

Introduction

Problem 1

求

$$\max |\alpha z^n + \beta| \quad |z| \leq r$$

Solution

$$\max |\alpha z^n + \beta| = |\alpha| \max |z^n + \frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max |z^n + \alpha| \quad |z| \leq r$$

有

$$|z^n + \alpha| \leq |z^n| + |\alpha| \leq r^n + |\alpha|$$

取最大值 \Leftrightarrow 等号成立 $\Leftrightarrow z^n$ 与 α 同向 $\Leftrightarrow z^n = \lambda \alpha \quad \lambda > 0$

当 $\alpha = 0$ 时

$$\begin{aligned} |z^n| &= |z|^n = r^n \\ |z| &= r \\ \max |\alpha z^n + \beta| &= r^n \quad z = r e^{i\theta} \quad \theta \in [0, 2\pi) \end{aligned}$$

当 $\alpha \neq 0$ 时

$$\max |z^n + \alpha| = r^n + |\alpha|$$

此时

$$\begin{aligned} r^n &= |z^n| = |\lambda \alpha| \\ \lambda &= \frac{|r|^n}{|\alpha|} \\ z^n &= \lambda \alpha = \frac{|r|^n}{|\alpha|} \alpha \end{aligned}$$

设 $\alpha = r_0 e^{i\theta_0}$

$$\begin{aligned} z^n &= \frac{|r|^n}{|\alpha|} \alpha \\ &= \frac{|r|^n}{|\alpha|} r_0 e^{i\theta_0} \\ &= |r|^n e^{i\theta_0} \end{aligned}$$

则取最大值时

$$z = z_k = r e^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \dots, n$$

综上,

$$\max |z^n + \alpha| = \begin{cases} r^n & z = r e^{i\theta} \quad \theta \in [0, 2\pi) & \alpha = 0 \\ r^n + |\alpha| & z = r e^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \dots, n & \alpha \neq 0 \end{cases}$$

Problem 2

$$|z|^2 = z\bar{z} \quad z^2 = |z|^2$$

等号成立的条件是?

Solution

$$z^2 = |z|^2 = z\bar{z} \Leftrightarrow z(z - \bar{z}) = 0 \Leftrightarrow z = \bar{z}$$

即 $z \in \mathbb{R}$ 时等号成立。

Problem 3

证明

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

并说明几何意义

Solution

$$\begin{aligned} |z_1 + z_2|^2 + |z_1 - z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= 2(z_1\bar{z}_1 + z_2\bar{z}_2) \\ &= 2(|z_1|^2 + |z_2|^2) \end{aligned}$$

几何意义：平行四边形对角线平方和等于对边平方和

Problem 4

$|z_1| = |z_2| = |z_3| = |z_4| = r$ 且 $z_1 + z_2 + z_3 + z_4 = 0$ 则 z_1, z_2, z_3, z_4 满足什么条件时 $z_1 z_2 z_3 z_4$ 构成正方形?

Solution

定理. z_1, z_2, \dots, z_n 将圆 $|z| = \alpha$ 等分 $\Leftrightarrow z_k$ 是分圆多项式 $z^n + \alpha = 0$

当 z_1, z_2, z_3, z_4 构成正方形时, $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式。

又

$$\begin{aligned} &(z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= z^4 - \sum_{k=1}^4 z_k z^3 + (z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) z^2 \\ &\quad - (z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4) z + z_1 z_2 z_3 z_4 \end{aligned}$$

$(z - z_1)(z - z_2)(z - z_3)(z - z_4)$ 是分圆多项式

$$\Leftrightarrow \begin{cases} \sum_{k=1}^4 z_k &= 0 \\ z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 &= 0 \\ z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 &= 0 \\ z_1 z_2 z_3 z_4 &\neq 0 \end{cases}$$

而

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = z_1 z_2 z_3 z_4 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right)$$

$$z_k \overline{z_k} = |z_k|^2 = r^2$$

所以

$$\frac{1}{z_k} = \frac{\overline{z_k}}{r^2}$$

$$z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = \frac{z_1 z_2 z_3 z_4}{r^2} (\overline{z_1} + \overline{z_2} + \overline{z_3} + \overline{z_4}) = 0$$

