Introduction

Problem 1

求

$$\max|\alpha z^n + \beta| \qquad |z| \leq r$$

Solution

$$\max|\alpha z^n + \beta| = |\alpha|\max|z^n + \frac{\beta}{\alpha}|$$

因此下面只讨论

$$\max|z^n + \alpha| \qquad |z| \leq r$$

有

$$|z^n + \alpha| \leq |z^n| + |\alpha| \leq r^n + |\alpha|$$

取最大值 $\Leftrightarrow$ 等号成立 $\Leftrightarrow$ z^n与α同向 $\Leftrightarrow$ $z^n = \lambda\alpha \quad \lambda > 0$
当 $\alpha = 0$ 时

$$|z^n| = |z|^n = r^n$$
$$|z| = r$$
$$\max|\alpha z^n + \beta| = r^n \qquad z = re^i \quad \theta \in [0, 2\pi)$$

当 $\alpha \neq 0$ 时

$$\max|z^n + \alpha| = r^n + |\alpha|$$

此时

$$r^n = |z^n| = |\lambda\alpha|$$
$$\lambda = \frac{|r|^n}{|\alpha|}$$
$$z^n = \lambda\alpha = \frac{|r|^n}{|\alpha|}\alpha$$

设 $\alpha = r_0 e^{i\theta_0}$

$$\begin{aligned} z^n &= \frac{|r|^n}{|\alpha|}\alpha \\ &= \frac{|r|^n}{|\alpha|} r_0 e^{i\theta_0} \\ &= |r|^n e^{i\theta_0} \end{aligned}$$

则取最大值时

$$z = z_k = re^{\frac{i(\theta_0 + 2k\pi)}{n}} \qquad k = 1, 2, \cdots, n$$

综上，

$$\max|z^n + \alpha| = \begin{cases} r^n & z = re^{i\theta} \quad \theta \in [0, 2\pi) \qquad\qquad \alpha = 0 \\ r^n + |\alpha| & z = re^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 1, 2, \cdots, n \quad \alpha \neq 0 \end{cases}$$

Problem 2

$$|z|^2 = z\overline{z} \qquad z^2 = |z|^2$$

等号成立的条件是?

Solution

$$z^2 = |z|^2 = z\overline{z} \Leftrightarrow z(z-\overline{z}) = 0 \Leftrightarrow z = \overline{z}$$

即 $z \in \mathbb{R}$ 时等号成立。

Problem 3

证明

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$$

并说明几何意义

Solution

$$\begin{aligned}
|z_1+z_2|^2 + |z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2)(\overline{z_1}-\overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\
&= 2(z_1\overline{z_1} + z_2\overline{z_2}) \\
&= 2(|z_1|^2+|z_2|^2)
\end{aligned}$$

几何意义：平行四边形对角线平方和等于对边平方和

Problem 4

$|z_1| = |z_2| = |z_3| = |z_4| = r$且$z_1+z_2+z_3+z_4 = 0$ 则 z_1,z_2,z_3,z_4 满足什么条件时 $z_1z_2z_3z_4$ 构成正方形?

Solution

定理. $z_1, z_2, \cdots, z_n$ 将圆 $|z| = \alpha n$ 等分 $\Leftrightarrow z_k$ 是分圆多项式 $z^n + \alpha = 0$

当 z_1,z_2,z_3,z_4 构成正方形时，$(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ 是分圆多项式。

又

$$\begin{aligned}
&(z-z_1)(z-z_2)(z-z_3)(z-z_4) \\
=&z^4 - \sum_{k=1}^{4} z_k z^3 + (z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4)z^2 \\
&-(z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)z + z_1z_2z_3z_4
\end{aligned}$$

$(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ 是分圆多项式

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{4} z_k & = 0 \\ z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4 & = 0 \\ z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4 & = 0 \\ z_1z_2z_3z_4 & \neq 0 \end{cases}$$

Problem 4 continued on next page...

