Chapter 2 Analytic Function

Problem 1

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ 且 $u,v\in C^{(n)}$, 求 $f(z)$n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z)=A+iB$，则

$$\mathrm{d}f=f'(z)\mathrm{d}z=f'(z)(\mathrm{d}x+i\mathrm{d}y)\Leftrightarrow \mathrm{d}f \begin{aligned}[t] &=\mathrm{d}u+i\mathrm{d}v\\ &=\tfrac{\partial u}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y+i(\tfrac{\partial v}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y)\\ &=(A\mathrm{d}x-B\mathrm{d}y)+i(B\mathrm{d}x+A\mathrm{d}y)\end{aligned}$$

由上式得

$$\begin{cases} A\mathrm{d}x-B\mathrm{d}y=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\\ B\mathrm{d}x+A\mathrm{d}y=\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=A\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-B\end{cases}$$

即

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=F'(z)$$

而

$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z)=\frac{\partial^n u}{\partial x^n}+i\frac{\partial^n v}{\partial x^n}$$

u,v 需要满足 Cauchy-Riemann 条件

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$