BASIC FLUID EQUATIONS IN DIFFERENT COORDINATE SYSTEMS

BY

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CERTIFICATION

This is to certify that this project work was carried out by **Hassan Abdulrazaq Tijani** with matriculation number **16/56EB075** and approved as meeting the requirement for the award of the Bachelor of Science (B. Sc.) degree of the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria.

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DEDICATION

This project is dedicated to God,my beloved parents and Africa's underprivileged children.

Acknowledgement

All praises glorifications are for the almighty Allah, my creator and sustainer of the world, whom I owe every deep sense of gratitude over the years for sparing my life from the beginning to the end of my course in the prestigious University of Ilorin. My profound gratitude to Dr. E. O. Titiloye who has been the ideal project supervisor , for his advice, and patient encouragement aided the writing of the project in innumerable ways. I pray almighty God bless him. I express my thanks to Prof. K. A Rauf who is the Head of Department (H.O.D) for being a true father, creating accomodating environment for we students to excel in our studies. My sincere gratitute to my level adviser,Dr. Idayat. F. Usamot , for her fatherly advice on our academics, may Allah be with her and her family. I also appreciate the immeasuraable effort of my lecturers in the department including Prof. J. A Gbadeyan, Prof. O. T Opoola, Prof. O. M. Bamigbola, Prof M. O Ibrahim, Prof. O. A. Taiwo,Prof R. B. Adeniyi, Prof A. S. Idowu, Prof. M. S. Dada, Prof. K. O. Babalola, Dr. Olubunmi. A. Fadipe-Joseph,Dr. Yidiat. O. Aderinto, Dr. Catherine. N. Ejieji, Dr.B. M. Yisa, Dr J. U. Abubakar, Dr. K. A. Bello, Dr. Gata. N. Bakare, Dr T. O. Olotu, Dr. B.M. Ahmad, Dr O. A.Uwaheren, Mr Odetunde and all other members of staff of the department of mathematics, who contributed greatly to my academic excellence, obtained during my period of study in the department. May God bless them all.

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Abstract

This project studies the flow of fluids in three dimensional coordinate considering the continuity equation and Navier-strokes equation in Cartesian coordinate and moreover converting to spherical coordinate , these equations are not autonomous but subject to the Newtons law of motion

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Chapter 1

General Introduction

1.1 Background of Study

Fluid mechanics is the branch of science concerned with the mechanics of fluids (liquids, gases, and plasmas) and the forces acting on them. It deals with the study of all fluids. It has applications in a wide range of disciplines, including mechanical, civil, chemical and biomedical engineering, geophysics, astrophysics, and biology. It can be divided into fluid statics, the study of fluids at rest, and fluid dynamics, and the study of the effects of fluid motion. The study of fluid mechanics goes back at least to the days of ancient Greece , when Archimedes investigated fluid statics and buoyancy and formulated his famous law known now as the Archimedes' principle, which was published in his work "On Floating Bodies" generally considered to be the first major work on fluid mechanics. Fluid mechanics is also a branch of continuous mechanics which deals with the relationship between forces, motion and statical conditions in continuous materials. This study areas deals with many problems such as surface tensions , fluid statics , flow on enclosed bodies or round bodies(solid or otherwise) flow stability etc.

1.2 Scope of Study

This project work is subject to the law of motion even as fluid flow leads to the equation of motion. It also focuses on flow in spherical coordinate and its conversion from cartesian coordinate to sphrical coordinate.

1.3 Aim and Objectives

The aims of this research is to study the flow of fluids through different co-ordinate system and the conversion from cartesian to spherical coordinate.

1.4 Definition of Relevant Terms

1.4.1 Fluids

A fluid is any substance that flows or a substance that deforms continuously when subjected to an external shearing stress. This term is used for both liquids and gases. One can easily define Fluids in the manner of how they respond, i.e. deforms or flow when they are subjected to a force in a specific situation such as shear stress Edward N. (2007).

1.4.2 Fluid Mechanics

Fluid mechanics is the study of the behaviour of fluids under the influence of force. It is divided into two broad's topic namely: (i) Fluid static. (ii)Fluid Dynamics.

1.4.3 Fluid Statics

It is the study of Fluids at rest. It can also be define as the forces that holds fluids in static equilibrium. It is based on Newton's first law of motion. .i.e. there are no shear Stress present (a shear results from a velocity gradient). .

1.4.4 Fluids Dynamics

This is the study of fluids in motion and the forces that keeps them in motion.

1.4.5 Deformation

This is the change in shape of mass of fluid when a force is been applied on it . The shear stress exists during the deformation).

1.4.6 Continuous Deformation

This occurs when a fluid continues to deform for as long as the force is applied on it.

1.4.7 Types of Fluids

(1). Ideal Fluid A Fluid which is incompressible and have no viscosity and surface tension is known as an Ideal fluid. Ideal fluid is only an imaginary fluid, such fluids do not really exists.
(2). Real Fluid A fluid which possesses viscosity, surface tension, compressibility and density is known as real fluid. All the fluids in actual practice are real fluids and are actually available in nature.
(3). Newtonian Fluid
A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as Newtonian fluid.
(4). Non-Newtonian Fluid A real fluid in which the shear stress is not proportional to the rate of shear strain (or velocity gradient) is known as non-newtonian fluid.
(5). Uniform Fluids Fluids are said to be uniform if its properties are the same at all points .Fluids are not uniform excepts when temperature and pressure are the same throughout.
(6). Ideal Plastic Fluid A fluid in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient) is known as ideal plastic fluid.
(7). Invisid Fluid
An invisid fluid (Non-viscous) is a type of fluid even when in motion is incapable of sustaining a shear stress. In reality no fluid is non-viscous,

however the effects of viscosity is negligible or very small such as water and air.

1.4.8 Properties Of Fluids

(1). Viscosity: This is the fluid property which measures the resistance to flow. It is a quantitative measure of fluid resistance to flow. For example different fluid have different viscosity. i.e. the viscosity of mercury is higher than water while that of water is higher than air. It is denoted as μ .
(2). Surface Tension: This is one of the important fluid property. It is well observed that some insects could walk on water without their body getting wet while some can't. It is thereby defined as the force per unit length acting in the surface of the right angle to one side of a line drawn in the surface.
(3).Density: Density is defined as the ratio between mass and volume.. It is denoted as . The s.i unit is .
(4). Relative Density: It is defined as the ration of mass density of substance to some standard mass density. It is unit less.
(5). Specific Weight: This is defined as the weight per unit volume, which varies from one point to point dependent on the varational of gravity(g) .

