

## Problems

### Problem 1

求证

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

### Solution 1

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= 2\pi i \sum_{k=1}^n \operatorname{Res}\left[\frac{1}{1+z^n}, z_k\right] \\ &= 2\pi i \sum_{k=1}^n \frac{1}{nz_k^{n-1}} \\ &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \end{aligned}$$

注意到

$$z_k^n = 1$$

则

$$-z_k = \frac{1}{z_k^{n-1}}$$

因此

$$\begin{aligned} \oint_{|z|=r>1} \frac{1}{1+z^n} dz &= \frac{2\pi i}{n} \sum_{k=1}^n \frac{1}{z_k^{n-1}} \\ &= -\frac{2\pi i}{n} \sum_{k=1}^n z_k \end{aligned}$$

若  $n=1$ , 即

$$1+z^n = 1+z = 0$$

解得

$$z = -1$$

因此

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 2\pi i$$

否则

$$\sum_{k=1}^n z_k = 0$$

即

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = 0$$

又

$$\begin{aligned}\oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz &= \oint_{|z|=r>1} \frac{(z^{2n}-1)+1}{1+z^n} dz \\ &= \oint_{|z|=r>1} \frac{(z^n-1)(z^n+1)}{1+z^n} dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz \\ &= \oint_{|z|=r>1} (z^n-1) dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz\end{aligned}$$

由 Cauchy-Goursat 定理

$$\oint_{|z|=r>1} (z^n-1) dz = 0$$

所以

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz$$

综上,

$$\oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i & n=1 \\ 0 & n \geq 2 \end{cases}$$

### Solution 2

令  $z = re^{i\theta}$   $0 \leq \theta < \pi, t = \frac{1}{z}$ , 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

此时原积分

$$\begin{aligned}\oint_{|z|=r>1} \frac{1}{1+z^n} dz &= -\oint_{|t|=\frac{1}{r}<1} \frac{1}{1+\frac{1}{t^2}} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt\end{aligned}$$

只有一个奇点  $t=0$ 。因此

$$\begin{aligned}I_n &= \oint_{|t|=\frac{1}{r}<1} \frac{t^{n-2}}{1+t^n} dt \\ &= \begin{cases} 0 & n \geq 2 \\ \oint_{|t|=\frac{1}{r}<1} \frac{1}{t(t+1)} dt = 2\pi i & n=1 \end{cases} \quad (\text{Cauchy-Goursat 定理})\end{aligned}$$

### Problem 2

求积分

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz \quad n \in \mathbb{Z}$$

**Solution**

当  $n \leq 0$  时

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = 0$$

当  $n > 0$  时

$$\begin{aligned} \frac{1 - \cos 4z^3}{z^n} &= z^{-n} \left( 1 - \sum_{k=0}^n (-1)^k \frac{(4z^3)^{2k}}{(2k)!} \right) \\ &= z^{-n} \left( 1 - \sum_{k=0}^n (-1)^k \frac{4^{2k} z^{6k}}{(2k)!} \right) \\ &= z^{-n} \sum_{k=1}^n (-1)^{k-1} \frac{4^{2k} z^{6k}}{(2k)!} \end{aligned}$$

$\frac{1 - \cos 4z^3}{z^n}$  在奇点  $z = 0$  的 Laurent 级数  $\sum_{n=-\infty}^{+\infty} C_n z^n$  中,  $C_{-1}$  对应上式中

$$6k - n = -1$$

此时

$$\begin{aligned} C_{-1} &= (-1)^{k-1} \frac{4^{2k}}{(2k)!} \quad k = \frac{n-1}{6} \\ &= (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

所以  $n > 0$  时

$$\begin{aligned} \oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz &= 2\pi i \operatorname{Res}\left[\frac{1 - \cos 4z^3}{z^n}, 0\right] \\ &= 2\pi i C_{-1} \\ &= 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} \end{aligned}$$

综上,

$$\oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz = \begin{cases} 2\pi i (-1)^{\frac{n-7}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!} & n = 6k + 1, k \in \mathbb{N} \\ 0 & n \neq 6k + 1, k \in \mathbb{N} \end{cases}$$

**Problem 3**

求积分

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz$$

**Solution**

令  $t = \frac{1}{z}$ , 则

$$|t| = \frac{1}{r} < 1$$

$$t = \frac{1}{r} e^{-i\theta}$$

$$dz = d\left(\frac{1}{t}\right) = -\frac{1}{t^2} dt \quad 0 \leq \theta < \pi \quad \text{积分方向为顺时针}$$

原积分

$$\begin{aligned} \oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz &= - \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(t+1)} \left(-\frac{1}{t^2}\right) dt \\ &= \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^4(t+1)} dt \\ &= 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] \end{aligned}$$

又

$$\frac{e^t}{t^4(t+1)} = t^{-4} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) (1 - t + t^2 - t^3 + \dots)$$

其中,  $t^{-1}$  的系数为

$$-1 + 1 - \frac{1}{2!} + \frac{1}{3!} = -\frac{1}{3}$$

因此

$$\oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz = 2\pi i \operatorname{Res}\left[\frac{e^t}{t^4(t+1)}, 0\right] = 2\pi i \left(-\frac{1}{3}\right) = -\frac{2}{3}\pi i$$

