Midterm Exam

CS 231a Spring 2016-2017

Monday, May 22, 2017

This written exam is 80 minutes long and is open book and open notes, but not open to electronic devices. Answer the questions in the spaces provided in the exam booklet.

Question	Points	Score
1	10	
2	10	
3	25	
4	20	
5	15	
Total:	80	

Honor Code Statement

1. The Honor Code is an undertaking of the students, individually and collectively:

  1. that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

  2. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Honor Code Acknowledgement

I acknowledge and accept the Honor Code.

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True/False Questions

1. [10 points] The following statements are either true or false. Circle the correct answer to the left of the question. If you circle an incorrect answer, you will be penalized one point. If you do not circle an answer, you will not be penalized.

   1. True False The orthographic projection of two parallel lines in the world must be parallel in the image.

   Solution: True. Orthographic projections preserve parallelism.

   2. True False An image has at most three vanishing points.

   Solution: False. Vanishing lines are composed of many, different vanishing points.

   3. True False Given a single image of an 8×8 checkerboard, with knowledge that the checkerboard squares are 2.5 cm in width and that the camera has square pixels and no distortion, it is possible to determine the camera's intrinsic and extrinsic parameters.

   Solution: False. Recall from problem set one, this means that all the points lie on a single plane, meaning that we do not have enough degrees of freedom to solve for all parameters.

   4. True False HoG produces a local feature descriptor computed around automatically chosen points in an image.

   Solution: False. We use it as a global image descriptor, or a patch descriptor, but this "local feature descriptor computed around automatically chosen points in an image" is really describing SIFT.

   5. True False In practice, we use difference of Gaussians (DoG) to approximate a Laplacian because actually computing Laplacians can be very computationally expensive.

   Solution: True. Laplacians take a lot of time to compute. DoG approximates Laplacians instead of specifically computing them.

   6. True False If the camera has zero-distortion, then we can solve for the intrinsic matrix K using a linear system.

   Solution: True. Distortion changes the system of linear equations when solving for the camera matrix to become nonlinear. See section 6 of the first course notes for more detail.

   7. True False RANSAC is robust to outliers when the ratio of the number of inliers to the number of outliers is large.

Solution: True. We also accepted False, as some students pointed out that RANSAC also requires a certain number of observations for it to be successful.

8. True False One of the main advantages of using the Hough transform for fitting lines is that no parameter tuning is required.

Solution: False. Grid size needs to be tuned.

9. True False The runtime of voxel coloring is dependent on the number of colors of the object.

Solution: False. Voxel coloring is $O(LN^3)$ where L is number of images and N is the size of voxel

10. True False It is possible to reconstruct absolute scale of the scene only if the camera is calibrated.

Solution: False. Similarity ambiguity is unavoidable even for calibrated camera.

Multiple Choice Questions

2. [10 points] The following questions are multiple choice. You will only get the full point if you get the question correct, but zero points otherwise. There is no penalty for guessing an incorrect answer.

   1. Which of the following is not a part of the SIFT calculation pipeline? **Circle one.**
      - A. Difference of Gaussians
      - B. Histogram of Gradients
      - **C. Laplacian of Gaussians**
      - D. Orientation Histogram
      - E. Image Pyramid

      Note: Since this was the MC question that was most missed, we wanted to clarify it. SIFT never uses Laplacian of Gaussians because it takes too long to compute (instead using Difference of Gaussians). Image pyramids are used when constructing the different scale spaces and choices B and D are synonymous when considering the binning process of the gradient orientations.
   2. Which of the following are true for camera matrices? **Circle all that apply.**
      - **A.** $M = K[R\ t]$ **means the camera has a rotation** R **relative to the world and a translation** t **relative to the world.**
      - B. $M = K[0\ t]$ means the camera has no rotation relative to the world and a translation t relative to the world.
      - **C.** $M = K[R\ 0]$ **means the camera has a rotation** R **relative to the world and no translation relative to the world.**
      - D. $M = K[0\ 0]$ means the camera has no rotation relative to the world and no translation relative to the world.
   3. Recall from the first problem set that $\omega = (KK^T)^{-1}$, where K is the camera matrix. Which of the following is true about ω? **Circle all that apply.**
      - **A.** ω **is symmetric**
      - **B.** ω **is invertible**
      - C. ω is orthogonal
      - **D.** ω **is known up to a scale**
   4. Recall that the Harris corner detector slides a small window across the image and observes changes of intensity values of the pixels within that window. Which of the following is the most correct when sliding the window across a corner? **Circle one.**
      - A. We see no variation of intensity.
      - B. We see a little variation of intensity.
      - C. We see a large variation of intensity perpendicular to the edge, but not parallel to it.
      - **D. We see a large variation of intensity in at least two directions.**
   5. Which of the following are drawbacks of bag of words (BOW) models? **Circle all that apply.**
      - **A. They don't capture spatial information**
      - **B. They are not ideal for solving detection problems**
      - C. Creating BOW features is time-consuming.
      - D. They are hard to understand because their inner workings are opaque
   6. Which are the benefits of RANSAC compared to least squares method? **Circle all that apply.**
      - A. Able to fit multiple models.
      - **B. More robust to outliers.**
      - C. Faster to compute in all cases.
      - D. It handles measurements with small Gaussian noise better.

