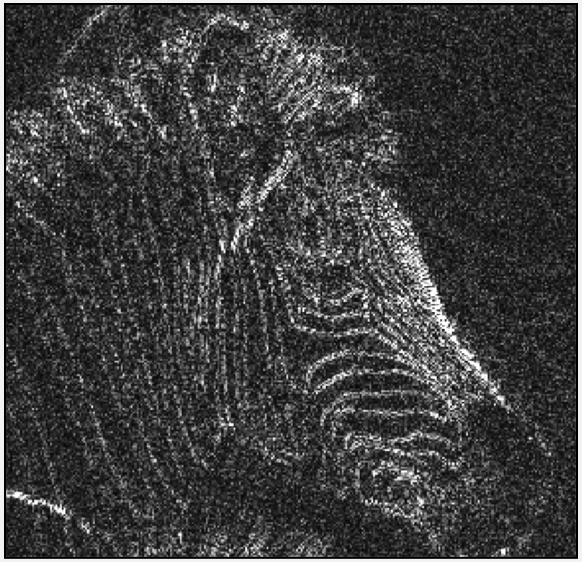
Department of Computer Science
University of Bristol

COMS30030 - Image Processing and Computer Vision



Week 02

Frequency Domain & Transforms

Majid Mirmehdi | majid@cs.bris.ac.uk

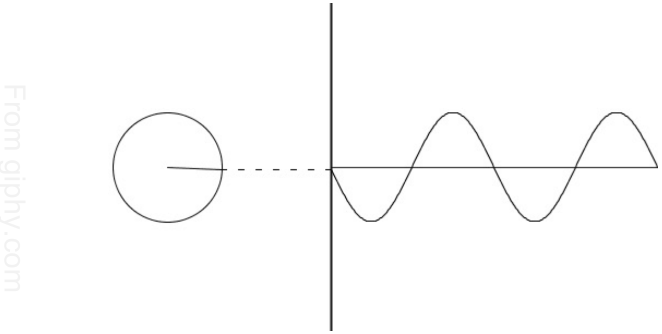
Signals as Functions

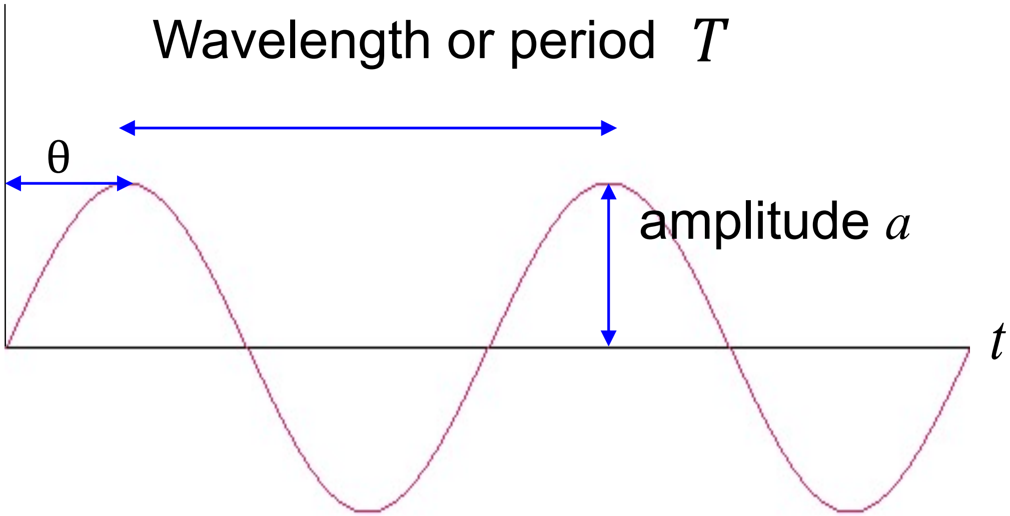
Frequency - allows us to characterise signals:

- Repeats over regular intervals with Frequency $u = \frac{1}{T}$ cycles/sec (Hz)
- Amplitude a (peak value)
- the Phase θ (shift in degrees)