**Problem 4**

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

**Solution**

令  $z = e^{i\theta}$ ,  $\theta \in [0, 2\pi)$ , 注意到:

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z} \\ d\theta &= \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} \end{aligned}$$

则原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{a + b\left(\frac{z^2+1}{2z}\right)} \\ &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b} \end{aligned}$$

当  $b = 0$  时, 原积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \int_0^{2\pi} \frac{d\theta}{a} = \frac{2\pi}{a}$$

当  $b \neq 0$  时, 原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2 + 2az + b} \\ &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1} \end{aligned}$$

对方程

$$z^2 + \frac{2a}{b}z + 1 = 0$$

其两根

$$z_1 z_2 = 1$$

且

$$z_{1,2} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

可设  $b > 0$ , 则

$$z_2 = -\frac{a}{b} \frac{\sqrt{a^2 - b^2}}{b} < -\frac{a}{b} < -1$$

即只有一个奇点  $z_1$ 。所以原积分

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{ib} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b}z + 1} \\ &= \frac{2}{ib} 2\pi i \frac{1}{2z_1 + \frac{2a}{b}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \end{aligned}$$

## Problem 5

求积分

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} \quad a > |b| \quad a, b \in \mathbb{R}$$

## Solution

**定理.** 若  $f(x)$  在  $[-1, 1]$  上可积, 则  $\int_0^{2\pi} f(\cos x) d\theta = \int_0^{2\pi} f(\sin x) d\theta$

所以

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

**Problem 6**

求积分

$$I_p = \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \quad p \in (-1, 1)$$

**Solution**

令

$$\begin{aligned} a &= 1 + p^2 \\ b &= -2p \end{aligned}$$

则

$$a > b \quad a, b \in (-1, 1)$$

因此

$$\begin{aligned} I_p &= \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} \\ &= \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \\ &= \frac{2\pi}{1 - p^2} \end{aligned}$$

**Problem 7**

求积分

$$I_{A,B} = \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \quad A, B \in \mathbb{R} > 0$$

**Solution**

令

$$\begin{aligned} \cos^2 \theta &= \frac{\cos 2\theta + 1}{2} \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

则

$$\begin{aligned}
 I_{A,B} &= \int_0^{2\pi} \frac{d\theta}{A^2 \cos^2 \theta + B^2 \sin^2 \theta} \\
 &= \int_0^{2\pi} \frac{d\theta}{A^2 \frac{\cos 2\theta + 1}{2} + B^2 \frac{1 - \cos 2\theta}{2}} \\
 &= \int_0^{4\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\
 &= 2 \int_0^{2\pi} \frac{dt}{(A^2 + B^2) + (A^2 - B^2) \cos t} \\
 &= 2 \frac{2\pi}{\sqrt{(A^2 + B^2)^2 - (A^2 - B^2)^2}} \\
 &= \frac{2\pi}{AB}
 \end{aligned}$$

## Problem 8

求积分

$$I_n = \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$$

## Solution

$\frac{1}{(x^2 + a^2)(x^2 + b^2)}$  是偶函数, 因此

$$\begin{aligned}
 I_n &= \int_0^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} \\
 &= \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)}
 \end{aligned}$$

而

$$\begin{aligned}
 &\lim_{R \rightarrow +\infty} \int_{-R}^{+R} \frac{1}{(x^2 + a^2)(x^2 + b^2)} + \int_{|z|=R, \text{Im} z > 0} \frac{1}{(z^2 + a^2)(z^2 + b^2)} dz \\
 &= 2\pi i (\text{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|a|] + \text{Res}[\frac{1}{(x^2 + a^2)(x^2 + b^2)}, i|b|]) \\
 &= 2\pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2 + b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2 + b^2) + ab})
 \end{aligned}$$

**定理.** 若  $P_n(z), Q_m(z)$  是多项式, 且  $\deg P_n = n \leq \deg Q_m - 2 = m - 2$ ,  $Q_m(z)$  在实轴  $z = x$  上没有零点, 即  $Q_m(x) \neq 0, \forall x \in \mathbb{R}$ , 则

$$\lim_{R \rightarrow +\infty} \int_{|z|=R, \text{Im} z > 0} \frac{P_n(z)}{Q_m(z)} dz = 0$$

所以

$$\begin{aligned} I_n &= \frac{1}{2} 2\pi i (\text{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|a|] + \text{Res}[\frac{1}{(x^2+a^2)(x^2+b^2)}, i|b|]) \\ &= \pi i (\frac{1}{4(i|a|)^3 + 2i|a|(a^2+b^2) + ab} + \frac{1}{4(i|b|)^3 + 2i|b|(a^2+b^2) + ab}) \end{aligned}$$

### Problem 9

求积分

$$I_n = \int_0^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N}$$

**Solution**

$$\begin{aligned} I_n &= \int_0^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\ &= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k] - \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{|z|=R, \text{Im}z > 0} \frac{1}{1+z^{2n}} dz \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k] \end{aligned}$$

