# Math 453

Selected Solutions to Assignment 11

**Problem 1:** Let $R$ be a ring. Show that the following hold in R:
(1) If $a \in R$, then $0 \cdot a = 0$.
(2) For all $a \in R$, $(-1) \cdot a = -a$.
(3) The multiplicative identity 1 is unique.

**Solution:** Let $a \in R$.

(1) By the definition of additive inverse and the distributive property, respectively, we have

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

By the additive cancellation property, therefore, we have $0 = a \cdot 0$, as desired.

(2) By part (1) and the definition of additive inverse, respectively, we have

$$0 = 0 \cdot a = (-1 + 1) \cdot a = (-1) \cdot a + 1 \cdot a$$

By the definition of additive inverse, therefore, we have $(-1) \cdot a = -(1 \cdot a)$, which by definition of multiplicative identity gives us $(-1) \cdot a = -a$, as desired.

(3) Suppose 1 and $e$ are multiplicative identities on $R$. By definition of multiplicative identity, we have $1 \cdot e = 1$ and $1 \cdot e = e$, so $1 = e$. Hence, the multiplicative identity on $R$ is unique.

**Problem 2:** Show that a finite integral domain is a field.

**Solution:** Let $R$ be a finite integral domain, and let $a \in R$ be a nonzero element. Define $f_a : R \to R$ by $f_a(r) = ar$ for each $r \in R$. (Note that we are simply defining $f_a$ as a set function without any additional structure; indeed, $f_a$ is **not** a ring homomorphism – why not?) Suppose $f_a(x) = f_a(y)$ for some $x, y \in R$, so $ax = ay$, so $ax - ay = 0$, so $a(x - y) = 0$. Since $R$ is an integral domain and $a \neq 0$, we thus have $x - y = 0$, so $x = y$. Hence, $f_a$ is injective. Since $R$ is finite, $f_a$ is surjective; in particular, $1 = f_a(s)$ for some $s \in R$, so $as = 1$. Thus, $a$ is a unit, so $R$ is a field.

**Problem 7:** Let $R$ be a ring. Show that there is exactly one ring homomorphism $\phi : \mathbb{Z} \to R$.

**Solution:** Let $R$ be a ring, and let $\phi : \mathbb{Z} \to R$ be a ring homomorphism. Let $a \in \mathbb{Z}$. If $a = 0$, then by definition of ring homomorphism, we have

$\phi(a) = 0_R$, where $0_R$ is the additive identity on $R$. If $a > 0$, then $a = \sum_{i=1}^{|a|} 1$, so again by definition of ring homomorphism

$$\phi(a) = \phi\left(\sum_{i=1}^{|a|} 1\right) = \sum_{i=1}^{|a|} \phi(1) = \sum_{i=1}^{|a|} 1_R,$$

where $1_R$ is the multiplicative identity on $R$. Similarly, if $a < 0$, we have $a = -\sum_{i=1}^{|a|} 1$, so

$$\phi(a) = \phi\left(-\sum_{i=1}^{|a|} 1\right) = \sum_{i=1}^{|a|} -\phi(1) = \sum_{i=1}^{|a|} -1_R$$

Hence, the action of $\phi$ on $R$ is uniquely determined.