COMPLETENESS OF ÅQVIST'S SYSTEMS **E** AND **F**

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Abstract. This paper tackles an open problem posed by Åqvist. It is the problem of whether his dyadic deontic systems **E** and **F** are complete with respect to their intended Hanssonian preference-based semantics. It is known that there are two different ways of interpreting what it means for a world to be best or top-ranked among alternatives. This can be understood as saying that it is optimal among them, or maximal among them. First, it is established that, under either the maximality rule or the optimality rule, **E** is sound and complete with respect to the class of all preference models, the class of those in which the betterness relation is reflexive, and the class of those in which it is total. Next, an analogous result is shown to hold for **F**. That is, it is established that, under either rule, **F** is sound and complete with respect to the class of preference models in which the betterness relation is limited, the class of those in which it is limited and reflexive, and the class of those in which it is limited and total.

§1. Introduction. This paper tackles an open problem posed by Åqvist. It is the problem of whether his dyadic deontic systems **E** and **F** are complete with respect to a preference-based semantics. In models of this kind, possible worlds are ranked in terms of a Hanssonian preference relation, viewed as a relation of comparative goodness or betterness. The label "Hanssonian" derives from Bengt Hansson, to whom the credit for the key novelty should go.[1] In that framework, the truth conditions for the dyadic deontic modalities are phrased in terms of best antecedent-worlds. Systems **E** and **F** are also known to be complete with respect to a selection function semantics in the style of Stalnaker (1968). However, the semantics in terms of preference relations is more natural and intuitive. The question of whether the above systems are also complete relative to their intended Hanssonian modelling does not appear to have been settled.

This paper is part of a larger project, whose aim is to characterize axiomatically the dyadic logics for conditional obligation that may be associated with a Hanssonian preference-based semantics. The main motivation for such a semantics has to do with the analysis of so-called contrary-to-duty (CTD) obligation sentences. They tell us what comes into force when some other (primary) obligations are violated. A number of researchers in deontic logic have accepted the idea that an appropriate semantics for contrary-to-duty obligation sentences calls for an ordering on possible worlds, in terms of preference or relative goodness.[2] The question of how to axiomatize the associated dyadic deontic logics has been the focus of much attention, starting with Spohn (1975), and continuing with

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[1] Cf. Hansson (1969, sec. 14–15). The label "Hanssonian" is found on p. 169 of Åqvist (1987), and on p. 244 of Åqvist (2002).

[2] For further background on the Hanssonian preference-based semantics, see Alchourrón (1993, sec. 3.4.2), Makinson (1993, sec. 7) and Hilpinen & McNamara (2013, p. 112ff.).

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Åqvist (1987, 1993, 2002), Hansen (1999), and Parent (2008, 2010). However, there are still gaps that need to be filled in. These are of two kinds.

First, results in the literature have so far mostly concerned classes of structures with strong conditions on the betterness relation. One such condition is the property of transitivity, which has been called into question by moral philosophers and economists.[3] One would like to know what happens when such a condition is relaxed.[4] What Lewis (1973) calls the limit assumption is another requirement that one would like to be able to drop. Roughly speaking, it says that a set of possible worlds should always have a best element. A number of deontic logicians objected to the limit assumption, Lewis (1973, pp. 97–98) among them.

Second, a number of researchers in so-called Rational Choice Theory have argued that the notion of best is ambiguous. It may be characterized in terms of either optimality or maximality.[5] For some item x to qualify as an optimal element of X, it must be as good as any other element in that set. For x to count as a maximal element, no other element in X must be strictly better than it. Thus, while the optimal elements are all equally good, the maximal elements are either equally good or incomparable. Depending on what notion of best is used, one gets different evaluation rules for the modalities, but also different forms of the limit assumption. Much work remains to be done to clarify their relationships.

The present paper is a sequel to a companion paper,[6] in which I make a first attempt at addressing the above issues. There are cases where, no matter what notion of best is used, the same dyadic system is sound and complete with respect to its intended modelling, given analogous properties for the betterness relation. This is shown with reference to Åqvist's stronger system **G**, and a system between **G** and **F**, which I call **F**+(CM). Thus, with regard to at least these two systems, the contrast between maximality and optimality is not as significant as one would expect, because the determined logic remains the same. The minimal constraint put on the betterness relation is a version of the limit assumption known as "smoothness". The general statement that the logic remains the same applies to the class of models meeting the latter condition, and three of the five sub-classes of models obtained by combining the following extra requirements: reflexivity, totalness and transitivity. Thus, the claimed symmetry between maximality and optimality is not total, but noteworthy enough to be mentioned.

The question arises as to whether the above findings exhaust what may be said about the contrast between maximality and optimality; and if not, how they may be extended and generalized. In this connection, this paper investigates what happens when other minimal properties are envisaged for the betterness relation, if any is envisaged at all. I start by considering models in which no assumption whatsoever is made about the structure of the betterness relation, and go on to investigate the effect(s) of assuming one (or two) of the following: reflexivity, totalness, and another form of the limit assumption called

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[3] Cf. Sen (1971) and Temkin (1987).

[4] In Parent (2010), an axiomatization result is reported for the non-necessarily transitive case. However, the axiomatization is carried out in a language in which the conditional obligation operator is not primitive but defined using a monadic modal operator, called a "choice operator" by Alchourrón (1993). In a deontic setting, the meaning of such an operator is less intuitive and natural than it is in the semantics for defeasible inference, where it is read as "normally, A".

