APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Laplace Method:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[y^n(t)] = s^n Y - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - s f^{n-2}(0) \cdots - y^{n-1}(0)$$

$$L[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\,dt \equiv U(x,s)$$

$$L[u_x(x,t)] = U_x(x,s)$$

$$L[u_t(x,t)] = sU(x,s) - U(x,0)$$

$$L[u_{tt}(x,t)] = s^2 U(x,s) - sU(x,0) - u_t(x,0)$$

If $F(s) = L\{f(t)\}$ then,

$$L\{u(t-a)\cdot g(t-a)\} = e^{-as} G(s).$$

Linear P.D.E Of Order 2

Examples: (i) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t)$, $0<x<2,\ t>0$

$u(0,t)=0,\ u(2,t)=0;\ u(x,0) = 3\sin(2\pi x)$.

Solution.

$u_{xx}(x,t) = u_t(x,t)$

taking the Laplace transform

$L[u_{xx}(x,t)] = L[u_t(x,t)]$

$U_{xx}(x,s) = sU(x,s) - u(x,0)$

Using the condition, $u(x,0) = 3\sin(2\pi x)$ we have;

$sU(x,s) - 3\sin(2\pi x) = U_{xx}(x,s)$

$\Rightarrow U_{xx}(x,s) - sU(x,s) = -3\sin(2\pi x)$

$$\frac{d^2u}{dx^2} - su = -3\sin(2\pi x)$$

Solving the Homogenous Problem

$$\frac{d^2u}{dx^2} - su = 0$$

The Characteristic equation is given by

$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$

The homogeneous solution is:

$$U_h(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

i.e $\frac{d^2u}{dx^2} - su = -3\sin(2\pi x)$ - - - - (*)

Let $u = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x)$ - - - - (a)

$u' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x)$ - - - (b)

$u'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x)$ - - - (c)

Substituting (a) and (c) in equation (*)

$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - s\Delta_1 \sin(2\pi x) - s\Delta_2 \cos(2\pi x) = -3\sin(2\pi x)$

$-4\pi^2\Delta_1 - s\Delta_1 = -3$ Also, $-4\pi^2\Delta_2 - s\Delta_2 = 0$

$-\Delta_1[4\pi^2 + s] = -3$ $\qquad \Delta_2[s + 4\pi^2] = 0$

$\Delta_1 = \frac{3}{s+4\pi^2}$ $\qquad \Delta_2 = 0$

∴ The particular solution is:

$$U_p(x,s) = \frac{3}{s+4\pi^2}\sin(2\pi x)$$

The general solution is given by: $U(x,s) = U_h(x,s) + U_p(x,s)$

$$U(x,s) = A_1 e^{\sqrt{s}x} + A_2 e^{-\sqrt{s}x} + \frac{3\sin(2\pi x)}{s+4\pi^2}$$

Applying the boundary Conditions

$u(0,t) = 0, \; u(2,t) = 0$

$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$u(2,s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$ [But $A_1 = -A_2$]

$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$

$A_2[e^{-2\sqrt{s}} - e^{2\sqrt{s}}] = 0 \Rightarrow A_2 = 0$

$\Rightarrow A_1 = 0$

$$U(x,s) = \frac{3\sin(2\pi x)}{s+4\pi^2}$$

Substituting (a) and (c) in equation (*)

$-c^2\pi^2 n_1 \sin(\pi x) - c^2\pi^2 n_2 \cos(\pi x) - s^2 n_1 \sin(\pi x) - s^2 n_2 \cos(\pi x) = -\frac{\sin \pi x}{s}$

$\Rightarrow -c^2\pi^2 n_1 - s^2 n_1 = -\frac{1}{s} \Rightarrow -n_1[s^2 + c^2\pi^2] = -\frac{1}{s}$

$n_1 = \frac{1}{s[s^2+c^2\pi^2]}$

Also, $-c^2\pi^2 n_2 \cos(\pi x) - s^2 n_2 \cos(\pi x) = 0$

$n_2[s^2+c^2\pi^2] = 0 \Rightarrow n_2 = 0$

Substituting 'n_1' and 'n_2' in (e)

$U_p(x,s) = \frac{\sin(\pi x)}{s[s^2+c^2\pi^2]}$

The general solution is given as:

