

Introduction to Complex Analysis: Recap

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Chapter 1 Introduction

公式. Euler 公式

$$e^{ix} = \cos x + i \sin x$$

Problem 1

求

$$\max |\alpha z^n + \beta| \quad |z| \leq r$$

Solution

$$\max |\alpha z^n + \beta| = |\alpha| \max |z^n + \frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max |z^n + \alpha| \quad |z| \leq r$$

有

$$|z^n + \alpha| \leq |z^n| + |\alpha| \leq r^n + |\alpha|$$

取最大值 \Leftrightarrow 等号成立 $\Leftrightarrow z^n$ 与 α 同向 $\Leftrightarrow z^n = \lambda \alpha \quad \lambda > 0$

当 $\alpha = 0$ 时

$$|z^n| = |z|^n = r^n$$

$$|z| = r$$

$$\max |\alpha z^n + \beta| = r^n \quad z = r e^{i\theta} \quad \theta \in [0, 2\pi)$$

当 $\alpha \neq 0$ 时

$$\max |z^n + \alpha| = r^n + |\alpha|$$

此时

$$r^n = |z^n| = |\lambda \alpha|$$

$$\lambda = \frac{|r|^n}{|\alpha|}$$

$$z^n = \lambda \alpha = \frac{|r|^n}{|\alpha|} \alpha$$

设 $\alpha = r_0 e^{i\theta_0}$

$$\begin{aligned} z^n &= \frac{|r|^n}{|\alpha|} \alpha \\ &= \frac{|r|^n}{|\alpha|} r_0 e^{i\theta_0} \\ &= |r|^n e^{i\theta_0} \end{aligned}$$

则取最大值时

$$z = z_k = r e^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \dots, n$$

综上,

$$\max |z^n + \alpha| = \begin{cases} r^n & z = r e^{i\theta} \quad \theta \in [0, 2\pi) & \alpha = 0 \\ r^n + |\alpha| & z = r e^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \dots, n & \alpha \neq 0 \end{cases}$$

Problem 2

$$|z|^2 = z\bar{z} \quad z^2 = |z|^2$$

等号成立的条件是?

Solution

$$z^2 = |z|^2 = z\bar{z} \Leftrightarrow z(z - \bar{z}) = 0 \Leftrightarrow z = \bar{z}$$

即 $z \in \mathbb{R}$ 时等号成立。

Problem 3

证明

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

并说明几何意义

Proof.

$$\begin{aligned} |z_1 + z_2|^2 + |z_1 - z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= 2(z_1\bar{z}_1 + z_2\bar{z}_2) \\ &= 2(|z_1|^2 + |z_2|^2) \end{aligned}$$

□

几何意义：平行四边形对角线平方和等于对边平方和

Problem 4

$|z_1| = |z_2| = |z_3| = |z_4| = r$ 且 $z_1 + z_2 + z_3 + z_4 = 0$ 则 z_1, z_2, z_3, z_4 满足什么条件时 $z_1 z_2 z_3 z_4$ 构成正方形?

Solution

定理. z_1, z_2, \dots, z_n 将圆 $|z| = \alpha$ 等分 $\Leftrightarrow z_k$ 是分圆多项式 $z^n + \alpha = 0$ 的根

当 z_1, z_2, z_3, z_4 构成正方形时, $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式。
又

$$\begin{aligned} &(z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= z^4 - \sum_{k=1}^4 z_k z^3 + (z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) z^2 \\ &\quad - (z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4) z + z_1 z_2 z_3 z_4 \end{aligned}$$

$(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式

$$\Leftrightarrow \begin{cases} \sum_{k=1}^4 z_k &= 0 \\ z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 &= 0 \\ z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 &= 0 \\ z_1 z_2 z_3 z_4 &\neq 0 \end{cases}$$

而

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = z_1 z_2 z_3 z_4 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right)$$

$$z_k \overline{z_k} = |z_k|^2 = r^2$$

所以

$$\frac{1}{z_k} = \frac{\overline{z_k}}{r^2}$$

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = \frac{z_1 z_2 z_3 z_4}{r^2} (\overline{z_1} + \overline{z_2} + \overline{z_3} + \overline{z_4}) = 0$$

所以 z_1, z_2, z_3, z_4 需满足

$$z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = 0$$

Problem 5

$f(z)$ 在 z_0 连续, $f(z_0) \neq 0$ 。求证 $\exists \delta > 0$, 当 $|z - z_0| < \delta$ 时有 $f(z) \neq 0$

Proof. 因 $f(z_0) \neq 0$, 有 $|f(z_0)| > 0$

令 $\varepsilon = \frac{1}{2}|f(z_0)| < |f(z_0)|$, 因为 $f(z)$ 在 z_0 连续, 存在 $\delta > 0$ 使得 $\forall z \quad |z - z_0| < \delta$

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \frac{1}{2}|f(z_0)|$$

即

$$\frac{1}{2}|f(z_0)| < |f(z)| < \frac{3}{2}|f(z_0)|$$

□

Chapter 2 Analytic Functions

定理. *Cauchy-Riemann* 定理 $f(z) = u(x, y) + iv(x, y)$, $u, v \in \mathbb{C}^{(1)}$, 则 $f(z)$ 在 $z_0 = x_0 + iy_0$ 点可导等价于

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

在 $z_0 = x_0 + iy_0$ 点成立, 且

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}$$

Problem 6

$f(z) = f(x+iy) = u(x, y) + iv(x, y)$ 且 $u, v \in C^{(n)}$, 求 $f(z)$ n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z) = A + iB$, 则

$$\begin{aligned} df = f'(z)dz = f'(z)(dx + idy) &\Leftrightarrow df = du + idv \\ &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\ &= (Adx - Bdy) + i(Bdx + Ady) \end{aligned}$$

