1 Electric Quantities

- Electric charge
  - carriers,e^-,h^+.
  - 1 *coulomb* of negative change contains 6.241×10^{18} electrons.
- Voltage
  - Voltage difference between two points is the work in *joules* required to move 1 coulomb of charge from one point to the other.

$$V(\text{volts}) = \frac{W(\text{joules})}{Q(\text{coulombs})}. \tag{1}$$

  - Potential Energy(capability to do work).

$$v = \frac{dw}{dq} \tag{2}$$

- Current
  - 1 *ampere* is a steady flow of 1 *coulomb* of change pass a given point in a conductor in 1 second.

$$I(\text{amperes}) = \frac{Q(\text{coulombs})}{t(\text{seconds})}. \tag{3}$$

  - Time varying current

$$i(t) = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t} = \frac{\Delta q}{\Delta t} \tag{4}$$

- Power
  - The rate at which something either absorbs or produces energy.
  - The power absorbed by an electric element is the product of the voltage and the current.

$$P(\text{Watts}) = \frac{W(\text{joules})}{t(\text{seconds})} = V(\text{volts}) \times I(\text{amperes}). \tag{5}$$

- Work

$$p = \frac{dw}{dt} = (\frac{dw}{dq}) * (\frac{dq}{dt}) = v_i \tag{6}$$

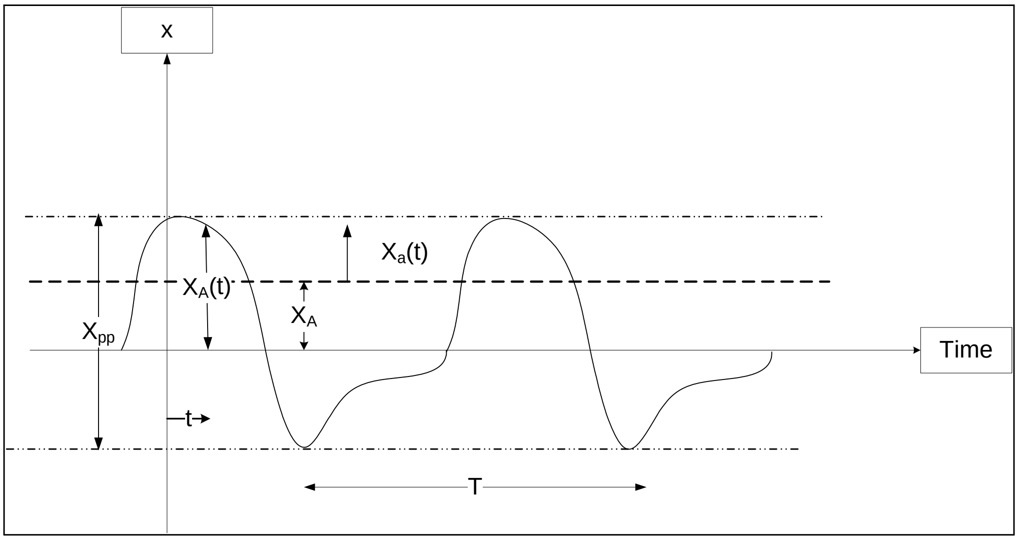


Figure 1:

2 Signal Components

- Signals
  - Quantity of interest.
  - manifested as either voltages or currents.

$$X_A(t) = X_A + X_a(t) \tag{7}$$

  - X_A is DC component.
  - $X_a(t)$ is AC component.
- DC Component
  - DC value(average value):

$$X_A = \frac{1}{T}\int_{t_0}^{t_0+T} x(\tau)d\tau \tag{8}$$

- AC Component
- Root-Mean-Square
  - Effective (RMS) Value(dissipate same amount of power accross a resister):

$$X_{eff} \equiv X_{rms} \triangleq \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(\tau)d\tau} \tag{9}$$

3 Circuit Elements

- Ideal voltage source
  - Voltage source provides a specified voltage across the two terminals and does not depend on the current flowing through the source.
  - The output impedance of ideal voltage source is zero.
    * The current flowing through an ideal voltage source is completely determined by the circuit connected to the source.
- Ideal current source
  - Current source provides a specified current and does not depend on the voltage across the source.
  - The output impedance of ideal current source is infinite.
    * The voltage across an ideal current source is completely determined by the circuit connected to the source.
- Resistor R
  - Resistance is the property of materials that resists the movement of electrons.
  - Ohm's Law

$$R(\text{ohms}) = \frac{V(\text{volts})}{I(\text{amperes})} \tag{10}$$

  - Conductance is the inverse of resistance.
  - For parallel resistors,

$$R_T = \frac{1}{1/R_1 + 1/R_2 + 1/R_3.... + 1/R_N} \tag{11}$$

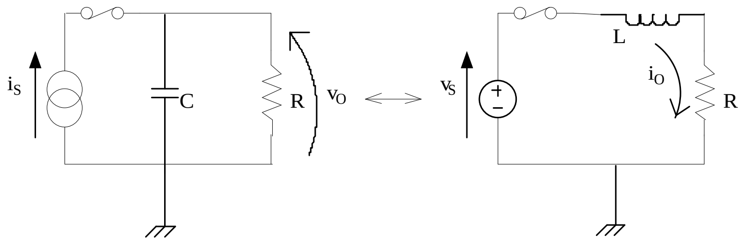
    * When two resistors are connected in parallel, the equivalent resistance is smaller than any of the two resistors.

$$\begin{aligned} R_T &= \frac{R_1R_2}{R_1+R_2} < R_1 \\ R_T &= \frac{R_1R_2}{R_1+R_2} < R_2 \end{aligned} \tag{12}$$

  - For series resistors,

$$R_T = R_1 + R_2 + R_3.... + R_N \tag{13}$$

- Capacitor C
  - Capacitance is the ability of a capacitor to store charges on its two conductors.
$$C(\text{farad}) = \frac{Q(\text{coulombs})}{V(\text{volts})} \tag{14}$$
  - For time-varying voltage $v_C(t)$ across the capacitor
    * The capacitor is an open circuit, i.e., $i_C(t) = 0$, when $v_C(t)$ is a constant.
    * The voltage across the capacitor can not jump. However, the current flowing through the capacitor does not have such constraint.
$$i_C(t) = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = C \times \frac{dv_C(t)}{dt} \tag{15}$$
$$v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i_C(\tau) d\tau \tag{16}$$
  - For parallel capacitors,
$$C_T = C_1 + C_2 + C_3.... + C_N \tag{17}$$
  - For series capacitors,
$$C_T = \frac{1}{1/C_1 + 1/C_2 + 1/C_3.... + 1/C_N} \tag{18}$$
- Inductor L
  - For time-varying current $i_L(t)$ flowing through the inductor
    * The inductor is a short circuit, i.e., $v_L(t) = 0$, when $i_L(t)$ is a constant.
    * The current flowing through the inductor can not jump. However, the voltage across the inductor does not have such constraint.
$$\begin{aligned} v_L(t) &= \lim_{\Delta t \to 0} \frac{\Delta N\phi(\text{flux linkages})}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta L i_L}{\Delta t} \\ &= \lim_{\Delta t \to 0} L \times \frac{i_L(t+\Delta t) - i_L(t)}{\Delta t} \\ &= L \times \frac{di_L(t)}{dt} \end{aligned} \tag{19}$$
$$i_L(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau \tag{20}$$



Current Source	$\leftrightarrow$	Voltage Source
Parallel	$\leftrightarrow$	Serial
v	$\leftrightarrow$	i
C	$\leftrightarrow$	L
R	$\leftrightarrow$	R

Figure 2: Similarity between capacitor and inductor.

- For parallel inductors,

$$L_T = \frac{1}{1/L_1 + 1/L_2 + 1/L_3.... + 1/L_N} \tag{21}$$

- For series capacitors,

$$L_T = L_1 + L_2 + L_3.... + L_N \tag{22}$$

- Dual circuit
  - Capacitor and Inductor formulas are the same except that the symbols differ.
  - Capacitor

$$i_S(t) = \frac{v_o(t)}{R} + C\frac{dv_o(t)}{dt} \tag{23}$$

  - Inductor

$$v_S(t) = i_o(t)R + L\frac{di_o(t)}{dt} \tag{24}$$

4 Electric Circuit, structure

- Electric Circuits/Network
  - Conceptually equivalent to Graphs in math.
  - A structure consists of Nodes (Vertices) and Edges (Branches)
- Edge : circuit part contains one electruc component/element.

- Node : (lossless) circuit part between two or more edges.
- Path : a successive sequence of connected nodes or edges.
- Loop : a closed path among edges in a circuit.
- Terminal : a node that does not belong to any loop in a circuits.
- Sources : Elements that provide input signals (voltage/current)
- Loads : Elements that dissipate output signals.
- Ground : Reference node of voltage level in a circuit.
  - Signal Ground = reference voltage level
  - System Ground : sink of leakage currents

5 Electric Circuit , Operation

- Classical Circuit Theory is based on Flow Model.
  - Treating current as flow of charges - Conservation of Charges
  - Guided by Circuit topology - Transmission in guided media.
  - Ignoring electro magnetic induction - except special deliberate conditions(e.g. transformers)
- Open & Short Circuits :
  - Open Circuit : connect two terminals with an element of infinite resistance. (no current flows between terminals)
  - Short Circuit : connect two terminals with an element of zero resistance (no voltage difference/drop between terminals)
  - Loaded Terminal : connect two terminals with an element of finite impedence (finite voltage drop & current flow between terminals)
- Parallel & Serial Connection
  - Parallel Connection : Edges share(split current flows) same voltage drop.(potential difference)
  - Serial Connection : Edges share(spilt voltage drops) same current flow.

6 Circuit Laws

- Kirchhoff's voltage law (KVL)
  - The algebraic sum of the voltages around any closed loop of a circuit is zero.
- Kirchhoff's current law (KCL)
  - The algebraic sum of the currents entering every node must be zero.

7 Network Theorems

Definition 1 *Linear circuit is formed by interconnecting the terminals of independent sources, controlled sources, and linear passive elements to form one or more closed paths.*

- Linear passive elements include Resistor, Capacitor, and Inductor.
  - The $i - v$ characteristics of these elements satisfy the conditions of linearity.

Theorem 2 *In a linear network containing multiple sources, the voltage across or current through any passive element may be found as the algebraic sum of the individual voltages or currents due to each of the independent sources action along, with all other independent sources deactivated.*

- Voltage source is deactivated by replacing it with a short circuit.
- Current source is deactivated by replacing it with an open circuit.
- Controlled sources remain active when the superposition theorem is applied.

