Homework 1: Due Tuesday, September 6

Reading: Chapter 2

Problem 1: Prove that for all positive integers $n > 0$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 2: Prove that for all positive integers $n > 1$,

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2.$$

Problem 3: Given a set A, an equivalence relation on A is a subset $K \subset A \times A$ such that the following three properties are satisfied:
(1) (Reflexive) If $a \in A$, then $(a, a) \in K$.
(2) (Symmetric) If $(a, b) \in K$, then $(b, a) \in K$.
(3) (Transitive) If (a, b) and (b, c) are both in K, then $(a, c) \in K$.
In this case, if $(a, b) \in K$, we say that a is equivalent to b and denote it by $a \sim b$. Note by the symmetric property this also means that b is equivalent to a. For each positive integer n and $a, b \in \mathbb{Z}$, we say that two integers a is congruent to b modulo n, denoted by $a \equiv b \bmod (n)$, if $n|(a-b)$. Prove that this is an equivalence relation on $\mathbb{Z}$ (i.e. the set $K = \{(a, b)|a \equiv b \bmod (n)\}$ satisfies the three properties above).

Problem 4: Prove that for integers n, if n^2 is odd, then n is odd.

For the next 4 problems, we are in the following setting. Consider the square with vertices labelled 1, 2, 3, and 4 (top right corner is 1, bottom right is 2, bottom left is 3, and top left is 4). Now let R denote the clockwise rotation which sends 1 to 2, 2 to 3, 3 to 4 and 4 to 1. one can think of this as a function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$. In fact all the symmetries of the square can be thought of as such a function. Let F be the flip which sends 1 to 2, 2 to 1, 3 to 4 and 4 to 3. Let I denote the identity symmetry.

Problem 5: Show that all the rotations preserving the square are given by I, R, $R^2 = R \circ R$ (i.e. composing the corresponding functions), and R^3.

Problem 6: Show that all the flips, including diagonal flips, are given by F, FR, FR^2, and FR^3.

Problem 7: Show that the above 8 rotations and flips is a complete list of all the symmetries of the square. Describe RF in terms of this list.

Problem 8: Determine the inverses of the rotations R, R^2, and R^3 in terms of the above list.