而

$$z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = z_1z_2z_3z_4(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4})$$
$$z_k\overline{z_k} = |z_k|^2 = r^2$$

所以

$$\frac{1}{z_k} = \frac{\overline{z_k}}{r^2}$$
$$z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = \frac{z_1z_2z_3z_4}{r^2}(\overline{z_1} + \overline{z_2} + \overline{z_3} + \overline{z_4}) = 0$$

所以 z_1,z_2,z_3,z_4 需满足

$$z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = 0$$

Problem 5

$f(z)$ 在 z_0 连续，$f(z_0) \neq 0$。求证 $\exists\delta > 0$，当 $|z - z_0| < \delta$ 时有 $f(z) \neq 0$

Solution

因 $f(z_0) \neq 0$，有 $|f(z_0)| > 0$

令 $\varepsilon = \frac{1}{2}|f(z_0)| < |f(z_0)|$，因为 $f(z)$ 在 z_0 连续，存在 $\delta > 0$ 使得 $\forall z \quad |z - z_0| < \delta$

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \frac{1}{2}|f(z_0)|$$

即

$$\frac{1}{2}|f(z_0)| < |f(z)| < \frac{3}{2}|f(z_0)|$$

Chapter 2 Analytic Function

Problem 6

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ 且 $u,v\in C^{(n)}$，求 $f(z)$n 阶可导的 Cauchy-Riemann 条件和 $f^{(n)}(z)$

Solution

设 $f'(z)=A+iB$，则

$$\begin{aligned}\mathrm{d}f=f'(z)\mathrm{d}z=f'(z)(\mathrm{d}x+i\mathrm{d}y)\Leftrightarrow \mathrm{d}f &= \mathrm{d}u+i\mathrm{d}v\\ &= \tfrac{\partial u}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y+i(\tfrac{\partial v}{\partial x}\mathrm{d}x+\tfrac{\partial u}{\partial y}\mathrm{d}y)\\ &= (A\mathrm{d}x-B\mathrm{d}y)+i(B\mathrm{d}x+A\mathrm{d}y)\end{aligned}$$

由上式得

$$\begin{cases} A\mathrm{d}x-B\mathrm{d}y=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\\ B\mathrm{d}x+A\mathrm{d}y=\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\end{cases}$$

解得

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=A\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-B\end{cases}$$

即

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=F'(z)$$

而

$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+i\frac{\partial^2 v}{\partial x^2}$$

由归纳法可证明

$$f^{(n)}(z)=\frac{\partial^n u}{\partial x^n}+i\frac{\partial^n v}{\partial x^n}$$

u,v 需要满足 Cauchy-Riemann 条件

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$

Problem 7

求 $\cos(x+iy)$ 的实部和虚部，其中 $x,y\in\mathbb{R}$

Solution

$$\begin{aligned}\cos(x+iy)&=\frac{1}{2}(e^{-y+ix}+e^{y-ix})\\ &=\frac{1}{2}(e^{-y}e^{ix}+e^{y}e^{-ix})\\ &=\frac{1}{2}[e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)]\\ &=\frac{1}{2}(e^{y}+e^{-y})\cos x+i\frac{1}{2}(-e^{y}+e^{-y})\sin x\end{aligned}$$

Problem 8

求证：$\forall A, B \in \mathbb{R}$ 存在 $z = x + iy$ 使得 $\cos(x+iy) = A + iB$（即 $\mathrm{Im}[\cos(z)] = \mathbb{C}$）

Solution

令

$$\begin{aligned} \frac{e^y + e^{-y}}{2}\cos x = A \\ \frac{e^{-y} - e^y}{2}\sin x = B \end{aligned} \tag{1}$$

1. 当 $B = 0$ 时，由式 (1) 知 $y = 0$ 或 $\sin x = 0$。
   $|A| \leq 1$ 时可令 $y = 0$，此时
   $$\cos x = A$$
   解得
   $$\begin{cases} x = \arccos A + 2k\pi & k \in \mathrm{Z} \\ y = 0 \end{cases}$$
   $|A| > 1$ 时，令 $\sin x = 0$ 得
   $$\cos x = \pm 1$$
   $$\frac{e^y + e^{-y}}{2} = |A| > 1$$
   考察函数 $f(y) = e^y + e^{-y}2 - 1$
   $$f(0) = 0$$
   $$\lim_{y \to +\infty} f(y) = \lim_{y \to -\infty} f(y) = +\infty$$
   且 $f(y)$ 连续。因此存在 y_A 使得 $\pm y_A$ 是方程 $\frac{e^y+e^{-y}}{2} = |A|$ 的解。
   此时
   $$\begin{cases} x = k\pi & k \in \mathrm{Z} \\ y = \pm y_A \end{cases}$$
2. 当 $B \neq 0$ 时，由式 (1) 知 $y \neq 0$。结合 $\cos^2 x + \sin^2 x = 1$ 得 $y \in (-\infty, 0) \cup (0, +\infty)$ 时
   $$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$
   令 $f_{A,B}(y) = 4A^2(e^{-y} + e^y)^2 + \frac{4B^2}{(e^{-y}-e^y)^2}$，$f_{A,B}(y)$ 是偶函数。
   $$\lim_{y \to 0^+} f_{A,B}(y) = +\infty$$
   $$\lim_{y \to +\infty} f_{A,B}(y) = 0$$
   因此 $\exists y_{A,B} > 0$，使得 $\pm y_{A_B}$ 是方程
   $$\frac{4A^2}{(e^{-y} + e^y)^2} + \frac{4B^2}{(e^{-y} - e^y)^2} = 1$$
   的根。将 $\pm y_{A,B}$ 代入式 (1) 可解出对应的 x。