Example 3 *Given the circuit in Figure 3, find the voltage across the resistor R_3 using the superposition theorem of linear network.*

1. The voltage across the resistor R_3 is the superposition of the voltage when each independent source actions alone, as shown in Figure 3 (b) and (c).

$$V = V_1 + V_2 \tag{25}$$

2. The contribution of the voltage source V_A.
   - The current source I_A is replaced with an open circuit.

$$V_1 = \frac{R_3}{R_1 + R_2 + R_3} V_A \tag{26}$$

3. The contribution of the current source I_A.
   - The voltage source V_A is replaced with an short circuit.

$$V_2 = \frac{R_1 R_3}{R_1 + R_2 + R_3} I_A \tag{27}$$

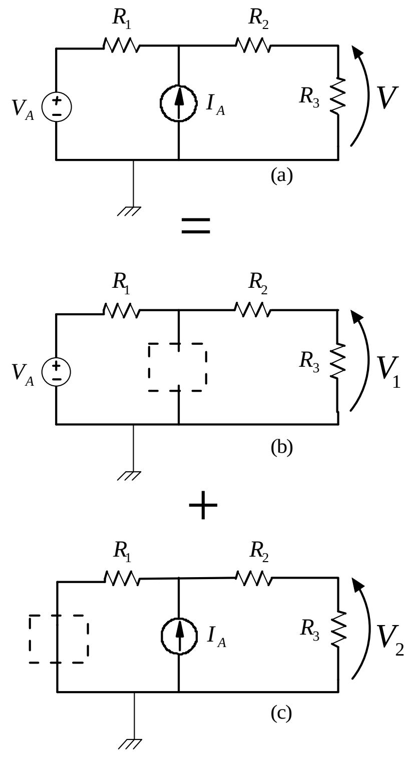


Figure 3: Example of linear circuit with multiple independent sources.

7.1 Equivalent Circuits of One-Port Networks

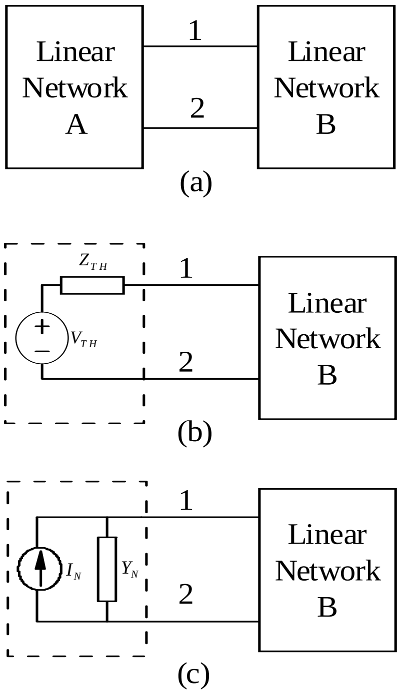


Figure 4: Equivalent circuits. (a) The original circuit. (b) Thevenin's equivalent. (c) Norton's equivalent.

- Equivalent circuit
  - A reduction of a complex linear circuit into a simpler form.
  - A model of a complex linear circuit contained in a black box.

Theorem 4 *Thevenin's theorem states that an arbitrary linear, one port network such as network A in Figure 4 (a) can be replaced at terminals* $1, 2$ *with an equivalent series-connected voltage source* V_{TH} *and impedance* Z_{TH} *as in Figure 4 (b).*

- V_{TH} is the open-circuit voltage of network A at terminals $1, 2$.
- Z_{TH} is the ratio of the open-circuit voltage over short circuit current determined at terminals $1, 2$.
  - The equivalent impedance looking into network A through terminals $1, 2$ with all independent sources deactivated.

* Voltage sources are replaced by short circuits.
* Current sources are replaced by open circuits.

Theorem 5 *Norton's theorem states that an arbitrary linear, one port network such as network A in Figure 4 (a) can be replaced at terminals* $1, 2$ *with an equivalent parallel-connected current source* I_N *and admittance* Y_N *as in Figure 4 (c).*

- I_N is the short-circuit current flowing through terminals 1, 2 due to network A.
- Y_N is the ratio of short-circuit current over open-circuit voltage at terminals $1, 2$.
- **Conversion of equivalent circuits**.
  - Any method for determining Z_{TH} is equally valid for finding Y_N.

$$I_N = I_{SC} = \frac{V_{TH}}{Z_{TH}} \tag{28}$$

$$V_{TH} = V_{OC} = I_N \times \frac{1}{Y_N} \tag{29}$$

$$Z_{TH} = \frac{1}{Y_N} \tag{30}$$

Example 6 *In Figure 5,* $V_A = 4V$*,* $I_A = 2A$*,* $R_1 = 2\,\Omega$*,* $R_2 = 3\,\Omega$*, find the Thevenin's equivalent circuit and Norton's equivalent circuit for the network to the left of terminals* $1, 2$.

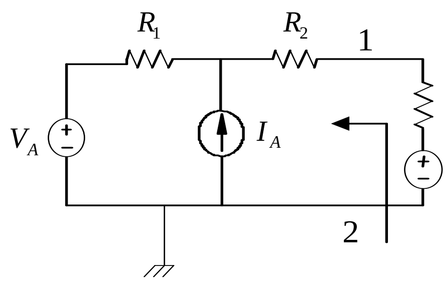


Figure 5: Examples of Thevenin's and Norton's equivalent circuits.

1. Thevenin's equivalent
   - V_{TH} is the open-circuit voltage at terminals $1, 2$.

$$V_{TH} = V_A + I_A \times R_1 = 4 + 4 = 8V. \tag{31}$$

- Z_{TH} is the ratio of the open-circuit voltage over short circuit current determined at terminals $1, 2$ with network B disconnected.
  - By the superposition of the short-circuit current caused by V_A and I_A, the short circuit current can be found.

$$\begin{aligned} I_{1,2} &= I(V_A) + I(I_A) \\ &= \frac{1}{R1+R2}V_A + \frac{R1}{R1+R2}I_A \\ &= \frac{8}{5}A. \end{aligned} \tag{32}$$

  - By definition, Z_{TH} can be derived as $V_{TH}/I_{1,2} = 5\,\Omega$.
- Alternatively, Z_{TH} can be found as the equivalent impedance for the circuit to the left of terminals $1, 2$.
  - V_A is replaced with short circuit.
  - I_A is replaced with open circuit.

$$Z_{TH} = R_1 + R_2 = 5\,\Omega. \tag{33}$$

2. Norton's equivalent
   - I_N is the short-circuit current at terminals $1, 2$, which can be derived as in Eq. (32).
   - Y_N is the ratio of the short-circuit current over the open-circuit voltage with network B disconnected. From Eq. (31), the open circuit voltage is $8V$. Thus, $Y_N = \frac{8}{5}A/8V = 1/5S$.
   - Alternatively, $Y_N = 1/Z_{TH} = 1/5 = 0.2S$.

7.2 Equivalent Circuits of Two-Port Networks

- A two-port network is an electrical circuit or device with two pairs of terminals.
  - Figure 6 depicts a two-port linear network.
  - Only two of the four variables V_1, V_2, I_1, I_2 can be independent.

7.2.1 Characterization of Two-Port Networks

- Z **parameters** (V_1, V_2 depend on I_1, I_2)
  - The four z_{ij} parameters represent impedance.
  - z_{12} and z_{21} are transfer impedances.

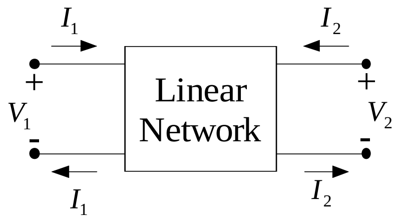


Figure 6: Two-port linear network.

- Each of the z_{ij} parameters can be evaluated by open-circuiting an appropriate port of the network.

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{34}$$

$$\begin{aligned} z_{11} &= \frac{V_1}{I_1}|_{I_2=0} \\ z_{12} &= \frac{V_1}{I_2}|_{I_1=0} \\ z_{21} &= \frac{V_2}{I_1}|_{I_2=0} \\ z_{21} &= \frac{V_2}{I_2}|_{I_1=0} \end{aligned} \tag{35}$$

- Z **parameters** in matrix form.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{36}$$

- Y **parameters** (I_1, I_2 depend on V_1, V_2)
  - The four y_{ij} parameters represent admittance.
  - y_{12} and y_{21} are transfer admittances.
  - Each of the y_{ij} parameters can be evaluated by short-circuiting an appropriate port of the network.

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned} \tag{37}$$

$$
\begin{aligned}
y_{11} &= \frac{I_1}{V_1}|_{V_2=0} \\
y_{12} &= \frac{I_1}{V_2}|_{V_1=0} \\
y_{21} &= \frac{I_2}{V_1}|_{V_2=0} \\
y_{21} &= \frac{I_2}{V_2}|_{V_1=0}
\end{aligned}
\tag{38}
$$

- Y **parameters** in matrix form.

$$
\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{39}
$$

- Relation between y_{ij} parameters and z_{ij} parameters.

$$
\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} = \frac{1}{z_{11}z_{22} - z_{12}z_{21}} \begin{bmatrix} z_{22} & -z_{21} \\ -z_{12} & z_{11} \end{bmatrix} \tag{40}
$$

- H **parameters** (V_1, I_2 depend on I_1, V_2)
  - The h_{11} represents impedance.
  - The h_{22} represents admittance.
  - Parameters h_{11} and h_{21} are obtained by short-circuiting port 2.
  - Parameters h_{12} and h_{22} are obtained by open-circuiting port 1.

$$
\begin{aligned}
V_1 &= h_{11}I_1 + h_{12}V_2 \\
I_2 &= h_{21}I_1 + h_{22}V_2
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
h_{11} &= \frac{V_1}{I_1}|_{V_2=0} \\
h_{12} &= \frac{V_1}{V_2}|_{I_1=0} \\
h_{21} &= \frac{I_2}{I_1}|_{V_2=0} \\
h_{22} &= \frac{I_2}{V_2}|_{I_1=0}
\end{aligned}
\tag{42}
$$

- H **parameters** in matrix form.

$$
\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \tag{43}
$$

Example 7 *In Figure 7, $R_1 = 10\,\Omega$, $R_2 = 6\,\Omega$, find the z parameters and the h parameters for the network.*

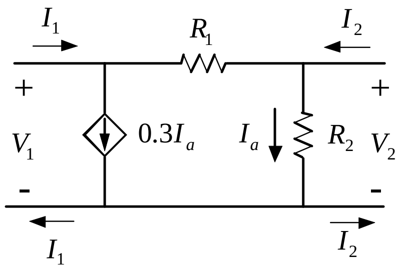


Figure 7: Example of two-port linear network.