      Note: We accepted both if you circled and did not circle choice A. We realized there are variants of RANSAC that enable this, even if we did not specifically cover them in class.

7. Which one of the following is NOT a direct application of image rectification? **Circle one.**
   A. Point triangulation
   B. Disparity map estimation
   C. Finding correspondences
   D. View morphing
   E. Line fitting
8. Which one of the following is a limitation of bundle adjustment compared to other reconstruction methods? **Circle one.**
   A. Assumption that the keypoints are not occluded in any views
   B. Does not extend easily to more than two cameras
   C. Too many parameters to optimize
   D. Only works on affine camera models.
   E. Requires that the correspondence matching never fails for every view
9. One of the crucial assumptions of voxel coloring is that the luminance values of the object we want to reconstruct are the same regardless of the viewpoint. What is the name of the surface that satisfies such assumption? **Circle one.**
   A. Lambertian reflection
   B. Minnaert reflection
   C. Phong reflection
   D. Flat shading
   E. Gouraud shading
10. What is the SIFT descriptor least robust against? **Circle one.**
    A. Affine illumination
    B. Rotation
    C. Scale
    D. Occlusion

Short Answers

3. [25 points] Please answer the following questions in no longer than 3 sentences.

(a) [3 points] You are given a well-trained classifier which gives you the likelihood of an image patch to be a pedestrian. Given this classifier, explain how you can recognize the locations of pedestrians from an image of a street full of pedestrians. What search strategy are you going to use? What if there are multiple false positives detections around a single pedestrian?

> **Solution:** Sliding windows search with non-maximum suppression.

(b) [3 points] At least how many correspondences do you need to solve a **projective** Structure-from-Motion of two cameras?

> **Solution:** For projective SfM, we have $2mn$ equations and $11m + 3n - 16$ unknowns where m is the number of cameras and n is the number of points. For the system to be solvable, $2mn > 11m + 3n - 16$. For $m = 2$, we need at $n \geq 7$ points.

(c) [3 points] Explain the difference between the fundamental matrix and essential matrix. How many degrees of freedom do they have and what do they encode?

> **Solution:** Fundamental matrix encodes the relationship of two views in the image coordinate with 7 degree of freedom. Essential matrix is a special case of fundamental matrix which only encodes the relative motion of two cameras in the camera coodinates, depending only on the relative rotation R and translation t, which results in degree of freedom of 5.

(d) [3 points] Why is bundle adjustment commonly used as the last step of the structure from motion? (We know that it boosts the accuracy of reconstruction. But why is it the **last** step of the reconstruction?)

> **Solution:** First, bundle adjustment requires good initial estimation just as many other nonlinear optimization techniques do. Second, bundle adjustment is more accurate but expensive operation. You want to run it as a last refinement step of the best approximate estimation of structure and motion you achieved.

(e) [3 points] We want to find a 3 x 4 camera matrix up to scale. How many unknown variables are there and what is the minimum number of corresponding 2D/3D points that we need to solve this problem?

> **Solution:** 11 unknowns and at least 6 pairs of 2D/3D points.

(f) [3 points] Explain the two motivations behind the Derivative of Gaussian filter in edge detection. (hint: the two motivations correspond to the **derivative** of the **Gaussian**).

