

Problems

Problem 1

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= 2\pi i \sum_{k=1}^n \operatorname{Res}\left[\frac{1}{1+z^n}, z_k\right] \\ &= 2\pi i \sum_{k=1}^n \frac{1}{nz_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \\ &= -\frac{2\pi i}{n} \sum_{k=1}^n z_k \end{aligned}$$

若 $n=1$, 即

$$1+z^n = 1+z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 2\pi i$$

否则

$$\sum_{k=1}^n z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 0$$

所以

$$\begin{aligned}\frac{z^{2n}}{1+z^n} &= \frac{(z^{2n}-1)+1}{1+z^n} \\ &= \frac{(z^n-1)(z^n+1)}{1+z^n} + \frac{1}{1+z^n} \\ &= z^n - 1 + \frac{1}{1+z^n}\end{aligned}$$

又

$$\oint_{|z|=r>1} z^n - 1 = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz$$

综上,

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 2

令 $z = re^{i\theta}$ $0 \leq \theta < \pi, t = \frac{1}{z}$, 则

$$\begin{aligned}|t| &= \frac{1}{r} < 1 \\ t &= \frac{1}{r} e^{-i\theta}\end{aligned}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n} dz &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^2}} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt\end{aligned}$$

只有一个奇点 $t=0$ 。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt \\ &= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)} dt = 2\pi i & n=1 \end{cases} \quad (\text{Cauchy-Goursat 定理})\end{aligned}$$

Problem 2

求积分

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz \quad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = 0$$

当 $n > 0$ 时

$$\begin{aligned}\frac{1 - \cos 4z^3}{z^n} &= z^{-n} \left(1 - \sum_{k=0}^n (-1)^k \frac{(4z^3)^{2k}}{(2k)!} \right) \\ &= z^{-n} \left(1 - \sum_{k=0}^n (-1)^k \frac{4^{2k} z^{6k}}{(2k)!} \right) \\ &= z^{-n} \sum_{k=1}^n (-1)^{k-1} \frac{4^{2k} z^{6k}}{(2k)!}\end{aligned}$$

$\frac{1 - \cos 4z^3}{z^n}$ 在奇点 $z = 0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中, C_{-1} 对应上式中

$$6k - n = -1$$

此时

$$\begin{aligned}C_{-1} &= (-1)^{k-1} \frac{4^{2k}}{(2k)!} \quad k = \frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

所以 $n > 0$ 时

$$\begin{aligned}\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz &= 2\pi i \operatorname{Res}\left[\frac{1 - \cos 4z^3}{z^n}, 0\right] \\ &= 2\pi i C_{-1} \\ &= 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

综上,

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = \begin{cases} 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k + 1, k \in \mathbb{N} \\ 0 & n \neq 6k + 1, k \in \mathbb{N} \end{cases}$$

Problem 3

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz$$

Solution

Problem 4

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta}$$

Problem 5

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta}$$

Problem 6

求积分

$$I_p = \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \quad p \in (-1, 1)$$

Problem 7

求积分

$$I_{A,B} = \int_0^{2\pi} d\theta A^2 \cos^2 \theta + B^2 \sin^2 \theta \quad A, B \in \mathbb{R} \text{ and } A, B > 0$$

Problem 8

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \quad A, B \in \mathbb{R} \text{ and } AB > 0$$

Problem 9

求积分

$$I_n = \int_0^{+\infty} \frac{dx}{1 + x^{2n}} \quad n \in \mathbb{N}$$

Problem 10

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \quad n \in \mathbb{N}$$

Problem 11

求积分

$$J_n = \int_0^{+\infty} \frac{dx}{(1 + x^2)^n} \quad n \in \mathbb{N}$$

Problem 12

求积分

$$J_n = \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \quad n \in \mathbb{N}$$

Problem 13

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

Problem 14

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx$$