APPLICATION OF LAPLACE TRANSFORM METHOD IN SOLVING SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Laplace Method:

$\mathcal{L}\left[f(t)\right] = F(S) = \int_0^{-\infty} e^{-st} f(t)\mathrm{dt}$
$\mathcal{L}\left[f^n(t)\right] = S^n Y - S^{n-1}Y(0) - S^{n-2}Y'(0)\cdots\cdots - Sf^{n-2}(0)\cdots\cdots - y^{n-1}(0)$
$\mathcal{L}\left[U_x(x,t)\right] = \int_0^{\infty} e^{-st}U(x,t)\mathrm{dt} \equiv U(x,s)$
$\mathcal{L}\left[U_x(x,t)\right] = U_x(x,s)$
$\mathcal{L}\left[U_x(x,t)\right] = U_x(x,s)$
$\mathcal{L}\left[U_t(x,t)\right] = SU(x,s) - U(x,0)$
$\mathcal{L}\left[U_t(x,t)\right] = S^2U(x,s) - SU(x,0) - U_t(x,0)$
$F(s) = \mathcal{L}\left[f(t)\right]$ then,
$\mathcal{L}\left[U(t-a)\cdot g(t-a)\right] = e^{-as}G(s)$

P.D.E of Order 2

Examples:

(1) $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial x}(x,t), 0 < x < 2, t > 0 \quad U(0,t) = 0, U(2,t) = 0, U(x,0) = 3\sin(2\pi x)$

Solution:
$U_{xx}(x,t) = U_t(x,t)$

taking the Laplace transform

$\mathcal{L}\left[U_{xx}(x,t)\right] = \mathcal{L}\left[U_t(x,t)\right]$

$U_{xx}(x,s) = sU(x,s) - U(x,0)$

Using the condition, $U(x,0) = 3\sin(2\pi x)$

$SU(x,s) - 3\sin(2\pi x) = U_{xx}(x,s)$

$\Rightarrow U_{xx}(x,s) - SU(x,s) = -3\sin(2\pi x)$

$\dfrac{d^2u}{dx^2} - SU = -3\sin(2\pi x)$

Solving the Homogenous Problem

$\dfrac{d^2u}{dx^2} - SU = 0$

The characteristic equation is given by

$m^2 - S = 0 \Rightarrow m = \pm\sqrt{S}$

The homogenous solution is:

$$U_A(x, S) = A_1 e^{\sqrt{S}x} + A_2 e^{-\sqrt{S}x}$$

Solving the non-homogenous problem using the method of Undetermined Coefficient

i.e $\dfrac{d^2u}{dx^2} - SU = -3\sin(2\pi x) - - - - - - - - - - - - - (*)$

Let

$U = \Delta_1 \sin(2\pi x) + \Delta_2 \cos(2\pi x) - - - - - - - - - - (a)$

$U' = 2\pi\Delta_1 \cos(2\pi x) - 2\pi\Delta_2 \sin(2\pi x) - - - - - - - - (b)$

$U'' = -4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - - - - - - (c)$

Substituting (a) and (c) in equation (*)

$-4\pi^2\Delta_1 \sin(2\pi x) - 4\pi^2\Delta_2 \cos(2\pi x) - S\Delta_1 \sin(2\pi x) - S\Delta_2 \cos(2\pi x) = -3\sin(2\pi x)$

$-4\pi^2\Delta_1 - S\Delta_1 = -3$ Also, $4\pi^2\Delta_2 - S\Delta_2 = 0$

$-\Delta_1 \left[4\pi^2 + S\right] = -3$ $\Delta_2 \left[S + 4\pi^2\right] = 0$

$\Delta_1 = \dfrac{3}{S + 4\pi^2}$ $\Delta_2 = 0$

The particular solution is:

$$U_p(x, S) = \frac{3}{S + 4\pi^2} \sin(2\pi x)$$

The general solution is given by:

$$U(x, s) = A_1 e^{\sqrt{sx}} + A_2 e^{-\sqrt{sx}} + \frac{3\sin(2\pi x)}{S + 4\pi^2}$$

Applying the boundary conditions $U(0, t) = 0,\ U(2, t) = 0$

$U(0, s) = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$

$U(2, s) = A_1 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0 \quad [\text{But } A_1 = -A_2]$

$-A_2 e^{2\sqrt{s}} + A_2 e^{-2\sqrt{s}} = 0$

$A_2 \left[e^{-2\sqrt{s}} - e^{2\sqrt{s}}\right] = 0 \quad \Rightarrow A_2 = 0, A_1 = 0$

$U(x, s) = \dfrac{3\sin(2\pi x)}{S + 4\pi^2}$