由上式得

$$\begin{cases} Adx - Bdy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ Bdx + Ady = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = A \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -B \end{cases}$$

即

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = F'(z)$$

而

$$F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z) = \frac{\partial^n u}{\partial x^n} + i \frac{\partial^n v}{\partial x^n}$$

u, v 需要满足 Cauchy-Riemann 条件

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Problem 7

求 $\cos(x + iy)$ 的实部和虚部, 其中 $x, y \in \mathbb{R}$

Solution

$$\begin{aligned}
\cos(x + iy) &= \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\
&= \frac{1}{2}(e^{-y}e^{ix} + e^ye^{-ix}) \\
&= \frac{1}{2}[e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x)] \\
&= \frac{1}{2}(e^y + e^{-y}) \cos x + i \frac{1}{2}(-e^y + e^{-y}) \sin x
\end{aligned}$$

Problem 8

求证: $\forall A, B \in \mathbb{R}$ 存在 $z = x + iy$ 使得 $\cos(x + iy) = A + iB$ (即 $\text{Im}[\cos(z)] = \mathbb{C}$)

Proof. 令

$$\begin{aligned}
\frac{e^y + e^{-y}}{2} \cos x &= A \\
\frac{e^{-y} - e^y}{2} \sin x &= B
\end{aligned} \tag{1}$$

1. 当 $B = 0$ 时, 由式 (1) 知 $y = 0$ 或 $\sin x = 0$ 。

$|A| \leq 1$ 时可令 $y = 0$, 此时

$$\cos x = A$$

解得

$$\begin{cases} x = \arccos A + 2k\pi & k \in \mathbb{Z} \\ y = 0 \end{cases}$$

$|A| > 1$ 时, 令 $\sin x = 0$ 得

$$\begin{aligned}
\cos x &= \pm 1 \\
\frac{e^y + e^{-y}}{2} &= |A| > 1
\end{aligned}$$

考察函数 $f(y) = \frac{e^y + e^{-y}}{2} - 1$

$$f(0) = 0$$

$$\lim_{y \rightarrow +\infty} f(y) = \lim_{y \rightarrow -\infty} f(y) = +\infty$$

且 $f(y)$ 连续。因此存在 y_A 使得 $\pm y_A$ 是方程 $\frac{e^y + e^{-y}}{2} = |A|$ 的解。

则 $A > 0$ 时

$$\begin{cases} x = 2k\pi & k \in \mathbb{Z} \\ y = \pm y_A \end{cases}$$

$A < 0$ 时

$$\begin{cases} x = 2k + 1\pi & k \in \mathbb{Z} \\ y = \pm y_A \end{cases}$$

2. 当 $B \neq 0$ 时, 由式 (1) 知 $y \neq 0$ 。结合 $\cos^2 x + \sin^2 x = 1$ 得 $y \in (-\infty, 0) \cup (0, +\infty)$ 时

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

令 $f_{A,B}(y) = \frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2}$, $f_{A,B}(y)$ 是偶函数。

$$\lim_{y \rightarrow 0^+} f_{A,B}(y) = +\infty$$

$$\lim_{y \rightarrow +\infty} f_{A,B}(y) = 0$$

因此 $\exists y_{A,B} > 0$, 使得 $\pm y_{A,B}$ 是方程

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

的根。将 $\pm y_{A,B}$ 代入式 (1) 可解出对应的 x 。

□

Problem 9

已知 $e^w = z \neq 0$, 求

$$w = \operatorname{Ln} z$$

Solution

设 $w = u + iv$ $u, v \in \mathbb{R}$, 则

$$e^w = e^{u+iv} = e^u e^{iv} = z = r e^{i\theta}$$

$$\theta = \arg z \in [0, 2\pi) \quad r = |z| > 0$$

则

$$e^u = r$$

$$\Rightarrow u = \ln r$$

且

$$e^{iv} = e^{i\theta}$$

$$\Rightarrow v = \theta + 2k\pi \quad k \in \mathbb{Z}$$

$$= \arg z$$

所以

$$w = u + iv = \operatorname{Ln} z$$

$$= \ln z + 2k\pi i \quad k \in \mathbb{Z}$$

$$= \ln |z| + i \arg z + 2k\pi i \quad k \in \mathbb{Z}$$

定理. *Picard* 小定理 若 $f(z)$ 是解析函数且 $f(z)$ 不是常数, 则除去最多一个例外 w_0 , 方程 $f(z) = A + iB = w$ 至少有一个解 z 。

Problem 10

求

$$\operatorname{Ln}(3 + 2i)$$

Solution

$$\begin{aligned}\operatorname{Ln}(3 + 2i) &= \ln(3 + 2i) + 2k\pi i & k \in \mathbb{Z} \\ &= \ln 13 + i \arg(3 + 2i) + 2k\pi i & k \in \mathbb{Z}\end{aligned}$$

Problem 11

求

$$\operatorname{Ln} z^n$$

Solution

$$\begin{aligned}\operatorname{Ln} z^n &= \ln z^n + 2k\pi i & k \in \mathbb{Z} \\ &= \ln |z^n| + i \arg z^n + 2k\pi i & k \in \mathbb{Z} \\ &= n \ln |z| + ni \arg z + 2k\pi i & k \in \mathbb{Z} \\ &= n \operatorname{Ln} z\end{aligned}$$

Problem 12

求

$$i^{\sqrt{3}i}$$

Solution

$$\begin{aligned}i^{\sqrt{3}i} &= e^{\sqrt{3}i \operatorname{Ln} i} \\ &= e^{\sqrt{3}i(\frac{\pi}{2}i + 2k\pi i)} \\ &= e^{-\sqrt{3}(\frac{1}{2} + 2k)\pi} & k \in \mathbb{Z}\end{aligned}$$

