

APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$$\begin{aligned}\mathcal{L}[f(t)] &= F(S) = \int_0^{-\infty} e^{-st} f(t) dt \\ \mathcal{L}[f^n(t)] &= s^n y - s^{n-1} y(0) - s^{n-2} y'(0) \dots \dots - s f^{n-2}(0) \dots \dots - y^{n-1}(0) \\ \mathcal{L}[u_x(x, t)] &= \int_0^{\infty} e^{-st} u_x(x, t) dt \equiv u(x, s) \\ \mathcal{L}[u_x(x, t)] &= u_x(x, s) \\ \mathcal{L}[u_x(x, t)] &= u_x(x, s) \\ \mathcal{L}[u_t(x, t)] &= s u(x, s) - u(x, 0) \\ \mathcal{L}[u_t(x, t)] &= s^2 u(x, s) - s u(x, 0) - u_t(x, 0) \\ F(s) &= \mathcal{L}[f(t)] \text{ then,} \\ \mathcal{L}[u(t-a) \cdot g(t-a)] &= e^{-as} G(s)\end{aligned}$$

P.D.E of Order 2

Examples:

$$(1) \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial u}{\partial x}(x, t), 0 < x < 2, t > 0 \quad u(0, t) = 0, u(2, t) = 0, u(x, 0) = 3 \sin(2\pi x)$$

Solution:

$$u_{xx}(x, t) = u_t(x, t)$$

taking the Laplace transform

$$\mathcal{L}[u_{xx}(x, t)] = \mathcal{L}[u_t(x, t)]$$

$$u_{xx}(x, s) = s u(x, s) - u(x, 0)$$

Using the condition, $u(x, 0) = 3 \sin(2\pi x)$

$$s u(x, s) - 3 \sin(2\pi x) = u_{xx}(x, s)$$

$$\implies u_{xx}(x, s) - s u(x, s) = -3 \sin(2\pi x)$$

$$\frac{d^2 u}{dx^2} - s u = -3 \sin(2\pi x)$$

Solving the Homogenous Problem

$$\frac{d^2 u}{dx^2} - su = 0$$

The characteristic equation is given by

$$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$$

The homogenous solution is:

$$u_A(x, s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

$$\text{i.e } \frac{d^2 u}{dx^2} - su = -3 \sin(2\pi x) \quad \text{--- (*)}$$

Let

$$U = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x) \quad \text{--- (a)}$$

$$U' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x) \quad \text{--- (b)}$$

$$U'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) \quad \text{--- (c)}$$

Substituting (a) and (c) in equation (*)

$$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - s\Delta_1 \sin(2\pi x) - s\Delta_2 \cos(2\pi x) = -3 \sin(2\pi x)$$

$$-4\pi^2\Delta_1 - s\Delta_1 = -3 \quad \text{Also, } 4\pi^2\Delta_2 - s\Delta_2 = 0$$

$$-\Delta_1 [4\pi^2 + s] = -3 \quad \Delta_2 [s + 4\pi^2] = 0$$

$$\Delta_1 = \frac{3}{s + 4\pi^2} \quad \Delta_2 = 0$$

The particular solution is:

$$u_p(x, s) = \frac{3}{s + 4\pi^2} \sin(2\pi x)$$

The general solution is given by:

$$u(x, s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x} + \frac{3 \sin(2\pi x)}{s + 4\pi^2}$$

Applying the boundary conditions $u(0, t) = 0$, $u(2, t) = 0$
 $u(0, s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$$u(2, s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0 \quad [\text{But } A_1 = -A_2]$$

$$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$$

$$A_2 [e^{-2\sqrt{s}} - e^{2\sqrt{s}}] = 0 \Rightarrow A_2 = 0, A_1 = 0$$

$$u(x, s) = \frac{3 \sin(2\pi x)}{s + 4\pi^2}$$

Substituting (a) and (c) in equation (*)

$$-c^2 \pi^2 n_1 \sin(\pi x) - c^2 \pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = \frac{-\sin(\pi x)}{s}$$

$$\Rightarrow -c^2 \pi^2 n_1 - s^2 n_1 = \frac{-1}{s} \Rightarrow -n_1 [s^2 + c^2 \pi^2] = \frac{-1}{s}$$

