Investor behaviors in financial markets: A Maxwell's demon approach

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WEMFIMAP, July 2022

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Measuring market efficiency

- Efficient market hypothesis (EMH): "A market in which prices always fully reflect available information is called efficient" (Fama)
- Three forms:
  - weak: the information set is just historical prices,
  - semi-strong form: includes public information (announcements of annual earnings, stock splits, etc.),
  - strong form: includes private information for some investors.

  In this work, we test weak-form efficiency.
- Among the several methods for measuring efficiency:
  - Hurst exponents,
  - information theory, which quantifies the uncertainty in a symbolic sequence:
    - permutation entropy (Bandt and Pompe, 2002): quantifies the uncertainty of the permutation between a series of consecutive prices and its sorted version,
    - entropy of a series of 0s (negative price return) and 1s (positive price returns) (Risso, 2008)
- Entropy is presented as a gradual indicator of efficiency: the market is more or less efficient. In this work, **we build a statistical test of efficiency.**

From prices to Shannon's entropy...

- From a time series of $n+1$ prices $P_0, \cdots, P_n$, we build a binary sequence $X_1, \cdots, X_n$:

$$X_i = \mathbf{1}_{\{P_i - P_{i-1}\} \geq 0} \tag{1}$$

- We assume that the random series (X_i) is stationary.
- Symbolic representation of a sequence of consecutive prices: for example $(X_1, X_2, X_3) = (0, 1, 1)$ represents a decrease followed by two daily price increases.
- For a given length $L < n$, 2^L sequences are possible, ordered with Gray's binary code. If $L = 3$, the 8 sequences are $(G_1^3, \cdots, G_8^3) = ((0,0,0),(0,0,1),(0,1,1),(0,1,0),(1,1,0),(1,0,0),(1,0,1),(1,1,1))$.
- The probability to draw a particular sequence of length L is $p_i^L = \mathbb{P}((X_., X_{.+1}, \cdots, X_{.+L}) = G_i^L)$.
- For the probabilistic choice system $(\{0,1\}^{2^L}, (p_i^L)_i)$, the **Shannon entropy** is the amount of uncertainty in $(\{0,1\}^{2^L}, (p_i^L)_i)$ and is defined by:

$$H^L = -\sum_{i=1}^{2^L} p_i^L \log_2(p_i^L) \tag{2}$$

... and to market information

- We decompose a sequence $(X_1, \cdots, X_{L+1})$ (observed in time L) in:
  - an observed prefix $(X_1, \cdots, X_L) = G_i^L$ of probability p_i^L,
  - a random suffix X_{L+1} whose conditional distribution is Bernoulli of parameter $\pi_i^L \in [0, 1]$.

The sequence $(X_1, \cdots, X_{L+1})$ is thus equal to $(G_i^L, 1)$ with probability $p_i^L \pi_i^L$ and $(G_i^L, 0)$ with probability $p_i^L(1 - \pi_i^L)$. The market entropy is then :

$$H^{L+1} = -\sum_{i=1}^{2^L} (p_i^L \pi_i^L \log_2(p_i^L \pi_i^L) + p_i^L(1 - \pi_i^L) \log_2(p_i^L(1 - \pi_i^L))) \tag{3}$$

- The EMH asserts that, conditionally on the prefix $(X_1, \cdots, X_L)$, the values of suffix $X_{L+1} = 1$ and $X_{L+1} = 0$ are equiprobable. It leads to a particular case with $\pi_i^L = \frac{1}{2}$, the **entropy of an efficient market** is then:

$$H_\star^{L+1} = -\sum_{i=1}^{2^L} p_i^L \log_2\left(\frac{p_i^L}{2}\right) = 1 + H^L \tag{4}$$

- We define the **market information** as the difference between the entropy according to the ideal EMH and the entropy of the market:

$$I^{L+1} = H_\star^{L+1} - H^{L+1} \tag{5}$$

Some insights on market information

- The theoretical value of the market information I^{L+1} is 0 if and only if the market is efficient.
- As soon as EMH does not hold, we have $I^{L+1} > 0$.
- We evaluate empirically the probabilities $p_1^L, \cdots, p_{2L}^L$ and $\pi_i^L, \cdots, \pi_{2L}^L$ and get the empirical information $\widehat{I}^{L+1}$.
- We still have $\widehat{I}^{L+1} \geq 0$, but if the EMH holds, we may have $\widehat{I}^{L+1}$ slightly higher than 0. Building a statistical test is thus mandatory for answering the question of the efficiency of the market.

