"Put Your Parachute on, and Jump Out!"
Input/Output Logics without Weakening

Xavier Parent and Leendert van der Torre

University of Luxembourg
x.parent@mail.com
leonvandertorre@uni.lu

Abstract. Makinson and van Torre [15] have devised a number of input/output (I/O) logics to reason about conditional norms. The key idea is to make obligations relative to a given set of conditional norms. The meaning of the normative concepts is, then, given in terms of a set of procedures yielding outputs for inputs. Using the same methodology, Stolpe [20,21] has developed some more I/O logics to include systems without the rule of weakening of the output (or principle of inheritance). We extend Stolpe's account in two directions. First, we show how to make it support reasoning by cases, a natural feature of human reasoning. Second, we show how to inject a new (as we call it, "aggregative") form of cumulative transitivity, which we think is more suitable for normative reasoning. The main outcomes of the paper are soundness and completeness theorems for the proposed systems with respect to their intended semantics.

Keywords: Input/output logic; output weakening; principle of inheritance; reasoning by cases; cumulative transitivity; axiomatization; completeness

1 Introduction

Makinson and van Torre [15] devised a number of input/output (I/O) logics to reason about conditional norms. The key idea is to make obligations relative to a given set of conditional norms. The meaning of the normative concepts is, then, given in terms of a set of procedures yielding outputs for inputs. A number of I/O operations are studied [15]. It is shown that they correspond to a series of proof systems of increasing strength. The ambition of the founders of I/O logic was to provide an alternative to modal logic. One reason why the latter has been so popular in deontic logic is that it is a general framework, which provides us with plenty of freedom to pick and choose the axiom schemata we think are right. One would like to know if, or to which extend, I/O logic can offer the same kind of flexibility.

In this paper, we focus on the so-called rule of Weakening of the Output (WO), which all the I/O operations defined by Makinson and van Torre [15] satisfy. The rule may be given the following form, where the conditional obligation for x given a is written as (a, x), and $\vdash$ stands for the deducibility relation in propositional logic:

$$\text{WO}\ \frac{(a, x) \qquad x \vdash y}{(a, y)}$$

This is also known as the "principle of inheritance". It has been called into question, mostly in connection with the deontic paradoxes [4,6,11,10,2] and the question of how to accommodate conflicts between obligations−see, e.g., [7,8]. This raises the question whether the framework may be generalized to include systems without output weakening; if yes, how.

An important step towards answering the above question has been made by Stolpe [20,21]. He considers two of the four standard I/O operations defined by Makinson and van der Torre [15], namely the so-called simple-minded I/O operation out_1, and the so-called reusable I/O operation out_3. Both develop output by detachment. While out_1 spells out the basic mechanism used to achieve this, out_3 extends it to cover iteration of successive detachments. For both operations, a suitable semantics is given, for which the rule WO fails. Each semantics comes with a sound and complete axiomatic characterization. The present paper extends Stolpe's account in two ways.

First, we show how to make it support reasoning by cases, a natural feature of human reasoning. We look at the I/O operation out_2, called "basic" by Makinson and van der Torre [15]. Its distinctive feature is that it validates the rule OR:

$$\text{OR}\ \frac{(a, x) \qquad (b, x)}{(a \vee b, x)}$$

The present paper shows how to incorporate such a rule. We provide a suitable semantics for the I/O operation, a proof system for it, and a completeness result linking the two.

Second, we show how to integrate other forms of cumulative transitivity. Stolpe uses the rule of (as he calls it) "mediated cumulative transitivity" (MCT):

$$\text{MCT}\ \frac{(a, x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a, y)}$$

As we will see in Section 4, given the other rules of his system, MCT turns out to be equivalent to the rule of cumulative transitivity (CT), as initially used by Makinson and van der Torre [15]:

$$\text{CT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

We look at the following alternative (call it "aggregative") variant, first introduced in [18]:

$$\text{ACT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)}$$

The counterexamples usually given to CT in the literature [17,12,14] no longer work, when ACT is used in place of CT. This is because they all rely on the intuition that the obligation of y ceases to hold when the obligation of (a, x) is violated. The following example, due to Broome [1, § 7.4], may be used to illustrate this point:

You ought to exercise hard everyday	$(\top, x)$
If you exercise hard everyday, you ought to eat heartily	(x, y)
?* You ought to eat heartily	?* $(\top, y)$

Intuitively, the obligation to eat heartily no longer holds, if you take no exercise. In this example, the correct conclusion is $(\top, x \wedge y)$, and not $(\top, y)$. Thus, ACT appears to be more suitable for normative reasoning, because it keeps track of what has been previously detached

The layout of this paper is as follows. Section 2 lays the groundwork, tackling out_1 in essentially the same way as Stolpe does. Section 3 extends the account so that reasoning by cases is supported. Section 4 shows how to inject the aggregative form of cumulative transitivity mentioned above. The main achievement of the paper is the establishment of soundness and completeness theorems for the proposed systems with respect to their intended semantics. In the body of the paper, the proofs are only outlined−the detailed proofs are given in the Appendix. Section 5 discusses some properties satisfied by the I/O operations defined in this paper.

2 Developing the Output by Detachment (out_1)

We start with the simple-minded I/O operation out_1. The I/O operation to be defined here is noted $\mathcal{O}_1$. It is essentially a variation on the I/O operation PN_1 put forth by Stolpe [20,21]. The main reason for including such an operation in our study is that the completeness result for it will be needed for subsequent developments.

First, some definitions are needed. A normative code is a set N of conditional obligations. A conditional obligation is a pair $(a,\ x)$, where a and x are two formulae of classical propositional logic. We use this notation instead of $\bigcirc(x \mid a)$, because the latter has distinct interpretations in the literature. In the notation $(a,\ x)$, the first element a is called the body of the rule, and is thought of as an input, representing some condition or situation. The second element x is called the head of the rule, and is thought of as an output, representing what the norm tells us to be obligatory in that situation. We use the standard notation $(\top, x)$ for the unconditional obligation of x, where $\top$ is a zero-place connective standing for 'tautology'. In I/O logic, the main construct has the form

$$x \in out(N, a)$$

Intuitively: given input a (state of affairs), x (obligation) is in the output under norms N. An equivalent notation is: $(a, x) \in out(N)$. The I/O operations to be defined in this paper will be denoted by the symbol $\mathcal{O}$ in order to avoid any confusion with *out* and $\bigcirc$.

Some further notation. $\mathcal{L}$ is the set of all formulae of classical propositional logic. Given an input $A \subseteq \mathcal{L}$, and a set N of norms, $N(A)$ denotes the image of N under A, i.e., $N(A) = \{x : (a,x) \in N \text{ for some } a \in A\}$. $Cn(A)$ denotes the set $\{x : A \vdash x\}$, where $\vdash$ is the deducibility relation used in classical propositional logic. The notation $x \dashv\vdash y$ is short for $x \vdash y$ and $y \vdash x$. We use PL as an abbreviation for (classical) propositional logic.

Definition 1 (Semantics). *$x \in \mathcal{O}_1(N,A)$ if and only if there is some finite $M \subseteq N$ such that*

- *$M(Cn(A)) \neq \emptyset$, and*
- *$x \dashv\vdash \wedge M(Cn(A))$*

Intuitively: x is equivalent to the conjunction of heads of rules in some $M \subseteq N$ that are all triggered by input A.

The main difference between $\mathcal{O}_1$ and PN_1 arises when A does not trigger any norm, viz. $M(Cn(A)) = \emptyset$ for all $M \subseteq N$. In this limiting case, PN_1 outputs the set of all tautologies, while $\mathcal{O}_1$ outputs nothing. Von Wright [23, pp. 152-4] argues, rightly in our view, that the obligation of $\top$ does not express a genuine prescription.

$\mathcal{O}_1$ is monotonic with respect to the input set. The latter claim requires a careful and detailed proof, because there is a pitfall to avoid.