Problem 9

已知 $e^w = z \neq 0$，求

$$w = \mathrm{Ln} z$$

Solution

设 $w = u + iv \quad u, v \in \mathbb{R}$，则

$$e^w = e^{u+iv} = e^u e^{iv} = z = re^{i\theta}$$
$$\theta = \arg z \in [0, 2\pi) \qquad r = |z| > 0$$

则

$$\begin{aligned} & e^u && = r \\ \Rightarrow \quad & u && = \ln r \end{aligned}$$

且

$$\begin{aligned} & e^{iv} && = e^{i\theta} \\ \Rightarrow \quad & v && = \theta + 2k\pi \qquad k \in \mathbb{Z} \\ & && = \arg z \end{aligned}$$

所以

$$\begin{aligned} w = u + iv \quad & = \mathrm{Ln} z \\ & = \ln z + 2k\pi i && k \in \mathbb{Z} \\ & = \ln |z| + i \arg z + 2k\pi i && k \in \mathbb{Z} \end{aligned}$$

定理. *Picard* 小定理 若 $f(z)$ 是解析函数且 $f(z)$ 不是常数，则除去最多一个例外 w_0，方程 $f(z) = A + iB = w$ 至少有一个解 z。

Problem 10

求

$$\mathrm{Ln}(3 + 2i)$$

Solution

$$\begin{aligned} \mathrm{Ln}(3 + 2i) \quad & = \ln(3 + 2i) + 2k\pi && k \in \mathbb{Z} \\ & = \ln 13 + i \arg(3 + 2i) + 2k\pi && k \in \mathbb{Z} \end{aligned}$$

Problem 11

求

$$\mathrm{Ln} z^n$$

Solution

$$\begin{aligned} \mathrm{Ln} z^n \quad & = \ln z^n + 2k\pi && k \in \mathbb{Z} \\ & = \ln |z^n| + i \arg z^n + 2k\pi && k \in \mathbb{Z} \\ & = n \ln |z| + ni \arg z + 2k\pi && k \in \mathbb{Z} \\ & = n \mathrm{Ln} z \end{aligned}$$

Problem 12

求

$$i^{\sqrt{3}i}$$

Solution

$$\begin{aligned} i^{\sqrt{3}i} &= e^{\sqrt{3}i\mathrm{Ln}i} \\ &= e^{\sqrt{3}i(\frac{\pi}{2}i+2k\pi i)} \\ &= e^{-\sqrt{3}(\frac{1}{2}+2k)\pi} \qquad k \in \mathbb{Z} \end{aligned}$$

Chapter 3 Complex Integral

定理. *Cauchy-Goursat* 定理 若 C 分段光滑，且 $f(z)$ 在 C 上连续，在 C 内处处可导，则 $\oint_C f(z)\mathrm{d}z = 0$

定理. *Cauchy* 高阶导数公式

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{C_r}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$$

这里 $C_r = |z - z_0| = r$

定理. *Lioville* 定理 有界的解析函数是常数

Problem 13

求证，$f(z)$ 是解析函数则

$$f(z)\text{的像是}\begin{cases}\text{二维区域} & f(z) \not\equiv C\\ \text{点} & f(z) \equiv C\end{cases}$$

Solution

有

$$J_{(x,y)} = \begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}_{(x,y)}$$

$$\frac{\Delta u\Delta v}{\Delta x\Delta y} = |detJ|_{(x,y)} = |\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}|_{(x,y)}$$

