NUMERICAL SOLUTION TO HIGHER ORDER INTIAL VALUE PROBLEMS USING ADAMS-BASHFORTH AND ADAMS-MOULTON METHOD AND RUNGE-KUTTA METHOD OF ORDER 4

Definition: An Initial Value Problem (often times abbreviated IVP) is a problem where we want to find a solution to some differential equation that satisfies a given initial value $y(x_0) = y_0$.

In the field of differential equation, an Initial Value Problem (also called a Cauchy Problem) is an Ordinary Differential Equation together with a specific value, called the Initial condition of the unknown function at a given point in the domain of the solution.

An Initial Value Problem is a differential equation $y'(t) = f(t, y(t))$ with $f : \Omega\ c \in \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ where (is an open set of $\mathbb{R} \times \mathbb{R}^n$, together with a point in the domain of $(t_0, y_0) \in \Omega$, called the initial condition.

A solution to an initial value problem is a function y that is a solution to the differrential equation and satisfies $y(t_0) = y_0$.

In higher dimension, the differential equation is replaced with family of equation $y_i'(t) = f_1(t, y_2(t), y_2(t), \cdots)$ and $y(t)$ is viewed as the Vector $(y_1(t), \cdots, y_n(t), \cdots)$, mot commonly associated with the position in space, such as Banach Spacce or Spaces distributions. Initial Value Problems are extended to higher Order by treating the derivatives in the same way as an indedependent fuction, e.g: $y''(t) = f(t, y(t), y'(t)$

Solving Initial Value Problems Anaytically
Example I: A simple example is to solve $y' = 0.85y$ and $y(0) = 19$.

Solution

We are trying to find a formular for $y(t)$ that satisfies these
start by noting that $y' = \frac{dy}{dt}$, so $\frac{dy}{dt} = 0.85y$

Now, we separate the valuables, so that y is on the left and t on the right.

$\frac{dy}{dt} = 0.85\text{dt}$
on integrating both sides, we have

$$\int \frac{dy}{y} = \int 0.85\text{dt}$$

$\Rightarrow \ln \mid y \mid = 0.85t + B$

Eliminate the ln by taking e of both sides
$e^{\ln|y|} = e^{0.85t}.\ e^{B}$
$\mid y \mid = e^{B}.e^{0.85t}$

Let C be a new unknown constant,

$C = \pm e^{B}$, so
$y = Ce^{0.85t}$

Now, we need to find a value for c use $y(0) = 19$ as given at the start and substitute 0 for t and 19 for y
$\Rightarrow 19 = Ce^{0.85(0)}$

$\Rightarrow C = 19$

This gives the final solution

$y(t) = 19e^{0.85t}$

Example II: The solution of $y' + 3y = 6t + 5, y(0) = 3$ can be found to be $y(t) = 2e^{-3t} + 2t + 1$

Indeed,

$$\begin{aligned}
(y' + 3y &= \frac{d}{dt}\left(2e^{-3t} + 2t + 1) + 3(2e^{-3t} + 2t + 1\right) \\
&= (-6e^{-3t} + 2) + 6e^{-3t} + 6t + 3 \\
&= -6e^{-3t} + 2 + 6e^{-3t} + 6t + 3 \\
&= 6t + 5
\end{aligned}$$

Example III: Solve the Initial Value Problem $\dfrac{dy}{dx} = 10 - x,\ y(0 = -1$

Solution

For an equation of the form $\dfrac{dy}{dx} = f(x)$, the Problmes can be solves in two step

Step 1:

$\dfrac{dy}{dx} = 10 - x \Rightarrow \text{dy} = (10 - x)\text{dx}$

$\displaystyle\int \text{dy} = \int (10 - x)\text{dx} \Rightarrow y = 10x - \frac{x^2}{2} + C$

Step 2:

When $x = 0, y = 1$

$- \mid 2 \mid 0(0) - \dfrac{0}{2} + C \Rightarrow C = 1$

Hence, the solution is: $y = 10x - \dfrac{x^2}{x} - 1$

Example IV: Solve the Initial Initial Value Problem $\dfrac{dy}{dx} = 9x^2 - 4x + 5,\ y(-1) = 0$

Solution

Step 1:

$\int \mathrm{dy} = \int (9x^2 - 4x + 5)\mathrm{dx} \Rightarrow y = \dfrac{9x^3}{3} - \dfrac{4x^2}{2} + 5x + C$