Statistical test of market efficiency

- **Null hypothesis:** the market is efficient and therefore $I^{L+1} = 0$.
- We only have access to the estimator $\widehat{I}^{L+1}$.
- We need the distribution of $\widehat{I}^{L+1}$ under the null hypothesis.
- We are able to express exactly all the moments of $\widehat{I}^{L+1}$ under the null hypothesis.
- In addition, the support of the distribution has a floor in zero (looks like a semi-infinite interval). Some well-chosen (non-Gaussian) distribution may be a good approximation.
- We focus on the case where the EMH holds, that is for $\pi_i^L = 1/2$ for all i, but for an empirical market information based on $\widehat{\pi}_i^L$ instead of π_i^L. The provided expression also assumes that we have access to the true probabilities of the prefixes: $\widehat{p}_i^L = p_i^L$.

Theorem

For $L \in \mathbb{N}^\star$, the moment-generating function of $\widehat{I}^{L+1}$, conditionally on the event $\mathcal{E} = \{\forall i \in [\![1, 2^L]\!], \widehat{p}_i^L = p_i^L, \pi_i^L = 1/2\}$, is:

$$M_{\widehat{I}^{L+1}} : t \mapsto \mathbb{E}\left[e^{t\widehat{I}^{L+1}} \middle| \mathcal{E} \right] = e^t \prod_{i=1}^{2^L} \sum_{j=0}^{n_i} C_{i,j}^L(t),$$

for value of t for which this quantity is defined and where

$$C_{i,j}^L(t) = \binom{n_i}{j} \frac{1}{2^{n_i}} \left(\frac{j}{n_i} \right)^{tp_i^L j/n_i \ln(2)} \left(1 - \frac{j}{n_i} \right)^{tp_i^L (1-j/n_i)/\ln(2)}.$$

Moments of the market information

We then deduce the moments, for $r \in \mathbb{N}$:

$$\mathbb{E}\left[\left(\widehat{I}^{L+1}\right)^r \middle| \mathcal{E}\right] = \sum_{m=0}^{r} \binom{r}{m} \sum_{j_1=0}^{n_1} \cdots \sum_{j_{2^L}=0}^{n_{2^L}} \alpha_{j_1,\dots,j_{2^L}}^m \prod_{i=1}^{2^L} \binom{n_i}{j_i} \frac{1}{2^{n_i}},$$

where

$$\alpha_{j_1,\dots,j_{2^L}} = \sum_{k=1}^{2^L} p_k^L \left(\frac{j_k}{n_k} \log_2\left(\frac{j_k}{n_k}\right) + \left(1 - \frac{j_k}{n_k}\right) \log_2\left(1 - \frac{j_k}{n_k}\right)\right).$$

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Asymptotic distribution of the market information estimator

We introduce the function g_j, defined, for $(t, x) \in \mathbb{R} \times (0, 1)$ and a given $j \in [\![1, 2^L]\!]$, by:

$$g_j(t,x) = \exp\left(it\left[p_j^L x \log_2(p_j^L x) + p_j^L(1-x)\log_2(p_j^L(1-x)) - p_j^L \log_2\left(\frac{p_j^L}{2}\right)\right]\right), \quad (6)$$

where i is the imaginary unit. As a summand of $\widehat{I}^{L+1}$, this functions appears in the characteristic function of the market information:

$$\varphi_{\widehat{I}^{L+1}} : t \in \mathbb{R} \mapsto \mathbb{E}\left[e^{it\widehat{I}^{L+1}} \middle| \mathcal{E}\right] = \prod_{j=1}^{2^L} \mathbb{E}\left[g_j(t, \widehat{\pi}_j)\right]. \quad (7)$$

