Math 453

Selected Solutions to Assignment 9

Problem 4: Are there any (non-trivial) homomorphisms from $\mathbb{Z}_8 \times \mathbb{Z}_2$ to $\mathbb{Z}_4 \times \mathbb{Z}_4$? Explain your answer.

Solution: Yes. There is at least one non-trivial homomorphism: let the map $\phi : \mathbb{Z}_8 \times \mathbb{Z}_2 \to \mathbb{Z}_4 \times \mathbb{Z}_4$ be defined by $\phi((i,j)) = (i \bmod 4, j \cdot 2 \bmod 4)$. Let $(i,j),(k,l) \in \mathbb{Z}_8 \times \mathbb{Z}_2$; then

$$\begin{aligned}
\phi((i,j)+(k,l)) &= \phi((i+j,k+l)) \\
&= ((i+j) \bmod 4, (k+l) \cdot 2 \bmod 4) \\
&= (i \bmod 4 + j \bmod 4, 2k \bmod 4 + 2l \bmod 4) \\
&= (i \bmod 4, 2k \bmod 4) + (j \bmod 4, 2l \bmod 4) \\
&= \phi((i,j)) + \phi((k.l)),
\end{aligned}$$

so ϕ is a homomorphism.

Problem 6: Compute the number of elements of order 2 and order 4 in each of the following groups: $\mathbb{Z}_{16}, \mathbb{Z}_8 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: In $\mathbb{Z}_{16}$, there is one element of order 2, namely 8; there are two elements of order 4, namely 4 and 12.

In $\mathbb{Z}_8 \times \mathbb{Z}_2$, there are three elements of order 2, namely $(4,0),(4,1),(0,1)$; there are four elements of order 4, namely $(2,0),(2,1),(6,0),(6,1)$.

In $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, there are 7 elements of order 2, namely $(2,0,0),(0,2,0)$, $(0,0,2),(2,2,0),(2,0,2),(0,0,2)$, and $(2,2,2)$. Note that each element of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ has order 1, 2, or 4 (why?), so the elements of order 4 are precisely the elements that do not have order 1 or 2. There is one element of order 1 (the identity), and we calculated above that there are 7 elements of order 2. Since $|\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4| = 64$, there are $64 - 7 - 1 = 56$ elements of order 4.

In $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, there are 7 elements of order 2, namely $(1,0,0),(0,1,0)$, $(0,0,1),(1,1,0),(1,0,1),(0,0,1)$, and $(1,1,1)$. Clearly, there are no elements of order 4.