COMPLEX ANALYSIS

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CERTIFICATION

This is to certify that this project was carried out by **ABIFARIN, Nafisat Busayo** with Matriculation Number 18/56EB111 in the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria, for the award of Bachelor of Science (B.Sc.) degree in Mathematics.

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ACKNOWLEDGMENTS

All praises, adoration and glorification are for Almighty Allah, the most beneficent, the most merciful, and the sustainer of the world. I pray may the peace of Allah and His blessings be upon the Noble Prophet(the last of all prophets), his companions, household and the entire Muslims. I give gratitude to Allah for His mercies and grace bestowed on me over the years vis-a-vis sparing my life from the beginning to the end of my course in the "Better by Far" University.

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DEDICATION

I would like to dedicate the project to God, for the grace and faithfulness of God thus far. For His mercies, guidance and protection throughout my years of study.

ABSTRACT

This project deals with

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Chapter 1

INTRODUCTION

1.1 Complex Analysis

Complex analysis, traditionally known as the theory of functions of complex variable is the branch of mathematical analysis that investigates functions of complex numbers together with their derivatives.

As a differentiable function of a complex variable is equal to its Taylor Series (that is, it is analytic). Complex analysis is particularly concerned with analytic functions of a complex variable(that is, holomorphic functions).

1.2 Historical Background of Complex Analysis

1.2.1 Introduction

In 18th Century, a reaching generalization of analysis was discovered, centred on the imaginary number of $i = \sqrt{-1}$. The name imaginary arises because squares of real numbers are always positive. Positive numbers have two distinct squares roots - one positive and one negative; zero has a single square root which is zero and negative have no real square roots at all. Numbers formed by combining real and imaginary components such as $2 + 3i$ are said to be complex.

1.2.2 Why do we study Complex Analysis?

Complex analysis serves as an effective capstone course for mathematics major and as a stepping stone to the pursuit of higher mathematics in graduate school.

Nearly any number we can think of is a real number. So what happens when we square a number and its gives a negative result, normally, this does not happen because:

1. When we square a positive number, we get a positive result.
2. When we square a negative number we get a positive result also. Example, $-3x - 3 = +9$.

Moving on to imaginary numbers, the unit for imaginary number is i, which is the square root of -1

$$i = \sqrt{-1} \tag{1.1}$$

Because when we square i, we get -1

$$i^2 = -1 \tag{1.2}$$

Ordinarily, equation of this form $a^2 + 9 = 0$ cannot be solved since our $a = \sqrt{-9}$ which does not have a solution. But with complex analysis, it does have a solution.

$$\begin{aligned} a^2 + 9 &= 0 \\ a^2 &= -9 \\ a &= \sqrt{-9} \\ \text{Since } i^2 &= -1 \\ a &= \sqrt{9i^2} \\ a &= \pm 3i \end{aligned}$$

1.2.3 Leonhard Euler

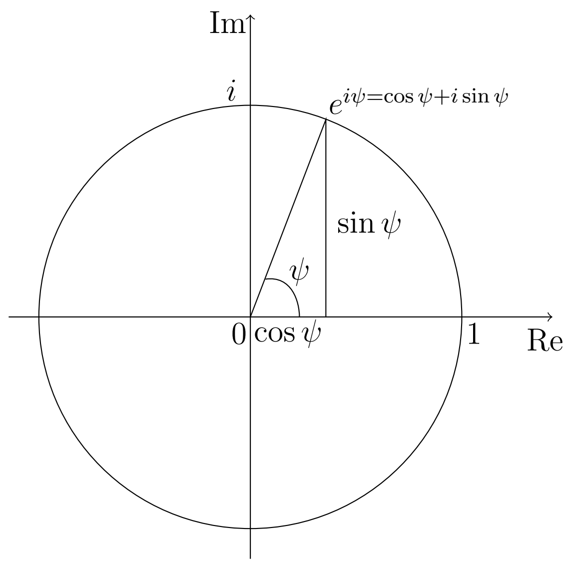
Leonhard Euler is the 18th-century Swiss mathematician and is among the most successful mathematicians in history.
Euler made important contributions to Complex analysis and he introduced the scientific notation. He discovered what is known now as **Euler's formular**, that for any real number ψ, the complex exponential equation satisfies

$$e^{i\psi} = \cos\psi + i\sin\psi \tag{1.3}$$

An this has been called the most remarkable formular in mathematics. Euler's identity is a special case of this:

$$e^{i\pi} + 1 = 0 \tag{1.4}$$

This identity is remarkable as it involves $e, \pi, i, 1$ and 0 and they are arguably the five most important constant in mathematics



1.2.4 Georg Friedrich Bernhard Riemann

He was a German mathematician born in 1826. His contribution to complex analysis include most notably, the introduction of Riemann surfaces breaking new ground in a natural, geometric treatment of complex analysis.