Chapter 3 Complex Integral

定理. *Cauchy-Goursat* 定理 若 C 分段光滑, 且 $f(z)$ 在 C 上连续, 在 C 内处处可导, 则 $\oint_C f(z)dz = 0$

定理. *Cauchy* 高阶导数公式

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

这里 $C_r = \{z \mid |z - z_0| = r\}$

定理. *Lioville* 定理 有界的解析函数是常数

Problem 13

求证, $f(z)$ 是解析函数则

$$f(z) \text{ 的像是 } \begin{cases} \text{二维区域} & f(z) \not\equiv C \\ \text{点} & f(z) \equiv C \end{cases}$$

Proof. 有

$$J_{(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \bigg|_{(x,y)}$$

$$\frac{\Delta u \Delta v}{\Delta x \Delta y} = |\det J|_{(x,y)} = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right|_{(x,y)}$$

因为 $f(z) = u + iv$ 是解析函数

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

所以

$$\frac{\Delta u \Delta v}{\Delta x \Delta y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(x)|^2 \geq 0$$

即

$$\Delta u \Delta v = |f'(x)|^2 \Delta x \Delta y$$

若 $f'(x) \not\equiv 0$, 那么 $f(x)$ 不是常数。此时假设 $f'(z_0) \neq 0$, 则

$$\exists \delta > 0 \text{ 使得 } |z - z_0| < \delta \text{ 时 } f'(z) \neq 0$$

$|z - z_0| < \delta$ 时

$$\Delta u \Delta v > 0$$

即像是二维区域。

当 $f'(z) \equiv 0$ 时, $f(z)$ 是常数, 这时 $f(z)$ 的像是一个点

□

Chapter 4 Series

定义. 幂级数

$$\sum_{n=0}^{+\infty} C_n (z - z_0)^n$$

定义. *Fourier* 级数

$$\sum_{n=0}^{+\infty} C_n e^{in\theta} = \sum_{n=0}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

定义. *Taylor* 级数

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

定义. *Laurent* 级数

$$\sum_{n=-\infty}^{+\infty} C_n (z - z_0)^n$$

定理. *Abel* 定理 若 $f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 在 z_0 收敛, 则 $\forall z$ 有 $|z| < |z_0|$ 时 $f(z)$ 绝对收敛. 若存在 z_0 , $f(z)$ 在 z_0 发散, 则 $\forall z$ 有 $|z| > |z_0|$ 时 $f(z)$ 发散. (即幂级数的收敛域是圆盘)

定义. 收敛半径 若存在常数 $R > 0$, 当 $|z| < R$ 时, $f(z)$ 绝对收敛, 而当 $|z| > R$ 时, $f(z)$ 发散, 这时 R 称为 $f(z)$ 的收敛半径.

定理. 若

$$\lim_{n \rightarrow +\infty} \left| \frac{C_n}{C_{n+1}} \right| = \lambda$$

则 $R = \lambda$

定理. 若

$$\lim_{n \rightarrow +\infty} \left| \frac{1}{\sqrt[n]{C_n}} \right| = \lambda$$

则 $R = \lambda$

定理. 若 $f(z)$ 只有有限个奇点, 则离原点最近的奇点 z_0 的模即为收敛半径.

定理. 若 $f(z)$ 在 z_0 处条件收敛, 则 $R = |z_0|$

定理. 若 $f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 满足 $C_n = a_n + b_n$ $a_n, b_n \in \mathbb{R}$ 且 $\sum_{n=0}^{+\infty} a_n z^n$ 的收敛半径是 R_1 , $\sum_{n=0}^{+\infty} b_n z^n$ 的收敛半径是 R_2 , 则 $R = \min\{R_1, R_2\}$

定理. 当 $|z| < R$ 时,

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$$

即在收敛圆内, $f(z)$ 处处满足 *Cauchy-Riemann* 条件. 根据 *Abel* 定理, 收敛圆上处处是奇点.

Problem 14

举出级数在其收敛圆上处处发散、既有发散的点也有收敛的点、处处收敛的例子。

Solution

1. 考察

$$f(z) = \sum_{n=0}^{+\infty} z^n = \frac{1}{1-z} \quad R=1$$

$\forall z, |z|=1$, $f(z)$ 不存在, 即收敛圆上处处发散。

2. 考察

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \quad R=1$$

$f(-1) = -\ln 2$ 但 $f(1) = +\infty$ 发散。更一般的, 对 $z = e^{i\theta}$ $\theta \in [0, 2\pi)$ 有

$$\begin{aligned} f(e^{i\theta}) &= \sum_{n=1}^{+\infty} \frac{\cos n\theta}{n} + i \sum_{n=1}^{+\infty} \frac{\sin n\theta}{n} \\ &= \frac{1}{2} \ln \frac{1}{2(1-\cos \theta)} + i \frac{\pi - \theta}{2} \end{aligned}$$

即收敛圆上除 $z=1$ 外都收敛。

3. 考察

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^2} \quad R=1$$

因为

$$\sum_{n=1}^{+\infty} \left| \frac{z^n}{n^2} \right| \leq \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$$

所以

$$\forall z : |z| \leq 1$$

有 $f(z)$ 绝对收敛

Problem 15

将 $\frac{1}{z-b}$ 在 $z_0 = a$ 处展成 Laurent 级数, $a \neq b$

Solution

$$\begin{aligned} \frac{1}{z-b} &= \frac{1}{-(b-a) + (z-a)} \\ &= \frac{1}{a-b} \frac{1}{1 - \frac{z-a}{b-a}} \\ &= \frac{1}{a-b} \sum_{n=0}^{+\infty} \frac{(z-a)^n}{(b-a)^n} \\ &= \frac{1}{a-b} \sum_{n=0}^{+\infty} \frac{(-1)^n (z-a)^n}{(b-a)^{n+1}} \end{aligned}$$

