The epidemic lambda-coalescent model

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1 Introduction

Superspreading in infectious disease epidemiology (Lloyd-Smith et al. 2005). For example SARS-CoV-2 superspreading (Wang et al. 2020; Lemieux et al. 2021; Gómez-Carballa et al. 2021). Coalescent model (Kingman 1982a,b). Work by Li and Fraser (Li et al. 2017; Fraser and Li 2017). Lambda-coalescent models (Pitman 1999; Sagitov 1999; Donnelly and Kurtz 1999). Beta-coalescent (Schweinsberg 2003) is a specific type of Lambda-coalescent. Was used in (Hoscheit and Pybus 2019) and (Menardo et al. 2021). David's paper on inference of multiple mergers while dating a pathogen phylogeny (Helekal et al. 2024).

2 Coalescence probabilities

2.1 General case

Discrete time t. Non-overlapping generations of infected individuals. At time t there are N_t infected individuals. Each of them creates a number $s_{t,i}$ of secondary infections at time $t + 1$, following the offspring distribution $\alpha_t(s)$. The mean of this distribution is the basic reproduction number R_t and the variance is V_t. We have:

$$N_{t+1} = \sum_{i=1}^{N_t} s_{t,i} \tag{1}$$

Let $p_{k,t}$ be the probability that k individuals at time $t + 1$ have the same infector at time t.

Inclusive Coalescence Probability

Inclusive coalescence probability $p_{k,t}(N_t, N_{t+1})$ is the probability that exactly k randomly-sampled individuals from generation $t + 1$ find a common ancestor in generation t *(conditioning on population sizes N_t and N_{t+1}.*

Given full information about offspring counts from individuals in generation t $\mathbf{s}_t = (s_{t,1}, \ldots s_{t,N_t})$, we have

Don't use all of these: -Ian

$$p_{k,t}(\mathbf{s}_t, N_t) = \sum_{i=1}^{N_t} \frac{\binom{s_{t,i}}{k}}{\binom{N_{t+1}}{k}} \tag{2}$$

$$= \sum_{i=1}^{N_t} \frac{s_{t,i}!}{(s_{t,i} - k)!} \frac{(N_{t+1} - k)!}{N_{t+1}!} \tag{3}$$

$$= \sum_{i=1}^{N_t} \frac{\Gamma(s_{t,i} + 1)\Gamma(N_{t+1} - k + 1)}{\Gamma(s_{t,i} - k + 1)\Gamma(N_{t+1})} \tag{4}$$

$$= \sum_{i=1}^{N_t} \frac{s_{t,i}(s_{t,i}-1)\dots(s_{t,i}-k+1)}{N_{t+1}(N_{t+1}-1)\dots(N_{t+1}-k+1)}. \tag{5}$$

Full information $\{s_{t,i}\}$ yields the population size N_{t+1} but is not feasible to observe in practice. We can instead express the inclusive coalescence probability conditioning on the next population size N_{t+1} by summing over possible offspring counts $\mathbf{s}_t = (s_{t,1}, \dots s_{t,N_t})$ conditional on the total generation size

To-do/To-discuss:

- Almost certainly should be an appendix not a main-body derivation, doesn't fundamentally change anything but I think this looks horrific

- Sum subscripts over $\mathbf{s}_t \in \mathbb{N}_0^{N_t}$ rather than $\mathbf{s}_t : \dots$ (c.f. exclusive probability derivation)? We have probability 0 if sum does not equal N_{t+1} anyway

- Change $\mathbf{s}_t \mapsto \mathbf{s}$ and $s_{t,i} \mapsto s_i$ for clarity/brevity? Should maintain $S_{t,i}$ etc. for consistent notation ($S_{t,i} = s_i$ etc.)

- Define $S_t^{-(1)} = (S_{t,2}, \dots, S_{t,N_t})$ notation

- Where to stop? Leave as conditional expectation of binomial coefficients, or convert to factorials etc? Factorials are slightly clearer that this is just a 'falling factorial moment' computed from probability generating function

$$p_{k,t}(N_t, N_{t+1}) = \sum_{\mathbf{s}_t : \sum_{i=1}^{N_t} s_{i,t} = N_{t+1}} \mathbb{P}\left[\mathbf{S}_t = \mathbf{s}_t \middle| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right] p_{k,t}(\mathbf{s}_t, N_t) \tag{6}$$

$$= \sum_{\mathbf{s}_t : \sum_{i=1}^{N_t} s_{i,t} = N_{t+1}} \mathbb{P}\left[\mathbf{S}_t = \mathbf{s}_t \middle| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right] \sum_{i=1}^{N_t} \frac{\binom{s_{t,i}}{k}}{\binom{N_{t+1}}{k}} \tag{7}$$

