Problems

Problem 1

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^n}, z_k] \\ &= 2\pi i \sum_{k=1}^{n} \frac{1}{nz_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}&\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z \\ =&\frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}} \\ =&-\frac{2\pi i}{n} \sum_{k=1}^{n} z_k\end{aligned}$$

若 $n=1$，即

$$1+z^n = 1+z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 2\pi i$$

否则

$$\sum_{k=1}^{n} z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 0$$

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所以

$$\begin{aligned}\frac{z^{2n}}{1+z^n} &= \frac{(z^{2n}-1)+1}{1+z^n} \\ &= \frac{(z^n-1)(z^n+1)}{1+z^n} + \frac{1}{1+z^n} \\ &= z^n - 1 + \frac{1}{1+z^n}\end{aligned}$$

又

$$\oint_{|z|=r>1} z^n - 1 = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 2

令 $z = re^{i\theta} \quad 0 \leq \theta < \pi, t = \frac{1}{z}$，则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^2}}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t\end{aligned}$$

只有一个奇点 $t=0$。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t \\ &= \begin{cases} 0 & n \geq 2 \qquad \text{(Cauchy-Goursat 定理)} \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)}\mathrm{d}t = 2\pi i & n = 1 \end{cases}\end{aligned}$$

Problem 2

求积分

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = 0$$

当 $n > 0$ 时

$$\begin{aligned}\frac{1-\cos 4z^3}{z^n} &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{(4z^3)^{2k}}{(2k)!}) \\ &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{4^{2k}z^{6k}}{(2k)!}) \\ &= z^{-n}\sum_{k=1}^{n}(-1)^{k-1}\frac{4^{2k}z^{6k}}{(2k)!}\end{aligned}$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z=0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中，C_{-1} 对应上式中

$$6k-n=-1$$

此时

$$\begin{aligned}C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k=\frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

所以 $n > 0$ 时

$$\begin{aligned}\oint_{|z|=r>0}\frac{1-\cos 4z^3}{z^n}\mathrm{d}z &= 2\pi i\mathrm{Res}[\frac{1-\cos 4z^3}{z^n},0] \\ &= 2\pi i C_{-1} \\ &= 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

综上，

$$\oint_{|z|=r>0}\frac{1-\cos 4z^3}{z^n}\mathrm{d}z = \begin{cases} 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n=6k+1, k\in\mathbb{N} \\ 0 & n\neq 6k+1, k\in\mathbb{N}\end{cases}$$

Problem 3

求积分

$$\oint_{|z|=r>1}\frac{z^3e^{\frac{1}{z}}}{1+z}\mathrm{d}z$$

Solution

令 $t=\frac{1}{z}$, 则

$$|t|=\frac{1}{r}<1$$

$$t=\frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z=\mathrm{d}(\frac{1}{t})=-\frac{1}{t^2}\mathrm{d}t \qquad 0\le\theta<\pi \quad \text{积分方向为顺时针}$$

Problem 3 continued on next page...

原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t \\ &= 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0]\end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4}(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots)(1-t+t^2-t^3+\dots)$$

其中，t^{-1} 的系数为

$$-1+1-\frac{1}{2!}+\frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z = 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

Problem 4

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}, \theta \in [0, 2\pi)$，注意到:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$
$$\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \frac{\mathrm{d}z}{iz}$$

则原积分

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} &= \oint_{|z|=1} \frac{\frac{\mathrm{d}z}{iz}}{a+b(\frac{z^2+1}{2z})} \\ &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{bz^2+2az+b}\end{aligned}$$

当 $b=0$ 时，原积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a} = \frac{2\pi}{a}$$

当 $b \neq 0$ 时，原积分

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{bz^2+2az+b} \\ &= \frac{2}{ib}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\end{aligned}$$

对方程

$$z^2 + \frac{2a}{b}z + 1 = 0$$

其两根

$$z_1 z_2 = 1$$

且

$$z_{1,2} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

可设 $b > 0$，则

$$z_2 = -\frac{a}{b}\frac{\sqrt{a^2 - b^2}}{b} < -\frac{a}{b} < -1$$

即只有一个奇点 z_1。所以原积分

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} &= \frac{2}{ib}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1} \\ &= \frac{2}{ib}2\pi i \frac{1}{2z_1 + \frac{2a}{b}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}}\end{aligned}$$

Problem 5

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1, 1]$ 上可积，则 $\int_0^{2\pi} f(\cos x)\mathrm{d}\theta = \int_0^{2\pi} f(\sin x)\mathrm{d}\theta$

所以

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 6

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Solution

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2p\cos\theta+p^2} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} \\ &= \frac{2\pi}{\sqrt{a^2-b^2}} \\ &= \frac{2\pi}{1-p^2} \end{aligned}$$

Problem 7

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} > 0$$

Solution

令

$$\begin{aligned} \cos^2\theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned}$$

则

$$\begin{aligned} I_{A,B} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\dfrac{\cos 2\theta+1}{2} + B^2\dfrac{1-\cos 2\theta}{2}} \\ &= \int_0^{4\pi} \frac{\mathrm{d}t}{(A^2+B^2)+(A^2-B^2)\cos t} \\ &= 2\int_0^{2\pi} \frac{\mathrm{d}t}{(A^2+B^2)+(A^2-B^2)\cos t} \\ &= 2\frac{2\pi}{\sqrt{(A^2+B^2)^2-(A^2-B^2)^2}} \\ &= \frac{2\pi}{AB} \end{aligned}$$

Problem 8

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}$$

Problem 9

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Problem 10

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 11

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 12

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} \mathrm{d}x$$

Problem 13

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} \mathrm{d}x$$