[5] Cf. Herzberger (1973), Suzumura (1976), and Sen (1997).

[6] Parent (2014).

"limitedness". This, with a view to appreciate whether the choice between maximality and optimality matters here.

This paper is organized as follows. In Section 2, I describe the systems being used. In Section 3, I show completeness. In Section 4, I conclude with some remarks.

§2. Syntax, semantics and proof theory. The syntax is generated by adding the following operators to the syntax of propositional logic: $\Box$ (for necessity); $\Diamond$ (for possibility); $\bigcirc(-/-)$ (for conditional obligation); $P(-/-)$ (for conditional permission). The set of well-formed formulas is defined in the straightforward way. Iterated modalities are allowed, and so are mixed formulas, e.g., $p \wedge \bigcirc(q/p)$. Prop denotes the set of all propositional letters.

A preference model is a structure

$$M = (W, \succeq, v)$$

in which

(i) $W \neq \emptyset$ (W is a non-empty set of "possible worlds");
(ii) $\succeq \subseteq W \times W$ (intuitively, $\succeq$ is a betterness or comparative goodness relation; "$a \succeq b$" can be read as "world a is at least as good as world b");
(iii) v : Prop $\to \mathcal{P}(W)$ (v is an assignment, which associates a set of possible worlds to each propositional letter p).

The definition of truth at a world in a model is as usual except for the modal operators. For the alethic modalities, we have

$$\begin{array}{lll} M, a \models \Box A & \text{iff} & \text{for all } b \text{ in } W : M, b \models A \\ M, a \models \Diamond A & \text{iff} & \text{for some } b \text{ in } W : M, b \models A \end{array}$$

The truth conditions for the deontic modalities have the following pattern, where $best(\|A\|^M)$ is a shorthand for the set of best (according to $\succeq$) worlds in which A is true:

$$\begin{array}{lll} M, a \models \bigcirc(B/A) & \text{iff} & best(\|A\|^M) \subseteq \|B\|^M \\ M, a \models P(B/A) & \text{iff} & best(\|A\|^M) \cap \|B\|^M \neq \emptyset \end{array}$$

As already mentioned, there are two ways to formalize the notion of best antecedent-worlds: one may do it using the notion of optimality, or the notion of maximality. They are not clearly distinguished in the deontic logic literature even though their differences can be significant.[7] They may be defined thus:

$$\mathrm{opt}_{\succeq}(\|A\|^M) = \{b \in \|A\|^M \mid \forall c \ \ (M, c \models A \to b \succeq c)\}$$
$$\mathrm{max}_{\succeq}(\|A\|^M) = \{b \in \|A\|^M \mid \forall c \ \ ((M, c \models A \ \& \ c \succeq b) \to b \succeq c)\}$$

It is easy to see that $\mathrm{opt}_{\succeq}(\|A\|^M) \subseteq \mathrm{max}_{\succeq}(\|A\|^M)$ although the converse inclusion may fail in general. Typically, it will fail if there are "gaps" in the ranking. For later reference, it is useful to note that totalness of $\succeq$ ("for all $a, b \in W$, $a \succeq b$ or $b \succeq a$") is a sufficient condition for the two notions to coincide:

[7] Goble (2013) makes a similar point in the context of a discussion of Kratzer's semantics for monadic obligation.

PROPOSITION 2.1. *If* $\succeq$ *is total, then* $\text{opt}_{\succeq}(\|A\|^M) = \max_{\succeq}(\|A\|^M)$.

Proof. The proof is straightforward, and is omitted. □

Thus, one gets two different pairs of evaluation rules depending on which of the following two equations is adopted:[8]

$$best(\|A\|^M) = \max_{\succeq}(\|A\|^M) \qquad \text{(max rule)}$$

$$best(\|A\|^M) = \text{opt}_{\succeq}(\|A\|^M) \qquad \text{(opt rule)}$$

I shall say that a model M applies the max rule or the opt rule, depending on whether, in M, deontic formulas are interpreted using the former or the latter. From Proposition 2.1, it immediately follows that, in a given model M with $\succeq$ total, the same deontic formulas are true at a given world whatever rule is applied. I shall usually drop reference to M when it is clear what model is intended. When it is pertinent, I will also use the subscripted notation $\models_o$ or $\models_m$ to indicate interpreting deontic formulas in terms of the opt rule or the max rule.

The comparative goodness relation $\succeq$ may be constrained by suitable conditions as desired. In this paper, I will be primarily interested in the following two constraints, besides totalness:

- reflexivity: for all $a \in W$, $a \succeq a$;
- limitedness: if $\|A\|^M \neq \emptyset$ then $best(\|A\|^M) \neq \emptyset$.[9]

A betterness relation $\succeq$ will be called "opt-limited" or "max-limited" depending on whether limitedness holds with respect to $\text{opt}_{\succeq}$ or $\max_{\succeq}$. Reflexivity and limitedness have rarely been studied in their own right. Neither has been the import of totalness, alone or combined with limitedness.