所以 z_1, z_2, z_3, z_4 需满足

$$z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = 0$$

Problem 5

$f(z)$ 在 z_0 连续, $f(z_0) \neq 0$ 。求证 $\exists \delta > 0$, 当 $|z - z_0| < \delta$ 时有 $f(z) \neq 0$

Solution

因 $f(z_0) \neq 0$, 有 $|f(z_0)| > 0$

令 $\varepsilon = \frac{1}{2}|f(z_0)| < |f(z_0)|$, 因为 $f(z)$ 在 z_0 连续, 存在 $\delta > 0$ 使得 $\forall z \quad |z - z_0| < \delta$

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \frac{1}{2}|f(z_0)|$$

即

$$\frac{1}{2}|f(z_0)| < |f(z)| < \frac{3}{2}|f(z_0)|$$

Chapter 2 Analytic Function

Problem 6

$f(z) = f(x+iy) = u(x, y) + iv(x, y)$ 且 $u, v \in C^{(n)}$, 求 $f(z)$ n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z) = A + iB$, 则

$$\begin{aligned} df = f'(z)dz = f'(z)(dx + idy) &\Leftrightarrow df = du + idv \\ &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy) \\ &= (Adx - Bdy) + i(Bdx + Ady) \end{aligned}$$

由上式得

$$\begin{cases} Adx - Bdy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ Bdx + Ady = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = A \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -B \end{cases}$$

即

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = F'(z)$$

而

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z) = \frac{\partial^n u}{\partial x^n} + i\frac{\partial^n v}{\partial x^n}$$

u, v 需要满足 Cauchy-Riemann 条件

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Problem 7

求 $\cos(x + iy)$ 的实部和虚部, 其中 $x, y \in \mathbb{R}$

Solution

$$\begin{aligned} \cos(x + iy) &= \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\ &= \frac{1}{2}(e^{-y}e^{ix} + e^ye^{-ix}) \\ &= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^y(\cos x - i\sin x)] \\ &= \frac{1}{2}(e^y + e^{-y})\cos x + i\frac{1}{2}(-e^y + e^{-y})\sin x \end{aligned}$$

Problem 8

求证: $\forall A, B \in \mathbb{R}$ 存在 $z = x + iy$ 使得 $\cos(x + iy) = A + iB$ (即 $\text{Im}[\cos(z)] = \mathbb{C}$)

Solution

令

$$\begin{aligned}\frac{e^y + e^{-y}}{2} \cos x &= A \\ \frac{e^{-y} - e^y}{2} \sin x &= B\end{aligned}\tag{1}$$

1. 当 $B = 0$ 时, 由式 (1) 知 $y = 0$ 或 $\sin x = 0$ 。

$|A| \leq 1$ 时可令 $y = 0$, 此时

$$\cos x = A$$

解得

$$\begin{cases} x = \arccos A + 2k\pi & k \in \mathbb{Z} \\ y = 0 \end{cases}$$

$|A| > 1$ 时, 令 $\sin x = 0$ 得

$$\begin{aligned}\cos x &= \pm 1 \\ \frac{e^y + e^{-y}}{2} &= |A| > 1\end{aligned}$$

考察函数 $f(y) = e^y + e^{-y} - 2$

$$f(0) = 0$$

$$\lim_{y \rightarrow +\infty} f(y) = \lim_{y \rightarrow -\infty} f(y) = +\infty$$

且 $f(y)$ 连续。因此存在 y_A 使得 $\pm y_A$ 是方程 $\frac{e^y + e^{-y}}{2} = |A|$ 的解。
此时

$$\begin{cases} x = k\pi & k \in \mathbb{Z} \\ y = \pm y_A \end{cases}$$

2. 当 $B \neq 0$ 时, 由式 (1) 知 $y \neq 0$ 。结合 $\cos^2 x + \sin^2 x = 1$ 得 $y \in (-\infty, 0) \cup (0, +\infty)$ 时

