Markov Decision Processes

Last Time

- What does "Markov" mean in "Markov Process"?

Guiding Questions

Guiding Questions

- What is a **Markov decision process**?

Guiding Questions

- What is a **Markov decision process**?
- What is a **policy**?

Guiding Questions

- What is a **Markov decision process**?
- What is a **policy**?
- How do we **evaluate** policies?

Decision Networks and MDPs

Decision Networks and MDPs

Decision Network

Chance node

Decision node

Utility node

Decision Networks and MDPs

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Chance node

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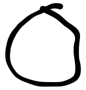
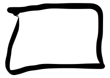
Utility node

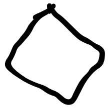
MDP Dynamic Decision Network

Decision Networks and MDPs

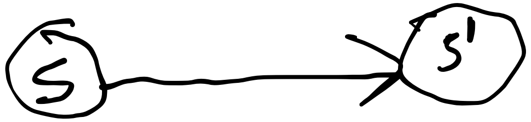
Decision Network

 Chance node

 Decision node

 Utility node

MDP Dynamic Decision Network



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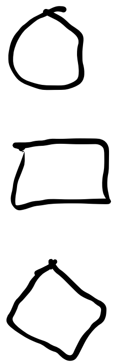

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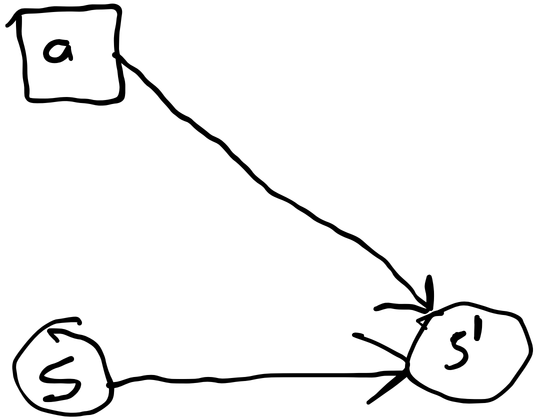

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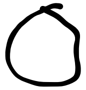
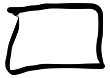
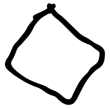
 Chance node

 Decision node

 Utility node

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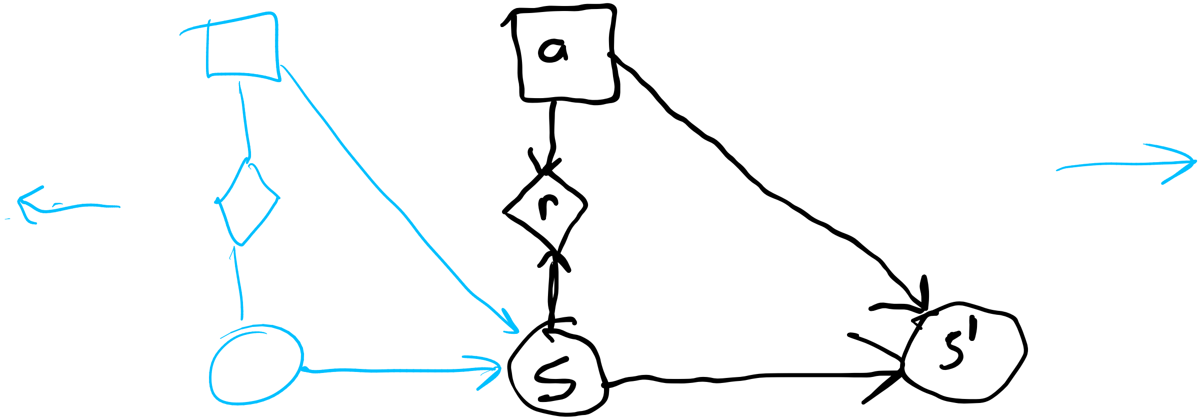

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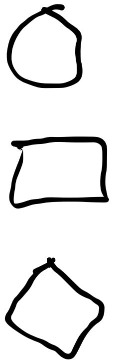