$U_g(x,s) = U_c(x,s) + U_p(x,s)$

$U_g(x,s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}} + \frac{\sin(\pi x)}{s[s^2+c^2\pi^2]}$ ——— (**)

Applying the boundary conditions

$u(0,t) = 0$ and $u(l,t) = 0$

$U(0,s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$U(l,s) = A_1 e^{\frac{s}{c}} + A_2 e^{-\frac{s}{c}} = 0 \Rightarrow A_2 = 0 \Rightarrow A_1 = 0$

Substituting 'A_1' and 'A_2' in eqn (**)

$U(x,s) = \frac{\sin(\pi x)}{s[s^2+c^2\pi^2]}$

Applying Inverse Laplace transform

$L^{-1}[U(x,s)] = \sin(\pi x)\, L^{-1}\left[\frac{1}{s[s^2+c^2\pi^2]}\right]$

Resolving $\frac{1}{s[s^2+c^2\pi^2]}$ into partial fractions

$\frac{1}{s[s^2+c^2\pi^2]} = \frac{A}{s} + \frac{Bs+D}{s^2+c^2\pi^2} = \frac{A[s^2+c^2\pi^2] + [Bs+D]s}{s[s^2+c^2\pi^2]}$

$1 = A[s^2+c^2\pi^2] + Bs^2 + Ds$

Taking the Inverse Laplace operation

$$L^{-1}[u(x,s)] = L^{-1}\left[\frac{3\sin(2\pi x)}{s+4\pi^2}\right]$$

$$u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x)$$

[2] $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) + \sin(\pi x)$; $0<x<1, t>0$

$u(x,0)=0$, $u_t(x,0)=0$

$u(0,t)=0$, $u(1,t)=0$

Solution

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) + \sin(\pi x)$$

Taking the Laplace transform

$$L[u_{tt}(x,t)] = c^2 L[u_{xx}(x,t)] + L[\sin(\pi x)]$$

$$s^2 U(x,s) - s u(x,0) - u_t(x,0) = c^2 U_{xx}(x,s) + \frac{\sin(\pi x)}{s}$$

Applying the initial Conditions ; $u(x,0)=0$ & $u_t(x,0)=0$

$$s^2 U(x,s) - c^2 U_{xx}(x,s) = \frac{\sin(\pi x)}{s}$$

Re-arranging

$$c^2 U_{xx}(x,s) - s^2 U(x,s) = -\frac{\sin(\pi x)}{s}$$

Solving the Homogenous problem

$$c^2 \frac{d^2 u}{dx^2}(x,s) - s^2 u(x,s) = 0$$

The Auxiliary equation is given by :

$$c^2 m^2 - s^2 = 0 \Rightarrow m^2 - \left(\frac{s}{c}\right)^2 = 0 \Rightarrow m = \pm\frac{s}{c}$$

The homogenous solution is : $U_h(x,s) = A_1 e^{\frac{sx}{c}} + A_2 e^{-\frac{sx}{c}}$

Solving the non-homogenous problem by Method of Undetermined Coefficient

i.e $c^2 \frac{d^2 u}{dx^2}(x,s) - s^2 u(x,s) = -\frac{\sin(\pi x)}{s}$ ---- (*)

Let $U(x,s) = n_1 \sin(\pi x) + n_2 \cos(\pi x)$ ---- (a)

$U_x(x,s) = \pi n_1 \cos(\pi x) - \pi n_2 \sin(\pi x)$ ---- (b)

$U_{xx}(x,s) = -\pi^2 n_1 \sin(\pi x) - \pi^2 n_2 \cos(\pi x)$ ---- (c)

Comparing or equating the Coefficients of both sides

s: $D = 0$

s^2: $A + B = 0 \Rightarrow A = -B$

Constant: $1 = A c^2\pi^2 \Rightarrow A = \frac{1}{c^2\pi^2} \Rightarrow B = \frac{-1}{c^2\pi^2}$

$$\therefore \frac{1}{s[s^2+c^2\pi^2]} = \frac{1}{s\, c^2\pi^2} - \frac{s}{(c^2\pi^2)(s^2+c^2\pi^2)}$$

$$= \frac{1}{c^2\pi^2}\left[\frac{1}{s} - \frac{s}{s^2+c^2\pi^2}\right]$$

$$U(x,t) = \frac{\sin(\pi x)}{c^2\pi^2} L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+(c\pi)^2}\right]$$

$$U(x,t) = \frac{\sin(\pi x)}{c^2\pi^2}\left[1 - \cos(c\pi t)\right]$$

<u>NEXT</u>: Solution of non-linear PDEs by the Combined Laplace transform and the new Modified Variational Iteration Method.

