# Value Iteration Convergence

# Review

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- How do we reason about the **future consequences** of actions in an MDP?

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- How do we reason about the **future consequences** of actions in an MDP?
- What are the basic **algorithms for solving MDPs**?

# Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?
- Can there be multiple optimal policies?
- Is there always a deterministic optimal policy?

# Value Iteration: The Bellman Operator

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$|S| = 3$

$V = \begin{bmatrix} 0.4 \\ 0.9 \\ 1.2 \end{bmatrix}$

Algorithm: Value Iteration

while $\|V - V'\|_\infty > \epsilon$

$\quad V \leftarrow V'$

$\quad V' \leftarrow B[V]$

return $V'$

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$$B[V](s) = \max_{a \in A} \left(R(s,a) + \gamma E\left[V(s')\right]\right)$$

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Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

# Metrics

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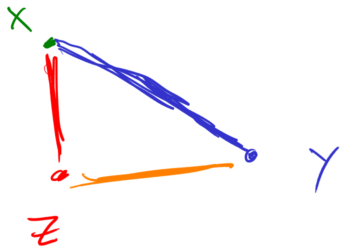

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$M = \mathbb{R}^2$

$$f(x) = \begin{bmatrix} \frac{x_2}{2} + 1 \\ \frac{x_1}{2} + \frac{1}{2} \end{bmatrix}$$

Script: contraction_mapping.jl

# Banach's Theorem

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Theorem (Banach): If $f$ is a contraction mapping on metric space $(M, d)$, then

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By Banach's theorem (part 1), $\hat{V} = B[\hat{V}]$. Since $\hat{V}$ satisfies Bellman's equation, it is optimal and $\hat{V} = V^*$.

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Theorem: Policy iteration converges to an optimal policy for a finite MDP in finite time.

Proof (sketch):

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4. By (1), (2), and (3), the policy will improve until it finds the optimal policy, and it will always find the optimal policy.

# Is there always a deterministic optimal policy?

# Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?
- Can there be multiple optimal policies?
- Is there always a deterministic optimal policy?

# Break

Conservation MDP