Chap 3 – Growth of Functions

3.1 Asymptotic notation

3.2 Standard notations and common functions

3.1 Asymptotic notation

- Discrete complexity function

  $f: \mathbb{N} \to \mathbb{R}^*$ $\quad \mathbb{N} =$ the set of natural numbers

  $\mathbb{R}^* =$ the set of nonnegative real numbers

  1 Discrete functions, but sometimes extend to continuous functions

  2 Non-negative function values, but sometimes extend to "asymptotically nonnegative", i.e.

  $\exists n_0 \geq 0$ such that $f(n) \geq 0 \;\; \forall n \geq n_0$

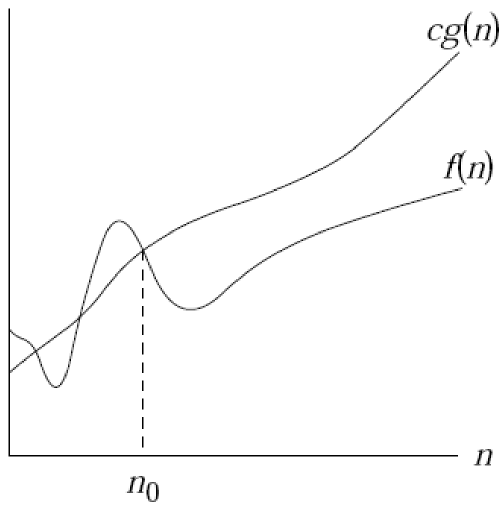
3.1 Asymptotic notation

- O-notation ($\leq$)

alternatively, $n_0 \geq 0$

$$O(g(n)) = \{f(n):\ \exists c > 0, n_0 > 0 \text{ such that } 0 \leq f(n) \leq cg(n)) \text{ for all } n \geq n_0\}$$

$\exists c$ large enough



$g(n)$ is an **asymptotic upper bound** for $f(n)$

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

3.1 Asymptotic notation

- Example
  - $10n^2 + 10n = O(n^2)$

    $\because 10n^2 + 10n \le 11n^2 \quad \forall n \ge 10$
  - $2n^2 = O(n^3)$

    $\because 2n^2 \le 2n^3 \quad \forall n \ge 0$

    or $2n^2 \le \frac{1}{8}n^3 \quad \forall n \ge 16$

    or $2n^2 \le cn^3 \quad \forall n \ge \lceil 2/c \rceil,\ c > 0$
  - Examples of functions in $O(n^2)$

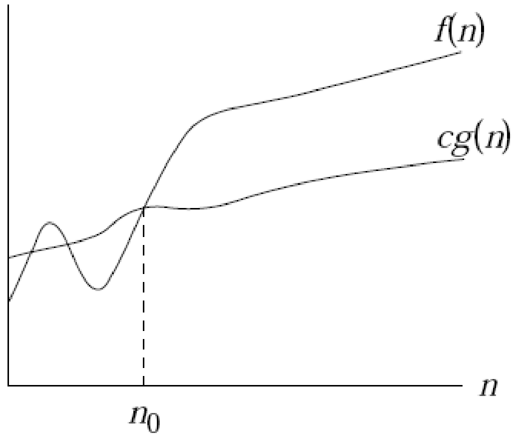
n^2	n
$n^2 + \lg n$	$n^{1.9999}$
$3n^2 + 10n$	$n^2/\lg\lg\lg n$

3.1 Asymptotic notation

- Ω-notation ($\geq$)

$$\Omega(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 \text{ such that } 0 \leq cg(n) \leq f(n)) \text{ for all } n \geq n_0\}$$

$\exists c$ small enough



$g(n)$ is an **asymptotic lower bound** for $f(n)$

3.1 Asymptotic notation

- Example
  - $0.5n^2 - n = \Omega(n^2)$

    $\because 0.4n^2 \le 0.5n^2 - n \quad \forall n \ge 10$
  - $\sqrt{n} = \Omega(\lg n)$

