# Simple Games

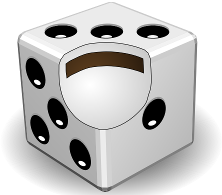
- Games: a mathematical formalism for rational interaction
- What is the best solution concept? (Nash Equilibrium)

# Types of Uncertainty

Alleatory

**Markov Decision Process**

Epistemic (Static)



**Reinforcement Learning**

Epistemic (Dynamic)

**POMDP**

Interaction

**Game**

# Normal Form Games

- Alice and Bob are working on a homework assignment.
- They can either **share** or **withhold** their knowledge.
- If one player shares knowledge, the other benefits greatly, but the sharer also benefits by getting to test their knowledge

Alice's Payoffs

| | | B | |
|---|---|---|---|
| | | **S** | **W** |
| A | **S** | 4 | 2 |
| | **W** | 3 | 1 |

Bob's Payoffs

| | | B | |
|---|---|---|---|
| | | **S** | **W** |
| A | **S** | 4 | 3 |
| | **W** | 2 | 1 |

| | | Bob | |
|---|---|---|---|
| | | **S** | **W** |
| Alice | **S** | 4, 4 | 2, 3 |
| | **W** | 3, 2 | 1, 1 |

Called a **Normal Form**, **Simple**, or **Bimatrix** Game

Question for today: What **solution concept** should we use for games?

# Dominant Strategies

Bob

Alice

| | **S** | **W** |
|---|---|---|
| **S** | 4, 4 | 2, 3 |
| **W** | 3, 2 | 1, 1 |

- **Dominant (Pure) Strategy**: Action $a$ is a dominant strategy if it is a best response to every action taken by the other player.
- **Dominant Strategy Equilibrium**: Every player plays a dominant strategy

**Definitions**

- Action $a^i \in A^i$
- Joint Action $a = (a^1, \ldots, a^k)$
- All Other Actions $a^{-i} = (a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^k)$
- Reward $R^i(a)$
- Joint Reward $R(a) = (R^1(a), \ldots, R^k(a))$

**Deterministic Best Response**:

Action $a^i$ is a deterministic best response to $a^{-i}$ if

$$R^i(a^i, a^{-i}) \geq R^i(a^{i'}, a^{-i}) \quad \forall a^{i'}$$

Is the dominant strategy equilibrium always the best outcome for the players?

# A more surprising example: The Prisoner's Dilemma

- 2 criminals are captured
- Each can either keep silent or testify
    - other keeps silent -> minor conviction (1 year)
    - other testifies -> major conviction: 4 years
    - testify -> 1 year removed from sentence

Player 2

Player 1

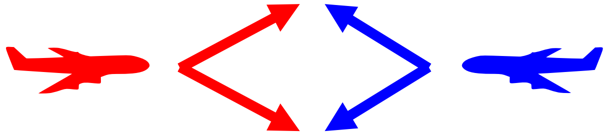
| | **S** | **T** |
|---|---|---|
| **S** | -1, -1 | -4, 0 |
| **T** | 0, -4 | -3, -3 |

- Dominant strategy for both players is to testify
- Dominant strategy equilibrium is a very bad social result (for the criminals)

Do all simple games have a dominant strategy equilibrium?

