

Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Utility

$$U(A) > U(B)$$

indicates A is preferable to B

$$U(A) = U(B)$$

" indifferent

Probability

$$P(A) > P(B)$$

" A is more plausible than B

$$P(A) = P(B)$$

" A and B are equally plausible

What is a Random Variable?

Today Only!

R.V. X

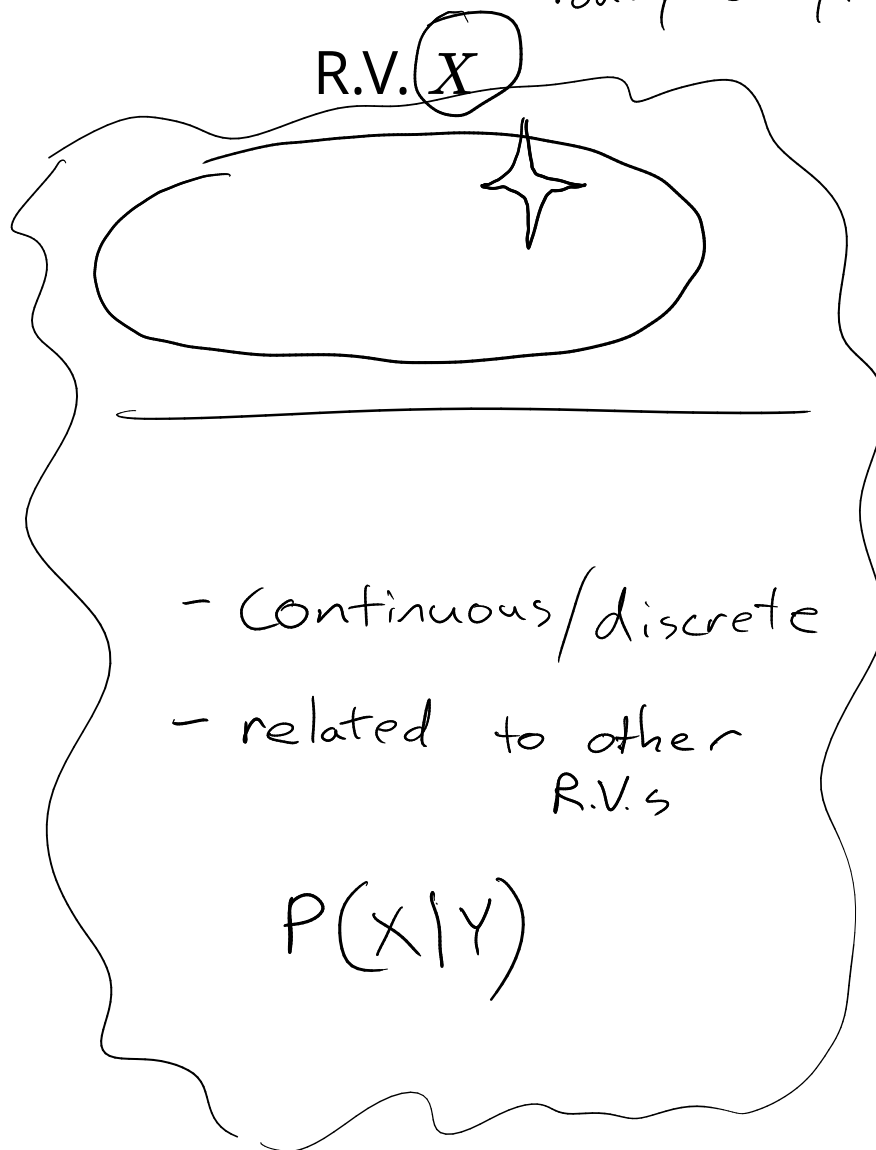
M

- Variable

- finite set of values

- probability

$$P(X=1) = 0.5$$



Blackbelly

(Ω, Σ, μ)

sample space

measure

σ algebra

$$X: \Omega \rightarrow E$$

Vocabulary/Notation

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example
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Vocabulary/Notation

Bernoulli(0.5)

Term

Definition

Coinflip Example

Uniform Example

Vocabulary/Notation

		Bernoulli(0.5)	$\mathcal{U}(0, 1)$
Term	Definition	Coinflip Example	Uniform Example

Vocabulary/Notation

Term	Definition	Bernoulli(0.5)	$\mathcal{U}(0, 1)$
Coinflip Example	Uniform Example	support(X)	