    $\because \sqrt{n} \ge \frac{1}{2}\lg n = \lg \sqrt{n} \quad \forall n \ge 1$

    or $\sqrt{n} \ge \lg n \quad \forall n \ge 16$

    $\text{N.B.} \sqrt{n} \ge \lg n, 1 \le n \le 4; \sqrt{n} < \lg n, 5 < n < 15$
  - Examples of functions in $\Omega(n^2)$

n^2	n^3
$n^2 + \lg n$	$n^{2.0001}$
$3n^2 + 10n$	$n^2 \lg\lg\lg n$

3.1 Asymptotic notation

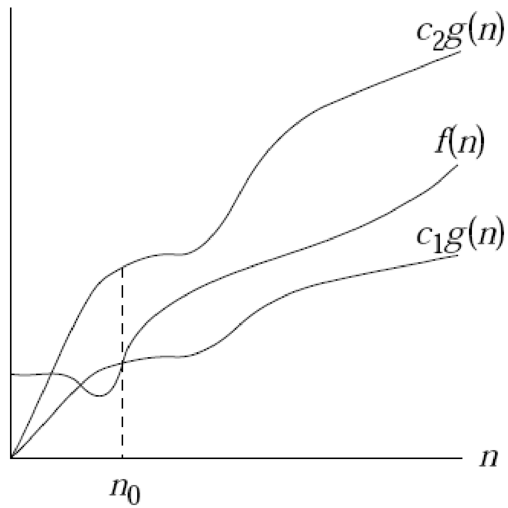
- Θ-notation (=)

$\Theta(g(n)) = \{f(n) :\ \exists c_1, c_2 > 0, n_0 > 0$ such that

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n))$ for all $n \geq n_0\}$

$\exists c_1$ small enough

$\exists c_2$ large enough



$g(n)$ is an **asymptotic tight bound** for $f(n)$

3.1 Asymptotic notation

- Example
  - $\frac{1}{2}(n^2 - n) = \Theta(n^2)$

    $\because \frac{1}{2}(n^2 - n) \leq \frac{1}{2}n^2 \quad \forall n \geq 0$

    $\frac{1}{4}n^2 \leq \frac{1}{2}(n^2 - n) \quad \forall n \geq 2$

    $\therefore \frac{1}{4}n^2 \leq \frac{1}{2}(n^2 - n) \leq \frac{1}{2}n^2 \quad \forall n \geq 2$
  - Examples of functions in $\Theta(n^2)$

    n^2

    $n^2 + \lg n$

    $3n^2 + 10n$
  - **THEOREM**

    $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

3.1 Asymptotic notation

- o-notation (<)

  $o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0$ such that
  $0 \le f(n) < cg(n))$ for all $n \ge n_0\}$

  no matter how small c is

  $f(n)$ is **asymptotically smaller** than $g(n)$.

Example of functions in $o(n^2)$	functions not in $o(n^2)$
n	n^2
$n^{1.9999}$	$n^2 + \lg n$
$n^2/\lg\lg\lg n$	$3n^2 + 10n$

3.1 Asymptotic notation

- o-notation ($<$)

  Example

  $\frac{1}{2}(n^2 - n) = o(n^3)$

  $\because \frac{1}{2}(n^2 - n) < cn^3 \Rightarrow 2cn^2 - n + 1 > 0, \ \text{if } n > 0$

  Let $f(n) = 2cn^2 - n + 1$

  Then,

  $f'(n) = 4cn - 1$

  $f''(n) = 4c > 0 \Rightarrow f$ has a minimum at $n = \frac{1}{4c}$

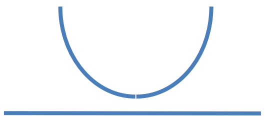
  Furthermore, f has no real roots if $1 - 8c < 0 \Rightarrow c > \frac{1}{8}$

3.1 Asymptotic notation

- o-notation (<)

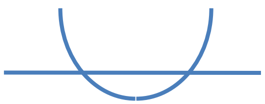
  Example (Cont'd)

  Case 1: $c > \frac{1}{8}$

  Pick $n_0 = 1$ ($\because n > 0$)

  Case 2: $c \leq \frac{1}{8}$

  The larger root is $\dfrac{1+\sqrt{1-8c}}{4c}$.