> **Solution:** The derivative gives a location with high gradient which well defines an edge and the smoothing using Gaussian filter reduces the noise prior to taking derivative.

(g) [3 points] A good descriptor should be able to describe a keypoint invariant to various transformations. What is one way to build a descriptor that is illumination invariant?

Solution: Normalize the intensity of each patch: for w, $w_n = \dfrac{(w - \bar{w})}{\|(w - \bar{w})\|}$

(h) [4 points] Write the *most specific* transformation type next to each matrix (e.g. if a matrix is both projective and affine, you must write affine) and give a one-sentence explanation for your decision. Your options are: Affine, Projective, Similarity, and Isometric.

1. $$\begin{bmatrix} 2/3 & 0 & 0 & 0 \\ 0 & -1/3 & 1/\sqrt{3} & -1/3 \\ 0 & -1/\sqrt{3} & -1/3 & 1/\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Similarity

2. $$\begin{bmatrix} 3/5 & 2/5 & 0 & 0 \\ -2/5 & 3/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Affine

3. $$\begin{bmatrix} 1 & 3/4 & -1/5 & 1 \\ -3/4 & 2 & 1/2 & 1 \\ 1/5 & -1/2 & 2 & \sqrt{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution: Projective

4. $$\begin{bmatrix} -1/\sqrt{3} & 0 & 3/5 & 7/5 \\ 4/\sqrt{3} & 1/2 & -4/5 & -1/3 \\ 1/\sqrt{3} & -1/2 & 1/5 & 1/\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Affine

Epipolar Geometry

4. [20 points] Ada and Charles, young, budding, and aptly-named computer science students, are looking to get involved in the field of computer vision. Naturally, they start by looking at the exciting problem of finding epipoles in images.

   NOTE: In this problem we will denote Ada with normal variables, and Charles with prime variables (e.g. Ada's camera matrix is $\mathbf{K}$, Charles' is $\mathbf{K}'$).

   (a) [5 points] They decide to use their cellphones to take images. Ada has the hot new "Goosung Y5" and Charles has the slightly older "Huapple C4". They download the phones' camera datasheets and see that:

   - Ada's camera has no distortion, no skew, focal length $\alpha = \beta = 1$, and principal offset point $c_x = c_y = 0$.
   - Charles' phone has all the same parameters, but a skew angle of $45°$.

   Before doing anything else, Ada and Charles figure that they should write out their camera intrinsic matrices, as it will be useful for later calculations. What are $\mathbf{K}$ and $\mathbf{K}'$?
   Hint: $\cot(45°) = 1$ and $\sin(45°) = 1/\sqrt{2}$.

   Solution:

$$\mathbf{K} = \begin{pmatrix} \alpha & 0 & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}' = \begin{pmatrix} \alpha & -\alpha\cot(\theta) & c_x \\ 0 & \beta/\sin(\theta) & c_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) [6 points] Now that they have their camera matrices written down, they go to take photos of 3D objects. Ada stays in one place and chooses her camera coordinate system to be the same as the world's. She tells Charles to walk a few meters away and up a hill for a better viewpoint. Since Charles kept track of his motion, he knows that his **R** and **T** relative to Ada are:

$$\mathbf{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Knowing this information, and your answer for their camera matrices, find their Fundamental Matrix **F**.

Hint: With block algebra, calculating inverses can be done like so: $\begin{pmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

Another Hint: To calculate the inverse of a 2×2 matrix, use the following formula:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Solution:

$$\mathbf{F} = \mathbf{K}^{-T} [T_\times] \mathbf{R} \mathbf{K}'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{K}'^{-1}$$
$$= \begin{pmatrix} -3 & 0 & 2 \\ 0 & -3 & -1 \\ 1 & 2 & 0 \end{pmatrix} \mathbf{K}'^{-1}$$

Now, to find $\mathbf{K}'^{-1}$. Thankfully, we know from block algebra that

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

which matches the form of $\mathbf{K}'^{-1}$. Further, we also know the simple formula to calculate a 2×2 matrix inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus,

$$\mathbf{K}'^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now multiply this with the other part of **F** we had to get:

$$\mathbf{F} = \begin{pmatrix} -3 & -\frac{3}{\sqrt{2}} & 2 \\ 0 & -\frac{3}{\sqrt{2}} & -1 \\ 1 & \frac{3}{\sqrt{2}} & 0 \end{pmatrix}$$