- From Eq. (35), the z_{ij} parameters can be obtained as follows.

$$
\begin{aligned}
z_{11} &= \frac{V_1}{I_1}|_{I_2=0} = \frac{(10+6)I_a}{I_a+0.3I_a} = \frac{16}{1.3} = 12.31\,\Omega \\
z_{12} &= \frac{V_1}{I_2}|_{I_1=0} = \frac{6I_a - 0.3I_a \times 10}{I_a+0.3I_a} = \frac{3}{1.3} = 2.31\,\Omega \\
z_{21} &= \frac{V_2}{I_1}|_{I_2=0} = \frac{6I_a}{I_a+0.3I_a} = \frac{6}{1.3} = 4.62\,\Omega \\
z_{22} &= \frac{V_2}{I_2}|_{I_1=0} = \frac{6I_a}{I_a+0.3I_a} = \frac{6}{1.3} = 4.62\,\Omega
\end{aligned}
$$

- From Eq. (42), the h_{ij} parameters can be calculated as follows.

$$
\begin{aligned}
h_{11} &= \frac{V_1}{I_1}|_{V_2=0} = \frac{I_1 \times R_1}{I_1} = R_1 = 10\,\Omega \\
h_{12} &= \frac{V_1}{V_2}|_{I_1=0} = \frac{I_a \times R_2 - 0.3I_a \times R_1}{I_a \times R_2} = \frac{3}{6} = 0.5 \\
h_{21} &= \frac{I_2}{I_1}|_{V_2=0} = -1 \\
h_{22} &= \frac{I_2}{V_2}|_{I_1=0} = \frac{I_a+0.3I_a}{I_a \times R_2} = \frac{1.3}{6} = 0.217S.
\end{aligned}
$$

7.2.2 Equivalent Circuits of Special Two-Port Networks

- T-Model network. (Figure 8 (a))
  - $Z1, Z2$,and $Z3$ can be derived from the Z **parameters** of a two-port networks.

$$
\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} Z_1+Z_3 & Z_3 \\ Z_3 & Z_2+Z_3 \end{bmatrix} \tag{44}
$$

- π-Model network. (Figure 8 (b))

– Za, Zb,and Zc can be derived from the Y **parameters** of a two-port networks.

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{Z_a} + \frac{1}{Z_c} & -\frac{1}{Z_c} \\ -\frac{1}{Z_c} & \frac{1}{Z_b} + \frac{1}{Z_c} \end{bmatrix} \tag{45}$$

- Transformation from π-Model to T-Model.

$$\begin{aligned} Z_1 &= \frac{Z_a Z_c}{Z_a + Z_b + Z_c} \\ Z_2 &= \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \\ Z_3 &= \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \end{aligned} \tag{46}$$

- Transformation from T-Model to π-Model.

$$\begin{aligned} Za &= \frac{Z1Z3 + Z1Z2 + Z2Z3}{Z2} \\ Zb &= \frac{Z1Z3 + Z1Z2 + Z2Z3}{Z1} \\ Zc &= \frac{Z1Z3 + Z1Z2 + Z2Z3}{Z3} \end{aligned} \tag{47}$$

8 Electric Circuit Analysis

8.1 Circuit Equations

- There are simpler ways to write circuit equations then using KVL/KCL blindly.
- Derived from basic electric properties

$$v_C(t) = v_R(t) = v(t) \tag{48}$$

$$i_C(t) + i_R(t) = i(t) \tag{49}$$

- For Capacitor C

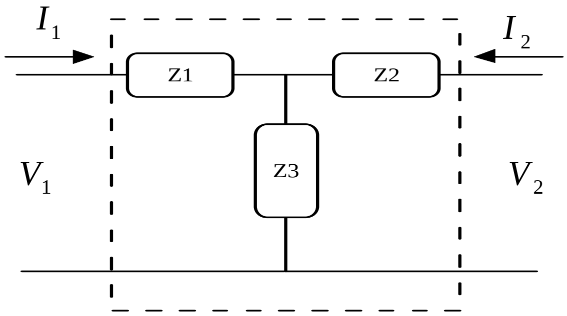
$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dv(t)}{dt} \tag{50}$$

- For Resistor R

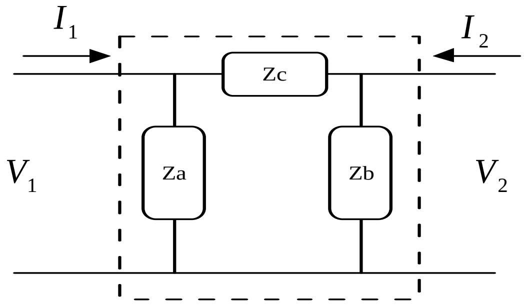
$$i_R(t) = \frac{v_R(t)}{R} = \frac{v(t)}{R} \tag{51}$$

- Circuit Equation for Parallel RC Circuit
  – From Eqn (49) : $i(t) = i_C(t) + i_R(t)$
  – Circuit Equation :

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R} \tag{52}$$



(a) T-Model Network



(b) $\pi-$Model Network

Figure 8: Equivalent circuits of two port networks. (a) T-Model Network. (b) Pi-Model Network.

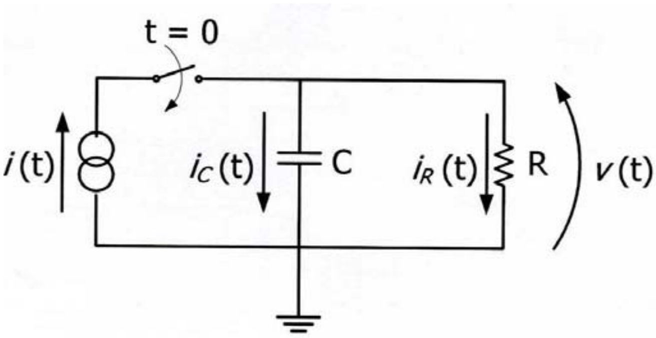


Figure 9: Parallel RC Circuit

8.2 Time Domain Electric Circuit Analysis

- Analysis of Linear Circuits is done by solving Linear Differential Equations (LDEs).
- Two Phases of Solving LDEs
  - Finding General Solutions of Homogeneous Equations with NULL Input. → Driving Free Behavior
  - Finding Particular Solution of Non-homogeneous Equations w.r.t. a Particular Input. → Driving Behavior

8.3 Driving Free Behaviors of First Order Electric Circuits

Definition 8 *Driving or Source Free Behavior of an Electric Circuit : Response of an electric circuit towards the initial conditions of its elements without the influence of input signals.*

Example 9 *For the Parallel RC Circuit as shown, its Circuit Equation is*

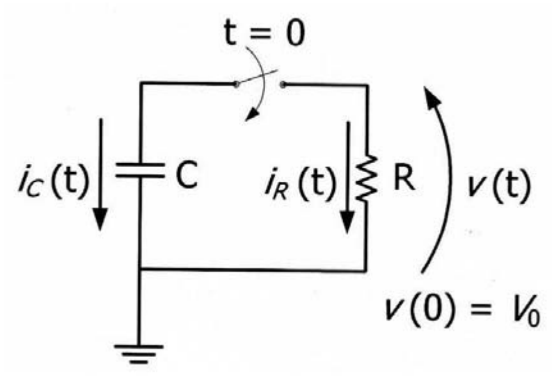


Figure 10:

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = 0 \tag{53}$$

Analysis of Driving Free Behavior ≡ Solution of Homogeneous Equation[1]

[1] The solution of a homogeneous linear differential equation is equivalent to the search for the Null Space of the corresponding linear operator.

Solve the equation by substitutng $v(t)$ with the expected solution function[2]:

$$v_h(t) = Ke^{st}$$

The LDE becomes

$$CK_1e^{st} + \frac{K_1}{R}e^{st} = 0$$

Since

$$\forall t < \infty, |e^{st}| \neq 0$$

we have

$$sCK_1 + \frac{K_1}{R} = 0 \Rightarrow s = -\frac{1}{RC}$$

Next, we shall find K_1 that satisifies the initial condition :

$$v_h(0) = K_1e^{st}|_{t=0} = K_1 = V_0$$

Therefore,

$$v_h(t) = V_0e^{-t/RC}$$

8.4 Driven (Steady State) Behaviors of First Order Electric Circuits

The steady-state behavior of an electric circuit is yielded as a particular solution of the non-homogeneous equation of the circuit as $t \to \infty$ (after the effects of initial conditions die down).

Example 10 *For an Parallel RC Circuit, the Circuit Equation is*

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = i(t) \Rightarrow \frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{i(t)}{C} \tag{54}$$

- Let $\alpha = 1/RC$ and $\beta = 1/C$, the equation becomes

$$\frac{dv(t)}{dt} + \alpha v(t) = \beta i(t) \tag{55}$$

Remark 11 *Basic Form of Steady State Solution : As the effects of initial conditions die away, the steady state solution of a linear system shall resemble its driving function or input.* $v(t)$ ˜ $f(t)$ *as* $t \to \infty$.

[2]The function e^{st} preserves its form under differentiation and integration:

$$\frac{d}{dt}e^{st} = se^{st}, \quad \int_0^t e^{s\tau}d\tau = \frac{e^{st}}{s}$$

for all s values (real or complex).
If s = real and positive, e^{st} is an exponential growth function.
If s = real and negative, e^{st} is an exponential decay function.
If s = imaginary ($s = j\omega$), $e^{j\omega t} = \cos\omega t + j\sin\omega t$ is a combination of sinusoidals.

