Minimal elements for the base b representations of the primes which are > b

Keywords

prime number, number theory, minimal element, partially ordered set, subsequence, formal language theory, positional notation, radix, algorithm, computer science, primality test, Miller–Rabin primality test, Baillie–PSW primality test, sieving, heuristic algorithm, conjecture, open problem, mathematical proof

Target of this article

Introduction

A <u>string</u> x is a <u>subsequence</u> of another string y, if x can be obtained from y by deleting zero or more of the <u>characters</u> (in this article, <u>digits</u>) in y. For example, 514 is a subsequence of 352148, "*string*" is a subsequence of "*Meistersinger*". In contrast, 758 is not a subsequence of 378259, since the <u>characters</u> (in this article, <u>digits</u>) must be in the same order. The <u>empty string</u> λ is a subsequence of every string. There are 2^n subsequences of a string with length n, e.g. the subsequences of 123456 are (totally $2^6 = 64$ subsequences):

λ, 1, 2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56, 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456, 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 3456, 12345, 12346, 12356, 12456, 13456, 23456, 123456

(In this article, we only consider the subsequences with length ≥2, and not consider the subsequences <u>beginning with 0</u> and/or <u>ending with 0</u>, e.g. for the string 123456, we have these subsequences: 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56, 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456, 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456,

3456, 12345, 12346, 12356, 12456, 13456, 23456, 123456, totally 57 subsequences, and for a string with length n with no character 0, there are 2^n-n-1 subsequences)

<u>Subsequence</u> should not be confused with <u>substring</u>, a substring is a contiguous sequence of characters within a string, e.g. 397 is a subsequence of 163975, "*string*" is a substring of "*substring*". In contrast, 514 is a subsequence of 352148, but not a substring. The <u>empty</u> <u>string</u> λ is a substring of every string. There are $n^*(n+1)/2+1$ substrings of a string with length n, e.g. the substrings of 123456 are (totally $6^*(6+1)/2+1 = 22$ substrings):

λ, 1, 2, 3, 4, 5, 6, 12, 23, 34, 45, 56, 123, 234, 345, 456, 1234, 2345, 3456, 12345, 23456, 123456

There are 64-22 = 42 subsequences of 123456 which are not substrings:

13, 14, 15, 16, 24, 25, 26, 35, 36, 46, 124, 125, 126, 134, 135, 136, 145, 146, 156, 235, 236, 245, 246, 256, 346, 356, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2346, 2356, 2456, 12346, 12356, 12456, 13456

Substring also called "subword", while subsequence also called "scattered subword".

(For the references of the difference between "subsequence" and "substring", see this post and this post, and see the list below)

Subsequence	Substring
<u>A071062</u>	<u>A033274</u>
<u>A130448</u>	<u>A238334</u>
<u>A039995</u>	<u>A039997</u>
<u>A039994</u>	<u>A039996</u>
<u>A094535</u>	<u>A093301</u>
<u>A350508</u>	<u>A038103</u>
https://primes.utm.edu/glossary/xpage/Mini malPrime.html	https://www.mersenneforum.org/showthread.php?p=235098#post235098
longest common subsequence problem	longest common substring problem

The <u>longest common subsequence problem</u> and the <u>longest common substring problem</u> are two hard problems on <u>strings</u>, the former is <u>NP-hard</u> and <u>NP-complete</u>, while the latter is not.

divisibility ordering	subset ordering	subsequence ordering	substring ordering
greatest common divisor of natural	intersection of sets	longest common subsequence of	longest common substring of strings

numbers	strings	
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Note: The comment by Charles R Greathouse IV in https://oeis.org/A062115 is wrong, it should be A033274 instead of A071062, however, A062115 is a 10-automatic sequence is really true, currently there is no analog of A062115 with subsequence instead of substring in OEIS (Searching of this sequence in OEIS), the first difference of such sequence and A062115 is that such sequence does not have the term 169 (since the prime number 19 is a subsequence but not a substring, of 169), but A062115 has.

(In this article, we only research <u>subsequence</u> and not research <u>substring</u>, the reason is the minimal set of <u>subsequence ordering</u> must be <u>finite</u> even if the set is <u>infinite</u> (by the theorem that there are no <u>infinite</u> <u>antichains</u> for the <u>subsequence ordering</u>), and hence we may find this set, but the minimal set of <u>substring ordering</u> may be <u>infinite</u>, and it is highly possible that we cannot find this set, e.g. the minimal set of subsequence ordering of the set of prime number digit strings with length ≥ 2 in decimal (<u>proofs for that this set is infinite</u>) is known to be finite and contain exactly 77 elements, and the largest element is $50^{28}27$, where 0^{28} means the string with 28 0's, but the minimal set of substring ordering of the set of prime number digit strings with length ≥ 2 in decimal is very likely to be infinite, since all primes of the form $1\{0\}3$ (10^n+3 , $10^$

The <u>set</u> of all <u>strings</u> ordered by <u>subsequence</u> (i.e. under the <u>binary relation</u> "is a subsequence of") is a <u>partially ordered set</u> (i.e. the binary relation "is a subsequence of" is a <u>partial order relation</u>, since this binary relation is <u>reflexive</u>, <u>antisymmetric</u>, and <u>transitive</u>), hence, any given (<u>finite</u> or <u>infinite</u>) set (e.g. the set of the "<u>prime numbers</u> > b" <u>strings</u> in <u>base</u> b, for $2 \le b \le 36$), which is the target of this article) of strings ordered by subsequence is also a partially ordered set, and thus we can draw its <u>Hasse diagram</u> and find its <u>greatest</u> <u>element</u>, <u>least element</u>, <u>maximal elements</u>, and <u>minimal elements</u>, however, the greatest element and least element may not exist, and for an infinite set, the maximal elements also may not exist, thus we are only interested on finding the <u>minimal elements</u> of such sets, and we define <u>minimal set</u> of a set as the set of the minimal elements of this set, under a given <u>partially ordered binary relation</u> (this binary relation is "is a subsequence of" in this article), and we use M(S) to denote the minimal set of the set S.

A partially ordered set is a <u>totally ordered set</u> if the elements in this set are pairwise <u>comparable</u>, two elements x and y are <u>comparable</u> with respect to a binary relation " \leq " if at least one of $x \leq y$ or $y \leq x$ is true, thus, under the binary relation "is a subsequence of", two strings x and y are <u>comparable</u> if either x is a subsequence of y, or y is a subsequence of x. A surprising result from <u>formal language theory</u> is that every set of pairwise incomparable (i.e. not comparable) strings is finite (note that this is not true for general <u>partially ordered binary relations</u>, e.g. the set of the <u>positive integers</u>, under the binary relation "is a <u>divisor</u> of", the <u>infinite set</u> of the <u>prime numbers</u> (<u>proofs for that this set is infinite</u>) is pairwise incomparable, in fact, this set is exactly the minimal set of the set of the <u>positive integers</u> >1 under this binary relation). This means that from any set of strings we can find its <u>minimal</u>

<u>elements</u>. A string x in a set of strings S is a *minimal string* (minimal element of a set of strings ordered by subsequence) if whenever y (an element of S) is a subsequence of x, we have y = x.

The set of all minimal strings of S is denoted M(S), M(S) is the **kernel** of the set S, and the set M(S) must be finite! Even if S is an infinite set, such as the set of prime number digit strings with length ≥2 in decimal (proofs for that this set is infinite) and the set of square number digit strings with length ≥2 in decimal, although the set of the minimal strings of the latter set is not known and extremely difficult to compute. The set of the minimal strings of the former set has exactly 77 elements, and it is {11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 227, 251, 257, 277, 281, 349, 409, 449, 499, 521, 557, 577, 587, 727, 757, 787, 821, 827, 857, 877, 881, 887, 991, 2087, 2221, 5051, 5081, 5501, 5581, 5801, 5851, 6469, 6949, 8501, 9001, 9049, 9221, 9551, 9649, 9851, 9949, 20021, 20201, 50207, 60649, 80051, 666649, 946669, 5200007, 22000001, 60000049, 66000049, prove that this set is complete, and the research of this set in other bases is exactly the target of this article. The set of the minimal strings of the latter set is {16, 25, 36, 49, 64, 81, 100, 121, 144, 289, 324, 400, 441, 484, 529, 576, 676, 729, 784, 900, 961, 1024, 1089, 2209, 2304, 2401, 2601, 2704, 3721, 3844, 4761, 5041, 5184, 6561, 6889, 7056, 7569, 7744, 7921, 21904, 22201, 28224, 29241, 29929, 31329, 35344, 38809, 46656, 47524, 55696, 62001, 63001, 69696, 79524, 80089, 80656, 82944, 88209, 88804, 91204, 91809, 97344, 97969, 98596, 99856, 138384, 139129, 173889, 182329, 199809, 300304, 301401, 304704, 305809, 332929, 339889, 345744, 374544, 393129, 473344, 505521, 515524, 558009, 559504, 567009, 589824, 595984, 657721, 660969, 665856, 683929, 695556, 702244, 719104, 743044, 777924, 779689, 842724, 850084, 876096, 877969, 896809, 898704, 929296, 935089, 1317904, 1557504, 1882384, 1898884, 2022084, 2027776, 2039184, 2070721, 2477476, 2802276, 2979076, 2999824, 3055504, 3073009, 3139984, 3323329, 3415104, 3794704, 4477456, 4545424, 4575321, 5053504, 5067001, 5071504, 5280804, 5303809, 5513104, 5527201, 5531904, 5574321, 5579044, 5707321, 5750404, 5755201, 5987809, 6517809, 6568969, 6620329, 6901129, 7006609, 7011904, 7033104, 7096896, 7177041, 7474756, 7551504, 7557001, 7573504, 7941124, 8020224, 8054244, 8282884, 8340544, 8508889, 8538084, 8620096, 8809024, 9229444, 9535744, 9809424, 9847044, 9935104, 9998244, 13118884, 13337104, 15038884, 15578809, 18939904, 19775809, 20903184, 20912329, 20994724, 23902321, 27709696, 29833444, 31102929, 31899904, 33039504, 33085504, 33315984, 33500944, 35533521, 35545444, 37797904, 38093584, 39980329, 40755456, 45535504, 47073321, 47444544, 50098084, 50566321, 50580544, 50608996, 50808384, 51151104, 53333809, 53993104, 55011889, 55517401, 55666521, 57501889, 57775201, 58247424, 58339044, 58859584, 59089969, 60575089, 60590656, 61199329, 65658609, 66650896, 66863329, 69072721, 69338929, 70006689, 70543201, 70997476, 71351809, 72233001, 73153809, 73994404, 74407876, 74632321, 75968656, 77668969, 77686596, 77757124, 77898276, 78907689, 78960996, 78978769, 79869969, 84052224, 85507009, 86992929, 88059456, 88096996, 88585744, 88868329, 89056969, 91833889, 94303521, ...}, although this set seems to be endless, but by the theorem that there are no infinite antichains for the subsequence ordering, this set must be finite, but this set is extremely difficult to found (reference), and it is also difficult to determine the number of elements in this set, and is much more difficult than that of the first set in every base $2 \le b \le 36$ (to find these two sets in bases $2 \le b \le 36$ (the prime or square = b (i.e. the prime or square "10") is also excluded when the base (b) is itself prime or square),

