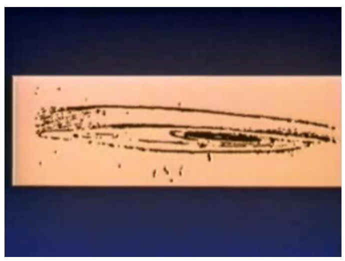
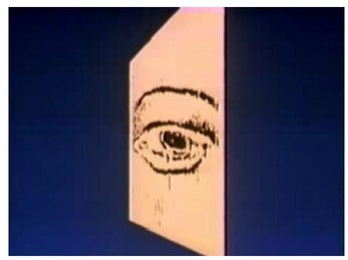
Cylinder Reflections
The Mathematics Behind the Images

Dr. Don Spickler[1] & Dr. Jennifer Bergner[2]

1 Anamorphic Art

Anamorphic art is created by distorting an image so that is only revealed from a single vantage point or from its reflection on a mirrored surface. This artistic process was first attempted during the Renaissance and became exceedingly popular during the Victorian Era. The earliest known examples come from the notebooks of Leonardo da Vinci. He successfully sketched an eyeball in 1485 that could only be discerned when looking at the drawing from a certain angle.

More modern artists using these techniques include Julian Beever, who creates three-dimensional illusions on sidewalks using chalk. Below are his Fountain and Rafting.

As you can see, Julian Beever likes to incorporate human subjects in all of his sidewalk art. This shows that not only can he create the perspective shift but he can do it to scale.

[1]despickler@salisbury.edu
[2]jabergner@salisbury.edu

Hans Hamngren and István Orosz use the mirrored cylinder technique. They achieve this illusion by either drawing the image on a distorted grid, similar to the way M. C. Escher created many of his illusions, or looking at the mirrored image while drawing on a flat surface. Pictured below is the work of István Orosz.

Below is a work by Hans Hamngren, an old fire hydrant in an old fire extinguisher.

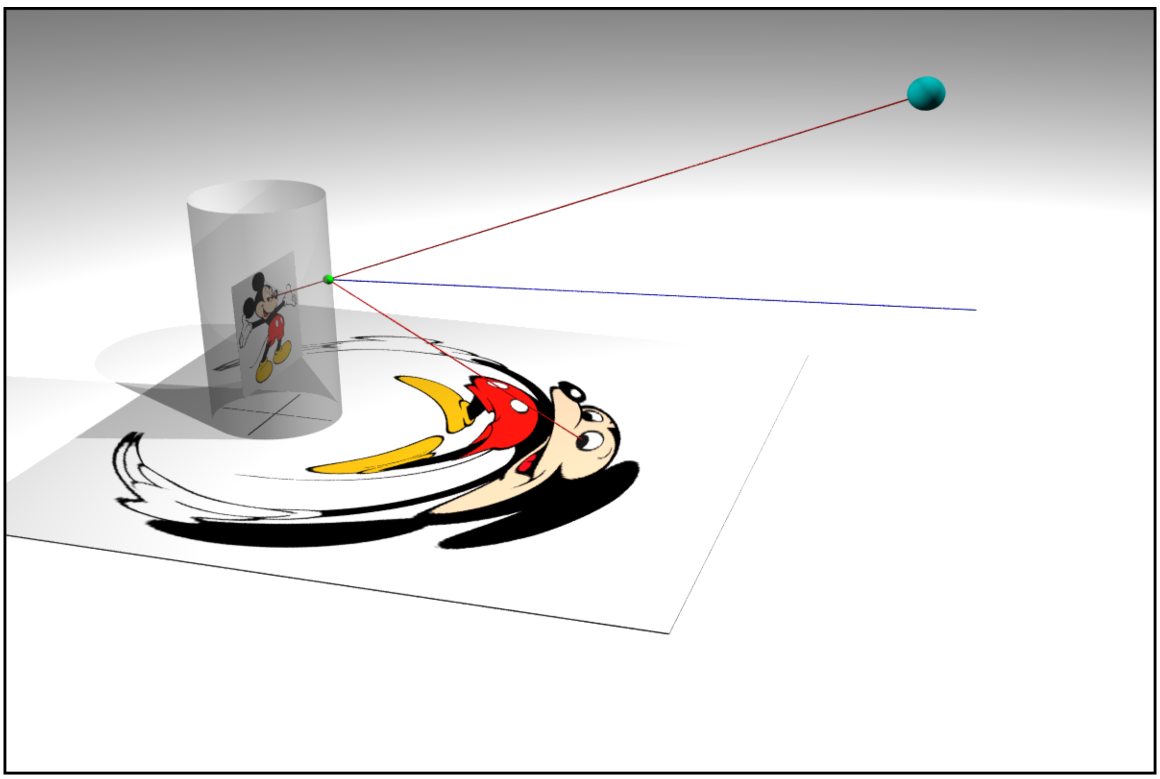
2 The Mathematics

2.1 Step 1: The Setup

We make the following assumptions on the setup to simplify the calculations. We assume that the base of the cylinder is on the xy-plane, the central axis passes through the origin, and the paper is on the xy-plane. We will let $\mathbf{P}$ represent any pixel on the original image and $\mathbf{V}$ be the position of the viewer. Also note that the equation of a vertical cylinder whose central axis passes through the origin is $r^2 = x^2 + y^2$ where r is the radius of the cylinder.

