A Hotelling-Downs game for strategic candidacy with binary issues

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Overview

Motivation

The Hotelling-Downs Model

Model of an election as a *political spectrum*. Existence of Equilibria in such a game has been widely studied (e.g. ice-cream vendors).

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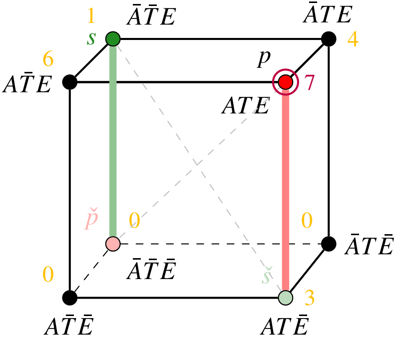
- **Judgment Aggregation:** Aggregating *logical propositions*. Inspiration for *binary issue setting* and *single-peakedness* notion (see [Puppe, 2018]).

The Model



1
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6
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7
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$\bar{A}T\bar{E}$
0
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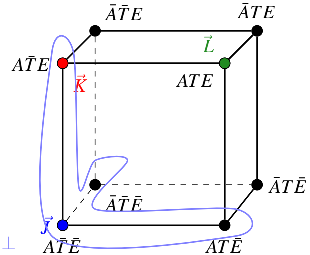

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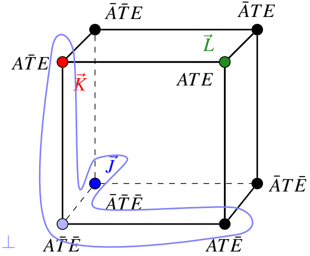

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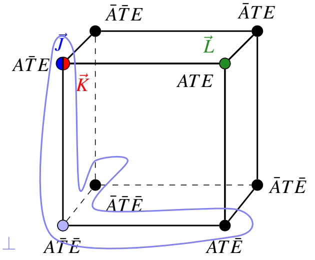

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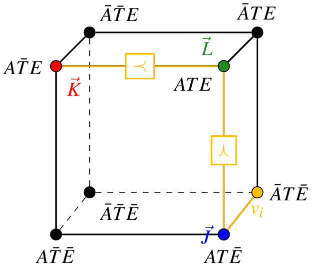

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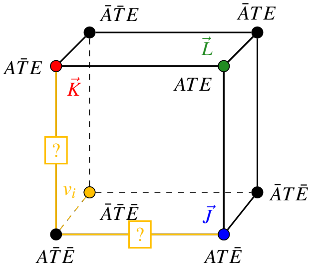

Payoff and Stability Notions

Candidates don't necessarily want to win !

- Each one has a **weak ranking** over its rivals with themselves at the top.
- A **better response** for c_i from $\mathbf{s}$ is $s_i' \in \mathscr{H}_i$ s.t. $\mathscr{F}((s_i', s_{-i})) \succ_{c_i} \mathscr{F}(\mathbf{s})$.

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Nash Equiibrium (NE)

A state $\mathbf{s} \in \Pi_{i=1}^{m} \mathscr{H}_i$ is a Nash equilibrium if there is no strategy $s'_i \in \mathscr{H}_i$ for a candidate $c_i \in C$ such that $\mathscr{F}((s'_i, s_{-i})) \succ_{c_i} \mathscr{F}(\mathbf{s})$.

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t-local Equilibrium

A state $\mathbf{s} \in \Pi_{i=1}^{m} \mathscr{H}_i$ is a t-local equilibrium if there is no strategy $s'_i \in \mathscr{H}_i$ for a candidate $c_i \in C$ such that $dist(s'_i, s_i) \leq t$ and $\mathscr{F}((s'_i, s_{-i})) \succ_{c_i} \mathscr{F}(\mathbf{s})$.

