Chapter 2 Analytic Function

Problem 1

$f(z) = f(x+iy) = u(x,y)+iv(x,y)$ 且 $u, v \in C^{(n)}$, 求 $f(z)$n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z) = A + iB$，则

$$\mathrm{d}f = f'(z)\mathrm{d}z = f'(z)(\mathrm{d}x + i\mathrm{d}y) \Leftrightarrow \mathrm{d}f \begin{aligned}[t] &= \mathrm{d}u + i\mathrm{d}v \\ &= \tfrac{\partial u}{\partial x}\mathrm{d}x + \tfrac{\partial u}{\partial y}\mathrm{d}y + i(\tfrac{\partial v}{\partial x}\mathrm{d}x + \tfrac{\partial u}{\partial y}\mathrm{d}y) \\ &= (A\mathrm{d}x - B\mathrm{d}y) + i(B\mathrm{d}x + A\mathrm{d}y) \end{aligned}$$

由上式得

$$\begin{cases} A\mathrm{d}x - B\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \\ B\mathrm{d}x + A\mathrm{d}y = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = A \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -B \end{cases}$$

即

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = F'(z)$$

而

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z) = \frac{\partial^n u}{\partial x^n} + i\frac{\partial^n v}{\partial x^n}$$

u, v 需要满足 Cauchy-Riemann 条件

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Problem 2

求 $\cos(x+iy)$ 的实部和虚部，其中 $x, y \in \mathbb{R}$

Solution

$$\begin{aligned} \cos(x+iy) &= \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\ &= \frac{1}{2}(e^{-y}e^{ix} + e^{y}e^{-ix}) \\ &= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)] \\ &= \frac{1}{2}(e^{y} + e^{-y})\cos x + i\frac{1}{2}(-e^{y} + e^{-y})\sin x \end{aligned}$$