Introduction to Complex Analysis: Recap

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目录

Chapter 1 Introduction

公式. *Euler* 公式

$$e^{ix} = \cos x + i \sin x$$

Problem 1

求

$$\max|\alpha z^n + \beta| \qquad |z| \le r$$

Solution

$$\max|\alpha z^n + \beta| = |\alpha|\max|z^n + \frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max|z^n + \alpha| \qquad |z| \le r$$

有

$$|z^n + \alpha| \le |z^n| + |\alpha| \le r^n + |\alpha|$$

取最大值 $\Leftrightarrow$ 等号成立 $\Leftrightarrow z^n$ 与 α 同向 $\Leftrightarrow z^n = \lambda\alpha \quad \lambda > 0$
当 $\alpha = 0$ 时

$$|z^n| = |z|^n = r^n$$
$$|z| = r$$
$$\max|\alpha z^n + \beta| = r^n \qquad z = re^i \quad \theta \in [0, 2\pi)$$

当 $\alpha \ne 0$ 时

$$\max|z^n + \alpha| = r^n + |\alpha|$$

此时

$$r^n = |z^n| = |\lambda\alpha|$$
$$\lambda = \frac{|r|^n}{|\alpha|}$$
$$z^n = \lambda\alpha = \frac{|r|^n}{|\alpha|}\alpha$$

设 $\alpha = r_0 e^{i\theta_0}$

$$z^n = \frac{|r|^n}{|\alpha|}\alpha$$
$$= \frac{|r|^n}{|\alpha|}r_0 e^{i\theta_0}$$
$$= |r|^n e^{i\theta_0}$$

则取最大值时

$$z = z_k = re^{\frac{i(\theta_0 + 2k\pi)}{n}} \qquad k = 1, 2, \cdots, n$$

综上，

$$\max|z^n + \alpha| = \begin{cases} r^n & z = re^{i\theta} \quad \theta \in [0, 2\pi) & \alpha = 0 \\ r^n + |\alpha| & z = re^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \cdots, n & \alpha \ne 0 \end{cases}$$

Problem 2

$$|z|^2 = z\overline{z} \qquad z^2 = |z|^2$$

等号成立的条件是?

Solution

$$z^2 = |z|^2 = z\overline{z} \Leftrightarrow z(z - \overline{z}) = 0 \Leftrightarrow z = \overline{z}$$

即 $z \in \mathbb{R}$ 时等号成立。

Problem 3

证明

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

并说明几何意义

Proof.

$$
\begin{aligned}
|z_1 + z_2|^2 + |z_1 - z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\
&= 2(z_1\overline{z_1} + z_2\overline{z_2}) \\
&= 2(|z_1|^2 + |z_2|^2)
\end{aligned}
$$

$\square$

几何意义：平行四边形对角线平方和等于对边平方和

Problem 4

$|z_1| = |z_2| = |z_3| = |z_4| = r$ 且 $z_1 + z_2 + z_3 + z_4 = 0$ 则 z_1, z_2, z_3, z_4 满足什么条件时 $z_1 z_2 z_3 z_4$ 构成正方形?

Solution

定理. $z_1, z_2, \cdots, z_n$ 将圆 $|z| = \alpha n$ 等分 $\Leftrightarrow z_k$ 是分圆多项式 $z^n + \alpha = 0$ 的根

当 z_1, z_2, z_3, z_4 构成正方形时，$(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式。
又

$$
\begin{aligned}
&(z - z_1)(z - z_2)(z - z_3)(z - z_4) \\
=& z^4 - \sum_{k=1}^{4} z_k z^3 + (z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) z^2 \\
& - (z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4) z + z_1 z_2 z_3 z_4
\end{aligned}
$$

$(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{4} z_k & = 0 \\ z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 & = 0 \\ z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 & = 0 \\ z_1 z_2 z_3 z_4 & \neq 0 \end{cases}$$

而

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = z_1 z_2 z_3 z_4 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right)$$

$$z_k \overline{z_k} = |z_k|^2 = r^2$$

所以

$$\frac{1}{z_k} = \frac{\overline{z_k}}{r^2}$$

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = \frac{z_1 z_2 z_3 z_4}{r^2} (\overline{z_1} + \overline{z_2} + \overline{z_3} + \overline{z_4}) = 0$$

所以 z_1, z_2, z_3, z_4 需满足

$$z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = 0$$

Problem 5

$f(z)$ 在 z_0 连续，$f(z_0) \neq 0$。求证 $\exists \delta > 0$，当 $|z - z_0| < \delta$ 时有 $f(z) \neq 0$

Proof. 因 $f(z_0) \neq 0$，有 $|f(z_0)| > 0$
令 $\varepsilon = \frac{1}{2}|f(z_0)| < |f(z_0)|$，因为 $f(z)$ 在 z_0 连续，存在 $\delta > 0$ 使得 $\forall z \quad |z - z_0| < \delta$

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \frac{1}{2}|f(z_0)|$$

即

$$\frac{1}{2}|f(z_0)| < |f(z)| < \frac{3}{2}|f(z_0)|$$

$\square$

Chapter 2 Analytic Functions

定理. *Cauchy-Riemann* 定理 $f(z) = u(x,y) + iv(x,y), \quad u,v \in \mathbb{C}^{(1)}$，则 $f(z)$ 在 $z_0 = x_0 + iy_0$ 点可导
等价于

$$\begin{cases} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{cases}$$

在 $z_0 = x_0 + iy_0$ 点成立，且

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Problem 6

$f(z) = f(x+iy) = u(x,y)+iv(x,y)$ 且 $u,v \in C^{(n)}$，求 $f(z)$ n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution
设 $f'(z) = A + iB$，则

$$\begin{aligned} \mathrm{d}f = f'(z)\mathrm{d}z = f'(z)(\mathrm{d}x + i\mathrm{d}y) \Leftrightarrow \mathrm{d}f &= \mathrm{d}u + i\mathrm{d}v \\ &= \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + i(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y) \\ &= (A\mathrm{d}x - B\mathrm{d}y) + i(B\mathrm{d}x + A\mathrm{d}y) \end{aligned}$$