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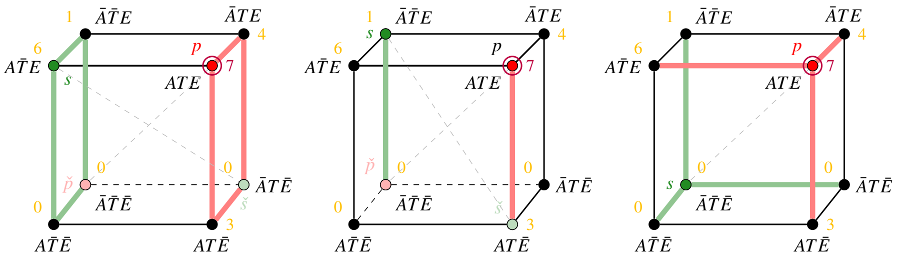
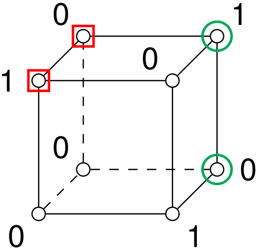
A Nash equilibrium is equivalent to a K-local equilibrium. Also, a t-local equilibrium is a t'-local equilibrium for every $1 \leq t' \leq t \leq K$.

Some Results



Negative Results in the general case

A 1-local equilibrium may fail to exist even with $m = 2,\ K = 3$



		$s_2 \in \mathscr{H}_2$		
		$(1,0,1)$		$(1,1,1)$
$s_1 \in \mathscr{H}_1$	$(0,1,0)$	$(1,\mathbf{2})$	$\xleftarrow{c_2}$	$(\mathbf{1.5},1.5)$
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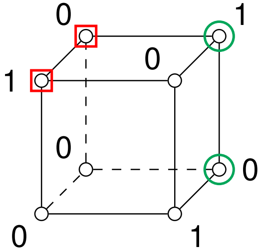

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NP-Hardness of the decision problem

Deciding whether there exists a t-local equilibrium is NP-hard, for $t \in \{2,\dots,K\}$, even under *narcissistic* preferences.

We make a reduction of *Exact Cover by 3-Sets (X3C)*, which is NP Complete;

Some structure of the $m = 2$ case

Influence Sets (for $\mathbf{s} = (s_1, s_2)$)

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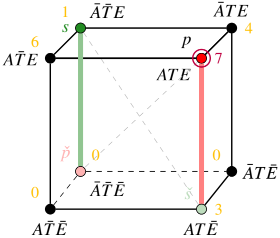
When doing a 1$-$local deviation, a candidate can only win/lose influence over positions that *agree/disagree* with him in the deviated issue.



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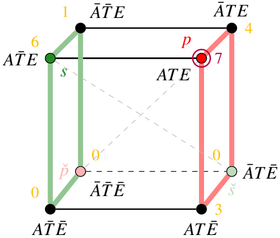

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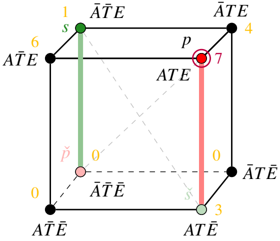

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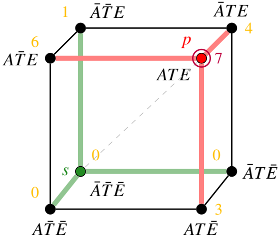

Antipodality

Influence sets are *antipodal*: $P_i^{\mathbf{s}} = \check{P}_{-i}^{\mathbf{s}}$

Some Positive Results

Let $p^m \in \mathscr{H}$ be defined, $\forall j \in \mathscr{H}$ as: $(p^m)_j = \arg\max_{e\in\{0,1\}} f_N(\mathscr{H}\big|_{j=e})$. i.e. p^m captures the *majoritarian view* on each issue.

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Existence of 2-local equilibria

There always exists a 2-local equilibrium in the BSC game when $m = 2$, n is odd, and $p^m \in \mathscr{H}_1 \cap \mathscr{H}_2$.

Unfortunately...

A 3-local equilibrium may not exist, even when $m = 2$, $K = 3$, and the sets of candidates' strategies coincide, contain p^m and are connected.

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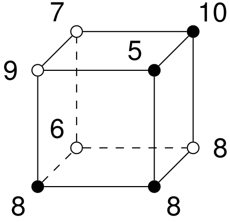
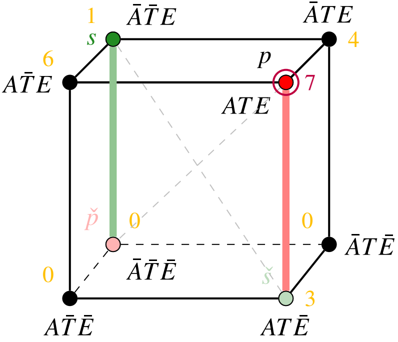