$$\Rightarrow n_1 = \frac{1}{s [s^2 + c^2 \pi^2]}$$

$$\text{Also, } -c^2 \pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$$

$$n_2 [s^2 + c^2 \pi^2] = 0$$

$$\Rightarrow n_2 = 0$$

Substituting 'n₁' and 'n₂' in (a)

$$u_p(x, s) = \frac{\sin(\pi x)}{s}$$

The general solution is given as:

$$u_g(x, s) = u_u(x, s) + u_p(x, s)$$

$$u_g(x, s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{\sin(\pi x)}{s [s^2 + c^2 \pi^2]} \text{-----} (**)$$

Apply the boundary conditions; $u(0, t) = 0$ and $u(1, t) = 0$

$$u(0, s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

$$u(1, s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$$

Substituting 'A₁' and 'A₂' in equation (**)

$$u(x, s) = \frac{\sin(\pi x)}{s [s^2 + c^2 \pi^2]}$$

Applying Inverse Laplace Transform

$$\mathcal{L}^{-1} [u(x, s)] = \sin(\pi x) \mathcal{L}^{-1} \left[\frac{1}{s [s^2 + c^2 \pi^2]} \right]$$

Resolving $\frac{1}{s [s^2 + c^2 \pi^2]}$ into partial fractions

$$\frac{1}{s [s^2 + c^2 \pi^2]} = \frac{A}{s} + \frac{Bs + D}{s^2 + c^2 \pi^2} = \frac{A [s^2 + c^2 \pi^2] + [Bs + D] s}{s [s^2 + c^2 \pi^2]}$$

$$1 = A [s^2 + c^2 \pi^2] + Bs^2 + Ds$$

Taking the Inverse Laplace equation

$$\mathcal{L}^{-1} [u(x, s)] = \mathcal{L}^{-1} \left[\frac{3 \sin(2\pi x)}{s + 4\pi^2} \right]$$

$$\mathbf{u}(\mathbf{x}, \mathbf{t}) = 3\mathbf{e}^{-4\pi^2 \mathbf{t}} \sin(2\pi \mathbf{x})$$

$$(2) \quad \frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t); 0 < x < 1, t > 0, \\ u(x, 0) = 0, u_t(x, 0) = 0, u(0, t) = 0, u(1, t) = 0$$

Solution:

$$u_{tt}(x, t) = c^2 u_{xx}(x, t) + \sin(\pi x)$$

Taking the Laplace transform

$$\mathcal{L} [u_{tt}(x, t)] = c^2 \mathcal{L} [u_{xx}(x, t)] + \mathcal{L} [\sin(\pi x)]$$

$$s^2 u(x, s) - su(x, 0) - u_t(x, 0) = c^2 u_{xx}(x, s) + \frac{\sin(\pi x)}{s}$$

Applying the initial conditions; $u(x, 0) = 0$ and $u_t(x, 0) = 0$

$$s^2 u(x, s) - c^2 u_{xx}(x, s) = \frac{\sin(\pi x)}{s}$$

Re-arranging

$$c^2 u_{xx}(x, s) - s^2 u(x, s) = -\frac{\sin(\pi x)}{s}$$

Solving the Homogenous problem

$$c^2 \frac{d^2 u}{dx^2}(x, s) - s^2 u(x, s) = 0$$

The Auxillary equation is given by:

$$c^2 m^2 - s^2 = 0 \implies m^2 - \left(\frac{s}{c}\right)^2 = 0 \implies m = \pm \frac{s}{c}$$

The homogenous solution is: $u(x, s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}}$

Solving the non-homogenous problem by method of Undetermined Coefficient:

$$\text{i.e } c^2 \frac{d^2 u}{dx^2}(x, s) - s^2 u(x, s) = -\frac{\sin(\pi x)}{s} \text{ --- --- --- --- --- } (*)$$