  So, pick $n_0 = 1 + \left\lfloor \dfrac{1+\sqrt{1-8c}}{4c} \right\rfloor$

3.1 Asymptotic notation

- ω-notation (>)

  $\omega(g(n)) = \{f(n) :\ \forall c > 0, \exists n_0 > 0$ such that

  $0 \le cg(n) < f(n))$ for all $n \ge n_0\}$

  no matter how large c is

  $f(n)$ is **asymptotically larger** than $g(n)$.

Example of functions in $\omega(n^2)$	functions not in $\omega(n^2)$
n^3	n^2
$n^{2.0001}$	$n^2 + \lg n$
$n^2 \lg\lg\lg n$	$3n^2 + 10n$

3.1 Asymptotic notation

- ω-notation (>)

  Example

  $\frac{1}{2}(n^2 - n) = \omega(n)$

  $$\because \frac{1}{2}(n^2 - n) > cn \Rightarrow n - 1 > 2c, \quad \text{if } n > 0$$
  $$\Rightarrow n > 1 + 2c \geq \lfloor 1 + 2c \rfloor$$
  $$\Rightarrow n \geq 1 + \lfloor 1 + 2c \rfloor = \lfloor 2 + 2c \rfloor$$

  So, pock $n_0 = \lfloor 2 + 2c \rfloor$

3.1 Asymptotic notation

- Comparing functions using limit
  - **THEOREM**

    $$f(n) = o(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

    Proof

    $$\forall c > 0\ \exists n_0 \text{ such that } f(n) < cg(n)\ \ \forall n \geq n_0$$

    $$\Leftrightarrow \forall c > 0\ \exists n_0 \text{ such that } \frac{f(n)}{g(n)} < c\ \ \forall n \geq n_0$$

  - **THEOREM**

    $$f(n) = \omega(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

3.1 Asymptotic notation

- Comparing functions using limit
  - Example

$$\frac{1}{2}(n^2 - n) = o(n^3) \because \lim_{n\to\infty} \frac{\frac{1}{2}(n^2-n)}{n^3} = 0$$

$$\frac{1}{2}(n^2 - n) = \omega(n) \because \lim_{n\to\infty} \frac{\frac{1}{2}(n^2-n)}{n} = \infty$$

$$\frac{1}{2}(n^2 - n) = \Theta(n^2) \because \lim_{n\to\infty} \frac{\frac{1}{2}(n^2-n)}{n^2} = \frac{1}{2}$$ (see next page)

3.1 Asymptotic notation

- Comparing functions using limit
  - **THEOREM**

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = a, 0 \le a < \infty \Rightarrow f(n) = O(g(n))$$

Proof

$$\because \forall\varepsilon > 0\ \exists n_0 \text{ such that } \left|\frac{f(n)}{g(n)} - a\right| < \varepsilon\ \ \forall n \ge n_0$$

So, let $\varepsilon = 1$, then $\exists n_0$ such that $\forall n \ge n_0$

$$\left|\frac{f(n)}{g(n)} - a\right| < 1 \Rightarrow \frac{f(n)}{g(n)} - a < 1$$

$$\Rightarrow f(n) < (1+a)g(n)$$

$$\Rightarrow f(n) \le (1+a)g(n)$$

3.1 Asymptotic notation

- Comparing functions using limit
  - **THEOREM**

    $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = a, 0 < a \le \infty \Rightarrow f(n) = \Omega(g(n))$$

    $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = a, 0 < a < \infty \Rightarrow f(n) = \Theta(g(n))$$

  - The converse isn't true, since big-O, Ω, and Θ asks for $\exists c$, but limit requires $\forall c$. For example, let

    $f(n) = 1$, if n is odd

    $\quad\quad = 2$, if n is even

    $g(n) = 1$, for all n

    Then, $f(n) = \Theta(g(n))$, but $\lim \frac{f(n)}{g(n)}$ doesn't exist.