Characteristic function of the empirical market information

The characteristic function of the empirical market information, conditionally on $\mathcal{E}$, is for $t \in \mathbb{R}$ and $p_j^L = n_j/n > 0$ whatever j,

$$\begin{array}{rcl} \varphi_{\widehat{I}^{L+1}}(t) & = & \prod_{j=1}^{2^L} \mathbb{E}\left[g_j(t, \widehat{\pi}_j)\right] \\ & \overset{n\to\infty}{\sim} & \prod_{j=1}^{2^L} \left(1 + \frac{it}{2\ln(2)n}\right) \\ & \overset{n\to\infty}{\sim} & \left(1 - \frac{it}{\ln(2)n}\right)^{-2^{L-1}} . \end{array} \quad (8)$$

- We recognize the characteristic function of the gamma distribution $\Gamma(k, \theta)$ of shape parameter $k = 2^{L-1}$ and scale parameter $\theta = 1/\ln(2)\, n$.
- The result does neither depend on p_j^L (non-conditional characteristic function) nor on n_j.

Distribution of market efficiency

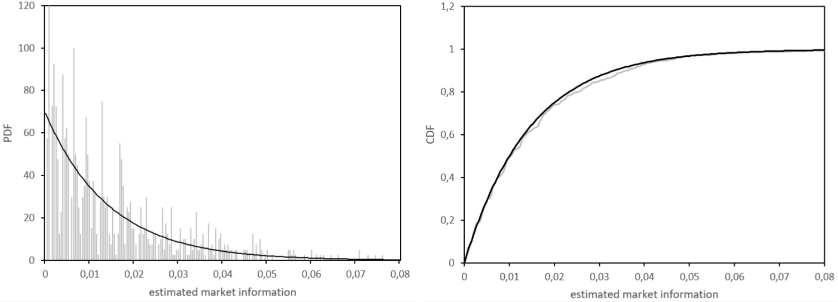

Figure: Asymptotic (black) and simulated (on 1,000 trajectories, grey) distributions of the estimated market information $\widehat{I^2}$ for $n = 100$.

Distribution of market efficiency

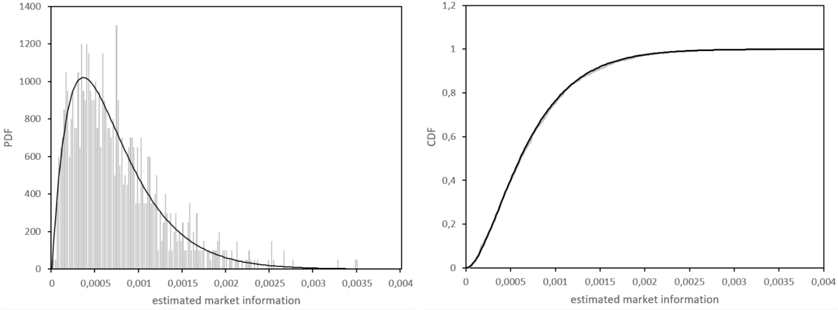

Figure: Asymptotic (black) and simulated (on 1,000 trajectories, grey) distributions of the estimated market information $\widehat{I^3}$ for $n = 4,000$.