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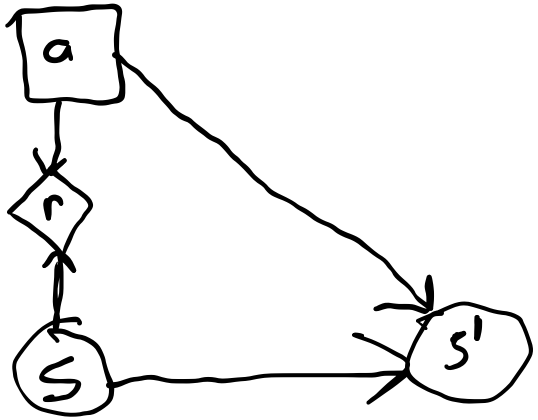

MDP Optimization problem

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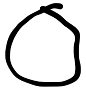
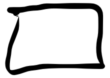
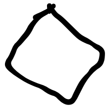
 Chance node

 Decision node

 Utility node

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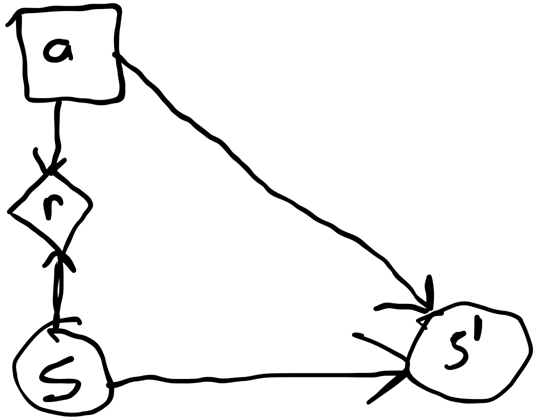

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$$\text{maximize} \quad \mathrm{E}\left[\sum_{t=0}^{\infty} r_t\right]$$

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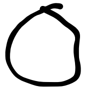
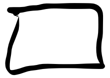
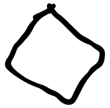
 Chance node

 Decision node

 Utility node

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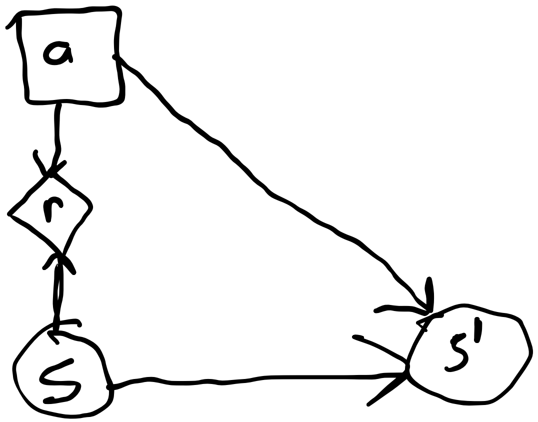

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Not well formulated!

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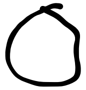
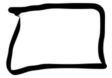
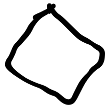
 Chance node

 Decision node

 Utility node

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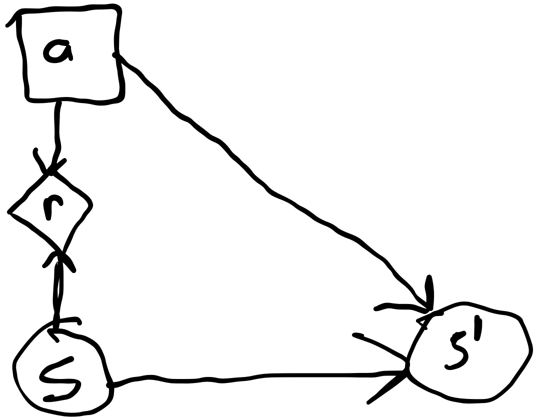

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Infinite

Finite MDP Objectives

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Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

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$s', r = G(s, a)$

- b: initial state distribution
- S_T: set of terminal states

MDP Example

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Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

- If you drive, you will have to pay $15 for parking; biking is free.
- On 1% of cold days, the ground is covered in ice and you will crash if you bike, but you can't discover this until you start riding. After your crash, you limp home with pain equivalent to losing $100.