因为 $f(z) = u + iv$ 是解析函数

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

所以

$$\frac{\Delta u\Delta v}{\Delta x\Delta y} = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 = |f'(x)|^2 \geq 0$$

即

$$\Delta u\Delta v = |f'(x)|^2\Delta x\Delta y$$

若 $f'(x) \not\equiv 0$，那么 $f(x)$ 不是常数。此时假设 $f'(z_0) \neq 0$，则

$$\exists\delta > 0\text{使得}|z - z_0| < \delta\text{时}f'(z) \neq 0$$

$|z - z_0| < \delta$ 时

$$\Delta u\Delta v > 0$$

即像是二维区域。
当 $f'(z) \equiv 0$ 时，$f(z)$ 是常数，这时 $f(z)$ 的像是一个点

Chapter 4 Series

定义. 幂级数

$$\sum_{n=0}^{+\infty} C_n(z-z_0)^n$$

定义. *Fourier* 级数

$$\sum_{n=0}^{+\infty} C_n e^{in\theta} = \sum_{n=0}^{+\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

定义. *Taylor* 级数

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

定义. *Laurent* 级数

$$\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n$$

定理. *Abel* 定理 若 $f(z)=\sum_{n=0}^{+\infty} C_n z^n$ 在 z_0 收敛，则 $\forall z$ 有 $|z|<|z_0|$ 时 $f(z)$ 绝对收敛。若存在 z_0，$f(z)$ 在 z_0 发散，则 $\forall z$ 有 $|z|>|z_0|$ 时 $f(z)$ 发散。(即幂级数的收敛域是圆盘)

定义. 收敛半径 若存在常数 $R>0$，当 $|z|<R$ 时，$f(z)$ 绝对收敛，而当 $|z|>R$ 时，$f(z)$ 发散，这时 R 称为 $f(z)$ 的收敛半径。

定理. 若

$$\lim_{n\to+\infty} |\frac{C_n}{C_{n+1}}| = \lambda$$

则 $R=\lambda$

定理. 若

$$\lim_{n\to+\infty} |\frac{1}{\sqrt{n}C_n}| = \lambda$$

则 $R=\lambda$

定理. 若 $f(z)$ 只有有限个奇点，则离原点最近的奇点 z_0 的模即为收敛半径。

定理. 若 $f(z)$ 在 z_0 处条件收敛，则 $R=|z_0|$

定理. 若 $f(z)=\sum_{n=0}^{+\infty} C_n z^n$ 满足 $C_n = a_n + b_n \quad a_n, b_n \in \mathbb{R}$ 且 $\sum_{n=0}^{+\infty} a_n z^n$ 的收敛半径是 R_1，$\sum_{n=0}^{+\infty} b_n z^n$ 的收敛半径是 R_2，则 $R=\min\{R_1, R_2\}$

Problem 14

将 $\frac{1}{z-b}$ 在 $z_0=a$ 处展成 Laurent 级数，$a \neq b$

Solution

Problem 14 continued on next page...

$$
\begin{aligned}
\frac{1}{z-b} &= \frac{1}{-(b-a)+(z-a)} \\
&= \frac{1}{a-b}\frac{1}{1-\frac{z-a}{b-a}} \\
&= \frac{1}{a-b}\sum_{n=0}^{+\infty}\frac{(z-a)^n}{(b-a)^n} \\
&= \frac{1}{a-b}\sum_{n=0}^{+\infty}\frac{(-1)(z-a)^n}{(b-a)^{n+1}}
\end{aligned}
$$

条件

$$|\frac{z-a}{b-a}| < 1$$

即

$$0 \le |z-a| < |b-a|$$

Problem 15

求

$$f(z) = \frac{1}{(z-1)(z-2)}$$

的 Laurent 级数

Solution

1. 当 $0 < |z-1| < 1$ 时

$$
\begin{aligned}
f(z) &= \frac{(z-1)-(z-2)}{(z-1)(z-2)} \\
&= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z-1-1} - \frac{1}{z-1} \\
&= -\frac{1}{1-(z-1)} - \frac{1}{z-1} \\
&= -\sum_{n=0}^{+\infty}(z-1)^n - \frac{1}{z-1} \\
&= \sum_{n=-1}^{+\infty} -(z-1)^n \qquad 0 < |z-1| < 1
\end{aligned}
$$

Problem 15 continued on next page...