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

令 $f_{A,B}(y) = 4A^2(e^{-y} + e^y)^2 + \frac{4B^2}{(e^{-y} - e^y)^2}$, $f_{A,B}(y)$ 是偶函数。

$$\lim_{y \rightarrow 0^+} f_{A,B}(y) = +\infty$$

$$\lim_{y \rightarrow +\infty} f_{A,B}(y) = 0$$

因此 $\exists y_{A,B} > 0$, 使得 $\pm y_{A,B}$ 是方程

$$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$

的根。将 $\pm y_{A,B}$ 代入式 (1) 可解出对应的 x 。

Chapter 5 Residues

Problem 9

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= 2\pi i \sum_{k=1}^n \operatorname{Res}\left[\frac{1}{1+z^n}, z_k\right] \\ &= 2\pi i \sum_{k=1}^n \frac{1}{nz_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned} &\oint_{|z|=r>1} \frac{1}{1+z^n} dz \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \\ &= -\frac{2\pi i}{n} \sum_{k=1}^n z_k \end{aligned}$$

若 $n=1$, 即

$$1+z^n = 1+z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 2\pi i$$

否则

$$\sum_{k=1}^n z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 0$$

又

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz &= \oint_{|z|=r>1} \frac{(z^{2n}-1)+1}{1+z^n} dz \\ &= \oint_{|z|=r>1} \frac{(z^n-1)(z^n+1)}{1+z^n} dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz \\ &= \oint_{|z|=r>1} (z^n-1) dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz\end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1} (z^n-1) dz = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz$$

综上,

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 2

令 $z = re^{i\theta}$ $0 \leq \theta < \pi, t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n} dz &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^2}} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt\end{aligned}$$

只有一个奇点 $t=0$ 。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt \\ &= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)} dt = 2\pi i & n=1 \end{cases} \quad (\text{Cauchy-Goursat 定理})\end{aligned}$$

Problem 10

求积分

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz \quad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = 0$$

当 $n > 0$ 时

$$\begin{aligned} \frac{1 - \cos 4z^3}{z^n} &= z^{-n} \left(1 - \sum_{k=0}^n (-1)^k \frac{(4z^3)^{2k}}{(2k)!} \right) \\ &= z^{-n} \left(1 - \sum_{k=0}^n (-1)^k \frac{4^{2k} z^{6k}}{(2k)!} \right) \\ &= z^{-n} \sum_{k=1}^n (-1)^{k-1} \frac{4^{2k} z^{6k}}{(2k)!} \end{aligned}$$

$\frac{1 - \cos 4z^3}{z^n}$ 在奇点 $z = 0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中, C_{-1} 对应上式中

$$6k - n = -1$$

此时

$$\begin{aligned} C_{-1} &= (-1)^{k-1} \frac{4^{2k}}{(2k)!} \quad k = \frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

所以 $n > 0$ 时

$$\begin{aligned} \oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz &= 2\pi i \operatorname{Res}\left[\frac{1 - \cos 4z^3}{z^n}, 0\right] \\ &= 2\pi i C_{-1} \\ &= 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

综上,

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = \begin{cases} 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k + 1, k \in \mathbb{N} \\ 0 & n \neq 6k + 1, k \in \mathbb{N} \end{cases}$$

Problem 11

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz$$

Solution

令 $t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned} \oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz &= - \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)} dt \\ &= 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] \end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) (1 - t + t^2 - t^3 + \dots)$$

其中, t^{-1} 的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz = 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] = 2\pi i \left(-\frac{1}{3}\right) = -\frac{2}{3}\pi i$$

Problem 12

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}$, $\theta \in [0, 2\pi)$, 注意到:

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z} \\ d\theta &= \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} \end{aligned}$$

则原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{a + b\left(\frac{z^2+1}{2z}\right)} \\ &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b} \end{aligned}$$

当 $b = 0$ 时, 原积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \int_0^{2\pi} \frac{d\theta}{a} = \frac{2\pi}{a}$$

当 $b \neq 0$ 时, 原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b} \\ &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1} \end{aligned}$$

对方程

$$z^2 + \frac{2a}{b}z + 1 = 0$$

其两根

$$z_1 z_2 = 1$$

且

$$z_{1,2} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

可设 $b > 0$, 则

$$z_2 = -\frac{a}{b} \frac{\sqrt{a^2 - b^2}}{b} < -\frac{a}{b} < -1$$

即只有一个奇点 z_1 。所以原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1} \\ &= \frac{2}{ib} 2\pi i \frac{1}{2z_1 + \frac{2a}{b}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \end{aligned}$$