the

SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE COMBINED LAPLACE TRANSFORM METHOD AND THE NEW MODIFIED VARIATIONAL ITERATION METHOD

Presenting a reliable combined Laplace transform and the new modified Variational Iteration method to solve some non-linear Partial Differential Equations. This method is more efficient and easy to handle non-linear PDEs.

Recall, $\mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0)$

$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

Where $f(x,s)$ is the Laplace transform of (x,t) [x is considered as a dummy variable and t, a parameter].

Illustrating the basic concept of He's Variational Iteration Method, we consider the following general differential equations:

$$Lu(x,t) + Nu(x,t) = g(x,t) \quad \text{--- (i)}$$

with the initial condition, $u(x,0) = h(x)$ --- (ii)

Where L is a linear operator of the first Order, N is a non-linear operator and $g(x,t)$ is non-homogenous term. According to Variational Iteration Method, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda\left[Lu(x,s) + N\tilde{u}(x,s) - g(x,s)\right]ds \quad n \geq 0 \quad \text{--- (iii)}$$

where λ is a Lagrange Multiplier ($\lambda = -1$), the subscript 'n' denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e $\delta\tilde{u}_n = 0$.

Equation (iii) is called a Correction Functional

Obtaining the Lagrange Multiplier 'λ' by using Integration by part of Equation (i), but the Lagrange Multiplier is of the form $\lambda = \lambda(x,t)$

then ⇒ taking Laplace transform of Equation (iii), then the correction functional will be in the form:

$$\mathcal{L}[u_{n+1}(x,t)] = \mathcal{L}[u_n(x,t)] + \mathcal{L}\left[\int_0^t \lambda(x,t)\left(Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)\right)ds\right], \quad n \geq 0 \quad \text{--- (iv)}$$

Therefore,

$$\mathcal{L}[u_{n+1}(x,t)] = \mathcal{L}[u_n(x,t)] + \lambda(x,t)\left[Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right] \quad \text{--- (v)}$$

To find the optimal Value of $\lambda(x,t)$, we first take the Variation with respect to $u_n(x,t)$

and in such a case, the integration is basically the single convolution with respect to t, and hence, Laplace transform is appropriate to use. — where $*$ is a single convolution w.r.t t.

$$= \mathcal{L}[u_n(x,t)] + \mathcal{L}[\lambda(x,t)]\,\mathcal{L}\left[Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right]$$

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$$\mathcal{L}\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

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$$U_{n+1} = U_n + \int_0^t \lambda\left[LU(x,s) + N\bar{U}(x,s) - g(x,s)\right]ds \quad n \geq 0 \quad -- (iii)$$

where λ is a Lagrange Multiplier $(\lambda = -1)$, the subscript 'n' denotes the nth approximation, $\bar{U}_n$ is considered as a restricted variation, i.e $\delta \bar{U}_n = 0$.

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$$\mathcal{L}\left[u_{n+1}(x,t)\right] = \mathcal{L}\left[u_n(x,t)\right] + \mathcal{L}\left[\int_0^t \lambda(x,t)\left(Lu_n(x,s) + N\bar{u}_n(x,s) - g(x,s)\right)ds\right], \quad n \geq 0 \quad \text{(iv)}$$

therefore,

$$\mathcal{L}\left[u_{n+1}(x,t)\right] = \mathcal{L}\left[u_n(x,t)\right] + \lambda(x,t) * \left[Lu_n(x,t) + N\bar{u}_n(x,t) - g(x,t)\right] \quad \text{(v)}$$

To find the optimal value of $\lambda(x,t)$, we first take the variation with respect to $u_n(x,t)$

... the single convolution with respect

and in such a case, the integration is basically the single convolution to t, and hence, Laplace transform is appropriate to use.

where $*$ is a single convolution wrt t.

$$\doteq \mathcal{L}\left[u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\mathcal{L}\left[L u_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right]$$