2. 当 $|z-1|>1$ 时

$$
\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z-1-1} - \frac{1}{z-1} \\
&= \frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} - \frac{1}{z-1} \\
&= \frac{1}{z-1}\sum_{n=0}^{+\infty}(\frac{1}{z-1})^n - \frac{1}{z-1} \\
&= \sum_{n=0}^{+\infty}(\frac{1}{z-1})^{n+1} - \frac{1}{z-1} \\
&= \sum_{n=1}^{+\infty}(\frac{1}{z-1})^{n+1} \\
&= \sum_{n=-\infty}^{-2}(\frac{1}{z-1})^{n}
\end{aligned}
$$

3. 在 $z=2$ 处展开同理

Chapter 5 Residues

定理. 对函数 $f(z)$，若 $f(z)$ 在 C 上连续，在 C 内有 n 个奇点 $z_1, z_2, \cdots, z^n$。设 $f(z)$ 在 z_k 附近可以展成 *Laurent* 级数

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n^{(k)}(z-z_k)^n$$

则

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} C_{-1}^{(k)}$$

称 $C_{-1}^{(k)}$ 为 $f(z)$ 在 z_k 点的留数，记作 $\mathrm{Res}[f, z_k]$。即

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f, z_k]$$

定理. 若 z_0 是 f 的一个一阶极点，且 $f(z) = \frac{P(z)}{Q(z)}$，其中 $P(z)$ 与 $Q(z)$ 在 z_0 点解析，$P(z_0) \neq 0, Q(z_0) = 0, Q'(z_0) \neq 0$，则 $\mathrm{Res}[f, z_0] = \frac{P(z_0)}{Q'(z_0)}$

Problem 16

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n}\mathrm{d}z = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

Solution 1

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}[\frac{1}{1+z^n}, z_k] \\ &= 2\pi i \sum_{k=1}^{n} \frac{1}{n z_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}}\end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned}&\oint_{|z|=r>1} \frac{1}{1+z^n}\mathrm{d}z \\ =&\frac{2\pi i}{n} \sum_{k=1}^{n} \frac{1}{z_k^{n-1}} \\ =&-\frac{2\pi i}{n} \sum_{k=1}^{n} z_k\end{aligned}$$

Problem 16 continued on next page...

若 $n=1$，即

$$1+z^n=1+z=0$$

解得

$$z=-1$$

因此

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=2\pi i$$

否则

$$\sum_{k=1}^{n}z_k=0$$

即

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=0$$

又

$$\begin{aligned}\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z&=\oint_{|z|=r>1}\frac{(z^{2n}-1)+1}{1+z^n}\mathrm{d}z\\&=\oint_{|z|=r>1}\frac{(z^n-1)(z^n+1)}{1+z^n}\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\\&=\oint_{|z|=r>1}(z^n-1)\mathrm{d}z+\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z\end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1}(z^n-1)\mathrm{d}z=0$$

所以

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z$$

综上，

$$\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z=\oint_{|z|=r>1}\frac{z^{2n}}{1+z^n}\mathrm{d}z=\begin{cases}2\pi i & n=1\\ 0 & n\geq 2\end{cases}$$

Solution 2

令 $z=re^{i\theta}\quad 0\leq\theta<\pi, t=\frac{1}{z}$，则

$$|t|=\frac{1}{r}<1$$

$$t=\frac{1}{r}e^{-i\theta}$$

$$\mathrm{d}z=\mathrm{d}(\frac{1}{t})=-\frac{1}{t^2}\mathrm{d}t\qquad 0\leq\theta<\pi\quad\text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1}\frac{1}{1+z^n}\mathrm{d}z&=-\oint_{|t|=\frac{1}{r}<1}\frac{1}{1+\frac{1}{t^2}}(-\frac{1}{t^2})\mathrm{d}t\\&=\oint_{|t|=\frac{1}{r}<1}\frac{t^{n-2}}{1+t^n}\mathrm{d}t\end{aligned}$$

Problem 16 continued on next page...