we can use some theorems in number theory, e.g. a digit in base b can be the last digit of a prime number > b if and only if this digit is coprime to b (i.e. this digit is in the reduced residue system mod b, there are eulerphi(b) such digits), and a digit in base b can be the last digit of a square number > b if and only if this digit is a quadratic residue mod b). For example, it is not even known whether there is a square composed of digits 6, 7, 8 (except 676 = 26²) (reference and reference and reference), also, it is not even known whether the non-simple family 3^m5ⁿ9^r44 contain a square or not, this situation usually not occur for primes in any base, i.e. every non-simple family which can not be ruled out as containing no primes > base usually contain a small prime > base, thus although the problem in this article (i.e. finding the minimal set of the primes > b in base b, for $2 \le b \le 36$) is hard, it is much easier than finding the minimal set of the squares > 10 in decimal (also finding the minimal set of the squares > b in base b for any base b > 4), thus the latter set is not discussed in this article. (another reason for we research the minimal strings of the prime numbers instead of the minimal strings of the square numbers is that the prime numbers behave similarly to a random sequence of numbers, while the square numbers do not, thus prime numbers are more mysterious than square numbers)

	the last digit of a prime number > b in base b	the last digit of a square number > b in base b
condition	coprime to b	a <u>quadratic residue</u> mod b
number of such digits	<u>A000010</u>	A000224
irregular triangle read by rows, row <i>b</i> is such digits in base <i>b</i>	A038566	<u>A096008</u>
bases <i>b</i> such that all such digits are (primes or 1, squares, respectively), thus the corresponding minimal set problems are easy to solve if single-digit numbers are not excluded, there are only finitely many such bases <i>b</i>	A048597 (2, 3, 4, 6, 8, 12, 18, 24, 30)	A254328 (2, 3, 4, 5, 8, 12, 16)

In this article, we want to find the <u>set</u> of the minimal strings of the "<u>prime number</u> > b" <u>digit strings</u> in <u>bases</u> $2 \le b \le 36$, since <u>decimal</u> (base 10) is not special in <u>mathematics</u>, there is no reason to only find this set in decimal (base 10), also, finding this set in decimal (base 10) is too easy to be researched in an article (only harder than bases 2, 3, 4, 6), thus it is necessary to research this set in other bases b.

Equivalently, a string x in a set of strings S is a minimal string if and only if any proper subsequence of x (subsequence of x which is unequal to x, like proper subset) is not in S.

The minimal set M(L) of a <u>language</u> L is interesting, this is because it allows us to compute two natural and related languages, defined as follows:

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sub(L) := \{x \in \Sigma^* : \text{there exists } y \in L \text{ such that } x \text{ is a subsequence of } y\};
 sup(L) := \{x \in \Sigma^* : \text{there exists } y \in L \text{ such that } y \text{ is a subsequence of } x\}.
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An amazing fact is that sub(L) and sup(L) are always regular. This follows from the following classical theorem:

Theorem: For every language *L*, there are only finitely many minimal strings. (Equivalently, there are no <u>infinite antichains</u> for the <u>subsequence ordering</u>) (references: https://books.google.com.tw/books?id=-

HrTBwAAQBAJ&pg=PA255&lpg=PA255&dq=every+set+of+pairwise+incomparable+strings +is+finite&source=bl&ots=U7D1b_pfao&sig=ACfU3U2_pcwWftogmSFA03C6D7_xR5uxg&hl=zh-

TW&sa=X&ved=2ahUKEwjP272ytqX2AhWMHKYKHfqVCOAQ6AF6BAgTEAM#v=onepage &q=every%20set%20of%20pairwise%20incomparable%20strings%20is%20finite&f=false https://www.jstor.org/stable/44161544 http://www.ams.org/mathscinet-getitem?mr=84g:05002 (article is not yet available) https://hal.archives-ouvertes.fr/hal-01888614/document

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TW&sa=X&ved=2ahUKEwjP272ytqX2AhWMHKYKHfqVCOAQ6AF6BAgREAM#v=onepage &q=every%20set%20of%20pairwise%20incomparable%20strings%20is%20finite&f=false https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.65.3806&rep=rep1&type=pdf http://www.combinatorics.org/Volume_7/PDF/v7i1n2.pdf https://www.researchgate.net/publication/233917563_Large_infinite_antichains_of_permutations http://www.lsv.fr/~phs/course1.pdf

Indeed, we have sup(L) = sup(M(L)) and $\Sigma^* - sub(L) = sup(M(\Sigma^* - sub(L)))$, and the superword language of a finite language is regular, since $sup(\{w_1, ..., w_n\}) = \bigcup_{i=1}^n \quad \Sigma^* w_{i,1} \Sigma^* ... \quad \Sigma^* w_{i,|w_i|} \Sigma^*$ where $w_i = w_{i,1} ... \quad w_{i,|w_i|}$ with $w_{i,j} \in \Sigma$.

Since there are no <u>infinite antichains</u> for the <u>subsequence ordering</u> of <u>strings</u> whose <u>characters belong to</u> a fixed <u>finite set</u> (e.g. the "<u>prime number</u> > b" <u>digit strings</u> in <u>positional numeral system</u> with <u>radix</u> b (which is exactly the target of this article), whose <u>characters</u> always belong to the set of the digits in base b: $\{0, 1, ..., b-1\}$, which is a <u>finite set</u> with b elements, note that the set must be <u>finite</u> (an easy counterexample for an infinite set S is the set of all strings with length 2 whose <u>characters</u> belong to the set S, which is clearly an <u>infinite antichain</u> for the <u>subsequence ordering</u>), thus, e.g. in <u>factorial base</u> there may exist infinitely many minimal primes, i.e. the minimal set of the prime strings of subsequence ordering may be infinite, since the set of the digits in factorial base is <u>infinite</u>, it includes *all* nonnegative integers, and thus this is not discussed in this article, just as the minimal set of substring ordering) (note that there can be <u>infinite</u> antichains for general <u>ordering</u>, e.g. the set of <u>primes</u> is an infinite antichain for the <u>divisibility</u> ordering (<u>proofs for that this set is infinite</u>), also, the set of strings {abc, abbc, abbbc, abbbbc, ...} is an infinite antichain for the <u>substring</u> ordering of strings whose characters are in a fixed finite set {a, b, c}), the set M(S) of minimal strings of any set S of strings must be <u>finite</u>.

Although the set M(S) of minimal strings is necessarily <u>finite</u>, determining it explicitly for a given S can be a difficult computational problem. We use some <u>numbertheoretic heuristics</u> to <u>compute</u> $M(L_b)$, where L_b is the <u>language</u> of <u>base-b</u> representations of the <u>prime numbers</u> which are > b, for $2 \le b \le 16$ (the set $M(L_b)$ can be called **b-kernel**, since this set is the kernel of the set L_b). (Also, I left as a challenge to readers the task of computing $M(L_b)$ for 17 $\le b \le 36$) (we stop at base 36 since this base is a maximum base for which it is possible to <u>write</u> the <u>numbers</u> with the <u>symbols</u> 0, 1, ..., 9 (the 10 <u>Arabic numerals</u>) and A, B, ..., Z (the 26 <u>Latin letters</u>) of the Latin alphabet, references: <u>http://www.tonymarston.net/php-mysql/converter.html</u> <u>https://www.dcode.fr/base-36-cipher</u> <u>https://docs.python.org/3/library/functions.html#int</u> <u>https://reference.wolfram.com/language/ref/BaseForm.html</u> <u>https://baseconvert.com/https://www.calculand.com/unit-converter/zahlen.php?og=Base+2-36&ug=1 https://linesegment.web.fc2.com/application/math/numbers/RadixConversion.html (in Japanese), also see https://primes.utm.edu/notes/words.html for English words which are prime numbers when viewed as a number base 36)</u>