Theorem 1 (Monotony). *$\mathcal{O}_1(N,A) \subseteq \mathcal{O}_1(N,B)$ whenever $A \subseteq B$.*

Proof. Assume $x \in \mathcal{O}_1(N,A)$ and $A \subseteq B$. From the former, there is $M_1 \subseteq N$ such that $M_1(Cn(A)) \neq \emptyset$, and

1. $x \dashv\vdash \wedge M_1(Cn(A))$

There is no guarantee that input set B does not trigger more pairs in M_1 than A does. To circumvent this problem, the argument takes a detour through the set

$$M_1^- = \{(c,y) \in M_1 : c \in Cn(A)\}$$

Thus, M_1^- is M_1 "stripped of" all the pairs that are not triggered by A. We have $M_1(Cn(A)) = M_1^-(Cn(A))$. We also have $M_1^-(Cn(A)) = M_1^-(Cn(B))$, viz.

$$\{y : (c,y) \in M_1^-, c \in Cn(A)\} = \{y : (c,y) \in M_1^-, c \in Cn(B)\}$$

The $\subseteq$-direction follows from $A \subseteq B$ and monotony for Cn. The $\supseteq$-direction follows from the definition of M_1^-. The argument may, then, be continued thus:

2. $x \dashv\vdash \wedge M_1^-(Cn(A))$
3. $x \dashv\vdash \wedge M_1^-(Cn(B))$

Thus, $x \in \mathcal{O}_1(N,B)$ as required. $\square$

We define $\mathcal{O}_1(N) = \{(A, x) : x \in \mathcal{O}_1(N, A)\}$.

The notion of derivation is defined as in standard I/O logic except that $(\top, \top)$ is not allowed to appear in a derivation unless it is explicitly given in the set N of assumptions.

Definition 2 (Proof system). *$(a, x) \in \mathcal{D}_1(N)$ if and only if there is a derivation of (a, x) from N using the rules {SI, EQ, AND}:*

$$SI\ \frac{(a,x) \qquad b \vdash a}{(b,x)} \qquad\qquad EQ\ \frac{(a,x) \qquad x \dashv\vdash y}{(a,y)}$$

$$AND\ \frac{(a,x) \qquad (a,y)}{(a, x \wedge y)}$$

Where A is a set of formulae, $(A, x) \in \mathcal{D}_1(N)$ means that $(a, x) \in \mathcal{D}_1(N)$, for some conjunction a of elements in A. $\mathcal{D}_1(N, A)$ is $\{x : (A, x) \in \mathcal{D}_1(N)\}$.

Theorem 2. *$\mathcal{O}_1$ validates the rules of $\mathcal{D}_1$ (for individual formulae a).*

Proof. The argument is relatively straightforward, and left to the reader. (For SI, the same trick as in the proof of Theorem 1 must be used.) □

Theorem 3 (Soundness). $\mathcal{D}_1(N, A) \subseteq \mathcal{O}_1(N, A)$

Proof. The proof is by induction on the length of the derivation, using Theorems 1 and 2. □

Theorem 4 (Completeness). $\mathcal{O}_1(N, A) \subseteq \mathcal{D}_1(N, A)$

Proof. We give an outline of the proof for the particular case where A is a singleton set.

Assume $x \in \mathcal{O}_1(N, a)$. Therefore, $\exists M \subseteq N$ such that $x \dashv\vdash \wedge_{i=1}^{n} x_i$, with $M(Cn(a)) = \{x_1, ..., x_n\}$. Below: a derivation of (a, x) from N.

$$\mathrm{EQ}\ \frac{\dfrac{\dfrac{(a_1,x_1)}{(a,x_1)}\,\mathrm{SI} \qquad \ldots\ldots \qquad \dfrac{(a_n,x_n)}{(a,x_n)}\,\mathrm{SI}}{(a, x_1 \wedge \ldots \wedge x_n)}\,\mathrm{AND}}{(a,x)}$$

The argument may be generalized to an input set of arbitrary cardinality. □

3 Reasoning by Cases

In this section, the account described in the previous section is extended to the basic operation out_2, which supports reasoning by cases. The I/O operation is denoted $\mathcal{O}_2$, and the corresponding proof system is called $\mathcal{D}_2$. We call a set of formulae complete if it is either maximal consistent[1] or equal to $\mathcal{L}$.

[1] The set is consistent, and none of its proper extensions is consistent.

Definition 3. $\mathcal{O}_2(N,A) = \cap\{\mathcal{O}_1(N,V) : A \subseteq V, V\, complete\}$.

Theorem 5. $\mathcal{O}_1(N,A) \subseteq \mathcal{O}_2(N,A)$.

Proof. Let $x \in \mathcal{O}_1(N,A)$. Let V be a complete set such that $A \subseteq V$. By Theorem 1, $x \in \mathcal{O}_1(N,V)$. By Definition 3, $x \in \mathcal{O}_2(N,A)$ as required. □

Theorem 6 (Factual monotony). $\mathcal{O}_2(N,A) \subseteq \mathcal{O}_2(N,B)$ *whenever* $A \subseteq B$.

Proof. Assume $x \in \mathcal{O}_2(N,A)$ and $A \subseteq B$. Let V be a complete set such that $B \subseteq V$. From this and the second opening assumption, $A \subseteq V$. From this and the first opening assumption, $x \in \mathcal{O}_1(N,V)$. Thus, $x \in \mathcal{O}_2(N,B)$. □

Definition 4. $(a,x) \in \mathcal{D}_2(N)$ *if and only if there is a derivation of* (a,x) *from* N *using the rules of* $\mathcal{D}_1$ *supplemented with*

$$OR\ \frac{(a,x) \qquad\qquad (b,x)}{(a \vee b, x)}$$

The next theorem appeals to the fact that $\mathcal{O}_1$ validates AND and EQ for an input set of arbitrary cardinality rather than just a singleton set.

Theorem 7. $\mathcal{O}_2$ *validates the rules of* $\mathcal{D}_2$ *(for individual formulae* a*).*

Proof. For (SI). Assume $x \in \mathcal{O}_2(N,a)$ with $b \vdash a$. Let V be a complete set such that $b \in V$. From $b \vdash a$, we get $a \in V$. By Definition 3, we infer $x \in \mathcal{O}_1(N,V)$. This shows that $x \in \mathcal{O}_2(N,b)$.

For (AND). Assume $x \in \mathcal{O}_2(N,a)$ and $y \in \mathcal{O}_2(N,a)$. Let V be a complete set such that $a \in V$. By Definition 3, $x \in \mathcal{O}_1(N,V)$ and $y \in \mathcal{O}_1(N,V)$. Since $\mathcal{O}_1$ validates AND, $x \wedge y \in \mathcal{O}_1(N,V)$. This shows that $x \wedge y \in \mathcal{O}_2(N,a)$.

For (OR). Assume $x \in \mathcal{O}_2(N,a)$ and $x \in \mathcal{O}_2(N,b)$. Let V be a complete set containing $a \vee b$. Since V is complete, either $a \in V$ or $b \in V$. Assume that the first applies. In that case, $x \in \mathcal{O}_1(N,V)$, by the first opening assumption and Definition 3. Assume the second applies. In that case $x \in \mathcal{O}_1(N,V)$, by the second opening assumption and Definition 3. Either way, $x \in \mathcal{O}_1(N,V)$, and thus $x \in \mathcal{O}_2(N,a \vee b)$ as required.

For (EQ), assume $x \in \mathcal{O}_2(N,a)$ and $x \dashv\vdash y$. Let V be a complete set containing a. By Definition 3, $x \in \mathcal{O}_1(N,V)$. Since $\mathcal{O}_1$ validates EQ, $y \in \mathcal{O}_1(N,V)$, and so $y \in \mathcal{O}_2(N,a)$ as required. □

Theorem 8 (Soundness). $\mathcal{D}_2(N,A) \subseteq \mathcal{O}_2(N,A)$.