由上式得

$$\begin{cases} A\mathrm{d}x - B\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \\ B\mathrm{d}x + A\mathrm{d}y = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = A \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -B \end{cases}$$

即

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = F'(z)$$

而

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z) = \frac{\partial^n u}{\partial x^n} + i\frac{\partial^n v}{\partial x^n}$$

u,v 需要满足 Cauchy-Riemann 条件

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Problem 7

求 $\cos(x + iy)$ 的实部和虚部，其中 $x,y \in \mathbb{R}$

5

Solution

$$\begin{aligned}
\cos(x+iy) &= \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\
&= \frac{1}{2}(e^{-y}e^{ix} + e^{y}e^{-ix}) \\
&= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)] \\
&= \frac{1}{2}(e^{y} + e^{-y})\cos x + i\frac{1}{2}(-e^{y} + e^{-y})\sin x
\end{aligned}$$

Problem 8

求证：$\forall A, B \in \mathbb{R}$ 存在 $z = x + iy$ 使得 $\cos(x+iy) = A + iB$（即 $\text{Im}[\cos(z)] = \mathbb{C}$）

Proof. 令

$$\begin{aligned}
\frac{e^{y} + e^{-y}}{2}\cos x &= A \\
\frac{e^{-y} - e^{y}}{2}\sin x &= B
\end{aligned} \tag{1}$$

1. 当 $B = 0$ 时，由式 (1) 知 $y = 0$ 或 $\sin x = 0$。
   $|A| \le 1$ 时可令 $y = 0$，此时

   $$\cos x = A$$

   解得

   $$\begin{cases} x = \arccos A + 2k\pi & k \in \mathbb{Z} \\ y = 0 \end{cases}$$

   $|A| > 1$ 时，令 $\sin x = 0$ 得

   $$\cos x = \pm 1$$
   $$\frac{e^{y} + e^{-y}}{2} = |A| > 1$$

   考察函数 $f(y) = \frac{e^{y}+e^{-y}}{2} - 1$

   $$f(0) = 0$$
   $$\lim_{y \to +\infty} f(y) = \lim_{y \to -\infty} f(y) = +\infty$$

   且 $f(y)$ 连续。因此存在 y_A 使得 $\pm y_A$ 是方程 $\frac{e^{y}+e^{-y}}{2} = |A|$ 的解。
   则 $A > 0$ 时

   $$\begin{cases} x = 2k\pi & k \in \mathbb{Z} \\ y = \pm y_A \end{cases}$$

   $A < 0$ 时

   $$\begin{cases} x = (2k+1)\pi & k \in \mathbb{Z} \\ y = \pm y_A \end{cases}$$

2. 当 $B \neq 0$ 时，由式 (1) 知 $y \neq 0$。结合 $\cos^2 x + \sin^2 x = 1$ 得 $y \in (-\infty, 0) \cup (0, +\infty)$ 时

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

令 $f_{A,B}(y) = \frac{4A^2}{(e^{-y}+e^y)^2} + \frac{4B^2}{(e^{-y}-e^y)^2}$，$f_{A,B}(y)$ 是偶函数。

$$\lim_{y \to 0^+} f_{A,B}(y) = +\infty$$

$$\lim_{y \to +\infty} f_{A,B}(y) = 0$$

因此 $\exists y_{A,B} > 0$，使得 $\pm y_{A_B}$ 是方程

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

的根。将 $\pm y_{A,B}$ 代入式 (1) 可解出对应的 x。

$\square$

Problem 9

已知 $e^w = z \neq 0$，求

$$w = \mathrm{Ln}z$$

Solution

设 $w = u + iv \quad u, v \in \mathbb{R}$，则

$$e^w = e^{u+iv} = e^u e^{iv} = z = re^{i\theta}$$
$$\theta = \arg z \in [0, 2\pi) \qquad r = |z| > 0$$

则

$$\begin{aligned} e^u \ &= r \\ \Rightarrow \ u \ &= \ln r \end{aligned}$$

且

$$\begin{aligned} e^{iv} \ &= e^{i\theta} \\ \Rightarrow \ v \ &= \theta + 2k\pi \qquad k \in \mathbb{Z} \\ &= \arg z \end{aligned}$$

所以

$$\begin{aligned} w = u + iv \ &= \mathrm{Ln}z \\ &= \ln z + 2k\pi i \qquad\qquad k \in \mathbb{Z} \\ &= \ln|z| + i\arg z + 2k\pi i \quad k \in \mathbb{Z} \end{aligned}$$

定理. *Picard 小定理 若 $f(z)$ 是解析函数且 $f(z)$ 不是常数，则除去最多一个例外 w_0，方程 $f(z) = A + iB = w$ 至少有一个解 z。*

Problem 10

求

$$\text{Ln}(3 + 2i)$$

Solution

$$
\begin{aligned}
\text{Ln}(3 + 2i) \quad &= \ln(3 + 2i) + 2k\pi i && k \in \mathbb{Z} \\
&= \ln 13 + i \arg(3 + 2i) + 2k\pi i && k \in \mathbb{Z}
\end{aligned}
$$

Problem 11

求

$$\text{Ln}z^n$$

Solution

$$
\begin{aligned}
\text{Ln}z^n \quad &= \ln z^n + 2k\pi i && k \in \mathbb{Z} \\
&= \ln |z^n| + i \arg z^n + 2k\pi i && k \in \mathbb{Z} \\
&= n \ln |z| + ni \arg z + 2k\pi i && k \in \mathbb{Z} \\
&= n\text{Ln}z
\end{aligned}
$$

Problem 12

求

$$i^{\sqrt{3}i}$$

Solution

$$
\begin{aligned}
i^{\sqrt{3}i} &= e^{\sqrt{3}i\text{Ln}i} \\
&= e^{\sqrt{3}i(\frac{\pi}{2}i + 2k\pi i)} \\
&= e^{-\sqrt{3}(\frac{1}{2} + 2k)\pi} \qquad k \in \mathbb{Z}
\end{aligned}
$$