Thus; $\frac{\delta}{\delta u_n} \mathcal{L}\left[u_{n+1}(x,t)\right] = \frac{\delta}{\delta u_n}\mathcal{L}\left[u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\frac{\delta}{\delta u_n}\mathcal{L}\left[Lu_n(x,t) + N\tilde{u}_n(x,t) - g(x,t)\right]$ ---- (vi)

Equation (vi) becomes,

$$\mathcal{L}\left[\delta u_{n+1}(x,t)\right] = \mathcal{L}\left[\delta u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\delta\mathcal{L}\left[Lu_n(x,t)\right] \quad \text{---- (vii)}$$

$\{$ i.e $\delta N\tilde{u}_n(x,t) = 0$ and $\delta g(x,t) = 0 \}$

We assume that L is a linear first-Order Partial Differential Operator in this chapter given by $\frac{\partial}{\partial t}$ then, equation (vii) can be written in the form

$$\mathcal{L}\left[\delta u_{n+1}(x,t)\right] = \mathcal{L}\left[\delta u_n(x,t)\right] + \mathcal{L}\left[\lambda(x,t)\right]\left[s\mathcal{L}\left[\delta u_n(x,t)\right]\right]$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t) = 0$

$$\Rightarrow \quad 0 = \mathcal{L}\left[\delta u_n(x,t)\right]\left[1 + s\mathcal{L}\left[\lambda(x,t)\right]\right]$$

$$\Rightarrow \quad 1 + s\mathcal{L}\left[\lambda(x,t)\right] = 0$$

$$s\mathcal{L}\left[\lambda(x,t)\right] = -1$$

$$\mathcal{L}\left[\lambda(x,t)\right] = \frac{-1}{s}$$

Taking the Laplace Inverse of both sides

$$\lambda(x,t) = \mathcal{L}^{-1}\left[\frac{-1}{s}\right]$$

$$\lambda(x,t) = -1$$

this implies $\lambda = -1$

Substituting $(\lambda = -1)$ in equation (iii)

$$u_{n+1} = u_n - \int_0^t \left[Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)\right]ds \quad \text{---- (viii)}$$

The successive approximation 'u_{n+1}' of the solution 'u' will be readily obtained by using the determined Lagrange multiplier and any selective function u_0 consequently, the solution is given by:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

Also, from equation (i)

i.e $Lu(x,t) + Nu(x,t) = g(x,t)$ ---- (*)

Taking the Laplace transform of both sides, we have

$$\mathcal{L}\left[Lu(x,t)\right] + \mathcal{L}\left[Nu(x,t)\right] = \mathcal{L}\left[g(x,t)\right]$$

[2]

Using the differentiation property of Laplace transform and initial condition (ii), we have:

$$s\mathcal{L}[u(x,t)] - h(x) = \mathcal{L}[g(x,t)] - \mathcal{L}[Nu(x,t)] \quad \text{------} \quad (**)$$

Applying the inverse Laplace transform on both sides of Equation (**), we find:

$$u(x,t) = G(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(Nu(x,t))\right] \quad \text{------} \quad (**_a*)$$

where $G(x,t)$ represents the terms arising from the source term and the prescribed initial condition $\left[\text{i.e. } G(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}[g(x,t)] + h(x)\right]\right\}\right]$

Taking the first partial derivative with respect to 't' of Equation (***) to obtain:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}G(x,t) - \frac{\partial}{\partial t}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu(x,t)]\right\} \quad \text{------} \quad (**_b*)$$

By the correction functional of the Variation Iteration Method

$$u_{n+1} = u_n - \int_0^t \left\{(u_n)_s(x,s) - \frac{\partial}{\partial s}G(x,s) + \frac{\partial}{\partial s}\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu(x,s)]\right\}\right\} ds$$

or

$$u_{n+1} = G(x,t) - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}[Nu_n(x,t)]\right\} \quad \text{------} \quad [****]$$

Equation [****] is the new modified correction functional of Laplace transform and the Variational iteration method, and the solution is given by:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

<u>Next:</u> Solving some non-linear PDEs by using the new modified Variational Iteration Laplace transform method:

<u>Examples:</u> [1] $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}$; $u(x,0) = x^2$

