

Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

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$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$

$$\hookleftarrow \text{Bernoulli}(0.5)$$

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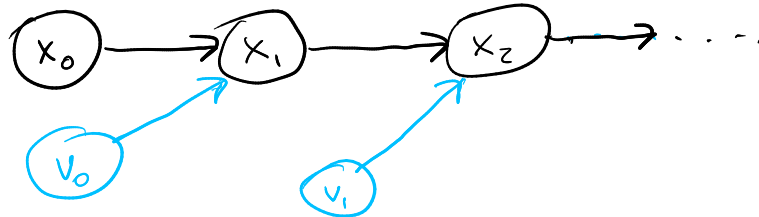
$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

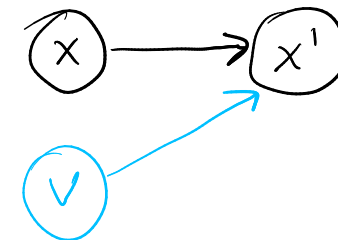
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Bayesian Network



Dynamic Bayesian Network



Stochastic Process

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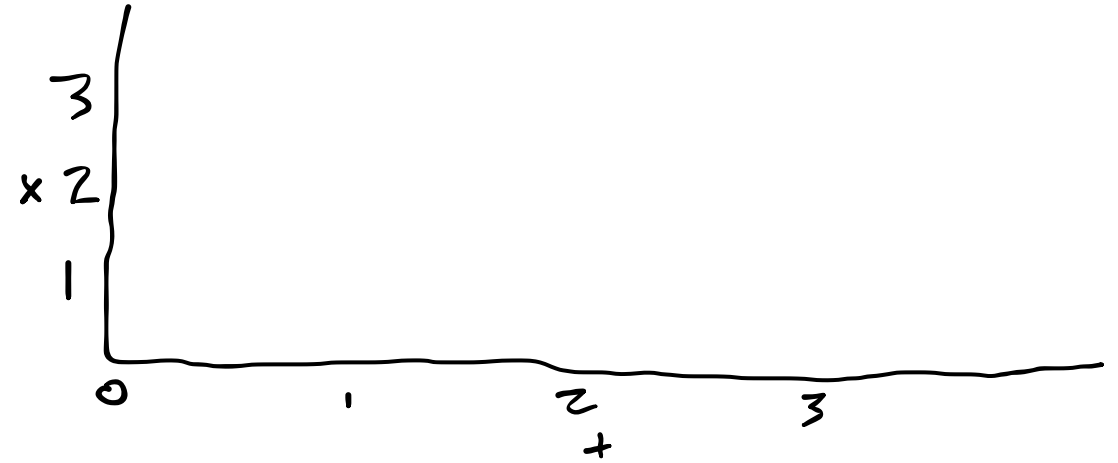
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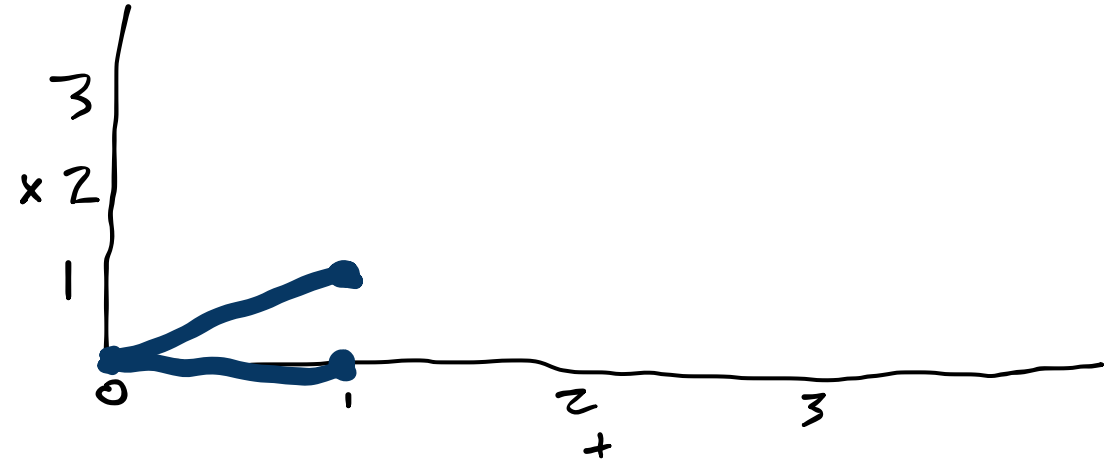