		$s_2 \in \mathscr{H}_2$			
		$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(1,1,1)$
$s_1 \in \mathscr{H}_1$	$(0,0,0)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$
	$(1,0,0)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$
	$(1,1,0)$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$
	$(1,1,1)$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$

Restricting the Setting



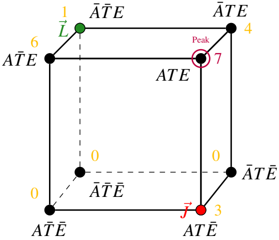
Distribution of Voters

SinglePeaked Distribution of voters

A distribution of voters $f : \mathscr{H} \rightarrow \mathbb{N}$ is SP if:
$\exists p^* \in \mathscr{H}; \forall x, y \in \mathscr{H}$:

$$y \in [x, p^*] \Longrightarrow f(x) \leq f(y)$$

General agreement around the *opinion* p^*.
(An example of a SP distribution is the *uniform*)



CentraleSupélec

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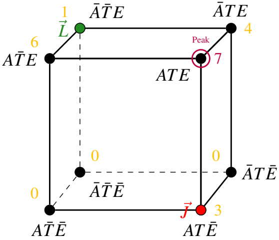

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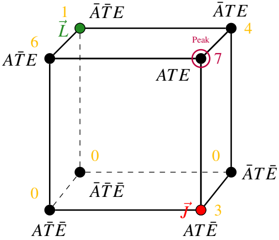

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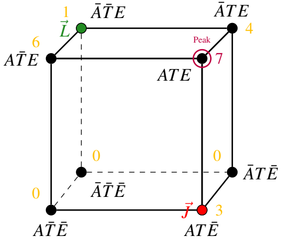

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It's actually stronger: c_1 **always wins** when taking the peak

[Betweenness Matching]

For any $\mathbf{s} = (s_1, s_2) \in H_1 \times H_2$ s.t. $s_{i_0} = p^* \neq s = s_{-i_0}$,

$$\exists \phi : \mathscr{H}_s \to \mathscr{H}_p \textbf{ bijection }, \textbf{ s.t.: } \forall x \in \mathscr{H}_i(\mathbf{s}),\ \phi(x) \in [x, p^*]$$

Every $x \in \mathscr{H}_s$ is mapped to exactly one $\phi(x) \in \mathscr{H}_p$ *between* x and the peak.

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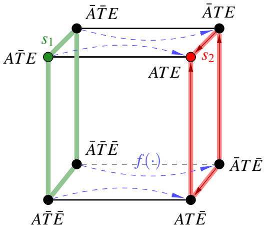

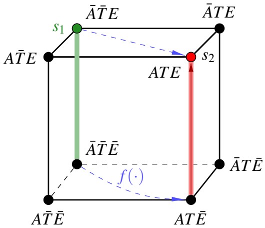

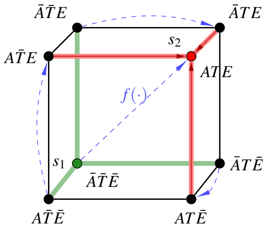

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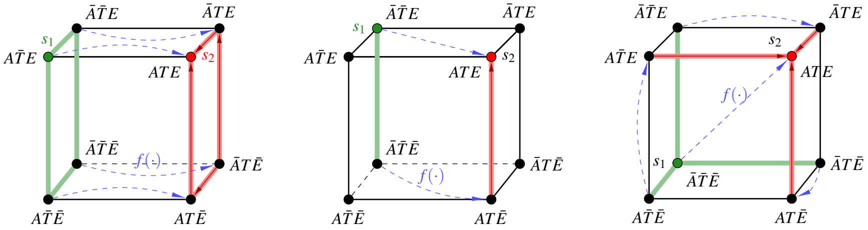

This is achieved using Hall's Theorem's Corollary for regular graphs. The problem is reduced to the *antipodal* case and decomposed by *layers*.

Unfortunately, NOT scalable !

NOT even a 1-local equilibrium !

A 1-local equilibrium may not exist in the populism game even when $m = 3$, $K = 2$, the candidates' preferences are fixed, and the distribution of voters is uniform.