Chapter 3 Complex Integral

定理. *Cauchy-Goursat* 定理 若 C 分段光滑, 且 $f(z)$ 在 C 上连续, 在 C 内处处可导, 则 $\oint_C f(z)\mathrm{d}z = 0$

定理. *Cauchy* 高阶导数公式

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$$

这里 $C_r = \{z| \ |z - z_0| = r\}$

定理. *Lioville* 定理 有界的解析函数是常数

Problem 13

求证，$f(z)$ 是解析函数则

$$f(z)\text{的像是} \begin{cases} \text{二维区域} & f(z) \not\equiv C \\ \text{点} & f(z) \equiv C \end{cases}$$

Proof. 有

$$J_{(x,y)} = \left(\begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right)\bigg|_{(x,y)}$$

$$\frac{\Delta u \Delta v}{\Delta x \Delta y} = |det J|_{(x,y)} = \left| \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right|_{(x,y)}$$

因为 $f(z) = u + iv$ 是解析函数

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

所以

$$\frac{\Delta u \Delta v}{\Delta x \Delta y} = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 = |f'(x)|^2 \geq 0$$

即

$$\Delta u \Delta v = |f'(x)|^2 \Delta x \Delta y$$

若 $f'(x) \not\equiv 0$, 那么 $f(x)$ 不是常数。此时假设 $f'(z_0) \neq 0$, 则

$$\exists \delta > 0 \ \text{使得} |z - z_0| < \delta \text{时} f'(z) \neq 0$$

$|z - z_0| < \delta$ 时

$$\Delta u \Delta v > 0$$

即像是二维区域。

当 $f'(z) \equiv 0$ 时，$f(z)$ 是常数，这时 $f(z)$ 的像是一个点 □

Chapter 4 Series

定义. 幂级数

$$\sum_{n=0}^{+\infty} C_n(z-z_0)^n$$

定义. *Fourier* 级数

$$\sum_{n=0}^{+\infty} C_n e^{in\theta} = \sum_{n=0}^{+\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

定义. *Taylor* 级数

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

定义. *Laurent* 级数

$$\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n$$

定理. *Abel* 定理 若 $f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 在 z_0 收敛，则 $\forall z$ 有 $|z| < |z_0|$ 时 $f(z)$ 绝对收敛。若存在 z_0，$f(z)$ 在 z_0 发散，则 $\forall z$ 有 $|z| > |z_0|$ 时 $f(z)$ 发散。(即幂级数的收敛域是圆盘)

定义. 收敛半径 若存在常数 $R > 0$，当 $|z| < R$ 时，$f(z)$ 绝对收敛，而当 $|z| > R$ 时，$f(z)$ 发散，这时 R 称为 $f(z)$ 的收敛半径。

定理. 若

$$\lim_{n\to+\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lambda$$

则 $R = \lambda$

定理. 若

$$\lim_{n\to+\infty}\left|\frac{1}{\sqrt[n]{C_n}}\right| = \lambda$$

则 $R = \lambda$

定理. 若 $f(z)$ 只有有限个奇点，则离原点最近的奇点 z_0 的模即为收敛半径。

定理. 若 $f(z)$ 在 z_0 处条件收敛，则 $R = |z_0|$

定理. 若 $f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 满足 $C_n = a_n + ib_n$ $a_n, b_n \in \mathbb{R}$ 且 $\sum_{n=0}^{+\infty} a_n z^n$ 的收敛半径是 R_1，$\sum_{n=0}^{+\infty} b_n z^n$ 的收敛半径是 R_2，则 $R = \min\{R_1, R_2\}$

定理. 当 $|z| < R$ 时，

$$f(z) = \sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}z^n$$

即在收敛圆内，$f(z)$ 处处满足 *Cauchy-Riemann* 条件。根据 *Abel* 定理，收敛圆上处处是奇点。

Problem 14

举出级数在其收敛圆上处处发散、既有发散的点也有收敛的点、处处收敛的例子。
Solution

1. 考察

$$f(z) = \sum_{n=0}^{+\infty} z^n = \frac{1}{1-z} \qquad R = 1$$

   $\forall z, |z| = 1$，$f(z)$ 不存在，即收敛圆上处处发散。

2. 考察

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \qquad R = 1$$

   $f(-1) = -\ln 2$ 但 $f(1) = +\infty$ 发散。更一般的，对 $z = e^{i\theta} \quad \theta \in [0, 2\pi)$ 有

$$f(e^{i\theta}) = \sum_{n=1}^{+\infty} \frac{\cos n\theta}{n} + i\sum_{n=1}^{+\infty} \frac{\sin n\theta}{n}$$
$$= \frac{1}{2}\ln\frac{1}{2(1-\cos\theta)} + i\frac{\pi-\theta}{2}$$

   即收敛圆上除 $z = 1$ 外都收敛。

3. 考察

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^2} \qquad R = 1$$

   因为

$$\sum_{n=1}^{+\infty}\left|\frac{z^n}{n^2}\right| \le \sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$$

   所以

$$\forall z : |z| \le 1$$

   有 $f(z)$ 绝对收敛

Problem 15

将 $\frac{1}{z-b}$ 在 $z_0 = a$ 处展成 Laurent 级数，$a \ne b$

Solution

$$\frac{1}{z-b} = \frac{1}{-(b-a) + (z-a)}$$
$$= \frac{1}{a-b}\frac{1}{1-\frac{z-a}{b-a}}$$
$$= \frac{1}{a-b}\sum_{n=0}^{+\infty}\frac{(z-a)^n}{(b-a)^n}$$
$$= \frac{1}{a-b}\sum_{n=0}^{+\infty}\frac{(-1)(z-a)^n}{(b-a)^{n+1}}$$

条件

$$|\frac{z-a}{b-a}| < 1$$

即

$$0 \le |z-a| < |b-a|$$

Problem 16

求

$$f(z) = \frac{1}{(z-1)(z-2)}$$

的 Laurent 级数

Solution

1. 当 $0 < |z-1| < 1$ 时

$$
\begin{aligned}
f(z) &= \frac{(z-1)-(z-2)}{(z-1)(z-2)} \\
&= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z-1-1} - \frac{1}{z-1} \\
&= -\frac{1}{1-(z-1)} - \frac{1}{z-1} \\
&= -\sum_{n=0}^{+\infty}(z-1)^n - \frac{1}{z-1} \\
&= \sum_{n=-1}^{+\infty} -(z-1)^n \qquad 0 < |z-1| < 1
\end{aligned}
$$

