Math 453

Selected Solutions to Assignment 8

Problem 4: Suppose H is a subgroup of G with index 2. Show that H is a normal subgroup.

Solution: Let $a \in G$. If $a \in H$, then $aH = H = Ha$. If $a \in G \setminus H$, since the left cosets of H partition G, and since $a \in aH$, we have that H and aH are disjoint cosets. Since $[G : H] = 2$, we have that $H \cup aH = G$. Hence, we have $aH = G \setminus H$. Since the right cosets of H also partition G, we similarly have $Ha = G \setminus H$. Hence, $aH = Ha$. Since for each $a \in G$, we have $aH = Ha$, H is a normal subgroup of G.

Problem 6: What is the order of the quotient group $\mathbb{Z}_{10} \times \mathbb{Z}_{10}^{\times}/\langle(2,9)\rangle$?

Solution: Note that $|\mathbb{Z}_{10}| = 10$, and $|\mathbb{Z}_{10}^{\times}| = 4$, so $|\mathbb{Z}_{10} \times \mathbb{Z}_{10}^{\times}| = 40$. (Recall that $\mathbb{Z}_{10}^{\times}$ is the subset of integers

$$\{n \in \mathbb{N} \mid 1 \leq n \leq 10,\ \exists k \in \mathbb{N} \text{ s.t. } kn \equiv 1 \bmod 10\},$$

with group operation given by multiplication modulo 10.) Then by direct computation, we have

$$\begin{aligned}
(2,9)^1 &= (2,9) & \qquad (2,9)^6 &= (2,1)\\
(2,9)^2 &= (4,1) & (2,9)^7 &= (4,9)\\
(2,9)^3 &= (6,9) & (2,9)^8 &= (6,1)\\
(2,9)^4 &= (8,1) & (2,9)^9 &= (8,9)\\
(2,9)^5 &= (0,9) & (2,9)^{10} &= (0,1) = e_{\mathbb{Z}_{10}\times\mathbb{Z}_{10}^{\times}}.
\end{aligned}$$

Hence, $|\langle(2,9)\rangle| = 10$, so by Lagrange's Theorem, we have

$$\left|\mathbb{Z}_{10} \times \mathbb{Z}_{10}^{\times}/\langle(2,9)\rangle\right| = \frac{|\mathbb{Z}_{10} \times \mathbb{Z}_{10}^{\times}|}{|\langle(2,9)\rangle|} = \frac{40}{10} = 4.$$