条件

$$\left| \frac{z-a}{b-a} \right| < 1$$

即

$$0 \leq |z-a| < |b-a|$$

Problem 16

求

$$f(z) = \frac{1}{(z-1)(z-2)}$$

的 Laurent 级数

Solution

1. 当 $0 < |z-1| < 1$ 时

$$\begin{aligned} f(z) &= \frac{(z-1) - (z-2)}{(z-1)(z-2)} \\ &= \frac{1}{z-2} - \frac{1}{z-1} \\ &= \frac{1}{z-1-1} - \frac{1}{z-1} \\ &= -\frac{1}{1-(z-1)} - \frac{1}{z-1} \\ &= -\sum_{n=0}^{+\infty} (z-1)^n - \frac{1}{z-1} \\ &= \sum_{n=-1}^{+\infty} -(z-1)^n \quad 0 < |z-1| < 1 \end{aligned}$$

2. 当 $|z - 1| > 1$ 时

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\ &= \frac{1}{z-1-1} - \frac{1}{z-1} \\ &= \frac{1}{z-1} \frac{1}{1 - \frac{1}{z-1}} - \frac{1}{z-1} \\ &= \frac{1}{z-1} \sum_{n=0}^{+\infty} \left(\frac{1}{z-1}\right)^n - \frac{1}{z-1} \\ &= \sum_{n=0}^{+\infty} \left(\frac{1}{z-1}\right)^{n+1} - \frac{1}{z-1} \\ &= \sum_{n=1}^{+\infty} \left(\frac{1}{z-1}\right)^{n+1} \\ &= \sum_{n=-\infty}^{-2} \left(\frac{1}{z-1}\right)^n \end{aligned}$$

3. 在 $z = 2$ 处展开同理

Chapter 5 Residues

定义. 对函数 $f(z)$, 若 $f(z)$ 在 C 上连续, 在 C 内有 n 个奇点 z_1, z_2, \dots, z_n . 设 $f(z)$ 在 z_k 附近可以展成 *Laurent* 级数

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n^{(k)} (z - z_k)^n$$

则

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n C_{-1}^{(k)}$$

称 $C_{-1}^{(k)}$ 为 $f(z)$ 在 z_k 点的留数, 记作 $\text{Res}[f, z_k]$. 即

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f, z_k]$$

定理. 若 z_0 是 f 的一个一阶极点, 且 $f(z) = \frac{P(z)}{Q(z)}$, 其中 $P(z)$ 与 $Q(z)$ 在 z_0 点解析, $P(z_0) \neq 0, Q(z_0) = 0, Q'(z_0) \neq 0$, 则 $\text{Res}[f, z_0] = \frac{P(z_0)}{Q'(z_0)}$

Problem 17

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

Proof.

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= 2\pi i \sum_{k=1}^n \text{Res}\left[\frac{1}{1+z^n}, z_k\right] \\ &= 2\pi i \sum_{k=1}^n \frac{1}{nz_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned} &\oint_{|z|=r>1} \frac{1}{1+z^n} dz \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \\ &= -\frac{2\pi i}{n} \sum_{k=1}^n z_k \end{aligned}$$

若 $n = 1$, 即

$$1 + z^n = 1 + z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 2\pi i$$

否则

$$\sum_{k=1}^n z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 0$$

又

$$\begin{aligned} \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz &= \oint_{|z|=r>1} \frac{(z^{2n}-1)+1}{1+z^n} dz \\ &= \oint_{|z|=r>1} \frac{(z^n-1)(z^n+1)}{1+z^n} dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz \\ &= \oint_{|z|=r>1} (z^n-1) dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz \end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1} (z^n-1) dz = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz$$

综上,

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

□

Solution 2

Proof. 令 $z = re^{i\theta}$ $0 \leq \theta < \pi, t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= - \oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^n}} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt \end{aligned}$$

只有一个奇点 $t = 0$ 。因此

$$\begin{aligned} I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt \\ &= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)} dt = 2\pi i & n = 1 \end{cases} \quad (\text{Cauchy-Goursat 定理}) \end{aligned}$$

□

Problem 18

求积分

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz \quad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时, 根据 Cauchy-Goursat 定理

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = 0$$

当 $n > 0$ 时

$$\begin{aligned} \frac{1 - \cos 4z^3}{z^n} &= z^{-n} \left(1 - \sum_{k=0}^{+\infty} (-1)^k \frac{(4z^3)^{2k}}{(2k)!} \right) \\ &= z^{-n} \left(1 - \sum_{k=0}^{+\infty} (-1)^k \frac{4^{2k} z^{6k}}{(2k)!} \right) \\ &= z^{-n} \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{4^{2k} z^{6k}}{(2k)!} \end{aligned}$$

$\frac{1 - \cos 4z^3}{z^n}$ 在奇点 $z = 0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中, C_{-1} 对应上式中

$$6k - n = -1$$

此时

$$\begin{aligned} C_{-1} &= (-1)^{k-1} \frac{4^{2k}}{(2k)!} \quad k = \frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

所以 $n > 0$ 且 $n = 6k + 1$ 时

$$\begin{aligned} \oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz &= 2\pi i \text{Res}\left[\frac{1 - \cos 4z^3}{z^n}, 0\right] \\ &= 2\pi i C_{-1} \\ &= 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

综上,

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = \begin{cases} 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n > 0 \text{ 且 } n = 6k + 1, k \in \mathbb{N} \\ 0 & n < 0 \text{ 或 } n \neq 6k + 1, k \in \mathbb{N} \end{cases}$$

Problem 19

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz$$

Solution

令 $t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned} \oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz &= - \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)} dt \\ &= 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] \end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) (1 - t + t^2 - t^3 + \dots)$$

其中, t^{-1} 的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz = 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] = 2\pi i \left(-\frac{1}{3}\right) = -\frac{2}{3}\pi i$$