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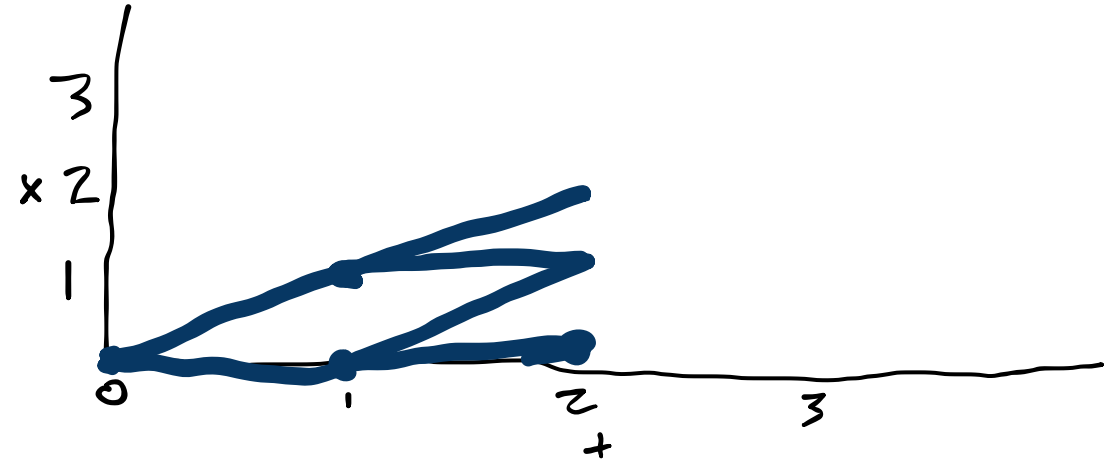


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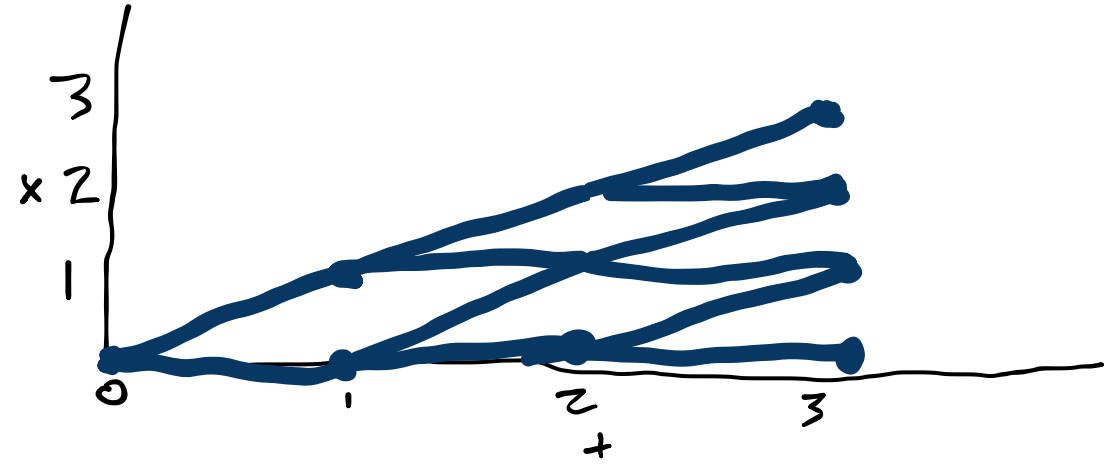


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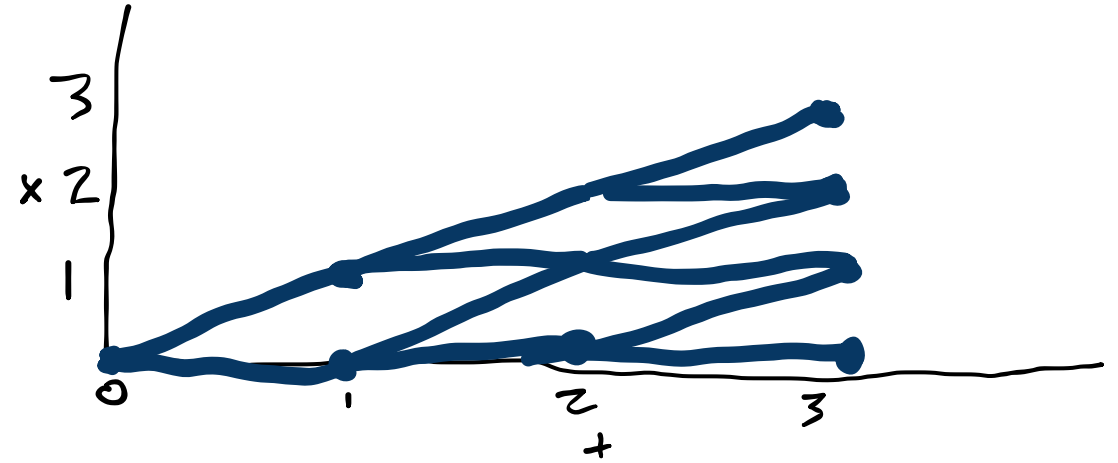
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$$P(x_{1:n}) = \prod_{t=1}^n P(x_t \mid \text{pa}(x_t))$$