1.4.9 Fluid Flow

This is defined as the movement of real fluids.

1.4.10 Types of Fluid Flow

(1). Steady and Unsteady Flows: Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc. at a point do not change with time.

Unsteady flow is that type of flow in which the velocity, pressure or density at a point changes with respect to time.

(2). Compressible and Incompressible Flows: Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid.

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible.

(3).Uniform and Non-uniform Flows: Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (i.e length of direction of the flow).

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space.

(4).Rotational and Irrotational Flows: Rotational flow is that type of flow in which the fluid particles while owing along streamlines also rotate about their own axis. And if the fluid particles while owing along streamlines, do not rotate about their own axis then that type of flow is called irrotational flow.

(5). Laminar and Turbulent Flow: Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or streamline and all the streamlines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is called streamline flow or viscous flow. Turbulent flow is that type of flow in which the fluid particles move in zig-zag way. Due to the movement of fluid particles in a zig-zag way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number VD/v called the Reynolds number. Where D = Diameter of pipe.

V = Mean velocity.

v = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar.

If the Reynold number is more than 4000, it is called turbulent flow.

If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent. . .

1.4.11 Types of Flowlines

(1). PATHLINE A pathline is a path followed by a fluid particle in motion. A pathline shows the direction of particular particle as it moves ahead. In general, this is the curve in 3-dimensional space. However, if the conditions are such that the ow is 2-dimensional, the curve becomes 2-dimensional..

(2). STREAKLINE The streakline is a curve which gives an instantaneous picture of the location of the fluid particles which have passed through a given point.

Examples.

(a) In an experimental work to trace the motion of fluid particles, a coloured dye may be injected into the following fluid and the resulting coloured lament lines at a given location given by the streaklines.

(3). STREAMLINE A streamline can be defined as an imaginary line within the flow so that the tangent at any point on it indicates the velocity at that point. Equation of a streamline in a 3-dimensional flow is given as;..

$$\frac{\partial x}{u} = \frac{\partial y}{v} = \frac{\partial z}{w}$$

.

EXAMPLE (a) For a 3-dimensional flow, the velocity distribution is given by u = -x; v = 3-y; w = 3-z. What is the equation of a streamline passing through (1; 2; 2)?. .

Solution Given: u =-x, v = 3-y, w = 3-z Equation of a streamline passing through (1; 2; 3): The streamlines are defined by,

. $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

.

Substituting for u, v and w, we get

$\frac{dx}{-x} = \frac{dy}{3-y} = \frac{dz}{3-z}$

Considering the expressions (i) and (ii) and integrating, we get

$$\int \frac{dx}{-x} = \int \frac{dy}{3-y} \tag{1.1}$$

-$\log_e x = -\log_e(3-y) + c_1$ (where C1 = constant of integration) Since the streamline passes through $x = 1, y = 2, C_1 = 0$.

$(x)^-1 = (3-y)^-1 or x = (3-y)$ . .

Considering the expressions (i) and (iii), and integrating, we get,

$$\int \frac{dx}{-x} = \int \frac{dz}{3-z} \tag{1.2}$$

. -$\log_e x = -\log_e(3-y) + c_2$

(where C_2 = constant of integration) Since the streamline passes through $x = 1, y = 2, C_2 = 0$.

$(x)^-1 = (3-z)^-1 or x = (3-z)$

The streamline equation is : $x = (3-y) = (3-z)$

1.4.12 What Is Heat Transfer

Heat transfer is a discipline of thermal engineering that concerns the generation, uses and exchange of thermal energy between physical system. Heat transfer is classified into various mechanics such as thermal convention, thermal conduction, thermal radiation and transfer of energy by phases changes. Heat transfer describe the flow of heat (thermal energy) due to temperature difference and the subsequent temperature differences and the subsequent temperature distribution and changes.

1.4.13 Types of Heat Transfers

(1).Conduction: It is the heat transfer through stationary matter by physical contact. E.g. Heat Transferred between the electric burner of a stove and the bottom of a pot is transferred by conduction
(2).Convention: It is the heat transfer by the macroscopic movement of fluid. This is the type of transfer that takes place in a forced air furnace and in weather systems.
(3).Radiation: Heat transfer by radiation occurs when micro-waves, infrared radiation, visible light or another form of electromagnetic radiation is emitted or absorbed. E.g warming of the earth by the sun.

1.5 Equations Of Motion

In Fluids mechanic , Equations of motion is a great subtpoic under fluid and it includes the continuity Equation,the Navier-stokes equation and conservation of thermal energy which yields the energy equation.

1.5.1 Continuity Equation

This continuity equation states that in case of a steady flow, the amount of fluids flowing past one point must be the same flowing past another points, or mass flow rate is constant. It is essentially a statement of the law of conversation of mass. The explicit formula of the continuity equation is $A_1V_1 = A_2V_2$.

consider a pipe having two Points A and B:

Let A_1 = Area of the pipe at point A

V_1 = Velocity of the pipe flowing through A

ρ_1 = Density of the fluidat point A

Let A_2 = Area of the pipe at point B

V_2 = Velocity of the pipe flowing through B

ρ_2 = Density of the fluid at point B

$\rho_1\ A_1\ V_1$ = The total quantity of fluid passing through point A

$\rho_1\ A_2V_2$ = and the total quantity of fluid passing through point B

From the theorem of continuity, we deduce that:

$$\rho_1 A_1 V_1 = \rho_1 A_2 V_2 \tag{1.3}$$

Equation (1.3) is applicable to the compressible as well as incompressible fluids. It is called Continuity Equation. In case of incompressible fluids,since there is not occurrence of change in density of fluid, that is, $\rho_1 = \rho_2$,the continuity equation reduces to:

$$A_1\ V_1 = A_2V_2$$

This proves that for an incompressible fluid is undergoing steady flow , the product of cross sectional area and the speed must be the same for any two random point and this is called the equation of continuity.

1.5.2 Navier stokes

The Navier stokes Equation is a fundamental differential equation that describe the flow of an incompressible fluids.