Limitedness implements what Lewis (1973) calls the "limit assumption". It guarantees the existence of a best A-world, whenever A is consistent. The limit assumption can be formulated using an alternative (albeit not necessarily equivalent) condition, called "smoothness" (or "stopperedness") in the non-monotonic logic literature: if $a \models A$, then either $a \in best(\|A\|^M)$ or $b \in best(\|A\|^M)$ for some b such that $b \succeq a$ and $a \not\succeq b$. Again, this can be spelled out in two different ways, depending on whether the max rule or the opt rule is used. In the companion paper mentioned in Section 1, these two variants of the smoothness condition are discussed in their own right, and so is their interplay with the two variants of limitedness as described above.[10] In this paper, I will put smoothness aside.

In Åqvist (1987, p. 181) and Åqvist (2002, p. 248), the following proof theory is given to **E**:

All truth functional tautologies (PL)

S5-schemata for $\Box$ and $\Diamond$ (S5)

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[8] Hansson (1969), Makinson (1993, sec. 7.1), Prakken & Sergot (1997), and Schlechta (1995) use the max rule. Alchourrón (1993, 1995), Åqvist (1987, 1993, 2002), Hansen (2005, sec. 6), McNamara (1995), and Spohn (1975) work with the opt rule. Neither Goldman (1977), nor Jackson (1985), nor Hilpinen (2001, sec. 8.5) specifies what notion of best is meant. (The last one uses "best" and "deontically optimal" interchangeably, but leaves optimality undefined.)

[9] I follow Åqvist's terminology. Alchourrón (1993) calls the constraint "limit expansion".

[10] The reader is invited to compare the discussion there with the paper by Goble (2013, sec. 1.2) mentioned above.

$$P(B/A) \leftrightarrow \neg \bigcirc (\neg B/A) \tag{DfP}$$
$$\bigcirc (B \rightarrow C/A) \rightarrow (\bigcirc(B/A) \rightarrow \bigcirc(C/A)) \tag{COK}$$
$$\bigcirc (B/A) \rightarrow \Box \bigcirc (B/A) \tag{Abs}$$
$$\Box A \rightarrow \bigcirc(A/B) \tag{Nec}$$
$$\Box(A \leftrightarrow B) \rightarrow (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \tag{Ext}$$
$$\bigcirc (A/A) \tag{Id}$$
$$\bigcirc (C/A \wedge B) \rightarrow \bigcirc(B \rightarrow C/A) \tag{Sh}$$
$$\text{If } \vdash A \text{ and } \vdash A \rightarrow B \text{ then } \vdash B \tag{MP}$$
$$\text{If } \vdash A \text{ then } \vdash \Box A \tag{N}$$

The abbreviations (PL), (S5), (MP) and (N) are self-explanatory. (DfP) introduces "P" as the dual of "$\bigcirc$" in the usual way. (COK) is the conditional analogue of the familiar distribution axiom K. (Abs) is the absoluteness axiom of Lewis (1973), and reflects the fact that the ranking is not world-relative. (Nec) is the deontic counterpart of the familiar necessitation rule. (Ext) permits the replacement of equivalent sentences in the antecedent of deontic conditionals. (Id) is the deontic analogue of the identity principle. (Sh) is named after Shoham (1988), who seems to have been the first to discuss it.

The basis of **F** is that of **E** with the single extra axiom:

$$\Diamond A \rightarrow (\bigcirc(B/A) \rightarrow P(B/A)) \tag{D*}$$

(D*) is the conditional analogue of the familiar axiom D. Its import is simply that conflicts of obligations are ruled out, for consistent (or possible) antecedents.

In Section 1 above, there was a brief mention of two extensions of **F** of increasing strength: **F**+(CM) and **G**. They are obtained by supplementing **F** with (CM) and (Sp), respectively:

$$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \rightarrow \bigcirc(C/A \wedge B) \tag{CM}$$
$$(P(B/A) \wedge \bigcirc(B \rightarrow C/A)) \rightarrow \bigcirc(C/A \wedge B) \tag{Sp}$$

(CM) is the principle of cautious monotony from the non-monotonic logic literature.[11] (Sp) is sometimes called the "Spohn axiom", after the logician who first introduced it.[12] It is equivalent to the so-called principle of rational monotony.[13] These two systems will be put aside.

The following applies:

THEOREM 2.2. *Under the opt rule or the max rule,* **E** *is sound with respect to:*

(i) *the class of all preference models;*
(ii) *the class of preference models in which $\succeq$ is reflexive;*
(iii) *the class of preference models in which $\succeq$ is total.*

Proof. The argument is straightforward, and is omitted. None of the axioms depend on reflexivity or totalness. □

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[11] Cf. Kraus et al. (1990).
[12] Cf. Spohn (1975).
[13] Cf. Lehmann & Magidor (1992).

THEOREM 2.3. *Under the opt rule,* **F** *is sound with respect to:*

(i) *the class of preference models in which* $\succeq$ *is opt-limited;*
(ii) *the class of preference models in which* $\succeq$ *is opt-limited and reflexive;*
(iii) *the class of preference models in which* $\succeq$ *is opt-limited and total.*

Proof. It suffices to verify that opt-limitedness validates (D*). Assume $a \models \Diamond A$ and $a \models \bigcirc(B/A)$. From the first, $\|A\|^M \neq \emptyset$. By opt-limitedness, there is some $b \in \mathrm{opt}_{\succeq}(\|A\|^M)$. From the second opening assumption, $b \models B$. Hence, $a \models P(B/A)$ as required. □

THEOREM 2.4. *Under the max rule,* **F** *is sound with respect to:*

(i) *the class of preference models in which* $\succeq$ *is max-limited;*
(ii) *the class of preference models in which* $\succeq$ *is max-limited and reflexive;*
(iii) *the class of preference models in which* $\succeq$ *is max-limited and total.*

Proof. A similar argument as in the proof of Theorem 2.3 may be used to show that max-limitedness validates (D*). □

Completeness is addressed in the next section.