只有一个奇点 $t=0$。因此

$$
\begin{aligned}
I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n}\mathrm{d}t \\
&= \begin{cases} 0 & n \geq 2 \qquad \text{(Cauchy-Goursat 定理)} \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)}\mathrm{d}t = 2\pi i & n = 1 \end{cases}
\end{aligned}
$$

Problem 17

求积分

$$
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z \qquad n \in \mathbb{Z}
$$

Solution

当 $n \leq 0$ 时

$$
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = 0
$$

当 $n > 0$ 时

$$
\begin{aligned}
\frac{1-\cos 4z^3}{z^n} &= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{(4z^3)^{2k}}{(2k)!}) \\
&= z^{-n}(1-\sum_{k=0}^{n}(-1)^k\frac{4^{2k}z^{6k}}{(2k)!}) \\
&= z^{-n}\sum_{k=1}^{n}(-1)^{k-1}\frac{4^{2k}z^{6k}}{(2k)!}
\end{aligned}
$$

$\frac{1-\cos 4z^3}{z^n}$ 在奇点 $z=0$ 的 Laurent 级数 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 中，C_{-1} 对应上式中

$$
6k-n=-1
$$

此时

$$
\begin{aligned}
C_{-1} &= (-1)^{k-1}\frac{4^{2k}}{(2k)!} \qquad k = \frac{n-1}{6} \\
&= (-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

所以 $n > 0$ 时

$$
\begin{aligned}
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z &= 2\pi i \mathrm{Res}[\frac{1-\cos 4z^3}{z^n}, 0] \\
&= 2\pi i C_{-1} \\
&= 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}
\end{aligned}
$$

综上，

$$
\oint_{|z|=r>0} \frac{1-\cos 4z^3}{z^n}\mathrm{d}z = \begin{cases} 2\pi i(-1)^{\frac{n-7}{6}}\frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k+1, k \in \mathbb{N} \\ 0 & n \neq 6k+1, k \in \mathbb{N} \end{cases}
$$

Problem 18

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z$$

Solution

令 $t = \frac{1}{z}$, 则

$$|t| = \frac{1}{r} < 1$$
$$t = \frac{1}{r} e^{-i\theta}$$
$$\mathrm{d}z = \mathrm{d}(\frac{1}{t}) = -\frac{1}{t^2}\mathrm{d}t \qquad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z &= -\oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)}(-\frac{1}{t^2})\mathrm{d}t \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)}\mathrm{d}t \\ &= 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0]\end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)(1 - t + t^2 - t^3 + \dots)$$

其中，t^{-1} 的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} \mathrm{d}z = 2\pi i \mathrm{Res}[\frac{e^t}{t^4(t+1)}, 0] = 2\pi i(-\frac{1}{3}) = -\frac{2}{3}\pi i$$

Problem 19

求积分

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \qquad a > |b| \quad a, b \in \mathbb{R}$$

Solution

令 $z = e^{i\theta}, \theta \in [0, 2\pi)$，注意到:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$
$$\mathrm{d}\theta = \frac{\mathrm{d}z}{ie^{i\theta}} = \frac{\mathrm{d}z}{iz}$$

Problem 19 continued on next page...

则原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\oint_{|z|=1}\frac{\frac{\mathrm{d}z}{iz}}{a+b(\frac{z^2+1}{2z})}\\&=\frac{2}{i}\oint_{|z|=1}\frac{\mathrm{d}z}{bz^2+2az+b}\end{aligned}$$

当 $b=0$ 时，原积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a}=\frac{2\pi}{a}$$

当 $b\neq 0$ 时，原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{i}\oint_{|z|=1}\frac{\mathrm{d}z}{bz^2+2az+b}\\&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\end{aligned}$$

对方程

$$z^2+\frac{2a}{b}z+1=0$$

其两根

$$z_1z_2=1$$

且

$$z_{1,2}=-\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b}$$

可设 $b>0$，则

$$z_2=-\frac{a}{b}\frac{\sqrt{a^2-b^2}}{b}<-\frac{a}{b}<-1$$

即只有一个奇点 z_1。所以原积分

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}&=\frac{2}{ib}\oint_{|z|=1}\frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1}\\&=\frac{2}{ib}2\pi i\frac{1}{2z_1+\frac{2a}{b}}\\&=\frac{2\pi}{\sqrt{a^2-b^2}}\end{aligned}$$

Problem 20

求积分

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}\qquad a>|b|\quad a,b\in\mathbb{R}$$

Solution

定理. 若 $f(x)$ 在 $[-1,1]$ 上可积，则 $\int_0^{2\pi}f(\cos x)\mathrm{d}\theta=\int_0^{2\pi}f(\sin x)\mathrm{d}\theta$

所以

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$$

Problem 21

求积分

$$I_p = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \qquad p \in (-1, 1)$$

Solution

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

Problem 22

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \qquad A, B \in \mathbb{R} > 0$$

Solution

令

$$\begin{aligned} \cos^2\theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2\theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

Problem 22 continued on next page...