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{u}) = o \tag{1.4}$$

$$\frac{\partial (\rho \overrightarrow{u})}{\partial t} + \overrightarrow{\nabla} \cdot [\rho \overline{\overline{u \otimes u}}] = -\overrightarrow{\nabla p} + \overrightarrow{\nabla} \cdot \overline{\overline{\tau}} + \rho \overrightarrow{f} \tag{1.5}$$

.

$$\frac{\partial}{\rho e} \partial t + \overrightarrow{\nabla} \cdot ((\rho e + p) \overrightarrow{u}) = \overrightarrow{\nabla} \cdot (\overline{\overline{\tau}} \cdot \overrightarrow{u}) + \rho \overrightarrow{f} \overrightarrow{u} + \overrightarrow{\nabla} \cdot (\overrightarrow{\dot{q}}) + r \tag{1.6}$$

1.5.3 Energy Equation

Energy is the quantitative property that must be transferred to an object in order to perform work on, or to heat, the object. Energy is a conserved quantity, the law of conservation of energy state that energy can be converted in form, but not created or destroyed. The SI unit of energy is the joule, which is the energy transferred to an object by the work of moving it a distance of 1 metre against a force of 1 newton Smith, Crosbie (1998).

In fluid mechanics one of the basic equations is energy equation. The different types of energy of flowing liquids considered in fluid are as follows:Potential energy: In relational concept with the generally known form of potential energy, it is due to configuration or position above some suitable datum (or elevation) line. It is usually denoted by z.

Velocity head or kinetic energy: This is due to velocity of flow liquid and is measured as $\frac{V^2}{2g}$ where, V is the velocity of flowing

and g is the acceleration due to gravity (g = 9.81) .

pressure energy: This is due to the pressure of liquid and reckoned as $\frac{p}{w}$where,

p is the pressure, and w is the weight density of the liquid.
Total head/energy: Total head of a liquid particle in motion is the sum of its potential head,kinetic head and pressure head. Mathematically

$$\text{Total Head(H)} = z + \frac{V^2}{2g} + \frac{p}{w} \text{ m of a liquid.}$$

Chapter 2

Literature Review

2.1 Introduction

Fluid mechanics is a wide ranged field that revolves around laws of applied mechanics and results from involving the physical laws of momentum Conservation of mass and energy. where the conservation of mass yields the Navier-stokes equation and conservation of thermal energy yields the energy equation.

All these equation are known as equation of motion .Moreover through these equations of motion flow of fluids through a particular co-ordinate can be converted to another coordinate. (spherical, Cartesian and cylindrical). Which will be treated further in the course of this project work.

2.2 Coordinate System

A coordinate system is a rule for mapping pairs of numbers to points in a plane.

In geometry, a coordinate system is a system that uses one or more numbers or coordinate to uniquely determine the position of the points or other geometric element on a manifold such as Euclidean space. A coordinate system is not just a set of axis, it is a set of rules for mapping a pair of numbers onto a points in plane Eric W.(1999).

Different coordinate system correspond to different rules. The polar coordinate has rules that are different than the rules of the XY coordinate system. Other coordinate system yet have another rules. The use of a coordinate system allows problem in geometry to be translated into problems about numbers and vice versa.

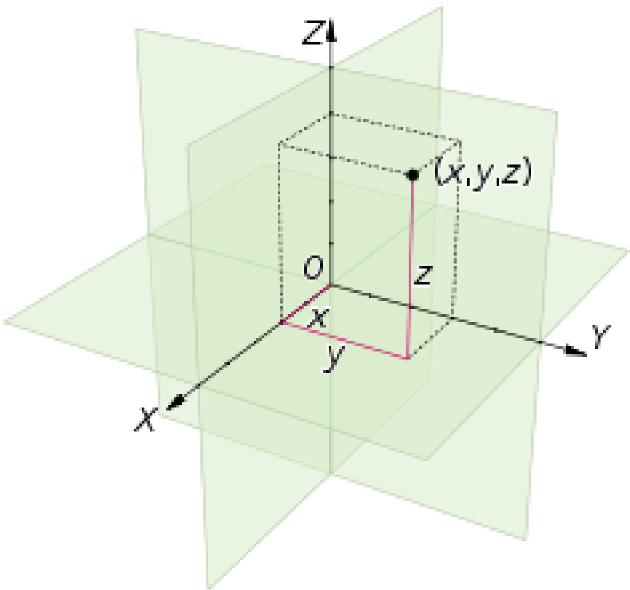
2.3 Common coordinate system

1. Number line: The simplest example of a coordinate system is the identification of points on a line with real numbers using the number line. In this system, an arbitrary point O (the origin) is chosen on a given line. The coordinate of a point P is defined as the signed distance from O to P, where the signed distance is the distance taken as positive or negative depending on which side of the line P lies. Each point is given a unique coordinate and each real number is the coordinate of a unique point.

2. Polar Coordinate: Another common coordinate system for the plane is the polar coordinate system. A point is chosen as the pole and a ray from this point is taken as the polar axis. For a given angle θ, there is a single line through the pole whose angle with the polar axis is θ (measured counter clockwise from the axis to the line). Then there is a unique point on this line whose signed distance from the origin is r for given number r. For a given pair of coordinates (r, θ) there is a single point, but any point is represented

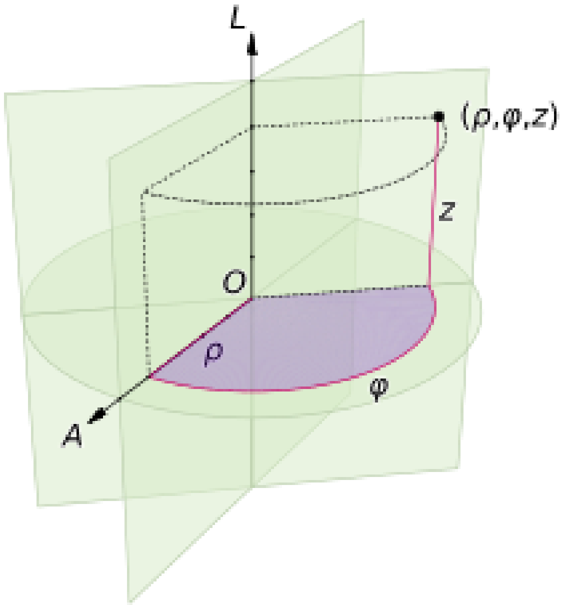
by many pairs of coordinates. For example, (r, θ), (r, $\theta + 2\pi$) and (r ,$\theta + \pi$) are all polar coordinates for the same point. The pole is represented by (0,θ) for any value of θ.