# Collision Avoidance Game

## Example: Airborne Collision Avoidance



**Player 2**

**Player 1**

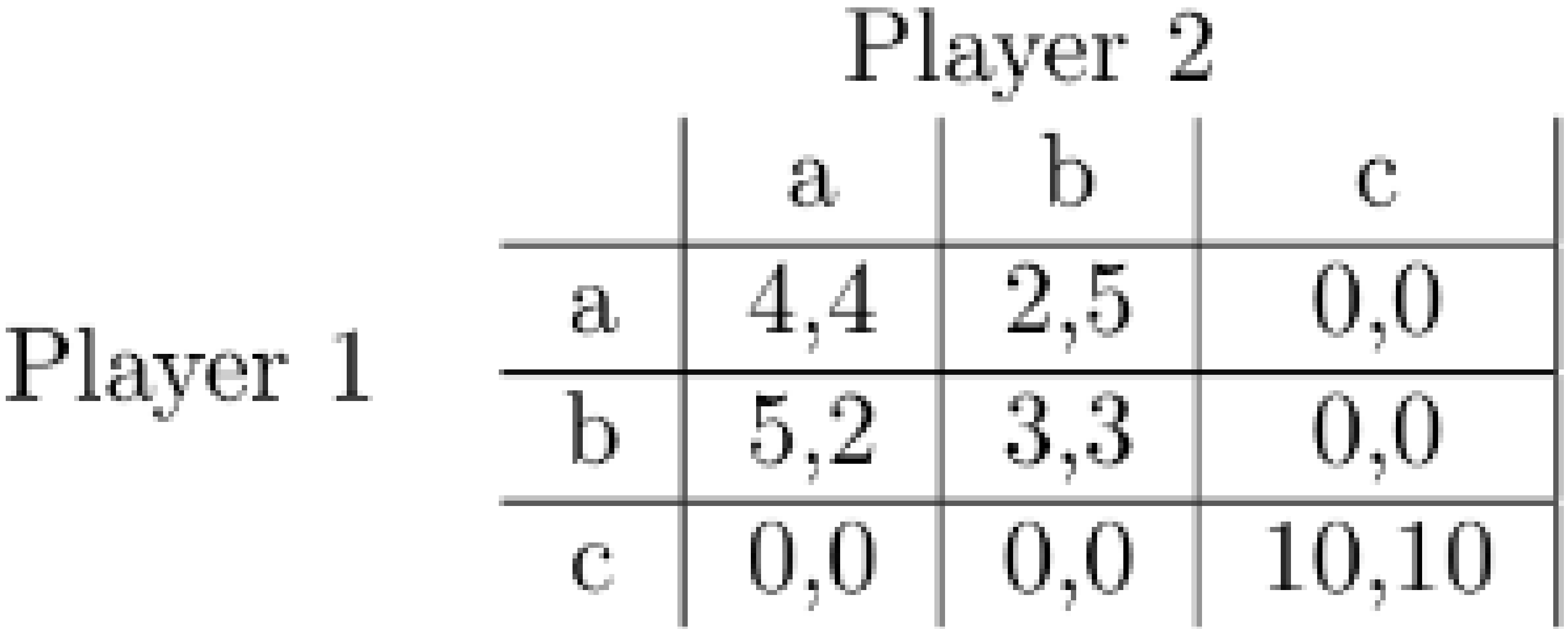
| | Up | Down |
|---|---|---|
| Up | Collision<br>-5, -5 | -1, 0 |
| Down | 0, -1 | Collision<br>-4, -4 |

**Pure Nash Equilibrium**: All players play a deterministic best response.

Which equilibrium is better?

Do all simple games have a pure Nash equilibrium?

# Practice: Find Pure Nash Equilibria

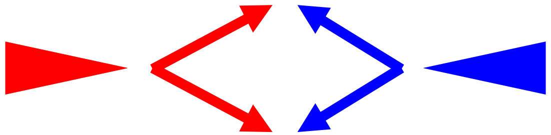


Player 2

| | | a | b | c |
|---|---|---|---|---|
| Player 1 | a | 4,4 | 2,5 | 0,0 |
| | b | 5,2 | 3,3 | 0,0 |
| | c | 0,0 | 0,0 | 10,10 |

# Missile Defense Game

**Missile Defense** (simplified)



| | | Defender | |
|---|---|---|---|
| | | Up | Down |
| Attacker | Up | Collision<br>-1, 1 | 1, -1 |
| | Down | 1, -1 | Collision<br>-1, 1 |

No Pure Nash Equilibrium!

Need a broader solution concept: Mixed Nash equilibrium.

# Vocabulary and Notation for Mixed Strategies

| | Single Player | Joint |
|---|---|---|
| Action | $a^i \in A^i$ | $a \in A$ |
| Policy (strategy) | $\pi^i(a^i)$ | $\pi(a) = \prod_i \pi^i(a^i)$ |
| Reward | $R^i(a)$ | $R(a)$ |
| Utility | $U^i(\pi) = \sum_a R^i(a)\pi(a)$ | $U(\pi) = \sum_a R(a)\pi(a)$ |

**Two Player Zero Sum:**

$$R^1(a) + R^2(a) = 0 \quad \forall a$$

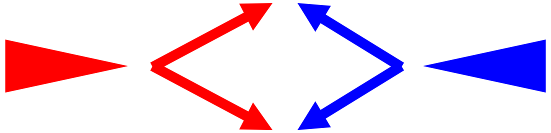
**Best Response**: Given a joint policy of all other agents, $\pi^{-i}$, a best response is a policy $\pi^i$ that satisfies

$$U^i\left(\pi^i, \pi^{-i}\right) \geq U^i\left(\pi^{i'}, \pi^{-i}\right)$$

for all other $\pi^{i'}$.