其中

$$z_k^{2n} + 1 = 0, \text{Im}z_k > 0$$

则

$$\begin{aligned} I_n &= \int_0^{+\infty} \frac{dx}{1+x^{2n}} \\ &= \frac{1}{2} 2\pi i \sum_{k=0}^i \text{Res}[\frac{1}{1+z^{2n}}, z_k] \\ &= \pi i \sum_{k=1}^n \text{Res}[\frac{1}{1+z^{2n}}, z_k] \\ &= \pi i \sum_{k=1}^n \frac{1}{2nz_k^{2n-1}} \\ &= -\frac{\pi i}{2n} \sum_{k=1}^n z_k \end{aligned}$$

由

$$z_k^{2n} = -1 = e^{\pi i}$$

得

$$z_k = e^{\frac{\pi i + 2(k-1)\pi i}{2n}} = \frac{e^{k\pi i} n}{e^{\pi i} 2n} \quad k = 1, 2, \dots, 2n$$



所以

$$\begin{aligned}
 I_n &= -\frac{\pi i}{2n} \sum_{k=0}^i z_k \\
 &= -\frac{\pi i}{2n} \frac{\sum_{k=1}^n e^{\frac{k\pi i}{n}}}{e^{\frac{\pi i}{2n}}} \\
 &= -\frac{\pi i}{2n} \frac{e^{\frac{\pi i}{n}}(1 - e^{\pi i})}{e^{\frac{\pi i}{2n}}(1 - e^{\frac{\pi i}{n}})} \\
 &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}}
 \end{aligned}$$

令  $\theta = \frac{\pi i}{2n}$ , 则

$$\begin{aligned}
 \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} &= \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\
 &= \frac{\cos \theta + i \sin \theta}{2 \sin^2 \theta - 2i \sin \theta \cos \theta} \\
 &= \frac{1}{-2i \sin \theta}
 \end{aligned}$$

所以

$$\begin{aligned}
 I_n &= -\frac{\pi i}{n} \frac{e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \\
 &= -\frac{\pi i}{n} \frac{\cos \theta + i \sin \theta}{1 - \cos 2\theta - i \sin 2\theta} \\
 &= -\frac{\pi i}{n} \frac{1}{-2i \sin \theta} \\
 &= \frac{\frac{\pi}{2n}}{\sin \frac{\pi}{2n}}
 \end{aligned}$$

## Problem 10

求积分

$$I_{n,r} = \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \quad n \in \mathbb{N}$$

**Solution**

$$\begin{aligned}
 I_{n,r} &= \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} \\
 &= \frac{1}{r^{2n}} \int_0^{+\infty} \frac{dx}{1 + \left(\frac{x}{r}\right)^{2n}} \\
 &= \frac{1}{r^{2n+1}} \int_0^{+\infty} \frac{d\left(\frac{x}{r}\right)}{1 + \left(\frac{x}{r}\right)^{2n}} \\
 &= \frac{1}{r^{2n+1}} I_n
 \end{aligned}$$

**Problem 11**

求积分

$$J_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \quad n \in \mathbb{N}$$

**Solution**

$$\begin{aligned}
 J_n &= \int_0^{+\infty} \frac{dx}{(1+x^2)^n} \\
 &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^n} \\
 &= \lim_{R \rightarrow +\infty} \frac{1}{2} \int_{-R}^{+R} \frac{dx}{(1+x^2)^n} \\
 &= \frac{1}{2} 2\pi(i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i] - \lim_{R \rightarrow +\infty} \int_{|z|=R, \operatorname{Im} z > 0} \frac{1}{(1+z^2)^n} dz) \\
 &= \frac{1}{2} (2\pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]) \\
 &= \pi i \operatorname{Res}[\frac{1}{(1+z^2)^n}, i]
 \end{aligned}$$

而

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n}$$

$$\text{令 } f(z) = \frac{1}{(z+i)^n}$$

$$\begin{aligned}
 \frac{1}{(1+z^2)^n} &= \frac{1}{(z+i)^n(z-i)^n} \\
 &= (z-i)^{-n} \sum_{k=0}^{+\infty} \frac{f^{(k)}(i)}{k!} (z-i)^k
 \end{aligned}$$

要求  $(z-i)^{-1}$  对应的系数  $C_{-1}$ , 对应于  $k = n-1$ 

$$\begin{aligned}
 C_{-1} &= \frac{f^{(n-1)}(i)}{(n-1)!} \\
 &= (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2}
 \end{aligned}$$

因此

$$\begin{aligned}
 J_n &= \pi i C_{-1} \\
 &= \pi i (-1)^{n-1} \frac{(2n-2)!(2i)^{-2n+1}}{[(n-1)!]^2} \\
 &= \frac{\pi(2n-2)!}{[(n-1)!]^2 2^{2n-1}}
 \end{aligned}$$

**Problem 12**

求积分

$$J_{n,r} = \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} \quad n \in \mathbb{N}$$

**Problem 13**

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx$$

**Problem 14**

求积分

$$I_{a,b,k} = \int_0^{+\infty} \frac{x^2 \cos kx}{(x^2 + a^2)(x^2 + b^2)} dx$$