§3. Completeness. In Åqvist (1987, pp. 179, 181) and Åqvist (2002, pp. 247, 249), the question of whether **E** and **F** are complete with respect to their intended Hanssonian modelling is left as an open issue. For both systems, a semantics applying the opt rule is used. For **E**, $\succeq$ is required to be reflexive. For **F**, $\succeq$ is in addition required to be opt-limited. Below I will tackle Åqvist's problem, but I will allow for the kind of variations hinted at in Section 2. With the opt rule replaced with the max rule, and max-limitedness in place of opt-limitedness, one gets closer to Hansson's own picture of his systems DSDL1 and DSDL2. One would like to know what happens then.

The proof of completeness is intricate, and takes a detour through the selection function semantics stemming from Stalnaker (1968) and generalized by Chellas (1975). Such a semantics has been adapted to the present setting in Åqvist (2002). The proposed approach is related to the two-step methodology used by Schlechta (1997, chap. 2) when discussing representation problems for non-monotonic structures. What follows is an attempt to recast the method into a modal logic framework. I shall call these new structures "selection function models", to distinguish them from those described above. In models of this new sort, the betterness relation $\succeq$ is swapped for a selection function $\mathfrak{f}$, which associates with each sentence A a subset of $\|A\|^M$, heuristically the subset of those A-worlds that are best. The evaluation rules for the dyadic modalities are rephrased thus:

$$M, a \models \bigcirc(B/A) \quad \text{iff} \quad \mathfrak{f}(A) \subseteq \|B\|^M$$
$$M, a \models P(B/A) \quad \text{iff} \quad \mathfrak{f}(A) \cap \|B\|^M \neq \emptyset$$

The relevant constraints for $\mathfrak{f}$ are:

$$\text{If } \|A\|^M = \|B\|^M \text{ then } \mathfrak{f}(A) = \mathfrak{f}(B) \qquad \text{(Syntax-independence)}$$
$$\mathfrak{f}(A) \subseteq \|A\|^M \qquad \text{(Inclusion)}$$
$$\mathfrak{f}(A) \cap \|B\|^M \subseteq \mathfrak{f}(A \wedge B) \qquad \text{(Chernoff)}$$
$$\text{If } \|A\|^M \neq \emptyset \text{ then } \mathfrak{f}(A) \neq \emptyset \qquad \text{(Consistency-preservation)}$$

The third constraint is named after Chernoff, who uses it in Chernoff (1954).

The following applies:

THEOREM 3.1. *System* **E** *is sound and complete with respect to the class of selection function models* $M = (W, \mathfrak{f}, v)$ *in which* $\mathfrak{f}$ *meets syntax-independence, inclusion and Chernoff.*

Proof. The proof is the same as for Åqvist's system **G** in Åqvist (2002, Theorem 77, p. 251). □

THEOREM 3.2. *System* **F** *is sound and complete with respect to the class of selection function models* $M = (W, \mathfrak{f}, v)$ *in which* $\mathfrak{f}$ *meets syntax-independence, inclusion, Chernoff, and consistency-preservation.*

Proof. As per above. □

The following will come in handy later:

THEOREM 3.3. *For every preference model* $M = (W, \succeq, v)$*, there is a preference model* $M' = (W', \succeq', v')$ *in which* $\succeq'$ *is total, and such that, under the opt rule,* M' *is equivalent to* M*. On that model,* $\succeq'$ *is reflexive. Furthermore, if* $\succeq$ *is opt-limited, then so is* $\succeq'$*.*

Proof. Let $M = (W, \succeq, v)$. Define $M' = (W', \succeq', v')$ as follows:

- $W' = \{\langle a, n\rangle \mid a \in W, n \in \omega\}$
- $\langle a, n\rangle \succeq' \langle b, m\rangle$ iff $a \succeq b$ or $n \geq m$
- $v'(p) = \{\langle a, n\rangle \mid a \in v(p)\}$

Each world a in W has infinitely (albeit countably) many "duplicates" in W'. Thus, $W' \neq \emptyset$ whenever $W \neq \emptyset$. $\succeq'$ is total, since $\geq$ is.

To show equivalence amounts to showing that, for all $a \in W$, and all $n \in \omega$, $a \models A$ iff $\langle a, n\rangle \models A$. The proof is by induction on A. I just do the case in which A is $\bigcirc(C/B)$.

From left-to-right, assume $a \models \bigcirc(C/B)$. Consider any $\langle b, m\rangle \in W'$ such that $\langle b, m\rangle \in \text{opt}_{\succeq'}(\|B\|^{M'})$. We have $\langle b, m\rangle \models B$. By the inductive hypothesis, $b \models B$. Suppose for *reductio* that $b \notin \text{opt}_{\succeq}(\|B\|^{M})$. Then, there is some $c \in W$ such that $c \models B$ and $b \not\succeq c$. But $\langle c, m\rangle \in W'$, and hence $\langle c, m+1\rangle \in W'$. By the inductive hypothesis, $\langle c, m+1\rangle \models B$. By definition of $\succeq'$, $\langle b, m\rangle \not\succeq' \langle c, m+1\rangle$ – contradiction. Thus, $b \in \text{opt}_{\succeq}(\|B\|^{M})$, and hence $b \models C$. By the inductive hypothesis, $\langle b, m\rangle \models C$, which suffices for $\langle a, n\rangle \models \bigcirc(C/B)$.