Case[1] : Input is a Step Function

$$i(t) = \left\{ \begin{array}{ll} 0 & , t < 0 \\ 1 & , t \geq 0 \end{array} \right\}$$

Corresponding Particular Solution : $v_p(t) = K_2$
Substitude it into (55) :

$$\frac{d}{dt}K_2 + \alpha K_2 = \beta I \quad , \forall t \geq 0 \quad \Rightarrow \quad K_2 = \frac{\beta I}{\alpha} = IR$$

Hence,

$$v_p(t) = IR$$

General Solution of Parallel RC Circuit :

$$v(t) = v_h(t) + v_p(t) \tag{56}$$

Remark 12 *For all Linear Systems, their General Solutions are always composed of their Homogeneous Solutions and Partucular Solutions (corresponding to specific inputs).*

From (56) :

$$v(t) = K_1 e^{-t/RC} + IR$$

- – According to initial condition :

$$v(0) = V_0 = K_1 + IR \Rightarrow K_1 = V_0 - IR$$

- Case[2] : Input is a Sinusoidal Function

$$i(t) = \left\{ \begin{array}{ll} 0 & , t < 0 \\ Ie^{j\omega_0 t} & , t \geq 0 \end{array} \right\}$$

  – Corresponding Steady State Solution : $v_p(t) = Ke^{j(\omega_0 t + \varphi)}$
  Because

$$\frac{dv_p}{dt} + \frac{v_p}{RC} = \frac{i}{C}$$

  time varying form of all these terms should be same and non-zero as $t \to 0$ (or $t >> 0$).
  Expand the equation :

$$K(j\omega_0 t)e^{j(\omega_0 t + \varphi)} + \frac{K}{RC}e^{j(\omega_0 t + \varphi)} = \frac{I}{C}e^{j\omega_0 t}$$

$$\Rightarrow Ke^{j\varphi} = \frac{I}{C} \times \frac{1}{\frac{1}{RC} + j\omega_0 t} = \frac{IR}{1 + jRC\omega_0 t} = \frac{IR(1 - jRC\omega_0 t)}{1 + R^2C^2\omega_0^2 t^2}$$

$$\Rightarrow K = \left\| \frac{IR}{1 + jRC\omega_0 t} \right\| \quad , \quad \varphi = \tan^{-1}(-RC\omega_0 t)$$

  as $\omega_0 \to 0$ $\quad K \to IR$, $\varphi \to 0^-$
  as $\omega_0 \to \infty$ $\quad K \to 0$, $\varphi \to -\frac{\pi}{2}$
  as $\omega_0 = \frac{1}{RC}$ $\quad K = \frac{IR}{\sqrt{2}}$, $\varphi = -\frac{\pi}{4}$

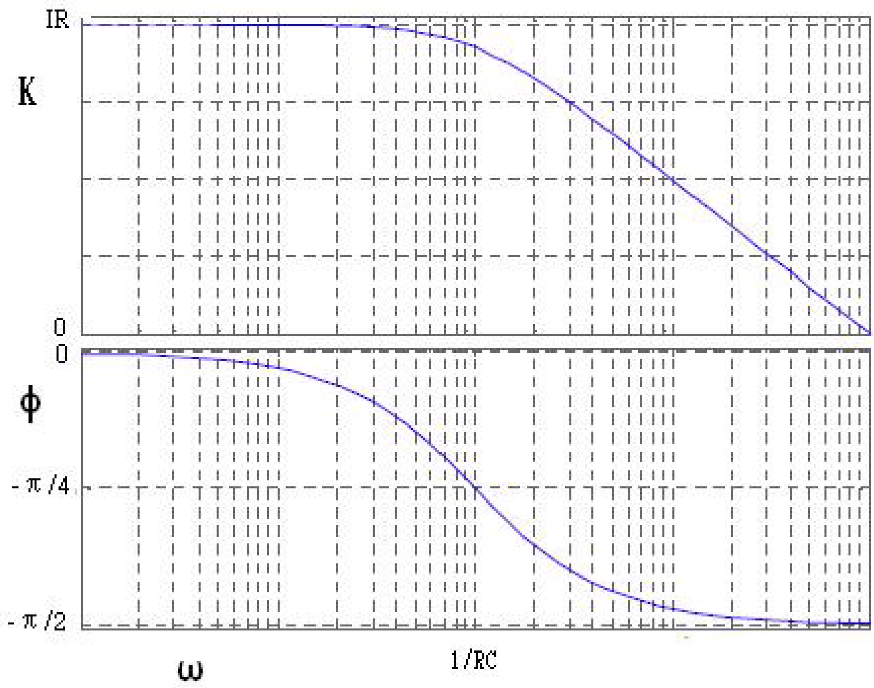


Figure 11:

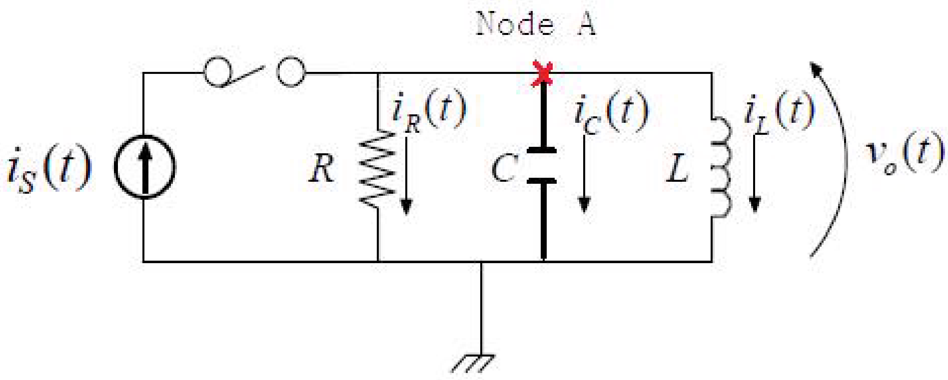


Figure 12:

8.5 Analyze 2^{nd} Order Electric for Driving Free Behavior

- Step[1] Writing Circuit Equations (Two Forms)

According to KCL rule:

$$i_C + i_L + i_R = i \tag{57}$$

also
$v_C = v_L = v_R = v$
Subs, $i - v$ characteristic of passive elements into (57), we get an integral-differential circuit equation:

$$C\frac{dV}{dt} + \frac{1}{L}\int_0^t V d\tau + \frac{V}{R} = i \tag{58}$$

$i - v$ characteristic of passive elements:

$$i_R = \frac{V_R}{R} = \frac{V}{R} \tag{59}$$

$$i_C = C\frac{dV_C}{dt} = C\frac{dV}{dt} \tag{60}$$

$$i_L = \frac{1}{L}\int_0^t V_L(\tau) = \frac{1}{L}\int_0^t V d\tau \tag{61}$$

we can tranform Equation(58) into a 2^{nd} order LDE by change of variable:
Let $u = \int_0^t v d\tau$ and hence $v = \frac{du}{dt}$ and $\frac{dv}{dt} = \frac{d^2u}{dt^2}$

Equation(2) becomes a 2^{nd} order LDE:

$$C\frac{d^2u}{dt^2} + \frac{1}{R}\frac{du}{dt} + \frac{1}{L}u = i \tag{62}$$

The Circuit Equations can also be written as a System of 1^{st} order LDE$_s$

Let $\overrightarrow{x} = \begin{bmatrix} u \\ v \end{bmatrix}$

Than$\frac{d\overrightarrow{x}}{dt} = \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ i - \frac{u}{LC} - \frac{v}{RC} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$
$A\overrightarrow{x} + \overrightarrow{d}$

Hence

$$\frac{d\overrightarrow{x}}{dt} = A\overrightarrow{x} + \overrightarrow{d} \tag{63}$$

- Step[2] Solve Homogeneous Equation (Driving Free Condition)

Approach[a]: Based on 2^{nd} Order Homogeneous Equation:

$$C\frac{d^2u}{dt^2} + \frac{1}{R}\frac{du}{dt} + \frac{1}{L}u = 0 \tag{64}$$

Again, u should be of the form:

$$u(t) = Ke^{st} \tag{65}$$

Substitute it into Equation(63):

$$s^2CKe^{st} + \frac{s}{R}Ke^{st} + \frac{K}{L}e^{st} = 0 \tag{66}$$

Since $e^{st} \neq 0$ for $t \geqslant 0$ in general, than

$$s^2C + \frac{s}{R} + \frac{1}{L} = 0 \tag{67}$$

Charateristic Equation as follow:

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \tag{68}$$

Solutions:

$$s = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} + \frac{4}{LC}}}{2} \tag{69}$$

Let $\omega_0 = \frac{1}{\sqrt{LC}}$ and $2\zeta\omega_0 = \frac{1}{RC}$,than

$$\zeta = \frac{1}{2}\sqrt{\frac{L}{R^2C}} \tag{70}$$

Than System Characteristic *Root* (s):

$$s = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_0 \tag{71}$$

ζ	Characteristic roots	Description
$\zeta > 1$	real, negative, unequal	overdamped
$\zeta = 1$	real, negative, equal	critically damped
$0 < \zeta < 1$	complex conjugates;real parts negative	underdamped
$\zeta = 0$	imaginary conjugates	undamped

- Step[3] Solve Homogenous Equation of First Order LDE

Approach $[b]$:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{72}$$

Form of $\mathbf{x}$:

$$\mathbf{x} = e^{\lambda t}\boldsymbol{\varepsilon} = e^{\lambda t}\begin{bmatrix} \varepsilon_u \\ \varepsilon_v \end{bmatrix} \tag{73}$$

Substitute $\mathbf{x}$ into Equation(72):

$$\lambda e^{\lambda t}\boldsymbol{\varepsilon} = \mathbf{A}e^{\lambda t}\boldsymbol{\varepsilon} \tag{74}$$

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x} \tag{75}$$

Characteristic Equation General Matrix Form:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0} \tag{76}$$

9 Laplace Transforms

- Laplace Transform
  - A mathematical tool for representing system transfer functions of causal LTI systems.
  - A generalization of Fourier transform.
    - The result of Fourier transform can be obtained by evaluating the Laplace transform along the imaginary axis.

Definition 13 *Given $f(t)$ with $f(t) = 0$ for $t < 0$, its Laplace Transform, which is defined as follows, is a complex function over a complex-number domain.*

$$F(s) \equiv \mathcal{L}[f(t)] \triangleq \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} \left(f(t)e^{-\sigma t}\right) e^{-j\omega t}dt. \tag{77}$$

Definition 14 *Correspondingly, the Inverse Laplace Transform is as follows:*

$$f(t) \equiv \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \oint_c F(s)e^{st}ds. \tag{78}$$

Example 15 *Given $f(t)$ & $g(t)$ ($f(t) = g(t) = 0$ for $t < 0$) and their Laplace Transforms $F(s)$ & $G(s)$,the following shows the properties of Laplace Transform.*

- **Linearity**

$$y(t) = af(t) + bg(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = aF(s) + bG(s) \tag{79}$$

- **Time Derivatives**

$$y(t) = \frac{df(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = sF(s) - f(0^-) \tag{80}$$

$$y(t) = \frac{d^n f(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} \ldots\ldots\ldots - d^{n-1} \frac{f(0^-)}{dt}$$

- **Time Integrals**

$$y(t) = \int_{0^-}^{t} f(\tau)d\tau + y(0^-) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{F(s)}{s} + \frac{y(0^-)}{s} \tag{81}$$

- **Time Scaling**

$$y(t) = f(\alpha t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{\alpha} F(\frac{s}{\alpha}), \text{ where } \alpha > 0. \tag{82}$$

- **Time Delay**

$$y(t) = f(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = e^{-st_0} F(s), \text{ where } t_0 > 0. \tag{83}$$

- t **Multiplication**

$$y(t) = t^n f(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = (-1)^n \frac{d^n F(s)}{ds^n} \tag{84}$$

- s **Shift**

$$y(t) = e^{at} f(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = F(s - a) \tag{85}$$

- **Convolution**

$$y(t) = x(t) * h(t) \stackrel{\mathcal{L}}{\longleftrightarrow} Y(s) = X(s)H(s) \tag{86}$$

- **Product**

$$y(t) = f(t)x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{1}{2\pi j}\oint_c F(s)X(s-\lambda)d\lambda. \tag{87}$$