$$= \sum_{i=1}^{N_t} \sum_{\mathbf{s}_t : \sum_{i=1}^{N_t} s_{i,t} = N_{t+1}} \frac{\binom{s_{t,i}}{k}}{\binom{N_{t+1}}{k}} \mathbb{P}\left[S_{t,1} = s_{t,1}, \mathbf{S}_t^{-(1)} = \mathbf{s}_t^{-(1)} \middle| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right] \tag{8}$$

$$= \frac{N_t}{\binom{N_{t+1}}{k}} \sum_{\mathbf{s}_t : \sum_{i=1}^{N_t} s_{i,t} = N_{t+1}} \binom{s_{t,1}}{k} \mathbb{P}\left[S_{t,1} = s_{t,1} \middle| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right]$$

$$\times \mathbb{P}\left[\mathbf{S}_t^{-(1)} = \mathbf{s}_t^{-(1)} \middle| S_{t,1} = s_{t,1}, \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right] \tag{9}$$

$$= \frac{N_t}{\binom{N_{t+1}}{k}} \sum_{s_{t,1}=0}^{N_{t+1}} \binom{s_{t,1}}{k} \mathbb{P}\left[S_{t,1} = s_{t,1} \middle| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1}\right]$$

$$\times \underbrace{\sum_{\mathbf{s}_t^{-(1)} : \sum_{i=2}^{N_t} s_{i,t} = N_{t+1} - s_{1,t}} \mathbb{P}\left[\mathbf{S}_t^{-(1)} = \mathbf{s}_t^{-(1)} \middle| \sum_{i=2}^{N_t} S_{t,i} = N_{t+1} - s_{1,t}\right]}_{=1} \tag{10}$$

$$= \frac{N_t}{\binom{N_{t+1}}{k}} \mathbb{E}\left[\binom{S_{t,1}}{k} \Bigg| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1} \right] \tag{11}$$

$$= N_t \frac{(N_{t+1} - k)!}{N_{t+1}!} \mathbb{E}\left[\frac{S_{t,1}!}{(S_{t,1} - k)!} \Bigg| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1} \right]. \tag{12}$$

The falling factorial moments $\mathbb{E}\left[\frac{S_{t,1}!}{(S_{t,1}-k)!} \big| \sum_{i=1}^{N_t} S_{t,i} = N_{t+1} \right]$ in (12) can be readily obtained by differentiating the probability generating function of $S_{t,1} | (\sum_{i=1}^{N_t} S_{t,i} = N_{t+1})$.

Exclusive Coalescence Probability

Generally, we observe a sample of individuals from each generation rather than the entire population. In this case, we are interested in the exclusive coalescence probability $p_{nkt}(N_t, N_{t+1})$ that exactly k individuals from a sample of n arose from a common ancestor one generation in the past given knowlege of the total population sizes N_t and N_{t+1}.

Given full information about offspring counts of the parents of sampled individuals at the present, $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,N_t})$, we have

$$p_{nkt}(\mathbf{x}_t, N_t) = \sum_{i=1}^{N_t} \frac{\binom{x_{t,i}}{k}}{\binom{n}{k}} \mathbb{I}\{x_{t,i} = k\} \tag{13}$$

$$= \sum_{i=1}^{N_t} \frac{x_{t,i}!}{(x_{t,i} - k)!} \frac{(n-k)!}{n!} \mathbb{I}\{x_{t,i} = k\}. \tag{14}$$

Similarly to the exclusive coalescence probability, we can use this to evaluate the exclusive probability given N_t and N_{t+1} by summing over possible parent offspring configurations (for $k \leq n$),

$$p_{nkt}(N_t, N_{t+1}) = \sum_{\mathbf{x}_t \in \mathbb{N}_0^{N_t}} \mathbb{P}\left[\mathbf{X}_t = \mathbf{x}_t \Bigg| \sum_{i=1}^{n} X_{t,i} = n \right] p_{n,k,t}(N_t, N_{t+1}) \tag{15}$$

$$= \sum_{\mathbf{x}_t \in \mathbb{N}_0^{N_t}} \mathbb{P}\left[\mathbf{X}_t = \mathbf{x}_t \Bigg| \sum_{i=1}^{n} X_{t,i} = n \right] \sum_{i=1}^{N_t} \frac{\binom{x_{t,i}}{k}}{\binom{n}{k}} \mathbb{I}\{x_{t,i} = k\} \tag{16}$$