<u>Solution</u>

Given $u_t = u_x^2 + uu_{xx}$; $u(x,0) = x^2$

Taking Laplace transform, subject to the initial Condition, we have:

$$\mathcal{L}[u_t] = \mathcal{L}[u_x^2] + \mathcal{L}[uu_{xx}]$$

$$s\mathcal{L}[u(x,t)] - u(x,0) = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$s\mathcal{L}[u(x,t)] - x^2 = \mathcal{L}[u_x^2 + uu_{xx}]$$

$$\mathcal{L}[u(x,t)] = \frac{x^2}{s} + \frac{1}{s}\mathcal{L}[u_x^2 + uu_{xx}]$$

Taking the inverse Laplace transform to obtain:

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{x^2}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(u_x^2 + uu_{xx}\right)\right]$$

$$u(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[u_x^2 + u u_{xx}\right]\right\}$$

The new correction functional is given as:

$$u_{n+1}(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_n)_x^2 + u_n (u_n)_{xx}\right]\right\} \qquad n \geq 0$$

The solution in series form is given by:

$$u_0(x,t) = x^2 \quad \left[\text{or } u_0(x,t) = u(x,0)\right]$$

$$u_1(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_0)_x^2 + (u_0)(u_0)_{xx}\right]\right\}$$

$$u_1(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[4x^2 + 2x^2\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s}\left[\frac{4x^2}{s} + \frac{2x^2}{s}\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{6x^2}{s^2}\right\}$$

$$u_1(x,t) = x^2 + 6x^2 t$$

$$u_2(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(u_1)_x^2 + (u_1)(u_1)_{xx}\right]\right\}$$

$$u_2(x,t) = x^2 + L^{-1}\left\{\frac{1}{s} L\left[(2x+12xt)^2 + (x^2+6x^2t)(2+12t)\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s} L\left[6x^2 + 72x^2t + 216x^2t^2\right]\right\}$$

$$= x^2 + L^{-1}\left\{\frac{1}{s}\left[\frac{6x^2}{s} + \frac{72x^2}{s^2} + \frac{216x^2 \cdot 2}{s^3}\right]\right\}$$

$$= x^2 + L^{-1}\left[\frac{6x^2}{s^2} + \frac{72x^2}{s^3} + \frac{432x^2}{s^4}\right]$$

$$u_2(x,t) = x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3$$

The series solution is given by:

$$u(x,t) = x^2 + 6x^2t + 36x^2t^2 + 72x^2t^3 + \cdots$$

$$= x^2\left[1 + 6t + 36t^2 + 72t^3 + \cdots\right]$$

$$u(x,t) = \frac{x^2}{1-6t}$$

(2) — Nonlinear

$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2t^2, \quad u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = x$ (*)

(a)

$L[u_{tt}] = L[x^2t^2] + L[u_{xx} - u^2]$

$s^2 L[u(x,t)] - s\,u(x,0) - \frac{\partial u(x,0)}{\partial t} = \frac{2x^2}{s^3} + L[u_{xx} - u^2]$ $4\times3\times2$

$s^2 L[u(x,t)] = \frac{2x^2}{s^3} + x + L[u_{xx} - u^2]$ $\frac{2x^2 +}{s^3}$ $4!$

$L[u(x,t)] = \frac{2x^2}{s^5} + \frac{x}{s^2} + \frac{1}{s^2} L[u_{xx} - u^2]$

$u(x,t) = \frac{x^2t^4}{12} + xt + L^{-1}\left\{\frac{1}{s^2} L[u_{xx} - u^2]\right\}$

$u_{n+1}(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[(u_n)_{xx} - (u_n)^2]\right\}$

$u_0(x,t) = u(x,0) + t\frac{\partial u(x,0)}{\partial t} = \underline{xt}.$

$u_1(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[-x^2t^2]\right.$ $\frac{2x^2}{s^3} = \frac{2x^2}{s^5}$

$u_1(x,t) = xt + \frac{x^2t^4}{12} - \frac{x^2t^4}{12} = xt$

$u_2(x,t) = xt + \frac{x^2t^4}{12} + L^{-1}\left\{\frac{1}{s^2} L[-x^2t^2]\right\}$

$= \underline{\underline{xt}}.$

$u(x,t) = \underline{\underline{xt}}.$