Homework 5: Due Tuesday, September 27

Reading: Chapters 7.

Problem 1: Find a collection of distinct subgroups

$$< a_1 >, < a_2 >, \ldots, < a_n >$$

of $\mathbb{Z}_{240}$ such that $< a_1 > \subset < a_2 > \subset \cdots \subset < a_n >$ with n as large as possible.

Problem 2: Let G be a group. Consider the set

$$Aut(G) := \{f : G \to G | f \text{ is an isomorphism }\}.$$

Elements of $Aut(G)$ are just isomorphisms from G to itself and are usually called automorphisms of G. Show that composition of functions gives a group structure on $Aut(G)$.

Problem 3: Prove or disprove the following:
(a) $Aut(\mathbb{Z}_8)$ is abelian.
(b) $Aut(\mathbb{Z}_8)$ is cyclic.

Problem 4: Compute the order of $Aut(\mathbb{Z}_n)$.

Problem 5: Give an example of an infinite group that has exactly two elements of order 4.

Problem 6: Find the orders of the following permutations:
(a) (14)
(b) (14762)
(c) (124) (35)

Problem 7: Let $f : G \to G'$ be an isomorphism. Prove that the orders of $g \in G$ and $f(g) \in G'$ are the same.

Problem 8: Show that the map $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $f(A) := (A^t)^{-1}$ is an isomorphism. Here A^t denote the transpose of A.