# Missile Defense Game

**Missile Defense** (simplified)



**Defender**

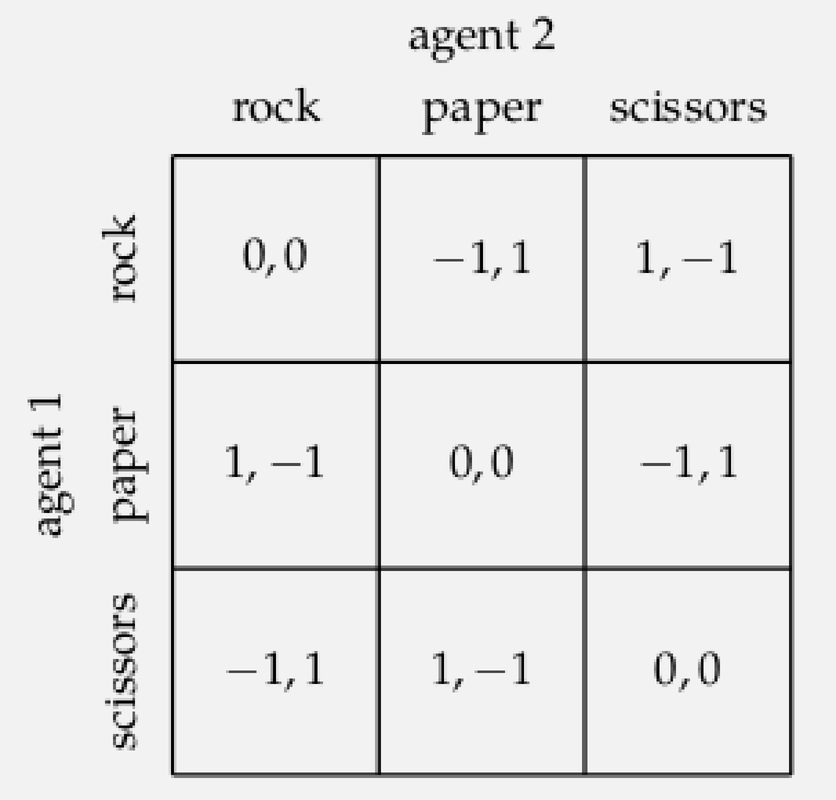
**Attacker**

| | Up | Down |
|---|---|---|
| Up | Collision<br>-1, 1 | 1, -1 |
| Down | 1, -1 | Collision<br>-1, 1 |

- A ***Nash equilibrium*** is a joint policy in which all agents are following a best response