则

$$\begin{aligned}
I_{A,B} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\cos^2\theta + B^2\sin^2\theta} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{A^2\dfrac{\cos 2\theta + 1}{2} + B^2\dfrac{1-\cos 2\theta}{2}} \\
&= \int_0^{4\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\int_0^{2\pi} \frac{\mathrm{d}t}{(A^2+B^2) + (A^2-B^2)\cos t} \\
&= 2\frac{2\pi}{\sqrt{(A^2+B^2)^2 - (A^2-B^2)^2}} \\
&= \frac{2\pi}{AB}
\end{aligned}$$

Problem 23

求积分

$$I_n = \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

$\frac{1}{(x^2+a^2)(x^2+b^2)}$ 是偶函数，因此

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} \\
&= \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)}
\end{aligned}$$

而

$$\begin{aligned}
&\lim_{R\to+\infty}\int_{-R}^{+R} \frac{1}{(x^2+a^2)(x^2+b^2)} + \int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(z^2+a^2)(z^2+b^2)}\mathrm{d}z \\
=&2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\
=&2\pi i(\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2) + ab})
\end{aligned}$$

定理. 若 $P_n(z), Q_m(z)$ 是多项式，且 $\mathrm{deg}P_n = n \le \mathrm{deg}Q_m - 2 = m-2$，$Q_m(z)$ 在实轴 $z = x$ 上没有零点，即 $Q_m(x) \neq 0, \forall x \in \mathbb{R}$，则

$$\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{P_n(z)}{Q_m(z)}\mathrm{d}z = 0$$

Problem 23 continued on next page...

所以

$$\begin{aligned}I_n &= \frac{1}{2}2\pi i(\mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \mathrm{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\ &= \pi i(\frac{1}{4(i|a|)^3+2i|a|(a^2+b^2)} + \frac{1}{4(i|b|)^3+2i|b|(a^2+b^2)}) \\ &= \frac{\pi}{2(|b|-|a|)|a||b|}\end{aligned}$$

Problem 24

求积分

$$I_n = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\ &= \lim_{R\to+\infty}\frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{1+x^{2n}} \qquad n \in \mathbb{N} \\ &= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2}\lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0}\frac{1}{1+z^{2n}}\mathrm{d}z \\ &= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k]\end{aligned}$$

其中

$$z_k^{2n}+1=0, \mathrm{Im}z_k > 0$$

则

$$\begin{aligned}I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}} \\ &= \frac{1}{2}2\pi i\sum_{k=0}^{i}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] \\ &= \pi i\sum_{k=1}^{n}\mathrm{Res}[\frac{1}{1+z^{2n}}, z_k] \\ &= \pi i\sum_{k=1}^{n}\frac{1}{2nz_k^{2n-1}} \\ &= -\frac{\pi i}{2n}\sum_{k=1}^{n} z_k\end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

Problem 24 continued on next page...

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i} n}{e^{\pi i} 2n} \qquad k = 1, 2, \cdots, 2n$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{2n} \sum_{k=0}^{i} z_k \\
&= -\frac{\pi i}{2n} \frac{\sum_{k=1}^{n} e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\
&= -\frac{\pi i}{2n} \frac{e^{\frac{\pi i}{n}}(1 - e^{\pi i})}{e^{\frac{\pi i}{2n}}(1 - e^{\frac{\pi i}{n}})} \\
&= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}}
\end{aligned}$$

令 $\theta = \frac{\pi i}{2n}$，则

$$\begin{aligned}
\frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin\theta} \\
&= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta} \\
&= \frac{1}{-2i\sin\theta}
\end{aligned}$$

所以

$$\begin{aligned}
I_n &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\
&= -\frac{\pi i}{n} \frac{\cos\theta + i\sin\theta}{1 - \cos 2\theta - i\sin\theta} \\
&= -\frac{\pi i}{n} \frac{1}{-2i\sin\theta} \\
&= \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}
\end{aligned}$$

Problem 25

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
I_{n,r} &= \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}} \\
&= \frac{1}{r^{2n}} \int_0^{+\infty} \frac{\mathrm{d}x}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} \int_0^{+\infty} \frac{\mathrm{d}(\frac{x}{r})}{1 + (\frac{x}{r})^{2n}} \\
&= \frac{1}{r^{2n+1}} I_n
\end{aligned}$$

Problem 25 continued on next page...