Example: sine function

$$f(t) = a\, sin 2\pi u t$$





Fourier's Theorem

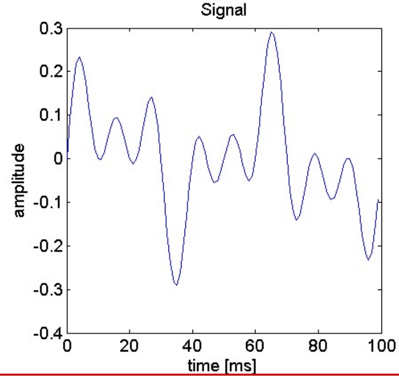
$$f(x) = \int a_n \cos(nx) + b_n \sin(nx) \;\; \delta n$$

Jean-Baptiste Joseph Fourier

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

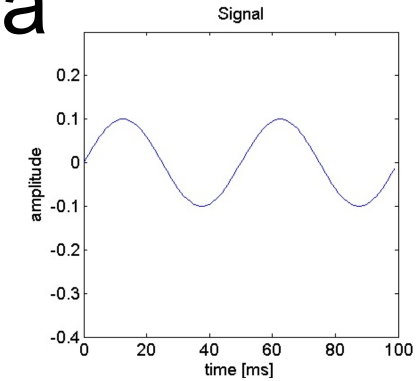
Intuition I: Simple 1D example

d = a + b + c

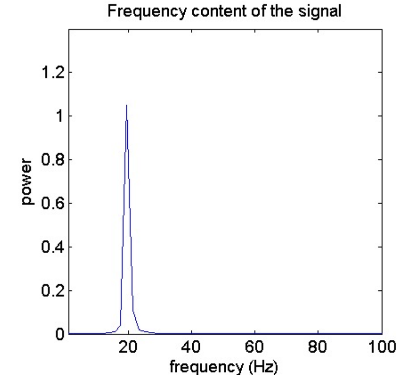


time domain

a

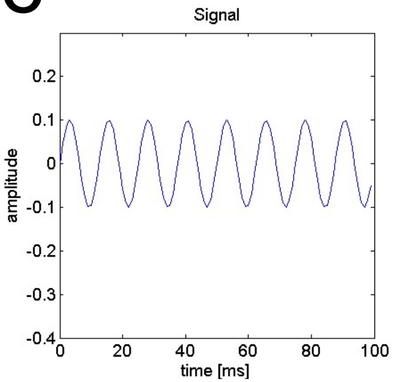


time domain

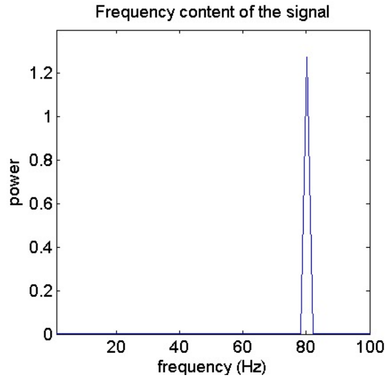


frequency domain

c

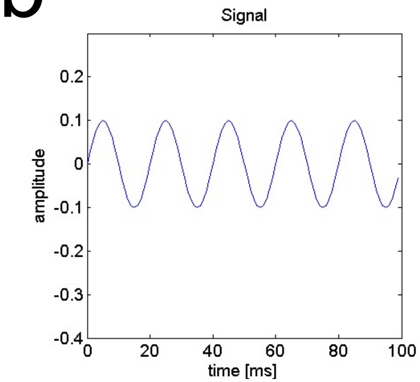


time domain

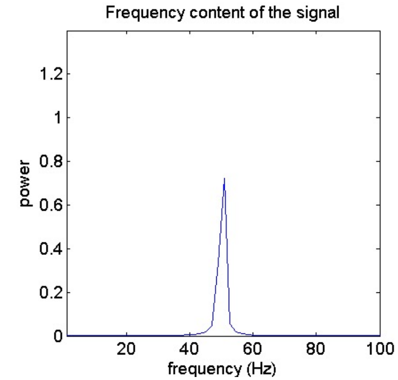


frequency domain

b



time domain



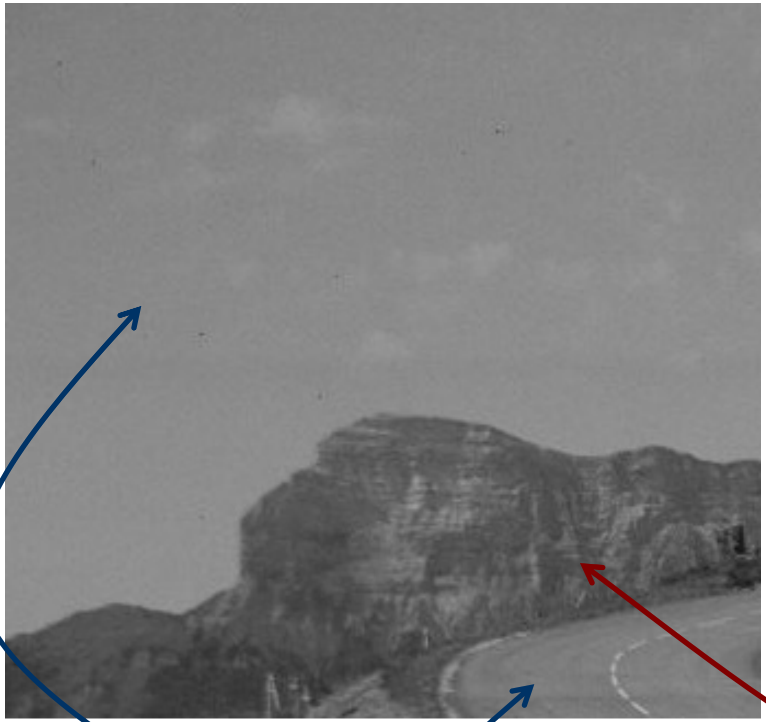
frequency domain