Problem 20

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}$, $\theta \in [0, 2\pi)$, 注意到:

$$\begin{aligned}\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z} \\ d\theta &= \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}\end{aligned}$$

则原积分

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{a + b(\frac{z^2+1}{2z})} \\ &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b}\end{aligned}$$

当 $b = 0$ 时, 原积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \int_0^{2\pi} \frac{d\theta}{a} = \frac{2\pi}{a}$$

当 $b \neq 0$ 时, 原积分

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b} \\ &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1}\end{aligned}$$

对方程

$$z^2 + \frac{2a}{b}z + 1 = 0$$

其两根

$$z_1 z_2 = 1$$

且

$$z_{1,2} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

可设 $b > 0$, 则

$$z_2 = -\frac{a}{b} \frac{\sqrt{a^2 - b^2}}{b} < -\frac{a}{b} < -1$$

即只有一个奇点 z_1 。所以原积分

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1} \\ &= \frac{2}{ib} 2\pi i \frac{1}{2z_1 + \frac{2a}{b}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}}\end{aligned}$$

Problem 21

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1, 1]$ 上可积, 则 $\int_0^{2\pi} f(\cos x) d\theta = \int_0^{2\pi} f(\sin x) d\theta$

所以

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 22

求积分

$$I_p = \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \quad p \in (-1, 1)$$

Solution

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \\ &= \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

Problem 23

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \quad A, B \in \mathbb{R} > 0$$

Solution

注意到

$$\begin{aligned}\cos^2 \theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2}\end{aligned}$$

则

$$\begin{aligned}I_{A,B} &= \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \\ &= \int_0^{2\pi} \frac{d\theta}{A^2 \frac{\cos 2\theta + 1}{2} + B^2 \frac{1 - \cos 2\theta}{2}} \\ &= \int_0^{4\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\ &= 2 \int_0^{2\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\ &= 2 \frac{2\pi}{\sqrt{(A^2 + B^2)^2 - (A^2 - B^2)^2}} \\ &= \frac{2\pi}{AB}\end{aligned}$$

Problem 24

求积分

$$I_n = \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数, 因此

$$\begin{aligned}I_n &= \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} \\ &= \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)}\end{aligned}$$

而

$$\begin{aligned}&\lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)} + \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{(z^2 + a^2)(z^2 + b^2)} dz \\ &= 2\pi i (\operatorname{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|a|] + \operatorname{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|b|]) \\ &= 2\pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2 + b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2 + b^2) + ab})\end{aligned}$$

定理. 若 $P_n(z), Q_m(z)$ 是多项式, 且 $\deg P_n = n \leq \deg Q_m - 2 = m - 2$, $Q_m(z)$ 在实轴 $z = x$ 上没有零点, 即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$, 则

$$\lim_{R \rightarrow +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{P_n(z)}{Q_m(z)} dz = 0$$

所以

$$\begin{aligned} I_n &= \frac{1}{2} 2\pi i (\operatorname{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|a|] + \operatorname{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|b|]) \\ &= \pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2 + b^2)} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2 + b^2)}) \\ &= \frac{\pi}{2(|a| + |b|)|a||b|} \end{aligned}$$

Problem 25

求积分

$$I_n = \int_0^{+\infty} \frac{dx}{1 + x^{2n}} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned} I_n &= \int_0^{+\infty} \frac{dx}{1 + x^{2n}} \quad n \in \mathbb{N} \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{1 + x^{2n}} \quad n \in \mathbb{N} \\ &= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{1 + x^{2n}} \quad n \in \mathbb{N} \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^i \operatorname{Res}[\frac{1}{1 + z^{2n}}, z_k] - \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{1 + z^{2n}} dz \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^i \operatorname{Res}[\frac{1}{1 + z^{2n}}, z_k] \end{aligned}$$

其中

$$z_k^{2n} + 1 = 0, \operatorname{Im} z_k > 0$$

则

$$\begin{aligned} I_n &= \int_0^{+\infty} \frac{dx}{1 + x^{2n}} \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^n \operatorname{Res}[\frac{1}{1 + z^{2n}}, z_k] \\ &= \pi i \sum_{k=1}^n \operatorname{Res}[\frac{1}{1 + z^{2n}}, z_k] \\ &= \pi i \sum_{k=1}^n \frac{1}{2n z_k^{2n-1}} \\ &= -\frac{\pi i}{2n} \sum_{k=1}^n z_k \end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i n}}{e^{\pi i 2n}} \quad k = 1, 2, \dots, 2n$$

所以

$$\begin{aligned} I_n &= -\frac{\pi i}{2n} \sum_{k=0}^i z_k \\ &= -\frac{\pi i}{2n} \frac{\sum_{k=1}^n e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\ &= -\frac{\pi i}{2n} \frac{e^{\frac{\pi i}{n}} (1 - e^{\pi i})}{e^{\frac{\pi i}{2n}} (1 - e^{\frac{\pi i}{n}})} \\ &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \end{aligned}$$

令 $\theta = \frac{\pi i}{2n}$, 则

$$\begin{aligned} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\ &= \frac{\cos \theta + i \sin \theta}{2 \sin^2 \theta - 2i \sin \theta \cos \theta} \\ &= \frac{1}{-2i \sin \theta} \end{aligned}$$

所以

$$\begin{aligned} I_n &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\ &= -\frac{\pi i}{n} \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\ &= -\frac{\pi i}{n} \frac{1}{-2i \sin \theta} \\ &= \frac{\frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \end{aligned}$$

Problem 26

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_{n,r} &= \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \\
&= \frac{1}{r^{2n}} \int_0^{+\infty} \frac{dx}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} \int_0^{+\infty} \frac{d(\frac{x}{r})}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} I_n
\end{aligned}$$