Step 2: When $x = -1, y = 0$

$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + C \Rightarrow 0 = -3 - 2 - 5 + C \Rightarrow C = 10$

SOLUTION: $y = 3x^3 - 2x^2 + 5x + 10$

Example V: Sovle the Initial Value Problem $\dfrac{ds}{dt} = \cos t + \sin t,\quad S(\pi) = 1$

Solution

Step 1:

$\int \mathrm{ds} = \int (\cos t + \sin t)\mathrm{dt} \Rightarrow S = \sin t - \cos t + C$

Step 2: When $t = \pi,\ S = 1$

$1 = \sin \pi - \cos \pi + C$

$1 = 0 - (-1) + C$

$\Rightarrow C = 0$

SOLUTION: $S = \sin t - \cos t$

Example VI: Solve the Initial Value Problem $\dfrac{d^2y}{dx^2} = 2 - 6x,\ y'(0) = 4,\ y(0) = 1$

SOLUTION: We will have to do the two steps twice to find the solution to this Initial Value Problem . The first time through will give us y' and the second time through will ngive us y

Step 1:

$y'' = (2 \cdot 6x)$

$\Rightarrow y' = \int (2 - 6x)\mathrm{dx}$

$\Rightarrow y' = 2x \cdot \frac{6x^2}{2} + C$

Step 2: When $x = 0, y' = 4$

$4 = 2(0) - 3(0)^2 + C \Rightarrow C = 4$

$y' = 2x - 3x^2 + 4$

Step 1:

$y = \int (2x - 3x^2 + 4)\mathrm{dx} \Rightarrow y = \frac{2x^2}{2} - \frac{3x^3}{3} + 4x + C$

Step 2: When $x = 0, y = 1$

$1 = 0^2 - 0^3 + 4(0) + C \Rightarrow C = 1$

SOLUTION: $y = x^2 - x^3 + 4x + 1$

Exampe VII: Sovle the Initial Value Problem $y^{(4)} = -\sin t + \cos t,\ y'''(0) = 7,\ y''(0) = y'(0) = -1, y(0) = 0$

Solution

Since we are working with the fourth derivatives, we will have to go through the two steps four times

Step 1:

$y^{(4)} = y^{iv} = -\sin t + \cos t$

$\Rightarrow y''' = \int (-\sin t + \cos t)\mathrm{dt}$

$\Rightarrow y''' = \cos t + \sin t + C$

Step 2: When $t = 0\ \ y''' = 7$

$7 = \cos(0) + \sin(0) + C$
$\Rightarrow 7 = 1 + C \Rightarrow C = 6$

$y''' = \cos t + \sin t + 6$

Again,

<u>**Step 1:**</u>

$y'' = \int(\cos t + \sin t + 6)\mathrm{dt}$
$\Rightarrow y'' = \sin t - \cos t + 6t + C$

<u>**Step 2:**</u> When $t = 0, \quad y'' = -1$

$-1 = \sin(0) - \cos(0) + 6(0) + C$
$\Rightarrow -1 = -1 + C \Rightarrow C = 0$
$y'' = \sin t - \cos t + 6t$

Again,

<u>**Step 1:**</u>

$y' = \int(\sin t - \cos t + 6t)\mathrm{dt} \Rightarrow y' = -\cos t - \sin t + 6\frac{t^2}{2} + C$

<u>**Step 2:**</u> When $t = 0, \quad y' = -1$

$-1 = \cos(0) - \sin(0) + 3(0) + C \Rightarrow -1 = -1 + C \Rightarrow C = 0$
$y' = -\cos t - \sin t + 3t^2$

Again,

<u>**Step 1:**</u>

$y = \int(-\cos t - \sin t + 3t^2)\mathrm{dt}$
$\Rightarrow y = -\sin t + \cos t + 3\frac{3t^3}{3} + C$

<u>**Step 2:**</u> When $t = 0, \quad y = 0$

$0 = -\sin(0) + \cos(0) + 0^3 + C$
$\Rightarrow 0 = 1 + C \Rightarrow C = -1$

SOLUTION: $y = -\sin t + \cos t + t^3 - 1$

Example VIII: Given the Velocity,

$$V = \frac{ds}{dt} = 32t - 2$$

and the Initial position of the body as $S(\frac{1}{2}) = 4$. Find the body's position at the time t.