Vocabulary/Notation

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support(X)	All the values that X can take		

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Coinflip Example	Uniform Example		
support(X)	All the values that X can take		
$x \in X$			
$X \in [0, 1]$			

Vocabulary/Notation

Term	Definition	Bernoulli(0.5) Coinflip Example	$\mathcal{U}(0, 1)$ Uniform Example
support(X) $x \in X$ $X \in [0, 1]$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	

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Distribution

- Discrete: PMF
- Continuous: PDF

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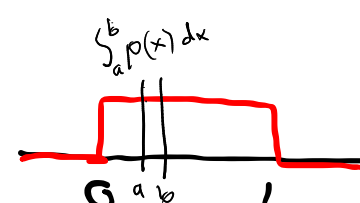
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Distributions of related R.V.s

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Joint Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Distributions of related R.V.s

Joint Distribution

Conditional Distribution

$$P(X, Y, Z)$$

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0	0	1	0.31
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Conditional Distribution

$$P(X \mid Y, Z)$$

Distributions of related R.V.s

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1	1	1	0.07

Conditional Distribution

$$\rightarrow P(X | Y, Z)$$

(Distribution - valued function)

X	$P(X Y=1, Z=1)$
0	0.84
1	0.16

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
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1	1	1	0.07

Conditional Distribution

$$P(X | Y, Z)$$

(Distribution - valued function)

X	$P(X Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
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0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
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1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
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Conditional Distribution

$$P(X | Y, Z)$$

(Distribution - valued function)

X	$P(X Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X) \quad P(Y) \quad P(Z)$$

X	$P(X)$	Y	$P(Y)$
0	0.85	0	0.45
1	0.15	1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

3 Rules

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

3 Rules (Burrito-level)

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

3 Rules

(Burrito-level)

(Blackbelly Level: Axioms of Probability)

AXIOM 1. STRUCTURE OF UNKNOWN REAL NUMBERS AND PLAUSIBLE VALUE. We assume a set T of unknown numbers is a partially ordered commutative algebra over \mathbb{R} with identity, 1.

We assume in addition a given sub-Boolean algebra E of $E(T)$ with $0, 1 \in E$ and denote by E_0 the set of non-zero members of E . We assume that the partial ordering in $E(T)$ as a Boolean algebra coincides with the ordering that $E(T)$ inherits from the algebra T . Finally, we assume a function $PV : T \times E_0 \rightarrow \mathbb{R}$, called **PLAUSIBLE VALUE**, whose value on the pair (x, e) is denoted $PV(x|e)$.

not on exam

AXIOM 2. STRONG RESCALING FOR PLAUSIBLE VALUE. If a, b belong to \mathbb{R} , if x belongs to T , and if e belongs to E_0 , then

$$PV(ax + b|e) = aPV(x|e) + b. \quad (2)$$

AXIOM 3. ORDER CONSISTENCY FOR PLAUSIBLE VALUE. If $x, y \in T$ and if $e \in E_0$, implies that $x \leq y$, then $PV(x|e) \leq PV(y|e)$.

Notice that if $e \in E(T)$, then $0 \leq e \leq 1$, in T , as it is true in the lattice ordering of $E(T)$.

AXIOM 4. THE COX AXIOM FOR PLAUSIBLE VALUE: If e, c are fixed in E , with $ec \in E_0$, if x_1, x_2 are in T , if $PV(x_1|ec) = PV(x_2|ec)$, then $PV(x_1|c) = PV(x_2|c)$. That is, we assume that as a function of x , the plausible value $PV(x|c)$ depends only on $PV(x|ec)$.

AXIOM 5. RESTRICTED ADDITIVITY OF PLAUSIBLE VALUE. For each fixed $y \in T$ and $e \in E_0$, the plausible value $PV(x + y|e)$ as a function of $x \in T$ depends only on $PV(x|e)$, which is to say that if $x_1, x_2 \in T$ and $PV(x_1|e) = PV(x_2|e)$, then $PV(x_1 + y|e) = PV(x_2 + y|e)$.