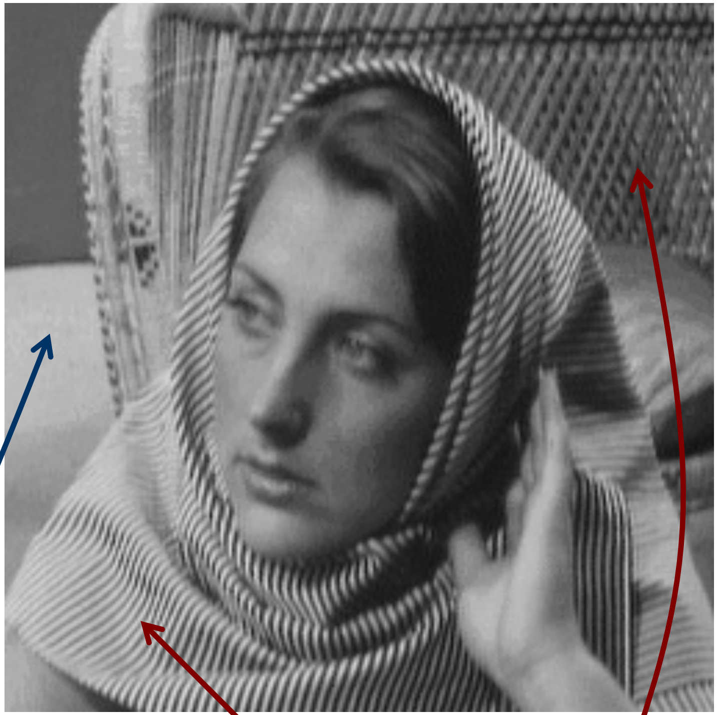
Intuition II: Simple 1D example

Animation by Lucas V Barbosa

Intuition III: Concept of Frequency in Images

Rate of change of intensity



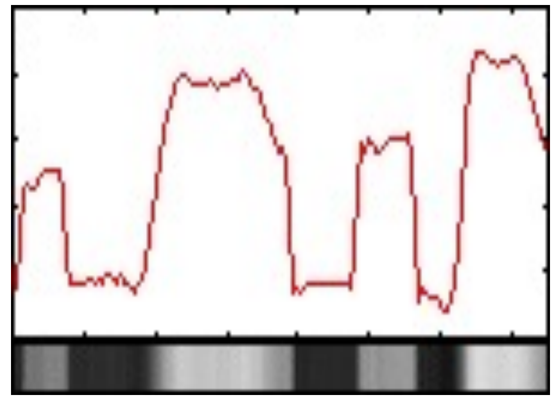


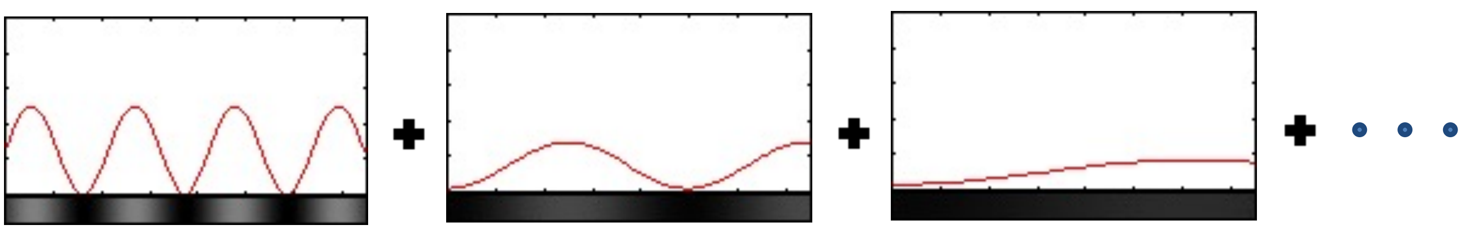
Slowly changing → low frequency

Rapidly changing → high frequency

Intuition IV: Images as waves!?

Take a single row or column of pixels from an image → a 1D signal





From ImageNagik

2D Fourier Transform: Continuous Form

- The Fourier Transform of a continuous function of two variables *f(x,y)* is:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\ e^{-i2\pi(ux+vy)}\ dxdy$$