Problem 26

求积分

$$J_n = \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \qquad n \in \mathbb{N}$$

Solution

$$\begin{aligned}
J_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{+R} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \frac{1}{2}2\pi(i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R\to+\infty}\int_{|z|=R,\mathrm{Im}z>0} \frac{1}{(1+z^2)^n}\mathrm{d}z) \\
&= \frac{1}{2}(2\pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]) \\
&= \pi i\mathrm{Res}[\frac{1}{(1+z^2)^n}, i]
\end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

令 $f(z) = \frac{1}{(z+i)^n}$

$$\begin{aligned}
\frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
&= (z-i)^{-n}\sum_{k=0}^{+\infty}\frac{f^{(k)}(i)}{k!}(z-i)^k
\end{aligned}$$

要求 $(z-i)^{-1}$ 对应的系数 C_{-1}，对应于 $k = n-1$

$$\begin{aligned}
C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
&= (-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
\end{aligned}$$

因此

$$\begin{aligned}
J_n &= \pi i C_{-1} \\
&= \pi i(-1)^{n-1}\frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
&= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
\end{aligned}$$

Problem 27

求积分

$$J_{n,r}=\int_0^{+\infty}\frac{\mathrm{d}x}{(r^2+x^2)^n}\qquad n\in\mathbb{N}$$

Solution

$$\begin{aligned}J_{n,r}&=\int_0^{+\infty}\frac{\mathrm{d}x}{(r^2+x^2)^n}\\&=\int_0^{+\infty}\frac{1}{r^{2n}}\frac{r\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n}\\&=\frac{1}{r^{2n-1}}\int_0^{+\infty}\frac{\mathrm{d}(\frac{x}{r})}{(1+(\frac{x}{r})^2)^n}\\&=\frac{1}{r^{2n-1}}J_n\end{aligned}$$

Problem 28

求积分

$$I_{a,b,k}=\int_0^{+\infty}\frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

设 $a\neq b$ 则

$$\begin{aligned}I_{a,b,k}&=\int_0^{+\infty}\frac{x\sin kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\\&=\frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\end{aligned}$$

而

$$\begin{aligned}\int_0^{+\infty}\frac{xe^{x}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x&=2\pi i\{\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)},ai]+\mathrm{Res}[\frac{ze^{ikz}}{(z^2+a^2)(z^2+b^2)},bi]\}\\&=2\pi i[\frac{aie^{-ka}}{4(ai)^3+2ai(a^2+b^2)}+\frac{bie^{-kb}}{4(bi)^3+2bi(a^2+b^2)}]\\&=\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})\end{aligned}$$

所以

$$\begin{aligned}I_{a,b,k}&=\frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{kx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x\\&=\frac{1}{2}\mathrm{Im}[\frac{\pi i}{b^2-a^2}(e^{-ka}-e^{-kb})]\\&=\frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb})\end{aligned}$$

Problem 28 continued on next page...

$a = b$ 时

$$\begin{aligned} I_{a,b,k} &= \lim_{a\to b}\frac{\pi}{2(b^2-a^2)}(e^{-ka}-e^{-kb}) \\ &= \frac{-k\pi e^{-kb}}{-4b} \\ &= \frac{k\pi}{4ae^{ka}} = \frac{k\pi}{4be^{kb}} \end{aligned}$$

Problem 29

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x$$

Solution

$a \neq b$ 时

$$\begin{aligned} I_{a,b,k} &= \int_0^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2\cos kx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\ &= \frac{1}{2}\mathrm{Re}\int_{-\infty}^{+\infty} \frac{x^2 e^{ikx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x \\ &= \frac{(be^{-kb}-ae^{-ka})\pi}{2(b^2-a^2)} \end{aligned}$$

当 $a = b$ 时

$$I_{a,b,k} = \frac{(1-ka)\pi}{4ae^{ka}} = \frac{(1-kb)\pi}{4be^{kb}}$$