Solution

Step 1:

$\dfrac{ds}{dt} = 32t - 2$
$\text{ds} = (32t - 2)\text{dt}$
$\int \text{ds} = \int (32t - 2)\text{dt}$
$S = \dfrac{32t^2}{2} - 2t + C$

Step 2: When $t = \dfrac{1}{2}, S = 4$

$4 = 16(\dfrac{1}{2})^2 - 2(\dfrac{1}{2}) + C$

$\Rightarrow 4 = 4 - 1 + C \Rightarrow C = 1$

SOLUTION: $S = 16t^2 - 2t + 1$

Example VIX: Given the aceleration $a = \frac{d^2s}{dt^2} = -4\sin 2t$. Initial Velocity $V(0) = 2$, and the Initial position of the body as $S(0) - 3$. Find the body's position at time t

Solution

Step 1:

$V = \int -4\sin 2t dt \Rightarrow V = \dfrac{4\cos 2t}{2} + C$

Step 2: When $t = 0,\ V = 2$

$\Rightarrow 2 = 2\cos(0) + C \Rightarrow 2 = 2 + C\ \Rightarrow C = 0$

$\therefore V = 2\cos 2t$

Step 1:

$S = \int 2\cos 2t dt \Rightarrow S = \dfrac{2\sin 2t}{2} + C$

Step 2: When $t = 0,\ S = -3$

$-3 = \sin(0) + C \Rightarrow C = -3$

SOLUTION: $S = \sin 2t - 3$

REDUCING N-ORDER ODEs INTO SYSTEM OF FIRST-ORDER ORDINARY DIFFERENTIAL

In this section we show how to reduce higher-order Ordinary Differential Equationinto systems of first order Ordinary Differential Equation(O.D.Es).

Example I: Reduce the differential equation into its equivalent system of first-order O.D.Es
$y''' + 6y'' + 11y' + 6y = 0$ – – – – – – – – – – (1.0)
$y(0) = 1$ – – – – – – – – – – – – – – – – – – (1.1)
$y'(0) = 0$ – – – – – – – – – – – – – – – – –(1.2)
$y''(0) = 0$ – – – – – – – – – – – – – – – – –(1.3)

Let $y = y_1$
(1.0) becomes, $y_1''' + 6y_1'' + 11y_1' + 6y_1 = 0$

$y_1' = y_2 \Rightarrow y_1'' = y_2,\ y_1(0) = 1$

$y_1'' = y_2' = y_3 \Rightarrow y_2' = y_3,\ y_2(0) = 0$

$y_1''' = y_2'' = y_3'$

$\Rightarrow y_3' + 6y_3 + 11y_2 + 6y_1 = 0$

$\Rightarrow y_3' = -6y_3 - 11y_2 - 6y_1, y_3(0) = 0$

$\therefore$ The third-Order Ordinary Differential Equationin its equivalent system of first Order ODEs is:

$y_1' = y_2, y_1(0) = 1$
$y_2' = y_3, y_2(0) = 0$
$y_3' = -6y_1 - 11y_2 - 6y_3, y_3(0) = 0$

which matrice form gives,

$$\underline{y}' \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example II: Reduce the differential equation into its equivalent System of First-Order O.D.Es

$y'' + 2y' + 6y = 0$ – – – – – – – – – – (1.0)

$y(0) = 0,\ y'(0) = 1$

Let $y = y_1$
(1.0) becomes

$y_1'' + 2y_1' + 6y_1 = 0$
$y_1' = y_2 \Rightarrow y_1' = y_2,\ y_1(0) = 0$
$y_1'' = y_2'$
$\Rightarrow y_2' + 2y_2 + 6y_1 = 0 \Rightarrow y_2' = -6y_1 - 2y_2, y_2(0) = 1$

Hence, the Second-Order Ordinary Differential Equationin its equivalent System of First Order ODEs is:

$y_1' = y_2,\ y_1(0) = 0$

$y_2' = -6y_1 - 2y_2,\ y_2(0) = 1$

Example III: Reduce the following higher Order Initial Value Problems to the equivalence Order System

(1) $y''' + 2y'' - y' - 2y = e^x,$ for $0 \leq x \leq 1,$ for which $y(0) = 1,\ y'(0) = 2$ and $y''(0) = 0$

