Introduction

Problem 1

求

$$\max|\alpha z^n + \beta| \qquad |z| \le r$$

Solution

$$\max|\alpha z^n + \beta| = |\alpha|\max|z^n + \frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max|z^n + \alpha| \qquad |z| \le r$$

有

$$|z^n + \alpha| \le |z^n| + |\alpha| \le r^n + |\alpha|$$

取最大值 $\Leftrightarrow$ 等号成立 $\Leftrightarrow$ z^n与α同向 $\Leftrightarrow$ $z^n = \lambda\alpha \quad \lambda > 0$
当 $\alpha = 0$ 时

$$|z^n| = |z|^n = r^n$$
$$|z| = r$$
$$\max|\alpha z^n + \beta| = r^n \qquad z = re^i \quad \theta \in [0, 2\pi)$$

当 $\alpha \neq 0$ 时

$$\max|z^n + \alpha| = r^n + |\alpha|$$

此时

$$r^n = |z^n| = |\lambda\alpha|$$
$$\lambda = \frac{|r|^n}{|\alpha|}$$
$$z^n = \lambda\alpha = \frac{|r|^n}{|\alpha|}\alpha$$

设 $\alpha = r_0 e^{i\theta_0}$

$$\begin{aligned} z^n &= \frac{|r|^n}{|\alpha|}\alpha \\ &= \frac{|r|^n}{|\alpha|} r_0 e^{i\theta_0} \\ &= |r|^n e^{i\theta_0} \end{aligned}$$

则取最大值时

$$z = z_k = re^{\frac{i(\theta_0 + 2k\pi)}{n}} \qquad k = 1, 2, \cdots, n$$

综上，

$$\max|z^n + \alpha| = \begin{cases} r^n & z = re^{i\theta} \quad \theta \in [0, 2\pi) & \alpha = 0 \\ r^n + |\alpha| & z = re^{\frac{i(\theta + 2k\pi)}{n}} \quad k = 1, 2, \cdots, n & \alpha \neq 0 \end{cases}$$

Residues

Problem 2

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^n}, z_k] \\ &= 2\pi i \sum_{k=1}^{n} \frac{1}{n z_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}&\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z \\ =&\frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}} \\ =&-\frac{2\pi i}{n} \sum_{k=1}^{n} z_k\end{aligned}$$

若 $n=1$，即

$$1+z^n = 1+z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 2\pi i$$

否则

$$\sum_{k=1}^{n} z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = 0$$

Problem 2 continued on next page...

又

$$\begin{aligned}\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z &= \oint_{|z|=r>1}\frac{(z^{2n}-1)+1}{1+z^n}\mathrm{d}z\\ &= \oint_{|z|=r>1}\frac{(z^n-1)(z^n+1)}{1+z^n}\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\\ &= \oint_{|z|=r>1}(z^n-1)\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1}(z^n-1)\mathrm{d}z=0$$

所以

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z=\begin{cases}2\pi i & n=1\\ 0 & n\geq 2\end{cases}$$

Solution 2

令 $z=re^{i\theta}\quad 0\leq\theta<\pi, t=\frac{1}{z}$，则

$$|t|=\frac{1}{r}<1$$
$$t=\frac{1}{r}e^{-i\theta}$$
$$\mathrm{d}z=\mathrm{d}(\frac{1}{t})=-\frac{1}{t^2}\mathrm{d}t\qquad 0\leq\theta<\pi\quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1}\frac{1}{1+\frac{1}{t^2}}(-\frac{1}{t^2})\mathrm{d}t\\ &= \oint_{|t|=\frac{1}{r}<1}\frac{t^{n-2}}{1+t^n}\mathrm{d}t\end{aligned}$$

只有一个奇点 $t=0$。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1}\frac{t^{n-2}}{1+t^n}\mathrm{d}t\\ &= \begin{cases}0 & n\geq 2 \qquad (\text{Cauchy-Goursat 定理})\\ \oint_{|t|=\frac{1}{r}<1}\frac{1}{t(t+1)}\mathrm{d}t=2\pi i & n=1\end{cases}\end{aligned}$$

Problem 3

求积分

$$\oint_{|z|=r>0}\frac{1-\cos 4z^3}{z^n}\mathrm{d}z\qquad n\in\mathbb{Z}$$

Problem 3 continued on next page...