$$= \frac{N_t}{\binom{n}{k}} \sum_{\mathbf{x}_t \in \mathbb{N}_0^{N_t}} \binom{x_{t,1}}{k} \mathbb{P}\left[\mathbf{X}_t = \mathbf{x}_t \Bigg| \sum_{i=1}^{N_t} X_{t,i} = n \right] \mathbb{I}\{x_{t,1} = k\} \tag{17}$$

$$= \frac{N_t}{\binom{n}{k}} \sum_{\mathbf{x}_t^{-(1)} : \sum_{i=2}^{n} x_{t,i} = n-k} \binom{k}{k} \mathbb{P}\left[X_{t,1} = k, \mathbf{X}_t^{-(1)} = \mathbf{x}_t^{-(1)} \Bigg| \sum_{i=1}^{N_t} X_{t,i} = n \right] \tag{18}$$

$$= \frac{N_t}{\binom{n}{k}} \mathbb{P}\left[X_{t,1} = k \Bigg| \sum_{i=1}^{N_t} X_{t,i} = n \right] \sum_{\mathbf{x}_t^{-(1)} : \sum_{i=2}^{n} x_{t,i} = n-k} \mathbb{P}\left[\mathbf{X}_t^{-(1)} = \mathbf{x}_t^{-(1)} \Bigg| \sum_{i=1}^{N_t} X_{t,i} = n, X_{t,1} = k \right] \tag{19}$$

$$= \frac{N_t}{\binom{n}{k}} \mathbb{P}\left[X_{t,1} = k \middle| \sum_{i=1}^{N_t} X_{i,t} = n\right]. \tag{20}$$

2.2 Poisson case

Here the offspring distribution is Poisson(R_t). In this case, we have

$$\sum_{i=1}^{N_t} S_{t,i} \sim \text{Poisson}(N_t R_t)$$

and

$$S_{t,1}|(\sum_{i=1}^{N_t} S_{t,i} = N_{t+1}) \sim \text{Binomial}(N_{t+1}, \frac{1}{N_t}).$$

Analogously to the Wright-Fisher model, individuals select a parent uniformly at random from the previous generation. The inclusive probability of coalescence for two lines is

$$p_{2,t} = \frac{1}{N_t}, \tag{21}$$

and more generally the inclusive probability of coalescence for n lines is

$$p_{k,t} = \frac{1}{N_t^{k-1}}. \tag{22}$$

The exclusive probability of coalescence for two lines from a sample of n $(n \geq 2)$ is

$$p_{n,2,t} = \frac{(N_t - 1)^{n-2}}{N_t^n}, \tag{23}$$

and more generally the exclusive probability of coalescence for k lines from a sample of n $(n \geq k)$ is

$$p_{n,k,t} = \frac{(N_t - 1)^{n-k}}{N_t^n}. \tag{24}$$

2.3 Negative-Binomial case

Here the offspring distribution is Negative-Binomial with mean R_t and variance V_t. The parameters of this distribution are $r = R_t^2/(V_t - R_t)$ and $p = R_t/V_t$.

The inclusive probability of coalescence for two lines is:

$$p_{2,t} = \frac{r+1}{N_t r + 1} \tag{25}$$

The inclusive probability of coalescence for k lines is:

$$p_{k,t} = \prod_{i=1}^{k-1} \frac{r+i}{N_t r + i} \tag{26}$$

2.4 Example

Let the offspring distribution have a mean of $R_t = 2$. In the Poisson case the offspring distribution is Poisson(2). We consider NegBin cases with the same mean and varying dispersion parameters r. When r is high the dispersion is low and the NegBin behaves almost like a Poisson. See Figure 1.

3 Lambda-coalescent

A lambda-coalescent model is defined by a probability measure $\Lambda(\mathrm{d}x)$ on the interval $[0,1]$, from which we can deduce the rate $\lambda_{n,k}$ at which any subset of k lineages within a set of n observed lineages coalesce:

$$\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\,\Lambda(\mathrm{d}x) \tag{27}$$

3.1 Beta-coalescent

Let the Beta function be denoted as $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The Beta$(2-\alpha,\alpha)$-coalescent model (Schweinsberg 2003) has a single parameter $\alpha \in [0,2]$ and is defined as:

$$\Lambda(\mathrm{d}x) = \frac{x^{1-\alpha}(1-x)^{\alpha-1}}{\mathrm{B}(2-\alpha,2)}\mathrm{d}x \tag{28}$$

from which we can deduce that:

$$\lambda_{n,k} = \frac{\mathrm{B}(k-\alpha, n-k+\alpha)}{\mathrm{B}(2-\alpha,\alpha)} \tag{29}$$