Problem 27

求积分

$$J_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_n &= \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^n} \\
&= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{(1+x^2)^n} \\
&= \frac{1}{2} 2\pi(i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R \rightarrow +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{(1+z^2)^n} dz) \\
&= \frac{1}{2} (2\pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]) \\
&= \pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]
\end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

令 $f(z) = \frac{1}{(z+i)^n}$

$$\begin{aligned}
\frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
&= (z-i)^{-n} \sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!} (z-i)^k
\end{aligned}$$

要求 $(z-i)^{-1}$ 对应的系数 C_{-1} , 对应于 $k = n-1$

$$\begin{aligned}
C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
&= (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
\end{aligned}$$

因此

$$\begin{aligned}
 J_n &= \pi i C_{-1} \\
 &= \pi i (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
 &= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
 \end{aligned}$$

Problem 28

求积分

$$J_{n,r} = \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
 J_{n,r} &= \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \\
 &= \int_0^{+\infty} \frac{1}{r^{2n}} \frac{r d(\frac{x}{r})}{(1 + (\frac{x}{r})^2)^n} \\
 &= \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{d(\frac{x}{r})}{(1 + (\frac{x}{r})^2)^n} \\
 &= \frac{1}{r^{2n-1}} J_n
 \end{aligned}$$

Problem 29

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution

设 $a \neq b$ 则

$$\begin{aligned}
 I_{a,b,k} &= \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{x e^{kx}}{(x^2 + a^2)(x^2 + b^2)} dx
 \end{aligned}$$

而

$$\begin{aligned}
 \int_{-\infty}^{+\infty} \frac{x e^{kx}}{(x^2 + a^2)(x^2 + b^2)} dx &= 2\pi i \{ \operatorname{Res}[\frac{z e^{ikz}}{(z^2 + a^2)(z^2 + b^2)}, ai] + \operatorname{Res}[\frac{z e^{ikz}}{(z^2 + a^2)(z^2 + b^2)}, bi] \} \\
 &= 2\pi i \left[\frac{a i e^{-ka}}{4(ai)^3 + 2ai(a^2 + b^2)} + \frac{b i e^{-kb}}{4(bi)^3 + 2bi(a^2 + b^2)} \right] \\
 &= \frac{\pi i}{b^2 - a^2} (e^{-ka} - e^{-kb})
 \end{aligned}$$

所以

$$\begin{aligned}
 I_{a,b,k} &= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{x e^{kx}}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \operatorname{Im} \left[\frac{\pi i}{b^2 - a^2} (e^{-ka} - e^{-kb}) \right] \\
 &= \frac{\pi}{2(b^2 - a^2)} (e^{-ka} - e^{-kb})
 \end{aligned}$$

$a = b$ 时

$$\begin{aligned}
 I_{a,b,k} &= \lim_{a \rightarrow b} \frac{\pi}{2(b^2 - a^2)} (e^{-ka} - e^{-kb}) \\
 &= \frac{-k\pi e^{-kb}}{-4b} \\
 &= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}}
 \end{aligned}$$

Problem 30

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution

$a \neq b$ 时

$$\begin{aligned}
 I_{a,b,k} &= \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{x^2 e^{ikx}}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{(be^{-kb} - ae^{-ka})\pi}{2(b^2 - a^2)}
 \end{aligned}$$

当 $a = b$ 时

$$I_{a,b,k} = \frac{(1 - ka)\pi}{4ae^{ka}} = \frac{(1 - kb)\pi}{4be^{kb}}$$

Chapter 6 Analytic Mappings

定义. $f(z) = \frac{az+b}{cz+d}$ 其中 $a, b, c, d \in \mathbb{C}$ 且 $ad - bc \neq 0$

定理. 分式线性映射保广义圆。(圆 \cup 直线映射到圆 \cup 直线)

定理. 分式线性映射保对称点。即对广义圆，圆心映射到圆心，圆心的对称点映射到无穷，边界映射到边界(可由最大模原理得出)

Problem 31

求分式线性映射 $w = \frac{az+b}{cz+d}$ 将单位圆盘 $|z| < 1$ 映射到单位圆盘 $|w| < 1$ 且使 z_1 映射到 0, 这里 $|z_1| < 1$

Solution

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \frac{a(z - (-\frac{b}{a}))}{c(z - (-\frac{d}{c}))} \end{aligned}$$

根据题意

$$w(z_1) = 0$$

根据保对称性, z_1 的对称点 $z'_1 = \frac{1}{\bar{z}_1}$ 满足

$$w(z'_1) = \infty$$

记 $\frac{a}{c} = a'$ 则此时

$$\begin{aligned} w &= a' \frac{z - z_1}{z - \frac{1}{\bar{z}_1}} \\ &= (-\bar{z}_1 a') \frac{z - z_1}{1 - \bar{z}_1 z} \\ &= A \left(\frac{z - z_1}{1 - \bar{z}_1 z} \right) \end{aligned}$$

根据最大模原理, $|w| = 1$ 时 $|z| = 1$, 且 $|z| = 1$ 时

$$1 = \bar{z}z = |z|^2$$

所以 $|w| = 1$ 时

$$\begin{aligned} 1 &= |A| \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| = |A| \left| \frac{z - z_1}{z\bar{z} - \bar{z}_1 z} \right| \\ &= \frac{|A| |z - z_1|}{|z| |\bar{z} - \bar{z}_1|} \\ &= \frac{|A|}{|z|} \\ &= |A| \end{aligned}$$

所以

$$A = e^{i\theta} \quad \theta \in [0, 2\pi)$$

所以

$$w = e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z} \quad \theta \in [0, 2\pi), |z_1| < 1$$

推论. $z_1 = 0$ 时 $w = e^{i\theta}z$ 对应逆时针旋转 θ 角

推论. 不变式 从单位圆盘到单位圆盘的映射满足

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}$$

Proof. 因为从单位圆盘到单位圆盘的映射

$$w = \frac{az + b}{cz + d} = e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z}$$