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

3 Rules (Burrito-level)

1)

Distributions of related R.V.s

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3 Rules (Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

Distributions of related R.V.s

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- 2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

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Joint \rightarrow Marginal

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- 3) Definition of Conditional Probability

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

Joint \rightarrow Marginal

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Joint \rightarrow Marginal

Joint + Marginal \rightarrow Conditional

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Joint \rightarrow Marginal

Joint + Marginal \rightarrow Conditional

Marginal + Conditional \rightarrow Joint

$$P(X, Y) = P(X \mid Y) P(Y)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

3 Rules

- 1) a) $0 \leq P(X | Y) \leq 1$
 b) $\sum_{x \in X} P(x | Y) = 1$

- 2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

- 3) Definition of Conditional Probability

$$P(X | Y) = \frac{P(X, Y)}{P(Y)}$$

Conditional Distribution

$$P(X | Y, Z)$$

X	Y	Z	P(X, Y, Z)
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Marginal Distribution

$$P(X) \quad P(Y) \quad P(Z)$$

X	P(X)
---	------

$$P(X=0) = 0.08 + 0.31 + 0.09 + 0.37 = 0.85$$

$$P(X=1) = 0.01 + 0.05 + 0.02 + 0.07 = 0.15$$

$$P(X=0) = \sum_{y \in Y, z \in Z} P(X=0, Y=y, Z=z)$$

$$P(X=0 | Y=1, Z=1) = \frac{P(X=0, Y=1, Z=1)}{P(Y=1, Z=1)}$$

$$= \frac{0.37}{0.37 + 0.07}$$

Joint → Marginal

Joint + Marginal → Conditional

Marginal + Conditional → Joint

$$P(X, Y) = P(X | Y) P(Y)$$

1) a) $0 \leq P(X | Y) \leq 1$

→ b) $\sum_{x \in X} P(x | Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

$P(X, Y) = P(X|Y) P(Y)$

Break

- $P \in \{0, 1\}$: Powder Day
- $C \in \{0, 1\}$: Pass Clear
- 1 in 5 days is a powder day
- The pass is clear 8 in 10 days
- If it is a powder day, there is a 50% chance the pass is blocked

$P(P=1) = 0.2$

$P(C=1) = 0.8$

$P(C=0 | P=1) = 0.5$

$P(C=1|P=1) + P(C=0|P=1) = 1.0$
 $\uparrow 0.5$

$P(C=1) = P(C=1, P=1) + P(C=1, P=0)$
 $\rightarrow 0.8 \rightarrow 0.1$

P	C	
0	0	0.1
0	1	0.7
1	0	0.1
1	1	0.1

$P(C=0|P=1) P(P=1)$
 $P(C=1|P=1) P(P=1)$
0.5

- Write out the joint probability distribution for P and C.
- Suppose it is a non-powder day, what is the probability that the pass is blocked?

$P(C=0 | P=0) = \frac{P(C=0, P=0)}{P(P=0)} = \frac{0.1}{0.8} = 0.125$

Bayes Rule

- Know: $P(B | A)$, $P(A)$, $P(B)$
- Want: $P(A | B)$

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

$$P(B|A) = \frac{P(A,B)}{P(A)}$$

$$P(A|B) P(B) = P(A,B) = P(B|A) P(A)$$

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$

$$P(A|B,C) = \frac{P(B|A,C) P(A|C)}{P(B|C)}$$

Independence

Independence

Definition: X and Y are *independent* iff $P(X, Y) = P(X) P(Y)$

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$$X \perp Y$$

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$$X \perp Y$$

$$P(X|Y) = P(X)$$

Definition: X and Y are *conditionally independent* given Z iff

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

Independence

Definition: X and Y are *independent* iff $P(X, Y) = P(X) P(Y)$

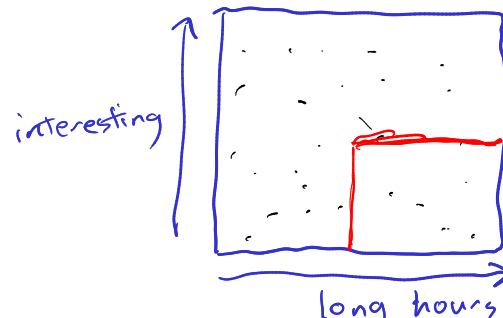
$$X \perp Y$$

$$P(X|Y) = P(X)$$

Definition: X and Y are *conditionally independent* given Z iff

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

$$X \perp Y | Z$$



crash on I-70 crash on US 40 bad storm ?

