Analytic Mappings

定义. $f(z) = \frac{az+b}{cz+d}$ 其中 $a, b, c, d \in \mathbb{C}$ 且 $ad - bc \neq 0$

定理. 分式线性映射保广义圆。(圆 $\cup$ 直线映射到圆 $\cup$ 直线)

定理. 分式线性映射保对称点。即对广义圆，圆心映射到圆心，圆心的对称点映射到无穷，边界映射到边界（可由最大模原理得出）

Problem 1

求分式线性映射 $w = \frac{az+b}{cz+d}$ 将单位圆盘 $|z| < 1$ 映射到单位圆盘 $|w| < 1$ 且使 z_1 映射到 0，这里 $|z_1| < 1$

Solution

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \frac{a(z - (-\frac{b}{a}))}{b(z - (-\frac{d}{c}))} \end{aligned}$$

根据题意

$$w(z_1) = 0$$

根据保对称性，z_1 的对称点 $z_1' = \frac{1}{\overline{z_1}}$ 满足

$$w(z_1') = \infty$$

记 $\frac{a}{c} = a'$ 则此时

$$\begin{aligned} w &= a' \tfrac{z - z_1}{z - \frac{1}{\overline{z_1}}} \\ &= (-\overline{z_1} a') \tfrac{z - z_1}{1 - \overline{z_1} z} \\ &= A(\tfrac{z - z_1}{1 - \overline{z_1} z}) \end{aligned}$$

根据最大模原理，$|w| = 1$ 时 $|z| = 1$，且 $|z| = 1$ 时

$$1 = \overline{z} z = |z|^2$$

所以 $|w| = 1$ 时

$$\begin{aligned} 1 = |A| \left| \frac{z - z_1}{1 - \overline{z_1} z} \right| &= |A| \left| \frac{z - z_1}{z\overline{z} - \overline{z_1} z} \right| \\ &= \frac{|A|}{|z|} \frac{|z - z_1|}{|\overline{z} - \overline{z_1}|} \\ &= \frac{|A|}{|z|} \\ &= |A| \end{aligned}$$

所以

$$A = e^{i\theta} \qquad \theta \in [0, 2\pi)$$

所以

$$w = e^{i\theta} \frac{z - z_1}{1 - \overline{z_1} z} \qquad \theta \in [0, 2\pi), |z_1| < 1$$

推论. $z_1 = 0$ 时 $w = e^{i\theta} z$ 对应逆时针旋转 θ 角

推论. 不变式 从单位圆盘到单位圆盘的映射满足

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$$

Proof. 因为从单位圆盘到单位圆盘的映射

$$w = \frac{az+b}{cz+d} = e^{i\theta}\frac{z-z_1}{1-\overline{z_1}z}$$

所以

$$\begin{aligned} a &= e^{i\theta} \\ b &= -z_1 e^{i\theta} \\ c &= -\overline{z_1} \\ d &= 1 \end{aligned}$$

又

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{ad-bc}{(cz+b)^2}$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1-|z_1|^2}{|1-\overline{z_1}z|^2} > 0$$

而

$$\begin{aligned} 1-|w|^2 &= 1-w\overline{w} \\ &= 1-e^{i\theta}\frac{z-z_1}{1-\overline{z_1}z}e^{-i\theta}\frac{\overline{z}-\overline{z_1}}{1-z_1\overline{z}} \\ &= \frac{(1-\overline{z_1}z)(1-z_1\overline{z})-(z-z_1)(\overline{z}-\overline{z_1})}{(1-\overline{z_1}z)(1-z_1\overline{z})} \\ &= \frac{1-z_1\overline{z}-\overline{z_1}z+|z|^2|z_1|^2-z\overline{z}+z\overline{z_1}+z_1\overline{z}-|z_1|^2}{|z-zz_1|^2} \\ &= \frac{1-|z_1|^2-|z|^2+|zz_1|^2}{|z-zz_1|^2} \\ &= \frac{(1-|z|^2)(1-|z_1|^2)}{|z-zz_1|^2} \end{aligned}$$

所以

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \frac{1-|z_1|^2}{|1-\overline{z_1}z|^2} = \frac{1-|z_1|^2}{|1-zz_1|^2}$$

所以

$$\frac{|\mathrm{d}w|}{1-|w|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}$$

□

Problem 2

求分式线性映射使 $|z-z_0|<r \longrightarrow |w-w_0|<R$ 且使 $z_1 \to w_0$。这里 $|z_1-z_0|<r$。

Solution

$|z-z_0|<r$ 到以 z_0 为圆心的单位圆的映射为

$$z' = \frac{z-z_0}{r}$$

以 z_0 为圆心的单位圆到以 z_1 为圆心到单位圆的映射为

$$w' = e^{i\theta}\frac{z'-z_1'}{1-\overline{z_1'}z'}$$

从 $|w-w_0|<R$ 到以 z_1 为圆心到单位圆的映射为

$$w' = \frac{w-w_0}{R}$$

所以

$$w' = \frac{w-w_0}{R} = e^{i\theta}\frac{\frac{z-z_0}{r}-\frac{z_1-z_0}{r}}{1-\overline{(\frac{z_1-z_0}{r})}(\frac{z_1-z_0}{r})}$$

解得

$$w = w_0 + e^{i\theta}Rr\frac{z-z_1}{r^2-\overline{(z_1-z_0)}(z-z_0)} \qquad \theta\in[0,2\pi) \quad |z_1-z_0|<r$$