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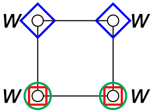

$$
\begin{array}{lccccc}
c_1: & c_1 & \succ & c_3 & \succ & c_2 \\
c_2: & c_2 & \succ & c_1 & \succ & c_3 \\
c_3: & c_3 & \succ & c_2 & \succ & c_1
\end{array}
$$

Seeing the Game Table, we notice that players have a 1-local deviation from every possible state:

$s_3 = (0,1)$:

$s_1 \in \mathscr{H}_1$ \ $s_2 \in \mathscr{H}_2$	$(0,0)$		$(1,0)$
$(0,0)$	$(w, w, \mathbf{2w})$	$\xrightarrow{c_2}$	$(w, \mathbf{\frac{3}{2}w}, \frac{3}{2}w)$
	$c_1 \downarrow$		$\downarrow c_1$
$(1,0)$	$(\mathbf{\frac{3}{2}w}, w, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(\frac{7}{6}w, \frac{7}{6}w, \mathbf{\frac{5}{3}w})$

$s_3 = (1,1)$:

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$(0,0)$	$(\frac{7}{6}w, \frac{7}{6}w, \mathbf{\frac{5}{3}w})$	$\xrightarrow{c_2}$	$(\mathbf{\frac{3}{2}w}, w, \frac{3}{2}w)$
	$c_1 \uparrow$		$\uparrow c_1$
$(1,0)$	$(w, \mathbf{\frac{3}{2}w}, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(w, w, \mathbf{2w})$

Candidates' Strategies

Theorem: 1-local by following your rival

There always exists a 1-local equilibrium in the BSC game when $m = 2$ and $\mathscr{H}_2 \subseteq \mathscr{H}_1$. Such an equilibrium can be found in polynomial time.

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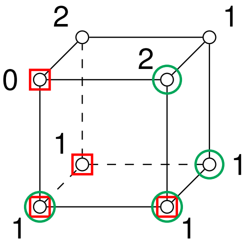
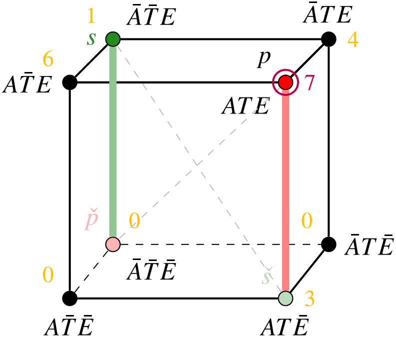
2-local eq. may NOT exist with $m = 2$, $K = 3$, and $\mathscr{H}_i$ balls of radius one

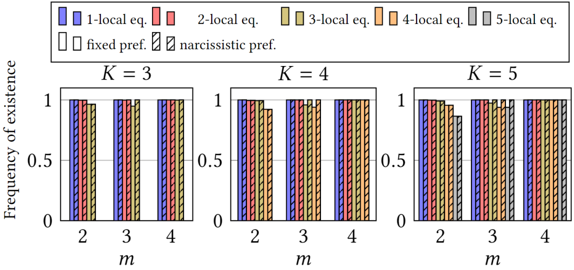


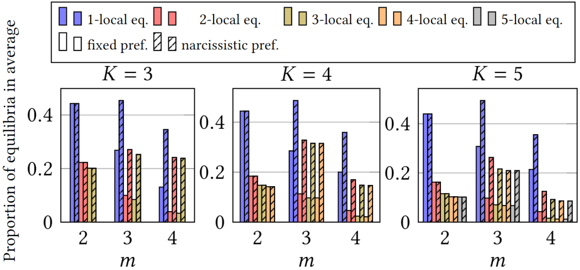
	$s_2 \in \mathscr{H}_2$: $(1,0,0)$	$(0,0,0)$	$(1,0,1)$	$(1,1,0)$
$s_1 \in \mathscr{H}_1$: $(0,0,0)$	$(4,\mathbf{5})$	$(\mathbf{4.5},4.5)$	$(4,\mathbf{5})$	$(4,\mathbf{5})$
$(1,0,0)$	$(\mathbf{4.5},4.5)$	$(\mathbf{5},4)$	$(4,\mathbf{5})$	$(4,\mathbf{5})$
$(0,1,0)$	$(\mathbf{4.5},4.5)$	$(\mathbf{5},4)$	$(\mathbf{5},4)$	$(4,\mathbf{5})$
$(0,0,1)$	$(\mathbf{4.5},4.5)$	$(\mathbf{5},4)$	$(4,\mathbf{5})$	$(\mathbf{5},4)$

Some Empirical Insight



Empirical Insight: Local Equilibria





Iterative Games : t-local Dynamic

By turns, candidates change their current strategy for a t-local improving move. It will stop at a t-local equilibrium!