$$\begin{aligned}
\mathbf{P} &= \langle p_x, p_y, p_z \rangle \\
\mathbf{V} &= \langle v_x, v_y, v_z \rangle \\
r^2 &= x^2 + y^2
\end{aligned}$$

We also take the original image and imagine that it is completely inside the cylinder with its vertical center line on the central axis of the cylinder.



2.2 Step 2: Draw a line from pixel to viewer.

To do this we take the starting position to be the pixel point $\mathbf{P}$ and the direction to be toward the viewer is $\mathbf{V} - \mathbf{P}$. The corresponding formulas are as follows where t represents the "distance" we travel from $\mathbf{P}$ to $\mathbf{V}$.

$$\begin{aligned}
\mathbf{L}(t) &= \mathbf{P} + t(\mathbf{V} - \mathbf{P}) \\
&= \langle p_x, p_y, p_z \rangle + t(\langle v_x, v_y, v_z \rangle - \langle p_x, p_y, p_z \rangle) \\
&= \langle p_x + t(v_x - p_x), p_y + t(v_y - p_y), p_z + t(v_z - p_z) \rangle
\end{aligned}$$

As t goes from 0 to 1 we trace out the line segment from $\mathbf{P}$ to $\mathbf{V}$.

2.3 Step 3: Find the intersection with cylinder.

To do this we take the line from the pixel to the viewer and plug the x and y expressions into the equation of the cylinder and solve for t. Note that we get a quadratic equation so to simplify the notation we rewrite it as $r^2 = at^2 + bt + c$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (p_x + t(v_x - p_x))^2 + (p_y + t(v_y - p_y))^2 \\ &= p_x^2 + p_y^2 + t(-2p_x^2 - 2p_y^2 + 2p_xv_x + 2p_yv_y) \\ &\qquad + t^2(p_x^2 + p_y^2 - 2p_xv_x + v_x^2 - 2p_yv_y + v_y^2) \\ &= at^2 + bt + c \end{aligned}$$

where

$$\begin{aligned} a &= p_x^2 + p_y^2 - 2p_xv_x + v_x^2 - 2p_yv_y + v_y^2 \\ b &= -2p_x^2 - 2p_y^2 + 2p_xv_x + 2p_yv_y \\ c &= p_x^2 + p_y^2 \end{aligned}$$

Now we solve the quadratic equation,

$$at^2 + bt + c - r^2 = 0$$

for t by simply using the quadratic formula. This gives the two solutions,

$$t = \frac{-b \pm \sqrt{b^2 - 4a(c - r^2)}}{2a}$$

The fact that we get two solutions should make sense since we just calculated the intersection of a line and cylinder. A line will pass through two points on a cylinder unless the line is tangent to the cylinder, in which case we get only one solution since $b^2 - 4a(c - r^2) = 0$, or if the line and cylinder do not intersect, a case we need not consider since we put the original image completely inside the cylinder. The next question is which solution do we want? We clearly do not want the point of intersection that is behind the image. There are several ways to determine which of the two solutions for t we want. One way is to calculate the distance from the viewer to each of the two points and take the closer one. A faster way is to recognize that, by the way we set up the lines, the line from the pixel to the viewer starts at the pixel and moves toward the viewer as t increases from 0 to 1. So the t value for the point we want is between 0 and 1, in particular, it is positive. The other point of intersection will correspond to a t value that is negative. Hence we need only consider the positive solution. Also, by the way we set this up, we know that a is positive and hence the positive solution is

$$t_i = \frac{-b + \sqrt{b^2 - 4a(c - r^2)}}{2a}$$

Now we substitute this value, t_i, in for t in the line equations to get the coordinates of the point of intersection which we will denote as $\langle I_x, I_y, I_z \rangle$.

$$\begin{aligned} \mathbf{L}(t_i) &= \mathbf{P} + t_i(\mathbf{V} - \mathbf{P}) \\ &= \langle p_x + t_i(v_x - p_x), p_y + t_i(v_y - p_y), p_z + t_i(v_z - p_z) \rangle \\ &= \langle I_x, I_y, I_z \rangle \end{aligned}$$

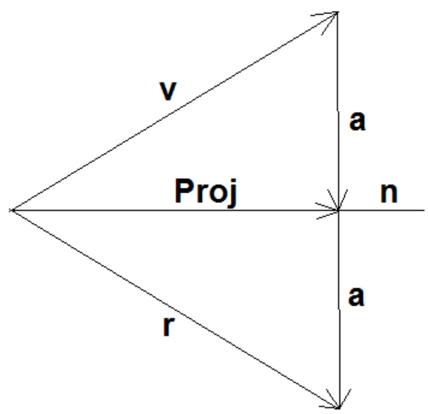
2.4 Step 4: Find the normal vector from intersection.