       12

2. 当 $|z - 1| > 1$ 时

$$
\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z-1-1} - \frac{1}{z-1} \\
&= \frac{1}{z-1} \frac{1}{1 - \frac{1}{z-1}} - \frac{1}{z-1} \\
&= \frac{1}{z-1} \sum_{n=0}^{+\infty} (\frac{1}{z-1})^n - \frac{1}{z-1} \\
&= \sum_{n=0}^{+\infty} (\frac{1}{z-1})^{n+1} - \frac{1}{z-1} \\
&= \sum_{n=1}^{+\infty} (\frac{1}{z-1})^{n+1} \\
&= \sum_{n=-\infty}^{-2} (\frac{1}{z-1})^n
\end{aligned}
$$

3. 在 $z = 2$ 处展开同理

Chapter 5 Residues

定义. 对函数 $f(z)$，若 $f(z)$ 在 C 上连续，在 C 内有 n 个奇点 $z_1, z_2, \cdots, z^n$。设 $f(z)$ 在 z_k 附近可以展成 *Laurent* 级数

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n^{(k)}(z - z_k)^n$$

则

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} C_{-1}^{(k)}$$

称 $C_{-1}^{(k)}$ 为 $f(z)$ 在 z_k 点的留数，记作 $\mathrm{Res}[f, z_k]$。即

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f, z_k]$$

定理. 若 z_0 是 f 的一个一阶极点，且 $f(z) = \frac{P(z)}{Q(z)}$，其中 $P(z)$ 与 $Q(z)$ 在 z_0 点解析，$P(z_0) \neq 0, Q(z_0) = 0, Q'(z_0) \neq 0$，则 $\mathrm{Res}[f, z_0] = \frac{P(z_0)}{Q'(z_0)}$

Problem 17

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

Proof.

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^n}, z_k]$$

$$= 2\pi i \sum_{k=1}^{n} \frac{1}{nz_k^{n-1}}$$

$$= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z$$

$$= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}$$

$$= -\frac{2\pi i}{n} \sum_{k=1}^{n} z_k$$

若 $n = 1$，即

$$1 + z^n = 1 + z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z = 2\pi i$$

否则

$$\sum_{k=1}^{n} z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z = 0$$

又

$$
\begin{aligned}
\oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} \mathrm{d}z &= \oint_{|z|=r>1} \frac{(z^{2n}-1)+1}{1+z^n} \mathrm{d}z \\
&= \oint_{|z|=r>1} \frac{(z^n-1)(z^n+1)}{1+z^n} \mathrm{d}z + \oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z \\
&= \oint_{|z|=r>1} (z^n-1) \mathrm{d}z + \oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z
\end{aligned}
$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1} (z^n-1) \mathrm{d}z = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} \mathrm{d}z$$

综上，

$$\oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} \mathrm{d}z = \left\{ \begin{array}{ll} 2\pi i & n=1 \\ 0 & n \geq 2 \end{array} \right.$$

$\square$

Solution 2

Proof. 令 $z = re^{i\theta}$ $0 \leq \theta < \pi, t = \frac{1}{z}$，则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2} \mathrm{d}t \qquad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$
\begin{aligned}
\oint_{|z|=r>1} \frac{1}{1+z^n} \mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^n}} (-\frac{1}{t^2}) \mathrm{d}t \\
&= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} \mathrm{d}t
\end{aligned}
$$

只有一个奇点 $t = 0$。因此

$$
\begin{aligned}
I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t \\
&= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)}\mathrm{d}t = 2\pi i & n = 1 \end{cases} \qquad \text{(Cauchy-Goursat 定理)}
\end{aligned}
$$

$\square$

Problem 18

求积分

$$
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}
$$

Solution

当 $n \leq 0$ 时，根据 Cauchy-Goursat 定理

$$
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = 0
$$

当 $n > 0$ 时

$$
\begin{aligned}
\frac{1-\cos 4z^3}{z^n} &= z^{-n}\left(1 - \sum_{k=0}^{+\infty}(-1)^k\frac{(4z^3)^{2k}}{(2k)!}\right) \\
&= z^{-n}\left(1 - \sum_{k=0}^{+\infty}(-1)^k\frac{4^{2k}z^{6k}}{(2k)!}\right) \\
&= z^{-n}\sum_{k=1}^{+\infty}(-1)^{k-1}\frac{4^{2k}z^{6k}}{(2k)!}
\end{aligned}
$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z = 0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty}C_n z^n$ 中，C_{-1} 对应上式中

$$
6k - n = -1
$$

此时

$$
\begin{aligned}
C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k = \frac{n-1}{6} \\
&= (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

所以 $n > 0$ 且 $n = 6k+1$ 时

$$
\begin{aligned}
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z &= 2\pi i \mathrm{Res}[\frac{1-\cos 4z^3}{z^n}, 0] \\
&= 2\pi i C_{-1} \\
&= 2\pi i (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

16

综上，

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = \begin{cases} 2\pi i(-1)^{\frac{n-7}{6}}\dfrac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n>0 \text{ 且 } n=6k+1, k\in\mathbb{N} \\ 0 & n\leq 0 \text{ 或 } n\neq 6k+1, k\in\mathbb{N} \end{cases}$$

Problem 19

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z$$

Solution

令 $t=\frac{1}{z}$，则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0\leq\theta<\pi \quad \text{积分方向为顺时针}$$

原积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z = -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t$$

$$= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t$$

$$= 2\pi i\mathrm{Res}[\frac{e^t}{t^4(t+1)},0]$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4}(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots)(1-t+t^2-t^3+\dots)$$

其中，t^{-1} 的系数为

$$-1+1-\frac{1}{2!}+\frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z = 2\pi i\mathrm{Res}[\frac{e^t}{t^4(t+1)},0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

