Review Problems for Final exam

(1) How many elements of order 8 in the cyclic group of order 640,000?
(2) State the number of generators of $\mathbb{Z}$.
(3) How many different isomorphism classes of abelian groups of order 56? List them all up to isomorphism.
(4) Let G be the subgroup of elements $\pi \in S_5$ such that $\pi(3) = 3$. What is the index (i.e. $[S_5 : G]$) of G in S_5?
(5) What is the center of $\mathbb{Z}_5 \times \mathbb{Z}_{15}$?
(6) How many distinct non-trivial subgroups does $\mathbb{Z}_{12}$ have?
(7) How many distinct non-trivial subgroups does $\mathbb{Z}_3 \times \mathbb{Z}_3$ have?
(8) Is $Aut(\mathbb{Z}_8)$ cyclic?
(9) Let $Z(G)$ denote the center of G. Is $G/Z(G)$ an abelian group?
(10) Is it true that every element of S_n can be written as a product of cycles of length 3?
(11) Same question as (10) but for A_n.
(12) Determine the group S_n/A_n up to isomorphism.
(13) Let $n \geq 2$, and H be a subgroup of S_n such that $|H|$ is odd. Show that H is in fact a subgroup of A_n.
(14) If H is a subgroup of G and N is a normal subgroup of G, show that $H \cap N$ is a normal subgroup of H.
(15) Is it true that the image of a group homomorphism $\phi : G \to G'$ is always a normal subgroup of G'? If yes, then prove it. Otherwise, give an explicit counterexample.
(16) Is the image of a cyclic group under a homomorphism $\phi : G \to G'$ always a normal subgroup of G'?
(17) Let $f : G \to G'$ denote a group homomorphism where G is infinite and G' is finite. Can $ker(f)$ be a finite subgroup of G?
(18) Is it true that all groups of order 9 are abelian?
(19) Are all groups of order 9 cyclic?
(20) Suppose G is cyclic, and H is a subgroup of G. Is it true that both H and G/H are cyclic groups?
(21) Is $\mathbb{Z}_9$ an integral domain?
(22) Suppose R is an integral domain. Is $R \times R$ always an integral domain?
(23) Let $R = \mathbb{Z}[x]$ and let $A \subset R$ be the subset of polynomials in which only even powers of x occur. Is A an ideal?
(24) Is $(2, X)$ a prime ideal in $\mathbb{Z}[X]$? Is it a maximal ideal?
(25) Let $R = \mathbb{R}[x, y]$ and $I = \{f \in R | f(1, 0) = f(0, 1) = 0\}$. Is I an ideal?

Is I a prime ideal? Is I a maximal ideal?
(26) Prove that the only ideal in a field are the zero ideal and the field F itself.
(27) Let R be a ring. An element $e \in R$ is called an idempotent if $e^2 = e$. Let e be an idempotent in R. Show that $f = 1 - e$ is also an idempotent. Show that $ef = 0$. Show that R is isomorphic $R/(e) \times R/(f)$.
(28) Let $f : R \to R'$ denote a ring homomorphism. Is it true that the image of f is an ideal of R'?
(29) Let R be a ring, and I, J denote two ideals in R such that $I + J = R$. Is it true that $IJ = I \cap J$?
(30) Let k be a finite field. Is it true that the number of irreducible polynomials in $k[x]$ is also finite?
(31) Is the polynomial $2x^4 + 2$ irreducible in $\mathbb{Q}[x]$? How about in $\mathbb{Z}[x]$?
(32) Show that $x^4 + 10x^2 + 1$ is irreducible in $\mathbb{Q}[x]$. Show the same for $x^5 + 1$. (Hint: Reduce this modulo p for an appropriate p and use the theorem from class.
(33) What are the possible rational roots of $2x^4 + 13x + 4$?
(34) Show that each of the following are irreducible over $\mathbb{Q}[x]$: $3x^4 - 8x^2 + 6x^2 - 4x + 6$, $x^4/5 - x^3/3 - 2x/3 + 1$. (Hint: Use Eisenstein's criterion.)
(35) If $a(x) \in F[x]$ and $b(x) \in F[x]$ (F is a field) have the same roots in F, are they necessarily associates?
(36) Prove that for any prime p, $x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$ is irreducible.
(37) Give an example of a subring of $\mathbb{Z}_3 \times \mathbb{Z}_3$ which is not an ideal.
(38) Let $f : R \to R'$ be a ring homomorphism. Show that the image of a unit in R under f is a unit in R'.
(39) Show that $p(x, y) = x^2 - 7y^2 - 24 \in \mathbb{Z}[x, y]$ has no integer solutions. (Hint: Suppose a and b are integers such that $p(a, b) = 0$. What can you say about the the solutions of the image of $p(x, y)$ in $\mathbb{Z}_7[x, y]$?)
(40) Prove that $x^2 + 10y^2 = n$ has no integer solutions if $n = 2, 3, 7, 8$. (Hint: Use the same strategy as in (39)).
(41) Determine the number of reducible monic degree two polynomials in $\mathbb{Z}_5[x]$. (Hint: Show that any such polynomial can be written as a product $(x - a)(x - b)$ where $a, b \in \mathbb{Z}_5$.)
(42) Determine the number of reducible degree two polynomials in $\mathbb{Z}_5[x]$. How many irreducible quadratics in $\mathbb{Z}_5[x]$?