This problem is very hard, since determining M(L) for arbitrary L is in general unsolvable and can be difficult even when L is relatively simple, the set M(L) is an antichain of L for the subsequence ordering (although may not be the "maximum antichain" (an antichain that has cardinality at least as large as every other antichain), which may not exist even for the subsequence ordering, although there cannot be an infinite antichains for the subsequence ordering), the problems in this article (i.e. determining $M(L_b)$ for $2 \le b \le 36$) are very hard open problems in number theory when b is large (say > 16) and may be $\frac{NP-complete}{D}$ or $\frac{NP-complete}{D}$ hard or an undecidable problem, or an example of Gödel's incompleteness theorems (like the continuum hypothesis and the halting problem, in fact, if the halting problem can be solved, then the problem in this article can also be solved (we only need to write a computer program for this problem, since this problem is discrete), however, the halting problem is known to be undecidable, i.e. a general algorithm to solve the halting problem for all possible program-input pairs cannot exist) (even in the weaker case that probable primes are allowed in place of proven primes, i.e. not including primality proving of the probable primes in $M(L_b)$), or as hard as the unsolved problems in mathematics, such as the Riemann hypothesis and the abc conjecture (which are the two famous hard problems in number theory), determining $M(L_b)$ is much harder when b > 24 and/or eulerphi(b) is larger, since eulerphi(b) is the number of possible last digits of a prime number > b in base b (these digits are exactly the base b digits coprime to b, all these bases are possible and for all such digits, there are infinitely many such primes (by Dirichlet's theorem), and for digits not coprime to b (let d be the greatest common divisor (GCD) of the digit and b), all such numbers are <u>divisible</u> by d and $d \le b$, thus cannot be primes > b). Maybe only God knows the set $M(L_b)$ when b > 24 (and only God knows the largest element in the set $M(L_b)$ when b > 24, and only God knows the number of the elements in the set $M(L_b)$ when b > 24). We can imagine an alien force, vastly more powerful than us, landing on Earth and demanding $M(L_b)$ for b =17 (or 18, 19, 20, 21, 22, 23, 24, 28, 30, 36) (including primality proving of all primes in this set) or they will destroy our planet. In that case, I claim, we should marshal all our computers and all our mathematicians and attempt to find the set and to prove the primality of all numbers in this set. But suppose, instead, that they ask for $M(L_b)$ for b = 25 (or 26, 27, 29, 31, 32, 33, 34, 35). In that case, I believe, we should attempt to destroy the aliens. (like Paul <u>Erdős for the Ramsey numbers</u>, I do not think that finding $M(L_b)$ for b > 16 is easier than finding the Ramsey numbers R(m,n) for m > 4, n > 4)

There are many unsolved problems (open problems) in number theory:

- * Grand Riemann hypothesis
- ** Extended Riemann hypothesis
- *** Generalized Riemann hypothesis
- **** Riemann hypothesis
- * *n* conjecture
- ** abc conjecture
- *** Fermat-Catalan conjecture
- **** Beal conjecture
- *** Lander, Parkin, and Selfridge conjecture
- *** Pillai's conjecture
- ** Szpiro's conjecture

and unsolved problems (<u>open problems</u>) about the prime numbers, which are related to this article:

- * Are there infinitely many <u>Mersenne primes</u> (the Lenstra–Pomerance–Wagstaff conjecture)? (Equivalently, are there infinitely many even <u>perfect numbers</u>?)
- * Are there infinitely many Wagstaff primes?
- * Are there infinitely many repunit primes?
- * Is every Fermat number $2^{2^n} + 1$ composite for n > 4?
- * Is every generalized half Fermat number $(3^{2^n} + 1)/2$ composite for n > 6?
- * Is every double Mersenne number $2^{2^{n-1}} 1$ composite for n > 7?
- * Are all Mersenne numbers of prime index square-free?
- * Are all Wagstaff numbers of prime index square-free?
- * Are all Fermat numbers square-free?
- * For any given natural number $b \ge 2$ which is not <u>perfect power</u>, are there infinitely many <u>generalized repunit primes</u> in base b (primes of the form $(b^n-1)/(b-1)$)? (If b is <u>perfect power</u>, then generalized repunits in base b can be factored algebraically, and thus there is at most one generalized repunit prime in base b, reference: https://oeis.org/A128164 https://oeis.org/A096059 https://oeis.org/A126589)
- * For any given natural number $b \ge 2$ which is neither perfect odd power (<u>A070265</u>) nor of the form $4m^4$ (<u>A141046</u>), are there infinitely many generalized Wagstaff primes in base b (primes of the form $(b^n+1)/(b+1)$)? (If b is either perfect odd power (<u>A070265</u>) or of the form $4m^4$ (<u>A141046</u>), then generalized Wagstaff numbers in base b can be factored algebraically, and thus there is at most one generalized Wagstaff prime in base b)
- * For any given even natural number $b \ge 2$, are there only finitely many generalized Fermat primes in base b (primes of the form $b^{2^n} + 1$)? (If b is odd, then all generalized Fermat numbers in base b are divisible by 2, and if b is either perfect odd power (A070265), then generalized Fermat numbers in base b can be factored algebraically, and thus there is no generalized Fermat prime in base b)
- * For any given odd natural number $b \ge 3$, are there only finitely many generalized half Fermat primes in base b (primes of the form $(b^{2^n} + 1)/2$)? (If b is either perfect odd power (A070265), then generalized half Fermat numbers in base b can be factored algebraically, and thus there is no generalized half Fermat prime in base b)

- * For any given natural number $b \ge 2$, are there infinitely Williams primes of the first kind base b (primes of the form $(b-1)^*b^n-1$)?
- * For any given natural number $b \ge 2$, are there infinitely Williams primes of the second kind base b (primes of the form $(b-1)*b^n+1$)?
- * For any given natural number $b \ge 2$, are there infinitely Williams primes of the third kind base b (primes of the form $(b+1)^*b^n-1$)?
- * For any given natural number $b \ge 2$ which is not == 1 mod 3, are there infinitely <u>Williams</u> <u>primes of the fourth kind</u> base b (primes of the form $(b+1)*b^n+1$)? (If b is == 1 mod 3, then all Williams numbers of the fourth kind in base b are divisible by 3, and thus there is no Williams primes of the fourth kind in base b)
- * Is 78557 the lowest Sierpiński number (the Selfridge conjecture)?
- * Is 509203 the lowest Riesel number?
- * Is 125050976086 the lowest Sierpiński number to base 3?
- * Is 63064644938 the lowest Riesel number to base 3?
- * Is 66741 the lowest Sierpiński number to base 4?
- * Is 39939 the lowest Riesel number to base 4 which is not square (for square k, $k*4^n-1$ can be factored algebraically)?
- * Is 159986 the lowest Sierpiński number to base 5?
- * Is 346802 the lowest Riesel number to base 5?
- * Is 174308 the lowest Sierpiński number to base 6?
- * Is 1597 a Riesel number to base 6? (Equivalently, is 84687 the lowest Riesel number to base 6?)

(for the generalization of the lowest Sierpiński numbers and the lowest Riesel numbers to other bases, see <u>CRUS pages</u> and <u>this article</u>)

other unsolved problems (open problems) about the prime numbers:

- * Goldbach conjecture
- * Twin prime conjecture
- * Polignac's conjecture
- * First Hardy-Littlewood conjecture
- * Fourth Landau problem
- * Bunyakovsky conjecture
- * Dickson's conjecture
- * Schinzel's hypothesis H
- * Are there any odd perfect numbers?
- * Are there any almost perfect numbers other than powers of 2?
- * Are there any quasiperfect numbers?
- * For any given natural number $n \ge 2$, are there infinitely many <u>n-perfect numbers</u>?
- * For any given natural number $n \ge 2$, are there infinitely many n-hyperperfect numbers?
- * Find the set of <u>friendly numbers</u>, especially, are 10, 14, 15, 20, 22, 26, 33, 34, ... friendly? (they are conjectured to be solitary, i.e. not friendly, but if friendly, their smallest friends are large numbers, like the status for the number 24, although 24 is friendly, its smallest friend is 91963648)
- * Are there any odd weird numbers?
- * Are there infinitely many amicable numbers?
- * Are there any pairs of amicable numbers which have opposite parity?

- * Are there any pairs of relatively prime amicable numbers?
- * Are there infinitely many betrothed numbers?
- * Are there any pairs of betrothed numbers which have the same parity?
- * Are there infinitely many sociable number cycles?
- * Are there any sociable number cycles with length 3?
- * Are there any sociable number cycles such that not all numbers have the same parity?
- * Are there any quasi-sociable number cycles with odd length?
- * Are there any numbers n such that eulerphi(x) = n has exactly one solution?
- * Are there any composite numbers n such that *eulerphi*(n) divides n-1?
- * Artin's conjecture on primitive roots
- * For any given integer a which is not a <u>square</u> and does not equal to −1, are there infinitely many primes with a as a <u>primitive root</u>?
- * For any given positive integer *b* which is not a <u>perfect power</u>, are there infinitely many primes with *b* as smallest positive <u>primitive root</u>?
- * For any given negative integer *b* which is not a <u>perfect power</u>, are there infinitely many primes with *b* as largest negative <u>primitive root</u>?
- * Are there infinitely many <u>Sophie Germain primes</u> (<u>A005384</u>)? (Equivalently, are there infinitely many safe primes (<u>A005385</u>))?
- * Are there infinitely many <u>Sophie Germain primes</u> of the second kind (<u>A005382</u>)? (Equivalently, are there infinitely many safe primes of the second kind (<u>A005383</u>))?
- * Are there infinitely many Proth primes (A080076)?
- * Are there infinitely many Proth primes of the second kind (A112715)?
- * Are there infinitely many Pierpont primes (A005109)?
- * Are there infinitely many Pierpont primes of the second kind (A005105)?
- * Are there infinitely many Cullen primes (primes of the form n^*2^n+1)?
- * Are there infinitely many Woodall primes (primes of the form $n^2 1$)?
- * Are there any primes p such that p^2^p+1 is also prime?
- * For any given natural number $b \ge 2$, are there infinitely generalized Cullen primes in base b (primes of the form n^*b^n+1)?
- * For any given natural number $b \ge 2$, are there infinitely generalized Woodall primes in base b (primes of the form n^*b^n-1)?
- * Are there infinitely many Carol primes (primes of the form $(2^n-1)^2-2$)?
- * Are there infinitely many Kynea primes (primes of the form $(2^n+1)^2-2$)?
- * For any given even natural number $b \ge 2$, are there infinitely <u>generalized Carol primes</u> in base b (primes of the form n^*b^n+1)? (If b is odd, then all generalized Carol numbers in base b are divisible by 2, and thus there is no generalized Carol prime in base b)
- * For any given even natural number $b \ge 2$, are there infinitely generalized Kynea primes in base b (primes of the form n^*b^n-1)? (If b is odd, then all generalized Kynea numbers in base b are divisible by 2, and thus there is no generalized Kynea prime in base b)

