{dy}{dx} = f(x) \tag{2.2}$$

$$\frac{dy}{dx} = f(x, y) \tag{2.3}$$

$$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = u \tag{2.4}$$

The first two classes contain only ordinary derivative of one or more dependent variable, with respect to a single independent variable, and are known today as ordinary differential equation. The third class involves partial derivatives of one dependent variable and today its called partial differential equation.

In the 20^{th} century, Ordinary differential equation(ODE) is widely applied in many field and the numerical solution has made a great development.

Many of the problems in the field of engineering is expressed in terms of boundary value problem (BVP) which are boundary differential equation with boundary conditions and also in terms of initial value problem (IVP) which are ODE with initial condition of the unknown function at a given domain of solution. Hilderbrand (1974).

Billingham and King (2003) studied mathematical modelling and state the importance of ODE in modelling dynamics system. Saying it gives the conceptual skills to formulate, develop, solve, evaluate and validate such system. Many physical, chemical and biological system can be described using mathematical model. Once the model is formulated, we need differential equation in order to predict and quantify the features of the system modelled. Tiza

(2016).

Numerical techniques are used to solved mathematical models in engineering, many branch of physics, human physiology, applied mathematics etc. Some of the mathematical techniques or method used in solving modelling problems are Cubic Spline Method, Finite Difference Methods, Taylor's Series Method, Runge-Kuta Method, Shooting Method, Pertubation method, Adomian Decomposition Method, Euler Method etch. The main method to be considered is Adomian Decomposition method, but we shall briefly discuss some of the method listed above i.e Euler Method, Runge-Kutta Method and Taylor's Series Method.

2.2 Runge-Kutta Method

This method was devise by two German Mathematician Runge and Kutta around 1900.

The method is used to find numerical solution to a first order differential equation, the method uses four value of k at four point in one step. Consider the equation

$$y' = f(x, y) \tag{2.5}$$

and $y(x_0) = y_0$

The function f and (x_0, y_0) are given, now choose a step size of $h > 0$ and define

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.6}$$

$$x_{n+1} = x_{n+h} \tag{2.7}$$

$$\forall n = 1, 2, 3, \ldots \tag{2.8}$$

using

$$k_1 = f(x_n, y_n) \tag{2.9}$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \tag{2.10}$$

$$k_3 = f(x_n + \frac{h}{2}, y_n \frac{h}{2}k_2) \tag{2.11}$$

$$k4 = f(x_n + h, y + hk_3) \tag{2.12}$$

Each $k_i, i = 1, 2, 3, \ldots$ represent k^{th} order Runge-Kutta method, the forth order is the most stable and easy to implement.

2.3 Taylor's Series Method

Considering the one-dimensional initial value problem $y'(x) = f(x, y)$ and $y(x_0) = y_0$ where f is a function of two variables x and y and (x_0, y_0) is a known point on the solution curve(initial points). If the existence of all higher order partial derivatives is assumed for y at $x = x_0$ then by Taylor's series, the value of y at an neighbouring point $x + h$ is written as

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots \tag{2.13}$$

where $y'(x)$ represent the derivative of y with respect to x since at x_0, y_0 is unknown, $y'(x_0)$ is obtained by computing $f(x_0, y_0)$. Similarly higher derivatives of y at x_0 can also be computed by using the relation.

$$y' = f(x,y) \tag{2.14}$$

$$y'' = f_x + f_y y' \tag{2.15}$$

$$y''' = f_{xx} + 2f_{xy}y' + f_{yy}(y')^2 + f_y y'' \tag{2.16}$$

and so on. Then

$$y(x_0+h) = y(x_0) + hf + \frac{h^2}{2!}(f_x + f_x y') + \frac{h^3}{3!}(f_{xx} + 2f_{xy}y\prime + f_{yy}(y')^2 + f''_{yy}) \tag{2.17}$$

Hence, the value of y at any neighbouring point $x_0 + h$ can be obtained by summing the above infinite series.

However, in any practical computation, the summation has to be terminated after some finite number of terms. If the series has been terminated after the p^{th} derivative term then the approximated formula is called the Taylor Series approximation of y of order p.

2.4 Euler's Method

Euler's method assumed our solution is written in the form of a Taylor's series. i.e,

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots \tag{2.18}$$

This gives us a reasonably good approximation if we take plenty terms, and if the value of h is reasonably small.

