Задачи с одним свободным концом

$J(y)=\int_a^b F(x,y,y')dx \qquad y(a)=A$

доп. экстремаль:

$\frac{\partial F}{\partial y}=\frac{d}{dx}\frac{\partial F}{\partial y'}, \quad y(a)=A$

$\left.\frac{\partial F}{\partial y'}\right|_{x=b}=0$ — условие трансверсальности

Пр. $J(y)=\int_0^1 (y+xy'+y'^2)dx, \quad y(0)=0$

$1=(x+2y')' \ ; \ y''=0$

$\Rightarrow y=C_1x+C_2 \qquad x+2y'|_{x=1}=0 \qquad y=C_1x$

$y(0)=C_2=0 \qquad 2y'(1)=0 \qquad C_1=-1/2$

$y'(1)=-1/2 \qquad \Rightarrow$ доп. экстремаль $y_0=-x/2$

$h\in C^1[0,1]$

$h(0)=0$

$\Delta J=J(y_0+h)-J(y_0)=\int_0^1\left(y_0+h+x(y_0'+h')+(y_0'+h')^2-y_0-xy_0'-y_0'^2\right)dx=$

$=\int_0^1(\underbrace{h+xh'}_{(xh)'}+2y_0'h'+h'^2)dx=\int_0^1 h'^2dx+xh\Big|_0^1-\int_0^1 h'dx=\int_0^1 h'^2dx+\cancel{h(1)-h(1)+h(0)}^{\,0}=$

$=\int_0^1 h'^2dx\geq 0$

$\Delta J=0: \int_0^1 h'^2dx=0 \ ; \ h=C, \ h(0)=0 \Rightarrow h\equiv 0$

$\Rightarrow y_0$ — строгий мин.

Пр. $J(y)=\int_1^3(8yy'\ln x-xy'^2+6xy')dx, \quad y(3)=15$

$8y'\ln x=(8y\ln x-2xy'+6x)' \ ; \ \cancel{8y'\ln x}=\cancel{8y'\ln x}+\frac{8y}{x}-2y'-2xy''+6$

$x^2y''+xy'-4y=3x$ — ур-е Эйлера

$y''-y'+y'-4y=3e^t \qquad x=e^t \quad y'_x=e^{-t}y'_t \quad y''_{xx}=e^{-2t}(y''_{tt}-y'_t)$

$y''-4y=3e^t$

ОУ: $\lambda^2-4=0 \ ; \ \lambda=\pm 2$

$y=C_1e^{2t}+C_2e^{-2t} \ ;$ ЧР: $y=Ae^t \overset{A=-1}{=} -e^t \Rightarrow y=C_1e^{2t}+C_2e^{-2t}-e^t$

$\Rightarrow y=C_1x^2+\frac{C_2}{x^2}-x$

$y(3) = 9C_1 + C_2/9 - 3 = 15$

$\frac{\partial F}{\partial y'} = 8y \ln x - 2xy' + 6x$; $\frac{\partial F}{\partial y'}\Big|_{x=1} = -2y' + 6 = 0 \Rightarrow y'(1) = 3$

$y'(1) = 2C_1 x - 2\frac{C_2}{x^3} - 1 = 2C_1 - 2C_2 - 1 = 3 \Rightarrow y_0 = 2x^2 - x$

$C_1 - C_2 = 2$

непр. диф

$h \in C^1[1,3]$, $h(3) = 0$

$$\Delta J = J(y_0+h) - J(y_0) = \int_1^3 \Big(8(y_0+h)(y_0'+h')\ln x - x(y_0'+h')^2 + 6x(y_0'+h') -$$

$$- 8y_0y_0'\ln x - xy_0'^2 - 6xy_0'\Big)dx = \int_1^3 \Big(8(\overbrace{y_0h' + y_0'h + hh'}^{(y_0h + h^2/2)'})\ln x - x(2y_0'h' + h'^2) +$$

$$+ 6xh'\Big)dx = -\int_1^3 xh'^2 dx + 8\cancel{(y_0h + h^2/2)\ln x}^{\,0}\Big|_1^3 - \int_1^3 8\frac{(y_0h + h^2/2)}{x}dx +$$

$$+ \int_1^3 (6xh' - 2xy_0'h')dx = -\int_1^3 \left(xh'^2 + \frac{4}{x}h^2\right)dx + \int_1^3 \big((8x - 8x^2)h' + (8 - 16x)h\big)dx =$$