3. Cartesian: The prototypical example of a coordinate system is the Cartesian coordinate system. In the plane, two perpendicular lines are chosen and the coordinates of a point are taken to be the signed distances to the lines. Depending on the direction and order of the coordinate axes, the three-dimensional system may be a right-handed or a left-handed system. This is one of many coordinate systems.

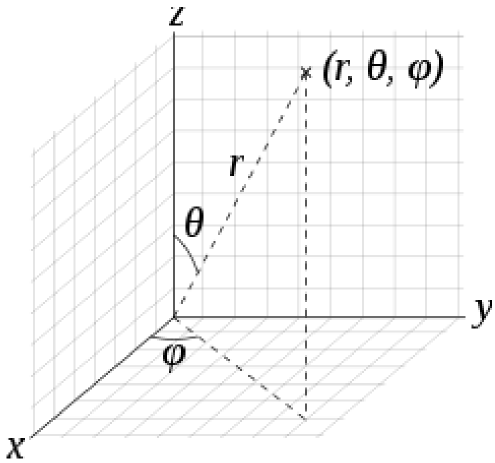


4. Cylindrical: A Cylindrical coordinate is a three dimensional coordinate system that specifies points position by the distance from a chosen references axis, the direction from the axis relative to a chosen reference direction and the distance from a chosen reference plane perpendicular to the axis. Cylindrical coordinate are useful in connection with objects and phenomenum

that have some rotational symmetry about the longitudinal axis , such as water flow in a straight pipe with round cross section, heat distribution in a metal cylinder , electromagnetic fields produced by an electric current in along straight wire, and so on. They are sometimes called Cylindrical polar coordinate and polar cylindrical coordinate.



5. Spherical coordinate System: A Spherical coordinate System is a coordinate System for three dimensional space where the position of a point is specified by three numbers, the radial distance of that point from a fixed origin, its polar angle measured from a fixed direction and the angle of its orthogonal projection on a reference plane that passes through the origin.

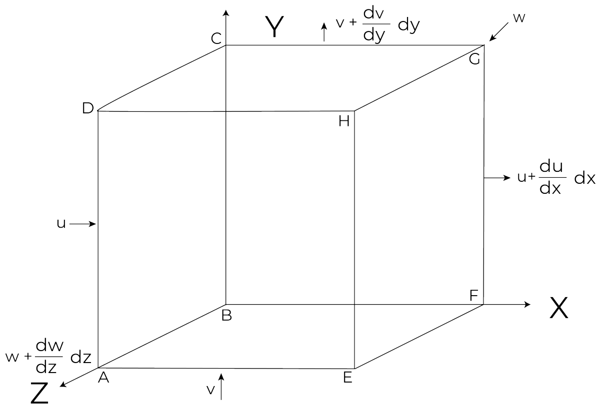


z
(r, θ, φ)
r
θ
y
φ
x

Chapter 3

Derivation Of Equation Of Motion in Coordinate System

3.1 Deriving continuity Equation in Cartesian Coordinate.



Mass rate entering through $ABCD$ in the diagram relating with time $=$ $\rho u \partial y \partial z - - - - - - - - - - - - - (1)$

Mass rate existing through $EFGH = \rho(u + \dfrac{\partial u}{\partial x} dx)\partial y \partial z - - - - - - - - - - - (2)$

Net Mass rate in X direction

equation(2) - equation (1)

$$
\begin{aligned}
\rho(u + \frac{\partial u}{\partial x}dx)\partial y \partial z - \rho u \, \partial y \, \partial z &= \rho \frac{\partial u}{\partial x} dx \, \partial y \, \partial z \\
&= \rho \frac{\partial u}{\partial x} du
\end{aligned}
$$

Y-Direction

Doing the Same Production for Y-Direction we have Net Mass rate in Y-Direction as $\rho \frac{\partial v}{\partial y} dv$

Z-Direction

Doing the Same Production for Z-Direction we have Net Mass rate in Z-Direction as $\rho \frac{\partial w}{\partial z} dw$

Therefore Net rate of mass leaving in the control volume = 0

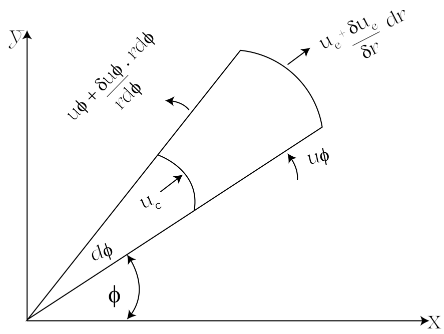
where control volume = X-direction + Y-direction + Z-direction

$$\rho \frac{\partial u}{\partial x} du + \rho \frac{\partial v}{\partial y} dv + \rho \frac{\partial w}{\partial z} dw = 0$$

For cases of incompressible fluid $\rho =$ constant, we get the continuity equation in Cartesian coordinate.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

3.1.1 Continuity Equation in Cylindrical Coordinate (r, ϕ, z)



Mass flow rate entering radially radius (r). $= U_c \times \rho r\ \partial\phi\ \partial z - - - - - - - - - (1)$

Mass flow rate leaving radially radius $(r + \partial r)$

$\left(u_c + \dfrac{\partial u_c}{\partial r}dr\right) \times \rho(r + \partial r)\partial\phi\ \partial z - - - (2)$

$= u_c\ \rho(r + \partial r)\partial\phi\ \partial z + \dfrac{\partial u_c}{\partial r}dr \times \rho(r + \partial r)\partial\phi\ \partial z$

$= \rho\ u_c\ r\ \partial\phi\ \partial z + \rho\ u_c\ \partial r\ \partial\phi\ \partial z + \dfrac{\partial u_c}{\partial r}\partial r\ \partial\phi\ \partial z\ \rho\ r + \dfrac{\partial u_c}{\partial r}dr\ \partial r\ \partial\phi\ \rho$

Because $\dfrac{\partial u_c}{\partial r}dr\ \partial r\ \partial\phi\ \rho$ is small, we neglect its cause $0.1 \times 0.1 = 0.01$ is smaller

Net mass rate in the radial direction = eqn(2) - eqn(1)

$u_c\ \rho\ \partial r\ \partial\phi\ \partial z + \dfrac{\partial u_c}{\partial r}\rho\ r\ \partial\phi\ \partial r\ \partial z$

$$\text{Since } \partial\phi\partial r\partial z = \partial v$$

$$= u_c\ \rho\ \partial v + \frac{\partial u_c}{\partial r}\rho\ r\ \partial v$$

$$= \rho\, \partial v \left(u_c + \frac{\partial u_c}{\partial r} \cdot r \right)$$