Problem 20

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} \qquad a>|b| \quad a,b\in\mathbb{R}$$

17

Solution

令 $z = e^{i\theta}, \theta \in [0, 2\pi)$，注意到：

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

$$\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \frac{\mathrm{d}z}{iz}$$

则原积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \oint_{|z|=1} \frac{\frac{\mathrm{d}z}{iz}}{a + b\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{2}{i} \oint_{|z|=1} \frac{\mathrm{d}z}{bz^2 + 2az + b}$$

当 $b = 0$ 时，原积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a} = \frac{2\pi}{a}$$

当 $b \neq 0$ 时，原积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \frac{2}{i} \oint_{|z|=1} \frac{\mathrm{d}z}{bz^2 + 2az + b}$$

$$= \frac{2}{ib} \oint_{|z|=1} \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1}$$

对方程

$$z^2 + \frac{2a}{b}z + 1 = 0$$

其两根

$$z_1 z_2 = 1$$

且

$$z_{1,2} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

可设 $b > 0$，则

$$z_2 = -\frac{a}{b}\frac{\sqrt{a^2 - b^2}}{b} < -\frac{a}{b} < -1$$

即只有一个奇点 z_1。所以原积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \frac{2}{ib} \oint_{|z|=1} \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1}$$

$$= \frac{2}{ib} 2\pi i \frac{1}{2z_1 + \frac{2a}{b}}$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 21

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1, 1]$ 上可积，则 $\int_0^{2\pi} f(\cos x)\mathrm{d}\theta = \int_0^{2\pi} f(\sin x)\mathrm{d}\theta$

所以

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 22

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Solution

令

$$a = 1 + p^2$$
$$b = -2p$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned}
I_p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \\
&= \frac{2\pi}{\sqrt{a^2 - b^2}} \\
&= \frac{2\pi}{1 - p^2}
\end{aligned}$$

Problem 23

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} > 0$$

Solution

注意到

$$\cos^2 \theta = \frac{\cos 2\theta + 1}{2}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

则

$$
\begin{aligned}
I_{A,B} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2 \dfrac{\cos 2\theta + 1}{2} + B^2 \dfrac{1 - \cos 2\theta}{2}} \\
&= \int_0^{4\pi} \frac{\mathrm{d}t}{(A^2 + B^2) + (A^2 - B^2)\cos t} \\
&= 2\int_0^{2\pi} \frac{\mathrm{d}t}{(A^2 + B^2) + (A^2 - B^2)\cos t} \\
&= 2\frac{2\pi}{\sqrt{(A^2 + B^2)^2 - (A^2 - B^2)^2}} \\
&= \frac{2\pi}{AB}
\end{aligned}
$$

Problem 24

求积分

$$I_n = \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数，因此

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} \\
&= \frac{1}{2}\lim_{R \to +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)}
\end{aligned}
$$

而

$$
\begin{aligned}
&\lim_{R \to +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)} + \int_{|z|=R, \mathrm{Im} z > 0} \frac{1}{(z^2 + a^2)(z^2 + b^2)}\mathrm{d}z \\
&= 2\pi i\left(\mathrm{Res}\left[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|a|\right] + \mathrm{Res}\left[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|b|\right]\right) \\
&= 2\pi i\left(\frac{1}{4(i|a|)^3 + 2i|a|(a^2 + b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2 + b^2) + ab}\right)
\end{aligned}
$$

     20

定理. 若 $P_n(z), Q_m(z)$ 是多项式，且 $\deg P_n = n \leq \deg Q_m - 2 = m - 2$，$Q_m(z)$ 在实轴 $z = x$ 上没有零点，即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$，则

$$\lim_{R \to +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{P_n(z)}{Q_m(z)} \mathrm{d}z = 0$$

所以

$$
\begin{aligned}
I_n &= \frac{1}{2} 2\pi i (\operatorname{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \operatorname{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
&= \pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2)} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2)}) \\
&= \frac{\pi}{2(|a|+|b|)|a||b|}
\end{aligned}
$$

Problem 25

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \lim_{R \to +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \frac{1}{2} 2\pi i \sum_{k=0}^{i} \operatorname{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2} \lim_{R \to +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{1+z^{2n}} \mathrm{d}z \\
&= \frac{1}{2} 2\pi i \sum_{k=0}^{i} \operatorname{Res}[\frac{1}{1+z^{2n}}, z_k]
\end{aligned}
$$

其中

$$z_k^{2n} + 1 = 0, \operatorname{Im} z_k > 0$$

则

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \\
&= \frac{1}{2} 2\pi i \sum_{k=0}^{n} \operatorname{Res}[\frac{1}{1+z^{2n}}, z_k] \\
&= \pi i \sum_{k=1}^{n} \operatorname{Res}[\frac{1}{1+z^{2n}}, z_k] \\
&= \pi i \sum_{k=1}^{n} \frac{1}{2n z_k^{2n-1}} \\
&= -\frac{\pi i}{2n} \sum_{k=1}^{n} z_k
\end{aligned}
$$

21

由

$$z_k^{2n} = -1 = e^{\pi i}$$

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \qquad k = 1, 2, \cdots, 2n$$

所以

$$
\begin{aligned}
I_n &= -\frac{\pi i}{2n} \sum_{k=0}^{i} z_k \\
&= -\frac{\pi i}{2n} \frac{\sum_{k=1}^{n} e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\
&= -\frac{\pi i}{2n} \frac{e^{\frac{\pi i}{n}}(1 - e^{\pi i})}{e^{\frac{\pi i}{2n}}(1 - e^{\frac{\pi i}{n}})} \\
&= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}}
\end{aligned}
$$

令 $\theta = \frac{\pi i}{2n}$, 则

$$
\begin{aligned}
\frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin 2\theta} \\
&= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta} \\
&= \frac{1}{-2i\sin\theta}
\end{aligned}
$$

所以

$$
\begin{aligned}
I_n &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\
&= -\frac{\pi i}{n} \frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin 2\theta} \\
&= -\frac{\pi i}{n} \frac{1}{-2i\sin\theta} \\
&= \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}
\end{aligned}
$$