3.1 Asymptotic notation

- Comparisons of functions
  - Transitivity

    $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$

    Same for $O, \Omega, o,$ and ω
  - Reflexivity

    $f(n) = \Theta(f(n))$

    Same for O and Ω
  - Symmetry

    $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$

3.1 Asymptotic notation

- Comparisons of functions
  - Transpose symmetry

    $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$

    $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$
  - Antisymmetry

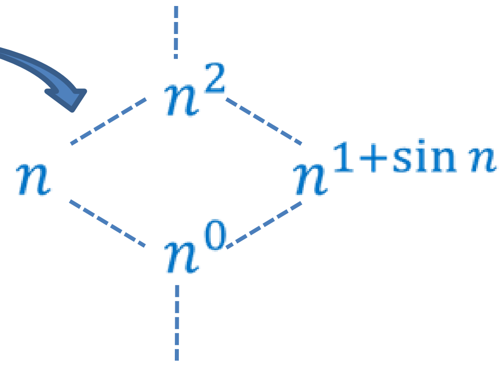
    $f(n) = O(g(n))$ and $g(n) = O(f(n)) \Rightarrow f(n) = \Theta(g(n))$

    $f(n) = \Omega(g(n))$ and $g(n) = \Omega(f(n)) \Rightarrow f(n) = \Theta(g(n))$

3.1 Asymptotic notation

- Comparisons of functions
  - Θ defines an equivalence relation on functions.
    Each equivalence class is called a complexity class.
    Each complexity class is usually represented the simplest function, e.g. we write $\Theta(n)$, rather than $\Theta(3n+2)$.
  - O and Ω define a partial order on these complexity classes

n^2
$n^{1.\alpha}$
$n \lg^k n$
$n \lg \ldots \lg n$
n



Hasse diagram

3.1 Asymptotic notation

- Comparisons of functions
  - Note that O and Ω are not total order, i.e. there are incomparable functions.

    E.g. The following pairs of functions are incomparable:

    1 n

    $n^{1+\sin n}, \because\ n^0 \leq n^{1+\sin n} \leq n^2$

    2 $\sin n$

    $\cos n$

    3 $f(n) = 1$, if n is odd; $= 0$, if n is even

    $\mathrm{g}(n) = 0$, if n is odd; $= 1$, if n is odd

3.1 Asymptotic notation

- Asymptotic notation in equation
  - One-way equality

    $2n^2 = \Theta(n^2)$ means $2n^2 \in \Theta(n^2)$

    We never write $\Theta(n^2) = 2n^2$, for otherwise

    $2n^2 = \Theta(n^2)$ and $\Theta(n^2) = 3n^2 \Rightarrow 2n^2 = 3n^2$???
  - L.H.S means $\forall$ and R.H.S means $\exists$

    $\exists f(n) \in \Theta(n)$ such that $2n^2 - 3n = 2n^2 - f(n)$

    $2n^2 - 3n = 2n^2 - \Theta(n) = \Theta(n^2)$

    $\forall f(n) \in \Theta(n)\ \exists g(n) \in \Theta(n^2)$ such that $2n^2 - f(n) = g(n)$

    i.e. $2n^2 - 3n \in 2n^2 - \Theta(n) \subset \Theta(n^2)$

3.1 Asymptotic notation

- Asymptotic notation in equation
  - Another example

    The recurrence of merge sort

    $T(n) = 2T(n/2) + \Theta(n)$

    tells us that there exists $f(n) \in \Theta(n)$ such that

    $T(n) = 2T(n/2) + f(n)$

    Since it is understood that all the lower-order terms are absorbed in $\Theta(n)$, we may let