$6x - 2xy_0' = 6x - 2x(4x-1) = 8x - 8x^2$; $\frac{8y_0h}{x} = 8(2x-1)h$

$$= -\int_1^3 \left(xh'^2 + \frac{4}{x}h^2\right)dx + \cancel{(8x - 8x^2)h\Big|_1^3}^{\,0} = -\int_1^3 \left(xh'^2 + \frac{4}{x}h^2\right)dx \le 0$$

$\Delta J = 0$: $h' = h \equiv 0 \Rightarrow y_0$ – строгий макс.

Пр. $J(y) = \int_0^1 (y'^2 + yy')dx$, $y(0) = 0$

$y' = (2y' + y)'$; $2y' + y = y + C$; $y' = C$; $y = Cx + C_1$; $y(0) = 0 \to C_1 = 0$

$2y' + y\big|_{x=1} = 0$; $2y'(1) + y(1) = 0$; $2C + C = 0$, $C = 0 \Rightarrow$ доп. экстремаль $y_0 = 0$

$h \in C^1[0,1]$, $h(0) = 0$

$$\Delta J = \int_0^1 \big((y_0'+h')^2 + (y_0'+h')(y_0+h) - y_0'^2 - y_0y_0'\big)dx = \int_0^1 (2y_0'h' + h'^2 + y_0h' +$$

$$+ y_0'h + hh')dx = \int_0^1 (h'^2 + hh')dx = \int_0^1 h'^2 dx + \frac{h^2}{2}\Big|_0^1 = \int_0^1 h'^2 dx + \frac{h^2(1)}{2} \ge 0$$

$\Delta J = 0$: $h' = 0$ $h(1) = 0$ h – const $\Rightarrow h \equiv 0$ – строгий макс.

Задачи без огранич.
(оба конца свободны)

В обоих концах – усл. [illegible],...
Внеинт. членов обращ. в ноль в обоих концах за сч. ф-б

Изопериметрические задачи

$J(y)=\int_a^b F(x,y,y')dx$, $y(a)=A$, $y(b)=B$

аналогия с усл. экстремумом

$\int_a^b G(x,y,y')dx = l = const$

Ф-я Лагранжа: $\mathcal{L}(x,y,y') = F(x,y,y') + \lambda G(x,y,y')$

$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}$ – ур-е Эйлера

$y(a)=A,\ y(b)=B$

$\int_a^b G(x,y,y') = l$

Пр. $J(y) = \int_1^2 x y'^2 dx$ $y(1)=0,\ y(2)=12,\ \int_1^2 xy\,dx = 9$

$$\begin{cases} \mathcal{L}(x,y,y') = xy'^2 + \lambda xy \\ \lambda x = (2xy')' \Rightarrow 2xy' = \frac{\lambda x^2}{2} + C;\quad y = \frac{\lambda x^2}{8} + \frac{C}{2}\ln x + C_1 \\ \int_1^2 xy\,dx = 9 \end{cases}$$

$$\left.\begin{array}{l} y(1) = \lambda/8 + C_1 = 0 \\ y(2) = \lambda/2 + C/2\ln 2 + C_1 = 12 \end{array}\right| \Rightarrow \begin{array}{l} C_1 = -\lambda/8 \\ C/2 \ln 2 = 12 + \lambda/8 - \lambda/2 = 12 - 3\lambda/8 \\ \Rightarrow C = \frac{2(12 - 3\lambda/8)}{\ln 2} \end{array}$$

$\int_1^2 \left(\frac{\lambda x^3}{8} + \frac{C}{2}x\ln x + C_1 x\right)dx = 9 \ominus$

$u = \ln x,\ dv = x\,dx$

$\int_1^2 x\ln x\,dx = \ln x\cdot x^2/2\Big|_1^2 - \int_1^2 \frac{x}{2}dx = 2\ln 2 - \frac{x^2}{4}\Big|_1^2 = 2\ln 2 - 3/4$

$\ominus\ \frac{\lambda x^4}{32}\Big|_1^2 + \frac{C}{2}(2\ln 2 - 3/4) + \frac{C_1 x^2}{2}\Big|_1^2 = \frac{\lambda}{32}\cdot 15 + C\ln 2 - \frac{3}{8}C + \frac{C_1}{2}\cdot 3 = 9$

...