Let $y = y_1$

$y_1''' + 2y_1'' - y_1 - 2y_1 = e^x$

$y_1'' = y_2 \Rightarrow y_1' = y_2,\ y_1(0) = 1$

$y_1'' = y_2' = y_3 \Rightarrow y_2' = y_2, y_2(0) = 2$

$y_1''' = y_2'' = y_3'$

$\Rightarrow y_3' + 2y_3 - y_2 - 2y_1 = e^x$

$y_3' = 2y_1 + y_2 - 2y_3 + e^x,\ y_3(0) = 0$

Hence the third-order Ordinary Differential Equationin its equivalent System of first Order ODEs is:

$y_1' = y_2,\ y_1(0) = 1$

$y_2' = y_3,\ y_2(0) = 2y_3' = 2y_1 + y_2 + 2y_3 + e^x,\ y_3(0) = 0$

In matrice form the ODE is expressed as;

$$\underline{y}' \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \ e^x, \ \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} \ = \ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Example IV: Reduce the following Second-Order Initial Value Problem to the equivalence Order System

$(1) y'' + 7y' + 5y = 0, y(0) = 1, y'(0) = 7$

$(2) y^{(iv)} + 3y''' - 2y'' + y' - 4y = 0, y(0) = 1, y'(0) = 2, y''(0) = 3, y'''(0) = 4$

Solution

$y'' + 7y' + 5y = 0$

let $y = y_1$

$y_1'' + 7y_1 + 5y1 = 0$

$y_1' = y_2 \Rightarrow y_1', y(0) = 1$

$y_1'' = y_2'$

$\Rightarrow y_2' + 7y_2 + 5y_1 = 0$

$\Rightarrow y_2' = -5y_1 - 7y_2, y_2(0) = 7$

Hence, the Second-Order Ordinary Differential Equationin the equivalent System of first Order ODEs is;

$y_1' = y_2, y(0) = 1$

$y_2' = -5y_1 - 7y_2, y_2(0) = 7$

In matrice form, we have;

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$y^{iv} + 3y''' - 2y'' + y' - 4y = 0, y(0) = 1, y'(0) = 2, y''(0) = 3, y'''(0) = 4$

Let $y = y_1$

$y_1^{iv} + 3y_1''' - 2y_1'' + y_1' - 4y_1 = 0$

$y_1' = y_2 \Rightarrow y_1(0) = 1$

$y_1'' = y_2' = y_3 \Rightarrow y_2' = y_3, y_2(0) = 2$

$y_1''' = y_2'' = y_3' = y_4 \Rightarrow y_3' = y_4, y_3(0) = 3$

$y_1^{iv} = y_2''' = y_3'' = y_4'$

$\Rightarrow y_4' + 3y_4 - 2y_3 + y_2 - 4y_1 = 0$

$y_4' = 4y_1 - y_2 + 2y_3 - 3y_4, y_4(0) = 4$

Hence, the fourth Order Ordinary Differential Equationin its equivalent System of first Order ODEs is;

$y_1' = y_2, y_1(0) = 1$

$y_2' = y_3, y_2(0) = 2$

$y_3' = y_4, y_3(0) = 3$

$y_4' = 4y_1 - y_2 + 2y_3 - 3y_4, y_4(0) = 4$

In matrice form we have;

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -1 & 2 & -3 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

SOLVING HIGHER ORDER INITIAL VALUE PROBLEM BY METHOD OF EIGNE VALUES AND EIGEN VECTORS

Here, we will learn how to solve Second-Order and Third-Order Initial Value Problems using eigen Value/eigen Vectors approach

Example I: Solve $y''' + 6y'' + 11y' + 6y = 0$ by reducing to a system of first Order ODEs. $y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$

Solution

$y_1''' + 6y_1'' + 11y_1' + 6y_1 = 0$

$y_1' = y_2,\ y_1(0) = 1$

$y_1'' = y_2' = y_3,\ y_2(0) = 0$

$y_1''' = y_2'' = y_3',\ y_3(0) = 0$

$\Rightarrow y_3' + 6y_3 + 11y_2 + 6y_1 = 0$

$y_3' = -6y_1 - 11y_2 - 6y_3,\ y_3(0) = 0$

Hence we have, $y_1' = y_2,\ y_1(0) = 1$

$y_2' = y_3,\ y_2(0) = 0$

$y_3' = -6y_1 - 11y_2 - 6y_3,\ y_3(0) = 0$

$\therefore$ In matrice form, we have;