(c) [6 points] Finally, after taking photos of the 3D object and labeling matching points, they have the following correspondences:

$$p_1 = \begin{pmatrix} 1/\sqrt{2} \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad p_1' = \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$$
$$p_2 = \begin{pmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \end{pmatrix} \quad \text{and} \quad p_2' = \begin{pmatrix} 10\sqrt{2} \\ 0 \end{pmatrix}$$

To reiterate, image points p_1 and p_2 in Ada's image are matched with p_1' and p_2', respectively, in Charles' image. Find the location of the epipole in Ada's image plane (ie. its x and y coordinates).

NOTE: As much as you'd like to check your $\mathbf{F}$ in (b) with these points, don't. They were chosen irrepsective of the answer to the previous section.

Hint: Don't get bogged down by complicated expressions. If you find yourself doing lots of arithmetic/algebra, just assign dummy variables to complex expressions and carry on from there. Getting a correct numerical final value should be a minor concern, showing that you understand the process required to get there is much more important.

Solution: NOTE: There's two answers we accepted for full credit. The first is using a nice little shortcut, ignoring all the information we gave in this question:

$$e = M \begin{bmatrix} -R^T T \\ 1 \end{bmatrix} = K[I\ 0] \begin{bmatrix} -R^T T \\ 1 \end{bmatrix} = -R^T T = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} \text{ in the image.}$$

The second answer is the more direct intersection of the two epipolar lines. This could have also been computed by going to homogeneous coordinates, computing the cross product of l_1 and l_2, then dividing by the third element to bring it back to the image coordinate space. In the solutions we did it this way since it's the most basic way of doing it (assumes no knowledge of the cross product, even though you should have known it).

We know from the course slides that $l_1 = \mathbf{F}p_1'$ and that $l_2 = \mathbf{F}p_2'$. The intersection of these lines is the location of the epipole in Ada's image plane. The other points p_1 and p_2 aren't used at all (since we only asked for Ada's epipole).

Remember, we must work in homogenous coordinates! As such, we append a 1 to the bottom of the given image points.

$$p_1' = \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \\ 1 \end{pmatrix} \qquad p_2' = \begin{pmatrix} 10\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$$

Now we compute l_1 and l_2:

$$l_1 = \begin{pmatrix} -3 & -\frac{3}{\sqrt{2}} & 2 \\ 0 & -\frac{3}{\sqrt{2}} & -1 \\ 1 & \frac{3}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -9\sqrt{2} - 10 \\ -13 \\ 3\sqrt{2} + 12 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$
$$l_2 = \begin{pmatrix} -3 & -\frac{3}{\sqrt{2}} & 2 \\ 0 & -\frac{3}{\sqrt{2}} & -1 \\ 1 & \frac{3}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 10\sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -30\sqrt{2} + 2 \\ -1 \\ 10\sqrt{2} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Now, we rewrite this in a more easy-to-work-with fashion:

$$l_1 \Rightarrow a_1 x + a_2 y + a_3 = 0$$
$$l_2 \Rightarrow b_1 x + b_2 y + b_3 = 0$$

Since we're looking for an intercept (that's where the epipole is), we rearrange the equations to get:

$$l_1 \Rightarrow y = \frac{1}{a_2}\left(-a_1 x - a_3\right)$$
$$l_2 \Rightarrow y = \frac{1}{b_2}\left(-b_1 x - b_3\right)$$

and equate the y's so that we can solve for x (and plug the result back in to find y).

$$\begin{aligned}
\frac{1}{a_2}\left(-a_1 x - a_3\right) &= \frac{1}{b_2}\left(-b_1 x - b_3\right) \\
\frac{a_1}{a_2}x + \frac{a_3}{a_2} &= \frac{b_1}{b_2}x + \frac{b_3}{b_2} \\
\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right)x &= \frac{b_3}{b_2} - \frac{a_3}{a_2} \\
\therefore x &= \frac{\frac{b_3}{b_2} - \frac{a_3}{a_2}}{\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right)}
\end{aligned}$$

Finally, we substitute this back into one of the l equations above to find the y coordinate (we chose to do l_1).