Policies and Simulation

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- A *policy*, denoted with $\pi(a_t \mid s_t)$, is a conditional distribution of actions given states.
- $a_t = \pi(s_t)$ is used as shorthand when a policy is deterministic.
- When a policy is combined with an MDP, it becomes a Markov stochastic process with

$$P(s' \mid s) = \sum_{a_t} T(s' \mid s, a_t)\, \pi(a_t \mid s_t)$$

Break

- Suggest a policy that you think is optimal for the icy day problem

Policy Evaluation

Naive Policy Evaluation not on Exam

Monte Carlo Policy Evaluation

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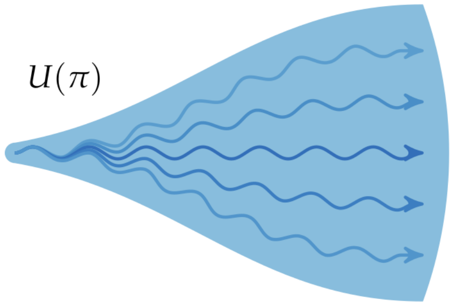

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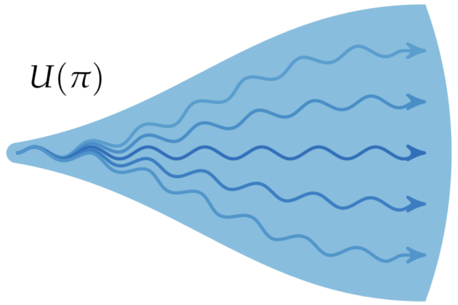
Let $\tau = (s_0, a_0, r_0, s_1, \ldots, s_T)$ be a *trajectory* of the MDP



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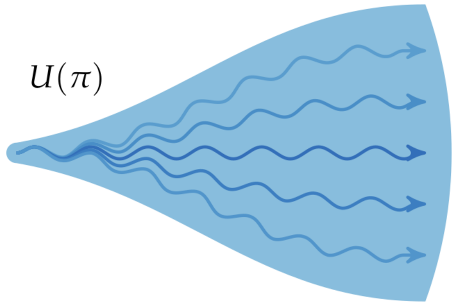

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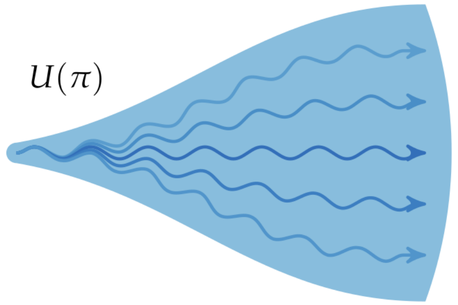

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where $\hat{u}^{(i)}$ is generated by a rollout simulation

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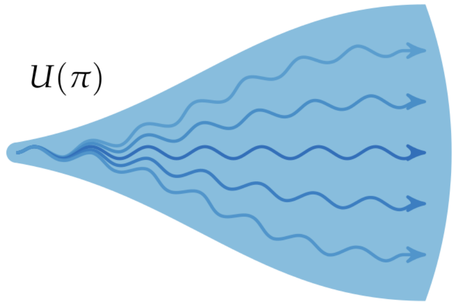

How can we quantify the accuracy of $\bar{u}_m$?

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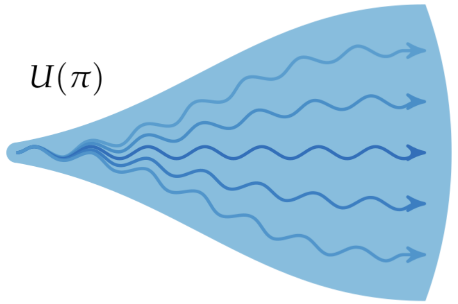

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Value Function-Based Policy Evaluation

Guiding Questions

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- What is a **Markov decision process**?
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- How do we **evaluate** policies?