Math 453

Selected Solutions to Assignment 2

Problem 2: Prove that for all positive integers $n > 1$,

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2.$$

Solution: We have $1^2 = 1 < \frac{2^3}{3} = \frac{8}{3} < 5 = 1^2 + 2^2$, so the inequalities hold in the case $n = 2$.

Now, assume that the inequalities hold for some $n \in \mathbb{Z}_{\geq 2}$; then we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n-1)^2 + n^2 < \frac{n^3}{3} + (n)^2 &= \frac{n^3 + 3n^2}{3} \\ &< \frac{n^3 + 3n^2 + 3n + 1}{3} \\ &= \frac{(n+1)^3}{3}, \end{aligned}$$

with the last inequality due to the fact that $n > 1$, so in particular $n > 0$. Thus, we have $1^2 + 2^2 + \cdots + n^2 < \frac{(n+1)^3}{3}$. We also have

$$\begin{aligned} \frac{(n+1)^3}{3} = \frac{n^3 + 3n^2 + 3n + 1}{3} &< 1^2 + 2^2 + \cdots + n^2 + \frac{3n^2 + 3n + 1}{3} \\ &= 1^2 + 2^2 + \cdots + n^2 + n(n+1) + \frac{1}{3} \\ &< 1^2 + 2^2 + \cdots + n^2 + n(n+1) + (n+1) \\ &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2, \end{aligned}$$

again with the last inequality due to the fact that $n > 1$, so in particular $n + 1 > \frac{1}{3}$. Thus, we have $\frac{(n+1)^3}{3} < 1^2 + 2^2 + \cdots + (n+1)^2$. Combining the last two displays, we have

$$1^2 + 2^2 + \cdots + n^2 < \frac{(n+1)^3}{3} < 1^2 + 2^2 + \cdots + n^2 + (n+1)^2,$$

so the inequalities hold for $n + 1$. Therefore, the inequalities hold for all positive integers $n > 1$ by induction on n.

Problem 4: Prove that for integers n, if n^2 is odd, then n is odd.

Solution 1: *(Assuming that every integer is exactly one of even or odd.)*

Let $n \in \mathbb{Z}$ be not odd. Therefore, n is even, so we may write $n = 2k$ for some $k \in \mathbb{Z}$. Then we have $n^2 = (2k)^2 = 2 \cdot (2k^2)$, and since $2k^2 \in \mathbb{Z}$, we have that n^2 is even. Therefore, n^2 is not odd.

The above shows that if $n \in \mathbb{Z}$ is not odd, then n^2 is not odd. Taking the contrapositive, we have that if $n \in \mathbb{Z}$ such that n^2 is odd, then n is odd.

Solution 2: *(Assuming that the product of two even integers is even, that the product of two odd integers is odd, and that the product of an even integer with an odd integer is even.)*

Let $n \in \mathbb{Z}$ be such that n^2 is odd. Then we may write $n^2 = 2k + 1$ for some $k \in \mathbb{Z}$; in particular, we have $n^2 - 1 = 2k$. But then $(n-1)(n+1) = 2k$, so $n - 1$ or $n + 1$ (or both) must be even. If $n - 1$ is even, then $n - 1 = 2l$ for some $l \in \mathbb{Z}$, so $n = 2l + 1$. Since $l \in \mathbb{Z}$, n is odd. If $n + 1$ is even, then $n + 1 = 2l'$ for some $l' \in \mathbb{Z}$. Then $n = 2l' - 1$, so $n = 2(l' - 1) + 1$; since $l' - 1 \in \mathbb{Z}$, n is odd.