    $f(n) = cn$

    and write

    $T(n) = 2T(n/2) + cn$ for some constant c

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Let A, B be two sets of functions, define

    $$A \oplus B = \{h(n) \mid \exists f \in A, g \in B, n_0 \geq 0 \text{ such that } h(n) = f(n) \oplus g(n)\ \forall n \geq n_0\}$$

  - **Theorem** (Rule of sum)

    $$\Theta(f(n)) + \Theta(g(n)) = \Theta(f(n) + g(n)) = \Theta(\max(f(n), g(n)))$$

  - Example

    for $i = 1$ **to** $2n$ **do** $S1; \quad \cdots \Theta(n)$

    for $i = 1$ **to** $3n^2$ **do** $S2; \cdots \Theta(n^2)$

    $$\Theta(n) + \Theta(n^2) = \Theta(n + n^2) = \Theta(\max(n, n^2)) = \Theta(n^2)$$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - **THEOREM** (Rule of product)

    $f(n)\Theta(g(n)) = \Theta(f(n))\Theta(g(n)) = \Theta(f(n)g(n))$

    where $f(n)\Theta(g(n))$ means $\{f(n)\} \cdot \Theta(g(n))$
  - **COROLLARY**

    $c \cdot \Theta(f(n)) = \Theta(f(n))$ for any constant $c > 0$
  - Example

    for $i = 1$ **to** $2n$ **do** $\cdots \Theta(n)$

        for $j = 1$ **to** $3n^2$ **do** S; $\cdots \Theta(n^2)$

    $\Theta(n)\Theta(n^2) = \Theta(n^3)$

    $2n \cdot \Theta(n^2) = \Theta(2n)\Theta(n^2) = \Theta(2n^3) = \Theta(n^3)$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example

    $n^2 \cdot \Theta(n) = \Theta(n^2)\Theta(n) = \Theta(n^3)$

    $n^2 + \Theta(n) \neq \Theta(n^2) + \Theta(n) = \Theta(n^2 + n) = \Theta(n^2)$

    This is a different interpretation.

    The equalities here are two-way set equalities.

    Each two-way set equality includes two one-way equalities.

    For example,

    $n^2 \cdot \Theta(n) = \Theta(n^2)\Theta(n)$

    consists of the following two one-way equalities:

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Cont'd)

    $n^2 \cdot \Theta(n) = \Theta(n^2)\Theta(n)$

    $\forall f(n) \in \Theta(n)\ \exists g(n) \in \Theta(n^2)\ \exists h(n) \in \Theta(n)$ such that

    $n^2 \cdot f(n) = g(n)h(n)$

    E.g. Let $g(n) = n^2$ and $h(n) = f(n)$

    $\Theta(n^2)\Theta(n) = n^2 \cdot \Theta(n)$

    $\forall f(n) \in \Theta(n^2)\ \forall g(n) \in \Theta(n)\ \exists h(n) \in \Theta(n)$ such that

    $f(n)g(n) = n^2 \cdot h(n)$

    E.g. $(n^2 + n^{1.9}) \cdot n = n^2 \cdot (n + n^{0.9})$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - **PROOF OF THEOREM** (Rule of sum, Big-O)

    We shall prove three one-way equalities:
  - $O(f(n)) + O(g(n)) = O(f(n) + g(n))$

    Proof

    $h(n) = O(f(n)) + O(g(n))$

    $\Rightarrow h(n) = f'(n) + g'(n) \;\; \forall n \geq n_0$

    where $f'(n) = O(f(n))$, $g'(n) = O(g(n))$

    $\Rightarrow h(n) \leq c_1 f(n) + c_2 g(n)$ for sufficient large n

    $\Rightarrow h(n) \leq \max(c_1, c_2)(f(n) + g(n))$

    $\Rightarrow h(n) = O(f(n) + g(n))$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - $O(f(n) + g(n)) = O(\max(f(n), g(n)))$