# Rock-paper scissors

1. Guess the Nash Equilibrium argument
2. Make a qualitative argument that this is an NE based on best responses

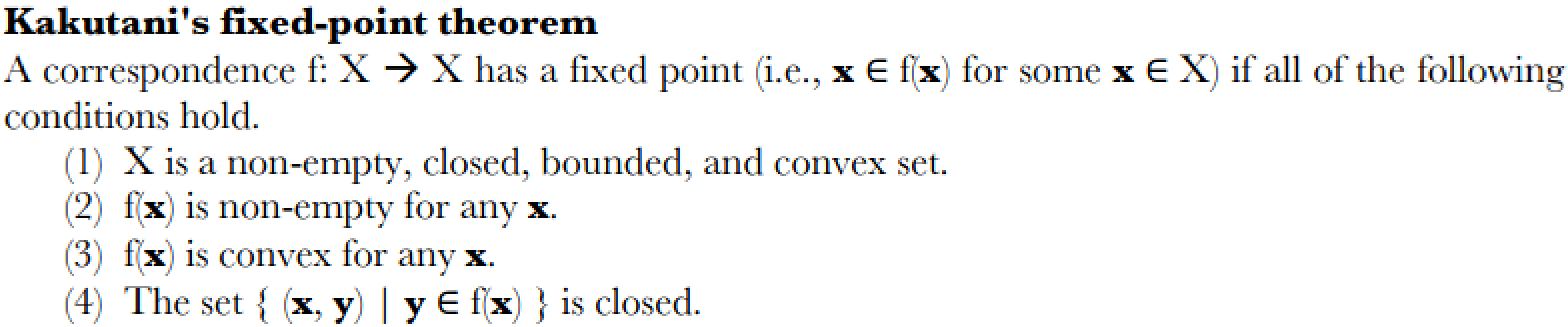


| | | agent 2 | | |
|---|---|---|---|---|
| | | rock | paper | scissors |
| agent 1 | rock | 0, 0 | −1, 1 | 1, −1 |
| | paper | 1, −1 | 0, 0 | −1, 1 |
| | scissors | −1, 1 | 1, −1 | 0, 0 |

Do all simple games have at least one Nash equilibrium?

Yes!! (might be mixed)

# Every finite game has a Nash Equilibrium



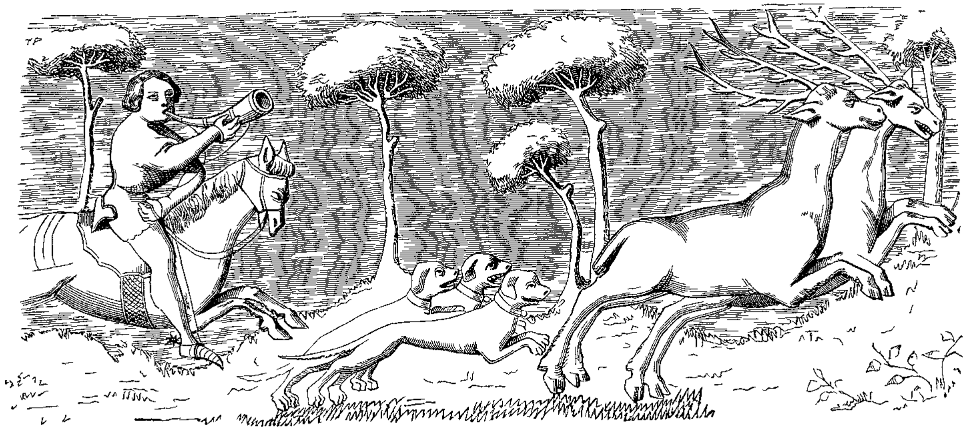
**Kakutani's fixed-point theorem**

A correspondence f: X → X has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

(1) X is a non-empty, closed, bounded, and convex set.
(2) $f(\mathbf{x})$ is non-empty for any $\mathbf{x}$.
(3) $f(\mathbf{x})$ is convex for any $\mathbf{x}$.
(4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let $x$ be a strategy profile, $\pi$.
- Let $f$ be $BR$, that is, the best response operator
- A fixed point of $BR$ is a Nash Equilibrium
- The $BR$ operator and policy space for finite games meet the conditions above
- $BR$ has a fixed point for every finite game, i.e. every finite game has a Nash Equilibrium

