Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$ $\qquad\qquad$ $x_{t+1} = x_t + v_t$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$$x_0 = 0 \qquad\qquad x_{t+1} = x_t + v_t$$

$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$

Stochastic Process

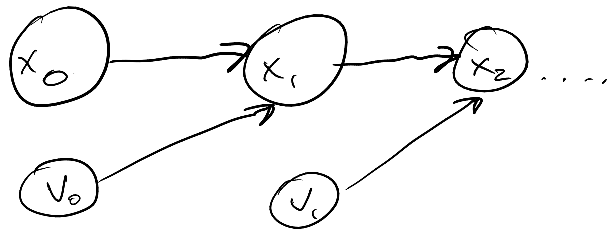
- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$$x_0 = 0 \qquad x_{t+1} = x_t + v_t$$

$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$

Bayes Net



Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$ $\qquad$ $x_{t+1} = x_t + v_t$

$v_t \sim \mathcal{U}(\{0, 1\})$ (i.i.d.)

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$

$x_{t+1} = x_t + v_t$

$v_t \sim \mathcal{U}(\{0,1\})$ (i.i.d.)

Shorthand:

$x' = x + v$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$ $\quad$ $x_{t+1} = x_t + v_t$ $\quad$ Shorthand:

$v_t \sim \mathcal{U}(\{0, 1\})$ (i.i.d.) $\quad$ $x' = x + v$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

Dynamic Bayes Net (DBN)

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$$x_0 = 0 \qquad x_{t+1} = x_t + v_t$$

$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$

Shorthand:

$$x' = x + v$$

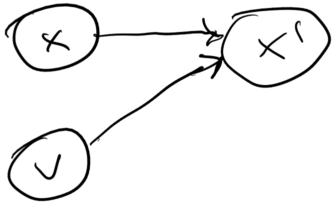
$$P(x' \mid x)$$

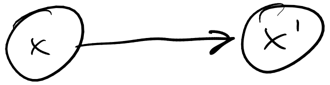
$$= \text{SparseCat}([x, x+1], [0.5, 0.5])$$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

Dynamic Bayes Net (DBN)

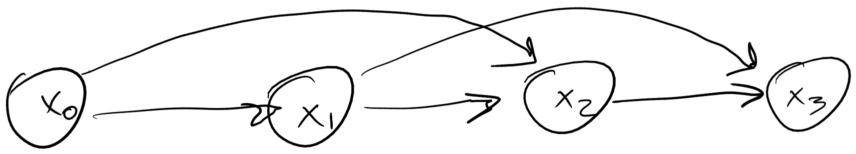




Causal Stochastic Processes

In general, stochastic processes may have connections between any times in their Bayesian Network.

In a *causal* stochastic process, x_t may depend on any x_τ with $\tau < t$.



Simulating a Causal Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

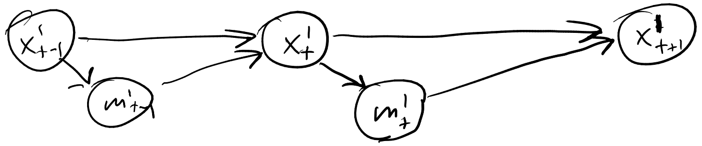
Markov Process

- A stochastic process $\{s_t\}$ is *Markov* if

$$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$

$$s_{t+1} \perp s_{t-\tau} \mid s_t \quad \forall \tau \in 1:t$$

Markov Process



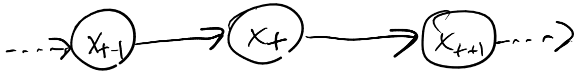
- A stochastic process $\{s_t\}$ is *Markov* if

$$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$

$$s_{t+1} \perp s_{t-\tau} \mid s_t \quad \forall \tau \in 1 : t$$

- s_t is called the "state" of the process

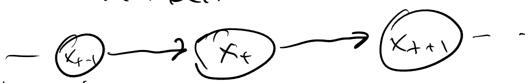
Positive Uniform Random Walk



Is $\{x_t\}$ Markov?

$x_{t+1} \perp x_{t-\tau} \mid \underline{x_t}$ ✓

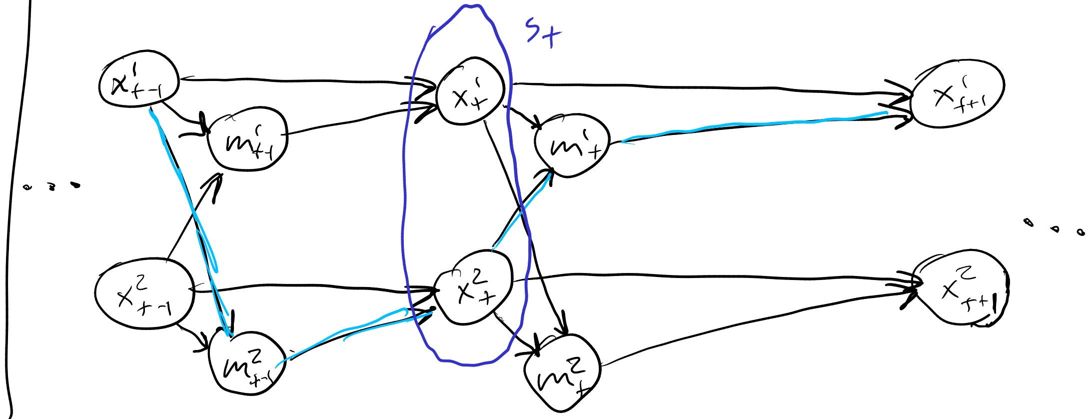
all paths would contain



all paths d-sep

Airborne Collision Avoidance

x^1: physical state of A/C 1
x^2: physical state of A/C 2
m^1: maneuver of A/C 1
m^2: " " " 2