Special cases include $\alpha = 2$ corresponding to the Kingman coalescent, $\alpha = 1$ which is known as the Bolthausen-Sznitman coalescent and $\alpha = 0$ for which the phylogeny is always star-shaped.
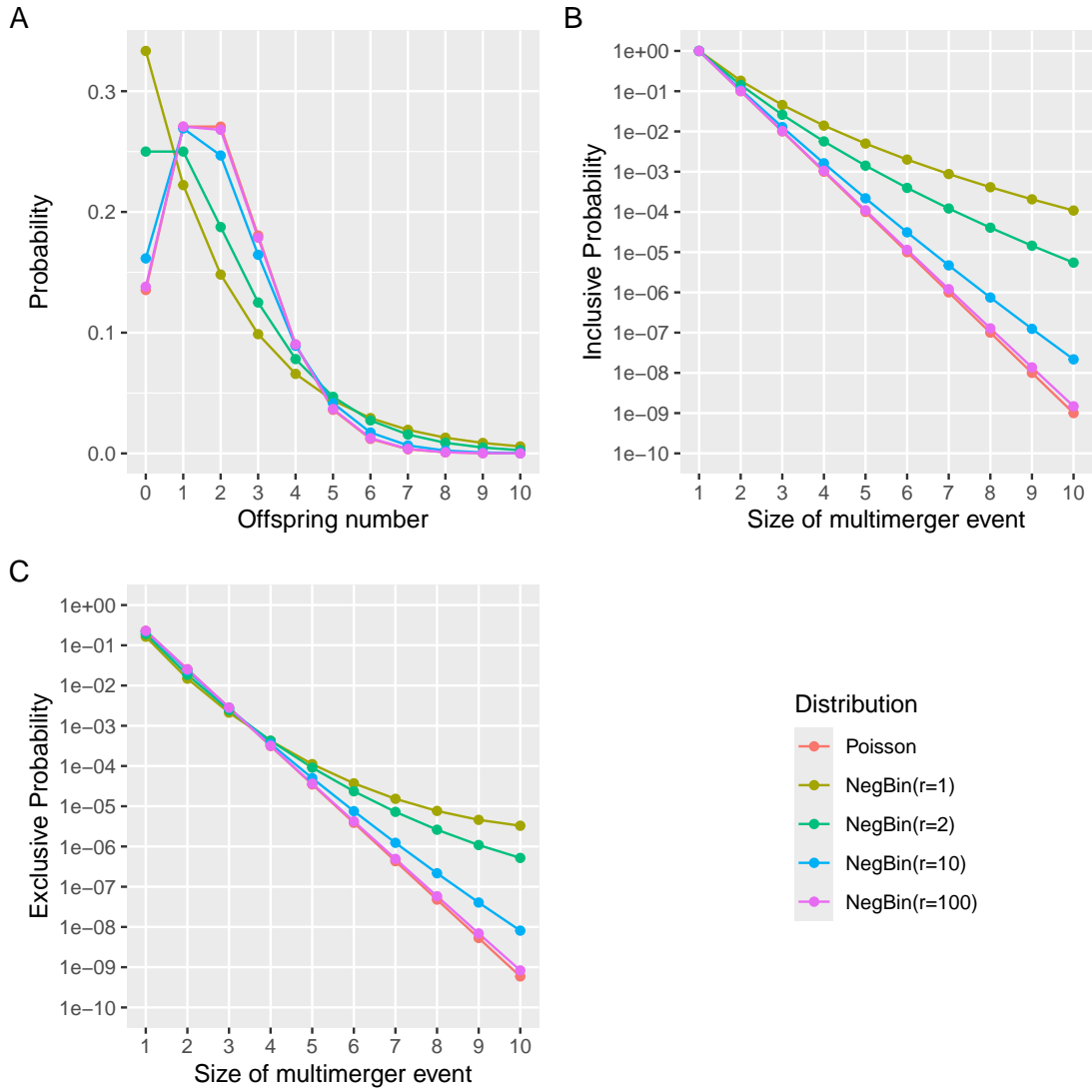
Figure 1: (A) Offspring distribution. (B) Inclusive probability of coalescence. (C) Exclusive probability of coalescence.

4   Implementation

We implemented the analytical methods described in this paper in a new R package entitled *EpiLambda* which is available at `https://github.com/xavierdidelot/EpiLambda` for R version 3.5 or later. All code and data needed to replicate the results are included in the "run" directory of the *EpiLambda* repository.

5 Discussion

Acknowledgements

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