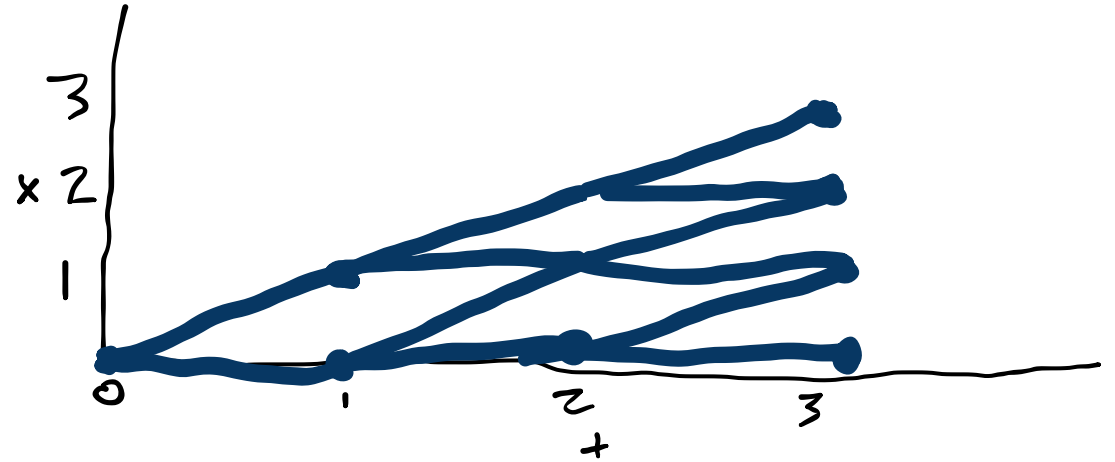


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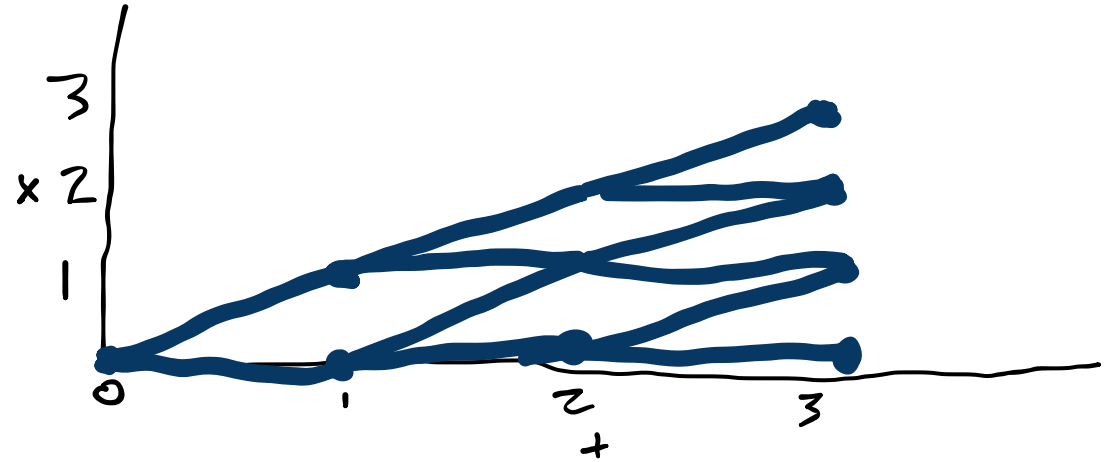
$$P(x_t \mid x_{t-1}) = \begin{cases} 0.5 & \text{if } x_t = x_{t-1} \\ 0.5 & \text{if } x_t = x_{t-1} + 1 \end{cases}$$