- Conversely, given *F(u,v)*, we can obtain *f(x,y)* by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)\ e^{i2\pi(ux+vy)}\ dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform: Discrete Form

- The FT of a discrete function of two variables, *f(x,y),* is:

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \underbrace{\boxed{f(x,y)}}_{\text{image}} \underbrace{\boxed{e^{-i2\pi\left(\frac{ux+vy}{N}\right)}}}_{\text{kernels (probing functions)}} \quad \text{for } u,v = 0,1,2,\ldots,N-1\,.$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse FT:*

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\ e^{i2\pi\left(\frac{ux+vy}{N}\right)} \quad \text{for } x,y = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

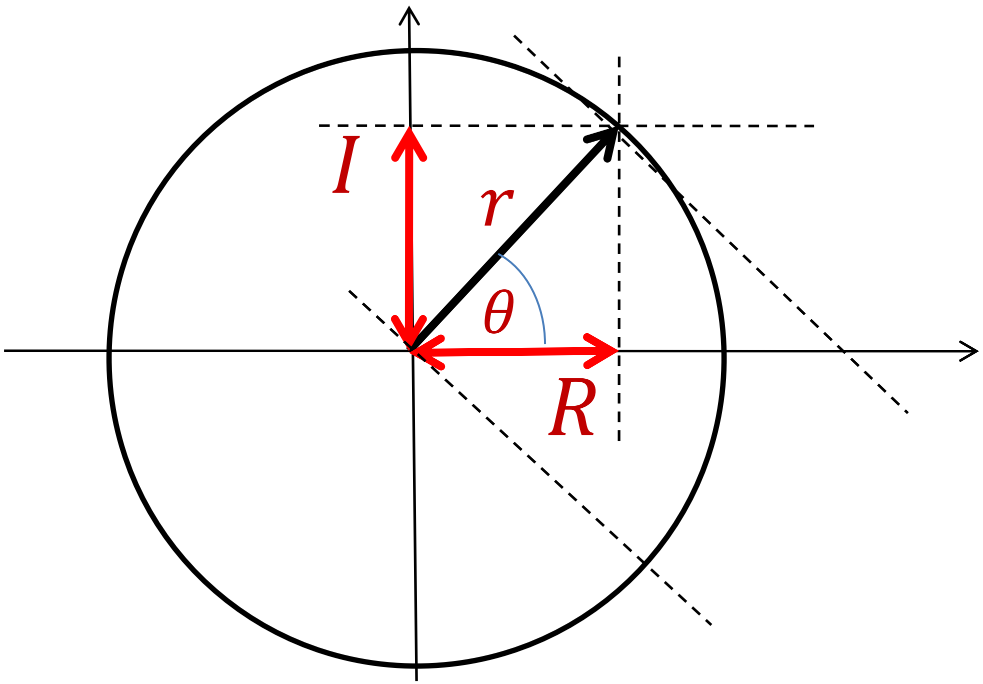
Note, they constitute a lossless representation of data.

