Math 453

Selected Solutions to Assignment 6

Problem 2: Let $\alpha := (\alpha_1 \cdots \alpha_s) \in S_n$ be a cycle, and $\pi \in S_n$. Show that $\pi\alpha\pi^{-1}$ is the cycle $(\pi(\alpha_1) \cdots \pi(\alpha_s))$. Here we think of π as a permutation of $\{1, ..., n\}$. Therefore, it makes sense to consider the elements $\pi(\alpha_i) \in \{1, ..., n\}$.

Solution: Suppose $a \in \{1, ..., n\}$ such that $a = \pi(\alpha_i)$ for some $1 \leq i \leq s$. Then

$$(\pi\alpha\pi^{-1})(\pi(\alpha_i)) = \pi(\alpha(\alpha_i)) = \pi(\alpha_{(i+1 \bmod s)+1}).$$

On the other hand, suppose $a \in \{1, ..., n\}$ such that $a \neq \pi(\alpha_i)$ for all $1 \leq i \leq s$. Since π is a permutation, $a = \pi(b)$ for some $b \in \{1, ..., n\}$. If $\alpha(b) \neq b$, then $b = \alpha_j$ for some $1 \leq j \leq s$, so $a = \pi(b) = \pi(\alpha_j)$, a contradiction. Hence, α fixes b, so

$$(\pi\alpha\pi^{-1})(a) = (\pi(\alpha(b))) = \pi(b) = a,$$

so $\pi\alpha\pi^{-1}$ fixes a. Therefore, by definition of cycle, $\pi\alpha\pi^{-1}$ is the cycle $(\pi(\alpha_1) \cdots \pi(\alpha_s))$.

Problem 3: Given α and π as in the previous exercise, the cycle $\pi\alpha\pi^{-1}$ is called a conjugate of α. We say that a cycle β is a conjugate of α if there is an element π such that $\beta = \pi\alpha\pi^{-1}$. Note that this already implies that there exists a π' such that $\beta = \pi'\alpha\pi'^{-1}$. Therefore, α is a conjugate of β if and only if β is a conjugate of α. In this case, we simply say that α and β are conjugates (or conjugate to one another). Use the last exercise to show that any two cycles of the same length are conjugates of each other.

Solution: Let α, β be cycles as in the previous problem, and suppose they have the same length s. Let $\pi : \{1, ..., n\} \to \{1, ..., n\}$ be defined as follows: if $a = \alpha_i$ for some $1 \leq i \leq s$, $\pi(a) = \beta_i$; and if $a \neq \alpha_i$ for all $1 \leq i \leq s$, then $\pi(a) = a$. Note that this map is well-defined since each element of a cycle appears exactly once in a cycle. Also, note that this map is injective: suppose $\pi(a) = \pi(b)$. If $a = \alpha_i$ and $b = \alpha_j$ for some $1 \leq i, j \leq s$, then $i = j$ by definition of π, so $a = b$; if $a, b \neq \alpha_i$ for all $1 \leq i \leq s$, then again $a = b$ by definition of π; if $a = \alpha_i$ for some $1 \leq i \leq s$ and $b \neq \alpha_j$ for all $1 \leq j \leq s$, then $b = \alpha_i$, a contradiction; and the last case follows similarly to the previous one. Since $\{1, ..., n\}$ is finite, we have that π is surjective, so $\pi \in S_n$. Therefore, by the previous problem, $\beta = (\beta_1 \cdots \beta_s) = (\pi(\alpha_1) \cdots \pi(\alpha_s)) = \pi\alpha\pi^{-1}$, so any two cycles of the same length are conjugates of each other.

Problem 5 Recall, given a group G, its center $Z(G) := \{g \in G | gh = hg \ \forall h \in G\}$. You saw on the midterm that this is a subgroup. Show that $Z(S_n) = \{e\}$ (i.e. it consists only of the identity permutation).

Solution: First, note that there is a mistake in the prompt. The question should restrict n to $n \geq 3$, as otherwise, you may calculate directly that $Z(S_2) = S_2$ (and this group is in fact isomorphic to $\mathbb{Z}_2$).

Now, let $\pi \in S_n$ with $\pi \neq e$, so for some $i \in \{1, ..., n\}$, $\pi(i) = j$ with $i \neq j$. Since $n \geq 3$, we may take $k \in \{1, ..., n\}$ such that $i \neq k \neq j$. Clearly, $e \in Z(S_n)$. Let $\alpha \in S_n$ be the permutation taking j to k, k to j, and fixing all other elements. (See the previous exercise for the reason why this is a well-defined permutation.) Then $(\pi \circ \alpha)(i) = \pi(i) = j$ and $(\alpha \circ \pi)(i) = \alpha(j) = k$; since we assumed $j \neq k$, we have that $\pi \circ \alpha \neq \alpha \circ \pi$, so $\pi \notin Z(S_n)$. Hence, $Z(S_n) = \langle e \rangle$.