$\Rightarrow \lambda = 32,\ C_1 = -4,\ C = 0$

усл. экстр.

$\Rightarrow y_0 = 4x^2 - 4$

$h \in C^1[1,2]$, $h(1)=h(2)=0$, $\int_1^2 xh\,dx=0$

$\Delta J = J(y_0+h)-J(y_0)=\int_1^2 (x(y_0'+h')^2 - xy_0'^2)dx = \int_1^2 (2xy_0'h'+h'^2)dx =$

$=\int_1^2 h'^2dx+\int_1^2 16x^2h'dx = \left[\int_1^2 16x^2h'dx = \int_1^2 6x^2dh = \cancel{6x^2h}\Big|_1^2 - \int_1^2 \cancel{12xh}dx\right] = \int_1^2 h'^2dx \geqslant 0$

$\Delta J = 0 \Leftrightarrow h'=0 \quad h=c \to h=0$

$\Rightarrow y_0$ - строгий мин.

Задача с высшими производными

$J(y)=\int_a^b F(x,y,y',\ldots y^{(n)})dx$, $y(a)=A_0$ $\quad y(b)=B_0$
непр. фун.
$y'(a)=A_1 \quad y'(b)=B_1$

$\ldots$

$y^{(n-1)}(a)=A_{n-1} \quad y^{(n-1)}(b)=B_{n-1}$

Ур-е Эйлера:

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}+\frac{d^2}{dx^2}\frac{\partial F}{\partial y''}-\ldots+(-1)^n\frac{d^n}{dx^n}\frac{\partial F}{\partial y^{(n)}}=0$$

Пр. $J(y)=\int_0^1(2e^xy-y''^2)dx$, $\quad y(0)=y'(0)=1$

$y(1)=e$, $y'(1)=2e$

$\frac{\partial F}{\partial y}+\frac{d^2}{dx^2}\frac{\partial F}{\partial y''}=0$

$2e^x+(-2y'')''=0$, $\quad y^{(4)}=e^x$

$y=e^x+C_1x^3+C_2x^2+C_3x+C_4 \qquad y(0)=1+C_4=0 \Rightarrow C_4=-1$

$y'(0)=e^x+3C_1x^2+2C_2x+C_3$; $y'(0)=1+C_3=1 \Rightarrow C_3=0$

$y(1)=e+C_1+C_2=e$, $\quad C_1+C_2=0 \qquad C_1=e$

$y'(1)=e+3C_1+2C_2=2e$; $3C_1+2C_2=e \Rightarrow C_2=-e$

$h\in C^2[0,1]$, $h(0)=h(1)=h'(0)=h'(1)=0$

$\Delta J=J(y_0+h)-J(y_0)=\int_0^1(2e^x(y_0+h)-(y_0''+h'')^2-2e^xy_0+y_0''^2)dx=$

$=\int_0^1(2e^xh-2y_0''h''-h''^2)dx=\int_0^1(2e^xh-h''^2)dx-2\int_0^1 y_0''d(h')dx=$

$=\int_0^1(2e^xh-h''^2)dx-2(\cancel{y_0''h'}\Big|_0^1-\int_0^1 y_0'''h'dx)=\int_0^1(2e^xh-h''^2)dx+2(\cancel{y_0'''h}\Big|_0^1-$

$-\int_0^1 y_0^{(4)}hdx)=\int_0^1 2h(\cancel{e^x-y_0^{(4)}})dx-\int_0^1 h''^2dx=-\int_0^1 h''^2dx\leq 0$

$\Delta J=0 \to h=0 \Rightarrow y_0$ - строгий макс.