Example 16 *Laplace Transform Pairs*[3]

$$\begin{aligned}
f(t) &\stackrel{\mathcal{L}}{\longleftrightarrow} && F(s)\\
\delta(t) &\stackrel{\mathcal{L}}{\longleftrightarrow} && 1\\
A &\stackrel{\mathcal{L}}{\longleftrightarrow} && \frac{A}{s}\\
At^n &\stackrel{\mathcal{L}}{\longleftrightarrow} && A(\frac{n!}{s^{n+1}})\\
Ae^{at} &\stackrel{\mathcal{L}}{\longleftrightarrow} && \frac{A}{s-a}\\
\cos\omega_0 t &\stackrel{\mathcal{L}}{\longleftrightarrow} && \frac{s}{s^2+\omega_0^2}\\
\sin\omega_0 t &\stackrel{\mathcal{L}}{\longleftrightarrow} && \frac{\omega_0}{s^2+\omega_0^2}\\
2Ae^{at}\cos(\omega_0 t+\theta) &\stackrel{\mathcal{L}}{\longleftrightarrow} && \frac{Ae^{j\theta}}{s-(a+j\omega_0)} + \frac{Ae^{-j\theta}}{s-(a-j\omega_0)} = \frac{2A\cos\theta(s-a-\omega_0\tan\theta)}{(s-a)^2+\omega_0^2}
\end{aligned}$$

10 Equivalent Circuits in Laplace Domain

- Capacitor
  - $i - v$ characteristic in Laplace domain.
    - Admittance of capacitor in Laplace domain is sC.

$$i_C(t) = C\frac{dv_C(t)}{dt} \stackrel{\mathcal{L}}{\longleftrightarrow} I_C(s) = C(sV_C(s) - v_C(0^-)) \tag{88}$$

    - Figure 13 (a) depicts the KCL equivalent circuit in Laplace domain.
  - $v - i$ characteristic in Laplace domain.
    - Impedance of capacitor in Laplace domain is $1/sC$.

$$\begin{aligned}
v_C(t) &= \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau \qquad (89)\\
&= \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0^-) \stackrel{\mathcal{L}}{\longleftrightarrow} V_C(s) = \frac{1}{sC}I_C(s) + \frac{1}{s}v_C(0^-)
\end{aligned}$$

[3] Footnote

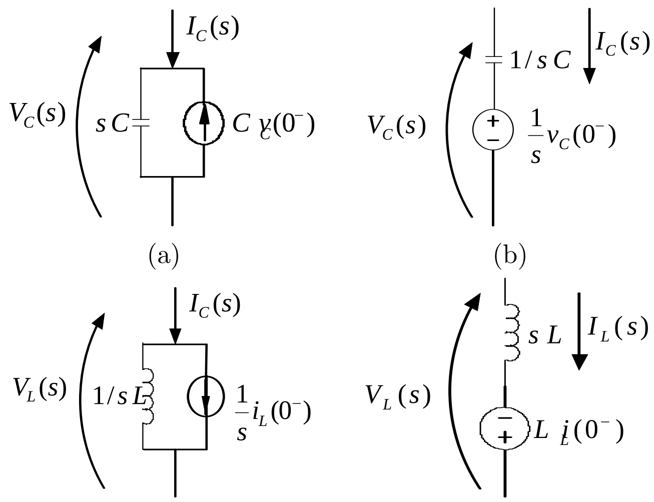


Figure 13: (a) KCL equivalent circuit of a capacitor in Laplace domain. (b) KVL equivalent circuit of a capacitor in Laplace domain. (c) KCL equivalent circuit of an inductor in Laplace domain. (b) KVL equivalent circuit of an inductor in Laplace domain.

    * Figure 13 (b) depicts the KVL equivalent circuit in Laplace domain.

- Inductor
  - $i - v$ characteristic in Laplace domain.
    * Admittance of inductor in Laplace domain is $\underline{1/sL}$.

$$\begin{aligned} i_L(t) &= \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau && (90) \\ &= \frac{1}{L}\int_{0}^{t} v_L(\tau)d\tau + i_L(0^-) \overset{\mathcal{L}}{\longleftrightarrow} I_L(s) = \frac{1}{sL}V_L(s) + \frac{1}{s}i_L(0^-) \end{aligned}$$

    * Figure 13 (c) depicts the KCL equivalent circuit in Laplace domain.
  - $v - i$ characteristic in Laplace domain.
    * Impedance of inductor in Laplace domain is $\underline{sL}$.

$$v_L(t) = L\frac{di_L(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} V_L(s) = L(sI_L(s) - i_L(0^-)) \qquad (91)$$

    * Figure 13 (d) depicts the KVL equivalent circuit of an inductor in Laplace domain.

11 System Transfer Function in Time and Laplace Domains

- Two ways to deduce the system transfer function in time and Laplace domains.
  - Start in time domain and then take the Laplace transform.
  - Start in Laplace domain and use Inverse Laplace transform to return to time domain.

Example 17 *Consider the RC circuit in Figure 14, the input is the current source $i_s(t)$ and the output $v_o(t)$ is the voltage across the RC components. Find out the system transfer function and the outputs w.r.t. different inputs.*

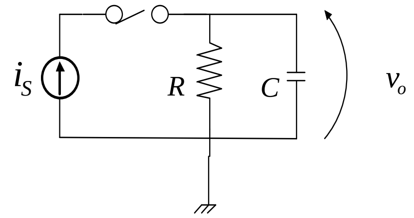


Figure 14: The RC circuit to be solved in time domain and Laplace domain.

- By writing the node equation with KCL in time domain, we obtain the system equation.

$$i_S(t) = \frac{v_o(t)}{R} + C\frac{dv_o(t)}{dt} \tag{92}$$

- By taking Laplace transform to both sides, the system equation in Laplace domain can be derived.

$$I_S(s) = \frac{V_o(s)}{R} + C(sV_o(s) - v_o(0^-)) \tag{93}$$

- The system transfer function in Laplace domain.

$$H(s) = \frac{V_o(s)}{I_S(s)}|_{v_o(0^-)=0} = \frac{1}{sC + (1/R)} = \frac{R}{1 + sRC} \tag{94}$$

11.0.1 Initial Value (Driving Free) Response

- Given $i_S(t) = 0$, $I_S(s) = 0$, the output can be obtained from Eq. (93).

$$\begin{aligned} 0 &= \frac{V_o(s)}{R} + C(sV_o(s) - v_o(0^-)) \\ \Rightarrow \quad V_o(s) &= \frac{v_o(0^-))}{s + (1/RC)} \end{aligned} \tag{95}$$

- The time domain response can be calculated by taking Inverse Laplace transform.

$$v_o(t) = v_o(0^-)e^{-t/RC} \tag{96}$$

11.0.2 Impulse Response

- Given $i_S(t) = \delta(t)$; $I_S(s) = 1$, the output can be obtained from Eq. (94).
  - For simplicity, we do not consider initial condition, i.e., $v_o(0^-) = \lim_{t\to 0^-} v_o(t) = 0$.

$$V_o(s) = \frac{R}{1 + sRC} \tag{97}$$

- The time domain response can be calculated by taking Inverse Laplace transform.

$$v_o(t) = \frac{1}{C}e^{-t/RC} \tag{98}$$

11.0.3 Step Response

- Given $i_S(t) = u(t)$; $I_S(s) = \frac{1}{s}$, the output can be obtained from Eq. (94).
  - For simplicity, we do not consider initial condition, i.e., $v_o(0^-) = \lim_{t\to 0^-} v_o(t) = 0$.

$$V_o(s) = \frac{\frac{1}{C}}{s(s + (1/RC))} = \frac{R}{s} - \frac{R}{s + (1/RC)} \tag{99}$$

- The time domain response can be calculated by taking Inverse Laplace transform.
  - The capacitor is like an open circuit once it is fully charged.

$$v_o(t) = R - Re^{-t/RC} = R(1 - e^{-t/RC}) \tag{100}$$

11.0.4 Sinusoidal Response

- Given $i_S(t) = \cos(\omega_0 t)\, u(t)$; $I_S(s) = s/\left(s^2 + \omega_0^2\right)$, the output can be obtained from Eq. (94).
  - For simplicity, we do not consider initial condition, i.e., $v_o(0^-) = \lim_{t\to 0^-} v_o(t) = 0$.

$$V_o(s) = \frac{R}{1 + sRC} \times \frac{s}{s^2 + \omega_0^2} = \frac{A}{1 + sRC} + \frac{Bs + D}{s^2 + \omega_0^2} \tag{101}$$

- The time domain response can be calculated by taking Inverse Laplace transform.

– The term $A/(1+sRC)$ stands for transient response, which can be transformed into exponentially decaying time function.

  * $A = -R^2C/\left(1+(\omega_0 RC)^2\right)$.

– The term $(Bs+D)/\left(s^2+\omega_0^2\right)$ represent the steady state response, which shall yield the time-domain signal.

$$\begin{aligned}\frac{Bs+D}{s^2+\omega_0^2} &= \frac{Bs}{s^2+\omega_0^2}+\frac{D}{s^2+\omega_0^2}\\ \Rightarrow v_o(t) &= B\cos(\omega_0 t)+\frac{D}{\omega_0}\sin(\omega_0 t)\\ &= K'\cos(\omega_0 t-\phi)\end{aligned} \tag{102}$$

  * $K' = \sqrt{B^2+\left(\frac{D}{\omega_0}\right)^2}$.
  * $\phi = \tan^{-1}(\frac{D}{\omega_0 B})$.
  * $B = R/\left(1+\left(\frac{\omega_0}{1/RC}\right)^2\right)$.
  * $D = \frac{1}{C}\left(\frac{\omega_0}{1/RC}\right)^2/\left(1+\left(\frac{\omega_0}{1/RC}\right)^2\right)$.

– $\omega_0 \ll \frac{1}{RC}$
  * $B \simeq R$, $D \simeq 0$, $v_o(t) = R\cos(\omega_0 t)$.
  * At low frequency, the capacitor is similar to an open circuit.

– $\omega_0 \gg \frac{1}{RC}$
  * $B \simeq 0$, $D \simeq \frac{1}{C}$, $v_o(t) = \frac{1}{\omega_0 C}\sin(\omega_0 t) = \frac{1}{\omega_0 C}\cos\left(\omega_0 t-\frac{\pi}{2}\right)$.
  * At high frequency, the capacitor is similar to a short circuit.

– The circuit acts as a low-pass filter.