$$X \perp Y | Z \Rightarrow X \perp Y \quad \text{Not true}$$

$$X \perp Y \Rightarrow X \perp Y | Z \quad \text{Not true}$$

↑ considering

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X | Y) \leq 1$

b) $\sum_{x \in X} P(x | Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

$$P(X, Y) = P(X | Y) P(Y)$$

Continuous

1)

Rules for Continuous RVs

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- 1) a) $0 \leq P(X | Y) \leq 1$
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- 2) $P(X) = \sum_{y \in Y} P(X, y)$
- 3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$
 $P(X, Y) = P(X | Y) P(Y)$

Continuous

- 1) $0 \leq p(X | Y)$

Rules for Continuous RVs

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$$P(X, Y) = P(X | Y) P(Y)$$

Continuous

1) $0 \leq p(X | Y)$
 $\int_X p(x|Y) dx = 1$

Rules for Continuous RVs

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1) a) $0 \leq P(X | Y) \leq 1$

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3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

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Continuous

1) $0 \leq p(X | Y)$

$$\int_X p(x|Y) dx = 1$$

2)

$$p(X) = \int_Y p(X, y) dy$$

Rules for Continuous RVs

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$$p(X) = \int_Y p(X, y) dy$$

3) $p(X | Y) = \frac{p(X, Y)}{p(Y)}$

$$p(X, Y) = p(X | Y) p(Y)$$

Multivariate Gaussian Distribution

$$x = [x_1, x_2]$$

$$\mathcal{N}(\mu, \Sigma)$$

Joint Distribution

Conditional Distribution

Marginal Distribution

$$p(x) = \mathcal{N}(x | \mu, \Sigma)$$

$$= \frac{\exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)}{(2\pi)^{n/2} |\Sigma|^{1/2}}$$

$$\mu = [3, 3] \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$p(x_1 | x_2)$$

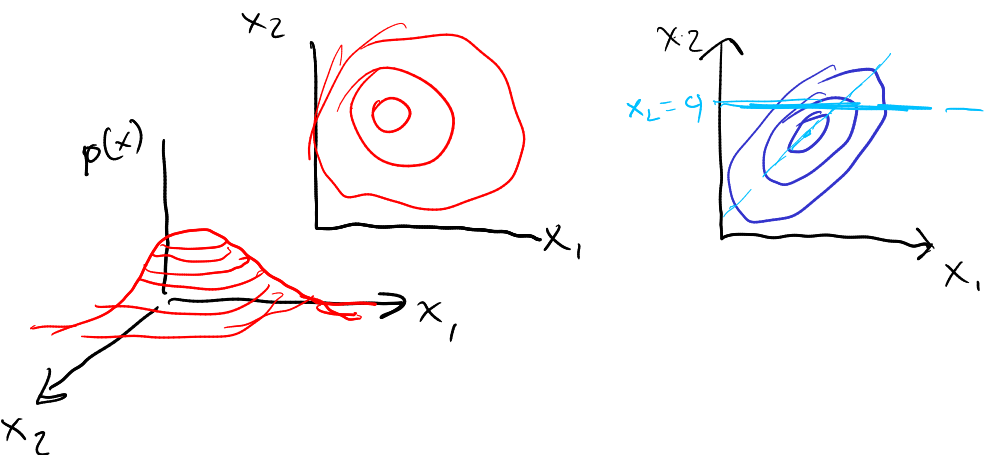
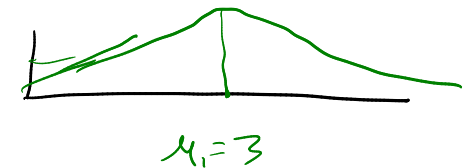
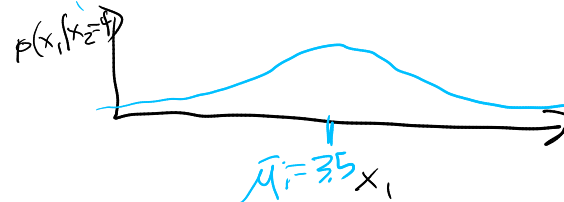
$$= \mathcal{N}(x_1 | \bar{\mu}_1, \bar{\Sigma}_1)$$

$$\bar{\mu}_1 = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$$

$$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

$$p(x_1) = \mathcal{N}(x_1 | \mu_1, \Sigma_{11})$$

$$p(x_1 | x_2 = 4)$$



Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables