Introduction

Problem 1

求

$$\max|\alpha z^n+\beta| \qquad |z|\le r$$

Solution

$$\max|\alpha z^n+\beta| = |\alpha|\max|z^n+\frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max|z^n+\alpha| \qquad |z|\le r$$

有

$$|z^n+\alpha|\le|z^n|+|\alpha|\le r^n+|\alpha|$$

取最大值 $\Leftrightarrow$ 等号成立 $\Leftrightarrow$ z^n与α同向 $\Leftrightarrow$ $z^n=\lambda\alpha \quad \lambda>0$
当 $\alpha=0$ 时

$$|z^n|=|z|^n=r^n$$
$$|z|=r$$
$$\max|\alpha z^n+\beta|=r^n \qquad z=re^i \quad \theta\in[0,2\pi)$$

当 $\alpha\neq 0$ 时

$$\max|z^n+\alpha|=r^n+|\alpha|$$

此时

$$r^n=|z^n|=|\lambda\alpha|$$
$$\lambda=\frac{|r|^n}{|\alpha|}$$
$$z^n=\lambda\alpha=\frac{|r|^n}{|\alpha|}\alpha$$

设 $\alpha=r_0e^{i\theta_0}$

$$\begin{aligned} z^n &= \frac{|r|^n}{|\alpha|}\alpha \\ &= \frac{|r|^n}{|\alpha|}r_0e^{i\theta_0} \\ &= |r|^ne^{i\theta_0} \end{aligned}$$

则取最大值时

$$z=z_k=re^{\frac{i(\theta_0+2k\pi)}{n}} \qquad k=1,2,\cdots,n$$

综上，

$$\max|z^n+\alpha|=\begin{cases} r^n & z=re^{i\theta} \quad \theta\in[0,2\pi) & \alpha=0 \\ r^n+|\alpha| & z=re^{\frac{i(\theta_0+2k\pi)}{n}} \quad k=1,2,\cdots,n & \alpha\neq 0 \end{cases}$$

Problem 2

$$|z|^2 = z\overline{z} \qquad z^2 = |z|^2$$

等号成立的条件是?
Solution

$$z^2 = |z|^2 = z\overline{z} \Leftrightarrow z(z-\overline{z}) = 0 \Leftrightarrow z = \overline{z}$$

即 $z \in \mathbb{R}$ 时等号成立。

Problem 3

证明

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

并说明几何意义
Solution

$$\begin{aligned}
|z_1+z_2|^2 + |z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2)(\overline{z_1}-\overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\
&= 2(z_1\overline{z_1} + z_2\overline{z_2}) \\
&= 2(|z_1|^2 + |z_2|^2)
\end{aligned}$$

几何意义：平行四边形对角线平方和等于对边平方和

Problem 4

$|z_1| = |z_2| = |z_3| = |z_4| = r$且$z_1+z_2+z_3+z_4 = 0$ 则 z_1,z_2,z_3,z_4 满足什么条件时 $z_1z_2z_3z_4$ 构成正方形?
Solution

定理. $z_1, z_2, \cdots, z_n$ 将圆 $|z| = \alpha n$ 等分 $\Leftrightarrow z_k$ 是分圆多项式 $z^n + \alpha = 0$

当 z_1,z_2,z_3,z_4 构成正方形时，$(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ 是分圆多项式。
又

$$\begin{aligned}
&(z-z_1)(z-z_2)(z-z_3)(z-z_4) \\
=&z^4 - \sum_{k=1}^{4} z_k z^3 + (z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4)z^2 \\
&-(z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4)z + z_1z_2z_3z_4
\end{aligned}$$

$(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ 是分圆多项式

$$\Leftrightarrow \begin{cases}
\sum_{k=1}^{4} z_k & = 0 \\
z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 & = 0 \\
z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 & = 0 \\
z_1z_2z_3z_4 & \neq 0
\end{cases}$$

Problem 4 continued on next page...

而

$$z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4=z_1z_2z_3z_4(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\frac{1}{z_4})$$
$$z_k\overline{z_k}=|z_k|^2=r^2$$

所以

$$\frac{1}{z_k}=\frac{\overline{z_k}}{r^2}$$
$$z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4=\frac{z_1z_2z_3z_4}{r^2}(\overline{z_1}+\overline{z_2}+\overline{z_3}+\overline{z_4})=0$$

所以 z_1,z_2,z_3,z_4 需满足

$$z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4=0$$

Residues

Problem 5

求证

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z=\begin{cases}2\pi i & n=1\\ 0 & n\geq 2\end{cases}$$

Solution 1

$$\begin{aligned}\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z&=2\pi i\sum_{k=1}^{n}\mathrm{Res}[\frac{1}{1+z^n},z_k]\\&=2\pi i\sum_{k=1}^{n}\frac{1}{nz_k^{n-1}}\\&=\frac{2\pi i}{n}\sum_{k=1}^{n}\frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n=1$$

则

$$-z_k=\frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}&\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\\=&\frac{2\pi i}{n}\sum_{k=1}^{n}\frac{1}{z_k^{n-1}}\\=&-\frac{2\pi i}{n}\sum_{k=1}^{n}z_k\end{aligned}$$

若 $n=1$，即

$$1+z^n=1+z=0$$

解得

$$z=-1$$

因此

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=2\pi i$$

否则

$$\sum_{k=1}^{n}z_k=0$$

即

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=0$$

又

$$\begin{aligned}\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z&=\oint_{|z|=r>1}\frac{(z^{2n}-1)+1}{1+z^n}\mathrm{d}z\\&=\oint_{|z|=r>1}\frac{(z^n-1)(z^n+1)}{1+z^n}\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\\&=\oint_{|z|=r>1}(z^n-1)\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1}(z^n-1)\mathrm{d}z=0$$

所以

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z=\begin{cases}2\pi i & n=1\\0 & n\geq 2\end{cases}$$

Solution 2

令 $z=re^{i\theta}\quad 0\leq\theta<\pi, t=\frac{1}{z}$，则

$$|t|=\frac{1}{r}<1$$
$$t=\frac{1}{r}e^{-i\theta}$$
$$\mathrm{d}z=\mathrm{d}(\frac{1}{t})=-\frac{1}{t^2}\mathrm{d}t\qquad 0\leq\theta<\pi\quad\text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z&=-\oint_{|t|=\frac{1}{r}<1}\frac{1}{1+\frac{1}{t^2}}(-\frac{1}{t^2})\mathrm{d}t\\&=\oint_{|t|=\frac{1}{r}<1}\frac{t^{n-2}}{1+t^n}\mathrm{d}t\end{aligned}$$

只有一个奇点 $t = 0$。因此

$$
\begin{aligned}
I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t \\
&= \begin{cases} 0 & n \geq 2 \qquad \text{(Cauchy-Goursat 定理)} \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)}\mathrm{d}t = 2\pi i & n = 1 \end{cases}
\end{aligned}
$$

Problem 6

求积分

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = 0$$

当 $n > 0$ 时

$$
\begin{aligned}
\frac{1-\cos 4z^3}{z^n} &= z^{-n}(1 - \sum_{k=0}^{n}(-1)^k \frac{(4z^3)^{2k}}{(2k)!}) \\
&= z^{-n}(1 - \sum_{k=0}^{n}(-1)^k \frac{4^{2k}z^{6k}}{(2k)!}) \\
&= z^{-n}\sum_{k=1}^{n}(-1)^{k-1} \frac{4^{2k}z^{6k}}{(2k)!}
\end{aligned}
$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z = 0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中，C_{-1} 对应上式中

$$6k - n = -1$$

此时

$$
\begin{aligned}
C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k = \frac{n-1}{6} \\
&= (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

所以 $n > 0$ 时

$$
\begin{aligned}
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z &= 2\pi i \mathrm{Res}[\frac{1-\cos 4z^3}{z^n}, 0] \\
&= 2\pi i C_{-1} \\
&= 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

综上，

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = \begin{cases} 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k+1, k \in \mathbb{N} \\ 0 & n \neq 6k+1, k \in \mathbb{N} \end{cases}$$

Problem 7

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z$$

Solution

令 $t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$
$$t = \frac{1}{r} e^{-i\theta}$$
$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t \\ &= 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0]\end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)(1 - t + t^2 - t^3 + \dots)$$

其中，t^{-1} 的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z = 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

Problem 8

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}, \theta \in [0, 2\pi)$，注意到:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$
$$\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \frac{\mathrm{d}z}{iz}$$

Problem 8 continued on next page...