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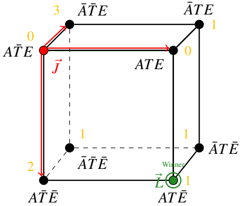

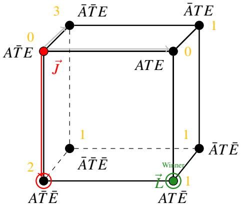

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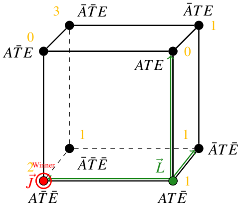

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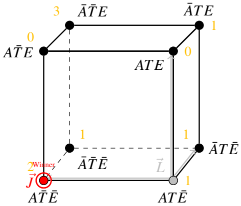
Empirical Insight: Dynamics



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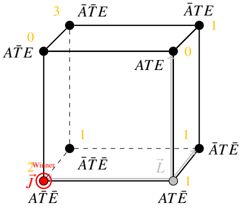

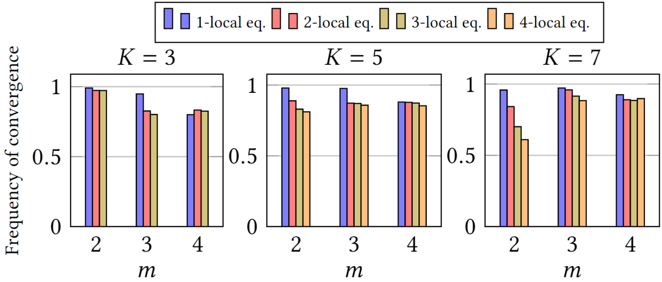

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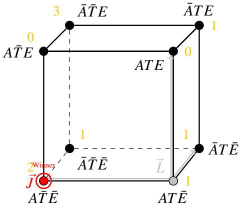

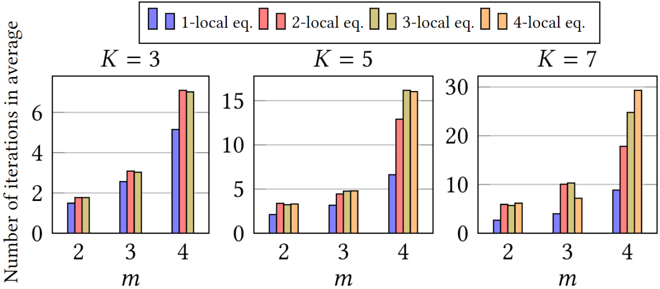

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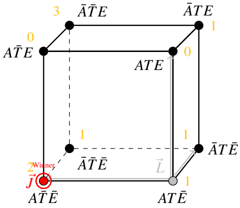

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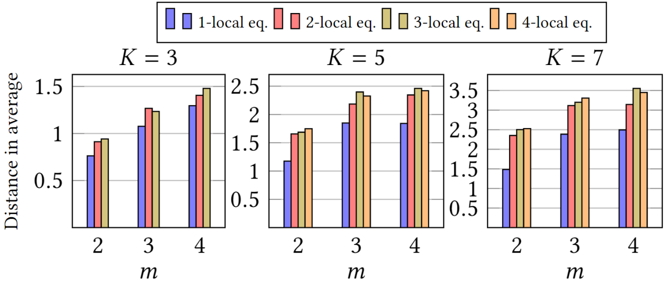

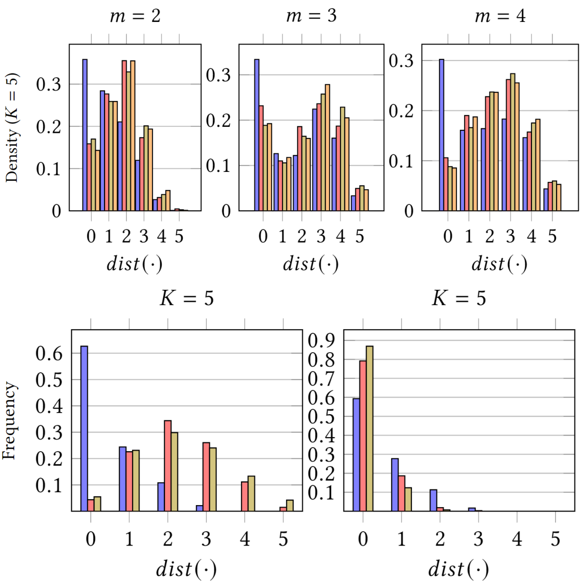
m = 2
m = 3
m = 4
Density (K = 5)
0.3
0.2
0.1
0
0 1 2 3 4 5
dist(·)
K = 5
K = 5
Frequency
0.6
0.5
0.4
0.3
0.2
0.1
0.9
0.8
0.7
0 1 2 3 4 5
dist(·)