Let:

$$u(x, s) = n_1 \sin(\pi x) + n_2 \cos(\pi x) \text{ --- --- --- --- --- } (a)$$

$$u_x(x, s) = \pi n_1 \cos(\pi x) - \pi n_2 \sin(\pi x) \text{ --- --- --- --- --- } (b)$$

$$u_{xx}(x, s) = -\pi^2 n_1 \sin(\pi x) - \pi^2 n_2 \cos(\pi x) \text{ --- --- --- --- --- } (c)$$

Comparing or Equating the Coefficients of boths sides:

$$s : D = 0$$

$$s^2 : A + B = 0 \implies A = -B$$

$$\text{constant} : 1 = Ac^2\pi^2 \implies A = \frac{1}{c^2\pi^2} \implies B = \frac{-1}{c^2\pi^2}$$

$$\therefore \frac{1}{s[s^2 + c^2\pi^2]} = \frac{1}{sc^2\pi^2} - \frac{s}{(c^2\pi^2)(s^2 + c^2\pi^2)} = \frac{1}{c^2\pi^2} \left[\frac{1}{s} - \frac{s}{s^2 + c^2\pi^2} \right]$$

$$u(x, t) = \frac{\sin(\pi x)}{c^2\pi^2} \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{s}{s^2 + (c\pi)^2} \right]$$

$$\mathbf{u}(\mathbf{x}, \mathbf{t}) = \frac{\sin(\pi \mathbf{x})}{\mathbf{c}^2 \pi^2} [1 - \cos(\mathbf{c} \pi \mathbf{t})]$$

NEXT:

Solution of non-linear PDE, by the combined Laplace transform and the new Modified Variational Iteration Method.

SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATION BY THE COMBINED LAPLACE TRANSFORM METHOD AND THE NEW MODIFIED VARIATIONAL ITERATION METHOD

Presenting a reliable combined Laplace transform and the new modified variational Iteration method to solve some non-linear Partial Differential Equations. This method is more efficient and easy to handle non-linear PDEs.

Recall,

$$\mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0)$$

$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

Where $F(x,s)$ is the Laplace transform of (x,t) [x is considered as a dummy variable and t , a parameter].

Illustrating the basic concept of **He's Variational Iteration Method**, we consider the following general differential equations:

$$Lu(x,t) + Nu(x,t) = g(x,t) \text{ --- (i)}$$

$$\text{with the initial condition, } u(x,0) = G(x) \text{ --- (ii)}$$

Where L is a linear operator of the first Order, N is a non-linear operator and $g(x,t)$ is non-homogenous term. According to Variational Iteration Method, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda [LU(x,s) + N(x,s) - g(x,s)] ds \text{ --- (iii)}$$

where λ is a Lagrange Multiplier ($\lambda = -1$), the subscripts ' n ' denotes the n th approximation, \bar{U}_n is considered as a restricted variation, i.e $\partial \bar{U}_n = 0$.

Equation (iii) is called a **Correction Functional**

Obtaining the Lagrange Multiplier ' λ ' by using Integration by Part of Equation (i), but the Lagrange Multiplier is of the form $\lambda = \lambda(x,t)$.

Then, Laplace transform of equation (iii), then the correction functional will be in the form:

$$\mathcal{L}[u_{n+1}(x,t)] = \mathcal{L}[u_n(x,t)] + \mathcal{L}\left[\int_0^t \lambda(x,t) (Lu_n(x,s) + Nu_n(x,s) - g(x,s)) ds\right], n \geq 0 \text{ --- (iv)}$$

Therefore,

$$\mathcal{L}[u_{n+1}(x,t)] = \mathcal{L}[u_n(x,t)] + \lambda [Lu_n(x,t) + Nu_n(x,t) - g(x,t)] \text{ --- (v)}$$

To find the optimal value of $\lambda(x, t)$, we first take the Variation with respect to $u_n(x, t)$ and in such a case, the integration is basically the single convolution to t , and hence, Laplace transform is appropriate to use.

thus;

$$\frac{\partial}{\partial u_n} \mathcal{L}[u_n(x, t)] + \mathcal{L}[\lambda(x, t)] \frac{\partial}{\partial u_n} \mathcal{L}[Lu_n(x, t) + Nu_n(x, t) - g(x, t)] - - - (vi)$$