则原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\oint_{|z|=1}\frac{\frac{\mathrm{d}z}{iz}}{a+b(\frac{z^2+1}{2z})}\\&=\frac{2}{i}\oint_{|z|=1}\frac{\mathrm{d}z}{bz^2+2az+b}\end{aligned}$$

当 $b=0$ 时，原积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a}=\frac{2\pi}{a}$$

当 $b\neq 0$ 时，原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{i}\oint_{|z|=1}\frac{\mathrm{d}z}{bz^2+2az+b}\\&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\end{aligned}$$

对方程

$$z^2+\frac{2a}{b}z+1=0$$

其两根

$$z_1z_2=1$$

且

$$z_{1,2}=-\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b}$$

可设 $b>0$，则

$$z_2=-\frac{a}{b}\frac{\sqrt{a^2-b^2}}{b}<-\frac{a}{b}<-1$$

即只有一个奇点 z_1。所以原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\\&=\frac{2}{ib}2\pi i\frac{1}{2z_1+\frac{2a}{b}}\\&=\frac{2\pi}{\sqrt{a^2-b^2}}\end{aligned}$$

Problem 9

求积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}\qquad a>|b|\quad a,b\in\mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1,1]$ 上可积，则 $\int_0^{2\pi}f(\cos x)\mathrm{d}\theta=\int_0^{2\pi}f(\sin x)\mathrm{d}\theta$

所以

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$$

Problem 10

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Solution

令

$$a = 1 + p^2$$
$$b = -2p$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

Problem 11

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} > 0$$

Solution

令

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Problem 11 continued on next page...

则

$$
\begin{aligned}
I_{A,B} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\dfrac{\cos 2\theta + 1}{2} + B^2\dfrac{1-\cos 2\theta}{2}} \\
&= \int_0^{4\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\int_0^{2\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\frac{2\pi}{\sqrt{(A^2+B^2)^2 - (A^2-B^2)^2}} \\
&= \frac{2\pi}{AB}
\end{aligned}
$$

Problem 12

求积分

$$
I_n = \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x
$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数，因此

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} \\
&= \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)}
\end{aligned}
$$

而

$$
\begin{aligned}
&\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)} + \int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(z^2+a^2)(z^2+b^2)}\mathrm{d}z \\
=&2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
=&2\pi i(\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2) + ab})
\end{aligned}
$$

定理. 若 $P_n(z), Q_m(z)$ 是多项式，且 $\deg P_n = n \le \deg Q_m - 2 = m - 2$，$Q_m(z)$ 在实轴 $z = x$ 上没有零点，即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$，则

$$
\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{P_n(z)}{Q_m(z)}\mathrm{d}z = 0
$$

Problem 12 continued on next page...

所以

$$\begin{aligned}
I_n &= \frac{1}{2}2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|])\\
&= \pi i(\frac{1}{4(i|a|)^3+2i|a|(a^2+b^2)} + \frac{1}{4(i|b|)^3+2i|b|(a^2+b^2)})\\
&= \frac{\pi}{2(|b|-|a|)|a||b|}
\end{aligned}$$

Problem 13

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}\\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}\\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}\\
&= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2}\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0}\frac{1}{1+z^{2n}}\mathrm{d}z\\
&= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k]
\end{aligned}$$

其中

$$z_k^{2n}+1=0, \mathrm{Im}z_k>0$$

则

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}}\\
&= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k]\\
&= \pi i\sum_{k=1}^{n}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k]\\
&= \pi i\sum_{k=1}^{n}\frac{1}{2nz_k^{2n-1}}\\
&= -\frac{\pi i}{2n}\sum_{k=1}^{n} z_k
\end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i}n}{e^{\pi i}2n} \qquad k = 1, 2, \cdots, 2n$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{2n}\sum_{k=0}^{i} z_k \\
&= -\frac{\pi i}{2n}\frac{\sum_{k=1}^{n} e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\
&= -\frac{\pi i}{2n}\frac{e^{\frac{\pi i}{n}}(1-e^{\pi i})}{e^{\frac{\pi i}{2n}}(1-e^{\frac{\pi i}{n}})} \\
&= -\frac{\pi i}{n}\frac{e^{\frac{\pi i}{2n}}}{1-e^{\frac{\pi i}{n}}}
\end{aligned}$$

