Chapter 2 Analytic Function

Problem 1

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ 且 $u,v\in C^{(n)}$, 求 $f(z)$n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z)=A+iB$，则

$$\begin{aligned}\mathrm{d}f=f'(z)\mathrm{d}z=f'(z)(\mathrm{d}x+i\mathrm{d}y)\Leftrightarrow \mathrm{d}f &= \mathrm{d}u+i\mathrm{d}v\\ &= \tfrac{\partial u}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y+i(\tfrac{\partial v}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y)\\ &= (A\mathrm{d}x-B\mathrm{d}y)+i(B\mathrm{d}x+A\mathrm{d}y)\end{aligned}$$

由上式得

$$\begin{cases} A\mathrm{d}x-B\mathrm{d}y=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\\ B\mathrm{d}x+A\mathrm{d}y=\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\end{cases}$$

解得

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=A\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-B\end{cases}$$

即

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=F'(z)$$

而

$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z)=\frac{\partial^n u}{\partial x^n}+i\frac{\partial^n v}{\partial x^n}$$

u,v 需要满足 Cauchy-Riemann 条件

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$

Problem 2

求 $\cos(x+iy)$ 的实部和虚部，其中 $x,y\in\mathbb{R}$

Solution

$$\begin{aligned}\cos(x+iy)&=\frac{1}{2}(e^{-y+ix}+e^{y-ix})\\ &=\frac{1}{2}(e^{-y}e^{ix}+e^{y}e^{-ix})\\ &=\frac{1}{2}[e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)]\\ &=\frac{1}{2}(e^{y}+e^{-y})\cos x+i\frac{1}{2}(-e^{y}+e^{-y})\sin x\end{aligned}$$

Problem 3

求证：$\forall A, B \in \mathbb{R}$ 存在 $z = x + iy$ 使得 $\cos(x+iy) = A + iB$（即 $\mathrm{Im}[\cos(z)] = \mathbb{C}$）

Solution

令

$$\begin{aligned}\frac{e^y + e^{-y}}{2}\cos x &= A \\ \frac{e^{-y} - e^y}{2}\sin x &= B\end{aligned} \tag{1}$$

1. 当 $B = 0$ 时，由式 (1) 知 $y = 0$ 或 $\sin x = 0$。
   $|A| \le 1$ 时可令 $y = 0$，此时
   $$\cos x = A$$
   解得
   $$\begin{cases} x = \arccos A + 2k\pi \qquad k \in \mathrm{Z} \\ y = 0 \end{cases}$$
   $|A| > 1$ 时，令 $\sin x = 0$ 得
   $$\cos x = \pm 1$$
   $$\frac{e^y + e^{-y}}{2} = |A| > 1$$
   考察函数 $f(y) = e^y + e^{-y}2 - 1$
   $$f(0) = 0$$
   $$\lim_{y \to +\infty} f(y) = \lim_{y \to -\infty} f(y) = +\infty$$
   且 $f(y)$ 连续。因此存在 y_A 使得 $\pm y_A$ 是方程 $\frac{e^y + e^{-y}}{2} = |A|$ 的解。
   此时
   $$\begin{cases} x = k\pi \qquad k \in \mathrm{Z} \\ y = \pm y_A \end{cases}$$
2. 当 $B \neq 0$ 时，由式 (1) 知 $y \neq 0$。结合 $\cos^2 x + \sin^2 x = 1$ 得 $y \in (-\infty, 0) \cup (0, +\infty)$ 时
   $$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$
   令 $f_{A,B}(y) = 4A^2(e^{-y} + e^y)^2 + \frac{4B^2}{(e^{-y} - e^y)^2}$，$f_{A,B}(y)$ 是偶函数。
   $$\lim_{y \to 0^+} f_{A,B}(y) = +\infty$$
   $$\lim_{y \to +\infty} f_{A,B}(y) = 0$$
   因此 $\exists y_{A,B} > 0$，使得 $\pm y_{A_B}$ 是方程
   $$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$
   的根。将 $\pm y_{A,B}$ 代入式 (1) 可解出对应的 x。

Problem 4

已知 $e^w = z \neq 0$，求

$$w = \text{Ln}z$$

Solution

设 $w = u + iv \quad u, v \in \mathbb{R}$，则

$$e^w = e^{u+iv} = e^u e^{iv} = z = re^{i\theta}$$
$$\theta = \arg z \in [0, 2\pi) \qquad r = |z| > 0$$

则

$$\begin{aligned} & e^u &= r \\ \Rightarrow \quad & u &= \ln r \end{aligned}$$

且

$$\begin{aligned} & e^{iv} &= e^{i\theta} \\ \Rightarrow \quad & v &= \theta + 2k\pi \qquad k \in \mathbb{Z} \\ & &= \arg z \end{aligned}$$

所以

$$\begin{aligned} w = u + iv \quad &= \text{Ln}z \\ &= \ln z + 2k\pi i \qquad && k \in \mathbb{Z} \\ &= \ln |z| + i \arg z + 2k\pi i \quad && k \in \mathbb{Z} \end{aligned}$$

定理. *Picard* 小定理 若 $f(z)$ 是解析函数且 $f(z)$ 不是常数，则除去最多一个例外 w_0，方程 $f(z) = A + iB = w$ 至少有一个解 z。

Problem 5

求

$$\text{Ln}(3 + 2i)$$

Solution

$$\begin{aligned} \text{Ln}(3 + 2i) \quad &= \ln(3 + 2i) + 2k\pi \qquad && k \in \mathbb{Z} \\ &= \ln 13 + i \arg(3 + 2i) + 2k\pi \quad && k \in \mathbb{Z} \end{aligned}$$

Problem 6

求

$$\text{Ln}z^n$$

Solution

$$\begin{aligned} \text{Ln}z^n \quad &= \ln z^n + 2k\pi \qquad && k \in \mathbb{Z} \\ &= \ln |z^n| + i \arg z^n + 2k\pi \quad && k \in \mathbb{Z} \\ &= n \ln |z| + ni \arg z + 2k\pi \quad && k \in \mathbb{Z} \\ &= n\text{Ln}z \end{aligned}$$

Problem 7

求

$$i^{\sqrt{3}i}$$

Solution

$$\begin{aligned} i^{\sqrt{3}i} &= e^{\sqrt{3}i\mathrm{Ln}i} \\ &= e^{\sqrt{3}i(\frac{\pi}{2}i+2k\pi i)} \\ &= e^{-\sqrt{3}(\frac{1}{2}+2k)\pi} \qquad k \in \mathbb{Z} \end{aligned}$$