For the converse direction, let $\langle a, n\rangle \models \bigcirc(C/B)$. Let $b \in W$ be such that $b \in \text{opt}_{\succeq}(\|B\|^{M})$. We have $b \models B$. We also have $\langle b, n\rangle \in W'$. By the inductive hypothesis, $\langle b, n\rangle \models B$. Let $\langle c, m\rangle \in W'$ be such that $\langle c, m\rangle \models B$. By the inductive hypothesis, $c \models B$. But $b \succeq c$, since $b \in \text{opt}_{\succeq}(\|B\|^{M})$. By definition of $\succeq'$, $\langle b, n\rangle \succeq' \langle c, m\rangle$. So $\langle b, n\rangle \in \text{opt}_{\succeq'}(\|B\|^{M'})$, and hence $\langle b, n\rangle \models C$. By the inductive hypothesis, $b \models C$, which suffices for $a \models \bigcirc(C/B)$.

Reflexivity of $\succeq'$ follows from totalness of $\succeq'$. The fact that opt-limitedness for $\succeq$ implies opt-limitedness for $\succeq'$ follows from the definition of $\succeq'$ and the equivalence between models. Assume there is some $\langle a, n\rangle \in W'$ such that $\langle a, n\rangle \models A$. By the above, $a \models A$. Since $\succeq$ is opt-limited, there is some $b \in W$ such that $b \in \text{opt}_{\succeq}(\|A\|^{M})$. Hence, $b \models A$. But $\langle b, n\rangle \in W'$. By the above, $\langle b, n\rangle \models A$. Let $\langle c, m\rangle \in W'$ be such that $\langle c, m\rangle \models A$. By the equivalence between models again, $c \models A$, and so $b \succeq c$, since $b \in \text{opt}_{\succeq}(\|A\|^{M})$. By definition of $\succeq'$, $\langle b, n\rangle \succeq' \langle c, m\rangle$, which suffices for $\langle b, n\rangle \in \text{opt}_{\succeq'}(\|A\|^{M'})$. □

COROLLARY 3.4. *For every preference model $M = (W, \succeq, v)$ applying the opt rule, there is a preference model $M' = (W', \succeq', v')$ in which $\succeq'$ is total, applying the max rule, and such that M' is equivalent to M. On that model, $\succeq'$ is reflexive. Furthermore, if $\succeq$ is opt-limited, then $\succeq'$ is max-limited.*

Proof. This follows from Theorem 3.3. Consider the model M' as defined in the proof of this theorem. On that model, $\succeq'$ is total. An easy induction establishes that $\langle a, n\rangle \models_o A$ iff $\langle a, n\rangle \models_m A$, using Proposition 2.1. Furthermore, in the argument for opt-limitedness, $\langle b, n\rangle \in \max_{\succeq'}(\|A\|^{M'})$. □

The next step is to show that, with **E** and **F**, the selection function semantics matches the preference-based semantics. It is known that each preference model applying the opt rule generates an equivalent selection function model in which $\mathfrak{f}$ meets the relevant conditions.[14] It remains to establish that the converse also holds. To this end, I adapt to the present setting a construction used by Schlechta (1997) in the context of the study of non-monotonic reasoning.

As usual, a function is understood as a relation that associates each element of one set with a unique element of another set.[15] A family of sets refers to any function g on a domain I (called an index set) such that, for each $i \in I$, $g(i)$ is a set. Writing $g(i)$ as X_i, the range of this function is the collection $\{X_i\}_{i\in I}$. Whenever convenient, we use the latter (more compact) notation to refer to the family itself. When $I = \emptyset$, we speak of the empty family, $\emptyset$. Let $\{X_i\}_{i\in I}$ be a family of sets. A function $g : I \to \bigcup_{i\in I} X_i$ is called a choice function on the family $\{X_i\}_{i\in I}$ if $(\forall i \in I)(g(i) \in X_i)$. In other words, g picks up a representative from each X_i. The set of all choice functions on the family $\{X_i\}_{i\in I}$ is called the general cartesian product of $\{X_i\}_{i\in I}$, and is denoted by $\prod_{i\in I}\{X_i\}$. This is the set

$$\{g : I \to \bigcup_{i\in I} X_i \mid (\forall i \in I)(g(i) \in X_i)\}$$

Note that, if $I = \emptyset$, then the only such g is $\emptyset$.

Now comes the result to be established.

THEOREM 3.5. *For every selection function model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets syntax-independence, inclusion and Chernoff, there is a preference model $M' = (W', \succeq, v')$ such that, under the opt rule, M' is equivalent to M.*

Proof. Let $M = (W, \mathfrak{f}, v)$. For all $a \in W$, define $\mathcal{Y}_a = \{\|C\|^M \subseteq W \mid a \in \|C\|^M - \mathfrak{f}(C)\}$. Note that $\emptyset \notin \mathcal{Y}_a$ although it may well be that $\mathcal{Y}_a = \emptyset$. Put $F_a := \prod \mathcal{Y}_a$. Define $M' = (W', \succeq, v')$ as follows:

- $W' = \{\langle a, g\rangle \mid a \in W, g \in F_a\}$
- $\langle a, g\rangle \succeq \langle b, g'\rangle$ iff $b \notin \mathrm{Rng}(g)$
- $v'(p) = \{\langle a, g\rangle \mid a \in v(p)\}$

Here $\mathrm{Rng}(g)$ denotes the range of g, viz $\{c \mid \langle i, c\rangle \in g$ for some $i \in I\}$.