Problem 13

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1, 1]$ 上可积, 则 $\int_0^{2\pi} f(\cos x) d\theta = \int_0^{2\pi} f(\sin x) d\theta$

所以

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 14

求积分

$$I_p = \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \quad p \in (-1, 1)$$

Solution

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \\ &= \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

Problem 15

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \quad A, B \in \mathbb{R} > 0$$

Solution

令

$$\begin{aligned} \cos^2 \theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

则

$$\begin{aligned}
 I_{A,B} &= \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \\
 &= \int_0^{2\pi} \frac{d\theta}{A^2 \frac{\cos 2\theta + 1}{2} + B^2 \frac{1 - \cos 2\theta}{2}} \\
 &= \int_0^{4\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\
 &= 2 \int_0^{2\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\
 &= 2 \frac{2\pi}{\sqrt{(A^2 + B^2)^2 - (A^2 - B^2)^2}} \\
 &= \frac{2\pi}{AB}
 \end{aligned}$$

Problem 16

求积分

$$I_n = \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数, 因此

$$\begin{aligned}
 I_n &= \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} \\
 &= \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)}
 \end{aligned}$$

而

$$\begin{aligned}
 &\lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)} + \int_{|z|=R, \text{Im}z>0} \frac{1}{(z^2 + a^2)(z^2 + b^2)} dz \\
 &= 2\pi i (\text{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|a|] + \text{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|b|]) \\
 &= 2\pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2 + b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2 + b^2) + ab})
 \end{aligned}$$

定理. 若 $P_n(z), Q_m(z)$ 是多项式, 且 $\deg P_n = n \leq \deg Q_m - 2 = m - 2$, $Q_m(z)$ 在实轴 $z = x$ 上没有零点, 即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$, 则

$$\lim_{R \rightarrow +\infty} \int_{|z|=R, \text{Im}z>0} \frac{P_n(z)}{Q_m(z)} dz = 0$$

所以

$$\begin{aligned}
 I_n &= \frac{1}{2} 2\pi i (\text{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \text{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
 &= \pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2)} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2)}) \\
 &= \frac{\pi}{2(|b|-|a|)|a||b|}
 \end{aligned}$$

Problem 17

求积分

$$I_n = \int_0^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
 I_n &= \int_0^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\
 &= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\
 &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{|z|=R, \text{Im}z > 0} \frac{1}{1+z^{2n}} dz \\
 &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k]
 \end{aligned}$$

其中

$$z_k^{2n} + 1 = 0, \text{Im}z_k > 0$$

则

$$\begin{aligned}
 I_n &= \int_0^{+\infty} \frac{dx}{1+x^{2n}} \\
 &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k] \\
 &= \pi i \sum_{k=1}^n \text{Res}[\frac{1}{1+z^{2n}}, z_k] \\
 &= \pi i \sum_{k=1}^n \frac{1}{2n z_k^{2n-1}} \\
 &= -\frac{\pi i}{2n} \sum_{k=1}^n z_k
 \end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i} n}{e^{\pi i} 2n} \quad k = 1, 2, \dots, 2n$$

所以

$$\begin{aligned} I_n &= -\frac{\pi i}{2n} \sum_{k=0}^i z_k \\ &= -\frac{\pi i}{2n} \frac{\sum_{k=1}^n e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\ &= -\frac{\pi i}{2n} \frac{e^{\frac{\pi i}{n}} (1 - e^{\pi i})}{e^{\frac{\pi i}{2n}} (1 - e^{\frac{\pi i}{n}})} \\ &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \end{aligned}$$

令 $\theta = \frac{\pi i}{2n}$, 则

$$\begin{aligned} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\ &= \frac{\cos \theta + i \sin \theta}{2 \sin^2 \theta - 2i \sin \theta \cos \theta} \\ &= \frac{1}{-2i \sin \theta} \end{aligned}$$