1.2.5 Augustin-Louis Cauchy

He was a French mathematician born in 1789. He made pioneering contributions to several branches of mathematics. Cauchy is most famous for his single-handed development of complex function theory. The first pivotal theorem proved by Cauchy, now known as Cauchy's Integral Theorem, is the following

$$\oint_c f(z)dz = 0 \tag{1.5}$$

where $f(z)$ is a complex valued function on and within the non-self closed curve c(contour) lying in the complex plane. The contour integral is taken along the contour c. In 1826, Cauchy gave a formal definition of a residue of a function. If the complex-valued function $f(z)$ can be expanded in the neighbourhood of a singularity as

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z) \tag{1.6}$$

In 1831, Cauchy proposed the formula known as Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz \tag{1.7}$$

where $f(z)$ is analytic on c and within the region bounded by the contour c and the complex number a is somewhere in this region.

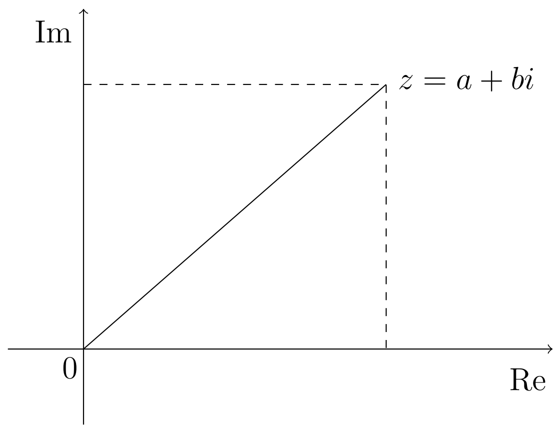
Clearly, the integrand has a simple pole at $z = a$. He also presented the residue theorem.

$$\frac{1}{2\pi i} \oint_c f(z)dz = \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z) \tag{1.8}$$

where the sum is over the n poles of $f(z)$ and within the contour c. These results of Cauchy's still form the core of complex function theory.

1.3 Complex Number

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is a symbol called the imaginary unit and satisfying the equation $i^2 = -1$. For the complex number $a + bi$, a is called the real part and b is called the imaginary part. The set of complex number is denoted by $\mathbb{C}$



Addition, Subtraction and Subtraction of complex numbers can be naturally defined by using the rule $i^2 = -1$ combined with the associative, commutative and distributive laws.

1.4 Algebraic Properties of Complex Numbers

1. When $a + ib = 0$ and a, b, c are the real numbers, the value of both $a, b = 0$ that is $a = 0$ and $b = 0$

Proof

$a + ib = 0$

$0 + 1 \cdot 0 = 0$

Hence, the value of a and b is zero.

2. When a, b, c are real numbers and $a + ib = c + id$, then $a = c$ and $b = d$

   Proof

   When a, b, c, d exist as real numbers, then

   $a + ib = c + id$

   $a = c + id - ib$

   $\therefore a = c$

3. For z_1, z_2 and z_3 complex numbers, the set must be satisfying the associative, commutative and distributive laws

   Commutative law for multiplication

   $z_1 \cdot z_2 = z_2 \cdot z_1$

   Associative law for multiplication

   $(z_1 z_2) z_3 = z_1 (z_2 z_3) = z_2 (z_1 z_3)$

   Commutative law for addition

   $z_1 + z_2 = z_2 + z_1$

   Associative law for addition

   $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Distributive law

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

4. The product and the sum of two complex conjugate quantities both exist as "real"

Proof

Let us assume $z = x + iy$ is the complex number where the real values are x and y. Accordingly, the conjugate of Z is equal to $\bar{z} = x - iy$

Now,

$(x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2 = z \cdot \bar{z}$ (multiplication)

which is real

And,

$x + iy + x - iy = z + \bar{z}$ (Addition)

$= 2x$

Hence, multiplication and addition of two complex conjugates are real.

5. If the product and sum of two complex numbers exist as real, then these complex numbers will be conjugate to each other.

Proof

Let us assume $z_1 = a + ib$ and $z_2 = c + id$, these are the two quantities where the real values are a, b, c, d and $b \neq 0$ and $d \neq 0$ also.

So by the theory of assumption, $z_1 + z_2 = a + ib + c + id$

$z_1 + z_2 = a + ib + c + id$

$z_1 + z_2 = (a + c) + i(b + d)$

Therefore, $b + d = 0 \implies d = -b$

And

$$
\begin{aligned}
z_1 \cdot z_2 &= (a + ib)(c + id) \\
&= (ac - bd) + i(ad + bc) \text{ exists for real.}
\end{aligned}
$$

Hence, $ad + bc = 0, -ab + bc = 0$

Therefore,

$ad + bc = 0$ or $-ab + bc = 0$ (since $d = -b$)

Thus,

$z_2 = c + id = a + i(-b) = a - ib = \bar{z}$, which verifies that the values of z_1 and z_2 are conjugates of each other.