[14] Cf. Åqvist (1987, sec. 21).

[15] Cf. e.g. Halmos (1960, p. 31). A function whose domain is empty is referred to as the empty function. It is denoted by $\emptyset : \emptyset \to Y$ or briefly by $\emptyset$. (Y may or may not be empty.)

LEMMA 3.6. *For all $a \in W$, there exists some g such that $g \in F_a$.*

Proof. Let $a \in W$. Either i) $\mathcal{Y}_a = \emptyset$ or ii) $\mathcal{Y}_a \neq \emptyset$. In case i), $F_a = \{\emptyset\}$. In case ii), $\mathcal{Y}_a$ is a collection of non-empty sets $\{X_i\}_{i \in I}$, with $I \neq \emptyset$. By the axiom of choice, their cartesian product too is non-empty; that is, there exists a function $g : I \to \bigcup_{i \in I} X_i$ such that $(\forall i \in I)(g(i) \in X_i)$. □

Non-emptiness of W' follows immediately, as required by our definition of what counts as a preference model:

COROLLARY 3.7. $W' \neq \emptyset$.

Proof. $W \neq \emptyset$. Hence, there exists some $a \in W$. By Lemma 3.6, there is some g such that $g \in F_a$. Hence, $\langle a, g\rangle \in W'$, and so $W' \neq \emptyset$ as required. □

LEMMA 3.8. $a \in \mathfrak{f}(B) \Leftrightarrow a \in \|B\|^M$ *and* $(\exists g \in F_a)$ $(\mathrm{Rng}(g) \cap \|B\|^M) = \emptyset$.

Proof. ($\Leftarrow$) Assume $a \notin \mathfrak{f}(B)$. If $a \notin \|B\|^M$, then we are done. So assume $a \in \|B\|^M$. It then follows that $a \in \|B\|^M - \mathfrak{f}(B)$. By construction, there is some $i \in I$ such that $X_i \in \mathcal{Y}_a$ and $X_i = \|B\|^M$. Let $g \in F_a$. By definition, $g(i) \in \|B\|^M$, and so $\mathrm{Rng}(g) \cap \|B\|^M \neq \emptyset$. Thus, either $a \notin \|B\|^M$ or $\nexists g \in F_a$ $(\mathrm{Rng}(g) \cap \|B\|^M) = \emptyset$, as required.

($\Rightarrow$) Assume $a \in \mathfrak{f}(B)$. By inclusion, $a \in \|B\|^M$.

Suppose $\mathcal{Y}_a = \emptyset$. Then, $F_a = \{\emptyset\}$. Trivially $\mathrm{Rng}(\emptyset) \cap \|B\|^M = \emptyset$.

Suppose $\mathcal{Y}_a \neq \emptyset$. Let $\mathcal{Y}_a$ be of the form $\{X_i\}_{i \in I}$, where, for all $i \in I$, $X_i \neq \emptyset$.

The first step is to show that, for all $i \in I$, there is some $b \in X_i$ such that $b \notin \|B\|^M$. Suppose otherwise. Then, for some $i \in I$, and some C, $X_i = \|C\|^M \subseteq \|B\|^M$. We have $\|C\|^M \in \mathcal{Y}_a$. So, by construction, $a \in \|C\|^M$ and $a \notin \mathfrak{f}(C)$. But $a \in \mathfrak{f}(B) \cap \|C\|^M$. Chernoff, then, yields $a \in \mathfrak{f}(B \wedge C)$. But $\|B \wedge C\|^M = \|C\|^M$, since $\|C\|^M \subseteq \|B\|^M$. So, by syntax-independence, $a \in \mathfrak{f}(C)$, a contradiction.

The next step is to form the n-tuple of all these b's. Consider the family $\{Y_i\}_{i \in I}$, where $Y_i = X_i - \|B\|^M$, for each $i \in I$. By the above, this is a collection of non-empty sets. By the axiom of choice, their cartesian product too is non-empty; that is, there exists a function $g : I \to \bigcup_{i \in I} Y_i$ such that $(\forall i \in I)(g(i) \in Y_i)$. For all $i \in I$, $Y_i \subseteq X_i$. So g meets the requirements for membership in F_a:

- $g \subseteq I \times \bigcup_{i \in I} X_i$;
- For all $i \in I$, there is a unique c in $\bigcup_{i \in I} X_i$, such that $\langle i, c\rangle \in g$;
- For all $i \in I$, $g(i) \in X_i$.