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n} \mathrm{d}z = 0$$

当 $n > 0$ 时

$$\begin{aligned}\frac{1-\cos 4z^3}{z^n} &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k \frac{(4z^3)^{2k}}{(2k)!}) \\ &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k \frac{4^{2k}z^{6k}}{(2k)!}) \\ &= z^{-n}\sum_{k=1}^{n}(-1)^{k-1} \frac{4^{2k}z^{6k}}{(2k)!}\end{aligned}$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z=0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中，C_{-1} 对应上式中

$$6k - n = -1$$

此时

$$\begin{aligned}C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k = \frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

所以 $n > 0$ 时

$$\begin{aligned}\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n} \mathrm{d}z &= 2\pi i \mathrm{Res}[\frac{1-\cos 4z^3}{z^n}, 0] \\ &= 2\pi i C_{-1} \\ &= 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

综上，

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n} \mathrm{d}z = \begin{cases} 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k+1, k \in \mathbb{N} \\ 0 & n \neq 6k+1, k \in \mathbb{N} \end{cases}$$

Problem 4

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z$$

Problem 4 continued on next page...

Solution

令 $t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$
$$t = \frac{1}{r}e^{-i\theta}$$
$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0 \le \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t \\ &= 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0]\end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)(1 - t + t^2 - t^3 + \dots)$$

其中，t^{-1} 的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z = 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

Problem 5

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}, \theta \in [0, 2\pi)$，注意到:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$
$$\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \frac{\mathrm{d}z}{iz}$$

则原积分

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} &= \oint_{|z|=1} \frac{\frac{\mathrm{d}z}{iz}}{a + b(\frac{z^2+1}{2z})} \\ &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{bz^2 + 2az + b}\end{aligned}$$

Problem 5 continued on next page...

当 $b=0$ 时，原积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a}=\frac{2\pi}{a}$$

当 $b\neq 0$ 时，原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{i}\oint_{|z|=1}\frac{\mathrm{d}z}{bz^2+2az+b}\\&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\end{aligned}$$

对方程

$$z^2+\frac{2a}{b}z+1=0$$

其两根

$$z_1z_2=1$$

且

$$z_{1,2}=-\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b}$$

可设 $b>0$，则

$$z_2=-\frac{a}{b}\frac{\sqrt{a^2-b^2}}{b}<-\frac{a}{b}<-1$$

即只有一个奇点 z_1。所以原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\\&=\frac{2}{ib}2\pi i\frac{1}{2z_1+\frac{2a}{b}}\\&=\frac{2\pi}{\sqrt{a^2-b^2}}\end{aligned}$$

Problem 6

求积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}\qquad a>|b|\quad a,b\in\mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1,1]$ 上可积，则 $\int_0^{2\pi}f(\cos x)\mathrm{d}\theta=\int_0^{2\pi}f(\sin x)\mathrm{d}\theta$

所以

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$$

Problem 7

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Solution

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

Problem 8

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} > 0$$

Solution

令

$$\begin{aligned} \cos^2\theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2\theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

Problem 8 continued on next page...

则

$$
\begin{aligned}
I_{A,B} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\dfrac{\cos 2\theta + 1}{2} + B^2\dfrac{1-\cos 2\theta}{2}} \\
&= \int_0^{4\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\int_0^{2\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\frac{2\pi}{\sqrt{(A^2+B^2)^2 - (A^2-B^2)^2}} \\
&= \frac{2\pi}{AB}
\end{aligned}
$$

Problem 9

求积分

$$
I_n = \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x
$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数，因此

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} \\
&= \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)}
\end{aligned}
$$

而

$$
\begin{aligned}
&\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)} + \int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(z^2+a^2)(z^2+b^2)}\mathrm{d}z \\
=&2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
=&2\pi i(\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2) + ab})
\end{aligned}
$$

定理. *若 $P_n(z), Q_m(z)$ 是多项式，且 $\mathrm{deg}P_n = n \leq \mathrm{deg}Q_m - 2 = m - 2$，$Q_m(z)$ 在实轴 $z = x$ 上没有零点，即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$，则*

$$
\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{P_n(z)}{Q_m(z)}\mathrm{d}z = 0
$$

所以

$$\begin{aligned}
I_n &= \frac{1}{2}2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
&= \pi i(\frac{1}{4(i|a|)^3+2i|a|(a^2+b^2)} + \frac{1}{4(i|b|)^3+2i|b|(a^2+b^2)}) \\
&= \frac{\pi}{2(|b|-|a|)|a||b|}
\end{aligned}$$

Problem 10

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\
&= \frac{1}{2}2\pi i \sum_{k=0}^{i} \mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2}\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{1}{1+z^{2n}}\mathrm{d}z \\
&= \frac{1}{2}2\pi i \sum_{k=0}^{i} \mathrm{Res}[\frac{1}{1+z^{2n}}, z_k]
\end{aligned}$$

其中

$$z_k^{2n} + 1 = 0, \mathrm{Im} z_k > 0$$

则

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \\
&= \frac{1}{2}2\pi i \sum_{k=0}^{i} \mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] \\
&= \pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] \\
&= \pi i \sum_{k=1}^{n} \frac{1}{2nz_k^{2n-1}} \\
&= -\frac{\pi i}{2n}\sum_{k=1}^{n} z_k
\end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

Problem 10 continued on next page...