所以

$$\begin{aligned} a &= e^{i\theta} \\ b &= -z_1 e^{i\theta} \\ c &= -\bar{z}_1 \\ d &= 1 \end{aligned}$$

又

$$\frac{dw}{dz} = \frac{ad - bc}{(cz + b)^2}$$

所以

$$\left| \frac{dw}{dz} \right| = \frac{1 - |z_1|^2}{|1 - \bar{z}_1 z|^2} > 0$$

而

$$\begin{aligned} 1 - |w|^2 &= 1 - w\bar{w} \\ &= 1 - e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z} e^{-i\theta} \frac{\bar{z} - \bar{z}_1}{1 - z_1 \bar{z}} \\ &= \frac{(1 - \bar{z}_1 z)(1 - z_1 \bar{z}) - (z - z_1)(\bar{z} - \bar{z}_1)}{(1 - \bar{z}_1 z)(1 - z_1 \bar{z})} \\ &= \frac{1 - z_1 \bar{z} - \bar{z}_1 z + |z|^2 |z_1|^2 - z \bar{z} + z \bar{z}_1 + z_1 \bar{z} - |z_1|^2}{|z - z z_1|^2} \\ &= \frac{1 - |z_1|^2 - |z|^2 + |z z_1|^2}{|z - z z_1|^2} \\ &= \frac{(1 - |z|^2)(1 - |z_1|^2)}{|z - z z_1|^2} \end{aligned}$$

所以

$$\left| \frac{dw}{dz} \right| = \frac{1 - |z_1|^2}{|1 - \bar{z}_1 z|^2} = \frac{1 - |z_1|^2}{|1 - z z_1|^2} = \frac{1 - |w|^2}{1 - |z|^2}$$

所以

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}$$

□

Problem 32

求分式线性映射使 $|z - z_0| < r \longrightarrow |w - w_0| < R$ 且使 $z_1 \rightarrow w_0$ 。这里 $|z_1 - z_0| < r$ 。

Solution

$|z - z_0| < r$ 到以原点为圆心的单位圆的映射为

$$z' = \frac{z - z_0}{r}$$

以原点为圆心的单位圆到以 z_1 为圆心到单位圆的映射为

$$w' = e^{i\theta} \frac{z' - z_1'}{1 - \overline{z_1'} z'}$$

从 $|w - w_0| < R$ 到以 z_1 为圆心到单位圆的映射为

$$w' = \frac{w - w_0}{R}$$

所以

$$w' = \frac{w - w_0}{R} = e^{i\theta} \frac{\frac{z - z_0}{r} - \frac{z_1 - z_0}{r}}{1 - \left(\frac{z_1 - z_0}{r}\right)\left(\frac{z_1 - z_0}{r}\right)}$$

解得

$$w = w_0 + e^{i\theta} R r \frac{z - z_1}{r^2 - (z_1 - z_0)(z - z_0)} \quad \theta \in [0, 2\pi) \quad |z_1 - z_0| < r$$

Problem 33

求证上半复平面 $\text{Im}z > 0$ 映射到单位圆盘 $|w| < 1$ 的分式线性映射为

$$w = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}} \quad \theta \in [0, 2\pi) \quad \text{Im}z_1 > 0$$

Proof. 根据保对称性, 由于 z_1 映射到 0, 所以 $z_1' = \overline{z_1}$ 映射到 ∞ 。所以

$$w = A \frac{z - z_1}{z - \overline{z_1}}$$

当 $z = x \in \mathbb{R}$ 时, 根据最大模原理

$$|w(z)| = 1$$

因此

$$\begin{aligned} 1 = |w| &= |A| \left| \frac{x - z_1}{x - \overline{z_1}} \right| \\ &= |A| \left| \frac{x - z_1}{\overline{x - z_1}} \right| \\ &= |A| \end{aligned}$$

即 $|A| = 1$, $A = e^{i\theta}$ $\theta \in [0, 2\pi)$ 所以

$$w = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}} \quad \theta \in [0, 2\pi) \quad \text{Im}z_1 > 0$$

□

Problem 34

求证当 $a, b, c, d \in \mathbb{R}$ 时, 分式线性映射使得 $\text{Im}z > 0$ 映射到 $\text{Im}w > 0$ 的充要条件是 $ad - bc > 0$

Proof. 令

$$\begin{aligned} z &= x + iy \\ x, y &\in \mathbb{R} \\ w &= \frac{az + b}{cz + d} \end{aligned}$$

则

$$\begin{aligned} w &= u + iv \\ &= \frac{a(x + iy) + b}{c(x + iy) + d} \\ &= \frac{(ax + b) + iay}{(cx + d) + icy} \\ &= \frac{[(ax + b) + iay][(cx + d) - icy]}{[(cx + d) + icy][(cx + d) - icy]} \end{aligned}$$

而

$$v = \text{Im}w = \frac{(ad - bc)y}{(cx + d)^2 + c^2y^2}$$

y 与 v 同号 $\Leftrightarrow ad - bc > 0$

□

Problem 35

$L = ax + b$ 是与 x 轴交于 x_0 点, 与 x 轴夹角为 α 的直线, 其中 $\alpha \in [0, \pi)$, $x_0 \in \mathbb{R}$ 。 $M = \{z = u + iv | v > ua + b\}$, 求将半平面 M 映到 $|w - w_0| < R$ 的分式线性映射。