On the other hand, $\mathrm{Rng}(g) \cap \|B\|^M = \emptyset$. □

Equivalence between models may be stated thus: for all $a \in W$, and all $g \in F_a$, $a \models A$ iff $\langle a, g\rangle \models A$. I just do the case where $A := \bigcirc(C/B)$. To this end, it is enough to show that, under the inductive hypothesis, we have that

LEMMA 3.9. $a \in \mathfrak{f}(B) \Leftrightarrow (\exists g \in F_a)$ $(\langle a, g\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'}))$

Proof. ($\Rightarrow$) Suppose $a \in \mathfrak{f}(B)$. By Lemma 3.8, $a \models B$ and $(\exists g \in F_a)$ $(\mathrm{Rng}(g) \cap \|B\|^M) = \emptyset$. By the inductive hypothesis, $\langle a, g\rangle \models B$. Consider any $\langle b, g'\rangle$ such that $\langle b, g'\rangle \models B$. By the inductive hypothesis, $b \models B$. Since $\mathrm{Rng}(g) \cap \|B\|^M = \emptyset$, $b \notin \mathrm{Rng}(g)$, and so $\langle a, g\rangle \succeq \langle b, g'\rangle$. Hence $\langle a, g\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'})$.

($\Leftarrow$) Assume $(\exists g \in F_a)$ $(\langle a, g\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'}))$. We have $\langle a, g\rangle \models B$. By the inductive hypothesis, $a \models B$. Suppose for *reductio* that $\mathrm{Rng}(g) \cap \|B\|^M \neq \emptyset$. Then, there is some

$b \in \|B\|^M$ with $b \in \mathrm{Rng}(g)$. By Lemma 3.6, there exists some $g' \in F_b$, and hence $\langle b, g'\rangle \in W'$. By the inductive hypothesis, $\langle b, g'\rangle \models B$. Since $\langle a, g\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'})$, $\langle a, g\rangle \succeq \langle b, g'\rangle$, and thus $b \notin \mathrm{Rng}(g)$, a contradiction. So, $\mathrm{Rng}(g) \cap \|B\|^M = \emptyset$. By Lemma 3.8, $a \in \mathfrak{f}(B)$, as required. □

To see the equivalence, assume $a \models \bigcirc(C/B)$. Consider any $\langle b, g'\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'})$. By construction, $g' \in F_b$. By Lemma 3.9, $b \in \mathfrak{f}(B)$, and so $b \models C$. By the inductive hypothesis, $\langle b, g'\rangle \models C$, which suffices for $\langle a, g\rangle \models \bigcirc(C/B)$.

For the converse, assume $\langle a, g\rangle \models \bigcirc(C/B)$. Consider any $b \in \mathfrak{f}(B)$. Using Lemma 3.9, it follows that $(\exists g' \in F_b)$ $(\langle b, g'\rangle \in \mathrm{opt}_{\succeq}(\|B\|^{M'}))$. So, $\langle b, g'\rangle \models C$. By the inductive hypothesis, $b \models C$, which suffices for $a \models \bigcirc(C/B)$. □

COROLLARY 3.10. *For every selection function model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets syntax-independence, inclusion, Chernoff and consistency-preservation, there is a preference model $M' = (W', \succeq, v')$ in which $\succeq$ is opt-limited, and such that, under the opt rule, M' is equivalent to M.*

Proof. This follows from Theorem 3.5. Note that Lemma 3.9 implies that $\succeq$ is opt-limited whenever $\mathfrak{f}$ satisfies consistency-preservation. □

This yields weak and strong completeness with respect to preference models.

THEOREM 3.11. *Under the opt rule or the max rule,* **E** *is complete with respect to:*

(i) *the class of all preference models;*
(ii) *the class of preference models in which $\succeq$ is reflexive;*
(iii) *the class of preference models in which $\succeq$ is total.*

Proof. For the sake of conciseness, I give the argument for weak completeness only.

For (i), opt rule. Suppose that A is not derivable in **E**. By Theorem 3.1, there is a selection function model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets syntax-independence, inclusion, and Chernoff, and some point $a \in W$, such that $a \not\models A$. By Theorem 3.5 and Lemma 3.6, there is a preference model $M' = (W', \succeq, v')$, and some $\langle a, g\rangle \in W'$, such that $\langle a, g\rangle \not\models_o A$. Hence, it is not the case that, under the opt rule, A is valid with respect to the class of all preference models.

For (ii) and (iii), opt rule. It suffices to re-run the proof of (i) under the opt rule, and apply Theorem 3.3 to the model M' as described above.

For (i), (ii) and (iii), max rule. It suffices to re-run the proof of (i) under the opt rule, and apply Corollary 3.4. □

THEOREM 3.12. *Under the opt rule,* **F** *is complete with respect to:*

(i) *the class of preference models in which $\succeq$ is opt-limited;*
(ii) *the class of preference models in which $\succeq$ is opt-limited and reflexive;*
(iii) *the class of preference models in which $\succeq$ is opt-limited and total.*

Proof.

For (i). Suppose that A is not derivable in **F**. By Theorem 3.2, there is a selection function model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets syntax-independence, inclusion, Chernoff and consistency-preservation, and some point $a \in W$, such that $a \not\models A$. By Corollary 3.10 and Lemma 3.6, there is a preference model $M' = (W', \succeq, v')$ in which $\succeq$ is opt-limited, and some $\langle a, g\rangle \in W'$, such that $\langle a, g\rangle \not\models_o A$. Hence, it is not the case that, under the opt rule, A is valid with respect to the class of preference models in which $\succeq$ is required to be opt-limited.