Example 18 *Consider the RLC circuit in Figure 15, the input is the current source $i_s(t)$ and the output $v_o(t)$ is the voltage across the RLC components. Find out the system transfer function and the outputs w.r.t. different inputs.*

Start in Time Domain

- By writing the node equation with KCL in time domain, we obtain

$$i_S(t) = i_R(t)+i_C(t)+i_L(t), \tag{103}$$

  where

$$\begin{aligned} i_R(t) &= \frac{v_o(t)}{R}\\ i_C(t) &= C\frac{dv_o(t)}{dt}\\ i_L(t) &= \frac{1}{L}\int_{-\infty}^{t} v_o(\tau)d\tau = \frac{1}{L}\int_0^t v_o(\tau)d\tau + i_L(0^-)\end{aligned} \tag{104}$$

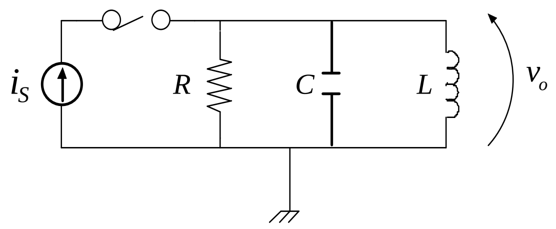


Figure 15: The RLC circuit to be solved in time domain and Laplace domain.

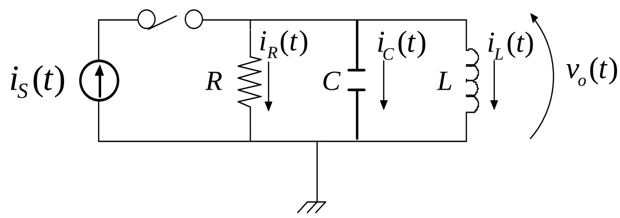


Figure 16: Time domain analysis.

- The system equation in time domain.

$$i_S(t) = \frac{v_o(t)}{R} + C\frac{dv_o(t)}{dt} + \frac{1}{L}\int_0^t v_o(\tau)d\tau + i_L(0^-) \tag{105}$$

- Apply the Laplace transform to both sides of Eq.(105).

$$I_S(s) = \frac{V_o(s)}{R} + C(sV_o(s) - v_o(0^-)) + \frac{1}{sL}V_o(s) + \frac{1}{s}i_L(0^-), \tag{106}$$

where $I_S(s) \triangleq \mathcal{L}[i_S(t)]$ and $V_o(s) = \mathcal{L}[v_o(t)]$.

- The system transfer function in Laplace domain.
  - Assume no initial condition, i.e., $v_o(0^-) = \lim_{t\to 0^-} v_o(t) = 0$ and $i_L(0^-) = \lim_{t\to 0^-} i_L(t) = 0$.

$$\frac{V_o(s)}{I_S(s)} = 1/(\frac{1}{R} + sC + \frac{1}{sL}) \tag{107}$$

Start in Laplace Domain

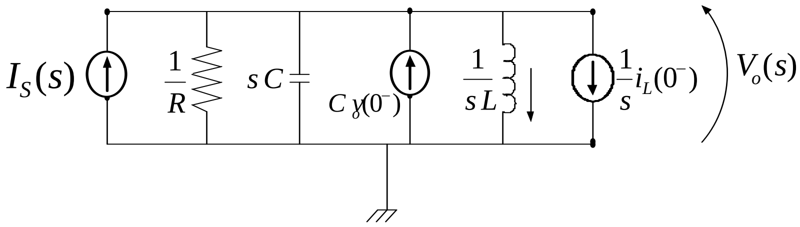


Figure 17: Laplace domain analysis.

- Replace all the circuit elements by their KCL equivalent circuits in Laplace domain.

$$\begin{aligned} R &\longrightarrow \frac{1}{R} \\ C &\longrightarrow sCV_o(s) - Cv_o(0^-) \\ L &\longrightarrow \frac{1}{sL}V_o(s) + \frac{1}{s}i_L(0^-) \end{aligned} \tag{108}$$

$$\begin{aligned} I_S(s) &= I_R(s) + I_C(s) + I_L(s) \\ &= \frac{V_o(s)}{R} + \left(sCV_o(s) - Cv_o(0^-)\right) + \left(\frac{1}{sL}V_o(s) + \frac{1}{s}i_L(0^-)\right) \end{aligned} \tag{109}$$

- The system transfer function.

$$\begin{aligned} I_S(s) &= I_R(s) + I_C(s) + I_L(s)|_{v_o(0^-)=0, i_L(0^-)=0} \\ &= \frac{V_o(s)}{R} + sCV_o(s) + \frac{1}{sL}V_o(s) \\ \Rightarrow \frac{V_o(s)}{I_S(s)} &= \frac{1}{(\frac{1}{R} + sC + \frac{1}{sL})} \end{aligned} \tag{110}$$

11.0.5 Impulse Response

- Impulse response is the system output w.r.t an input signal of impulse function. It is also the transfer function of the system which can be used to find system poles and zeros.
  - $i_S(t) = \delta(t)$; $\delta(t) \xrightarrow{\mathcal{L}} 1$.
  - Assume no initial condition.

$$\begin{aligned} V_o(s) &= \frac{1}{(\frac{1}{R} + sC + \frac{1}{sL})} I_S(s) \\ &= \frac{1}{(\frac{1}{R} + sC + \frac{1}{sL})} \times 1 \\ &= \frac{sRL}{s^2RLC + sL + R} \end{aligned} \tag{111}$$

- System poles and zeros are the roots of denominator and numerator of the system transfer function.
  - Zeros $\triangleq$ the roots of $N(s)$.
  - Poles $\triangleq$ the roots of $D(s)$.

$$H(s) = \frac{V_o(s)}{I_S(s)} \triangleq \frac{N(s)}{D(s)} \tag{112}$$

  - The simplest way to obtain $H(s)$ is by analyzing the system impulse response in Laplace domain.

$$\begin{aligned} H(s) &= \frac{V_o(s)}{1} = V_o(s) \\ &= \frac{sRL}{s^2RLC + sL + R} \\ &= \frac{N(s)}{D(s)} \end{aligned} \tag{113}$$

- The time-domain expression of $v_o(t) \triangleq \mathcal{L}^{-1}[V_o(s)]$ depends on the nature of the system poles, which are the roots of denominator.
  - Let the system characteristic equation be equal to 0.

$$s^2RLC + sL + R = 0 \tag{114}$$

  - Rewrite it as

$$as^2 + bs + c = 0 \text{ with } a = RLC,\ b = L,\ c = R \tag{115}$$

  - The roots are

$$\begin{aligned} s_P \text{ and } s'_P &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{4ac}{4a^2}} \\ &= -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}} \end{aligned} \tag{116}$$

  - The nature of the roots $(s_P,\ s'_P)$ shall depend on the value of the critical expression.

$$\begin{aligned} \lambda^2 &\triangleq (\frac{b}{2a})^2 - \frac{c}{a} \\ &= (\frac{L}{2RLC})^2 - (\frac{R}{RLC}) \\ &= (\frac{1}{2RC})^2 - \frac{1}{LC} \end{aligned} \tag{117}$$

– Traditionally, we define two more parameters.

  ∗ Neper frequency α.

$$\alpha \triangleq \frac{b}{2a} = \frac{1}{2RC} \tag{118}$$

  ∗ Resonant frequency ω_0.

$$\omega_0 \triangleq \sqrt{\frac{c}{a}} = \frac{1}{\sqrt{LC}} \tag{119}$$

– We can now rewrite the roots of the characteristics equation.

$$s_P \text{ and } s'_P = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{120}$$

• Three distinct cases are possible with respect to s_P and s'_P depending on the values of α and ω_0.

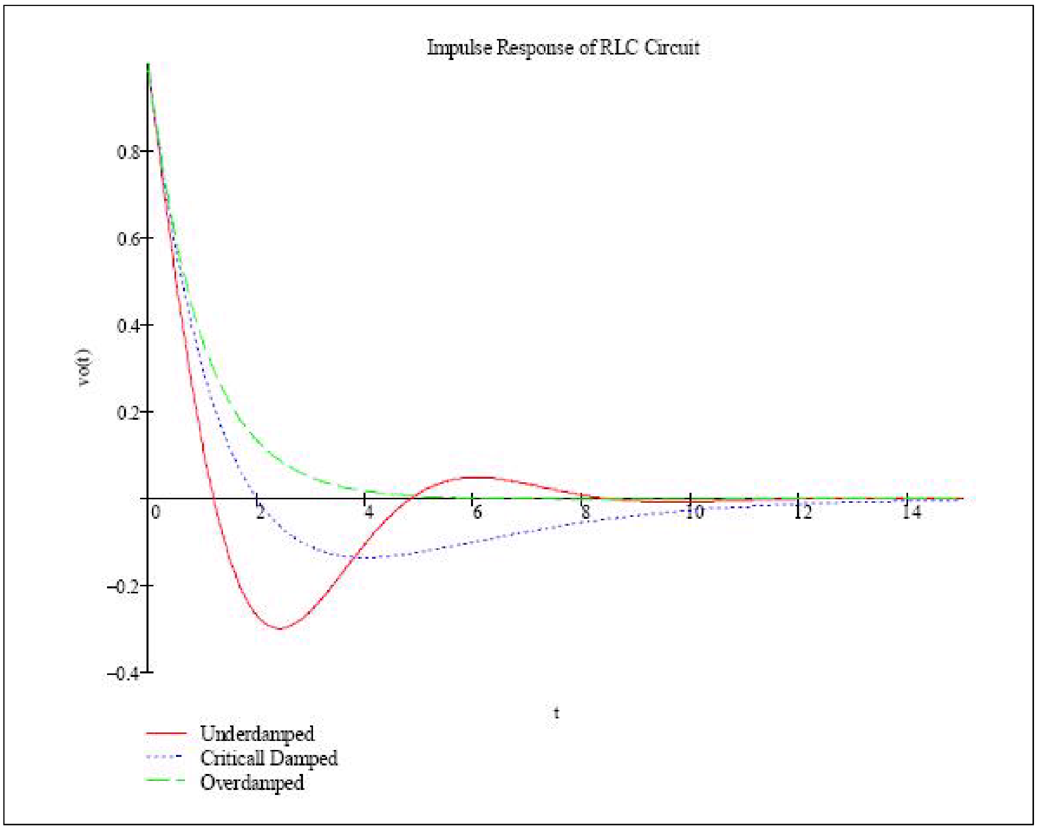


Figure 18:

- **Overdamped**
  - $\lambda \triangleq \sqrt{\alpha^2 - \omega_0^2}$ is real, i.e., $\alpha^2 > \omega_0^2$.
  - s_P and s'_P are two real, distinct and negative values.
  - $V_o(s)$ can be factorized as follows.
    - $A = RL/(1 - \frac{s'_P}{s_P})$ and $B = RL/(1 - \frac{s_P}{s'_P})$.
    - Both A and B are real numbers.

$$\begin{aligned} V_o(s) &= \frac{sRL}{RLC(s - s_P)(s - s'_P)} \\ &= \frac{A}{s - s_P} + \frac{B}{s - s'_P} \\ &= \frac{A}{s + |s_P|} + \frac{B}{s + |s'_P|} \end{aligned} \tag{121}$$

  - By taking the inverse Laplace transform, the response of $v_o(t)$ (without initial condition) can be formulized as follows:

$$v_o(t) = Ae^{-|s_P|t} + Be^{-|s'_P|t} \tag{122}$$

  - Figure **??** depicts the $v_o(t)$ in the overdamped case.