Solution

将 $M = \{z = u + iv | v > ua + b\}$ 映射到 $z > 0$ 的映射为

$$z_1 = (z - x_0)e^{-i\alpha}$$

设半平面 M 中 z_0 点被所求分式线性映射映射到 w_0 , z_0 被 z_1 映射到 z_1^0

将 $|w - w_0| < R$ 映射到单位圆的映射为

$$w_1 = \frac{w - w_0}{R}$$

将 $z > 0$ 映射到单位圆盘的映射为

$$w_1 = e^{i\theta} \frac{z_1 - z_1^0}{z_1 - \overline{z_1^0}}$$

即

$$\frac{w - w_0}{R} = e^{i\theta} \frac{(z - x_0)e^{-i\alpha} - (z_0 - x_0)e^{-i\alpha}}{(z - x_0)e^{-i\alpha} - \overline{(z_0 - x_0)e^{-i\alpha}}}$$

即

$$w = w_0 + Re^{i\theta} \frac{z - z_0}{(z - x_0) - \overline{(z_0 - x_0)}e^{2i\alpha}}$$

这里 $\theta \in [0, 2\pi)$, $z_1 \in M$ 。

取 $\theta = 0$, 则

$$w = w_0 + R \frac{z - z_0}{(z - x_0) - (z_0 - x_0)e^{2i\alpha}}$$

Problem 36

M 是与 x 轴夹角为 θ , 与 x 轴交于 x_1, x_2 点的条带, 其中 $x_1 < x_2$, $0 < \theta < 2\pi$ 。求一个单值可导映射将 M 映射到单位圆盘。

Solution

将 M 映射到 $N = \{z | z \in \mathbb{C}, \operatorname{Im} z > 0 \wedge \operatorname{Im} z < h = (x_2 - x_1) \sin \theta\}$ 且将 x_2 映射到原点的映射为

$$z_1 = (z - x_2)e^{-i\theta}$$

将 N 映射到 $O = \{z | z \in \mathbb{C}, \operatorname{Im} z > 0 \wedge \operatorname{Im} z < \pi\}$ 的映射为

$$z_2 = \frac{z_1 \pi}{h} = \frac{(z - x_2)e^{-i\theta} \pi}{(x_2 - x_1) \sin \theta}$$

将 O 映射到 $\operatorname{Im} z > 0$ 的映射为

$$z_3 = e^{z_2}$$

将 $\operatorname{Im} z > 0$ 映射到单位圆盘的一个映射为

$$\begin{aligned} w &= \frac{z_3 - i}{z_3 + i} \\ &= \frac{e^{z_2} - i}{e^{z_2} + i} \\ &= \frac{e^{\frac{(z-x_2)e^{-i\theta}\pi}{(x_2-x_1)\sin\theta}} - i}{e^{\frac{(z-x_2)e^{-i\theta}\pi}{(x_2-x_1)\sin\theta}} + i} \end{aligned}$$

Problem 37

求区域 $D = \{|z-a| > a \text{ 与 } |z-b| < b\}$ 之间的部分, 这里 $0 < a < b$ 到单位圆盘 $|w| < 1$ 的单值解析映射。

Solution

根据分式映射的保圆性, 映射

$$z_1 = \frac{z - 2a}{z}$$

将 D 映射到 $E = \{z | z \in \mathbb{C}, \operatorname{Re} z > 0 \wedge \operatorname{Re} z < \frac{b-a}{b}\}$, 且将 $z = 2a$ 映射到原点。而

$$z_2 = iz_1 \frac{\pi}{\frac{b-a}{b}} = \frac{b-i\pi}{b-a} \left(\frac{z-2a}{z} \right)$$

将 E 映射到 $F = \{z | z \in \mathbb{C}, \operatorname{Im} z > 0 \wedge \operatorname{Im} z < \pi\}$

将 F 映射到 $\operatorname{Im} z > 0$ 的映射为

$$z_3 = e^{z_2}$$

将 $\text{Im}z > 0$ 映射到单位圆盘的一个映射为

$$w = \frac{z_3 - i}{z_3 + i} = \frac{e^{\frac{b-i\pi}{b-a}(\frac{z-2a}{z})} - i}{e^{\frac{b-i\pi}{b-a}(\frac{z-2a}{z})} + i}$$

Problem 38

区域 $D = \{\text{弦切角为}\alpha, \text{弦为}AB\text{的扇形, 其中}0 < \alpha < \pi\}$, 求将 D 映射到单位圆盘 $|w| < 1$ 的单值解析映射。

Solution

分式线性映射

$$z_1 = -\frac{z - A}{z - B}$$

将 D 映射为 $E = \{z = re^{i\theta} | z \in \mathbb{C}, r > 0 \wedge 0 < \theta < \alpha\}$ 。而映射

$$z_2 = z_1^{\frac{\pi}{\alpha}}$$

将 E 映射为 $\text{Im}z > 0$ 。将 $\text{Im}z > 0$ 映射为单位圆盘的一个映射为

$$w = \frac{z_2 - i}{z_2 + i} = \frac{\left(-\frac{z-A}{z-B}\right)^{\frac{\pi}{\alpha}} - i}{\left(-\frac{z-A}{z-B}\right)^{\frac{\pi}{\alpha}} + i}$$

Problem 39

求将区域 $D = \{z = re^{i\theta} | z \in \mathbb{C}, r > 0 \wedge 0 < \theta < \alpha, 0 < \alpha < \pi\}$ 映射到圆盘 $\{w | |w - w_0| < R\}$ 的一个单值解析映射。

Solution

映射

$$z_1 = z^{\frac{\pi}{\alpha}}$$

将 D 映射到上半复平面。映射

$$w_1 = \frac{z_1 - i}{z_1 + i}$$

将上半复平面映射到单位圆盘。而

$$w_1 = \frac{w - w_0}{R}$$

将圆盘 $\{w | |w - w_0| < R\}$ 映射到单位圆盘。因此

$$w = w_0 + R \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}$$

特例：由复平面第一象限 ($\alpha = \frac{\pi}{2}$) 到圆盘 $\{w | |w - w_0| < R\}$ 的一个单值解析映射为

$$w = w_0 + R \frac{z^2 - i}{z^2 + i}$$