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ Using the characteristic equation $|A - \lambda I| = 0,$ we obtain the eigen values

$$\left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0 \Rightarrow \left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right|$$

$$\left| \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -6 & -11 & -6-\lambda \end{pmatrix} \right| = 0$$

$-\lambda(-\lambda(-6-\lambda)+11)-1(0+6)+0(0-6\lambda)=0$

$-\lambda(6\lambda+\lambda^2+11)-1(6)=0$

$\Rightarrow -\lambda^3-6\lambda^2-11\lambda-6=0$

$\Rightarrow \lambda^3+6\lambda^2+11\lambda+6=0$

By trial and error method,

When $\lambda=-1$, i.e $\lambda=-1$ is a zero of the polynomial,

$\Rightarrow \lambda+1$ is a factor.

By long division method,

$$\begin{array}{rrrl|rrrrrrr}
 & & & & \lambda^2 & + & 5\lambda & + & 6 & & \\
\cline{5-11}
\lambda & + & 1 & & \lambda^3 & + & 6\lambda^2 & + & 11\lambda & + & 6 \\
 & & & & \lambda^3 & + & \lambda^2 & & & & \\
\cline{5-11}
 & & & & & & 5\lambda^2 & + & 11\lambda & + & 6 \\
 & & & & & & 5\lambda^2 & + & 5\lambda & & \\
\cline{7-11}
 & & & & & & & & 6\lambda & + & 6 \\
 & & & & & & & & 6\lambda & + & 6 \\
\cline{9-11}
 & & & & & & & & & -- & \\
\cline{9-11}
\end{array}$$

$\therefore (\lambda+1)(\lambda^2+5\lambda+6) \equiv (\lambda+1)(\lambda+2)(\lambda+3)$
$\Rightarrow \lambda=-1,-2,-3$

Hence the eigen Values are $\lambda_1=-1, \lambda_2=-2, \lambda_3=-3$

Now, we obtain the eigen Vectors, using $(A-\lambda I)\underline{y}=0$

When $\lambda=-1$

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow (y_1, y)2, y_3) = (1,-1,1)$$

$$\begin{array}{rrcl} \Rightarrow & y_1 & = & 1 \\ & y_2 & = & -1 \\ & y_3 & = & 1 \end{array}$$

$\therefore$ The eigen Vector corresponding to the eigen Value $\lambda = -1$ is $k_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = -2$

Now, we obtain the eigen Vector using $(A - \lambda I)\underline{y} = 0$

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2y_1 + y_2 = 0 - - - - - - - - - - - - - - - - - (1)$

$2y_2 + y_3 = 0 - - - - - - - - - - - - - - - - - (2)$

$-6y_1 - 11y_2 - 4y_3 = 0 - - - - - - - - - - - - - (3)$

From (1) $2y_1 = -y_2 - - - - - - - - - - - - - - (4)$

From (2) $2y_2 = -y_3 - - - - - - - - - - - - - - (5)$

$\Rightarrow y_1 = -\frac{1}{2}y_2 = \frac{1}{4}y_3$

$y_1 = \frac{1}{4}y_3$, $y_2 = -\frac{1}{2}y_3$, $y_3 = y_3$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}y_3 \\ -\frac{1}{2}y_3 \\ y_3 \end{pmatrix} = y_3\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{4}y_3\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Set $\frac{1}{4}y_3 = 1$

$\therefore k_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

Where $k_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ is the eigen Vector corresponding to $\lambda = -2$

Also, When $\lambda = -3$

We obtain the eigen Vector, using $(A - \lambda I)\underline{y} = 0$

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right]\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3y_1 + y_2 = 0 - - - - - - - - - - - - - - - - (1)$

$3y_2 + y_3 = 0 - - - - - - - - - - - - - - - - (2)$

$-6y_1 - 11y_2 - 3y_3 = 0 - - - - - - - - - - - - (3)$

$3y_1 = -y_2 - - - - - - (4)$

$3y_2 = -y_3 - - - - - - (5)$

$\Rightarrow y_1 = -\frac{1}{3}y_2 \ , \ y_2 = -\frac{1}{3}y_3$

$y_1 = -\frac{1}{3}y_3 = \frac{1}{9}y_3$

$y_1 = \frac{1}{9}y_3 \ , \ y_2 = -\frac{1}{3}y_3 \ , \ y_3 = y_3$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{9}y_3 \\ -\frac{1}{3}y_3 \\ y_3 \end{pmatrix} = y_3\begin{pmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{pmatrix} = \frac{1}{9}y_3\begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$$