Problem 26

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Solution

     22

$$
\begin{aligned}
I_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \\
&= \frac{1}{r^{2n}} \int_0^{+\infty} \frac{\mathrm{d}x}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} \int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} I_n
\end{aligned}
$$

Problem 27

求积分

$$
J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \qquad n \in \mathbb{N}
$$

Solution

$$
\begin{aligned}
J_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \\
&= \lim_{R \to +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{\mathrm{d}x}{(1 + x^2)^n} \\
&= \frac{1}{2} 2\pi (i \operatorname{Res}[\frac{1}{(1 + z^2)^n}, i] - \lim_{R \to +\infty} \int_{|z| = R, \operatorname{Im} z > 0} \frac{1}{(1 + z^2)^n} \mathrm{d}z) \\
&= \frac{1}{2} (2\pi i \operatorname{Res}[\frac{1}{(1 + z^2)^n}, i]) \\
&= \pi i \operatorname{Res}[\frac{1}{(1 + z^2)^n}, i]
\end{aligned}
$$

而

$$
\frac{1}{(1 + z^2)^n} = \frac{1}{(z + i)^n (z - i)^n}
$$

令 $f(z) = \frac{1}{(z+i)^n}$

$$
\begin{aligned}
\frac{1}{(1 + z^2)^n} &= \frac{1}{(z + i)^n (z - i)^n} \\
&= (z - i)^{-n} \sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!} (z - i)^k
\end{aligned}
$$

要求 $(z - i)^{-1}$ 对应的系数 C_{-1}，对应于 $k = n - 1$

$$
\begin{aligned}
C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
&= (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
\end{aligned}
$$

     23

因此

$$\begin{aligned}
J_n &= \pi i C_{-1} \\
&= \pi i (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
&= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
\end{aligned}$$

Problem 28

求积分

$$J_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2+x^2)^n} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2+x^2)^n} \\
&= \int_0^{+\infty} \frac{1}{r^{2n}} \frac{r\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n} \\
&= \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n} \\
&= \frac{1}{r^{2n-1}} J_n
\end{aligned}$$

Problem 29

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

设 $a \neq b$ 则

$$\begin{aligned}
I_{a,b,k} &= \int_0^{+\infty} \frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\mathrm{Im} \int_{-\infty}^{+\infty} \frac{xe^{ikx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x
\end{aligned}$$

而

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{xe^{ix}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x &= 2\pi i\{\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)}, ai] + \mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)}, bi]\} \\
&= 2\pi i[\frac{aie^{-ka}}{4(ai)^3 + 2ai(a^2+b^2)} + \frac{bie^{-kb}}{4(bi)^3 + 2bi(a^2+b^2)}] \\
&= \frac{\pi i}{b^2-a^2}(e^{-ka} - e^{-kb})
\end{aligned}$$

所以

$$I_{a,b,k} = \frac{1}{2}\text{Im}\int_{-\infty}^{+\infty}\frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

$$= \frac{1}{2}\text{Im}[\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})]$$

$$= \frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb})$$

$a = b$ 时

$$I_{a,b,k} = \lim_{a\to b}\frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb})$$

$$= \frac{-k\pi e^{-kb}}{-4b}$$

$$= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}}$$

Problem 30

求积分

$$I_{a,b,k} = \int_0^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

$a \neq b$ 时

$$I_{a,b,k} = \int_0^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

$$= \frac{1}{2}\text{Re}\int_{-\infty}^{+\infty}\frac{x^2 e^{ikx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

$$= \frac{(be^{-kb}-ae^{-ka})\pi}{2(b^2-a^2)}$$

当 $a = b$ 时

$$I_{a,b,k} = \frac{(1-ka)\pi}{4ae^{ka}} = \frac{(1-kb)\pi}{4be^{kb}}$$

Chapter 6 Analytic Mappings

定义. $f(z) = \frac{az+b}{cz+d}$ 其中 $a, b, c, d \in \mathbb{C}$ 且 $ad - bc \neq 0$

定理. 分式线性映射保广义圆。(圆 $\cup$ 直线映射到圆 $\cup$ 直线)

定理. 分式线性映射保对称点。即对广义圆，圆心映射到圆心，圆心的对称点映射到无穷，边界映射到边界（可由最大模原理得出）

Problem 31

求分式线性映射 $w = \frac{az+b}{cz+d}$ 将单位圆盘 $|z| < 1$ 映射到单位圆盘 $|w| < 1$ 且使 z_1 映射到 0，这里 $|z_1| < 1$

Solution

$$
\begin{aligned}
w &= \frac{az + b}{cz + d} \\
&= \frac{a(z - (-\frac{b}{a}))}{c(z - (-\frac{d}{c}))}
\end{aligned}
$$

根据题意

$$w(z_1) = 0$$

根据保对称性，z_1 的对称点 $z_1' = \frac{1}{\overline{z_1}}$ 满足

$$w(z_1') = \infty$$

记 $\frac{a}{c} = a'$ 则此时

$$
\begin{aligned}
w &= a' \frac{z - z_1}{z - \frac{1}{\overline{z_1}}} \\
&= (-\overline{z_1} a') \frac{z - z_1}{1 - \overline{z_1} z} \\
&= A\left(\frac{z - z_1}{1 - \overline{z_1} z}\right)
\end{aligned}
$$

根据最大模原理，$|w| = 1$ 时 $|z| = 1$，且 $|z| = 1$ 时

$$1 = \overline{z} z = |z|^2$$

所以 $|w| = 1$ 时

$$
\begin{aligned}
1 = |A| \left| \frac{z - z_1}{1 - \overline{z_1} z} \right| &= |A| \left| \frac{z - z_1}{z\overline{z} - \overline{z_1} z} \right| \\
&= \frac{|A|}{|z|} \frac{|z - z_1|}{|\overline{z} - \overline{z_1}|} \\
&= \frac{|A|}{|z|} \\
&= |A|
\end{aligned}
$$

所以

$$A = e^{i\theta} \qquad \theta \in [0, 2\pi)$$

所以

$$w = e^{i\theta} \frac{z - z_1}{1 - \overline{z_1} z} \qquad \theta \in [0, 2\pi), |z_1| < 1$$

