Introduction to Complex Analysis: Recap

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Chapter 6 Analytic Mappings

定义. $f(z) = \frac{az+b}{cz+d}$ 其中 $a, b, c, d \in \mathbb{C}$ 且 $ad - bc \neq 0$

定理. 分式线性映射保广义圆。(圆 $\cup$ 直线映射到圆 $\cup$ 直线)

定理. 分式线性映射保对称点。即对广义圆，圆心映射到圆心，圆心的对称点映射到无穷，边界映射到边界（可由最大模原理得出）

Problem 1

求分式线性映射 $w = \frac{az+b}{cz+d}$ 将单位圆盘 $|z| < 1$ 映射到单位圆盘 $|w| < 1$ 且使 z_1 映射到 0，这里 $|z_1| < 1$

Solution

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \frac{a(z-(-\frac{b}{a}))}{b(z-(-\frac{d}{c}))} \end{aligned}$$

根据题意

$$w(z_1) = 0$$

根据保对称性，z_1 的对称点 $z_1' = \frac{1}{\overline{z_1}}$ 满足

$$w(z_1') = \infty$$

记 $\frac{a}{c} = a'$ 则此时

$$\begin{aligned} w &= a' \tfrac{z-z_1}{z-\frac{1}{\overline{z_1}}} \\ &= (-\overline{z_1}a') \tfrac{z-z_1}{1-\overline{z_1}z} \\ &= A(\tfrac{z-z_1}{1-\overline{z_1}z}) \end{aligned}$$

根据最大模原理，$|w| = 1$ 时 $|z| = 1$，且 $|z| = 1$ 时

$$1 = \overline{z}z = |z|^2$$

所以 $|w| = 1$ 时

$$\begin{aligned} 1 = |A| \left| \frac{z-z_1}{1-\overline{z_1}z} \right| &= |A| \left| \frac{z-z_1}{z\overline{z}-\overline{z_1}z} \right| \\ &= \frac{|A|}{|z|} \frac{|z-z_1|}{|\overline{z}-\overline{z_1}|} \\ &= \frac{|A|}{|z|} \\ &= |A| \end{aligned}$$

所以

$$A = e^{i\theta} \qquad \theta \in [0, 2\pi)$$

所以

$$w = e^{i\theta} \frac{z-z_1}{1-\overline{z_1}z} \qquad \theta \in [0, 2\pi), |z_1| < 1$$

推论. $z_1 = 0$ 时 $w = e^{i\theta}z$ 对应逆时针旋转 θ 角

推论. 不变式 从单位圆盘到单位圆盘的映射满足

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$$

Proof. 因为从单位圆盘到单位圆盘的映射

$$w = \frac{az+b}{cz+d} = e^{i\theta}\frac{z-z_1}{1-\overline{z_1}z}$$

所以

$$\begin{aligned} a &= e^{i\theta} \\ b &= -z_1e^{i\theta} \\ c &= -\overline{z_1} \\ d &= 1 \end{aligned}$$

又

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{ad-bc}{(cz+b)^2}$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1-|z_1|^2}{|1-\overline{z_1}z|^2} > 0$$

而

$$\begin{aligned} 1-|w|^2 &= 1-w\overline{w} \\ &= 1-e^{i\theta}\frac{z-z_1}{1-\overline{z_1}z}e^{-i\theta}\frac{\overline{z}-\overline{z_1}}{1-z_1\overline{z}} \\ &= \frac{(1-\overline{z_1}z)(1-z_1\overline{z})-(z-z_1)(\overline{z}-\overline{z_1})}{(1-\overline{z_1}z)(1-z_1\overline{z})} \\ &= \frac{1-z_1\overline{z}-\overline{z_1}z+|z|^2|z_1|^2-z\overline{z}+z\overline{z_1}+z_1\overline{z}-|z_1|^2}{|z-zz_1|^2} \\ &= \frac{1-|z_1|^2-|z|^2+|zz_1|^2}{|z-zz_1|^2} \\ &= \frac{(1-|z|^2)(1-|z_1|^2)}{|z-zz_1|^2} \end{aligned}$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1-|z_1|^2}{|1-\overline{z_1}z|^2} = \frac{1-|z_1|^2}{|1-zz_1|^2}$$

所以

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$$

□

Problem 2

求分式线性映射使 $|z - z_0| < r \longrightarrow |w - w_0| < R$ 且使 $z_1 \to w_0$。这里 $|z_1 - z_0| < r$。

Solution

$|z - z_0| < r$ 到以 z_0 为圆心的单位圆的映射为

$$z' = \frac{z - z_0}{r}$$

以 z_0 为圆心的单位圆到以 z_1 为圆心到单位圆的映射为

$$w' = e^{i\theta}\frac{z' - z_1'}{1 - \overline{z_1'}z'}$$

从 $|w - w_0| < R$ 到以 z_1 为圆心到单位圆的映射为

$$w' = \frac{w - w_0}{R}$$

所以

$$w' = \frac{w - w_0}{R} = e^{i\theta}\frac{\frac{z-z_0}{r} - \frac{z_1-z_0}{r}}{1 - \overline{(\frac{z_1-z_0}{r})}(\frac{z_1-z_0}{r})}$$

解得

$$w = w_0 + e^{i\theta}Rr\frac{z - z_1}{r^2 - \overline{(z_1 - z_0)}(z - z_0)} \qquad \theta \in [0, 2\pi) \quad |z_1 - z_0| < r$$

Problem 3

求证上半复平面 $\mathrm{Im}z > 0$ 映射到单位圆盘 $|w| < 1$ 的分式线性映射为

$$w = e^{i\theta}\frac{z - z_1}{z - \overline{z_1}} \qquad \theta \in [0, 2\pi) \quad \mathrm{Im}z_1 > 0$$

Proof. 根据保对称性，由于 z_1 映射到 0，所以 $z_1' = \overline{z_1}$ 映射到 ∞。所以

$$w = A\frac{z - z_1}{z - \overline{z_1}}$$

当 $z = x \in \mathbb{R}$ 时，根据最大模原理

$$|w(z)| = 1$$

因此

$$\begin{aligned} 1 = |w| &= |A|\left|\frac{x - z_1}{x - \overline{z_1}}\right| \\ &= |A|\left|\frac{x - z_1}{\overline{x - z_1}}\right| \\ &= |A| \end{aligned}$$

即 $|A| = 1$，$A = e^{i\theta} \quad \theta \in [0, 2\pi)$ 所以

$$w = e^{i\theta}\frac{z - z_1}{z - \overline{z_1}} \qquad \theta \in [0, 2\pi) \quad \mathrm{Im}z_1 > 0$$

□

Problem 4

求证当 a, b, c, $d \in \mathbb{R}$ 时，分式线性映射使得 $\mathrm{Im}z > 0$ 映射到 $\mathrm{Im}w > 0$ 的充要条件是 $ad - bc > 0$

Proof. 令

$$z = x + iy$$
$$x, y \in \mathbb{R}$$
$$w = \frac{az + b}{cz + d}$$

则

$$\begin{aligned} w &= u + iv \\ &= \frac{a(x+iy)+b}{c(x+iy)+d} \\ &= \frac{(ax+b)+iay}{(cx+d)+icy} \\ &= \frac{[(ax+b)+iay][(cx+d)-icy]}{[(cx+d)+icy][(cx+d)-icy]} \end{aligned}$$

而

$$v = \mathrm{Im}w = \frac{ad - bc}{(cx+d)^2 + c^2y^2}$$

y 与 v 同号

$$\Leftrightarrow ad - bc > 0$$

□