Similarly for ϕ direction

$$\left\{ u\phi + \frac{\partial u\phi}{r\,\partial\phi} r\,\partial\phi \right\} \times \rho\,\partial r\,\partial z - u\phi\,\rho\,\partial r\,\partial z$$

$$= \frac{1}{r}\frac{\partial u\phi}{\partial\phi}\,\rho\partial v$$

Similarly for Z direction

$$\left\{ u_z + \frac{\partial u_z}{\partial z}\partial z \right\}\ \rho\,\partial r\,\partial\phi - u_z\,\rho\,\partial r \times r \times \partial\phi$$

$$= u_z\rho\,\partial r\,\partial\phi + \frac{\partial u_z}{\partial z}\partial z\,\rho\,\partial r \cdot \partial\phi - u_z\rho\,\partial r\,\partial\phi$$

$$= \frac{\partial u_z}{\partial z}\,\rho \cdot \partial r \cdot \partial z \cdot \partial\phi$$

Since $\partial r \cdot \partial z \cdot \partial\phi = \partial v\ \frac{\partial u_z\ z}{\partial\phi}\ \rho\partial v$

$= \frac{\partial u_z}{\partial z}\rho\partial v$ Therefore, Net efflux across the cylindrical control volume is zero i.e $(r + \phi + z)$ direction $= 0$

$$\left\{ \frac{u_c}{r} + \frac{\delta u_c}{\delta r} \right\} + \frac{1}{r}\frac{\delta u\phi}{\delta\phi} \cdot \rho\delta v + \frac{\delta u_z}{\delta z}\ rho\delta v = 0$$

$$\frac{u_c}{r} + \frac{\delta u_c}{\delta r} + \frac{1}{r}\frac{\delta u\phi}{\delta\phi} + \frac{\delta u_z}{\delta z} = 0$$

.

3.1.2 DEFINITION OF THE CONTINUITY EQUATION IN SPHERICAL COORDINATION

Selecting a spherical volume dv. This is given as

$$dv = r^2 sin\theta dr d\theta d\phi$$

Where r, θ, ϕ stands for the radius, polar and azimuthal angles respectively , the azimuthal is also reffered to as co-latitude angle.

The differential mass is

$$dM = \rho r^2 sin\theta dr d\theta d\phi$$

The velocity field is represented as

$U = U_{er} + V_{e\theta} + W_{e\theta}$ in an eulerian reference frame mass conservation is represented by accumulation,net flow and source term in a central volume.

Accumulation is given as the time rate of change of mass

We therefore have

$$\frac{\partial pr^2}{\partial t} sin\theta dr d\theta d\phi$$

The net flow through the control volume can be divided into that corresponding to each direction.

3.1.3 Radial Flow,(r)

We have Min $=\rho$U A_{in} the in flow are A_{in} is a trapezoidal whose area is given by

$$Ain = \frac{1}{2}\left[r sin\theta d\phi + r sin(\theta + d\theta) d\phi\right] r d\theta$$

$$\Rightarrow \sin(\theta + d\theta) = \sin\theta \cos d\theta + \cos\theta \sin d\theta \approx r^2 \sin\theta \cos\theta d\theta$$

Substituting in A_{in} yields

$$A_{in} = r^2 sin\theta d\theta d\phi + \frac{1}{2} r^2 cos\theta d^2\theta d\phi \approx r^2 sin\theta d\theta d\phi$$

The out flow in radial direction is

$$M_{out} = \left(\rho U + \frac{\partial \rho U}{\partial r} dr\right) A_{out}$$

But ,A_{out}=midsegement $\times$ height

where mid Segment $=\frac{1}{2}\Bigg[(r+dr)sin\theta d\phi + (r+dr)sin(\theta+d\theta)d\phi\Bigg]$

Height=$(r+dr)d\theta$

$$A_{out} = r^2 sin\theta d\theta d\phi + 2rsin\theta drd\theta d\phi$$

The net flow in the radial direction is given by

$$M_{out} - M_{in} = 2\rho Ursin\theta drd\theta + \frac{\partial \rho U}{\partial r} r^2 sin\theta drd\theta d\phi$$

3.1.4 Polar Flow (θ)

The inflow in the Polar direction is

$$M_{in} = \rho V A_{in}$$

Where $A_{in} = rsin\theta drd\phi$

$$A_{out} = \frac{1}{2}\Bigg[rsin(\theta+d\theta)d\phi + (r+dr)sin(\theta+d\theta)d\theta\Bigg]dr$$

$$A_{out} \approx rcos\theta drd\theta d\phi + rsin\theta drd\phi$$

Finally, the net flow in the polar direction is

$$M_{out} - M_{in} = \rho V_r cos\theta drd\theta + \frac{\partial \rho V}{\partial \theta} rsin\theta drdr\theta\phi$$

3.1.5 Azimuthal Flow (ϕ)

The inflow in the azimuthal direction is given by as

M_{in}= ρ W A_{in}

with A_{in} = r dr d θ

The outflow in the azimuthal direction is given by as

$$M_{out} = (\rho W + \frac{\partial \rho W}{\partial \phi})A_{out}$$

and $A_{out} = rdrd\theta$

The net flow in polar direction is

$$M_{out} - M_{in} = \frac{\partial \rho W rdr}{\partial \phi} A_{out} d\theta d\phi$$

Now we have the continuity equation

$$\frac{\partial \rho}{\partial t} dv + 2\rho U \frac{d}{r} + \frac{\partial \rho u}{\partial r} + \rho V cos\theta \frac{dv}{rsin\theta} + \frac{\partial \rho V}{\partial d\theta}\frac{dv}{r} + \frac{\partial \rho W}{\partial \phi}\frac{dv}{rsin\theta} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{2}\frac{\partial \rho r^2 u}{\partial r} + \frac{1}{rsin\theta}\frac{\partial \rho v sin\theta}{\partial \theta} + \frac{1}{rsin\theta}\frac{\partial \rho W}{\partial \phi} = 0$$

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{1}{r^2}\frac{\partial r^2 u}{\partial r} + \frac{1}{rsin\theta}\frac{\partial v sin\theta}{\partial \theta} + \frac{1}{rsin\theta}\frac{\partial w}{\partial \phi}\right) = 0$$

3.2 Derivation Of Navier stokes Equation

The Navier stoke equation are the fundamental partial differential equation that describe the flow of incompressible fluids.

