Homework 7: Due Thursday, October 20

Reading: At this point, we have covered most of the following Chapters in the book: Chapters 1-13. If you have not done so already, please read these chapters.

As I mentioned in class, this problem set is meant to be an overall review of the concepts we have discussed thus far.

Of the following list, you only need to submit problems (3, 10, 12, 15, 20). However, I highly recommend the you do all the problems.

Problem 1 Determine $Aut(\mathbb{Z})$.
Problem 2 Let G be a group and $a, b \in G$. Show that the orders of ab and ba are the same.
Problem 3 Let $f : G \to H$ be a homomorphism of groups. Let $ker(f) := \{g \in G | f(g) = e\}$ where $e \in H$ is the identity of H. Show that $ker(f)$ is a subgroup of G. The subgroup $ker(f)$ is called the kernel of f.
Problem 4 Prove or disprove: $(\mathbb{Q}, +, 0)$ is a cyclic group.
Problem 5 Prove or disprove: A group with a finite number of subgroups is finite.
Problem 6: What is the order if $(2, 10) \in \mathbb{R}^{\times} \times \mathbb{Z}_{14}$. Recall, $\mathbb{R}^{\times} = \mathbb{R} \setminus 0$ is a group under multiplication of real numbers with identity given by 1.
Problem 7 Let $(G, *)$ be a group and $a_1, \cdots a_n \in G$. Prove that $(a_1 * a_2 * \cdots * a_n)^{-1} = a_n^{-1} * a_{n-1}^{-1} * \cdots * a_1^{-1}$.
Problem 8 Find all subgroups of $\mathbb{Z}_6$.
Problem 9 Let G be an abelian group. Prove that $H := \{h \in G | |h| < \infty\}$ is a subgroup of G.
Problem 10 Prove that a group G has exactly 3 subgroups if and only if G is cyclic with $|G| = p^2$ for a prime number p.
Problem 11 Let G be a group. Suppose that for each $a, b \in G$ we have that $a^3b^3 = (ab)^3$ and $a^5b^5 = (ab)^5$. Show that G is abelian.
Problem 12 How many elements in S_8 commute with the permutation $(123)(45678)$?
Problem 13 How many elements are there in S_8 with order 15?
Problem 14 Let G be a cyclic group of order 6. How many of its elements generate G? Same question for cyclic groups of order $5, 10$ and 8.
Problem 15 Find all subgroups of S_3. Hint: First use Lagrange's theorem

to determine the possible orders for such a subgroup.

Problem 16 Write the permutation $\alpha = (1235)(2467)(1872)(2946)$ as a product of disjoint cycles? For what n is α a permutation in S_n? What is the inverse of α? What is the order of α? Is α an even or an odd permutation?

Problem 17 Show that $[\mathbb{Q} : \mathbb{Z}]$ is not finite.

Problem 18 What is the order of $(2, 5, 1) \in \mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_3$?

Problem 19 Consider the function $det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ which sends a matrix A to its determinant $det(A)$. Show that this is a homomorphism. Let $SL_n(\mathbb{R}) := ker(det)$. This is exactly the subset of matrices whose determinant is 1. Conclude by Problem 3 that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

Problem 20 Let G be a group and $a \in G$. Consider the map $I_a : G \to G$ given by $I_a(g) := aga^{-1}$. Show that I_a is an automorphism of G. An automorphism $f : G \to G$ of G is called an inner automorphism if $f = I_a$ for some $a \in G$.