Proof. Same argument as before, but using Theorems 6 and 7. □

Theorem 9 (Completeness). $\mathcal{O}_2(N,A) \subseteq \mathcal{D}_2(N,A)$.

Proof. We give an outline of the proof for a singleton input set $\{a\}$. The proof may easily be generalized to an input set of arbitrary cardinality.

For ease of exposition, we write (SI,AND) to indicate an application of SI followed by that of AND. Thus, (SI,AND) abbreviates the following derived rule

$$(\text{SI,AND})\ \frac{(a_1, x_1) \qquad \ldots \qquad (a_n, x_n)}{(\wedge_{i=1}^n a_i, \wedge_{i=1}^n x_i)}$$

Case 1: a is inconsistent. In this case, there is exactly one complete set V containing a; it is $\mathcal{L}$. So $\mathcal{O}_2(N, a) = \mathcal{O}_1(N, \mathcal{L})$. Let $x \in \mathcal{O}_1(N, \mathcal{L})$. This means that $x \dashv\vdash \wedge_{i=1}^n x_i$, for $x_1, ..., x_n \in h(N)$. Let $a_1, ..., a_n$ be the body of the rules in question. We have $a \vdash \wedge_{i=1}^n a_i$. A derivation of (a, x) from N may, then, be obtained as shown below.

$$\text{SI}\ \dfrac{\dfrac{\dfrac{(a_1, x_1) \qquad \ldots \qquad (a_n, x_n)}{(\wedge_{i=1}^n a_i, \wedge_{i=1}^n x_i)}\ (\text{SI,AND}) \qquad \wedge_{i=1}^n x_i \dashv\vdash x}{(\wedge_{i=1}^n a_i, x)}\ \text{EQ} \qquad a \vdash \wedge_{i=1}^n a_i}{(a, x)}$$

Case 2: a is consistent. Assume (for reductio) that $x \in \mathcal{O}_2(N, a)$ and that $x \notin \mathcal{D}_2(N, a)$. From the former, $x \dashv\vdash \wedge_{i=1}^n x_i$, for $x_1, ..., x_n \in h(N)$. In order to derive the contradiction that $x \notin \mathcal{O}_2(N, a)$, we start by showing that $\{a\}$ can be extended to some "maximal" $V \supseteq \{a\}$ such that $x \notin \mathcal{D}_2(N, V)$. By maximal, we mean that for all $V' \supset V$, $x \in \mathcal{D}_2(N, V')$. Thus, V is amongst the "biggest" input sets V containing a and not making x derivable.

V is built from a sequence of sets $V_0, V_1, V_2, ...$ as follows. Consider an enumeration $x_1, x_2, x_3, ...$ of all the formulae. We define:

$$V_0 = \{a\}$$
$$V_n = \begin{cases} V_{n-1} \cup \{x_n\}, & \text{if } x \notin \mathcal{D}_2(N, V_{n-1} \cup \{x_n\}) \\ V_{n-1}, & \text{otherwise} \end{cases}$$
$$V = \cup\{V_n : n \geq 0\}$$

It is a straightforward matter to show the following:

Fact 1 *$x \notin \mathcal{D}_2(N, V_n)$, for all $n \geq 0$.*

Fact 2 *$V_n \subseteq V$, for all $n \geq 0$.*

Fact 3 *For every finite subset $V' \subseteq V$, $V' \subseteq V_n$, for some $n \geq 0$.*

By Fact 2, V includes $\{a\}$ (=V_0).The argument may be continued thus:

Claim 1 *$x \notin \mathcal{D}_2(N, V)$.*

Proof of the claim. Assume, to reach a contradiction, that $x \in \mathcal{D}_2(N, V)$. By compactness for $\mathcal{D}_2$, $x \in \mathcal{D}_2(N, V')$ for some finite $V' \subseteq V$. By Fact 3, $V' \subseteq V_n$ for some $n \geq 0$. By monotony in the right argument, $x \in \mathcal{D}_2(N, V_n)$. This contradicts Fact 1.

Claim 2 *For all $V' \supset V$, $x \in \mathcal{D}_2(N, V')$.*

Proof of the claim. Let $V' \supset V$. So, there is some y such that $y \in V'$ but $y \notin V$. Any such y is such that $y = x_n$, for some $n \geq 1$. By Fact 2 $V_n \subseteq V$. Therefore, $y \notin V_n$. By construction, $V_{n-1} = V_n$, and $x \in \mathcal{D}_2(N, V_{n-1} \cup \{y\}) = \mathcal{D}_2(V, V_n \cup \{y\})$. But $V_n \cup \{y\} \subseteq V \cup \{y\} \subseteq V'$. By monotony in the right argument for $\mathcal{D}_2$, we get that $x \in \mathcal{D}_2(N, V')$, as required.

Claim 3 *V is consistent.*

Proof of the claim. Assume not. Since $x \dashv\vdash \wedge_{i=1}^{n} x_i$, for $x_1, ..., x_n \in h(N)$, a derivation of (V, x) from N may be obtained by reiterating the argument under case 1, contradicting Claim 1.

Claim 4 *V is closed under $\vdash$ (i.e., if $V \vdash y$, then $y \in V$).*

Proof of the claim. Assume not. There is, then, some y such that $V \vdash y$ but $y \notin V$. From the former, there are $a_1, ..., a_n \in V$ such that $\wedge_{i=1}^{n} a_i \vdash y$, by compactness. From the latter, $x \in \mathcal{D}_2(N, V \cup \{y\})$, by Claim 2. Thus, the pair $(\wedge_{i=1}^{m} b_i \wedge y, x)$ is derivable from N, where $b_1, ..., b_m \in V$. By PL, $\wedge_{i=1}^{m} b_i \wedge (\wedge_{i=1}^{n} a_i) \vdash \wedge_{i=1}^{m} b_i \wedge y$. A derivation of (V, x) may be obtained thus, contradicting Claim 1.

$$\frac{(\wedge_{i=1}^{m} b_i \wedge y, x) \qquad \wedge_{i=1}^{m} b_i \wedge (\wedge_{i=1}^{n} a_i) \vdash \wedge_{i=1}^{m} b_i \wedge y}{(\wedge_{i=1}^{m} b_i \wedge (\wedge_{i=1}^{n} a_i), x)} \text{(SI)}$$

This is a derivation of (V, x), because $\wedge_{i=1}^{m} b_i \wedge (\wedge_{i=1}^{n} a_i)$ is a conjunction of elements in V.

Claim 5 *V is $\neg$-complete; that is, for all y, either $y \in V$ or $\neg y \in V$.*

Proof of the claim. Assume $y \notin V$ and $\neg y \notin V$ for some y. It, then, follows that $x \in \mathcal{D}_2(N, V_n \cup \{y\})$ and $x \in \mathcal{D}_2(N, V_m \cup \{\neg y\})$, for some $m, n \geq 1$. Either $V_n \subseteq V_m$ or $V_m \subseteq V_n$. Suppose the first applies−the argument for the other case is similar. By monotony in the right argument for $\mathcal{D}_2$, $x \in \mathcal{D}_2(N, V_m \cup \{y\})$. By OR, $x \in \mathcal{D}_2(N, V_m \cup \{y \vee \neg y\})$. By Claim 4, $y \vee \neg y \in V$. So, $x \in \mathcal{D}_2(N, V)$, in contradiction with Claim 1.

Taken together, claims 3 and 5 say that V is maximal consistent, and hence a complete set.

We are almost finished. We have $\mathcal{O}_1(N, V) = \mathcal{D}_1(N, V) \subseteq \mathcal{D}_2(N, V)$. So $x \notin \mathcal{O}_1(N, V)$. Hence, $x \notin \mathcal{O}_2(N, A)$. □

4 Aggregative Cumulative Transitivity

This section shows how to redefine Makinson and van der Torre's reusable output operation out_3 so that it validates neither WO nor CT but ACT:

$$\text{ACT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, x \wedge y)} \qquad\qquad \text{CT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

ACT and WO together imply CT.