Distribution of market efficiency

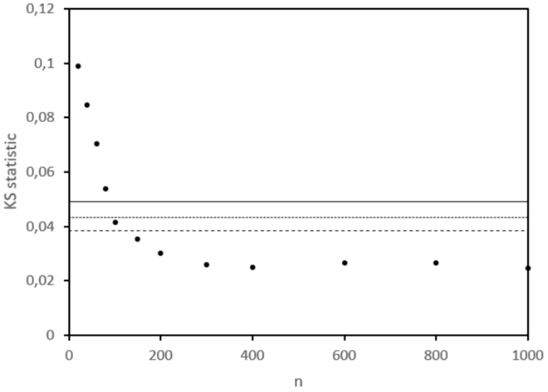

Figure: Kolmogorov-Smirnov statistic between the asymptotic and the simulated (on 1,000 trajectories) distributions of the estimated market information $\widehat{I^2}$ for various values of n. The three lines indicate the values of the statistics leading to a p-value of 5%, 1%, and 0.1% (from bottom to top). Asymptotic distribution validated for $n \gtrsim 100$.

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The fBm in finance

The fractional Brownian motion (fBm): a fractional model to depict fractal properties:

$$B_{\mathcal{H}}(t) = \frac{1}{\Gamma(\mathcal{H}+\frac{1}{2})} \int_{-\infty}^{\infty} \left((t-s)_+^{\mathcal{H}-1/2} - (-s)_+^{\mathcal{H}-1/2} \right) dW_s \tag{9}$$

- if $\mathcal{H} = 1/2$, the fBm is identical to a standard Brownian motion,
- if $\mathcal{H} > 1/2$, the increments are persistent, presence of long memory,
- if $\mathcal{H} < 1/2$, the increments are anti-persistent.

In econophysics, $\mathcal{H}$ is used to determine whether the market is **efficient**. We can use it directly to make **predictions** (Nuzman and Poor), exploiting the covariance matrix.
Is the market efficiency related to $\mathcal{H} = 1/2$ for all the dynamics, even multifractal or non-Gaussian extensions of the fBm ? $\rightarrow$ No, other stylized facts to be filtered:

- time-varying efficiency,
- non-Gaussian dynamic.

Market information and fBm

The log-price is supposed to follow an fBm of Hurst exponent $\mathcal{H}$: $\log(P_t) = B_t^{\mathcal{H}}$.
Considering one lagged return for forecasting a future return, the market information is:

$$I^2 = -\sum_{i=1}^{2} \left(p_i^1 \log_2\left(\frac{p_i^1}{2}\right) - p_i^1 \pi_i^1 \log_2(p_i^1 \pi_i^1) - p_i^1 (1-\pi_i^1) \log_2(p_i^1 (1-\pi_i^1))\right) \quad (10)$$

It relies on:

- $p_1^1 = \mathbb{P}(X_1 = 0) = \mathbb{P}(B_1^{\mathcal{H}} - B_0^{\mathcal{H}} \leq 0) = 1/2$,
- $p_2^1 = \mathbb{P}(X_1 = 1) = 1/2$,
- $\pi_1^1 = \mathbb{P}(X_2 = 1 | X_1 = 0) = \mathbb{P}(B_2^{\mathcal{H}} - B_1^{\mathcal{H}} > 0 | B_1^{\mathcal{H}} - B_0^{\mathcal{H}} \leq 0) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$ where $\rho = 2^{2\mathcal{H}-1} - 1$ is the theoretical correlation between $(B_2^{\mathcal{H}} - B_1^{\mathcal{H}})$ and $(B_1^{\mathcal{H}} - B_0^{\mathcal{H}})$,
- $\pi_2^1 = \mathbb{P}(X_2 = 1 | X_1 = 1) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$

Market information and fBm

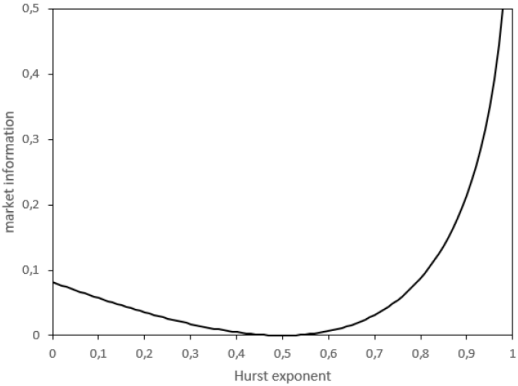

Figure: Theoretical market information I^2 for a dynamic of log-prices following an fBm, depending on its Hurst exponent.