Using the rate of stress and rate of strain tensors. It can be shown that the components F_j of a viscous force F on a non -rotating frame are given by

$$\frac{F_i}{V} = \frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} + \lambda \partial_{ij}\right)\nabla.u\right] \qquad (1)$$

$$= \frac{\partial}{\partial x_j}\left[\eta(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} - \frac{2}{3}\partial_{ij}\nabla.u\right] + \mu_\beta \partial_{ij}\nabla.u \qquad (2)$$

When η is the dynamics- viscosity. viscosity, λ is the second viscosity,∂_{ij} is the Kroneeker delta,∇ .u is the divergence,μ β is the bulk viscosity.

Now for an incompressible fluid, the divergrnce $\sum \nabla.u = 0$, so then the λ terms drop out. Taking the *eta* to be constant in spae and writing the reminder of (2) in vector form gives

$$\frac{F_{viscous}}{V} = \eta \nabla^2 u, \qquad (3)$$

Where $\nabla^2 u$ is the vector Laplacian.

There are two additional force acting on fluid parcels, namely

the pressure force.

$$\frac{F_{pressure}}{V} = -\nabla^2.P_1 \qquad (4)$$

where P is the pressure, and so called body force

$$F = \frac{F_{body}}{V} \qquad (5)$$

Adding the three forces (3),(4),(5) & equating them to Newton's Law of fluids yields the equation

$$\rho\frac{\partial u}{\partial t} + \rho u.\nabla u = -\nabla_p + \eta \nabla^2 u + F \qquad (6)$$

and diving through by the density ρ gives

$$\frac{\partial u}{\partial t} + u.\nabla u = \frac{-\nabla p}{\rho} + \frac{\eta \nabla^2 u}{\rho} + \frac{F}{\rho}$$

where Kinematic viscosity is $V = \frac{\eta}{\rho}$

$$\frac{\partial u}{\partial t} + u.\nabla u = \frac{-\nabla p}{\rho} + V\nabla^2 u + \frac{F}{\rho} \qquad (7) \qquad (3.1)$$

The Vector equation (7) are the Navier. Stokes Equation.

Now consider the irrotational Navier Stokes equation in Particular co-ordinate systems

(1) In Cartesian co-ordinate within the components of the velocity vector given by u=(u,v,w), the continuity equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and the Navier- Stokes are given by

$$\rho\left(\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x$$

$$\rho\left(\frac{\partial v}{\partial t} + \mu\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y$$

$$\rho\left(\frac{\partial w}{\partial t} + \mu\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z$$

(2) In Cylindrical Co-ordinate with the components of the velocity vector given by $U = (U_r, U_\theta, U_z)$. the continuity equation is

$$\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{1}{r}\frac{\partial u\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0,$$

and the Navier- stokes equation are given by

$$\rho\left(\frac{\partial u_r}{\partial t} + U_r\frac{\partial u_r}{\partial r} + \frac{U\phi}{r}\frac{\partial u_r}{\partial \phi} + U_z\frac{\partial u_r}{\partial z} - \frac{U^2\phi}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{U_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 U_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right) + F_r$$

$$\rho\left(\frac{\partial u_\phi}{\partial t} + U_r\frac{\partial u_\phi}{\partial r} + \frac{U_r U\phi}{r}\frac{U_\phi \partial u_\phi}{\partial \phi} + U_z\frac{\partial u_\phi}{\partial z}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \phi} + \eta\left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{U_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial U_\phi}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right) + F_\phi$$

$$\rho\left(\frac{\partial u_z}{\partial t} + U_r\frac{\partial u_z}{\partial r} + \frac{U\phi}{r}\frac{\partial u_z}{\partial \phi} + U_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + F_z$$

The Spherical co-ordinate with the compound of the velocity vector gives by $U = (U_r, U_\theta, U_\phi)$ the continuity equation

$$\frac{\partial U_r}{\partial r} + \frac{2U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_\theta cot\theta}{r} + \frac{1}{rsin\phi}\frac{\partial U_\phi}{\partial \phi} = 0$$

and the Navier - Stoke equation are given by

$$\rho\left(\frac{\partial u_r}{\partial t} + U_r\frac{\partial u_r}{\partial r} + \frac{U\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{U_\phi}{rsin\theta}\frac{\partial u_r}{\partial \phi} - \frac{U^2\theta}{r} - \frac{U^2\theta}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2U_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi} - \frac{2u_\theta}{r^2 sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + F_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial \theta} + U_r\frac{\partial u_\theta}{\partial r} + \frac{U_r U\theta}{r} + \frac{U_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{U_\phi}{rsin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{U^2\theta cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\eta\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} - \frac{U_\theta}{r^2 sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{cot\theta}{r^2 sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + F_\theta$$

$$\rho\left(\frac{\partial u_\phi}{\partial t} + U_r\frac{\partial u_\phi}{\partial r} + \frac{U_r U\phi}{r} + \frac{U_\phi}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{U_\theta cot\theta}{r} + \frac{U_\phi}{rsin\theta}\frac{\partial u_\phi}{\partial \phi}\right) = -\frac{1}{rsin\theta}\frac{\partial p}{\partial \phi}$$
$$+\eta\left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial u_\phi}{\partial r} - \frac{U_\phi}{r^2 sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial u_\phi}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{2}{r^2 sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2cot\theta}{r^2 sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + F_\phi$$

The Navier-Stokes equation with no body force $(i.eF =)$

$$i.e\rho\frac{\partial u}{\partial t} + \rho u.\nabla u. - \nabla p + \eta\nabla^2 u.$$

Chapter 4

CONVERSION OF EQUATION OF MOTION FROM CARTESIAN COORDINATE TO SPHERICAL

4.1 INTRODUCTION

We shall be dealing with the conversion process of the continuity equation and the Navier-Stokes equation from cartesian coordinate to spherical coordinate, that is from (X, Y, Z) into (r, θ, ϕ).