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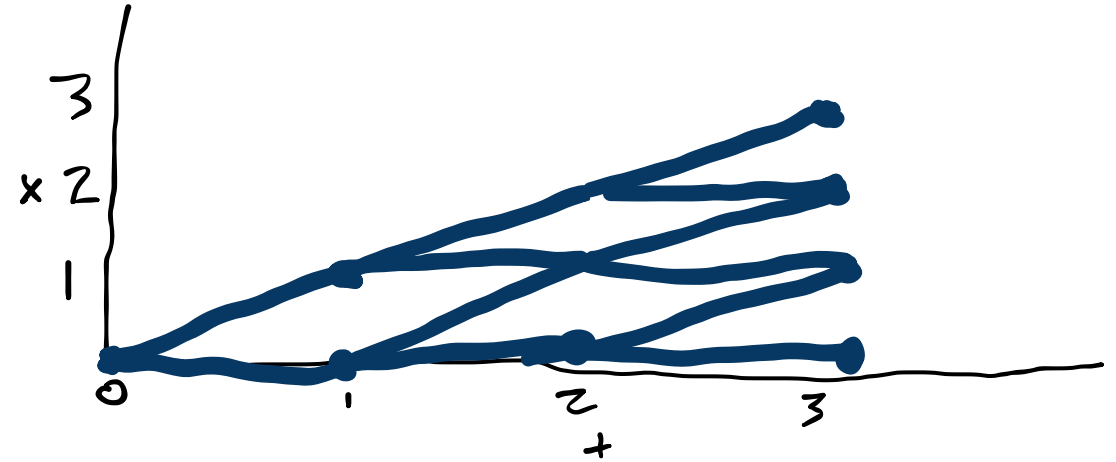
x0	x1	x2	P(x1, x2, x3)
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0	1	1	0.25
0	1	2	0.25

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For this particular process,

$$P(x_{1:n}) = \prod_{t=1}^n P(x_t \mid x_{t-1})$$

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For this particular process, since $\text{pa}(x_t) = x_{t-1}$, if $P(x_{t-1})$ is known,

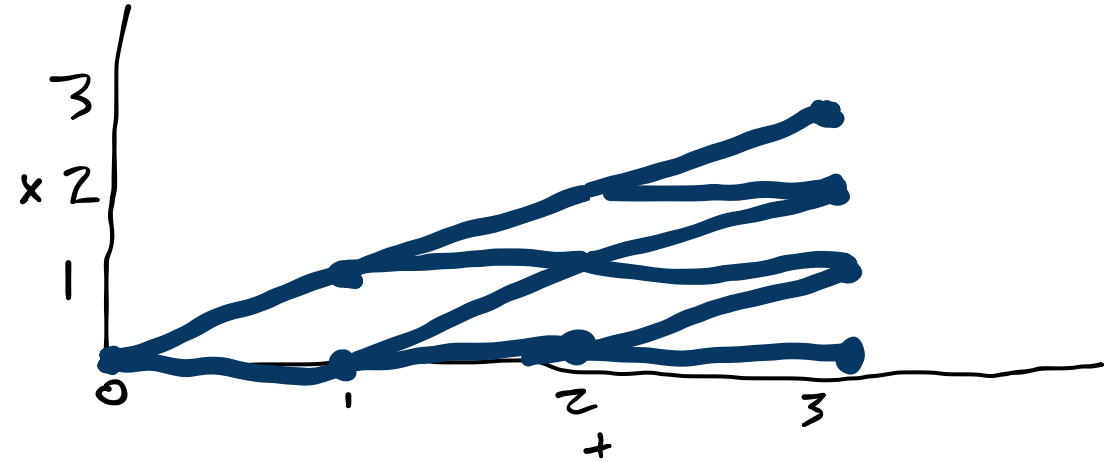
$$P(x_t) = \sum_{k \in x_{t-1}} P(x_t \mid x_{t-1} = k) P(x_{t-1} = k)$$

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$$\begin{aligned}
 P(x_t) &= \sum_{k \in x_{t-1}} P(x_t \mid x_{t-1} = k) P(x_{t-1} = k) \\
 &= 0.5 P(x_{t-1} = x_t - 1) + 0.5 P(x_{t-1} = x_t)
 \end{aligned}$$

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For this particular process, $x_t = \sum_{i=1}^t v_i$, so

$$E[x_t] = E \left[\sum_{i=1}^t v_i \right] = \sum_{i=1}^t E[v_i] = 0.5t$$

Stochastic Process

Expectation

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Expectation of a function
(such as reward)

$$E[f(x_t)] = \sum_{x \in x_t} f(x) P(x_t = x)$$

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Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

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- s_t is called the "state" of the process

Break

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Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

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- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

Hidden Markov Model

(Often you can't measure the whole state)

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$s_1 \dots s_n$

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$p_1 \dots p_n$

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Lottery

$[S_1 : p_1; \dots; S_n : p_n]$

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- $U([S_1 : p_1; \dots; S_n : p_n]) = \sum_{i=1}^n p_i U(S_i)$

Decision Networks

Markov Decision Process

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discount $\gamma \in [0, 1)$

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4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Guiding Question

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