And many hard problems in number theory which are either proved or disproved:

- * Fermat's Last Theorem (proved)
- ** Euler's sum of powers conjecture (disproved)
- * Catalan's conjecture (proved)
- * Dirichlet's theorem (proved)
- * length of primes in arithmetic progression has no upper bound (proved)

Notation

In what follows, if x is a <u>string</u> of <u>symbols</u> over the <u>alphabet</u> $\Sigma_b := \{0, 1, ..., b-1\}$ (the set of the base-b <u>digits</u>) we let $[x]_b$ denote the evaluation of x in the <u>positional numeral system</u> with <u>base (or radix)</u> b (starting with the <u>most significant digit</u>), and $[\lambda]_b := 0$ where λ is the <u>empty string</u>. This is extended to languages as follows: $[L]_b := \{[x]_b : x \in L\}$. We use <u>the convention</u> that A := 10, B := 11, C := 12, ..., C := 12, ..., C := 12, to conveniently represent strings of symbols in base C := 12. We let C := 12 be the <u>canonical representation</u> of C := 12 in base C := 12 be the representation without <u>leading zeros</u>. Finally, as usual, for a language C := 12 we let C := 12 be the usual C := 12 be usual.

Besides, we use M(S) to denote the minimal set (the <u>set</u> of the <u>minimal elements</u>) of the <u>set</u> S of <u>strings</u> for the <u>subsequence ordering</u>, and we use L_b to denote the <u>language</u> of <u>base-b representations</u> of the <u>prime numbers</u> which are > b (thus, L_b is a set of <u>strings</u>), this is a list for L_b for bases $2 \le b \le 36$:

Ь	L _b (using A–Z to represent digit values 10 to 35)
	Lb (using A 2 to represent digit values 10 to 33)
2	11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, 11111, 100101, 101001, 101011, 101111, 110101, 111001, 111011, 11101, 1000011, 1000111, 1001001, 1001111, 1010011, 1001001, 1001011, 1100101, 1100101, 1100101, 1100101, 1100101, 1001011, 10010101, 10010111, 10010101, 10010111, 10010111, 1000001, 10001011, 10100111, 10110111, 11100011, 10110111, 11000011, 11000111, 1100011, 11100011, 11100011, 11100011, 11100011, 11100011, 11100011, 110001011, 100011001, 100011011, 100011011, 100011011, 100011011, 10011001
3	12, 21, 102, 111, 122, 201, 212, 1002, 1011, 1101, 1112, 1121, 1202, 1222, 2012, 2021, 2111, 2122, 2201, 2221, 10002, 10022, 10121, 10202, 10211, 10222, 11001, 11012, 11201, 11212, 12002, 12011, 12112, 12121, 12211, 20001, 20012, 20102, 20122, 20201, 21002, 21011, 21022, 21101, 21211, 22021, 22102, 22111, 22122, 22212, 22221, 100022, 100112, 100202, 100222, 101001, 101021, 101102, 101111, 101212, 102101, 102112, 102121, 102202, 110021, 110111, 110212, 110221, 111002, 111022, 111121, 111211, 112001, 112012, 112102, 112201, 112212, 120011, 120112, 120121, 120222, 121001, 121021, 121102, 121122, 121221, 122002, 122011, 122022, 122002, 200001, 200012, 200111, 200122, 200212, 201022, 201101, 202001, 202021, 202122,
4	11, 13, 23, 31, 101, 103, 113, 131, 133, 211, 221, 223, 233, 311, 323, 331, 1003, 1013, 1021, 1033, 1103, 1121, 1201, 1211, 1213, 1223, 1231, 1301, 1333, 2003, 2021, 2023, 2111, 2113, 2131, 2203, 2213, 2231, 2303, 2311, 2333, 3001, 3011, 3013, 3103, 3133, 3203, 3211, 3221, 3233, 3301, 3323, 10001, 10013, 10031, 10033, 10111, 10121, 10123, 10211, 10303, 10313,

	10321, 10331, 11023, 11101, 11123, 11131, 11201, 11213, 11233, 11311, 11323, 11333, 12011, 12031, 12101, 12121, 12203, 12211, 12233, 12301, 12313, 12323, 13001, 13021, 13031, 13033, 13103, 13133, 13213, 13223, 13003, 13313, 13331, 20021, 20023, 20131, 20203, 20231,
<u>5</u>	12, 21, 23, 32, 34, 43, 104, 111, 122, 131, 133, 142, 203, 214, 221, 232, 241, 243, 304, 313, 324, 342, 401, 403, 412, 414, 423, 1002, 1011, 1022, 1024, 1044, 1101, 1112, 1123, 1132, 1143, 1204, 1211, 1231, 1233, 1242, 1244, 1321, 1343, 1402, 1404, 1413, 1424, 1431, 2001, 2012, 2023, 2034, 2041, 2102, 2111, 2113, 2133, 2212, 2221, 2223, 2232, 2311, 2322, 2342, 2344, 2403, 2414, 2432, 2443, 3004, 3013, 3024, 3042, 3101, 3114, 3134, 3141, 3211, 3213, 3224, 3233, 3244, 3312, 3321, 3323, 3332, 3404, 3422, 3431, 3444, 4003, 4014, 4041, 4043, 4131, 4142, 4212, 4223,
<u>6</u>	11, 15, 21, 25, 31, 35, 45, 51, 101, 105, 111, 115, 125, 135, 141, 151, 155, 201, 211, 215, 225, 241, 245, 251, 255, 301, 305, 331, 335, 345, 351, 405, 411, 421, 431, 435, 445, 455, 501, 515, 521, 525, 531, 551, 1011, 1015, 1021, 1025, 1035, 1041, 1055, 1105, 1115, 1125, 1131, 1141, 1145, 1151, 1205, 1231, 1235, 1241, 1245, 1311, 1321, 1335, 1341, 1345, 1355, 1411, 1421, 1431, 1435, 1445, 1501, 1505, 1521, 1535, 1541, 1555, 2001, 2011, 2015, 2025, 2041, 2045, 2051, 2055, 2115, 2131, 2135, 2151, 2155, 2205, 2225, 2231, 2301, 2311, 2325, 2335,
7	14, 16, 23, 25, 32, 41, 43, 52, 56, 61, 65, 104, 113, 115, 124, 131, 133, 142, 146, 155, 166, 203, 205, 212, 214, 221, 241, 245, 254, 256, 302, 304, 313, 322, 326, 335, 344, 346, 362, 364, 401, 403, 421, 436, 443, 445, 452, 461, 463, 506, 515, 524, 533, 535, 544, 551, 553, 566, 616, 623, 625, 632, 652, 661, 1004, 1006, 1013, 1022, 1033, 1042, 1051, 1055, 1064, 1105, 1112, 1123, 1136, 1141, 1154, 1156, 1165, 1202, 1211, 1222, 1226, 1231, 1235, 1253, 1264, 1301, 1312, 1316, 1325, 1343, 1345, 1402, 1411, 1424, 1433, 1442,
8	13, 15, 21, 23, 27, 35, 37, 45, 51, 53, 57, 65, 73, 75, 103, 107, 111, 117, 123, 131, 141, 145, 147, 153, 155, 161, 177, 203, 211, 213, 225, 227, 235, 243, 247, 255, 263, 265, 277, 301, 305, 307, 323, 337, 343, 345, 351, 357, 361, 373, 401, 407, 415, 417, 425, 431, 433, 445, 463, 467, 471, 475, 513, 521, 533, 535, 541, 547, 557, 565, 573, 577, 605, 615, 621, 631, 643, 645, 657, 661, 667, 673, 701, 711, 715, 717, 723, 737, 747, 753, 763, 767, 775, 1011, 1013, 1035, 1043, 1055, 1063, 1071,
9	12, 14, 18, 21, 25, 32, 34, 41, 45, 47, 52, 58, 65, 67, 74, 78, 81, 87, 102, 108, 117, 122, 124, 128, 131, 135, 151, 155, 162, 164, 175, 177, 184, 201, 205, 212, 218, 221, 232, 234, 238, 241, 254, 267, 272, 274, 278, 285, 287, 308, 315, 322, 328, 331, 337, 342, 344, 355, 371, 375, 377, 382, 407, 414, 425, 427, 432, 438, 447, 454, 461, 465, 472, 481, 485, 504, 515, 517, 528, 531, 537, 542, 548, 557, 562, 564, 568, 582, 601, 605, 614, 618, 625, 638, 641, 661, 667, 678, 685, 702,
10	11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541,