Stolpe [20,21] named "PN_3" his own variant of out_3. The distinctive rule of PN_3 is the rule MCT mentioned in the introduction:

$$\text{MCT}\ \frac{(a,x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a,y)}$$

We said that, given the other rules in Stolpe's system, MCT is equivalent to CT. This is easily checked. The other rules are: SI, AND and EQ. On the one hand, given reflexivity for $\vdash$, MCT entails CT. For assume (a,x) and $(a \wedge x, y)$. Since $x \vdash x$, a direct application of MCT yields (a,y). On the other hand, given SI, CT entails MCT:

$$\text{CT}\ \frac{(a,x') \qquad \dfrac{(a \wedge x, y) \qquad \dfrac{x' \vdash x}{a \wedge x' \vdash a \wedge x}}{(a \wedge x', y)}\ \text{SI}}{(a,y)}$$

As a matter of facts, the same will apply to our system, because SI, AND and EQ will remain. In the presence of these rules, ACT and the following aggregative variant of MCT are derivable from each other:

$$\text{AMCT}\ \frac{(a,x') \qquad x' \vdash x \qquad (a \wedge x, y)}{(a, x' \wedge y)}$$

In this respect, weakening has still a "ghostly" role to play for iteration of successive detachments.

For the sake of conciseness, we denote by $h(M)$ the set of all the heads of elements of M. Moreover, $\mathcal{B}_A^M$ denotes the set of all Bs such that $A \subseteq B = Cn(B) \supseteq M(B)$. Intuitively, $\mathcal{B}_A^M$ gathers all the Bs that contain A and are closed under both Cn and M.

Definition 5 (Semantics). *$x \in \mathcal{O}_3(N,A)$ if and only if there is some finite $M \subseteq N$ such that,*

- *$M(Cn(A)) \neq \emptyset$, and*
- *for all B, if $B \in \mathcal{B}_A^M$, then $x \dashv\vdash \wedge M(B)$.*

We do not single out any particular B as "proper". But we highlight two very useful such Bs, which we call the smallest and the largest: $\cap\mathcal{B}_A^M$; $\mathcal{L}$.

A subset M of N that makes $x \in \mathcal{O}_3(N,A)$ true is called an "A-witness for x". Unlike with $\mathcal{O}_1$, we have the guarantee that such a M does not contain any rule that is superfluous, viz. not required to get output x:

Theorem 10. *If M is an A-witness for x, then $x \dashv\vdash \wedge h(M)$.*

Proof. Let M be an A-witness for x. By Definition 5, $M(Cn(A)) \neq \emptyset$, and $x \dashv\vdash \wedge M(B)$ for all $B \in \mathcal{B}_A^M$. Consider $B = \mathcal{L}$. We have $x \dashv\vdash \wedge M(\mathcal{L})$. But $M(\mathcal{L}) = h(M)$, and thus $x \dashv\vdash \wedge h(M)$. □

Theorem 11 (Monotony). *$\mathcal{O}_3(N, A_1) \subseteq \mathcal{O}_3(N, A_2)$, whenever $A_1 \subseteq A_2$.*

Proof. Assume $x \in \mathcal{O}_3(N, A_1)$ and $A_1 \subseteq A_2$. From the first, we get: there is some finite $M_1 \subseteq N$ such that $M_1(Cn(A_1)) \neq \emptyset$ and, for all $B \in \mathcal{B}^{M_1}_{A_1}$,

$$M_1(B) = \{x_1, ..., x_n\} \text{ and } x \dashv\vdash \wedge^n_{i=1} x_i \tag{1}$$

Note that, by Theorem 10, $x \dashv\vdash \wedge h(M_1)$, and so the trick used for the proof of Theorem 1 is no longer needed.

From $A_1 \subseteq A_2$, we get $M_1(Cn(A_1)) \subseteq M_1(Cn(A_2))$, and so $M_1(Cn(A_2)) \neq \emptyset$. Now, let $B_1 \in \mathcal{B}^{M_1}_{A_2}$. From $A_1 \subseteq A_2$, $A_1 \subseteq B_1$, and hence $B_1 \in \mathcal{B}^{M_1}_{A_1}$. By (1), $x \dashv\vdash \wedge M_1(B_1) \dashv\vdash \wedge h(M_1)$. So, $x \in \mathcal{O}_3(N, A_2)$ as required. □

We define $\mathcal{O}_3(N) = \{(A, x) : x \in \mathcal{O}_3(N, A)\}$.

Example 1 shows that $\mathcal{O}_3$ does not validate the rule of deontic detachment, and hence does not validate CT.

Example 1 (Deontic Detachment). Consider $N = \{(\top, a), (a, x)\}$. We have $a \in \mathcal{O}_3(N, \top)$, since $M = \{(\top, a)\}$ is a $\top$-witness for a. We also have $x \in \mathcal{O}_3(N, a)$, since $M = \{(a, x)\}$ is an a-witness for x. But we do not have $x \in \mathcal{O}_3(N, \top)$. This may be verified in two steps. First, you identify all the non-empty subsets M of N that are triggered by the input, $M(Cn(\emptyset)) \neq \emptyset$. Next, you go through the list of all of these subsets, and check that, for none of them, the smallest relevant B outputs heads whose conjunction is equivalent to x:

M	B	$M(B)$
$\{(\top, a)\}$	$Cn(a)$	$\{a\}$
$\{(\top, a)(a, x)\}$	$Cn(a, x)$	$\{a, x\}$

We illustrate the account with two examples from the literature.

Example 2 ("Change your mind!"). Hansson [12] gives the following example, with credit to Pörn:

> "Consider the hoary example of the man who ought to go to a meeting on August 5 and who ought to send, on August 2, a note explaining his absence, if and only if he is in fact going to be absent." [12, p.425-6]

The example is structurally identical to the Chisholm example [3]. The norms involved may be rendered as $N = \{(\top, m), (m, s)(\neg m, \neg s)\}$, where m and s are for attending the meeting and sending a note, respectively. Given input $\top$, $m \wedge s$ is outputted, but not s. This is as it should be. The obligation of s will not be triggered unless the agent is going to fulfil his primary obligation of m. In the violation context $\neg m$, $m \wedge s$ is still outputted. If not, then the following intuitive deontic reasoning pattern would not be supported:

> "August 2 arrives, and though he is able to attend the meeting, he has no intention of doing so. He argues: 'I ought to change my mind, forbear note-writing, and attend the meeting.... My present fulfillment of this obligation will help make up for my sinfully staying at home on the fifth!'." [12, p. 426]

Example 3 (Paradox of the window). Consider the inference pattern: from "it ought to be the case that $x \wedge y$", infer "it ought to be the case that x". The so-called "paradox of the window" has been used by Weinberger to argue against such an inference pattern: when it ought to be that I leave the window closed and play the piano, it does not follow that I must play the piano: playing the piano when the window is open might be forbidden−the neighbors might have filed a complain at the police station about the noise nuisance. In his discussion of the paradox in the context of imperative logic, Hansen [10, p. 171] also gives the following example, due to Hare [13]: from the order "Put on your parachute and jump out!", one does not want to be able to infer the order "Jump!". As Hansen puts it, "the 'window paradox' seems to arise whenever the state of affairs mentioned in the imperative are only conjunctively desired by an authority"[10, ibidem].[2] Consider $N = \{(\top, x \wedge y)\}$. Given input $\top$, $x \wedge y$ is outputted, but not x.