    Proof

    $h(n) = O(f(n) + g(n))$

    $\Rightarrow h(n) \le c\big(f(n) + g(n)\big) \;\; \forall n \ge n_0$

    $\Rightarrow h(n) \le 2c \cdot \max\{f(n), g(n)\}$ for sufficient large n

    $\quad \mathrm{e.g.}\ \max(2n, n + 50) = 2n \;\; \forall n \ge 50$

    $\Rightarrow h(n) = O(\max(f(n), g(n)))$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - $O(\max(f(n), g(n))) = O(f(n)) + O(g(n))$

    Proof

    Suppose $\max(f(n), g(n)) = f(n)$ for sufficient large n

    $h(n) = O(\max(f(n), g(n)))$

    $\Rightarrow h(n) = O(f(n))$ for sufficient large n

    $h(n) = h(n) + 0$ where $h(n) = O(f(n)), 0 = O(g(n))$

    $\Rightarrow h(n) = O(f(n)) + O(g(n))$

    e.g.

    $n = O(\max(n^2, n^{-1}))$

    $\Rightarrow n = n + 0$ where $n = O(n^2), 0 = O(n^{-1})$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \Theta\left(\frac{n(n+1)}{2}\right) = \Theta\left(\max\left(\frac{n^2}{2}, \frac{n}{2}\right)\right)$$

$$= \Theta\left(\frac{n^2}{2}\right) = \Theta(n^2)$$

It is wrong to write

this isn't asymptotically nonnegative

$$\sum_{k=1}^{n} k = n^3 + \frac{n(n+1)}{2} - n^3$$

$$= \Theta\left(\max\left(n^3, \frac{n(n+1)}{2} - n^3\right)\right) = \Theta(n^3)$$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Cont'd)

    By the same reason, the 1st equation below is incorrect.

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = \Theta\left(\max\left(\frac{n^2}{2}, -\frac{n}{2}\right)\right)$$

$$= \Theta\left(\frac{n^2}{2}\right) = \Theta(n^2)$$

$$\sum_{k=1}^{n-1} k = \frac{n^2}{4} + \frac{n^2}{4} - \frac{n}{2} = \Theta\left(\max\left(\frac{n^2}{4}, \frac{n^2}{4} - \frac{n}{2}\right)\right)$$

$$= \Theta\left(\frac{n^2}{4}\right) = \Theta(n^2)$$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - **THEOREM** (Appendix A, p.1146; Rule of sum)

$$\sum_{k=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{k=1}^{n} f(k)\right)$$

$$\because h(n) = \sum_{k=1}^{n} \Theta(f(k)) \Leftrightarrow h(n) = \sum_{k=1}^{n} f'(k), f'(k) = \Theta(f(k))$$

$$\Leftrightarrow \sum_{k=1}^{n} c_2 f(k) \le h(n) \le \sum_{k=1}^{n} c_1 f(k) \quad \forall n \ge n_0$$

$$\Leftrightarrow c_2 \sum_{k=1}^{n} f(k) \le h(n) \le c_1 \sum_{k=1}^{n} f(k) \quad \forall n \ge n_0$$

$$\Leftrightarrow h(n) = \Theta\left(\sum_{k=1}^{n} f(k)\right)$$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \Theta(k) = \Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

Never write

$$\begin{aligned} \sum_{k=1}^{n} k &= 1 + 2 + \cdots + n \\ &= O(1) + O(2) + \cdots + O(n) \\ &= O(1 + 2 + \cdots + n) \\ &= O(n^2) \end{aligned}$$

n pairs of c and n_0

?

one pair of c and n_0

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Cont'd)

    It happens to work here:

    These O's have the same $c = 1$ and $n_0 = 0$.

    $$\begin{aligned} & 1 + 2 + \cdots + n \\ & = O(1) + O(2) + \cdots + O(n) \\ & \leq 1 \cdot 1 + 1 \cdot 2 + \cdots + 1 \cdot n \\ & = 1 \cdot (1 + 2 + \cdots + n) \\ & = O(1 + 2 + \cdots + n) \end{aligned}$$

    This O also has $c = 1$ and $n_0 = 0$.