4.2 CONVERSION OF THE CONTINUITY EQUATION FROM CARTESIAN TO SPHERICAL COORDINATE

The continuity equation in cartesian coordinate can be expressed as

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

Conventionally, the equation of continuity can be converted from the cartesian coordinate to the spherical coordinate by expressing (X, Y, Z) as (r, θ, ϕ)

Then

$X = r \sin\theta \cos\phi \quad r = \sqrt{x^2 + y^2 + z^2}$

$Y = r \sin\theta \sin\phi \quad r = ar\cos(\frac{z}{r})$

$Z = r\cos\theta \quad \phi = \arctan(\frac{y}{z})$

$P = r\sin\theta$

Now the position unit vector in the spherical coordinate is given as

$r = Z\cos\theta + v\sin\theta$

Where $v = x\cos\phi + y\sin\phi$

$$\begin{aligned}
\frac{dr}{d\theta} &= -Z\sin\theta + v\cos\theta && (4.1)\\
\frac{dv}{d\phi} &= x\sin\phi + y\cos\phi && (4.2)\\
V_r &= V_z\cos\theta + V_z\sin\theta\cos\phi + V_y\sin\theta\sin\phi && (4.3)\\
V_\theta &= \frac{d\theta}{dt} = -\frac{dz}{dt}\sin\theta + \frac{dx}{dt}\cos\phi\cos\theta + \frac{dy}{dt}\sin\phi\cos\theta && (4.4)\\
V_\theta &= -V_z\sin\theta + V_x\cos\phi\cos\theta + V_y\sin\phi\sin\theta && (4.5)
\end{aligned}$$

$dsp\frac{d\phi}{dt} = -\frac{dx}{dt}\sin\phi + \frac{dy}{dt}\cos\phi$

$$V_\phi = -V_x\sin\phi + V_y\cos\phi \qquad (4.6)$$

From **(4.3)** and **(4.4)** above we have

$$\begin{aligned}
V_r &= V_z\cos\theta + \sin\theta(V_x\cos\phi + V_y\sin\phi) && (4.7)\\
V_\theta &= -V_{\sin\theta} + \cos\theta(V_z\cos\phi + V_y\sin\phi) && (4.8)\\
V_\phi &= -V_x\sin\theta + V_y\cos\phi && (4.9)
\end{aligned}$$

Multiplying equation **(4.7)** and **(4.8)** through by $\cos\theta$ and $\sin\theta$ respectively, then we have

$$\begin{aligned}
V_r\cos\theta &= V_z\cos^2\theta + \cos\theta\sin\theta(V_x\cos\phi + V_y\sin\phi)\\
V_\theta\sin\theta &= -V_z\sin^2\theta + \sin\theta\cos\theta(V_x\cos\phi + V_y\sin\phi)
\end{aligned}$$

Now,

$$\begin{aligned}
V_r \cos\theta - V_\theta \sin\theta &= V_z \cos^2\theta + V_z \sin^2\theta \\
\implies V_r \cos\theta - V_\theta \sin\theta &= V_z(\cos^2\theta + \sin^2\theta) \\
\implies V_z &= V_r \cos\theta + V_\theta \sin\theta
\end{aligned}$$

Multiplying equation **(4.3)** and **(4.4)** by $\sin\theta$ respectively, to have $V_r \sin\theta = V_z \cos\theta \sin\theta + \sin^2\theta(V_z \cos\phi + V_y \sin\phi)$
$V_\theta \cos\theta = -V_z \sin\theta \cos\theta + \sin^2\theta(V_x \cos\phi + V_y \sin\phi)$

Now,

$$V_r \sin\theta + V_t heta \cos\theta = 2\sin^2\theta(V_x \cos\phi + V_y \sin\phi) \tag{4.10}$$

$\implies V_r \sin\theta + V_t heta \cos\theta = V_x \cos\phi + V_y \sin\phi$

Multiplying equation **(4.10)** and **(4.6)** by $\sin\phi$ and $\cos\phi$ respectively then we have

$V_r \sin\theta \sin\phi + V_\theta \sin\phi \cos\theta = V_x \sin\theta \cos\phi + V_y \sin^2\phi$

$V_\phi \cos\phi = -V_x \sin\theta \cos\phi + V_y \cos^2\phi$

$= V_r \sin\theta \sin\phi + V_\theta \cos\phi \cos\theta = V_y(\sin^2\phi + \cos^2\phi)$

$$\implies V_y = V_r \sin\theta \sin\phi + V_\theta \cos\phi + V_\theta \sin\phi \cos\theta \tag{4.11}$$

Multiplying equation **(4.10)** and **(4.6)** by $\cos\phi$ and $\sin\phi$ respectively then we have

$V_r \sin\theta \cos\phi + V_\theta \cos\theta \cos\phi = V_x \cos^2\theta + V_y \sin\phi \cos\phi$

$V_\phi \cos\phi = -V_x \sin^2\phi + V_y \cos\phi \sin\phi$

$$\implies V_x = V_r \sin\theta \cos\phi + V_\theta \cos\phi \sin\theta - V_\phi \sin\phi \tag{4.12}$$

By the general orthogonal curvilinear coordinates (U_1, U_2, U_3)

$$\nabla \cdot A = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial U_1}(h_1 h_2 A_1) + \frac{\partial}{\partial U_2}(h_1 h_2 A_2) + \frac{\partial}{\partial U_3}(h_1 h_2 A - 3)\right]$$

Where the scalar factors are given as

$h_1 = 1, h_2 = r, h_3 = r\sin\theta$

$U_1 = r, U_2 = 0, U_3 = \phi$

$A = V = V_r x + V_\theta y + V_\phi$

Where,

$$\nabla \cdot V = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta V_r) + \frac{\partial}{\partial\theta}(r\sin\theta V_\theta) + \frac{\partial}{\partial\phi}(rV_\phi)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[2r\sin\theta V_r + r^2\frac{\partial}{\partial r}\sin\theta V_r + r\cos\theta V\theta + \frac{\partial}{\partial\theta}r\sin\theta V_\theta + \frac{\partial}{\partial\phi}rV_\phi\right]$$

$$= \frac{2}{r}V_r + \frac{\partial}{\partial r}V_r + \frac{\cos\theta}{r\sin\theta}V_\theta + \frac{1}{r}(2V_r + \cot\theta V_\theta)$$

$$= \frac{\partial}{\partial r}V_r + \frac{1}{r}\frac{\partial}{\partial\theta}V_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r}(2V_r + \cot\theta V_\theta)$$

Here the continuity equation in spherical coordinate is expressed as

$$\nabla \cdot V = \frac{\partial}{\partial r}V_r + \frac{1}{r}\frac{\partial}{r\partial\theta} + \frac{\partial}{\partial\phi} + \frac{1}{r}(2V_r + \cot\theta V_\theta)$$