- **Underdamped**
  - $\lambda \triangleq \sqrt{\alpha^2 - \omega_0^2}$ is imaginary, i.e., $\alpha^2 < \omega_0^2$.
  - s_P and s'_P are two complex conjugate poles.
    - The damped resonant frequency $\omega_d \triangleq \sqrt{\omega_0^2 - \alpha^2}$ is real.

$$s_P \text{ and } s'_P = -\alpha \pm j\omega_d \tag{123}$$

  - $V_o(s)$ can be factorized as follows.
    - A and B are complex conjugates.

$$V_o(s) = \frac{sRL}{RLC\left(s - (-\alpha + \jmath\omega_d)\right)\left(s - (-\alpha - \jmath\omega_d)\right)} \tag{124}$$

$$= \frac{A}{s - (-\alpha + \jmath\omega_d)} + \frac{B}{s - (-\alpha - \jmath\omega_d)} \tag{125}$$

  - Taking inverse Laplace transform, $v_0(t)$ can be obtained.
    - Note that $(A + B)$ and $j(A - B)$ are real number since A and B are complex conjugates.

$$\begin{aligned} v_o(t) &= Ae^{(-\alpha+\jmath\omega_d)t} + Be^{(-\alpha-\jmath\omega_d)t} \\ &= e^{-\alpha t}\left(Ae^{\jmath\omega_d t} + Be^{-\jmath\omega_d t}\right) \\ &= e^{-\alpha t}\left[A\left(\cos\omega_d t + j\sin\omega_d t\right) + B\left(\cos\omega_d t - j\sin\omega_d t\right)\right] \\ &= e^{-\alpha t}\left[(A + B)\cos\omega_d t + j(A - B)\sin\omega_d t\right] \end{aligned} \tag{126}$$

– Figure **??** depicts the $v_o(t)$ in the underdamped case.

- **Critically damped**
  – $\lambda \triangleq \sqrt{\alpha^2 - \omega_0^2}$=0, i.e., $\alpha^2 = \omega_0^2$.
  – s_P and $s'_P = -\alpha$.

$$V_o(s) = \frac{sRL}{RLC(s-(-\alpha))^2} = \frac{A}{s+\alpha} + \frac{B}{(s+\alpha)^2} \tag{127}$$

  – Taking inverse Laplace Transform, $v_0(t)$ can be obtained.

$$v_0(t) = Ae^{-\alpha t} + Bte^{-\alpha t} \tag{128}$$

  – Figure **??** depicts the $v_0(t)$ in the case of critically damped.

- Summary
  – The positions of poles determine the nature of System Response.

$$s_P \text{ and } s'_P = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \text{ with } \alpha = \frac{1}{2RC} \text{ and } \omega_0 = \frac{!}{\sqrt{LC}}$$

  – Response is overdamped
    * if $\alpha^2 - \omega_0^2 > 0$ and s_P, s'_P are two distinct real numbers.
  – Response is underdamped
    * if $\alpha^2 - \omega_0^2 < 0$ and s_P, s'_P are complex conjugate numbers.
  – Response is critically damped
    * if $\alpha^2 = \omega_0^2$ and s_P, s'_P are real and the same.
  – The positions of system poles move along the path (known as Root Locus) shown in the following diagram.

11.0.6 Sinusoidal Response

- Sinusoidal response is the system output w.r.t an input signal of sinusoidal function.
  – $i_S(t) = \cos(\omega t)\, u(t)$ or $I_S(s) = \mathcal{L}[i_S(t)] = s/\left(s^2 + \omega^2\right)$.
  – For simplicity, we do not consider initial condition.

$$\begin{aligned} V_o(s) &= \frac{s^2RL}{(s^2+\omega^2)(s^2RLC + sL + R)} \qquad (129) \\ &= \frac{s^2RL}{RLC(s-s_P)(s-s'_P)(s^2+\omega^2)} = \frac{A}{(s-s_P)} + \frac{B}{(s-s'_P)} + \frac{Ds+E}{(s^2+\omega^2)} \end{aligned}$$

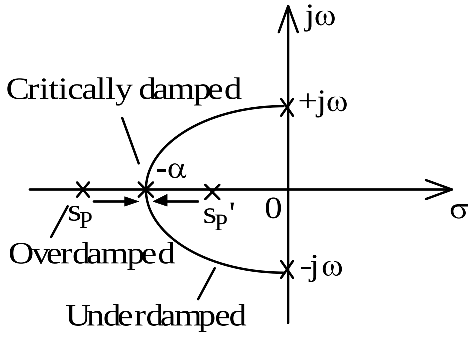


Figure 19: Locations of poles and zeros.

- Instead of solving for the coefficients A, B, D, E analytically by hand (which is a practically impossible task), we shall deduce the time and frequency domain responses by reasoning.
- Time-Domain Response
  * The terms $A/(s-s_P)$ and $B/(s-s'_P)$ stand for transient response. Both can be transformed into damped response in time with exponentially decaying envelop.

$$e^{-\alpha t} \text{ where } \alpha = \frac{1}{2RC} \tag{130}$$

  * The term $(Ds+E)/(s^2+\omega^2)$ represent the steady state response, which shall yield the time-domain signal $K'\cos(\omega t-\phi)$.
    · $K' = \sqrt{D^2+E^2}$.
    · $\phi = \tan^{-1}(\frac{E}{D})$.

$$\begin{aligned}\frac{Ds+E}{(s^2+\omega^2)} &= \frac{Ds}{(s^2+\omega^2)} + \frac{E}{(s^2+\omega^2)} \\ &\Rightarrow D\cos(\omega t)+E\sin(\omega t) \\ &= K'\cos(\omega t-\phi) \end{aligned} \tag{131}$$

12 Bode Plots

- A graph used to show the frequency response of an LTI system.
  - To plot the magnitude in decibels (dB) and use a log scale for ω.
    * The log scale helps to compress a wide range of data.
  - Both the magnitude and the phase of the transfer function versus the angular frequency ω.

- System transfer function $H(s)$ can be written as a product of factors of the following items.
  1. Constant factor K.
  2. Poles or zeros at the origin, $s^{\pm N}$.
  3. Real poles or zeros, $(Ts+1)^{\pm N}$.
  4. Complex-conjugate poles or zeros, $(T^2s^2+2T\zeta s+1)^{\pm N}$, where ζ is the damping ratio and $0<\zeta<1$.

$$H(s)=\frac{N(s)}{D(s)} \tag{132}$$

- The magnitude of $H(s)$ in dB allows us to plot the factors individually and sum the results to obtain the complete plot.

$$\left|H(s)\right|_{dB}=20\log_{10}\left|H(s)\right| \tag{133}$$

Given $H(s)$ as follows,

$$H(s)=K\frac{(1+T_{z1}s)(1+T_{z2}s)}{(1+T_{p1}s)(1+T_{p2}s)} \tag{134}$$

$\left|H(s)\right|_{dB}$ can be obtained by plotting each factor individually

$$\begin{aligned}\left|H(s)\right|_{dB} &= 20\log_{10}\left|H(s)\right| \\ &= 20\log_{10}\left|K\frac{(1+T_{z1}s)(1+T_{z2}s)}{(1+T_{p1}s)(1+T_{p2}s)}\right| \\ &= 20\log_{10}\left|K\right|+20\log_{10}\left|1+T_{z1}s\right|+20\log_{10}\left|1+T_{z2}s\right| \\ &\quad -20\log_{10}\left|1+T_{p1}s\right|-20\log_{10}\left|1+T_{p2}s\right|\end{aligned} \tag{135}$$

- Bode plot for each of the factors.
  1. Constant factor, K.
     - Magnitude
       * The magnitude $20\log_{10}|K|$ is <u>a constant</u>.
     - Phase
       * The phase is <u>a constant</u> and equal to $0\,°$ ($e^{j0}=1$) or $\pm180\,°$ ($e^{\pm j\pi}=-1$) depending on whether K is positive or negative, respectively.
  2. Poles or zeros at the origin, $s^{\pm N}$.
     - Magnitude
       * The magnitude is <u>a straight line</u> that intersects the ω axis (0dB) at $\omega=1$ and has a slope of <u>±20dB/decade.</u>

- Phase
  * The phase is a constant and equal to $\pm N 90°$.

$$H(j\omega) = (j\omega)^{\pm N} = (\omega e^{j\pi/2})^{\pm N} \tag{136}$$

$$\begin{aligned} |H(j\omega)|_{dB} &= 20\log_{10}\sqrt{(\omega^{\pm N})^2} \\ &= \pm 20N\log_{10}|\omega| \end{aligned} \tag{137}$$

$$\measuredangle H(j\omega) = \pm N(j\pi/2) = \pm N 90° \tag{138}$$

3. Real poles or zeros, $(Ts+1)^{\pm N}$.
   - Poles or zeros locate at $-\frac{1}{T}$.

$$H(j\omega) = (j\omega T + 1)^{\pm N} = \left(re^{j\theta}\right)^{\pm N} \tag{139}$$

$$\begin{aligned} |H(j\omega)|_{dB} &= r_{dB} \\ &= 20\log_{10}\left(\sqrt{1+(\omega T)^2}\right)^{\pm N} \\ &= \pm 10N\log_{10}\left(1+(\omega T)^2\right) \end{aligned} \tag{140}$$

   - Magnitude
     * Low frequency response ($\omega T \ll 1; \omega \ll \frac{1}{T}$): A horizontal line of 0dB.

$$|H(j\omega)|_{dB} = \pm 10N\log_{10}(1) = 0 \text{ dB} \tag{141}$$

     * High frequency response ($\omega T \gg 1; \omega \gg \frac{1}{T}$): A straight line of slope $\pm 20N$dB/decade that intersects 0dB when $\omega = \frac{1}{T}$.

$$\begin{aligned} |H(j\omega)|_{dB} &= \pm 10N\log_{10}(\omega T)^2 \\ &= \pm 20N\log_{10}\left(\frac{\omega}{1/T}\right) \\ &= \pm 20N\left(\log_{10}\omega - \log_{10}\frac{1}{T}\right) \end{aligned} \tag{142}$$

     * Corner frequency response $\omega = \frac{1}{T}$.

$$|H(j\omega)|_{dB} = \pm 10N\log_{10}2 = \pm 3N\text{dB} \tag{143}$$

$$|H(j\omega)| = 10^{\pm 3N/20} = 0.7079^N \text{ (negative) or } 1.4125^N \text{ (positive)} \tag{144}$$

   - Phase

$$\measuredangle H(j\omega) = \pm N\theta = \pm N\tan^{-1}\omega T \tag{145}$$

* Low frequency: $\measuredangle H(j\omega) \approx 0^\circ$.
* Corner frequency: $\measuredangle H(j\omega) \approx \pm N45^\circ$.
* High frequency: $\measuredangle H(j\omega) \approx \pm N90^\circ$.