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Cont'd)

    Since $n = O(n), 2n = O(n), \ldots,$ it follows that

    $$\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n) = O\left(\sum_{k=1}^{n} n\right) = O(n^2)$$

    One fallacious point: If $k = n,$ then $kn = n^2 \neq O(n)$

    Another fallacious point:

    $n + 2n + 3n + \cdots$

    $= \; O(n) + O(n) + O(n) + \cdots$

    $\leq 1n + 2n + 3n + \cdots$ These Os have different constants.

    Thus, they can't be replaced by a single O.

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Rule of product)

    $O(n^2) = O(n)O(n) = O(n)^2$

    But, $O(n^{-1}) \neq O(n)^{-1} = \Omega(n^{-1})$.

    $\because \quad h(n) = O(n)^{-1}$

    $\Leftrightarrow h(n) = f(n)^{-1}$ where $f(n) = O(n)$

    $\Leftrightarrow h(n) = \frac{1}{f(n)}$ where $f(n) \leq cn \; \forall n \geq n_0$

    $\Leftrightarrow h(n) = \frac{1}{f(n)}$ where $\frac{1}{f(n)} \geq \frac{1}{c} \cdot \frac{1}{n} \; \forall n \geq n_0$

    $\Leftrightarrow h(n) = \Omega(n^{-1})$

3.1 Asymptotic notation

- Operations on asymptotic notation
  - Example (Unary operation on asymptotic notation)

    $O(O(f(n))) = O(f(n))$

    because

    $$O(O(f(n))) = \bigcup_{g(n)\in O(f(n))} O(g(n))$$

    $$= \{h(n) \mid \exists g \in O(f(n)) \text{ such that } h(n) = O(g(n))\}$$

    $$= \{h(n) \mid h(n) = O(f(n))\} \quad \because O \text{ is transitive}$$

    $$= O(f(n))$$

    C.f. $$O(n)^{-1} = \bigcup_{f(n)\in O(n)} \{f(n)^{-1}\}$$

3.2 Standard notations and common functions

- Polynomials
  - A function $f(n)$ is polynomially bounded if $f(n) = O(n^k)$ for some k.
- Exponentials
  - Exponentials with base > 1 grow faster than polynomials.
  - $n^k = o(a^n), a > 1 \quad \because \lim_{n \to \infty} \frac{n^k}{a^n} = 0$
- Logarithms
  - Notations

Notations	$\lg n = \log_2 n$	binary logarithm
	$\ln n = \log_e n$	natural logarithm
	$\lg^k n = (\lg k)^n$	exponentiation
	$\lg \lg n = \lg(\lg n)$	composition

3.2 Standard notations and common functions

- Logarithms
  - Convention: Use lg

    ∵ The base of logarithm does NOT matter in asymptotic notation.

    $$\log_b n = \Theta(\log_a n) \quad \because \lim_{n\to\infty} \frac{\log_b n}{\log_a n} = \frac{1}{\log_a b}$$

  - A function $f(n)$ is polylogarithmically bounded if $f(n) = O(\lg^k n)$ for some k.
  - Polynomials grow faster than polylogarithms.

    $$\lg^k n = o(n^a), a > 0 \quad \because \lim_{n\to\infty} \frac{\lg^k n}{n^a} = \lim_{n\to\infty} \frac{(\lg n)^k}{(2^a)^{\lg n}} = 0$$

3.2 Standard notations and common functions

- Factorials
  - $n! = o(n^n)$
  - $n! = \omega(2^n)$
  - Stirling's formula

    $$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

  - $$\lg(n!) = \lg\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)\right)$$

    $$= \frac{1}{2}(\lg 2\pi + \lg n) + n(\lg n - \lg e) + \lg\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