	547, 557, 563, 569,
11	12, 16, 18, 21, 27, 29, 34, 38, 3A, 43, 49, 54, 56, 61, 65, 67, 72, 76, 81, 89, 92, 94, 98, 9A, A3, 106, 10A, 115, 117, 126, 128, 133, 139, 142, 148, 153, 155, 164, 166, 16A, 171, 182, 193, 197, 199, 1A2, 1A8, 1AA, 209, 214, 21A, 225, 227, 232, 236, 238, 247, 25A, 263, 265, 269, 281, 287, 296, 298, 2A1, 2A7, 304, 30A, 315, 319, 324, 331, 335, 342, 351, 353, 362, 364, 36A, 373, 379, 386, 38A, 391, 395, 3A6, 403, 407, 414, 418, 423, 434, 436, 452, 458, 467, 472, 478, 47A,
12	11, 15, 17, 1B, 25, 27, 31, 35, 37, 3B, 45, 4B, 51, 57, 5B, 61, 67, 6B, 75, 81, 85, 87, 8B, 91, 95, A7, AB, B5, B7, 105, 107, 111, 117, 11B, 125, 12B, 131, 13B, 141, 145, 147, 157, 167, 16B, 171, 175, 17B, 181, 18B, 195, 19B, 1A5, 1A7, 1B1, 1B5, 1B7, 205, 217, 21B, 221, 225, 237, 241, 24B, 251, 255, 25B, 267, 271, 277, 27B, 285, 291, 295, 2A1, 2AB, 2B1, 2BB, 301, 307, 30B, 315, 321, 325, 327, 32B, 33B, 347, 34B, 357, 35B, 365, 375, 377, 391, 397, 3A5, 3AB, 3B5, 3B7,
<u>13</u>	14, 16, 1A, 23, 25, 2B, 32, 34, 38, 41, 47, 49, 52, 56, 58, 61, 65, 6B, 76, 7A, 7C, 83, 85, 89, 9A, A1, A7, A9, B6, B8, C1, C7, CB, 104, 10A, 10C, 119, 11B, 122, 124, 133, 142, 146, 148, 14C, 155, 157, 164, 16A, 173, 179, 17B, 184, 188, 18A, 197, 1A8, 1AC, 1B1, 1B5, 1C6, 1CC, 209, 20B, 212, 218, 223, 229, 232, 236, 23C, 247, 24B, 256, 263, 265, 272, 274, 27A, 281, 287, 292, 296, 298, 29C, 2AB, 2B6, 2BA, 2C5, 2C9, 302, 311, 313, 328, 331, 33B, 344, 34A, 34C, 355,
14	13, 15, 19, 21, 23, 29, 2D, 31, 35, 3B, 43, 45, 4B, 51, 53, 59, 5D, 65, 6D, 73, 75, 79, 7B, 81, 91, 95, 9B, 9D, A9, AB, B3, B9, BD, C5, CB, CD, D9, DB, 101, 103, 111, 11D, 123, 125, 129, 131, 133, 13D, 145, 14B, 153, 155, 15B, 161, 163, 16D, 17D, 183, 185, 189, 199, 1A1, 1AB, 1AD, 1B3, 1B9, 1C3, 1C9, 1D1, 1D5, 1DB, 205, 209, 213, 21D, 221, 22B, 22D, 235, 239, 241, 249, 24D, 251, 255, 263, 26B, 271, 279, 27D, 285, 293, 295, 2A9, 2B1, 2BB, 2C3, 2C9, 2CB, 2D3,
<u>15</u>	12, 14, 18, 1E, 21, 27, 2B, 2D, 32, 38, 3E, 41, 47, 4B, 4D, 54, 58, 5E, 67, 6B, 6D, 72, 74, 78, 87, 8B, 92, 94, 9E, A1, A7, AD, B2, B8, BE, C1, CB, CD, D2, D4, E1, ED, 102, 104, 108, 10E, 111, 11B, 122, 128, 12E, 131, 137, 13B, 13D, 148, 157, 15B, 15D, 162, 171, 177, 182, 184, 188, 18E, 197, 19D, 1A4, 1A8, 1AE, 1B7, 1BB, 1C4, 1CE, 1D1, 1DB, 1DD, 1E4, 1E8, 1EE, 207, 20B, 20D, 212, 21E, 227, 22B, 234, 238, 23E, 24B, 24D, 261, 267, 272, 278, 27E, 281, 287,
<u>16</u>	11, 13, 17, 1D, 1F, 25, 29, 2B, 2F, 35, 3B, 3D, 43, 47, 49, 4F, 53, 59, 61, 65, 67, 6B, 6D, 71, 7F, 83, 89, 8B, 95, 97, 9D, A3, A7, AD, B3, B5, BF, C1, C5, C7, D3, DF, E3, E5, E9, EF, F1, FB, 101, 107, 10D, 10F, 115, 119, 11B, 125, 133, 137, 139, 13D, 14B, 151, 15B, 15D, 161, 167, 16F, 175, 17B, 17F, 185, 18D, 191, 199, 1A3, 1A5, 1AF, 1B1, 1B7, 1BB, 1C1, 1C9, 1CD, 1CF, 1D3, 1DF, 1E7, 1EB, 1F3, 1F7, 1FD, 209, 20B, 21D, 223, 22D, 233, 239, 23B, 241,
17	12, 16, 1C, 1E, 23, 27, 29, 2D, 32, 38, 3A, 3G, 43, 45, 4B, 4F, 54, 5C, 5G, 61, 65, 67, 6B, 78, 7C, 81, 83, 8D, 8F, 94, 9A, 9E, A3, A9, AB, B4, B6, BA, BC, C7, D2, D6, D8, DC, E1, E3, ED, F2, F8, FE, FG, G5, G9, GB, 104, 111, 115, 117, 11B, 128, 12E, 137, 139, 13D, 142, 14A, 14G, 155, 159, 15F, 166, 16A, 171, 17B, 17D, 186, 188, 18E, 191, 197, 19F, 1A2, 1A4, 1A8, 1B3, 1BB, 1BF, 1C6, 1CA, 1CG, 1DB, 1DD, 1EE, 1F3, 1FD, 1G2, 1G8, 1GA, 1GG, 209,

<u>18</u>	11, 15, 1B, 1D, 21, 25, 27, 2B, 2H, 35, 37, 3D, 3H, 41, 47, 4B, 4H, 57, 5B, 5D, 5H, 61, 65, 71, 75, 7B, 7D, 85, 87, 8D, 91, 95, 9B, 9H, A1, AB, AD, AH, B1, BD, C7, CB, CD, CH, D5, D7, DH, E5, EB, EH, F1, F7, FB, FD, G5, H1, H5, H7, HB, 107, 10D, 115, 117, 11B, 11H, 127, 12D, 131, 135, 13B, 141, 145, 14D, 155, 157, 15H, 161, 167, 16B, 16H, 177, 17B, 17D, 17H, 18B, 191, 195, 19D, 19H, 1A5, 1AH, 1B1, 1C1, 1C7, 1CH, 1D5, 1DB, 1DD, 1E1, 1EB,
19	14, 1A, 1C, 1I, 23, 25, 29, 2F, 32, 34, 3A, 3E, 3G, 43, 47, 4D, 52, 56, 58, 5C, 5E, 5I, 6D, 6H, 74, 76, 7G, 7I, 85, 8B, 8F, 92, 98, 9A, A1, A3, A7, A9, B2, BE, BI, C1, C5, CB, CD, D4, DA, DG, E3, E5, EB, EF, EH, F8, G3, G7, G9, GD, H8, HE, I5, I7, IB, IH, 106, 10C, 10I, 113, 119, 11H, 122, 12A, 131, 133, 13D, 13F, 142, 146, 14C, 151, 155, 157, 15B, 164, 16C, 16G, 175, 179, 17F, 188, 18A, 199, 19F, 1A6, 1AC, 1AI, 1B1, 1B7, 1BH, 1C4,
<u>20</u>	13, 19, 1B, 1H, 21, 23, 27, 2D, 2J, 31, 37, 3B, 3D, 3J, 43, 49, 4H, 51, 53, 57, 59, 5D, 67, 6B, 6H, 6J, 79, 7B, 7H, 83, 87, 8D, 8J, 91, 9B, 9D, 9H, 9J, AB, B3, B7, B9, BD, BJ, C1, CB, CH, D3, D9, DB, DH, E1, E3, ED, F7, FB, FD, FH, GB, GH, H7, H9, HD, HJ, I7, ID, IJ, J3, J9, JH, 101, 109, 10J, 111, 11B, 11D, 11J, 123, 129, 12H, 131, 133, 137, 13J, 147, 14B, 14J, 153, 159, 161, 163, 171, 177, 17H, 183, 189, 18B, 18H, 197, 19D,
21	12, 18, 1A, 1G, 1K, 21, 25, 2B, 2H, 2J, 34, 38, 3A, 3G, 3K, 45, 4D, 4H, 4J, 52, 54, 58, 61, 65, 6B, 6D, 72, 74, 7A, 7G, 7K, 85, 8B, 8D, 92, 94, 98, 9A, A1, AD, AH, AJ, B2, B8, BA, BK, C5, CB, CH, CJ, D4, D8, DA, DK, ED, EH, EJ, F2, FG, G1, GB, GD, GH, H2, HA, HG, I1, I5, IB, IJ, J2, JA, JK, K1, KB, KD, KJ, 102, 108, 10G, 10K, 111, 115, 11H, 124, 128, 12G, 12K, 135, 13H, 13J, 14G, 151, 15B, 15H, 162, 164, 16A, 16K, 175,
22	11, 17, 19, 1F, 1J, 1L, 23, 29, 2F, 2H, 31, 35, 37, 3D, 3H, 41, 49, 4D, 4F, 4J, 4L, 53, 5H, 5L, 65, 67, 6H, 6J, 73, 79, 7D, 7J, 83, 85, 8F, 8H, 8L, 91, 9D, A3, A7, A9, AD, AJ, AL, B9, BF, BL, C5, C7, CD, CH, CJ, D7, DL, E3, E5, E9, F1, F7, FH, FJ, G1, G7, GF, GL, H5, H9, HF, I1, I5, ID, J1, J3, JD, JF, JL, K3, K9, KH, KL, L1, L5, LH, 103, 107, 10F, 10J, 113, 11F, 11H, 12D, 12J, 137, 13D, 13J, 13L, 145, 14F, 14L,
23	16, 18, 1E, 1I, 1K, 21, 27, 2D, 2F, 2L, 32, 34, 3A, 3E, 3K, 45, 49, 4B, 4F, 4H, 4L, 5C, 5G, 5M, 61, 6B, 6D, 6J, 72, 76, 7C, 7I, 7K, 87, 89, 8D, 8F, 94, 9G, 9K, 9M, A3, A9, AB, AL, B4, BA, BG, BI, C1, C5, C7, CH, D8, DC, DE, DI, E9, EF, F2, F4, F8, FE, FM, G5, GB, GF, GL, H6, HA, HI, I5, I7, IH, IJ, J2, J6, JC, JK, K1, K3, K7, KJ, L4, L8, LG, LK, M3, MF, MH, 10C, 10I, 115, 11B, 11H, 11J, 122, 12C, 12I, 131,
24	15, 17, 1D, 1H, 1J, 1N, 25, 2B, 2D, 2J, 2N, 31, 37, 3B, 3H, 41, 45, 47, 4B, 4D, 4H, 57, 5B, 5H, 5J, 65, 67, 6D, 6J, 6N, 75, 7B, 7D, 7N, 81, 85, 87, 8J, 97, 9B, 9D, 9H, 9N, A1, AB, AH, AN, B5, B7, BD, BH, BJ, C5, CJ, CN, D1, D5, DJ, E1, EB, ED, EH, EN, F7, FD, FJ, FN, G5, GD, GH, H1, HB, HD, HN, I1, I7, IB, IH, J1, J5, J7, JB, JN, K7, KB, KJ, KN, L5, LH, LJ, MD, MJ, N5, NB, NH, NJ, 101, 10B, 10H, 10N,
25	14, 16, 1C, 1G, 1I, 1M, 23, 29, 2B, 2H, 2L, 2N, 34, 38, 3E, 3M, 41, 43, 47, 49, 4D, 52, 56, 5C, 5E, 5O, 61, 67, 6D, 6H, 6N, 74, 76, 7G, 7I, 7M, 7O, 8B, 8N, 92, 94, 98, 9E, 9G, A1, A7, AD, AJ, AL, B2, B6, B8, BI, C7, CB, CD, CH, D6, DC, DM, DO, E3, E9, EH, EN, F4, F8, FE, FM, G1, G9, GJ, GL, H6, H8, HE, HI, HO, I7, IB, ID, IH, J4, JC, JG, JO, K3, K9, KL, KN, LG, LM, M7, MD, MJ, ML, N2, NC, NI, NO,