Chapter 2

BASIC CONCEPTS IN COMPLEX ANALYSIS

2.1 Preamble

In this chapter, the basic concepts in Complex Analysis shall be discussed.

2.2 Function

Function is process or a relation that associates each element x of a set X, the domain of the function to a single element y of another set Y(possibly the same set), the codomain of the function.

If the function is called f, this relation is denoted by $y = f(x)$ where the element x is the input of the function and y is the value of the function or the image of x by f. A function is represented by the se of all pairs $(x, f(x))$, called the graph of the function. Hence, $f : X \to Y$ is a function such that for $x \in X$, there is a unique element $y \in Y$ such that $(x, y) \in f$.

2.2.1 Types of Functions

1. **Identity function**

   Let $\mathbb{R}$ be the set of real numbers. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = y = x$ for $x \in \mathbb{R}$, then the function is known as Identity function. The domain and the range being $\mathbb{R}$.

2. **Polynomial function**

   A polynomial function is defined by $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$. The highest power in the expression is the degree of the polynomial function. Polynomial functions are further classified based on their degree.

   - **Constant function:** If the degree is zero, polynomial function is a constant function.
   - **Linear function:** The polynomial function with degree one such as $y = x + 1$ or $y = x$ or $y = 2x - 5$.

3. **Constant function**

   If the function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = y = c$, for $x \in \mathbb{R}$ and c is a constant in $\mathbb{R}$, then such function is known as constant function. The domain of the function f is $\mathbb{R}$ and its range is a constant c.

4. **Rational function**

   A rational function is any function which can be represented by a rational fraction, say $\dfrac{f(x)}{g(x)}$ in which numerator $f(x)$ and denominator $g(x)$ are polynomial functions of x, where $g(x) \neq 0$.

5. **Modulus function**

   The absolute value of any number, c is represented in the form of $|c|$. If any function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x|$, is known as modulus function.

6. **Greatest Integer function**

   If a function $f\mathbb{R} \to \mathbb{R}$ is defined by $f(x) = [x], x \in X$. It round off to the integer less than the number. Suppose the given interval is in the form of $(m, m+1)$, the value of the greatest integer is m which is an integer.

7. **Quadratic function**

   If the degree of the polynomial function is two, then it is a quadratic function. It is expressed as $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants and x is a variable.

8. **Algebraic function**

   A function is called an algebraic function if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots.

2.3 Basic Definitions

1. **Function of Complex variable**

   Functions of (x, y) that depend only on the combination $(x + iy)$ are called functions of a complex variable. the function $f(z)$ of a complex variable z denoted by ω such that $z = x + iy$ where x and y presents the real and imaginary parts.

2. **Limits of Functions**

   The limit of a function at a point $''a''$ in its domain (if it exists) is the value that the function approaches as its argument approaches $''a''$. That is $\lim_{x \to a} f(x) = L$.

3. **Limit of function of a Complex Variable**

   Let a function f be defined at all point $z(f(z))$ in some neighborhood of z_0. The $f(z)$ is said to have a limit $''L''$ as z approaches z_0. i.e $\lim_{z \to z_0} f(z) = L \; \forall \; \epsilon > 0$, there exist $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z - z_0| < \delta$.

4. **Continuity**

   Let $f(z)$ be defined and single-valued in a neighborhood of z_0, then $f(z)$ is said to be continuous at $z = z_0$ if for every $\epsilon > 0$, there exist $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.
   For a function to be continuous, these conditions must be satisfied

   (a) $f(z)$ must be defined

   (b) $\lim f(z)$ exist at $z \to z_0$

   (c) $\lim_{z \to z_0} f(z) = f(z_0)$

5. **Differentiability**

   Let $f(z)$ be singled-valued in some region $\mathbb{R}$ of the z-plane and z_0 be any point in the derivative of $f(z)$ and it is defined as

$$f'(z) = \frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} \tag{2.1}$$

If a function satisfies the property at a point z_0, we say that the function is complex differentiable at z_0.

6. **Analyticity**

   A function $f(z)$ is said to be analytic in a region $\mathbb{R}$ of the complex plane if $f(z)$ has a derivative at each pint of $\mathbb{R}$ and if $f(z)$ is single valued. Pints at which a function $f(z)$ is not analytic are called singular points. A necessary condition for $f(z)$ to be analytic is

$$\frac{\partial f}{\partial z} = 0 \tag{2.2}$$

   Therefore, necessary condition for $f = u + iv$ to be analytic is that f depends only on z. In terms of the real and imaginary parts, u, v of f, condition (2.2) is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{2.3}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.4}$$

   Equations (2.3) and (2.4) are known as Cauchy-Riemann equations. They are a necessary condition for $f = u + iv$ to be analytic.

Chapter 3

Chapter 4

We consider the general second order linear defined as

Chapter 5

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