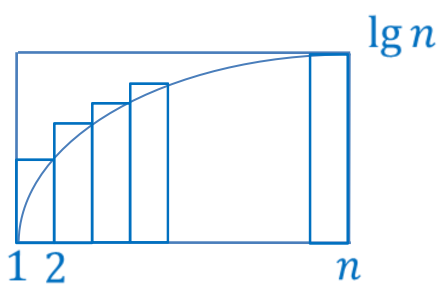
    $$= \Theta(n \lg n)$$

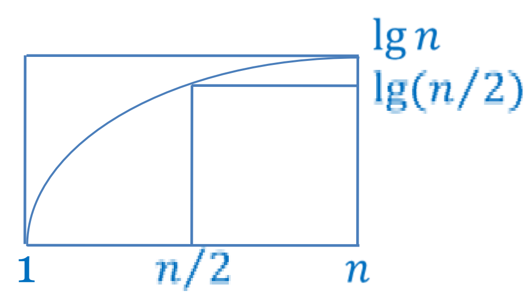
3.2 Standard notations and common functions

- Summation
  - Example

$$\sum_{i=1}^{n} \lg i = \lg(n!) = \Theta(n \lg n)$$

Another way to see the bound



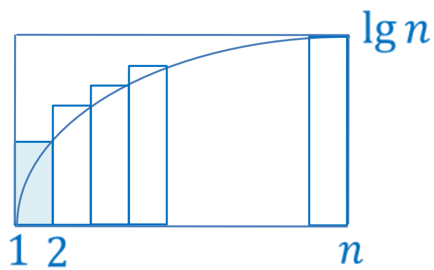


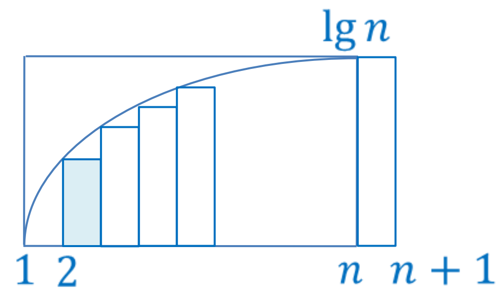
$$\frac{n}{2}(\lg n - 1) = \frac{n}{2}\lg\frac{n}{2} \leq \sum_{i=1}^{n} \lg i \leq (n-1)\lg n$$

3.2 Standard notations and common functions

- Summation (Appendix A)
  - Example (Cont'd)

    Yet another way to see the bound





$$\int_1^n \lg x \, dx \le \sum_{i=1}^{n} \lg i \le \int_1^{n+1} \lg x \, dx$$

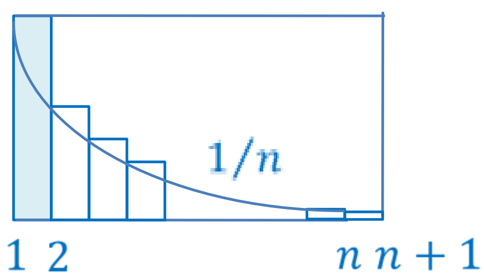
$$n \lg n - n \lg e + \lg e \le \sum_{i=1}^{n} \lg i$$

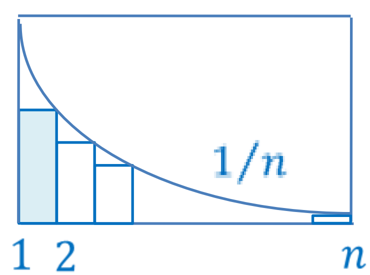
$$\le (n+1)\lg(n+1) - (n+1)\lg e + \lg e$$

3.2 Standard notations and common functions

- Summation (Appendix A)
  - Example (nth Harmonic number)

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$





$$\ln(n+1) \le \int_1^{n+1} \frac{dx}{x} \le \sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k}$$

$$\le 1 + \int_1^{n} \frac{dx}{x} = 1 + \ln n$$