Euler's Formula

$e^{i2\pi(\frac{ux+vy}{N})}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, a kernel is associated with a complex number (r, θ) in polar coordinates or $R(u,v), I(u,v)$ in standard complex notation.



2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function $f(x,y)$ multiplied by a particular kernel at a particular frequency and orientation specified by (u,v):

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - i\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

for $u, v = 0,1,2,\ldots,N-1$.

All kernels together form a new orthogonal basis for our image.

2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function *f(x,y)* multiplied by a particular kernel at a particular frequency and orientation specified by *(u,v)*:

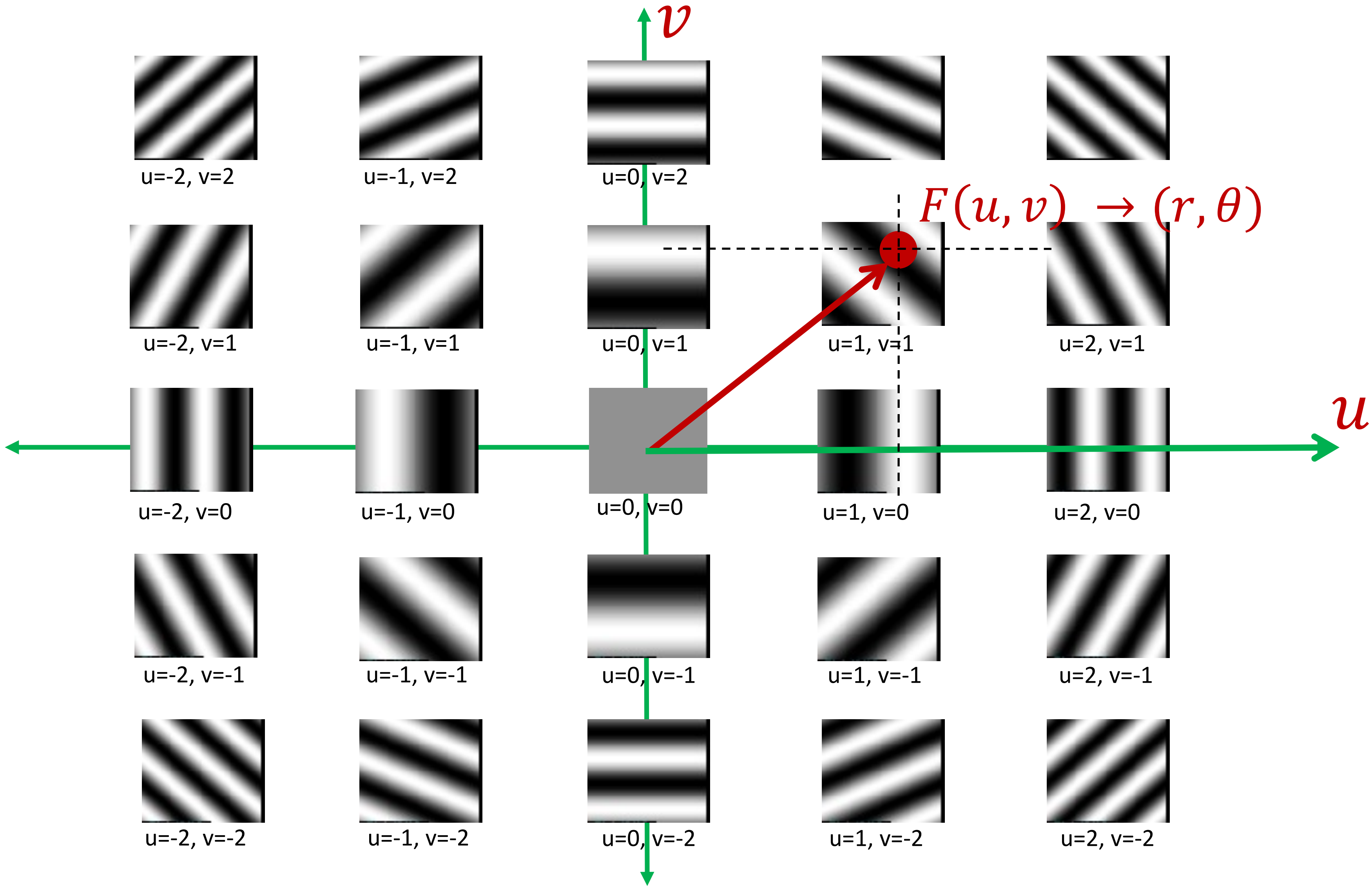
1 0

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