Definition 6 (Proof system). *$(a, x) \in \mathcal{D}_3(N)$ if and only if there is a derivation of (a, x) from N using the rules {SI, EQ, ACT}.*

$$ACT\,\frac{(a,x) \qquad (a \wedge x, y)}{(a, x \wedge y)}$$

We define $(A, x) \in \mathcal{D}_3(N)$ and $\mathcal{D}_3(N, A)$ as we did for $\mathcal{D}_1$.

Theorem 12. *$\mathcal{O}_3$ validates the rules of $\mathcal{D}_3$ (for individual formulae a).*

Proof. We show ACT only. Assume that $x \in \mathcal{O}_3(N, a)$, $y \in \mathcal{O}_3(N, a \wedge x)$ and $x \wedge y \notin \mathcal{O}_3(N, a)$. From the first two, it follows that there are finite $M_1, M_2 \subseteq N$ such that $M_1(Cn(a)) \neq \emptyset$, $M_2(Cn(a, x)) \neq \emptyset$, and

$$x \dashv\vdash \wedge M_1(B) \text{ for all } B \in \mathcal{B}_a^{M_1} \tag{2}$$

$$y \dashv\vdash \wedge M_2(B) \text{ for all } B \in \mathcal{B}_{a\wedge x}^{M_2} \tag{3}$$

By Theorem 10,

$$x \dashv\vdash \wedge h(M_1) \tag{4}$$

$$y \dashv\vdash \wedge h(M_2) \tag{5}$$

Therefore,

$$x \wedge y \dashv\vdash \wedge h(M_1) \wedge (\wedge h(M_2)) \tag{6}$$

$$\dashv\vdash \wedge h(M_3) \tag{7}$$

where $M_3 = M_1 \cup M_2$. From the third opening assumption, since $M_3(Cn(a)) \neq \emptyset$, it follows that there is some $B_1 \in \mathcal{B}_a^{M_3}$ such that

$$\text{not-}(\ x \wedge y \dashv\vdash \wedge M_3(B_1)\ ) \tag{8}$$

[2] Hansen also has an extensive discussion of the paradox in his Ph.D thesis [9, §1.5].

We have $M_1(B_1) \subseteq M_3(B_1)$, and so $B_1 \in \mathcal{B}_a^{M_1}$. Therefore $x \in B_1$, and hence $a \wedge x \in B_1$. So $B_1 \in \mathcal{B}_{a\wedge x}^{M_2}$ too, since $M_2(B_1) \subseteq M_3(B_1)$. Now,

$$M_3(B_1) = M_1(B_1) \cup M_2(B_1)$$

where $\wedge M_1(B_1) \dashv\vdash x$ and $\wedge M_2(B_1) \dashv\vdash y$. Thus, $\wedge M_3(B_1) \dashv\vdash x \wedge y$, a contradiction.

□

Theorem 13 (Soundness). $\mathcal{D}_3(N, A) \subseteq \mathcal{O}_3(N, A)$

Proof. Same argument as for Theorem 3 using Theorems 11 and 12. □

We now move to completeness.

Theorem 14 (Completeness). $\mathcal{O}_3(N, A) \subseteq \mathcal{D}_3(N, A)$

Proof. We give an outline of the proof for the particular case where A is a singleton set. Suppose that $x \in \mathcal{O}_3(N, a)$. To show: $x \in \mathcal{D}_3(N, a)$. From the former, there is some finite $M \subseteq N$ such that $M(Cn(a)) \neq \emptyset$ and, for all $B \in \mathcal{B}_a^M$, $x \dashv\vdash \wedge M(B)$.

Put $B_1 = Cn(a \cup \mathcal{D}_3(M, a))$. We have $a \in B_1 = Cn(B_1)$. We also have $M(B_1) \neq \emptyset$, because $Cn(a) \subseteq B_1$. A phasing result from [15] allows, then, to establish that $M(B_1) \subseteq B_1$, so that $B_1 \in \mathcal{B}_a^M$. The opening assumption, then, yields, $x \dashv\vdash \wedge M(B_1)$.

Based on this, one gets a derivation of (a, x) from N as follows. First, note that $M(B_1) \neq \emptyset$. By Definition 1, one gets $x \in \mathcal{O}_1(N, a \cup \mathcal{D}_3(M, a))$. By Theorem 4, $x \in \mathcal{D}_1(N, a \cup \mathcal{D}_3(M, a))$, and thus $x \in \mathcal{D}_3(N, a \cup \mathcal{D}_3(M, a))$. This means that $x \in \mathcal{D}_3(N, a \cup \{a_1, ..., a_n\})$, where, for each a_i, $a_i \in \mathcal{D}_3(M, a)$. By AND, $\wedge_{i=1}^n a_i \in \mathcal{D}_3(M, a)$. Since $M \subseteq N$, $\wedge_{i=1}^n a_i \in \mathcal{D}_3(N, a)$. A derivation of (a, x) from N is shown below.

$$\dfrac{\dfrac{(a, \wedge_{i=1}^n a_i) \qquad (a \wedge (\wedge_{i=1}^n a_i), x)}{(a, \wedge_{i=1}^n a_i \wedge x)}\,\text{ACT} \qquad \dfrac{x \vdash \wedge_{i=1}^n a_i}{\wedge_{i=1}^n a_i \wedge x \dashv\vdash x}}{(a, x)}\,\text{EQ}$$

The argument for $x \vdash \wedge_{i=1}^n a_i$ appeals to two lemmas:

- $x \dashv\vdash \wedge h(M)$, Theorem 10
- $h(M) \vdash a_i$, for all $1 \leq i \leq n$ – the proof of this is by induction on the length of the derivation of (a, a_i)

The argument may be generalized to an input set A of arbitrary cardinality. □

5 Properties

In a companion paper [18], we identify some desirable properties, which are all satisfied by $\mathcal{O}_3$. These are listed in Table 1.[3] They also hold for $\mathcal{O}_1$ and $\mathcal{O}_2$, when replacing out_3 by out_1 or out_2, respectively. On the left hand side of the table, exact factual detachment (EFD) and violation detection (VD) characterise what is special about *deontic* logic, while substitution (SUB), replacements of logical equivalents (RLE), implication (IMP) and paraconsistency (PC) say something about *logic*. We use the notation $x[\sigma]$ to denote a substitution instance of x. Thus, $x[\sigma]$ is obtained from x by replacing uniformly, in x, all occurrences of a propositional letter by the same propositional formula. $A[\sigma]$ and $N[\sigma]$ extend the notion of substitution instance to sets of formulae, and sets of norms in the straightforward way. We write $N \approx M$ whenever M is obtained from N, by replacing each $(b, y) \in N$ with some (c, z) such that b is equivalent with c, and y is equivalent with z. Implication makes use of the so-called materialisation $m(N)$ of a normative system N, which means that each norm (a, x) is interpreted as a material conditional $a \to x$, i.e. as the propositional sentence $\neg a \vee x$. We distinguish between violations $V(N, A) = \{x \in \mathcal{O}_3(N, A) \mid \neg x \in Cn(A)\}$ and non-violations (or cues for action) $\overline{V}(N, A) = \mathcal{O}_3(N, A) \setminus V(N, A)$.

