Problems

Problem 1

求证

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z=\begin{cases}2\pi i & n=1\\ 0 & n\geq 2\end{cases}$$

Solution 1

$$\begin{aligned}\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z&=2\pi i\sum_{k=1}^{n}\mathrm{Res}[\frac{1}{1+z^n},z_k]\\&=2\pi i\sum_{k=1}^{n}\frac{1}{nz_k^{n-1}}\\&=\frac{2\pi i}{n}\sum_{k=1}^{n}\frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n=1$$

则

$$-z_k=\frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}&\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\\=&\frac{2\pi i}{n}\sum_{k=1}^{n}\frac{1}{z_k^{n-1}}\\=&-\frac{2\pi i}{n}\sum_{k=1}^{n}z_k\end{aligned}$$

若 $n=1$，即

$$1+z^n=1+z=0$$

解得

$$z=-1$$

因此

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=2\pi i$$

否则

$$\sum_{k=1}^{n}z_k=0$$

即

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=0$$

Problem 1 continued on next page...

所以

$$\begin{aligned}\frac{z^{2n}}{1+z^n} &= \frac{(z^{2n}-1)+1}{1+z^n} \\ &= \frac{(z^n-1)(z^n+1)}{1+z^n} + \frac{1}{1+z^n} \\ &= z^n - 1 + \frac{1}{1+z^n}\end{aligned}$$

又

$$\oint_{|z|=r>1} z^n - 1 = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 2

令 $z = re^{i\theta} \quad 0 \leq \theta < \pi, t = \frac{1}{z}$，则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^2}}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t\end{aligned}$$

只有一个奇点 $t=0$。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t \\ &= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)}\mathrm{d}t = 2\pi i & n=1 \end{cases} \qquad \text{(Cauchy-Goursat 定理)}\end{aligned}$$

Problem 2

求积分

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}$$

Solution

当 $n \leq 0$ 时

$$\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = 0$$

Problem 2 continued on next page...

当 $n > 0$ 时

$$\begin{aligned}\frac{1-\cos 4z^3}{z^n} &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{(4z^3)^{2k}}{(2k)!}) \\ &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{4^{2k}z^{6k}}{(2k)!}) \\ &= z^{-n}\sum_{k=1}^{n}(-1)^{k-1}\frac{4^{2k}z^{6k}}{(2k)!}\end{aligned}$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z=0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中，C_{-1} 对应上式中

$$6k-n=-1$$

此时

$$\begin{aligned}C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k=\frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

所以 $n > 0$ 时

$$\begin{aligned}\oint_{|z|=r>0}\frac{1-\cos 4z^3}{z^n}\mathrm{d}z &= 2\pi i\mathrm{Res}[\frac{1-\cos 4z^3}{z^n},0] \\ &= 2\pi i C_{-1} \\ &= 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}\end{aligned}$$

综上，

$$\oint_{|z|=r>0}\frac{1-\cos 4z^3}{z^n}\mathrm{d}z = \begin{cases} 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n=6k+1, k\in\mathbb{N} \\ 0 & n\neq 6k+1, k\in\mathbb{N}\end{cases}$$

Problem 3

求积分

$$\oint_{|z|=r>1}\frac{z^3 e^{\frac{1}{z}}}{1+z}\mathrm{d}z$$

Solution

令 $t=\frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0\le\theta<\pi \quad \text{积分方向为顺时针}$$

原积分

$$
\begin{aligned}
\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t \\
&= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t \\
&= 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0]
\end{aligned}
$$

又

$$
\frac{e^t}{t^4(t+1)} = t^{-4}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)(1 - t + t^2 - t^3 + \dots)
$$

其中，t^{-1} 的系数为

$$
-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}
$$

因此

$$
\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z = 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i
$$

Problem 4

求积分

$$
\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta}
$$

Problem 5

求积分

$$
\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta}
$$

Problem 6

求积分

$$
I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)
$$

Problem 7

求积分

$$
I_{A,B} = \int_0^{2\pi} \mathrm{d}\theta A^2 \cos^2\theta + B^2 \sin^2\theta \qquad A, B \in \mathbb{R} \mathrm{and} A, B > 0
$$

Problem 8

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} \mathrm{and} AB > 0$$

Problem 9

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1 + x^{2n}} \qquad n \in \mathbb{N}$$

Problem 10

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Problem 11

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 12

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2 + x^2)^n} \qquad n \in \mathbb{N}$$

Problem 13

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x\sin kx}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x$$

Problem 14

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2\cos kx}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x$$