# Calculating Mixed Nash



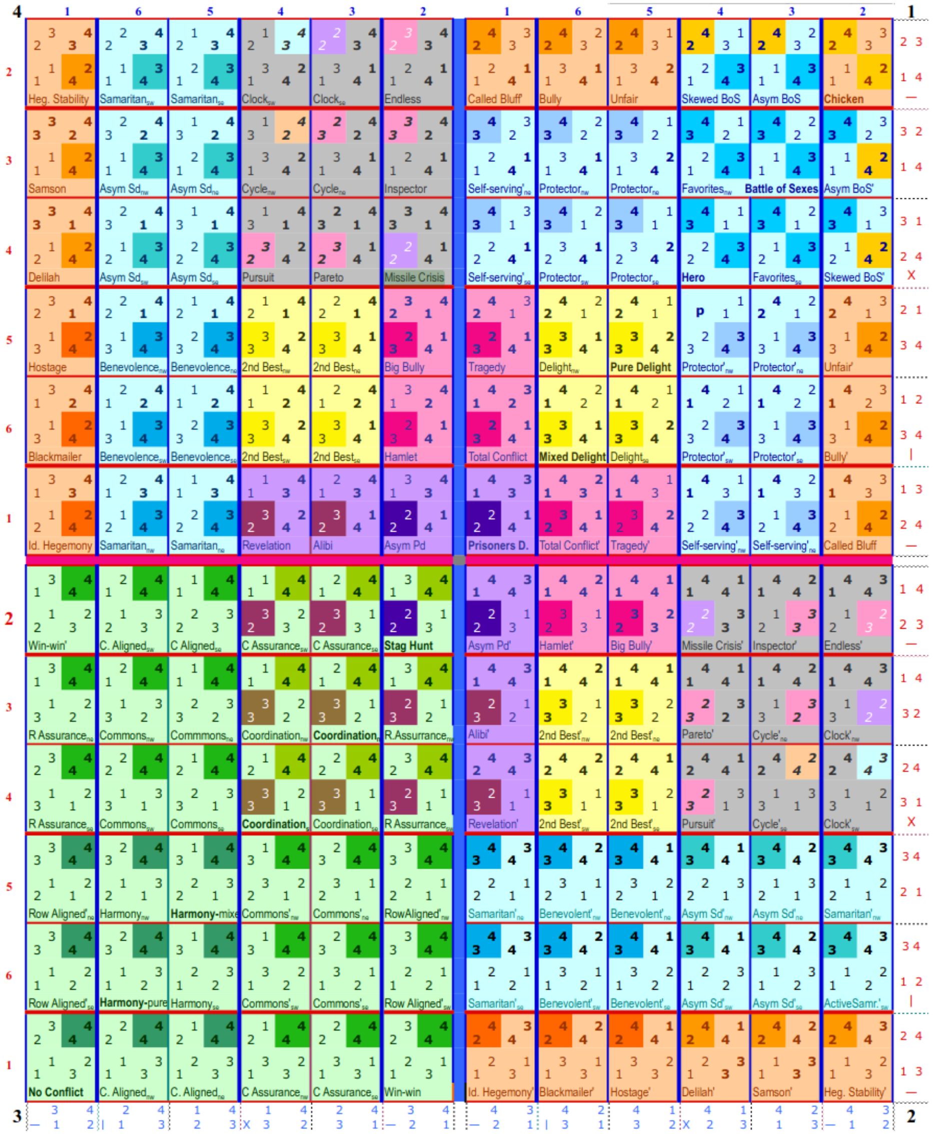
|  | **Stag** | **Hare** |
|---|---|---|
| **Stag** | 4, 4 | 1, 3 |
| **Hare** | 3, 1 | 2, 2 |

- In a Mixed Nash Equilibrium, players must be *indifferent* between two or more actions
- (In large games, finding the support of the mixed strategies is the hard part)

# General approach to find Nash Equilibria

$$\begin{aligned}
\underset{\pi, U}{\text{minimize}} \quad & \sum_i \left( U^i - U^i(\pi) \right) \\
\text{subject to} \quad & U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i \\
& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
\end{aligned}$$

Topology of bimatrix games:



# Algorithms that use best response

**Iterated Best Response**: randomly cycle between agents who play the best response for the current policy (converges to Nash for certain narrow classes of games)

**Fictitious Play**:

1. Estimate maximum likelihood policies for opponents:
$$\pi^j(a^j) \propto N(j, a^j)$$
2. Play best response to estimated policy

(converges to Nash for wider class of games, notably zero-sum)

# ~~Battle of the Sexes~~
# Bach or Stravinsky

- Two people want to go to a concert
- P1 prefers Bach, P2 Stravinsky

|  | **B** | **S** |
|---|---|---|
| **B** | 2, 1 | 0,0 |
| **S** | 0, 0 | 1, 2 |

**Correlated Equilibrium**

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent *i* can increase their expected utility by deviating from their current action to another.
- Easier to find than Nash equilibrium (Linear Program)

https://youtube.com/shorts/w3q77ZZIqwA
?si=J8H6L6W5kTRs-mUx

# Recap

- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that joint policy is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)
- Mixed Nash equillibria occur when players are indifferent between two outcomes