<u>26</u>	13, 15, 1B, 1F, 1H, 1L, 21, 27, 29, 2F, 2J, 2L, 31, 35, 3B, 3J, 3N, 3P, 43, 45, 49, 4N, 51, 57, 59, 5J, 5L, 61, 67, 6B, 6H, 6N, 6P, 79, 7B, 7F, 7H, 83, 8F, 8J, 8L, 8P, 95, 97, 9H, 9N, A3, A9, AB, AH, AL, AN, B7, BL, BP, C1, C5, CJ, CP, D9, DB, DF, DL, E3, E9, EF, EJ, EP, F7, FB, FJ, G3, G5, GF, GH, GN, H1, H7, HF, HJ, HL, HP, IB, IJ, IN, J5, J9, JF, K1, K3, KL, L1, LB, LH, LN, LP, M5, MF, ML, N1,
<u>27</u>	12, 14, 1A, 1E, 1G, 1K, 1Q, 25, 27, 2D, 2H, 2J, 2P, 32, 38, 3G, 3K, 3M, 3Q, 41, 45, 4J, 4N, 52, 54, 5E, 5G, 5M, 61, 65, 6B, 6H, 6J, 72, 74, 78, 7A, 7M, 87, 8B, 8D, 8H, 8N, 8P, 98, 9E, 9K, 9Q, A1, A7, AB, AD, AN, BA, BE, BG, BK, C7, CD, CN, CP, D2, D8, DG, DM, E1, E5, EB, EJ, EN, F4, FE, FG, FQ, G1, G7, GB, GH, GP, H2, H4, H8, HK, I1, I5, ID, IH, IN, J8, JA, K1, K7, KH, KN, L2, L4, LA, LK, LQ, M5,
28	11, 13, 19, 1D, 1F, 1J, 1P, 23, 25, 2B, 2F, 2H, 2N, 2R, 35, 3D, 3H, 3J, 3N, 3P, 41, 4F, 4J, 4P, 4R, 59, 5B, 5H, 5N, 5R, 65, 6B, 6D, 6N, 6P, 71, 73, 7F, 7R, 83, 85, 89, 8F, 8H, 8R, 95, 9B, 9H, 9J, 9P, A1, A3, AD, AR, B3, B5, B9, BN, C1, CB, CD, CH, CN, D3, D9, DF, DJ, DP, E5, E9, EH, ER, F1, FB, FD, FJ, FN, G1, G9, GD, GF, GJ, H3, HB, HF, HN, HR, I5, IH, IJ, J9, JF, JP, K3, K9, KB, KH, KR, L5, LB,
29	12, 18, 1C, 1E, 1I, 1O, 21, 23, 29, 2D, 2F, 2L, 2P, 32, 3A, 3E, 3G, 3K, 3M, 3Q, 4B, 4F, 4L, 4N, 54, 56, 5C, 5I, 5M, 5S, 65, 67, 6H, 6J, 6N, 6P, 78, 7K, 7O, 7Q, 81, 87, 89, 8J, 8P, 92, 98, 9A, 9G, 9K, 9M, A3, AH, AL, AN, AR, BC, BI, BS, C1, C5, CB, CJ, CP, D2, D6, DC, DK, DO, E3, ED, EF, EP, ER, F4, F8, FE, FM, FQ, FS, G3, GF, GN, GR, H6, HA, HG, HS, I1, IJ, IP, J6, JC, JI, JK, JQ, K7, KD, KJ, KL,
30	11, 17, 1B, 1D, 1H, 1N, 1T, 21, 27, 2B, 2D, 2J, 2N, 2T, 37, 3B, 3D, 3H, 3J, 3N, 47, 4B, 4H, 4J, 4T, 51, 57, 5D, 5H, 5N, 5T, 61, 6B, 6D, 6H, 6J, 71, 7D, 7H, 7J, 7N, 7T, 81, 8B, 8H, 8N, 8T, 91, 97, 9B, 9D, 9N, A7, AB, AD, AH, B1, B7, BH, BJ, BN, BT, C7, CD, CJ, CN, CT, D7, DB, DJ, DT, E1, EB, ED, EJ, EN, ET, F7, FB, FD, FH, FT, G7, GB, GJ, GN, GT, HB, HD, I1, I7, IH, IN, IT, J1, J7, JH, JN, JT, K1,
31	16, 1A, 1C, 1G, 1M, 1S, 1U, 25, 29, 2B, 2H, 2L, 2R, 34, 38, 3A, 3E, 3G, 3K, 43, 47, 4D, 4F, 4P, 4R, 52, 58, 5C, 5I, 5O, 5Q, 65, 67, 6B, 6D, 6P, 76, 7A, 7C, 7G, 7M, 7O, 83, 89, 8F, 8L, 8N, 8T, 92, 94, 9E, 9S, A1, A3, A7, AL, AR, B6, B8, BC, BI, BQ, C1, C7, CB, CH, CP, CT, D6, DG, DI, DS, DU, E5, E9, EF, EN, ER, ET, F2, FE, FM, FQ, G3, G7, GD, GP, GR, HE, HK, HU, I5, IB, ID, IJ, IT, J4, JA, JC, JI,
<u>32</u>	15, 19, 1B, 1F, 1L, 1R, 1T, 23, 27, 29, 2F, 2J, 2P, 31, 35, 37, 3B, 3D, 3H, 3V, 43, 49, 4B, 4L, 4N, 4T, 53, 57, 5D, 5J, 5L, 5V, 61, 65, 67, 6J, 6V, 73, 75, 79, 7F, 7H, 7R, 81, 87, 8D, 8F, 8L, 8P, 8R, 95, 9J, 9N, 9P, 9T, AB, AH, AR, AT, B1, B7, BF, BL, BR, BV, C5, CD, CH, CP, D3, D5, DF, DH, DN, DR, E1, E9, ED, EF, EJ, EV, F7, FB, FJ, FN, FT, G9, GB, GT, H3, HD, HJ, HP, HR, I1, IB, IH, IN, IP, IV,
33	14, 18, 1A, 1E, 1K, 1Q, 1S, 21, 25, 27, 2D, 2H, 2N, 2V, 32, 34, 38, 3A, 3E, 3S, 3W, 45, 47, 4H, 4J, 4P, 4V, 52, 58, 5E, 5G, 5Q, 5S, 5W, 61, 6D, 6P, 6T, 6V, 72, 78, 7A, 7K, 7Q, 7W, 85, 87, 8D, 8H, 8J, 8T, 9A, 9E, 9G, 9K, A1, A7, AH, AJ, AN, AT, B4, BA, BG, BK, BQ, C1, C5, CD, CN, CP, D2, D4, DA, DE, DK, DS, DW, E1, E5, EH, EP, ET, F4, F8, FE, FQ, FS, GD, GJ, GT, H2, H8, HA, HG, HQ, HW, I5, I7, ID,

