Problems

Problem 1

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^n}, z_k] \\ &= 2\pi i \sum_{k=1}^{n} \frac{1}{n z_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}} \\ &= -\frac{2\pi i}{n} \sum_{k=1}^{n} z_k\end{aligned}$$

若 $n=1$，即

$$1+z^n = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 2\pi i$$

否则

$$\sum_{k=1}^{n} z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 0$$

Problem 1 continued on next page...

所以

$$\begin{aligned}\frac{z^{2n}}{1+z^n} &= \frac{(z^{2n}-1)+1}{1+z^n} \\ &= \frac{(z^n-1)(z^n+1)}{1+z^n} + \frac{1}{1+z^n} \\ &= z^n - 1 + \frac{1}{1+z^n}\end{aligned}$$

又

$$\oint_{|z|=r>1} z^n - 1 = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Problem 2

求积分

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}$$

Problem 3

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z$$

Problem 4

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta}$$

Problem 5

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\sin\theta}$$

Problem 6

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Problem 7

求积分

$$I_{A,B} = \int_0^{2\pi} \mathrm{d}\theta A^2 \cos^2\theta + B^2 \sin^2\theta \qquad A, B \in \mathbb{R}\mathrm{and}A, B > 0$$

Problem 8

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2 \cos^2\theta + B^2 \sin^2\theta} \qquad A, B \in \mathbb{R}\mathrm{and}AB > 0$$

Problem 9

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1 + x^{2n}} \qquad n \in \mathbb{N}$$

Problem 10

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Problem 11

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 12

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 13

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x$$

Problem 14

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x$$