所以

$$\begin{aligned} I_n &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\ &= -\frac{\pi i}{n} \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\ &= -\frac{\pi i}{n} \frac{1}{-2i \sin \theta} \\ &= \frac{\frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \end{aligned}$$

Problem 18

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned} I_{n,r} &= \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \\ &= \frac{1}{r^{2n}} \int_0^{+\infty} \frac{dx}{1 + \left(\frac{x}{r}\right)^{2n}} \\ &= \frac{1}{r^{2n+1}} \int_0^{+\infty} \frac{d\left(\frac{x}{r}\right)}{1 + \left(\frac{x}{r}\right)^{2n}} \\ &= \frac{1}{r^{2n+1}} I_n \end{aligned}$$

Problem 19

求积分

$$J_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
 J_n &= \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^n} \\
 &= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{(1+x^2)^n} \\
 &= \frac{1}{2} 2\pi (i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R \rightarrow +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{(1+z^2)^n} dz) \\
 &= \frac{1}{2} (2\pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]) \\
 &= \pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]
 \end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

$$\text{令 } f(z) = \frac{1}{(z+i)^n}$$

$$\begin{aligned}
 \frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
 &= (z-i)^{-n} \sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!} (z-i)^k
 \end{aligned}$$

要求 $(z-i)^{-1}$ 对应的系数 C_{-1} , 对应于 $k = n-1$

$$\begin{aligned}
 C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
 &= (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
 \end{aligned}$$

因此

$$\begin{aligned}
 J_n &= \pi i C_{-1} \\
 &= \pi i (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
 &= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
 \end{aligned}$$

Problem 20

求积分

$$J_{n,r} = \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \quad n \in \mathbb{N}$$

Solution

$$\begin{aligned} J_{n,r} &= \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \\ &= \int_0^{+\infty} \frac{1}{r^{2n}} \frac{rd(\frac{x}{r})}{(1 + (\frac{x}{r})^2)^n} \\ &= \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{d(\frac{x}{r})}{(1 + (\frac{x}{r})^2)^n} \\ &= \frac{1}{r^{2n-1}} J_n \end{aligned}$$

Problem 21

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution设 $a \neq b$ 则

$$\begin{aligned} I_{a,b,k} &= \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{xe^{kx}}{(x^2 + a^2)(x^2 + b^2)} dx \end{aligned}$$

而

$$\begin{aligned} \int_0^{+\infty} \frac{xe^x}{(x^2 + a^2)(x^2 + b^2)} dx &= 2\pi i \{ \operatorname{Res}[\frac{ze^{ikz}}{(z^2 + a^2)(z^2 + b^2)}, ai] + \operatorname{Res}[\frac{ze^{ikz}}{(z^2 + a^2)(z^2 + b^2)}, bi] \} \\ &= 2\pi i [\frac{aie^{-ka}}{4(ai)^3 + 2ai(a^2 + b^2)} + \frac{bie^{-kb}}{4(bi)^3 + 2bi(a^2 + b^2)}] \\ &= \frac{\pi i}{b^2 - a^2} (e^{-ka} - e^{-kb}) \end{aligned}$$

所以

$$\begin{aligned} I_{a,b,k} &= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{xe^{kx}}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{1}{2} \operatorname{Im} [\frac{\pi i}{b^2 - a^2} (e^{-ka} - e^{-kb})] \\ &= \frac{\pi}{2(b^2 - a^2)} (e^{-ka} - e^{-kb}) \end{aligned}$$

$a = b$ 时

$$\begin{aligned} I_{a,b,k} &= \lim_{a \rightarrow b} \frac{\pi}{2(b^2 - a^2)} (e^{-ka} - e^{-kb}) \\ &= \frac{-k\pi e^{-kb}}{-4b} \\ &= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}} \end{aligned}$$

Problem 22

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

Solution

$a \neq b$ 时

$$\begin{aligned} I_{a,b,k} &= \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{1}{2} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{x^2 e^{ikx}}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{(be^{-kb} - ae^{-ka})\pi}{2(b^2 - a^2)} \end{aligned}$$

当 $a = b$ 时

$$I_{a,b,k} = \frac{(1 - ka)\pi}{4ae^{ka}} = \frac{(1 - kb)\pi}{4be^{kb}}$$