34	13, 17, 19, 1D, 1J, 1P, 1R, 1X, 23, 25, 2B, 2F, 2L, 2T, 2X, 31, 35, 37, 3B, 3P, 3T, 41, 43, 4D, 4F, 4L, 4R, 4V, 53, 59, 5B, 5L, 5N, 5R, 5T, 67, 6J, 6N, 6P, 6T, 71, 73, 7D, 7J, 7P, 7V, 7X, 85, 89, 8B, 8L, 91, 95, 97, 9B, 9P, 9V, A7, A9, AD, AJ, AR, AX, B5, B9, BF, BN, BR, C1, CB, CD, CN, CP, CV, D1, D7, DF, DJ, DL, DP, E3, EB, EF, EN, ER, EX, FB, FD, FV, G3, GD, GJ, GP, GR, GX, H9, HF, HL, HN, HT,
35	12, 16, 18, 1C, 1I, 1O, 1Q, 1W, 21, 23, 29, 2D, 2J, 2R, 2V, 2X, 32, 34, 38, 3M, 3Q, 3W, 3Y, 49, 4B, 4H, 4N, 4R, 4X, 54, 56, 5G, 5I, 5M, 5O, 61, 6D, 6H, 6J, 6N, 6T, 6V, 76, 7C, 7I, 7O, 7Q, 7W, 81, 83, 8D, 8R, 8V, 8X, 92, 9G, 9M, 9W, 9Y, A3, A9, AH, AN, AT, AX, B4, BC, BG, BO, BY, C1, CB, CD, CJ, CN, CT, D2, D6, D8, DC, DO, DW, E1, E9, ED, EJ, EV, EX, FG, FM, FW, G3, G9, GB, GH, GR, GX, H4, H6, HC,
<u>36</u>	11, 15, 17, 1B, 1H, 1N, 1P, 1V, 1Z, 21, 27, 2B, 2H, 2P, 2T, 2V, 2Z, 31, 35, 3J, 3N, 3T, 3V, 45, 47, 4D, 4J, 4N, 4T, 4Z, 51, 5B, 5D, 5H, 5J, 5V, 67, 6B, 6D, 6H, 6N, 6P, 6Z, 75, 7B, 7H, 7J, 7P, 7T, 7V, 85, 8J, 8N, 8P, 8T, 97, 9D, 9N, 9P, 9T, 9Z, A7, AD, AJ, AN, AT, B1, B5, BD, BN, BP, BZ, C1, C7, CB, CH, CP, CT, CV, CZ, DB, DJ, DN, DV, DZ, E5, EH, EJ, F1, F7, FH, FN, FT, FV, G1, GB, GH, GN, GP, GV,

The primes in $M(L_b)$ are called **minimal prime base b** in this article, although in fact this name should be used for L_b is the language of base-b representations of the prime numbers, where primes > b is not required (reference), this problem is an extension of the original minimal prime problem to include Conjectures 'R Us Sierpinski/Riesel conjectures base b with k-values < b, i.e. the smallest prime of the form k^*b^n+1 and k^*b^n-1 for all k < b. The original minimal prime base b puzzle does not cover CRUS Sierpinski/Riesel conjectures base b with CK < b, since in Riesel side, the prime is not minimal prime in original definition if either k-1 or b-1 (or both) is prime, and in Sierpinski side, the prime is not minimal prime in original definition if k is prime (e.g. $25*30^{34205}-1$ is not minimal prime in base 30 in original definition, since it is OT³⁴²⁰⁵ in base 30, and T (= 29 in decimal) is prime, but it is minimal prime in base 30 if only primes > base are counted), but this extended version of minimal prime base b problem does, this requires a restriction of prime > b, and the primes $\le b$ (including the k-1, b-1, k) are not allowed (i.e. only counting the primes > b, and we want to find the minimal set of "the primes > b" in base b), in fact, to include these conjectures, we only need to exclude the single-digit primes (i.e. the primes < b), also, in fact, I create this problem because I think that the single-digit primes are trivial (like strictly non-palindromic <u>number</u>, single-digit numbers are <u>trivially palindromic</u>, thus to test whether a number *n* is strictly non-palindromic, we do not consider the bases b > n, since in these bases, n is a single-digit number, thus trivially palindromic, note that all strictly non-palindromic numbers > 6 are primes), thus I do not count them (also see this forum post, there is someone else who also exclude the single-digit primes, but his research is about substring instead of subsequence), however, including the base (b) itself results in automatic elimination of all possible extension numbers with "0 after 1" from the set (when the base is prime, if the base is composite, then there is no difference to include the base (b) itself or not), which is quite restrictive (since when the base is prime, then the base (b) itself is the only prime ending with 0, i.e. having trailing zero, since in any base, all numbers ending with 0 (i.e. having trailing zero) are divisible by the base (b), thus cannot be prime unless it is equal the base (b), i.e. "10" in base b, note that the numbers cannot have <u>leading zero</u>, since typically this is not the way we write numbers (in any base), thus for all primes in our sets (i.e. all primes >

base (b)), all zero digits must be "between" other digits) (see this forum post, there is someone else who also excludes the prime = base), thus, we also exclude the prime = b (i.e. the prime "10") (you may ask me why we do not exclude the prime = b+1? Since b+1 is "11" in base b, this is a generalized repunit number base b, if we exclude it ("11" in base b), then we have the next question: should we exclude "111", "1111", "11111", etc. in base b? This is hard to answer, and if we exclude them all, the result will not be "primes > m" for some integer m, thus we do not exclude "11" in base b but exclude "10" in base b, we also exclude the single-digit primes (i.e. the primes < b) in base b), besides, this problem is better than the original minimal prime problem since this problem is regardless whether 1 is considered as prime or not, i.e. no matter 1 is considered as prime or not prime (in the beginning of the 20th century, 1 is regarded as prime) (reference of why 1 is not prime), the sets $M(L_b)$ in this problem are the same, while the sets $M(L_b)$ in the original minimal prime problem are different, e.g. in base 10, if 1 is considered as prime, then the set $M(L_b)$ in the original minimal prime problem is {1, 2, 3, 5, 7, 89, 409, 449, 499, 6469, 6949, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049}, while if 1 is not considered as 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049}, however, in base 10, the set $M(L_b)$ in this problem is always {11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 227, 251, 257, 277, 281, 349, 409, 449, 499, 521, 557, 577, 587, 727, 757, 787, 821, 827, 857, 877, 881, 887, 991, 2087, 2221, 5051, 5081, 5501, 5581, 5801, 5851, 6469, 6949, 8501, 9001, 9049, 9221, 9551, 9649, 9851, 9949, 20021, 20201, 50207, 60649, 80051, 666649, 946669, 5200007, 22000001, 60000049, 66000049, 66600049, 80555551, 555555555551, The third reason for excluding the primes $\leq b$ is that starting with b+1 makes the formula of the number of possible (first digit, last digit) combo of a minimal prime in base b more simple and smooth number, since if start with b, then when b is prime, there is an additional possible (first digit, last digit) combo: (1,0), and hence the formula will be $(b-1)^*$ eulerphi(b)+1 if b is prime, or $(b-1)^*\underline{eulerphi}(b)$ if b is composite (the fully formula will be $(b-1)^*\underline{eulerphi}(b) + \underline{isprime}(b)$ or $(b-1)^*\underline{eulerphi}(b) + \underline{floor}((b-\underline{eulerphi}(b)) / (b-1))$, which is more complex, and if start with 1 (i.e. the original minimal prime problem), the formula is much more complex, since the prime digits (i.e. the single-digit primes) should be excluded, and (for such prime >b) the first digit has b-1-primepi(b) choices, and the last digit has A048864(b) choices, by the rule of product, there are (b-1-primepi(b))*(A048864(b))choices of the (first digit, last digit) combo (if for such prime $\geq b$ instead of >b, then the formula will be (b-1-primepi(b))*(A048864(b))+1 if b is prime, or (b-1-primepi(b))*(A048864(b)) if b is composite), which is much more complex, (also, the possible (first digit, last digit) combo for a prime >b in base b are exactly the (first digit, last digit) combos which there are infinitely many primes have, while this is not true when the requiring is prime $\geq b$ or prime ≥ 2 instead of prime > b, since this will contain the prime factors of b, which are not coprime to b and hence there is only this prime (and not infinitely many primes) have this (first digit, last digit) combo) thus the problem in this forum (i.e. the minimal prime (start with b+1) problem) is much better than the original minimal prime problem.

For example, 857 is a minimal prime in decimal because there is no prime > 10 among the shorter subsequences of the digits: 8, 5, 7, 85, 87, 57. The subsequence does not have to consist of consecutive digits, so 149 is not a minimal prime in decimal (because 19 is prime and 19 > 10). But it does have to be in the same order; so, for example, 991 is still a minimal

prime in decimal even though a subset of the digits can form the shorter prime 19 > 10 by changing the order.

A summary of the results of our <u>algorithm</u> is presented in the table in the next section, I completely solved all bases up to 16 except for bases 14, 16, and the odd bases >6 (the <u>proofs</u> are at the end of this article), for bases 14, 16, and the odd bases >6, I only found all minimal primes up to certain limit (about 2³²) and some larger minimal primes (such as 3¹⁶1 in base 7 and 54¹¹ in base 9). I left as a challenge to readers the task of solving (finding all minimal primes and proving that these are all such primes) bases 7, 9, 11, 13, 14, 15, 16, and bases 17 through 36 (this will be a hard problem, e.g. base 23 has a minimal prime 9E⁸⁰⁰⁸⁷³, and base 30 has a minimal prime OT³⁴²⁰⁵).