The normal vector is perpendicular to the surface and is used in the calculation of the reflection line. For a cylinder, the normal vector will be parallel to the xy-plane and pass through the points $\langle I_x, I_y, I_z\rangle$ and $\langle 0, 0, I_z\rangle$. So our normal vector is the difference between these two points,

$$\mathbf{n} = \langle I_x, I_y, 0\rangle$$

2.5 Step 5: Find the reflection vector from intersection.

This is probably the most involved calculation in the process. From the diagram below notice that the reflection vector $\mathbf{r} = \mathbf{v} + 2\mathbf{a}$ where $\mathbf{v} = \mathbf{V} - \mathbf{P}$ is the vector from the pixel to the viewer.



The vector **Proj** is the projection vector of $\mathbf{v}$ onto $\mathbf{n}$. From the diagram,

$$\mathbf{Proj} = \mathbf{v} + \mathbf{a}$$

and so we have

$$\mathbf{a} = \mathbf{Proj} - \mathbf{v}$$

The derivation of the projection vector is as follows, where $(\cdot)$ represents the dot or scalar product of two vectors. For example, $\langle 2, 3, 4\rangle \cdot \langle 3, -1, 1\rangle = 2 \times 3 + 3 \times (-1) + 4 \times 1 = 7$.

$$\begin{aligned} \mathbf{Proj} &= \mathrm{Proj}_{\mathbf{n}}\mathbf{v} \\ &= \frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}|^2}\mathbf{n} \end{aligned}$$

So

$$\mathbf{a} = \frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}|^2}\mathbf{n} - \mathbf{v}$$

and the reflection vector is,

$$\begin{aligned} \mathbf{r} &= \mathbf{v} + 2\mathbf{a} \\ &= \mathbf{v} + 2\left(\frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}|^2}\mathbf{n} - \mathbf{v}\right) \\ &= \frac{2\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}|^2}\mathbf{n} - \mathbf{v} \\ &= \langle r_x, r_y, r_z\rangle \end{aligned}$$

We denote $\mathbf{r} = \langle r_x, r_y, r_z \rangle$ to make the subsequent calculations and notation easier to follow. The reflection line is the line that starts at the intersection point and moves in the direction of $\mathbf{r} = \langle r_x, r_y, r_z \rangle$. We again use t to represent the "distance" we travel along $\mathbf{r}$.

$$\begin{aligned} \mathbf{R}(t) &= \langle I_x, I_y, I_z \rangle + t \langle r_x, r_y, r_z \rangle \\ &= \langle I_x + tr_x, I_y + tr_y, I_z + tr_z \rangle \end{aligned}$$

2.6 Step 6: Find the intersection of reflection line and paper.

The paper is on the xy-plane so every three dimensional point on the paper has a z coordinate of 0. We can use this fact to find how far we must move down the reflection vector until we hit the paper, this is the value of t in the reflection line formula. So we simply set the z coordinate of the reflection line equal to 0 we can solve for t,

$$I_z + tr_z = 0$$

gives

$$t = -\frac{I_z}{r_z}$$

Substitute this value in for t in the reflection line equation and we get our intersection point.

$$\left\langle I_x - \frac{I_z}{r_z} r_x, I_y - \frac{I_z}{r_z} r_y, I_z - \frac{I_z}{r_z} r_z \right\rangle = \left\langle I_x - \frac{I_z}{r_z} r_x, I_y - \frac{I_z}{r_z} r_y, 0 \right\rangle$$

We plot this point on the paper in the same color as the original pixel and move on to the next pixel. When all of the points are plotted we have our transformed image.

3 Who did the work?

The calculations for this and several other anamorphic scenarios were done as undergraduate research projects by students in the Mathematics and Computer Science Department at Salisbury University.

- Jennifer Larson and Kristi Martini (2004) — Cylinder and Sphere
- Nicole Massarelli (2010) — Theoretical extensions to general convex surfaces. She also wrote the software for cylinder reflections.
- Angela Rose and Erika Gerhold (2011) — Tilted Cylinder