Homework 12: Due Thursday, December ?

Reading: Chapters 24, 25

Problem 1: Let I and J denote two ideals in R. Let $I + J := \{r \in R | r = i + j \text{ for some } i \in I, j \in J\}$. Let $IJ := \{r \in R | r = \Sigma_{i=1}^{k} a_i b_i \text{ where } a_i \in I, b_i \in J, k \geq 1\}$.
(a) Show that $I + J$ is an ideal of R.
(b) Show that IJ is an ideal of R.
(c) Show that $I \cap J$ is an ideal of R.

Problem 2: An ideal $I \subset R$ is said to be a *prime* ideal if whenver $ab \in I$ then either $a \in I$ or $b \in I$.
(1) What are the prime ideals in $\mathbb{Z}$?
(2) Show that an ideal $I \subset R$ is a prime ideal if and only if R/I is an integral domain.

Problem 3: An ideal $I \subset R$ is said to be a *maximal* ideal if for any ideal J containing I, either $J = R$ or $J = I$.
(1) Show that $(x) \subset \mathbb{Q}[x]$ is a maximal ideal.
(2) Show that an ideal $I \subset R$ is a maximal ideal if and only if R/I is a field.
(3) Conclude from (2) that any maximal ideal is a prime ideal.
(4) Given an example of an ideal in $\mathbb{Z}[x]$ which is a prime ideal, but not a maximal ideal.

Problem 4: Factor $x^4 - 4$ into irreducible factors over $\mathbb{Q}$, over $\mathbb{R}$, and over $\mathbb{C}$.

Problem 5: Find all irreducible polynomials of degree ≤ 4 in $\mathbb{Z}_2[x]$.

Problem 6: Consider the ring $\mathbb{Z}_{12}[x]$. Show that is is not an integral domain. Let $p(x) = x^2 - 4 \in \mathbb{Z}_{12}[x]$. Show that $p(x)$ has 4 distinct roots (i.e. there are four distinct elements in $\mathbb{Z}_{12}$ such that $p(x)$ vanishes at those points).

Problem 7: Let F be a field, and $J \subset F[x]$ be an ideal. Show that J is a prime ideal if and only if it has an irreducible generator.

Problem 8: In $\mathbb{Z}_6[x]$ factor each of following polynomials into a two polynomials of degree 1: x, $x+2$.