Thank you for your attention !

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Ideas for the proof of main theorem

Lemma: [d-regular bipartite graph matching] - Uses Hall's Theorem

A graph $G = (V, E)$ is said to be $d-$regular if $\forall v \in V$, $\mathbf{deg}_G(v) = d$ (i.e. all vertices have the same degree d).
We always have that **any $d-$regular bipartite graph has an X-perfect matching**.

Lemma

Let $r \in \{1, \ldots, K\}$. Let $s = \vec{0}_r$ and $p = \vec{1}_r$, both in the Hypercube $\mathscr{H}^r = \{0,1\}^r$. (Also, define $\mathscr{H}^r_s := \{x \in \mathscr{H}^r : d(x,s) < d(x,p)\}$, and $\mathscr{H}^r_p$ in a similar way)
*There **exists a bijection** $\phi : \mathscr{H}^r_s \to \mathscr{H}^r_p$ s.t. $\forall x \in \mathscr{H}^r_s$, $\phi(x) \in [x,p]$*

Let $\ell \in \{0,\ldots,c_r\}$, we consider
$\mathscr{H}_s^r|_\ell := \{x \in \mathscr{H}_s^r :\ d(x,s) = \ell\}$ **and** $\mathscr{H}_p^r|_\ell := \{y \in \mathscr{H}_p^r :\ d(y,p) = \ell\}$.
The GOOD bipartite graph to check: $G|_\ell = (X+Y, E)$:

$$X = \mathscr{H}_s^r|_\ell,\ Y = \mathscr{H}_p^r|_\ell,\ E = \{(x,y) \in X \times Y :\ (x,y,p) \in \tau_r\}$$

We can see that $G|_\ell$ is **regular!** The degree of nodes in $G|_\ell$ is:
For $x \in X = \mathscr{H}_s^r|_\ell$,
$\mathbf{deg}(x) = |\{(a,b) \in E :\ a = x\}| = |\{b \in Y : (x,b,p) \in \tau_r\}| = \left|[x,p] \cap \mathscr{H}_p^r|_\ell\right|$.
We see that, by definition, this set is:

$$\begin{aligned}
[x,p] \cap \mathscr{H}_p^r|_\ell &= \{z \in \mathscr{H}_p^r :\ d(z,\vec{1}_r) = \ell,\ \forall i \in \{1,\ldots,r\} :\ [x_i = 1 \implies z_i = 1]\} \\
&= \{y \in \mathscr{H}_p^r :\ \exists A \subseteq \mathbb{I}_{x\equiv 0}, |A| = r - 2\ell;\ y = 1_{\mathbb{I}_{x\equiv 1}} + \sum_{i\in A} e_i\}
\end{aligned}$$

So, we have: $\mathbf{deg}(x) = |[x,p] \cap \mathscr{H}_p^r|_\ell| = |\{A \subseteq \mathbb{I}_{x\equiv 0} :\ |A| = r - 2\ell\}| = \binom{r-\ell}{r-2\ell}$

Proof

Similarly (as $(a, y, \vec{1}_r) \in \tau_r \iff (y, a, \vec{0}_r) \in \tau_r$):
$\textbf{deg}(y) = |\{(a,b) \in E :\ b = y\}| = |\{a \in X : (a, y, p) \in \tau_r\}| = |[y, s] \cap \mathscr{H}_s^r|_\ell|$.
And, of course, we notice:

$$\begin{aligned}[y, s] \cap \mathscr{H}_s^r|_\ell &= \{x \in \mathscr{H}_s^r :\ d(x, \vec{0}_r) = \ell,\ \forall i \in \{1, \ldots, r\} :\ [y_i = 0 \Longrightarrow x_i = 0]\} \\ &= \{x \in \mathscr{H}_s^r :\ \exists A \subseteq \mathbb{I}_{y\equiv 1}, |A| = \ell;\ x = 0_{\mathbb{I}_{y\equiv 0}} + \sum_{i \in A} e_i\}\end{aligned}$$