Two equivalent indicators? No, the Hurst exponent is model-dependent and an irrelevant indicator of market efficiency when the log-prices are not an fBm.

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Action of Maxwell's demon

- Andrew Lo introduces the adaptive markets hypothesis **(AMH)** which states that statistical arbitrage is possible and that the EMH depends on the time scale.
- The statistical arbitrage of an investor reminds us of the action of **Maxwell's demon** on a thermodynamic system.
- Statistical arbitrage refers to trading strategies that use statistical techniques to make profit with an element of market risk reduction.

- In a closed thermodynamic system at uniform temperature, molecules are moving with velocities by no means uniform. The thermodynamic system is divided into two portions. The demon controls a door between these two portions. He has the information about the velocities of the molecules and decides whether or not the molecule can pass trough the door by knowing the velocities of the molecule and the mean of the velocities of all the molecules in the portion. He is then able to warm up one portion while cool down the other by moving molecules from the door. This action would decrease the entropy of the system and then violate the second law of thermodynamic.
- In this work, we study the evolution of the market entropy and the market efficiency with an **informed investor** acting as a Maxwell's demon by doing statistical arbitrage.

Brillouin's perspective

- Brillouin made a thought experiment about the Maxwell's demon to understand if he can operate on the thermodynamic system descripted by Maxwell. For him, if the thermodynamic system is at constant temperature at the beginning of the experiment then the radiation is the one of a **"blackbody"**, thus the demon is not able to see the molecules. He is then unable to violate the second principle. So Brillouin introduces a **source of light** so that the demon can see the molecules. But this action **increases the overall entropy.**

The experiment and model in financial markets

- We work with two symbolic representations of consecutive prices. One time series, M_t, corresponds to the successive **increments of the market price**. The other, A_t, corresponds to a series of **alternative data** that the Maxwell's demon is the only one to possess and which has a predictive power on M_{t+1}.
- We model the action of the demon in the market and simulate prices with the following model :

$$P_T = P_{T-\Delta t} \exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + (w_m(2M_t - 1) + w_d(2\alpha_t - 1))\sigma|G|\right) \qquad (11)$$

- With:
  - w_m: The weight of all the participants in the market without the knowledge of the alternative data
  - w_d: The weight of the Maxwell's demon
  - $G \sim \mathcal{N}(0, \Delta t)$
  - r: the risk free rate
  - σ: the volatility
  - $M_t \sim \mathcal{B}(\frac{1}{2})$: The Bernoulli variable corresponding to an efficient market implied by noise traders
  - $\alpha_t \in \{0, 1\}$ corresponding to the action of the demon on the market (0: he is shorting the security, 1 : he is buying the security): it depends on (M_{t-1}, A_{t-1}).

Simulation :

- We first create the sequence (A_i) corresponding to an efficient sequence meaning that $\pi_i^L = \frac{1}{2}$. Then we create the sequence (M_i) correlated to the sequence of the alternative data.
- In this section, we work with $L = 1$.
- The probability available for the informed investor is : $\mathbb{P}\left(M_t \middle| \begin{bmatrix} M_{t-1} \\ A_{t-1} \end{bmatrix}\right)$
- If we have $\mathbb{P}\left(M_t = 1 \middle| \begin{bmatrix} M_{t-1} \\ A_{t-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) > \frac{1}{2}$, then the demon is buying the security at the moment t-1 if $M_{t-1} = 0$ and $A_{t-1} = 1$ (so $\alpha_t = 1$).