Set $\frac{1}{9}y_3 = 1$

$\therefore k_3 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$

Where $k_3 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$ is the eigen Vector corresponding to $\lambda = -3$

$\therefore$ We have the solution in the form,

$$\gamma_1 = e^{\lambda_1 x}C_1 + e^{\lambda_2 x}C_2 + e^{\lambda_3 x}C_3$$

$$\gamma_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-x} + \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2x} + \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3x}$$

$$\gamma_1 = \begin{pmatrix} e^{-x} + e^{-2x} + e^{-3x} \\ -e^{-x} - 2e^{-2x} - 3e^{-3x} \\ e^{-x} + 4e^{-2x} + 9e^{-3x} \end{pmatrix} \gamma_1 = \gamma = e^{-x} + e^{-2x} + e^{-3x}$$

$y(x) = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}$

$y(0) = 1 \Rightarrow C_1 + C_2 + C_3 = 1$

$y'(0) = 0 \Rightarrow -C_2 - 2C_2 - 3C_3 = 0$

$y''(0) = 0 \Rightarrow C_1 + 4C_2 + 9C_3 = 0$

$\therefore$ We have 3×3 System of equation

$$\begin{array}{ccccccc} C_1 & + & C_2 & + & C_3 & = & 1 \\ -C_1 & - & 2C_2 & - & 3C_3 & = & 0 \\ C_1 & + & 4C_2 & + & 9C_3 & = & 0 \end{array}$$

Using Gaussian Elimination, i.e reducing to echolon form

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ -1 & -2 & -3 & \vdots & 0 \\ 1 & 4 & 9 & \vdots & 0 \end{array}\right) \sim \left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & 1 \\ 0 & 3 & 8 & \vdots & -1 \end{array}\right) \sim \left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 2 \end{array}\right)$$

$$\begin{array}{ccccccc} C_1 & + & C_2 & + & C_3 & = & 1 - - - - - (1) \\ & - & C_2 & - & 2C_3 & = & 1 - - - - - (2) \\ & & & & 2C_3 & = & 2 - - - - - (3) \end{array}$$

$\Rightarrow C_3 = 1, \ -C_2 - 2C_3 = 1 \Rightarrow -C_2 - 2 = 1 \Rightarrow -C_2 = 3 \Rightarrow C_2 = -3$
Also from (1) $C_1 + C_2 + C_3 = 1 \Rightarrow C_1 - 3 + 1 = 1 \Rightarrow C_1 = 3$

$\therefore y(x) = 3e^{-x} - 3e^{-2x} + e^{-3x}$

RUNGE-KUTTA METHODS

In Numerical Analysis, the **Runge-Kutta** methods are family of implicit and explicit iterative methods, which include the well known routine called the **Euler Method**, used in temporal discretization for the approximate solutions of Ordinary Differential Equations. These methods were developed around 1900 by the German mathematicians Carl Runge and Wilhelm Kutta.

Runge-Kutta methods belong to the class of one-step Integration for the numerical solution of Ordinary Differential Equations. Nonstiff Problems can be efficiency sovled with explicit Runge-Kutta methods, Stiff Problems with certain conflicit Runge-Kutta methods. The Runge-Kutta method is known as the **Predictor Method**.

The Euler's modified formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + h)\right] ------(1.0)$$

$$\begin{aligned} \text{Let} \quad k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left[x_n + h, y_n + hf(x_n, y_n)\right] \text{ or } k_2 = hf(x_n + h, y_n + k1) \end{aligned}$$

Substituting the values of k_1 and k_2 in 1.0 we get

$$\mathbf{y_{n+1} = y_n + \frac{1}{2}\left(k_1 + k_2\right)}$$

This is known as **Runge's Formula of Order 2**

The Runge's Formula of Order 3 is also given by;

$$\mathbf{y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_4 + k_3\right)}$$

where,

$$\begin{aligned} k_1 &= hf(x_0, y_0), \\ k_2 &= hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right) \\ k_3 &= hf(x_0 + h, y_0 + 2k_2 - k_1) \end{aligned}$$

The most widely known member of the Runge-Kutta family is generally referred to as **"RK4"** the **"Classic Runge-Kutta method"** Or simply as the **"Runge-Kutta method"**.