Table 1. Properties [18]

EFD	$(x, y) \in N \Rightarrow y \in \mathcal{O}_3(N, x)$	NM	$\mathcal{O}_3(N) \subseteq \mathcal{O}_3(N \cup M)$
VD	$(A, y) \Rightarrow (A \cup \{\neg y\}, y)$	NI	$M \subseteq \mathcal{O}_3(N) \Rightarrow$ $\mathcal{O}_3(N) = \mathcal{O}_3(N \cup M)$
SUB	$x \in \mathcal{O}_3(N, A) \Rightarrow x[\sigma] \in \mathcal{O}_3(N[\sigma], A[\sigma])$		
RLE	$N \approx M \Rightarrow \mathcal{O}_3(N) \subseteq \mathcal{O}_3(M)$	IO	$\mathcal{O}_3(N, A) \subseteq out_3(N, A)$
IMP	$\mathcal{O}_3(N, A) \subseteq Cn(m(N) \cup A)$	R	$outf_3(N, A) = outf_3(\mathcal{O}_3(N), A)$
PC	$x \in \overline{V}(N, A) \Rightarrow \exists M \subseteq N : x \in \mathcal{O}_3(M, A)$ and $\mathcal{O}_3(M, A) \cup A$ consistent	SR	$outf_3(N \cup M, A) =$ $outf_3(\mathcal{O}_3(N) \cup M, A)$

On the right hand side of the table, norm monotony (NM) and norm induction (NI) are called "norm change properties", because the normative system N is no longer held constant. Together, exact factual detachment, norm monotony and norm induction are equivalent to saying that $\mathcal{O}_3(N)$ is a closure operator. Finally, the reusability properties relate the system to standard I/O logic: inclusion in reusable output (IO), redundancy (R) and strong redundancy (SR). Their formulation appeals to some key notions of so-called constrained input/output logic, developed by Makinson and van der Torre [16] in order to reason about norm violation:

$$\begin{aligned} conf(N, A) &= \{N' \subseteq N \mid out(N', A) \cup A \text{ consistent}\} \\ maxf(N, A) &= \{N' \in conf(N, A) \mid N' \subseteq\text{-maximal}\} \\ outf(N, A) &= \{out(N.A) \mid N' \in maxf(N, A)\} \end{aligned}$$

[3] See the aforementioned paper for the motivation and a discussion of these properties.

It is worth recalling the reason why the founders of I/O logic introduced consistency checks in relation to contrary-to-duty reasoning: in unconstrained input/output logic, a violation leads to outputting the whole propositional language. This deontic explosion is not a property of the logics we introduce in this paper, as a direct consequence of the lack of the weakening rule. We believe that the unconstrained logics introduced in this paper can capture some aspects of contrary-to-duty-reasoning. A more detailed analysis of this point, in particular how to handle pragmatic oddities [19], is left for future research.

There is another property that acts as a bridge between the logics defined in this paper and the traditional input/output logics. It was not listed in [18] because it may not necessarily be considered a desirable property. This is the property: $out_1(N, A) = Cn(\mathcal{O}_1(N, A))$ and $out_3(N, A) = Cn(\mathcal{O}_3(N, A))$. Somewhat surprisingly, we do not have in general $out_2(N, A) = Cn(\mathcal{O}_2(N, A))$. For a counter-example, take $N = \{(a, x), (b, x \wedge y)\}$ and $A = \{a \vee b\}$. We leave it for future research to define a logic $\mathcal{O}'_2$ satisfying not only the properties in Table 1, but also the requirement $out_2(N, A) = Cn(\mathcal{O}'_2(N, A))$.

6 Summary

This paper has extended Stolpe's results on I/O logics without weakening in two directions. First, we have shown how to account for reasoning by cases. Second, we have shown how to inject a new ("aggregative") form of cumulative transitivity, which we think is more suitable for normative reasoning. Soundness and completeness theorems for the proposed systems have been reported.

Besides the topics left for future research at the end of Section 5, several other directions for future work can be taken. For one, it would be interesting to know if the two semantics proposed here may be merged to yield a new basic reusable operation out_4, with ACT, but not WO, amongst its primitive rules. This is a topic for future research. For quite another, Strasser et al. [22] investigate embeddings of so-called constrained I/O logic into adaptive logic, with a view to transferring complexity results back to I/O logic. Given the important role that WO plays in their embeddings, it would be interesting to know what happens when the latter rule goes away.

References

1. J. Broome. *Rationality Through Reasoning.* Wiley-Blackwell, West Sussex, UK, 2013.
2. F. Cariani. Ought and resolution semantics. *Noûs*, 47(3):534–558, 2013.
3. R.M. Chisholm. Contrary-to-duty imperatives and deontic logic. *Analysis*, 24:33–36, 1963.
4. J.W. Forrester. Gentle murder, or the adverbial Samaritan. *Journal of Philosophy*, 81:193–197, 1984.
5. D. Gabbay, J. Horty, X. Parent, R. van der Meyden, and L. van der Torre, editors. *Handbook of Deontic Logic and Normative Systems.* College Publications, London, UK, 2013.

6. L. Goble. A logic of good, should, and would: Part I. *Journal of Philosophical Logic*, 19:169–199, 1990.
7. L. Goble. A proposal for dealing with deontic dilemmas. In A. Lomuscio and D. Nute, editors, *Proceedings of the Seventh International Workshop on Deontic Logic in Computer Science (DEON04)*, pages 74–113. Springer, Berlin, 2004.
8. L. Goble. Prima facie norms, normative conflicts and dilemmas. In Gabbay et al. [5], pages 241–352.
9. J. Hansen. *Imperatives and Deontic Logic. On the Semantic Foundations of Deontic Logic.* PhD thesis, Faculty of Philosophy, University of Leipzig, 2008.
10. J. Hansen. Imperative logic and its problems. In Gabbay et al. [5], pages 499–544.
11. S. O. Hansson. Preference-based deontic logic (PDL). *Journal of Philosophical Logic*, 19:75–93, 1990.
12. S. O. Hansson. Situationist deontic logic. *Journal of Philosophical Logic*, 26(4):423–448, 1997.
13. R. M. Hare. Some alleged differences between imperatives and indicatives. *Mind*, 76(303):pp. 309–326, 1967.
14. D. Makinson. On a fundamental problem in deontic logic. In P. Mc Namara and H. Prakken, editors, *Norms, Logics and Information Systems*, Frontiers in Artificial Intelligence and Applications, pages 29–54. IOS Press, Amsterdam, 1999.
15. D. Makinson and L. van der Torre. Input/output logics. *Journal of Philosophical Logic*, 29(4):383–408, 2000.
16. D. Makinson and L. van der Torre. Constraints for input/output logics. *Journal of Philosophical Logic*, 30(2):155–185, 2001.
17. R. N. McLaughlin. Further problems of derived obligation. *Mind*, 64(255):400–402, 1955.
18. X. Parent and L. van Torre. Aggregative deontic detachment for normative reasoning, 2014. Short paper. To appear in *Proceedings of the 14th International Conference on Principles of Knowledge Representation and Reasoning.*
19. H. Prakken and M.J. Sergot. Contrary-to-duty obligations. *Studia Logica*, 57:91–115, 1996.
20. A. Stolpe. Normative consequence: The problem of keeping it whilst giving it up. In R. van der Meyden and L. van der Torre, editors, *Deontic Logic in Computer Science, 9th International Conference, DEON 2008*, volume 5076 of *Lecture Notes in Computer Science*, pages 174–188. Springer, 2008.
21. A. Stolpe. *Norms and Norm-System Dynamics.* PhD thesis, Department of Philosophy, University of Bergen, Norway, 2008.
22. C. Strasser, M. Beirlaen, and F. van de Putte. Dynamic proof theories for input/output logic, 2014. Under review.
23. G. von Wright. *Norm and Action: A Logical Enquiry.* Routledge & Kegan Paul PLC, 1963.

Appendix

In this appendix, the proofs of the soundness and completeness theorems, which were just outlined in the main text, are given in full.

I/O operation $\mathcal{O}_1$

Theorem 2. *$\mathcal{O}_1$ validates the rules of $\mathcal{D}_1$.*

Proof.

- For SI. The argument is virtually the same as in the proof of Theorem 1.
- For AND. Assume $x \in \mathcal{O}_1(N, a)$ and $y \in \mathcal{O}_1(N, a)$. Thus, there are $M_1, M_2 \subseteq N$ such that $M_1(Cn(a)) \neq \emptyset$, $M_2(Cn(a)) \neq \emptyset$ and
  1. $x \dashv\vdash \wedge M_1(Cn(a))$
  2. $y \dashv\vdash \wedge M_2(Cn(a))$
  3. $x \wedge y \dashv\vdash \wedge M_1(Cn(a)) \wedge M_2(Cn(a))$

  Put $M_3 = M_1 \cup M_2$. We have

  $$M_3(Cn(a)) = M_1(Cn(a)) \cup M_2(Cn(a))$$

  One, then, gets
  4. $x \wedge y \dashv\vdash \wedge M_3(Cn(a))$

  Thus, $x \wedge y \in \mathcal{O}_1(N, a)$ as required.
- For EQ, the argument is straightforward, and is omitted.