For Euler's method we take the first two terms only i.e $y(x+h) \approx y(x)+h'(x)$ The last term is just h multiply $\frac{dy}{dx}$, so we can write Euler's method as follows

$$y(x+h) \approx y(x) + hf(x,y) \tag{2.19}$$

In general, Euler's formula is given as

$$y_{i+1} = y_i + hf(x_i, y_i) \tag{2.20}$$

where y_i is the current values, y_{i+1} is the next estimated values, h is the interval between steps and $f(x_i, y_i)$ is the value of the derivative at the current (x_i, y_i) point.

Chapter 3

METHODOLOGY

3.1 Introduction

The Adomian Decomposition Method consist of decomposing the unknown function $U(x, y)$ of any equation into a sum of an infinite number of component defined by the decomposition series

$$U(x, y) = \sum_{n=0}^{\infty} U_n(x, y) \tag{3.1}$$

where the component $U_n(x, y), n \geq 0$ are to be determined in a recursive manner. The ADM is concern with finding the components $U_0, U_1, U_2, \ldots$ individually.

The ADM consist of splitting the given equation into linear and non-linear parts, inverting the highest order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary condition and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the non-linear function in terms of special polynomials called Adomian polynomials and finding the successive terms of the series solution by recurrent relation using Adomian polynomials. The solution is found as an infinite series in which each term can be easily determined and that converges quickly towards an accurate (exact) solution.

3.2 Method of Solution of ADM

Consider a differential equation

$$f(U(t)) = g(t) \tag{3.2}$$

where f represents a general non-linear ordinary differential operator including both linear and non-linear terms. Thurs, the equation may be written as:

$$LU + NU + RU = g \tag{3.3}$$

where L is the linear operator, N represent the non-linear operator and R represent the remaining linear part. Solve for LU, we obtained

$$LU = g - NU - RU \tag{3.4}$$

Then we defined the inverse operator of L as L^{-1} assuming it exist, we get

$$L^{-1}LU = L^{-1}g - L^{-1}NU - L^{-1}Ru \tag{3.5}$$

The Adomian Decomposition Method represents the $U(x,t)$ as an infinite series of the form

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \tag{3.6}$$

or equivalently,

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots \tag{3.7}$$

Also, ADM defines the non-linear term NU by the Adomian polynomials, which can be decomposed by an infinite series of polynomials given by

$$NU = \sum_{n=0}^{\infty} A_n \tag{3.8}$$

where A_n are the Adomian polynomials. Substituting equations (3.6) and (3.8) into equation (3.5), we get

$$\sum_{n=0}^{\infty} U(x,t) = \phi_0 + L^{-1}g(x) + L^{-1}R\sum_{n=0}^{\infty} U_n - L^{-1}\sum_{n=0}^{\infty} A_n \tag{3.9}$$

$$\phi_0 = \begin{cases} U(0) & \text{if } L\frac{d}{dx} \\ U(0) + xU'(0) & \text{if } L\frac{d^2}{dx^2} \\ U(0) + xU'(0) + \frac{x^2}{2!}U''(0) & \text{if } L\frac{d^3}{dx^3} \\ \vdots & \\ U(0) + xU'(0) + \frac{x^2}{2!}U''(0) + \cdots + \frac{x^n}{n!}U^n(0), & \text{if } L\frac{d^{n+1}}{dx^{n+1}} \end{cases} \tag{3.10}$$

Therefore, the formal recurrence algorithm could be defined as

$$\begin{cases} U_0 = \phi_0 + L^{-1}g(x), \\ U_1 = -L^{-1}RU_0 - L^{-1}A_0 \\ U_2 = -L^{-1}RU_1 - L^{-1}A_1 \\ \vdots \\ U_{n+1} = -L^{-1}RU_n - L^{-1}A_n, \quad n \geq 0 \end{cases} \tag{3.11}$$

where A_n are Adomian polynomials generated for each non-linear term so that A_0 depends only on U_0, A_1 depends only on U_0 and U_1, A_2 depends only on U_0, U_1 and U_2 and etc.