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i}n}{e^{\pi i}2n} \qquad k = 1, 2, \cdots, 2n$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{2n}\sum_{k=0}^{i} z_k \\
&= -\frac{\pi i}{2n}\frac{\sum_{k=1}^{n} e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\
&= -\frac{\pi i}{2n}\frac{e^{\frac{\pi i}{n}}(1 - e^{\pi i})}{e^{\frac{\pi i}{2n}}(1 - e^{\frac{\pi i}{n}})} \\
&= -\frac{\pi i}{n}\frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}}
\end{aligned}$$

令 $\theta = \frac{\pi i}{2n}$，则

$$\begin{aligned}
\frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin\theta} \\
&= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta} \\
&= \frac{1}{-2i\sin\theta}
\end{aligned}$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{n}\frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\
&= -\frac{\pi i}{n}\frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin\theta} \\
&= -\frac{\pi i}{n}\frac{1}{-2i\sin\theta} \\
&= \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}
\end{aligned}$$

Problem 11

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \\
&= \frac{1}{r^{2n}}\int_0^{+\infty} \frac{\mathrm{d}x}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}}\int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} I_n
\end{aligned}$$

Problem 11 continued on next page...

Problem 12

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2} 2\pi(i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(1+z^2)^n}\mathrm{d}z) \\
&= \frac{1}{2}(2\pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]) \\
&= \pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]
\end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

令 $f(z) = \frac{1}{(z+i)^n}$

$$\begin{aligned}
\frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
&= (z-i)^{-n}\sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!}(z-i)^k
\end{aligned}$$

要求 $(z-i)^{-1}$ 对应的系数 C_{-1}，对应于 $k = n-1$

$$\begin{aligned}
C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
&= (-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
\end{aligned}$$

因此

$$\begin{aligned}
J_n &= \pi i C_{-1} \\
&= \pi i(-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
&= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
\end{aligned}$$

Problem 13

求积分

$$J_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2+x^2)^n} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2+x^2)^n} \\
&= \int_0^{+\infty} \frac{1}{r^{2n}} \frac{r\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n} \\
&= \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n} \\
&= \frac{1}{r^{2n-1}} J_n
\end{aligned}$$

Problem 14

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x$$

Solution

设 $a \neq b$ 则

$$\begin{aligned}
I_{a,b,k} &= \int_0^{+\infty} \frac{x \sin kx}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x \\
&= \frac{1}{2} \mathrm{Im} \int_{-\infty}^{+\infty} \frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x
\end{aligned}$$

而

$$\begin{aligned}
\int_0^{+\infty} \frac{xe^{x}}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x &= 2\pi i \{\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)}, ai] + \mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)}, bi]\} \\
&= 2\pi i [\frac{aie^{-ka}}{4(ai)^3+2ai(a^2+b^2)} + \frac{bie^{-kb}}{4(bi)^3+2bi(a^2+b^2)}] \\
&= \frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})
\end{aligned}$$

所以

$$\begin{aligned}
I_{a,b,k} &= \frac{1}{2} \mathrm{Im} \int_{-\infty}^{+\infty} \frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)} \mathrm{d}x \\
&= \frac{1}{2} \mathrm{Im} [\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})] \\
&= \frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb})
\end{aligned}$$

Problem 14 continued on next page...

$a = b$ 时

$$\begin{aligned}
I_{a,b,k} &= \lim_{a\to b} \frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb}) \\
&= \frac{-k\pi e^{-kb}}{-4b} \\
&= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}}
\end{aligned}$$

Problem 15

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

$a \neq b$ 时

$$\begin{aligned}
I_{a,b,k} &= \int_0^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\mathrm{Re}\int_{-\infty}^{+\infty} \frac{x^2 e^{ikx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{(be^{-kb}-ae^{-ka})\pi}{2(b^2-a^2)}
\end{aligned}$$

当 $a = b$ 时

$$I_{a,b,k} = \frac{(1-ka)\pi}{4ae^{ka}} = \frac{(1-kb)\pi}{4be^{kb}}$$