So that: $\textbf{deg}(y) = |[y, s] \cap \mathscr{H}_s^r|_\ell| = |\{A \subseteq \mathbb{I}_{y\equiv 1} :\ |A| = \ell\}| = \binom{r-\ell}{\ell}$
As $G|_\ell$ is a **regular bipartite graph**, by the Lemma, we have a **perfect matching between** $\mathscr{H}_s|\ell$ **and** $\mathscr{H}_r|_\ell$ (call it ϕ^ℓ)
We can properly define the following to have the bijection we want:

$$\begin{aligned}\phi : \mathscr{H}_s^r &\to \mathscr{H}_p^r \\ x &\mapsto \phi^\ell(x) \textbf{ if } d(x, \vec{0}_r) = \ell\end{aligned}$$

(bijective cause surjective and pidgeon principle; and $(x, \phi(x), p) \in \tau_r$ cause every ϕ^ℓ satisfy that).

From General to Particular

Let $s, p \in \mathscr{H} = \{0,1\}^K$. We consider $\mathbb{I} := \{i : \ s_i = 1\}$ and define:

$$\psi : x \mapsto \psi(x) := (x_{\mathbb{I}^c}, (1 - x_i)_{\mathbb{I}})$$

This is such that: $\psi(s) = \vec{0}$, and for $\mathbb{J} = \{i : s_i = p_i\}$: $\psi(p) = (\vec{0}_{\mathbb{J}}, \vec{1}_{\mathbb{J}^c})$.
We can then partition the hypercube as:
$\mathscr{H} = \{0,1\}^K = \{0,1\}^{\mathbb{J}} \times \{0,1\}^{\mathbb{J}^c} = \dot{\bigcup}_{a \in \{0,1\}^{\mathbb{J}}} \{a\} \times \{0,1\}^{\mathbb{J}^c}$.
This is perfect to establish (as $\mathscr{H}_{\tilde{s}}|_{\{a\}\times\{0,1\}^{\mathbb{J}^c}} = \{a\} \times \mathscr{H}^r_{\vec{0}_r}$; ϕ from the lemma):

$$\varphi^a : \{a\} \times \mathscr{H}^r_{\vec{0}_r} \to \{a\} \times \mathscr{H}^r_{\vec{1}_r}$$
$$(a,z) \mapsto \varphi^a((a,z)) := (a, \phi(z))$$

And ultimately: $f : \mathscr{H}_{\tilde{s}} \to \mathscr{H}_{\tilde{p}}$ given by $x \mapsto \varphi^{x_{\mathbb{J}}}(x)$. By finally conjugating with our original bijection of the hypercube, we come back to:
$F := \psi^{-1} \circ f \circ \psi : \mathscr{H}_s \to \mathscr{H}_p$ which satisfies all the properties we want (**bijective** and $\forall x \in \mathscr{H}_s, \ F(x) \in [x, p]$)

Another way of reducing to the *antipodal* case:

We can restrict our attention to issues where $s_1 \neq s_2$. Let $r = |s| = dist(s_1, s_2)$ and focus on a new BSC game BSC^r, where the sets of voters and candidates are the same as in BSC, and taking the hypercube $\mathscr{H}^r := \{0,1\}^r$.
Assume, w.l.o.g., that s_1 and s_2 differ on the **first** r issues. We transform game BSC into game BSC^r by using the function $\phi : \mathscr{H} \to \mathscr{H}^r$ where $(\phi(p))_j = p_j$ for every $j \in \mathbf{s}$ and every position $p \in \mathscr{H}$.
It is easy to see that, by definition, $\phi(s_i) = \phi(\hat{s}_{-i})$ (**they are now antipodal!**) and $dist(\phi(s_i), \phi(s_{-i})) = r$. Also: $\forall p \in P_i^{\mathbf{s}}$, we have $\phi(p) \in P_i^{\phi(s)}$.
We denote by $w_p := f_N(p)$ and $w^r_{p^r} := f_N(p^r)$. In addition, we define $F(p^r) := \{p \in \mathscr{H} : \phi(p) = p^r\}$ for each $p^r \in \mathscr{H}^r$; and also w^r as follows: $w^r(p^r) := \sum_{p \in F(p^r)} w_p$. Hence, we have $sc(\phi(s_i)) = \sum_{p^r \in i\phi(s)} w_{p^r} = \sum_{p^r \in i\phi(s)} \sum_{p \in F(p^r)} w_p = sc(s_i)$ for every $i \in \{1,2\}$.
The distribution of voters in this new populism game is still single-peaked:

Lema

The distribution of voters in populism game BSC^r is single-peaked w.r.t. peak position $\phi(p^*)$.

Extra Slide: Details of the reduction

X3C

In an instance of X3C, we are given a set $X = \{x_1, x_2, \dots, x_{3q}\}$ and a set $S = \{S_1, S_2, \dots, S_r\}$ of 3-element subsets of X and we ask whether there exists an exact cover, i.e., a subset $S' \subseteq S$ such that every element of X occurs in exactly one member of S', in other words S' is a partition of X.

We construct our game as follows. We consider $K = 3q + 4$ issues, and we create $(3q + 10)w_p + 23$ voters, given an arbitrary integer w_p such that $w_p > 24$, where the voters are distributed as follows:

- w_p voters on each position $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ such that $e^i_i = 1$ and $e^i_j = 0$ for every $j \in [3q + 4] \setminus \{i\}$, for every $i \in [3q]$;
- $\frac{5}{2}w_p + 11$ voters on position $p_1 := (0, \dots, 0, 1, 1, 0, 0)$;
- 7 voters on position $p_2 := (0, \dots, 0, 0, 0, 1, 1)$;
- $\frac{5}{2}w_p + 3$ voters on position $p_3 := (0, \dots, 0, 0, 0, 1, 0)$;
- 2 voters on position $p_4 := (0, \dots, 0, 0, 0, 0, 1)$.

Idem

- We create $q+2$ candidates and denote the set of candidates by $C := C_S \cup \{c_a, c_b\}$, where the set $C_S := \bigcup_{j=1}^{q} c_j$ regroups the so-called subset-candidates.
- The sets of strategies are:
  - $\mathscr{H}_c := \mathscr{H}_S := \bigcup_{j=1}^{r} \{s^j = (s_1, \ldots, s_{3q}, 0, 0, 0, 0) \in \{0,1\}^K : \forall i \in [3q],\ s_i = 1 \text{ iff } x_i \in S_j\}$ for every $c \in C_S$;
  - $\mathscr{H}_{c_a} := \{s_a^1 := (0, \ldots, 0, 1, 0, 0, 1), s_a^2 := (0, \ldots, 0, 1, 1, 0, 0)\}$;
  - $\mathscr{H}_{c_b} := \{s_b^1 := (0, \ldots, 0, 0, 0, 1, 1), s_b^2 := (0, \ldots, 0, 1, 0, 1, 0)\}$.
- The candidates' truthful positions are arbitrary and their preferences are narcissistic.

One can prove that there exists a Nash equilibrium in the populism game iff there exists a subset of S that is a partition of X.

The idea is that only candidates c_a and c_b may have an incentive to deviate and they would do so only if there is a position e^i for $i \in [3q]$ not "covered" by the strategy position of a subset-candidate.

Idem

We report in the table the number of votes that candidates c_a and c_b can get from positions p_1, p_2, p_3, and p_4.

		$\mathscr{H}_{c_b}$	
		s_b^1	s_b^2
$\mathscr{H}_{c_a}$	s_a^1	$\left(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11\right)$	$\left(\frac{5}{4}w_p + 11, \frac{15}{4}w_p + 12\right)$
	s_a^2	$\left(\frac{5}{2}w_p + 11, \frac{5}{2}w_p + 12\right)$	$\left(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11\right)$

Table: Number of votes, from the voters whose truthful position is in $\{p_1, p_2, p_3, p_4\}$, that candidates c_a and c_b get according to all their possible strategies.

- A better response for candidate c_a or c_b would trigger a cycle of local deviations, preventing a Nash equilibrium to exist.
- Moreover, the only deviations that c_a or c_b can make are towards another strategy position at distance 2 from their previous strategy position.

It follows that the complexity result also holds for 2-local equilibria.