3.3 Adomian Polynomials

The main part of ADM is calculating the Adomian Polynomials. The ADM decomposes the solution of U and the non-linear terms NU into series

$$U = \sum_{n=0}^{\infty} U_n \qquad N(U) = \sum_{n=0}^{\infty} A_n \tag{3.12}$$

where A_n are the Adomian Polynomials.

To compute A_n, we take $NU = f(U)$ as a non-linear function in U where $U = U(x)$ and we consider the Taylor Series expansion of $f(U)$ about the initial function U_0.

$$\begin{aligned} f(U) = f(U_0) + f'(U_0)(U - U_0) + \frac{1}{2!} f''(U_0)(U - U_0)^2 \\ + \frac{1}{3!} f'''(U_0)(U - U_0)^3 + \cdots \end{aligned} \tag{3.13}$$

But $U = U_0 + U_1 + U_2 + \cdots$. Then,

$$\begin{aligned} f(U) = f(U_0) + f'(U_0)(U_1 + U_2 + U_3 + \cdots) + \frac{1}{2!} f''(U_0)(U_1 + U_2 + U_3 + \cdots)^2 \\ + \frac{1}{3!} f'''(U_0)(U_1 + U_2 + U_3 + \cdots)^3 + \cdots \end{aligned} \tag{3.14}$$

by expanding all term we get

$$\begin{aligned} f(U) = f(U) + f'(U_0)(U_1) + f'(U_0)(U_2) + f'(U_0)(U_3) + \cdots \\ + \frac{1}{2!} f''(U_0)(U_1)^2 + \frac{2}{2!} f''(U_0)(U_1 U_2) + \frac{1}{2!} f''(U_0)(U_1 U_3) + \cdots \\ + \frac{1}{3!} f'''(U_0)(U_1)^3 + \frac{3}{3!} f'''(U_0)(U_1^2 U_2) + \frac{1}{3!} f'''(U_0)(U_1^2 U_3) + \cdots \end{aligned} \tag{3.15}$$

The Adomian Polynomials are constructed in a certain way so that the polynomial A_1 consists of all terms in the expansion of order 1, A_2 consist of all terms of order 2, and so on.

In general, A_n consist of all terms of order n. Therefore, we have

$$
\begin{aligned}
A_0 &= f(U_0) \\
A_1 &= U_1 f'(U_0) \\
A_2 &= U_2 f'(U_0) + \frac{1}{2!} f''(U_0) \\
A_3 &= U_3 f'(U_0) + \frac{2}{2!} U_1 U_2 f''(U_0) + \frac{1}{3!} U_1^3 f''(U_0) \\
A_4 &= U_4 f'(U_0) + \left[\frac{1}{2!} U_2^2 + U_1 U_3\right] f''(U_0) + \frac{1}{2!} U_1^2 U_2 f'''(U_0) + \frac{1}{4!} U_1^4 f''''(U_0) \\
&\ \vdots
\end{aligned} \tag{3.16}
$$

Hence, A_n was defined via the general formula

$$A_n(U_0, U_1, \ldots, U_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{n=0}^{\infty} U_n \lambda^n \right) \right]_{\lambda=0} \quad n = 0, 1, 2, 3, \ldots (3.17)$$

To fin the A_n's by Adomian general formula, these polynomial will be computed as follows:

$$
\begin{aligned}
A_0 &= N(U_0) \\
A_1 &= \frac{d}{d\lambda} N(U_0 + U_1 \lambda)\Big|_{\lambda=0} = N(U_0)U1 \\
A_2 &= N'(U_0)U_2 + \frac{1}{2!} N''(U_0) U_1^2 - \frac{1}{2!} \frac{d^2}{dx^2} N(U_0 + \lambda U_1 + \lambda^2 U_2)\Big|_{\lambda=0} \\
A_3 &= N'(U_0)(U_3) + \frac{2}{2!} N''(U_0) U_1 U_2 + \frac{1}{3!} N'''(U_0) U_1^3 \\
&= \frac{1}{3!} \frac{d^2}{dx^2} N(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3)\Big|_{\lambda=0} \\
&\ \vdots
\end{aligned}
$$

Example 3.1

The Adomian polynomial of $f(U) = U^5$

Chapter 4

Chapter 5

SUMMARY AND CONCLUSION

5.1 Summary

5.2 Conclusion

REFERENCES