Is $\{x^1_t\}$ Markov?

Inconclusive given only Bayes Net Structure

(With reasonable pilots, No)

$s_t = (x^1_t, x^2_t)$

Is $\{s_t\}$ Markov?

Yes

Markov Process

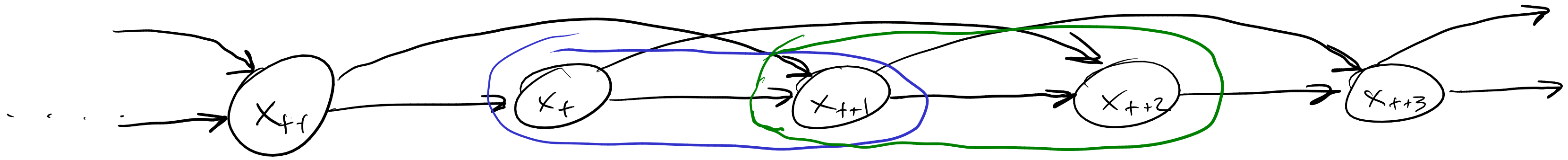
- A stochastic process $\{s_t\}$ is *Markov* if

$$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$

$$s_{t+1} \perp s_{t-\tau} \mid s_t \quad \forall \tau \in 1 : t$$

- s_t is called the "state" of the process

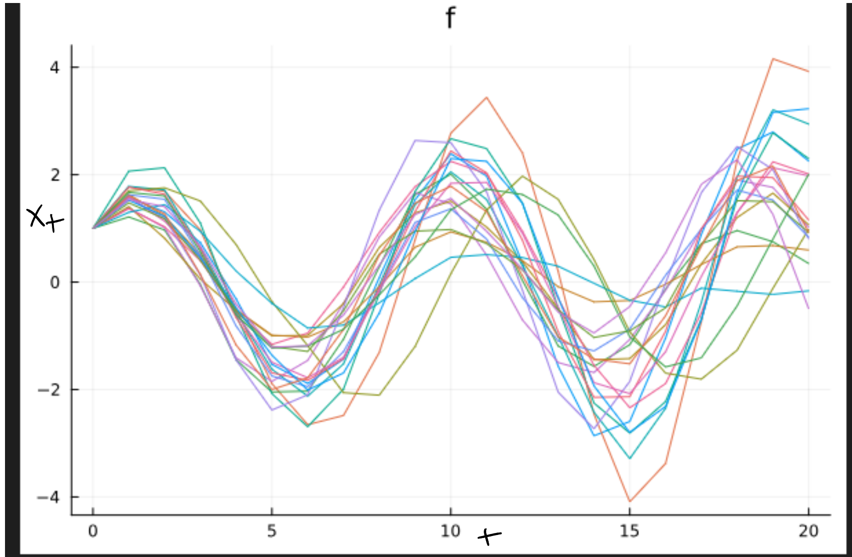
Another example



Cannot conclude $\{X_t\}$ is Markov based on structure

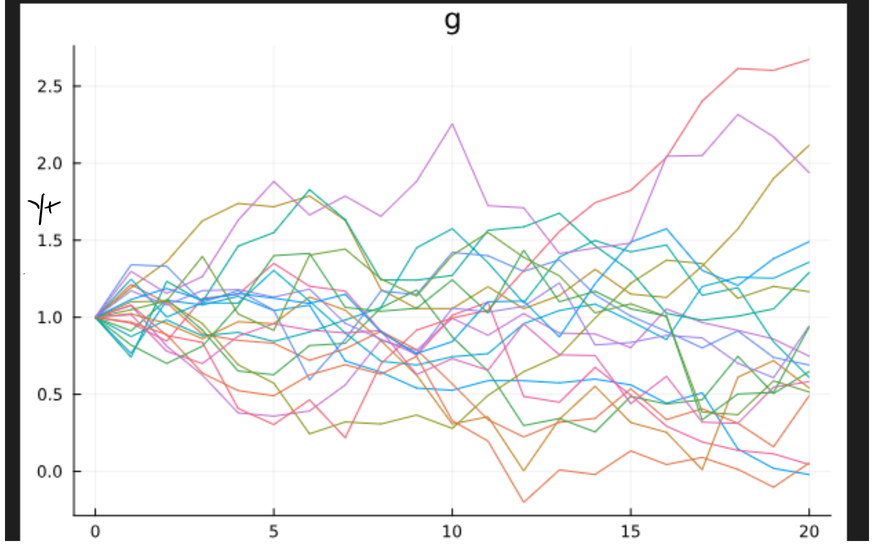
$s_t = (x_t, x_{t+1})$ $\quad$ $s_{t+1} = (x_{t+1}, x_{t+2})$ $\quad$ $P(s_{t+1} \mid s_t)$

