1 Problem

"Imagine taking a number and moving its last digit to the front. For example, 1,234 would become 4,123. What is the smallest positive integer such that when you do this, the result is exactly double the original number? (For bonus points, solve this one without a computer.)"

2 Solution

Let's begin by writing our numbers in the following form

$$10^N d_N + 10^{N-1} d_{N-1} + \cdots + 10^1 d_1 + 10^0 d_0 = \sum_{n=0}^{N} 10^n d_n \quad \text{where } d_n \in \{0, 1, 2, \ldots, 9\} \tag{1}$$

We are looking for a number that satisfies the following equality

$$2 \sum_{n=0}^{N} 10^n d_n = 10^N d_0 + \sum_{n=1}^{N} 10^{n-1} d_n \tag{2}$$

Rearranging we get

$$\sum_{n=1}^{N} (2 \cdot 10^n - 10^{n-1}) d_n = (10^N - 2 \cdot 10^0) d_0 \tag{3}$$

Note that $2 \cdot 10^n - 10^{n-1} = 19 \cdot 10^{n-1}$. As such we rewrite the equation as follows

$$\sum_{n=1}^{N} 10^{n-1} d_n = \frac{(10^N - 2) d_0}{19} \tag{4}$$

Since we know that the left side of the equation is an integer, we also know that the numerator of the right side has to be divisible by 19! As such, we know that one of the following must be true:

$$10^N - 2 \equiv 0 \ (mod\ 19) \tag{5}$$

or

$$d_0 \equiv 0 \ (mod\ 19) \tag{6}$$

Because we previously defined $d_n \in \{0, 1, 2, \ldots, 9\}$, we know that equation 6 is only true when $d_0 = 0$ but this solution is trivial. Therefore, we will impose equation 5 which we can solve for N to get

$$N = 18n + 17 \text{ where } n \in \mathbb{Z}_{\geq 0} \tag{7}$$

The smallest value that satisfies this equation is $n = 0$ which yields $N = 17$. Because out indexing begins at 0, we can deduce that the smallest possible number that satisfies the problem is 18 digits! Once we set $N = 17$ we observe that the right side of equation 4 becomes

$$\sum_{n=1}^{N} 10^{n-1} d_n = \frac{(10^{17} - 2)d_0}{19} \tag{8}$$

We know see that the left side of the equation can only take one of 10 different values (i.e. each of the 10 possible values for d_0) and represents the first 17 digits of the solution. Starting from 0, we find that $d_0 = 2$ is the first value that satisfies equation 1. This is shown below

$$\sum_{n=1}^{N} 10^{n-1} d_n = 1026315789473684 \tag{9}$$

Appending d_0 to the end of this number yields the following number and the solution to the riddle.

$$10263157894736842 \tag{10}$$

Move the 2 to the front of the number

$$21026315789473684 \tag{11}$$

and

$$\begin{aligned} & 2 \cdot 1026315789473684\ 2 \\ = \ & 2\ 1026315789473684 \end{aligned}$$