RUNGE-KUTTA METHOD OF ORDER 4

Let an Initial Value Problem be specified as follows

$\hat{y} = f(x, y)$, $y(x_0) = y_0$

Here, y is an unknown function (scalar or vector) of x, which we would like to approximate, we are told that $\hat{y}$, the rate at which y changes, is a function of x and of y itself. At the initial time x_0 the corresponding y value is y_0. The function f and the initial conditions x_0, y_0 are given.

Given the following inputs

- An Ordinary Differential Equationthat defines values of dy/dx in the form x and y.
- Initial Value of y, i.e, $y(0)$

Thus we are given below

$\frac{dy}{dx} = f(x, y), y(0) = y_0$

The task is to find value of unknown function y at a given point x.

The Runge-Kutta method finds approximate value of y for a given x. Only First Order Ordinary Differential Equationscan be solved by using the Runge-Kutta 4th Order Method.

The formula used to compute next value y_{n+1} from previous value y_n is given below. The value of n are $0, 1, 2, 3, \cdots \frac{(x - x_0)}{h}$. Here h is step height and $x_{n+1} = x_{n+h}$. Lower step size means more accuracy.

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k3)$

$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$

$$\Rightarrow y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

The formula basically computes next value y_{n+1} using current y_n plus weighted average of four increments.

- k_1 is the based on the slop of the the beginning of the interval, using y
- k_2 is the increment based on the slope at the midpoint of the interval, using $y + \dfrac{hk_1}{2}$
- k_3 is again the increment based on the slope at the midpoint, using $y + \dfrac{hk_2}{2}$
- k_4 is the increment based on the slope at the end of the interval, using $y + hk_3$

The method is a fourth-order method, meaning that the local trunction error is on the Order of $O(h^5)$, while the total accumulated error is Order $O(h^4)$. Example I: $\left\{ \begin{array}{rcl} y' & = & y - t^2 + 1 \\ y(0) & = & 0.5 \end{array} \right.$

The exact solution for this problem is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, and we are interested in the value of y for $0 \leq t \leq 2$.

1. We first solve this problem using RK4 with $h = 0.5$ from $t = 0$ to $t = 2$ with step size $h = 0.5$. It takes 4 steps:

$t_0 = 0,\ t_1 = 0.5,\ t_2 = 1,\ t_3 = 1.5,\ t_4 = 2$

Step 0: $t_0 = 0, y_0 = 0.5,$ since $y(t_0) = y_0$

Step I: $t_1 = 0.5, \text{but}$
$k_1 = hf(x_n, y_n)$
$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$
$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$
$k_4 = hf(x_n + h, y_n + k_3)$
$\Rightarrow$ at $t_1 = 0.5, n = 0$

$k_1 = hf(t_0, y_0) = 0.5f(0, 0.5) = 0.5(y_0 - t_0^2 + 1) = 0.5(0.5 - 0^2 + 1)$

$\Rightarrow k_1 = 0.5(1.5) = 0.75$

GENERALIZING ADAMS-BASHFORTH AND ADAMS-MOULTON FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

The Adams-Basforth for a single differential equation of the form; $y' = f(x, y), y(x_0) = y_0$ is given as
$y_{n+1} = y_n + \frac{h}{24}\big[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\big] - - - - - - - - (i)$

While the Adams-Bashforth for a System of Ordinary Differential Equationis given as;

$$y_{i,j+1} = y_{i,j} + \frac{h}{24}\big[55f_{i,j} - 59f_{i,j-1} - 37f_{i,j-2} - 9f_{i,j-3}\big] - - - - - - (ii)$$

(ii) is the Adams-Bashforth for a System where $i = 1,\ j = 3$

Also,
The Adams-Moulton for a single differential equation of the form; $y' = f(x, y),\ y(x_0) = y_0$ is given as

$$y_{n+1} = y_n + \frac{h}{24}\big[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\big] - - - - - - - - (iii)$$

While the Adams-Moulton for a System of Ordinary Differential Equationis given as,

$$y_{i,i+1} = y_{i,j} + \frac{h}{24}\big[9f_{i,j+1} + 19f_{i,j} - 5f_{i,j-1} + f_{i,j-2}\big] - - - - - - - - -(iv)$$

(iv) is the Adams-Moulton for a System, where $i = 1,\ j = 3$

for (i) and (iii) $f_n = f(x_n, y_n)$

And for (ii) and (iv) $f_{i,j} = f\big(x_J, y_{1,j}, y_{2,j}, \cdots, y_{n,j}\big)$