Then $\{s_t\}$ is Markov $\quad$ $P(s' \mid s)$




```
plot(0:20, sim(g, 20), title = "g", legend=nothing)
```

✓ 0.0s



$\{x_t\}$ is not Markov

$z_t = (x_t, x_{t-1})$

$\{z_t\}$ is Markov

cisions

$\{y_t\}$ is Markov

Simple Decisions

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Simple Decisions

Outcomes	Probabilities
$S_1 \ldots S_n$ or A, B, C	$p_1 \ldots p_n$

Simple Decisions

Outcomes

$S_1 \dots S_n$ or A, B, C

Probabilities

$p_1 \dots p_n$

Lottery

$[S_1 : p_1; \dots ; S_n :$

$p_n]$

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$

Simple Decisions

Outcomes	Probabilities	Lottery
$S_1 \dots S_n$ or A, B, C	$p_1 \dots p_n$	$[S_1 : p_1; \dots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1-p] \sim C$

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

Simple Decisions

Outcomes

$S_1 \dots S_n$ or A, B, C

Probabilities

$p_1 \dots p_n$

Lottery

$[S_1 : p_1; \dots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

Simple Decisions

Outcomes

$S_1 \dots S_n$ or A, B, C

Probabilities

$p_1 \dots p_n$

Lottery

$[S_1 : p_1; \dots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

Simple Decisions

Outcomes

$S_1 \dots S_n$ or A, B, C

Probabilities

$p_1 \dots p_n$

Lottery

$[S_1 : p_1; \dots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1-p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1-p] \succeq [B : p; C : 1-p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$

Simple Decisions

Outcomes

$S_1 \ldots S_n$ or A, B, C

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

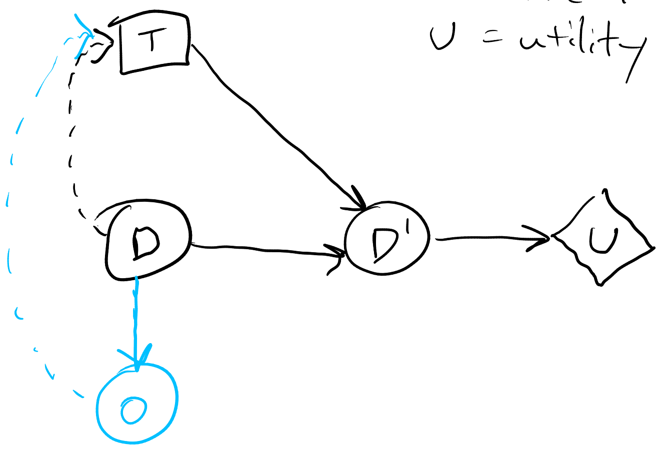
von Neumann - Morgenstern Axioms

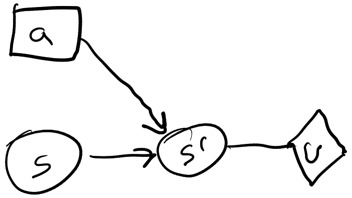
These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$
- $U([S_1 : p_1; \ldots; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

D = disease state
T = treatment
U = utility





$$a^* = \arg\max E[U(s') \mid a, s]$$

□ Action Node

○ Chance Node

◇ Utility

Conditional Edge ⟶ ends in ○

Functional Edge ⟶ ends in ◇

Informational Edge - - - -> ends in □

Markov Decision Process

Finite MDP Objectives

Finite MDP Objectives

1. Finite time

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

Finite MDP Objectives

1. Finite time $$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty}\mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty}\mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

typically 0.9, 0.95, 0.99

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \leq r_t \leq \bar{r}$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0,1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \leq r_t \leq \bar{r}$

then

$$\frac{\underline{r}}{1-\gamma} \leq \sum_{t=0}^{\infty} \gamma^t r_t \leq \frac{\bar{r}}{1-\gamma}$$

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \leq r_t \leq \bar{r}$

then

$$\frac{\underline{r}}{1-\gamma} \leq \sum_{t=0}^{\infty} \gamma^t r_t \leq \frac{\bar{r}}{1-\gamma}$$

4. Terminal States

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0,1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \leq r_t \leq \bar{r}$

then

$$\frac{\underline{r}}{1-\gamma} \leq \sum_{t=0}^{\infty} \gamma^t r_t \leq \frac{\bar{r}}{1-\gamma}$$

4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Break

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

n_t = number of people newly infected on day t

$$P(n_{15} \mid n_0 \ldots n_{14}) = \mathcal{N}(\mu_{14} n_0 + \mu_{13} n_1 \ldots \mid 14\sigma^2)$$

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?