Theorem 3 (Soundness). $\mathcal{D}_1(N, A) \subseteq \mathcal{O}_1(N, A)$

Proof. Let $x \in \mathcal{D}_1(N, A)$, i.e. $(A, x) \in \mathcal{D}_1(N)$. So there is a conjunction a of elements of A such that $(a, x) \in \mathcal{D}_1(N)$. We show that $x \in \mathcal{O}_1(N, a)$ by induction on the length n of the derivation.

For the basis of the induction, where $n = 1$, the argument is straightforward. By the definition of the notion of derivation, $(a, x) \in N$. Put $M = \{(a, x)\}$. We have $M(Cn(a)) = \{x\}$, and so $x \in \mathcal{O}_1(N, a)$ by Definition 1.

For the inductive part of the proof, we assume as an inductive hypothesis that the theorem holds for all $k < n$, and show that it holds, too, at n. Since some pairs preceding (a, x) may not have a as body, we formulate the inductive hypothesis as follows:

For all $k < n$, if $\alpha_k \in \mathcal{D}_1(N)$, then $\alpha_k \in \mathcal{O}_1(N)$.

The argument is split into cases, depending on which rule is used to derive (a, x):

- (a, x) is obtained using (SI). In this case, there is $k < n$ such that $\alpha_k = (b, x) \in \mathcal{D}_1(N)$ and $b \vdash a$. By the inductive hypothesis, $(b, x) \in \mathcal{O}_1(N)$, i.e. $x \in \mathcal{O}_1(N, b)$. By Theorem 2, $x \in \mathcal{O}_1(N, a)$ as required.
- (a, x) is obtained using (AND). In this case, x is of the form $x_1 \wedge x_2$, and there are $k, j < n$ such that $\alpha_k = (a, x_1) \in \mathcal{D}_1(N)$ and $\alpha_j = (a, x_2) \in \mathcal{D}_1(N)$. By the inductive hypothesis, $(a, x_1) \in \mathcal{O}_1(N)$ and $(a, x_2) \in \mathcal{O}_1(N)$, viz $x_1 \in \mathcal{O}_1(N, a)$ and $x_2 \in \mathcal{O}_1(N, a)$. By Theorem 2, $x_1 \wedge x_2 \in \mathcal{O}_1(N, a)$ as required.
- (a, x) is obtained using (EQ). In this case, there is $k < n$ such that $\alpha_k = (a, y) \in \mathcal{D}_1(N)$ and $x \dashv\vdash y$. By the inductive hypothesis, $(a, y) \in \mathcal{O}_1(N)$, i.e. $y \in \mathcal{O}_1(N, a)$. By Theorem 2, $x \in \mathcal{O}_1(N, a)$ as required.

Thus, $x \in \mathcal{O}_1(N, a)$. By monotony in the second argument, Theorem 1, $x \in \mathcal{O}_1(N, A)$.

Theorem 4 (Completeness). $\mathcal{O}_1(N, A) \subseteq \mathcal{D}_1(N, A)$

Proof. Assume $x \in \mathcal{O}_1(N, A)$. So there is some $M \subseteq N$ such that $x \dashv\vdash \wedge_{i=1}^n x_i$, with $M(Cn(A)) = \{x_1, ..., x_n\}$. For each x_i, there is some $a_i \in Cn(A)$ such that $(a_i, x_i) \in M$. For each a_i, there is also a conjunction b_i of elements in A such that $b_i \vdash a_i$. A derivation of (A, x) from M, and hence from N, is shown below.

$$\cfrac{\cfrac{(a_1,x_1)}{(\wedge_{i=1}^n b_i, x_1)}\,\text{SI} \qquad \cdots\cdots \qquad \cfrac{(a_n,x_n)}{(\wedge_{i=1}^n b_i,x_n)}\,\text{SI}}{\text{EQ}\ \cfrac{(\wedge_{i=1}^n b_i, x_1\wedge\ldots\wedge x_n)}{(\wedge_{i=1}^n b_i, x)}}\,\text{AND}$$

This is a derivation of (A, x), because $\wedge_{i=1}^n b_i$ is a conjunction of elements in A. □

I/O operation $\mathcal{O}_3$

Theorem 12. *$\mathcal{O}_3$ validates the rules of $\mathcal{D}_3$.*

Proof.

- For SI. The argument is virtually the same as in the proof of Theorem 11.
- For ACT. The proof is given in the main body of the paper.
- The argument for (EQ) is straightforward, and is omitted. □

For completeness, a few more lemmas are needed.

Lemma 1. *If $x \in \mathcal{D}_3(M, a)$, then $h(M) \vdash x$.*

Proof. By induction on the length n of the derivation of (a, x).

For the basis of the induction, where $n = 1$, the argument is straightforward. By the definition of the notion of derivation, $(a, x) \in M$. So, $h(M) \vdash x$ as required.

For the inductive part of the proof, we assume as an inductive hypothesis that the theorem holds for all $k < n$, and show that it holds, too, at n. We formulate the inductive hypothesis as follows:

For all $k < n$, if $\alpha_k \in \mathcal{D}_3(M)$, then $h(M) \vdash z$, where z is the head of α_k.

We consider the rules of $\mathcal{D}_3$ in turn:

- (a, x) is obtained using (SI). In this case, there is $k < n$ such that $\alpha_k = (b, x) \in \mathcal{D}_3(M)$ and $b \vdash a$. By the inductive hypothesis, $h(M) \vdash x$, as required.
- (a, x) is obtained using (ACT). In this case, x is of the form $x_1 \wedge x_2$, and there are $k, j < n$ such that $\alpha_k = (a, x_1) \in \mathcal{D}_3(M)$ and $\alpha_j = (a \wedge x_1, x_2) \in \mathcal{D}_3(M)$. By the inductive hypothesis, $h(M) \vdash x_1$ and $h(M) \vdash x_2$. By PL, $h(M) \vdash x_1 \wedge x_2$ as required.

– (a,x) is obtained using (EQ). In this case, there is $k<n$ such that $\alpha_k = (a,y) \in \mathcal{D}_3(M)$ and $x \dashv\vdash y$. By the inductive hypothesis, $h(M) \vdash y$. By PL, $h(M) \vdash x$ as required. □

Given some system with n derivation rules, we say that a derivation respects an order $R_1, ..., R_n$ of those rules iff a rule R_j is never applied in it before a rule R_i for $i<j$. We say that an order is universal iff whenever (a,x) is derivable then there is a derivation of (a,x) respecting that order.

Call $deriv_3^\star$ the system characterized by {SI, ACT, WO} or, equivalently, {SI, CT, AND, WO}. We use this star notation to make it clear that (unlike with the usual $deriv_3$) the pair $(\top,\top)$ is not allowed to appear in any derivation unless it is in N.