令 $\theta = \frac{\pi i}{2n}$，则

$$\begin{aligned}
\frac{e^{\frac{\pi i}{2n}}}{1-e^{\frac{\pi i}{n}}} &= \frac{\cos\theta + i\sin\theta}{1-\cos 2\theta - i\sin\theta} \\
&= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta} \\
&= \frac{1}{-2i\sin\theta}
\end{aligned}$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{n}\frac{e^{\frac{\pi i}{2n}}}{1-e^{\frac{\pi i}{n}}} \\
&= -\frac{\pi i}{n}\frac{\cos\theta + i\sin\theta}{1-\cos 2\theta - i\sin\theta} \\
&= -\frac{\pi i}{n}\frac{1}{-2i\sin\theta} \\
&= \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}
\end{aligned}$$

Problem 14

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n}+x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n}+x^{2n}} \\
&= \frac{1}{r^{2n}}\int_0^{+\infty} \frac{\mathrm{d}x}{1+(\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}}\int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{1+(\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} I_n
\end{aligned}$$

Problem 15

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2}2\pi(i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(1+z^2)^n}\mathrm{d}z) \\
&= \frac{1}{2}(2\pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]) \\
&= \pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]
\end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

令 $f(z) = \frac{1}{(z+i)^n}$

$$\begin{aligned}
\frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
&= (z-i)^{-n}\sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!}(z-i)^k
\end{aligned}$$

要求 $(z-i)^{-1}$ 对应的系数 C_{-1}，对应于 $k = n-1$

$$\begin{aligned}
C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
&= (-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
\end{aligned}$$

因此

$$\begin{aligned}
J_n &= \pi i C_{-1} \\
&= \pi i(-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
&= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
\end{aligned}$$

Problem 16

求积分

$$J_{n,r}=\int_0^{+\infty}\frac{\mathrm{d}x}{(r^2+x^2)^n}\qquad n\in\mathbb{N}$$

Solution

$$\begin{aligned}J_{n,r}&=\int_0^{+\infty}\frac{\mathrm{d}x}{(r^2+x^2)^n}\\&=\int_0^{+\infty}\frac{1}{r^{2n}}\frac{r\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n}\\&=\frac{1}{r^{2n-1}}\int_0^{+\infty}\frac{\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n}\\&=\frac{1}{r^{2n-1}}J_n\end{aligned}$$

Problem 17

求积分

$$I_{a,b,k}=\int_0^{+\infty}\frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

设 $a\neq b$ 则

$$\begin{aligned}I_{a,b,k}&=\int_0^{+\infty}\frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\\&=\frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\end{aligned}$$

而

$$\begin{aligned}\int_0^{+\infty}\frac{xe^{x}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x&=2\pi i\{\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)},ai]+\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)},bi]\}\\&=2\pi i[\frac{aie^{-ka}}{4(ai)^3+2ai(a^2+b^2)}+\frac{bie^{-kb}}{4(bi)^3+2bi(a^2+b^2)}]\\&=\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})\end{aligned}$$

所以

$$\begin{aligned}I_{a,b,k}&=\frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\\&=\frac{1}{2}\mathrm{Im}[\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})]\\&=\frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb})\end{aligned}$$

Problem 17 continued on next page...

$a = b$ 时

$$
\begin{aligned}
I_{a,b,k} &= \lim_{a\to b}\frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb}) \\
&= \frac{-k\pi e^{-kb}}{-4b} \\
&= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}}
\end{aligned}
$$

Problem 18

求积分

$$
I_{a,b,k} = \int_0^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x
$$

Solution

$a \neq b$ 时

$$
\begin{aligned}
I_{a,b,k} &= \int_0^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\mathrm{Re}\int_{-\infty}^{+\infty}\frac{x^2e^{ikx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{(be^{-kb}-ae^{-ka})\pi}{2(b^2-a^2)}
\end{aligned}
$$

当 $a = b$ 时

$$
I_{a,b,k} = \frac{(1-ka)\pi}{4ae^{ka}} = \frac{(1-kb)\pi}{4be^{kb}}
$$