4. Complex conjugate poles or zeros, $(T^2s^2 + 2\zeta Ts + 1)^{\pm N}$.
   - For simplicity, we consider only the case of a single pair of complex conjugate poles.
     * If the poles are repeated by N times, all coordinates on the curves will be multiplied by N.
     * If we have zeros instead of poles, curves are mirror images through the ω axis.

$$H(s) = \frac{1}{T^2s^2 + 2\zeta Ts + 1} \tag{146}$$

$$\begin{aligned} |H(j\omega)|_{dB} &= -20\log_{10}\left(\left(1 - T^2\omega^2\right)^2 + 4\zeta^2T^2\omega^2\right)^{1/2} \\ &= -10\log_{10}\left(\left(1 - T^2\omega^2\right)^2 + 4\zeta^2T^2\omega^2\right) \end{aligned} \tag{147}$$

   - Magnitude
     * Low frequency response ($\omega T \ll 1; \omega \ll \frac{1}{T}$): <u>A horizontal line</u> of 0dB.
     * High frequency response ($\omega T \gg 1; \omega \gg \frac{1}{T}$): <u>A straight line</u> of slope $\pm$40dB/decade that intersects 0dB when $\omega = \frac{1}{T}$.

$$\begin{aligned} |H(j\omega)|_{dB} &= -10\log_{10}\left(\left(1 - T^2\omega^2\right)^2 + 4\zeta^2T^2\omega^2\right) \\ &\simeq -10\log_{10}\left(\left(T^2\omega^2\right)^2 + 4\zeta^2T^2\omega^2\right) \simeq -40\log_{10}(T\omega) \\ &= -40\left(\log_{10}\omega - \log_{10}\frac{1}{T}\right) \end{aligned} \tag{148}$$

   - Phase
     * $0 < \omega < 1/T$.

$$\measuredangle H(j\omega) = -\tan^{-1}\frac{2T\zeta\omega}{1 - T^2\omega^2} \tag{149}$$

     * $\omega > 1/T$.

$$\measuredangle H(j\omega) = -180 + \tan^{-1}\frac{2T\zeta\omega}{T^2\omega^2 - 1} \tag{150}$$

Example 19 *Consider the transfer function $V_o(s) = R/(1 + sRC)$ where $R =$ $1\,\mathrm{k}\Omega$ and $C = \mu\mathrm{F}$, express its frequency responses including both the magnitude and the phase responses with Bode Plot.*

- $V_o(s)$ can be firstly written as follows.

$$V_o(s) = R \times \frac{1}{1 + sRC} \tag{151}$$

- The factor R.
  - Magnitude is a constant and equal to $20 \log_{10} 10^3 = 60$dB.
  - Phase is a constant and equal to $0\,^\circ$.
- The factor $1/(1 + sRC)$.
  - $\omega < 1/RC$
    - Magnitude is a constant and equal to 0dB.
    - Phase is around $0\,^\circ$.
  - $\omega \geq 1/RC$
    - Magnitude is a straight line of slope -20dB/decade.
    - Phase is around $-45\,^\circ$ when $\omega = 1/RC$.
    - Phase is around $-90\,^\circ$ when $\omega \gg 1/RC$.

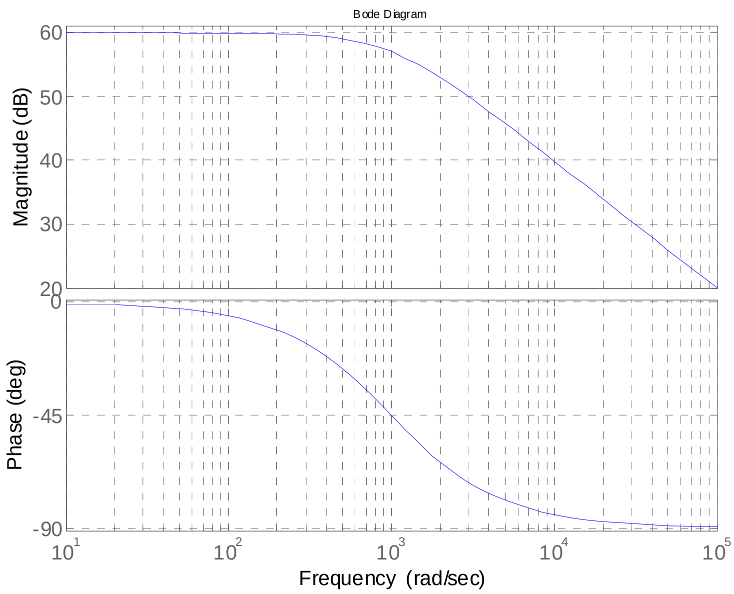


Figure 20: Bode plot for the transfer function $V_o(s) = R/(1 + sRC)$.

Example 20 *Consider the transfer function* $V_o(s) = sRC/(1+sRC)$ *where* $R = 1\,\mathrm{k}\Omega$ *and* $C = \mu\mathrm{F}$, *express its frequency response including both the magnitude and the phase responses with Bode Plot.*

- $V_o(s)$ can be firstly written as follows.

$$V_o(s) = RC \times s \times \frac{1}{1+sRC} \tag{152}$$

- The factor RC
  - Magnitude is a constant and equal to $20\log_{10} 10^{-3} = -60$dB.
  - Phase is a constant and equal to $0\,^\circ$.
- The factor s
  - Magnitude is a straight line of slope 20dB/decade and intersects the ω-axis at $\omega = 1$.
  - Phase is $90\,^\circ$.
- The factor $1/(1+sRC)$.
  - $\omega < 1/RC$
    * Magnitude is a constant and equal to 0dB.
    * Phase is around $0\,^\circ$.
  - $\omega \geq 1/RC$
    * Magnitude is a straight line of slope -20dB/decade.
    * Phase is around $-45\,^\circ$ when $\omega = 1/RC$.
    * Phase is around $-90\,^\circ$ when $\omega \gg 1/RC$.

Example 21 *Consider the transfer function* $V_o(s) = sRL/\left(s^2RLC + sL + R\right)$ *where* $R = 1\,\mathrm{k}\Omega$, $C = 1\,\mu\mathrm{F}$ *and* $L = 1H$, *express its frequency response including both the magnitude and the phase responses with Bode Plot.*

- $V_o(s)$ can be firstly written as follows.

$$V_o(s) = RL \times s \times \frac{1}{(s^2RLC + sL + R)} = L \times s \times \frac{1}{\left(s^2LC + s\frac{L}{R} + 1\right)} \tag{153}$$

- The factor L
  - Magnitude is a constant and equal to $20\log_{10} 1 = 0$dB.
  - Phase is a constant and equal to $0\,^\circ$.
- The factor s

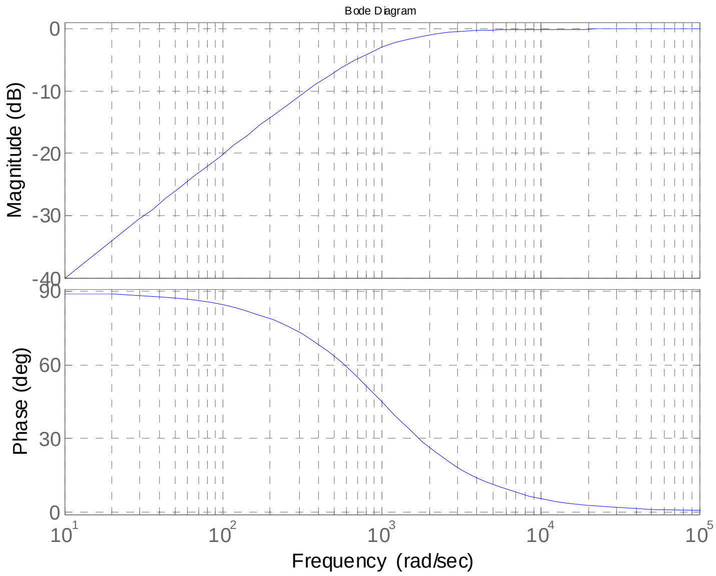


Figure 21: Bode plot for the system transfer function $V_o(s) = sRC/(1 + sRC)$.

- Magnitude is a straight line that intersects the ω axis (0dB) at $\omega = 1$ and has a slope of 20dB/decade.
- Phase is a constant and equal to 90°.

- The factor $1/(s^2LC + s\frac{L}{R} + 1)$
  - $\omega \ll \frac{1}{T} = \frac{1}{\sqrt{LC}}$
    - Magnitude is a horizontal line of 0dB.
    - Phase is $-\tan^{-1}\frac{2T\zeta\omega}{1-T^2\omega^2}$.
  - $\omega \gg \frac{1}{T} = \frac{1}{\sqrt{LC}}$
    - Magnitude is a straight line of slope -40dB/decade that intersects 0dB when $\omega = \frac{1}{T}$.
    - Phase is $-180 + \tan^{-1}\frac{2T\zeta\omega}{T^2\omega^2-1}$.

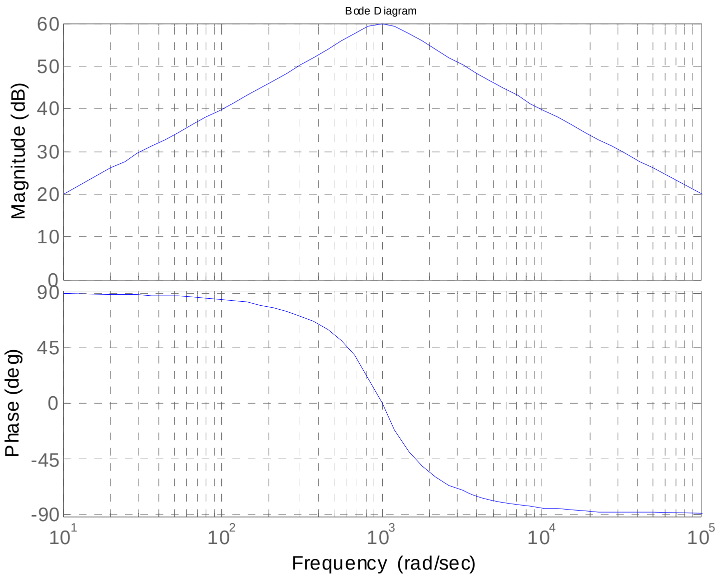


Figure 22: Bode plot for the system transfer function $V_o(s) = sRL/\left(s^2RLC + sL + R\right)$.