$$y = \frac{1}{a_2}\left(-a_1 \frac{\frac{b_3}{b_2} - \frac{a_3}{a_2}}{\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right)} - a_3\right) = \frac{\frac{a_1 a_3}{a_2} - \frac{a_1 b_3}{b_2}}{a_1 - \frac{a_2 b_1}{b_2}} - \frac{a_3}{a_2}$$

$$(x, y) = \left(\frac{\frac{b_3}{b_2} - \frac{a_3}{a_2}}{\frac{a_1}{a_2} - \frac{b_1}{b_2}} \;,\; \frac{\frac{a_1 a_3}{a_2} - \frac{a_1 b_3}{b_2}}{a_1 - \frac{a_2 b_1}{b_2}} - \frac{a_3}{a_2}\right)$$

where the a_i's and b_i's are the parameters of l_1 and l_2, respectively.

NOTE: The answer from this method and the answer from the first method above are different. This is completely fine since these point correspondences were chosen arbitrarily and not with respect to actual image values (if they were, then this answer would match the relatively nice values above).

(d) [3 points] Now ignore the previous parts. Assume Ada's camera takes pictures of size 100×150 pixels (width by height). If Charles only walked directly in front of Ada along the $+z$ axis of Ada's image plane and was careful not to rotate relative to Ada's axes, where would the epipole in Ada's image be? Where would the epipole in Charles' image be?

Solution: Since Charles simply walked forward (ie. performed only a z translation and no rotation), Ada's epipole will be at the center of her image plane, at $x = 50$, $y = 75$ pixels. Charles' epipole will be at the same position in his image plane, at $x = 50$, $y = 75$ pixels.

Fitting a Circle

5. [15 points] In class, we have discussed several methods for fitting a model. Now we are trying to fit a circle/circles on a 2D plane given a bunch of data points. Every data point is represented by (x_i, y_i) coordinates.

   (a) [2 points] Warm up questions: How many parameters do you need to describe any circle in a 2D plane? At least how many data points do you need to fit a circle?

   > **Solution:** Three. Three.

   (b) [6 points] Suppose that you have enough points to fit a circle. First, let's consider the least squares method (it seems a good default choice). Write the linear system in the form of $Ax = b$, whose solution minimizes the objective function in a least squares sense, and explicitly explain what parameters you are fitting in your linear system.

   Hint: The parameters you are fitting may not be exactly the same as parameters we usually use to describe a circle, but a substitution with an auxiliary variable z that is dependent on the circle parameters may be required.

   > **Solution:** Intuitively, we usually use (c_x, c_y, R) to describe a circle. For each (x_i, y_i), we want the property
   > $$(x_i - c_x)^2 + (y_i - c_y)^2 = R^2,$$
   > which can be rewritten as
   > $$2x_i c_x + 2y_i c_y + (R^2 - c_x^2 - c_y^2) = x_i^2 + y_i^2.$$
   > So we can use least squares to fit parameters $(c_x, c_y, R^2 - c_x^2 - c_y^2)$. Note that we can always recover (c_x, c_y, R) from this set of parameters. So the linear system to solve least squares is
   > $$\begin{pmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ ... & ... & ... \end{pmatrix} \begin{pmatrix} c_x \\ c_y \\ R^2 - c_x^2 - c_y^2 \end{pmatrix} = \begin{pmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ ... \end{pmatrix}$$

   (c) [3 points] Another fitting method we've covered in class is the Hough transform. Let's define the Hough space according to the parameters you defined in (b). What does a data point in original space correspond to in this Hough space? Briefly describe how we can use the Hough transform to fit observations from a single circle.

   > **Solution:** A plane. These planes will intersect at a point, which defines the equation of the circle.

   (d) [2 points] The Hough transform can also be used to fit more than one circle. Suppose that you are given 10 data points. Five of them could perfectly fit one circle, and the other five perfectly fit another circle. Briefly describe what the Hough space looks like according to our definition in (c).

   > **Solution:** There are 10 planes in the Hough space. Five of them intersect at one point, and the other five planes intersect at another point.

   (e) [2 points] Someone gives you several data points and they all lie on a line. Unfortunately, this person still wants you to fit a circle (impossible!). If you have to bite the bullet and use the Hough space defined above to fit these points, what would the Hough space look like?

Solution: Each pair of planes in the Hough space intersect at one line. All the intersection lines are parallel.