For (ii) and (iii). The argument recapitulates that for (i), using in addition Theorem 3.3. □

THEOREM 3.13. *Under the max rule,* **F** *is complete with respect to:*

(i) *the class of preference models in which $\succeq$ is max-limited;*
(ii) *the class of preference models in which $\succeq$ is max-limited and reflexive;*
(iii) *the class of preference models in which $\succeq$ is max-limited and total.*

Proof. This follows similarly from Theorem 3.12 (i) and Corollary 3.4. □

§4. Conclusion. Table 1 below summarizes the results established in this paper, putting them side by side with those reported in the companion paper mentioned in Section 1. The leftmost column shows the constraints placed on $\succeq$. The $\pm$ symbol stands for the two possibilities of presence and absence of a given property. For instance, the top row covers the class of all preference models $({}^{-}_{-})$, the class of those in which $\succeq$ is required to be reflexive $({}^{+}_{-})$, and the class of those in which $\succeq$ is required to be total, and hence reflexive $({}^{+}_{+})$. The other two columns show the corresponding systems, the middle column for models applying the max rule, and the rightmost one for models applying the opt rule. It is understood that smoothness (resp. limitedness) is defined for max in the max column, and for opt in the opt column.

The last two rows show the results reported in the aforementioned companion paper. The focus is on the stronger systems **F**+(CM) and **G**. The minimal property envisaged for

Table 1. *Soundness and completeness results*

<table>
<tr><th>Properties of $\succeq$</th><th>Max</th><th>Opt</th></tr>
<tr><td>$\pm$ reflexivity
$\pm$ totalness</td><td>E</td><td>E</td></tr>
<tr><td>limitedness
$\pm$ reflexivity
$\pm$ totalness</td><td>F</td><td>F</td></tr>
<tr><td>smoothness
$\pm$ reflexivity
$\pm$ totalness</td><td>F+(CM)</td><td>F+(CM)</td></tr>
<tr><td>smoothness
transitivity
$-$ totalness
$\pm$ reflexivity</td><td>?</td><td rowspan="2">G</td></tr>
<tr><td>$+$ totalness
(reflexivity)</td><td>G</td></tr>
</table>

the betterness relation is smoothness.[16] One can see that, with regard to at least these two systems, the contrast between maximality and optimality is not as significant as one would expect, because the determined logic remains the same. Thus, under the max rule or the opt rule, **F**+(CM) is sound and complete with respect to the class of models in which $\succeq$ is required to be smooth, the class of those in which it is required to be smooth and reflexive, and the class of those in which it is required to be smooth and total. Likewise, under either rule, **G** is sound and complete with respect to the class of models in which $\succeq$ is required to be smooth, transitive and total (and hence, reflexive). The special case when, under the max rule, totalness goes away is left open. This is indicated by a question mark in Table 1. I will get back to it in a moment.

The first two rows show the results that have just been established. We can see that the above point carries over to the different classes of models considered in this paper. First, under the max rule or the opt rule, **E** is sound and complete with respect to the class of all preference models, the class of those in which $\succeq$ is required to be reflexive, and the class of those in which it is required to be total. Second, under either rule, **F** is sound and complete with respect to the class of preference models in which $\succeq$ is required to be limited, the class of those in which it is required to be limited and reflexive, and the class of those in which it is required to be limited and total.

The completeness results for **E** and **F** were first established with respect to optimality, and then extended to maximality. The arguments made an essential use of the known completeness of the systems with respect to a selection function semantics. A key step was to show that, with **E** and **F**, the selection function semantics matches the preference semantics. Here the main difficulty was to show how to derive the latter from the former. This amounted to showing that, starting with a selection function model of the appropriate kind, one can always generate an equivalent preference model of the appropriate kind, which applies the opt rule. The completeness results were transferred from optimality to maximality, by showing that, under appropriate hypotheses regarding $\succeq$, the language of dyadic deontic logic cannot distinguish between a model applying the opt rule and one applying the max rule. That is, starting with a model of the appropriate kind, which applies the first rule, one can always find an equivalent model of the appropriate kind, which applies the second rule.

These results may be viewed negatively, as showing that the language of dyadic deontic logic lacks the expressive power to distinguish between optimality and maximality. However, there might be a larger story to tell about the contrast between the two when transitivity alone, or combined with limitedness, comes into the picture. The bottom row looks at transitivity in tandem with smoothness. Here a small difference between maximality and optimality already emerges. Under the max rule, the distinctive axiom of **G**, the Spohn axiom, requires both transitivity and totalness, while under the opt rule it requires transitivity only. In other words, under the opt rule, totalness alone has no import at all again, and neither has reflexivity. Thus, under the opt rule, **G** is also sound and complete with respect to the class of models in which $\succeq$ is required to be smooth and transitive, and the class of those in which it is also required to be reflexive. There is nothing similar for **G** under the max rule. Goble (2014) reports a characterization result for these two classes of models under the max rule. The determined system, which he calls DDL-3, happens to

[16] In the aforementioned paper, reflexivity and smoothness are in fact used in tandem, as is common practice in deontic logic. It can easily be extracted from the arguments given there that reflexivity is idle.

be very much like **F**+(CM), but without the alethic modalities. The betterness relation is here relativized to worlds. It remains to be seen whether the proof of completeness given there can be adapted to suit the present framework. Assuming that an analogous result is available for **F**+(CM), this would imply that, under the max rule, transitivity has no import when combined with smoothness. It would then be interesting to know if this still applies in the absence of smoothness.

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