The slowest varying frequency component, i.e. when $u=0, v=0$ → average image graylevel

All kernels together form a new orthogonal basis for our image.

'Fabric' of the 2D Fourier Space (as kernels)



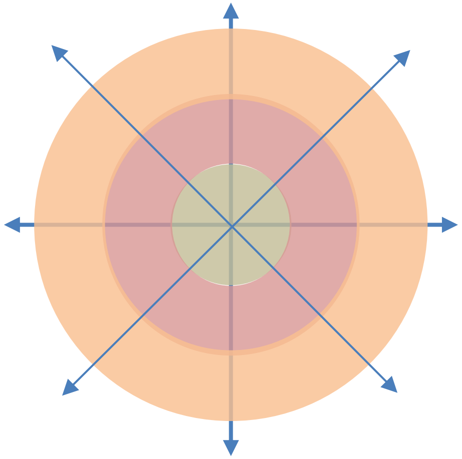
Power Spectrum and Phase Spectrum

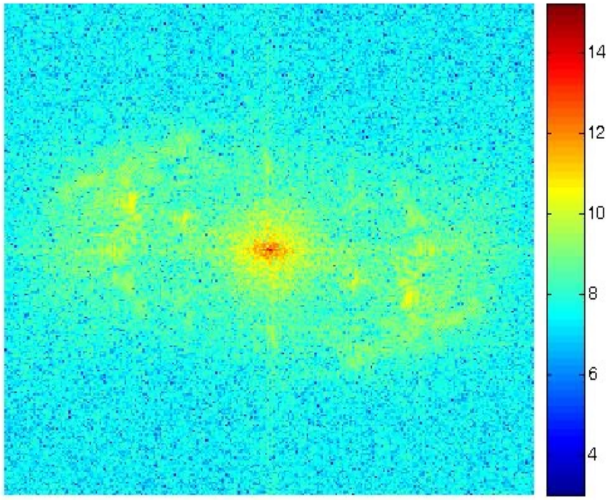
$f(x, y)$

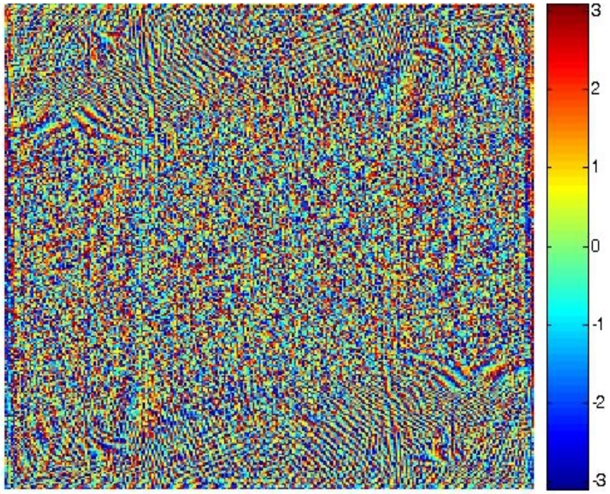
Low to Low-ish frequencies

Mid-range frequencies

High frequencies







$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

The Frequency Domain

- *F(u,v)* is a complex number and has real and imaginary parts:

$$F(u,v) = R(u,v) + iI(u,v)$$

- Magnitudes (forming the Power Spectrum):

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

- Phase Angles (forming the Phase Spectrum):

$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

- Expressing *F(u,v)* in polar coordinates *(r, θ)* :

$$F(u,v) = |F(u,v)|e^{i\theta(u,v)} = re^{i\theta}$$