推论. $z_1 = 0$ 时 $w = e^{i\theta}z$ 对应逆时针旋转 θ 角

推论. 不变式 从单位圆盘到单位圆盘的映射满足

$$\frac{|\mathrm{d}w|}{1 - |w|^2} = \frac{|\mathrm{d}z|}{1 - |z|^2}$$

Proof. 因为从单位圆盘到单位圆盘的映射

$$w = \frac{az + b}{cz + d} = e^{i\theta}\frac{z - z_1}{1 - \overline{z_1}z}$$

所以

$$a = e^{i\theta}$$
$$b = -z_1 e^{i\theta}$$
$$c = -\overline{z_1}$$
$$d = 1$$

又

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{ad - bc}{(cz + b)^2}$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1 - |z_1|^2}{|1 - \overline{z_1}z|^2} > 0$$

而

$$
\begin{aligned}
1 - |w|^2 &= 1 - w\overline{w} \\
&= 1 - e^{i\theta}\frac{z - z_1}{1 - \overline{z_1}z}e^{-i\theta}\frac{\overline{z} - \overline{z_1}}{1 - z_1\overline{z}} \\
&= \frac{(1 - \overline{z_1}z)(1 - z_1\overline{z}) - (z - z_1)(\overline{z} - \overline{z_1})}{(1 - \overline{z_1}z)(1 - z_1\overline{z})} \\
&= \frac{1 - z_1\overline{z} - \overline{z_1}z + |z|^2|z_1|^2 - z\overline{z} + z\overline{z_1} + z_1\overline{z} - |z_1|^2}{|z - zz_1|^2} \\
&= \frac{1 - |z_1|^2 - |z|^2 + |zz_1|^2}{|z - zz_1|^2} \\
&= \frac{(1 - |z|^2)(1 - |z_1|^2)}{|z - zz_1|^2}
\end{aligned}
$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1 - |z_1|^2}{|1 - \overline{z_1}z|^2} = \frac{1 - |z_1|^2}{|1 - zz_1|^2} = \frac{1 - |w|^2}{1 - |z|^2}$$

所以

$$\frac{|\mathrm{d}w|}{1 - |w|^2} = \frac{|\mathrm{d}z|}{1 - |z|^2}$$

$\square$

Problem 32

求分式线性映射使 $|z - z_0| < r \longrightarrow |w - w_0| < R$ 且使 $z_1 \to w_0$。这里 $|z_1 - z_0| < r$。

Solution

$|z - z_0| < r$ 到以原点为圆心的单位圆的映射为

$$z' = \frac{z - z_0}{r}$$

此时 z_1 被映射到 z_1'。将单位圆 $|z'| < 1$ 映射到另一个单位圆 $|w'| < 1$ 且使得 z_1' 被映射到原点的映射为

$$w' = e^{i\theta} \frac{z' - z_1'}{1 - \overline{z_1'} z'}$$

从 $|w - w_0| < R$ 到以 z_1 为圆心到单位圆的映射为

$$w' = \frac{w - w_0}{R}$$

所以

$$w' = \frac{w - w_0}{R} = e^{i\theta} \frac{\frac{z - z_0}{r} - \frac{z_1 - z_0}{r}}{1 - \left(\frac{z_1 - z_0}{r}\right)\left(\frac{z_1 - z_0}{r}\right)}$$

解得

$$w = w_0 + e^{i\theta} Rr \frac{z - z_1}{r^2 - \overline{(z_1 - z_0)}(z - z_0)} \qquad \theta \in [0, 2\pi) \quad |z_1 - z_0| < r$$

Problem 33

求证上半复平面 $\mathrm{Im} z > 0$ 映射到单位圆盘 $|w| < 1$ 的分式线性映射为

$$w = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}} \qquad \theta \in [0, 2\pi) \quad \mathrm{Im} z_1 > 0$$

Proof. 根据保对称性，由于 z_1 映射到 0，所以 $z_1' = \overline{z_1}$ 映射到 ∞。所以

$$w = A \frac{z - z_1}{z - \overline{z_1}}$$

当 $z = x \in \mathbb{R}$ 时，根据最大模原理

$$|w(z)| = 1$$

因此

$$\begin{aligned} 1 = |w| &= |A| \left| \frac{x - z_1}{x - \overline{z_1}} \right| \\ &= |A| \left| \frac{x - z_1}{x - z_1} \right| \\ &= |A| \end{aligned}$$

即 $|A| = 1,\ A = e^{i\theta} \quad \theta \in [0, 2\pi)$ 所以

$$w = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}} \qquad \theta \in [0, 2\pi) \quad \mathrm{Im} z_1 > 0$$

$\square$

Problem 34

求证当 a, b, c, $d \in \mathbb{R}$ 时，分式线性映射使得 $\mathrm{Im}z > 0$ 映射到 $\mathrm{Im}w > 0$ 的充要条件是 $ad - bc > 0$

Proof. 令

$$z = x + iy$$
$$x, y \in \mathbb{R}$$
$$w = \frac{az + b}{cz + d}$$

则

$$
\begin{aligned}
w &= u + iv \\
&= \frac{a(x + iy) + b}{c(x + iy) + d} \\
&= \frac{(ax + b) + iay}{(cx + d) + icy} \\
&= \frac{[(ax + b) + iay][(cx + d) - icy]}{[(cx + d) + icy][(cx + d) - icy]}
\end{aligned}
$$

而

$$v = \mathrm{Im}w = \frac{(ad - bc)y}{(cx + d)^2 + c^2 y^2}$$

y 与 v 同号 $\Leftrightarrow ad - bc > 0$ □

Problem 35

$L = ax + b$ 是与 x 轴交于 x_0 点，与 x 轴夹角为 α 的直线，其中 $\alpha \in [0, \pi)$，$x_0 \in \mathbb{R}$。$M = \{z = u + iv | v > ua + b\}$，求将半平面 M 映到 $|w - w_0| < R$ 的分式线性映射。