Equation (vi) becomes,

$$\mathcal{L}[\partial u_{n+1}(x, t)] = \mathcal{L}[\partial u_n(x, t)] + \mathcal{L}[\lambda(x, t)] \partial \mathcal{L}[Lu_n(x, t)] - - - - - (vii)$$

$$\{i.e \partial N\bar{u}_n(x, t) = 0 \text{ and } \partial g(x, t) = 0\}$$

We assume that L is a linear first-order Partial Differential Operator in this chapter given by $\frac{\partial}{\partial t}$ then, equation (vii) can be written in the form

$$\mathcal{L}[\partial u_{n+1}(x, t)] = \mathcal{L}[\partial u_n(x, t)] + \mathcal{L}[\lambda(x, t)] [s\mathcal{L}u_n(x, t)]$$

the extreme condition of $u_{n+1}(x, t)$ requires that $\partial u_{n+1}(x, t) = 0$

$$\implies 0 = \mathcal{L}[\partial u_n(x, t)] [1 + s\mathcal{L}[\lambda(x, t)]]$$

$$\implies 1 + s\mathcal{L}[\lambda(x, t)] = 0$$

$$\implies s\mathcal{L}[\lambda(x, t)] = -1$$

$$\implies \mathcal{L}[\lambda(x, t)] = \frac{-1}{s}$$

Taking the Laplace Inverse of both sides

$$\lambda(x, t) = \mathcal{L}^{-1} \left[\frac{-1}{s} \right]$$

$$\lambda(x, t) = -1$$

This implies $\lambda = -1$

Substituting ($\lambda = -1$) in equation (iii)

$$u_{n+1} = u_n - \int_0^t [Lu_n(x, s) + N\bar{u}_n(x, s) - g(x, s)] ds - - - - - (viii)$$

The successive approximation ' u_{n+1} ' of the solution ' u ' will be readily obtained by using the determined Lagrange Multiplier and any selective function u_n consequently, the solution is given by:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$$

Also, from equation (i)

$$i.e \quad LU(x, t) + Nu(x, t) = g(x, t) \quad \text{---} \quad (*)$$

taking the Laplace transform of both sides, we have

$$\mathcal{L}[LU(x, t)] + \mathcal{L}[Nu(x, t)] = \mathcal{L}[g(x, t)]$$

Using the differential property of Laplace transform and initial condition (ii), we have:

$$s\mathcal{L}[u_{x,t}] - h(x) = \mathcal{L}[g(x, t)] - \mathcal{L}[Nu(x, t)] \quad \text{---} \quad (**)$$

Applying the inverse Laplace transform on both side of the equation (**), we find:

$$u(x, t) = G(x, t) - \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L}[(Nu(x, t))] \right] \quad \text{---} \quad (***)$$

Where $G(x, t)$ represents the terms arising from the source term and the prescribed initial condition $\left[i.e \quad G(x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} [\mathcal{L}[g(x, t)] + h(x)] \right\} \right]$

Taking the first Partial derivatives with respect to 't' of equation (***) to obtain:

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial}{\partial t} G(x, t) - \frac{\partial}{\partial t} \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[Nu(x, t)] \right\} \quad \text{---} \quad (****)$$

By the correction functional of the Variation iteration method:

$$u_{n+1}(x, t) = u_n - \int_0^t \left\{ (u_n)_1(x, s) - \frac{\partial}{\partial s} G(x, s) + \frac{\partial}{\partial s} \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[Nu(x, s)] \right\} \right\} ds$$

or

$$u_{n+1}(x, t) = G(x, t) - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[Nu_n(x, t)] \right\} \quad \text{---} \quad (****)$$

Equation (****) is the new modified correction functional of Laplace transform and the Variational Iteration Method, and the solution is given by:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$$

Next: Solving some non-linear PDE, by using the new modified variational iteration Laplace transformation method:

Examples:

$$(1) \frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x} \right)^2 + u \frac{\partial^2 u}{\partial x^2} ; u(x, 0) = x^2$$

Solution:

Given $u_t = u_x^2 + uu_{xx}$; $u(x, 0) = x^2$

Taking Laplace transform, subject to the initial condition, we have:

$$\mathcal{L}[u_t] = \mathcal{L}[u_x^2] + \mathcal{L}[uu_{xx}]$$

$$s\mathcal{L}[u(x, t)] - u(x, 0) = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$s\mathcal{L}[u(x, t)] - x^2 = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$\mathcal{L}[u(x, t)] = \frac{x^2}{s} + \frac{1}{s}\mathcal{L}[u_x^2 + uu_{xx}]$$

Taking the Inverse Laplace transform to obtain:

$$u(x, t) = \mathcal{L}^{-1}\left[\frac{x^2}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x^2 + uu_{xx}]\right]$$

$$u(x, t) = x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[u_x^2 + uu_{xx}]\right\}$$

The new correction functional is given as:

$$u_{n+1}(x, t) = x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[(u_n)_x + u_n(u_n)_{xx}]\right\} \quad n \geq 0$$

The solution in series form is given by:

$$u_0(x, t) = x^2 \quad [\text{ or } u_0(x, t) = u(x, 0)]$$

$$u_1(x, t) = x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[(u_0)_x^2 + (u_0)(u_0)_{xx}]\right\}$$

$$u_1(x, t) = x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[4x^2 + 2x^2]\right\}$$

$$= x^2 + \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\frac{4x^2}{s} + \frac{2x^2}{s}\right]\right\}$$

$$= x^2 + \mathcal{L}^{-1}\left\{\frac{6x^2}{s^2}\right\}$$

$$u_1(x, t) = x^2 + 6x^2t$$

$$\begin{aligned}
u_2(x, t) &= x^2 + \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[(u_1)_x^2 + (u_1) (u_1)_{xx} \right] \right\} \\
u_2(x, t) &= x^2 + \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[(2x + 12xt)^2 + (x^2 + 6x^2t)(2 + 12t) \right] \right\} \\
&= x^2 + \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[6x^2 + 72x^2t + 216x^2t^2 \right] \right\} \\
&= x^2 + \mathcal{L}^{-1} \left\{ \frac{1}{s} \left[\frac{6x^2}{s} + \frac{72x^2}{s^2} + \frac{216x^2 \cdot 2}{s^3} \right] \right\} \\
&= x^2 + \mathcal{L}^{-1} \left[\frac{6x^2}{s^2} + \frac{72x^2}{s^3} + \frac{432x^2}{s^4} \right] \\
u_1(x, t) &= x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3
\end{aligned}$$

The series solution is given by:

$$\begin{aligned}
u(x, t) &= x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3 + - - - - - \\
u(x, t) &= x^2 [1 + 6t + 36t^2 + 72t^3 + - - - - -] \\
u(x, t) &= \frac{x^2}{1 - 6t}
\end{aligned}$$

$$(2) \quad u_{tt}(x, t) - u_{xx}(x, t) + u^2(x, t) = x^2t^2, \quad u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = x$$

$$\mathcal{L} [u_{tt}] = \mathcal{L} [x^2t^2] + \mathcal{L} [u_{xx} - u^2]$$

$$s^2 \mathcal{L} [u(x, t)] - su(x, 0) - \frac{\partial u(x, 0)}{\partial t} = \frac{2x^2}{s^3} + \mathcal{L} [u_{xx} - u^2]$$

$$\begin{aligned}
s^2 \mathcal{L} [u(x, t)] &= \frac{2x^2}{s^3} + x + \mathcal{L} [u_{xx} - u^2] \\
\mathcal{L} [u(x, t)] &= \frac{2x^2}{s^5} + \frac{x}{s^2} + \frac{1}{s^2} \mathcal{L} [u_{xx} - u^2] \\
u(x, t) &= \frac{x^2 t^4}{12} + xt + \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} [u_{xx} - u^2] \right\} \\
u_{n+1}(x, t) &= xt + \frac{x^2 t^4}{12} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} [(u_n)_{xx} - (u_n)^2] \right\} \\
u_1(x, t) &= xt + \frac{x^2 t^4}{12} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} [-x^2 t^2] \right\} \\
u_1(x, t) &= xt + \frac{x^2 t^4}{12} - \frac{x^2 t^4}{12} = xt \\
&= xt \\
u(x, t) &= xt
\end{aligned}$$