4.3 CONVERSION OF NAVIER-STOKES EQUATION FROM CARTESIAN TO SPHERICAL COORDINATE

The Navier-stokes equation can be in cartesian form generally as

$$\rho\left[\frac{\partial V_i}{\partial t} + V_i\frac{\partial V_j}{\partial x_i}\right] = \rho F_1 - \frac{\partial\rho}{\partial x_i} + \frac{\partial\tau_{ij}}{\partial x_j}. \tag{4.13}$$

or $\rho\left[\frac{\partial V_i}{\partial t}\nabla\cdot V_jV_i\right]=\rho F_i-\nabla P+\mu\nabla^2V_i$

$$\frac{\partial V_i}{\partial t}+\nabla\cdot VV_i=F_i-\frac{\nabla P}{\rho}+\upsilon\nabla^2\Omega \tag{4.14}$$

since $\upsilon=\frac{\mu}{\rho}$ By the general orthogonal coordinate (U_1,U_2,U_3)

$$\nabla^2=\frac{1}{h_1,h_2,h_3}\left[\frac{\partial}{\partial U_1}\left(\frac{h_1h_2}{h_1}\frac{\partial}{\partial U_2}\right)+\frac{\partial}{\partial U_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial U_2}\right)+\frac{\partial}{\partial U_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial U_3}\right)\right] \tag{4.15}$$

Where the scalar factors are

$h_1=1,h_2=r,h_3=r\sin\theta$

$U_1=r,U_2=\theta,U_3=\phi$

$$\nabla^2=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}+\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\pi}\left(\frac{r}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right] \tag{4.16}$$

$$=\frac{1}{r^2\sin\theta}\left[2r\sin\theta\frac{\partial}{\partial r}+r^2\sin\theta\frac{\partial}{\partial r^2}+\frac{r\cos\theta}{r}\frac{\partial}{\partial\theta}+\frac{r\sin\theta}{r}\frac{\partial^2}{\partial\theta^2}+\frac{r}{r\sin\theta}\frac{\partial^2}{\partial\theta^2}\right]$$

$$=\frac{2}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}$$

$$=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2}+\frac{1}{r}\left[2\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right]$$

$$\nabla\cdot V=\frac{2}{r}V_r+\frac{\partial}{\partial r}V_r+\frac{1}{r}\cot\theta V_\theta+\frac{1}{r}\frac{\partial}{\partial\theta}V_\theta+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_\theta$$

$$\nabla\cdot P=\left[\frac{2}{r}+\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\cdot P$$

Substituting $\nabla^2,\nabla\cdot P,\nabla\cdot V$ into the equation **(4.14)** where V_i has its velocity component as V_r,V_θ,V_ϕ of spherical coordinate.

In the r component

$$\frac{\partial V_r}{\partial r} + V_r\frac{2V_r}{r} + V_r\frac{\partial V_r}{\partial r} + \frac{V_r}{r}\cot\theta V_\theta + \frac{V_r}{r} + \frac{\partial}{\partial\theta}V_\theta + \frac{V_r}{r\sin\theta}\frac{\partial}{\partial\phi}V_\phi$$

$$= g_r - \frac{1}{\rho}\left(\frac{2}{r} + \frac{\partial}{\partial r}\right)P_r + \upsilon\left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{V_r}{r}\left(\frac{2\partial}{\partial r} + \frac{1}{r}\cot\theta\frac{\partial}{\partial\theta}\right)\right] + g_r \tag{4.17}$$

In the θ component

$$\frac{\partial V_\theta}{\partial t} + V_\theta\frac{2V_r}{r} + V_\theta\frac{\partial V_r}{\partial r} + \frac{V_\theta}{\partial r}\cot\theta V_\theta + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_\theta}{r\sin\theta}\frac{\partial}{\partial\theta}V_\phi$$

$$= -\frac{1}{\rho}\left(\frac{1}{r}\cot\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\right)P_\theta + \upsilon\left[\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial\theta^2} + \frac{1}{r\sin\theta}\frac{\partial^2 V_\theta}{\partial\phi^2} + \frac{V_\theta}{r}\left(\frac{2\partial}{\partial r} + \frac{1}{r}\cot\theta\frac{\partial}{\partial\theta}\right)\right] + g_\theta \tag{4.18}$$

In the ϕ component

$$\frac{\partial V_\phi}{\partial t} + V_\theta\frac{2}{r}V_r + V_\phi\frac{\partial}{\partial r} + \frac{V_\phi}{\partial r}\cot\theta V_\theta + \frac{V_\phi}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}V_\phi$$

$$= -\frac{1}{\rho}\left(\frac{1}{r}\cot\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\right)P_\theta + \upsilon\left[\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\phi}{\partial\theta^2} + \frac{1}{r\sin\theta}\frac{\partial^2 V_\phi}{\partial\phi^2} + \frac{V_\phi}{r}\left(\frac{2\partial}{\partial r} + \frac{1}{r}\cot\theta\frac{\partial}{\partial\theta}\right)\right] + g_\phi \tag{4.19}$$

Chapter 5

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 SUMMARY

From the study of this project, FLuids were defined and explained down to its types and properties,Coordinate system were defined and also explained down to its properties and also equation of motion is explained and talked about and also equation of motion were derived and have been converted from its cartesian to spherical coordinate and from cylindrical to spherical coordinate in three dimensional coordinate. Considering the Navier-stokes equation and the Continuity equation which were derived from the conservation of momentum and the conservation of mass respectively.

These equations can be expressed as

$$\frac{\Delta\rho}{\Delta t} + \rho V \cdot \overrightarrow{V} = 0$$

Which is the continuity equation and

$\rho\left(\frac{\Delta\overrightarrow{v}}{\Delta t}\right) = \rho\overrightarrow{F_i} - \nabla\overrightarrow{P} + \mu\nabla^2\overrightarrow{V}$ respectively

5.2 CONCLUSION

The fundamental equations of motion were converted from cartesian coordinate to spherical coordinate and this was possible by applying the general orthogonal curvilinear coordinate, which has quite shown the relationship between cartesian and spherical coordinate in three dimension flow.
In this study, we have employed the general orthogonal curvillinear coordinate systems to be able to convert the continuity equation and Navier-Stokes equation from cartesian coordinate to spherical coordinate.

5.3 RECOMMENDATION

It is hereby recommended that further study should be focused on how to use the general orthogonal curvillinear coordinate to solve the conversion process of the continuity and Navier-Stokes equation from cartesian coordinate to cyclindrical coordinate, that is from (X, Y, Z) into (r, θ, z).

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