Solution

将 $M = \{z = u + iv | v > ua + b\}$ 映射到 $z > 0$ 的映射为

$$z_1 = (z - x_0)e^{-i\alpha}$$

设半平面 M 中 z_0 点被所求分式线性映射映射到 w_0，z_0 被 z_1 映射到 z_1^0
将 $|w - w_0| < R$ 映射到单位圆的映射为

$$w_1 = \frac{w - w_0}{R}$$

将 z > 0 映射到单位圆盘的映射为

$$w_1 = e^{i\theta}\frac{z_1 - z_1^0}{z_1 - \overline{z_1^0}}$$

即

$$\frac{w - w_0}{R} = e^{i\theta}\frac{(z - x_0)e^{-i\alpha} - (z_0 - x_0)e^{-i\alpha}}{(z - x_0)e^{-i\alpha} - \overline{(z_0 - x_0)e^{-i\alpha}}}$$

即

$$w = w_0 + Re^{i\theta}\frac{z - z_0}{(z - x_0) - \overline{(z_0 - x_0)}e^{2i\alpha}}$$

这里 $\theta \in [0, 2\pi)$，$z_1 \in M$。
取 $\theta = 0$，则

$$w = w_0 + R\frac{z - z_0}{(z - x_0) - \overline{(z_0 - x_0)}e^{2i\alpha}}$$

Problem 36

M 是与 x 轴夹角为 θ，与 x 轴交于 x_1, x_2 点的条带，其中 $x_1 < x_2$，$0 < \theta < 2\pi$。求一个单值可导映射将 M 映射到单位圆盘。

Solution
将 M 映射到 $N = \{z | z \in \mathbb{C}, \operatorname{Im}z > 0 \wedge \operatorname{Im}z < h = (x_2 - x_1)\sin\theta\}$ 且将 x_2 映射到原点的映射为

$$z_1 = (z - x_2)e^{-i\theta}$$

将 N 映射到 $O = \{z | z \in \mathbb{C}, \operatorname{Im}z > 0 \wedge \operatorname{Im}z < \pi\}$ 的映射为

$$z_2 = \frac{z_1 \pi}{h} = \frac{(z - x_2)e^{-i\theta}\pi}{(x_2 - x_1)\sin\theta}$$

将 O 映射到 $\operatorname{Im}z > 0$ 的映射为

$$z_3 = e^{z_2}$$

将 $\operatorname{Im}z > 0$ 映射到单位圆盘的一个映射为

$$
\begin{aligned}
w &= \frac{z_3 - i}{z_3 + i} \\
&= \frac{e^{z_2} - i}{e^{z_3} + i} \\
&= \frac{e^{\frac{(z-x_2)e^{-i\theta}\pi}{(x_2-x_1)\sin\theta}} - i}{e^{\frac{(z-x_2)e^{-i\theta}\pi}{(x_2-x_1)\sin\theta}} + i}
\end{aligned}
$$

Problem 37

求区域 $D = \{|z-a| > a$ 与 $|z-b| < b\}$ 之间的部分，这里 $0 < a < b\}$ 到单位圆盘 $|w| < 1$ 的单值解析映射。

Solution
根据分式映射的保圆性，映射

$$z_1 = \frac{z - 2a}{z}$$

将 D 映射到 $E = \{z | z \in \mathbb{C}, \operatorname{Re}z > 0 \wedge \operatorname{Re}z < \frac{b-a}{b}\}$，且将 $z = 2a$ 映射到原点。而

$$z_2 = iz_1\frac{\pi}{\frac{b-a}{b}} = \frac{bi\pi}{b-a}\left(\frac{z-2a}{z}\right)$$

将 E 映射到 $F = \{z | z \in \mathbb{C}, \operatorname{Im}z > 0 \wedge \operatorname{Im}z < \pi\}$
将 F 映射到 $\operatorname{Im}z > 0$ 的映射为

$$z_3 = e^{z_2}$$

     30

将 $\mathrm{Im}z > 0$ 映射到单位圆盘的一个映射为

$$w = \frac{z_3 - i}{z_3 + i} = \frac{e^{\frac{bi\pi}{b-a}(\frac{z-2a}{z})} - i}{e^{\frac{bi\pi}{b-a}(\frac{z-2a}{z})} + i}$$

Problem 38

区域 $D = \{$弦切角为α，弦为AB的扇形，其中$0 < \alpha < \pi\}$，求将 D 映射到单位圆盘 $|w| < 1$ 的单值解析映射。

Solution
分式线性映射

$$z_1 = -\frac{z - A}{z - B}$$

将 D 映射为 $E = \{z = re^{i\theta} | z \in \mathbb{C},\ r > 0 \wedge 0 < \theta < \alpha\}$。而映射

$$z_2 = z_2^{\frac{\pi}{\alpha}}$$

将 E 映射为 $\mathrm{Im}z > 0$。将 $\mathrm{Im}z > 0$ 映射为单位圆盘的一个映射为

$$w = \frac{z_2 - i}{z_2 + i} = \frac{\left(-\frac{z-A}{z-B}\right)^{\frac{\pi}{\alpha}} - i}{\left(-\frac{z-A}{z-B}\right)^{\frac{\pi}{\alpha}} + i}$$

Problem 39

求将区域 $D = \{z = re^{i\theta} | z \in \mathbb{C},\ r > 0 \wedge 0 < \theta < \alpha\ 0 < \alpha < \pi\}$ 映射到圆盘 $\{w|\ |w - w_0| < R\}$ 的一个单值解析映射。

Solution
映射

$$z_1 = z^{\frac{\pi}{\alpha}}$$

将 D 映射到上半复平面。映射

$$w_1 = \frac{z_1 - i}{z_1 + i}$$

将上半复平面映射到单位圆盘。而

$$w_1 = \frac{w - w_0}{R}$$

将圆盘 $\{w|\ |w - w_0| < R\}$ 映射到单位圆盘。因此

$$w = w_0 + R\frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}$$

特例：由复平面第一象限（$\alpha = \frac{\pi}{2}$）到圆盘 $\{w|\ |w - w_0| < R\}$ 的一个单值解析映射为

$$w = w_0 + R\frac{z^2 - i}{z^2 + i}$$