The experiment and model in financial markets

- We introduce $p^1_{i,2}$ with $p^1_{i,2} = \mathbb{P}\left(\begin{bmatrix} M_t \\ A_t \end{bmatrix} = G^1_{i,2}\right)$
- With $(G^1_{1,2}, G^1_{2,2}, G^1_{3,2}, G^1_{4,2}) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
- We determine the probabilistic choice system $\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}, (p^1_{i,2})_i\right)$
- We then have the demon entropy :

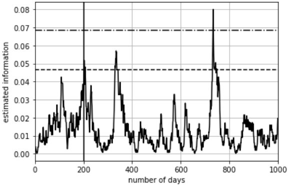
$$H^2_2 = -\sum_{i=1}^{4}(p^1_{i,2}\pi^1_{i,2}\log_2(p^1_{i,2}\pi^1_{i,2}) + p^1_{i,2}(1-\pi^1_{i,2})\log_2(p^1_{i,2}(1-\pi^1_{i,2})))$$

- We can then introduce $H^2_{\star,2}$ corresponding to a particular case with $\pi^1_{i,2} = \frac{1}{2}$
- The demon information is then : $I^2_2 = H^2_{\star,2} - H^2_2$
- We also introduce the correlation between A_{t-1} and M_t informing us about the correlation for the **prediction** of the time series of alternative data.

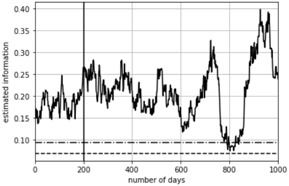
Simulations

- We take $w_m = 0.95$ and $w_d = 0.05$. We compute the market information with $n = 100$ and $L = 1$. We do the simulation for 1000 business days.
- For the first 200 days, the Maxwell's demon does not invest on the financial security ($w_d = 0$). We have $P_0 = 50\$$. We take $r = 0$ and $\sigma = 0.01$.

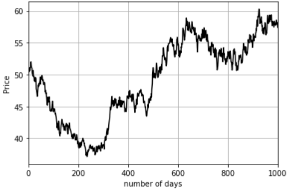
Simulations



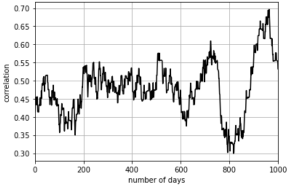
(a) Market information



(b) Demon's information



(c) Price



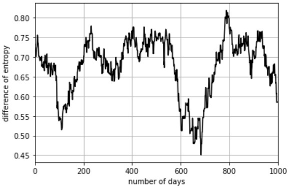
(d) Correlation between A_{t-1} and M_t

Figure: Market simulation.
The horizontal lines are the confidence intervals of the statistical test of absence of information, with probabilities of 95% and 99%

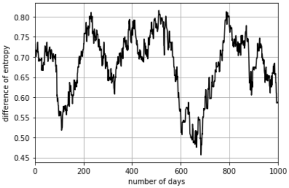
- We can see from those figures that the demon's information is strongly correlated with the correlation between A_{t-1} and M_t. Here the correlation is 0.92.
- If the correlation decreases then the informed investor is unable to predict well the future return, so **its information decreases.**
- Brillouin: If the intensity of light is not sufficient to determine the position and speed of molecules, the Maxwell's demon is **not able to move the molecules** in order to change the entropy of the thermodynamic system.
- In the long term, we always observe a **decrease** in the market information (so an increase of market entropy) as in physics with an increase of the entropy of the thermodynamic system.

Simulations

- We can not create order by adding disorder to disorder.



(a) $H_2^2 - H^2$



(b) $H_2^2 - H_d^2$

Figure: Difference of entropy for the two systems

H_d^2 corresponds to the demon's entropy while observing only the time series of alternative data

Simulations

- If the market is inefficient **(meaning arbitrage is possible)** then the participants and Maxwell's demon are taking advantage of the situation and the market quickly becomes efficient.
- It reminds us of the **Brillouin's cycle** meaning that after a peak of information, we lose it by intervening on the system.

Questions ?

1. A statistical test of market efficiency based on information theory
2. Asymptotic analysis
3. Market information on fractal dynamics
4. Maxwell's demon in financial markets