Lemma 2. *For $deriv_3^\star$, there are at least three universal orders, (SI, ACT), WO, where the parentheses indicate that every arrangement within them is counted.*

Proof. It suffices to show that any application of WO followed by an application of either ACT or SI is "reversible":

WO, SI ⇒ SI, WO

$$\text{SI}\;\frac{\dfrac{(a,x)}{(a,x\vee y)}\,\text{WO}}{(a\wedge b, x\vee y)}$$

$$\text{WO}\;\frac{\dfrac{(a,x)}{(a\wedge b,x)}\,\text{SI}}{(a\wedge b, x\vee y)}$$

WO, ACT (case 1) ⇒ ACT, WO

$$\text{ACT}\;\frac{(a,x)\qquad \dfrac{(a\wedge x, y\wedge z)}{(a\wedge x, y)}\,\text{WO}}{(a, x\wedge y)}$$

$$\text{WO}\;\frac{\dfrac{(a,x)\qquad (a\wedge x, y\wedge z)}{(a, x\wedge y\wedge z)}\,\text{ACT}}{(a, x\wedge y)}$$

WO, ACT (case 2) ⇒ SI, ACT, WO

$$\text{ACT}\ \dfrac{\dfrac{(a, x \wedge y)}{(a, x)}\,\text{WO} \qquad (a \wedge x, z)}{(a, x \wedge z)}$$

$$\text{WO}\ \dfrac{\dfrac{(a, x \wedge y) \qquad \dfrac{(a \wedge x, z)}{(a \wedge x \wedge y, z)}\,\text{SI}}{(a, x \wedge y \wedge z)}\,\text{ACT}}{(a, x \wedge z)}$$

□

Corollary 1 *$Cn(\mathcal{D}_3(N, A)) = deriv_3^\star(N, A)$.*

Proof. For the right-in-left direction, assume $x \in deriv_3^\star(N, A)$. By definition, $(a, x) \in deriv_3^\star(N)$, where a is a conjunction of elements of A. By Lemma 2, there is a derivation of (a, x) from N, in which WO is applied last. In this derivation, (a, x) is obtained from (a, y) with $y \vdash x$. Since the derivation of (a, y) involves only SI and ACT, such a pair is derivable in $\mathcal{D}_3$. That is, $y \in \mathcal{D}_3(N, a)$, and so $y \in \mathcal{D}_3(N, A)$. So, $x \in Cn(\mathcal{D}_3(N, A))$.

The argument for the left-in-right direction is straightforward, and left to the reader. □

Theorem 14 (Completeness). *$\mathcal{O}_3(N, A) \subseteq \mathcal{D}_3(N, A)$*

Proof. Suppose that $x \in \mathcal{O}_3(N, A)$. To show: $x \in \mathcal{D}_3(N, A)$. From the former, there is some finite $M \subseteq N$ such that $M(Cn(A)) \neq \emptyset$ and, for all $B \in \mathcal{B}_A^M$, $x \dashv\vdash \wedge M(B)$.

Put $B_1 = Cn(A \cup \mathcal{D}_3(M, A))$. We have $A \subseteq B_1 = Cn(B_1)$. First, we show that $M(B_1) \subseteq B_1$, so that $B_1 \in \mathcal{B}_A^M$.

The argument 'escalates' to $deriv_3^\star$. This move is made possible by

Sub-lemma 1 *$B_1 = Cn(A \cup deriv_3^\star(M, A))$.*

Proof of sub-lemma. For the $\subseteq$-direction, assume $A \cup \mathcal{D}_3(M, A) \vdash y$. We have $\mathcal{D}_3(M, A) \subseteq deriv_3^\star(M, A)$. By monotony for $\vdash$, $A \cup deriv_3^\star(M, A) \vdash y$ as required. For the $\supseteq$-direction, assume $A \cup deriv_3^\star(M, A) \vdash y$. By Corrolary 1, $A \cup Cn(\mathcal{D}_3(M, A)) \vdash y$. But $\mathcal{D}_3(M, A) \vdash b$ for all b in $Cn(\mathcal{D}_3(M, A))$. By the cut rule for $\vdash$, $A \cup \mathcal{D}_3(M, A) \vdash y$ as required. □

With this in hand, we can now show $M(B_1) \subseteq B_1$. Let $y \in M(B_1)$. So $(b, y) \in M$ for some $b \in B_1$. Hence $A \cup deriv_3^\star(M, A) \vdash b$, and thus $a, x_1, ..., x_n \vdash b$, where a is a conjunction of elements in A, and $x_1, ..., x_n \in deriv_3^\star(M, A)$. For all $i \leq n$, $x_i \in deriv_3^\star(M, b_i)$, where b_i is a conjunction of elements in A. By PL, $\wedge_{i=1}^n x_i \vdash a \rightarrow b$. The following is derivable (in $deriv_3^\star$):

$$\dfrac{\dfrac{\dfrac{(b_1, x_1)}{(\wedge_{i=1}^n b_i, x_1)}\text{SI} \quad ... \quad ... \quad \dfrac{(b_n, x_n)}{(\wedge_{i=1}^n b_i, x_n)}\text{SI}}{\text{WO}\ \dfrac{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n x_i)}{(\wedge_{i=1}^n b_i, a \to b)}}\text{AND}}{(\wedge_{i=1}^n b_i \wedge a, a \to b)}\text{SI}$$

We, then, get

$$\dfrac{(\wedge_{i=1}^n b_i \wedge a, a \to b) \qquad \dfrac{(b, y)}{(\wedge_{i=1}^n b_i \wedge a \wedge (a \to b), y)}\text{SI}}{(\wedge_{i=1}^n b_i \wedge a, y)}\text{CT}$$

Thus, $y \in deriv_3^\star(M, A)$, and so $y \in B_1$.

Using the opening assumption, it then follows that $x \dashv\vdash \wedge M(B_1)$. We now show how to get a derivation of (A, x) from N.

First, note that $M(B_1) \neq \emptyset$, since $M(Cn(A)) \neq \emptyset$ and $A \subseteq B_1$. From this and $x \dashv\vdash \wedge M(B_1)$, we get $x \in \mathcal{O}_1(N, A \cup \mathcal{D}_3(M, A))$, by Definition 1. By Theorem 4, $x \in \mathcal{D}_1(N, A \cup \mathcal{D}_3(M, A))$, and thus $x \in \mathcal{D}_3(N, A \cup \mathcal{D}_3(M, A))$. This means that $x \in \mathcal{D}_3(N, a \cup \{a_1, ..., a_n\})$, where a is a conjunction of elements of A and, for each a_i, $a_i \in \mathcal{D}_3(M, A)$. For each a_i, there is a conjunction b_i of elements in A such that $a_i \in \mathcal{D}_3(M, b_i)$. The following is derivable from M, and hence from N:

$$\dfrac{\dfrac{(b_1, a_1)}{(\wedge_{i=1}^n b_i, a_1)}\text{SI} \quad ... \quad ... \quad \dfrac{(b_n, a_n)}{(\wedge_{i=1}^n b_i, a_n)}\text{SI}}{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n a_i)}\text{AND}$$

The following is also derivable from N:

$$\dfrac{(a \wedge (\wedge_{i=1}^n a_i), x)}{(\wedge_{i=1}^n b_i \wedge a \wedge (\wedge_{i=1}^n a_i), x)}\text{SI}$$

By Theorem 10, $x \dashv\vdash \wedge h(M)$. By Lemma 1, for each a_i, $h(M) \vdash a_i$, and thus $h(M) \vdash \wedge_{i=1}^n a_i$. Hence, $x \vdash \wedge_{i=1}^n a_i$, and so $x \dashv\vdash x \wedge (\wedge_{i=1}^n a_i)$. The following may, then, be derived.

$$\text{ACT}\ \dfrac{\dfrac{(\wedge_{i=1}^n b_i, \wedge_{i=1}^n a_i)}{(\wedge_{i=1}^n b_i \wedge a, \wedge_{i=1}^n a_i)}\text{SI} \qquad (\wedge_{i=1}^n b_i \wedge a \wedge (\wedge_{i=1}^n a_i), x)}{\text{EQ}\ \dfrac{(\wedge_{i=1}^n b_i \wedge a, \wedge_{i=1}^n a_i \wedge x)}{(\wedge_{i=1}^n b_i \wedge a, x)}}$$

But $\wedge_{i=1}^n b_i \wedge a$ is a conjunction of elements in A. Therefore (A, x) is derivable from N.

This completes the proof. $\square$