Prime numbers are central in <u>number theory</u> because of the <u>fundamental theorem of arithmetic</u>: every <u>natural number greater than 1</u> is either a prime itself or can be <u>factorized</u> as a <u>product</u> of primes that is unique <u>up to</u> their order (<u>sociology</u> is applied <u>psychology</u>, <u>psychology</u> is applied <u>biology</u>, <u>biology</u> is applied <u>chemistry</u>, <u>chemistry</u> is applied <u>physics</u>, <u>physics</u> is applied <u>mathematics</u>, the basics of <u>mathematics</u> is the <u>numbers</u>, the basics of the <u>numbers</u> is the <u>natural numbers</u>, the researching of the <u>natural numbers</u> is <u>number theory</u>). Also see <u>The Prime Pages</u> (a website about the prime numbers). Also see <u>Primegrid</u>. Also see <u>the set of the primes</u> and <u>factoring into primes</u>.

addition	<u>multiplication</u>
subtraction	division
<u>0</u>	1
negation	reciprocal
the set {1}	the set of the <u>prime numbers</u>
less than	divides
1 + 1 + 1 + + 1 with <i>n</i> 1's	the <u>prime factorization</u> of n (e.g. $360 = 2^3 * 3^2 * 5$)

Addition and multiplication are the basic operations of arithmetic (which is also the basics of mathematics). In the addition operation, the identity element is $\underline{0}$, and all natural numbers > 0 can be written as the sum of many $\underline{1}$'s, and the number $\underline{1}$ cannot be broken up; in the multiplication operation, the identity element is $\underline{1}$, and all natural numbers > 1 can be written as the product of many prime numbers, and the prime numbers cannot be broken up. Also, primes are the natural numbers $n \ge \underline{1}$ such that if n divides x^*y (x and y are natural numbers), then n divides either x or y (or both). Also, prime numbers are the numbers n such that the ring of integers modulo n (Z_n) is a field (also is an integral domain, also is a division ring, also has no zero divisors other than 0 (for the special case that n = 1, see zero ring)). Also, many famous problems in number theory are related to the prime numbers, such as the Goldbach's conjecture, the twin prime conjecture, the , etc. and also many famous problems in number theory, although they do not have "prime number" in their , but solving them needs to using the prime numbers, such as the Fermat's Last Theorem, the Riemann hypothesis, the abc conjecture, etc. Besides, "the set $M(L_b)$ " to "the set of the prime numbers

(except b itself) <u>digit strings</u> with length > 1 in <u>base</u> b" to "the <u>partially ordered binary relation</u> by <u>subsequence</u>" is "the <u>set</u> of the prime numbers" to "the <u>set</u> of the <u>integers</u> > 1" to "the <u>partially ordered binary relation</u> by <u>divisibility</u>" (and indeed, the "> 1" in "the prime numbers (except b itself) <u>digit strings</u> with length > 1 in <u>base</u> b" can be corresponded to the "> 1" in "the integers > 1") (for the reason why b itself is excluded, see the sections above and <u>this</u> <u>forum post</u>), thus the problem in this article is very important and beautiful.

subsequence ordering	<u>divisibility</u> ordering
the <u>prime numbers</u> > <i>b</i> <u>digit strings</u> in <u>base</u> <i>b</i>	the <u>integers</u> > 1
the set $M(L_b)$	the <u>set</u> of the <u>prime numbers</u>

Recreations involving the decimal digits of primes have a long history. To give just a few examples, without trying to be exhaustive, Yates studied the "repunits", which are primes of the form 111...111. Caldwell and Dubner studied the "near-repdigits", which are primes with all like or repeated digits but one (e.g. 7877 and 333337). Card introduced prime numbers such as 37337999, in which every nonempty prefix is also a prime; he called them "snowball" primes. These were later studied by Angell & Godwin and Caldwell, who called them "right-truncatable" primes. They also studied the "left-truncatable" primes, such as 4632647, in which every nonempty suffix is prime (the left-truncatable primes are called "Russian doll primes" like that the right-truncatable primes are called "snowball primes", see this page). Kahan and Weintraub gave a list of all the left-truncatable primes (The list of all left-truncatable primes and right-truncatable primes are in http://primerecords.dk/lefttruncatable.txt and http://primerecords.dk/right-truncatable.txt, respectively, also see OEIS sequences A024785 and A024770). Huestis introduced the "recursively laminar primes". In this note, I discuss an apparently new problem on the decimal digits of primes, but one inspired from a classical theorem in formal language theory, i.e. there are only finitely many minimal elements for the subsequence ordering of any given set of strings (in fact, every set of pairwise incomparable strings (for the subsequence ordering) is finite).

(Important note: $\underline{\text{suffix}}$, $\underline{\text{prefix}} \subset \underline{\text{substring}} \subset \underline{\text{subsequence}}$, but $\underline{\text{subsequence}} \not\subset \underline{\text{substring}} \not\subset \underline{\text{substring}} \subset \underline{\text{substring}} \subset \underline{\text{substring}} \subset \underline{\text{subsequence}}$

However, there is no reason to only study these classes of primes in decimal, since the number 10 is not special in <u>mathematics</u>, <u>decimal</u> (<u>base</u> 10) is not special in <u>mathematics</u>, we use <u>decimal</u> (<u>base</u> 10) only because <u>humans</u> have 10 <u>fingers</u>, this fact does not have any <u>mathematical</u> significance, and if <u>humans</u> have 12 <u>fingers</u> instead of 10 <u>fingers</u>, we will use <u>duodecimal</u> (<u>base</u> 12) instead of <u>decimal</u> (<u>base</u> 10), e.g. in base 10 there are "<u>full reptend primes</u>", the primes p which the <u>repeating decimal</u> of k/p for integers $1 \le k \le p-1$ are the <u>cyclic permutation</u> of a (p-1)-digit number (e.g. p=7, the repeating decimal of k/7 for integers $1 \le k \le 6$ are the cyclic permutation of the 6-digit number <u>142857</u>: 142857, 285714, 428571, 571428, 714285, 857142), such number is called <u>cyclic number</u>, a prime p is a full reptend prime if and only if the period length of 1/p in decimal is p-1 (by <u>Fermat's little theorem</u>, for all primes p not dividing 10, the period length of 1/p in decimal always divides p-1, and if p divides 10, then 1/p terminates in decimal and does not give a repeating decimal), i.e. 10 is a <u>primitive root</u> mod p, and this can be generalized to other <u>bases</u> p, full

reptend primes in base b are the primes p which the "repeating base b" of k/p for integers $1 \le k \le p-1$ are the <u>cyclic permutation</u> of a (p-1)-digit number in base b, a prime p is a full reptend prime in base b if and only if the period length of 1/p in base b is p-1 (by Fermat's little theorem, for all primes p not dividing b, the period length of 1/p in base b always divides p-1, and if p divides b, then 1/p terminates in base b and does not give a "repeating base b"), i.e. b is a primitive root mod p, if b is an even square, then such prime p does not exist, and if b is an odd square, then the only such prime p is b, and the natural density of the primes b (over the set of the primes) such that b is a primitive root mod b for given base b is conjectured to be b0.373955813619..., if b1 is neither perfect power and b1 is a perfect b2. It is in b3.4085397, this is Artin's conjecture on primitive roots, if b3 is a perfect b4 is a perfect b5. The power with b7 prime (i.e. b7 divides b4. Then the natural density should be multiplied by b5. The prime b8 is b9 is b9. The prime b9 is b9 is b9 is b9. Then the natural density should be multiplied by b9. The prime b9 is b9 is b9 is b9. The prime b9 is b9 in b9 is b9 in b9 in b9 is b9 in b9

The smallest full reptend primes in base b for b = 2, 3, 4, ... 36 are (0 if no full reptend primes exist for this base b) 3, 2, 0, 2, 11, 2, 3, 2, 7, 2, 5, 2, 3, 2, 0, 2, 5, 2, 3, 2, 5, 2, 7, 2, 3, 2, 5, 2, 11, 2, 3, 2, 19, 2, 0 (*OEIS* sequence <u>A056619</u>)

The smallest base such that p is a full reptend prime for the first 100 primes p (i.e. p = 2, 3, 5, 7, ..., 541) are 3, 2, 2, 3, 2, 2, 3, 2, 5, 2, 3, 2, 6, 3, 5, 2, 2, 2, 2, 7, 5, 3, 2, 3, 5, 2, 5, 2, 6, 3, 3, 2, 3, 2, 2, 6, 5, 2, 5, 2, 2, 2, 19, 5, 2, 3, 2, 3, 2, 6, 3, 7, 7, 6, 3, 5, 2, 6, 5, 3, 3, 2, 5, 17, 10, 2, 3, 10, 2, 2, 3, 7, 6, 2, 2, 5, 2, 5, 3, 21, 2, 2, 7, 5, 15, 2, 3, 13, 2, 3, 2, 13, 3, 2, 7, 5, 2, 3, 2, 2 (*OEIS* sequence A001918)

The smallest prime p such that b is the smallest base such that p is a full reptend prime for b = 2, 3, 4, ... 36 are (0 if no such primes exist for this base b) 3, 7, 0, 23, 41, 71, 0, 0, 313, 643, 4111, 457, 1031, 439, 0, 311, 53173, 191, 107227, 409, 3361, 2161, 533821, 0, 12391, 0, 133321, 15791, 124153, 5881, 0, 268969, 48889, 64609, 0 (*OEIS* sequence A023048)

b	full reptend primes in base b (written in base 10)	OEIS sequence
2	3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83, 101, 107, 131, 139, 149, 163, 173, 179, 181, 197, 211, 227, 269, 293, 317, 347, 349, 373, 379, 389, 419, 421, 443, 461, 467, 491, 509, 523, 541,	<u>A001122</u>
3	2, 5, 7, 17, 19, 29, 31, 43, 53, 79, 89, 101, 113, 127, 137, 139, 149, 163, 173, 197, 199, 211, 223, 233, 257, 269, 281, 283, 293, 317, 331, 353, 379, 389, 401, 449, 461, 463, 487, 509, 521,	<u>A019334</u>
4	(not exist, since 4 is square number, thus 4 is <u>quadratic residue</u> mod all primes and cannot be <u>primitive root</u> mod any odd primes, this only remains to check the prime 2, but 2 divides 4)	A000000 (the empty sequence)
<u>5</u>	2, 3, 7, 17, 23, 37, 43, 47, 53, 73, 83, 97, 103, 107, 113, 137, 157, 167, 173, 193, 197, 223, 227, 233, 257, 263, 277, 283, 293, 307, 317, 347, 353, 373, 383, 397, 433, 443, 463, 467, 503, 523,	<u>A019335</u>
<u>6</u>	11, 13, 17, 41, 59, 61, 79, 83, 89, 103, 107, 109, 113, 127, 131,	<u>A019336</u>

	137, 151, 157, 179, 199, 223, 227, 229, 233, 251, 257, 271, 277, 347, 367, 373, 397, 401, 419, 443, 449, 467, 487, 491, 521,	
7	2, 5, 11, 13, 17, 23, 41, 61, 67, 71, 79, 89, 97, 101, 107, 127, 151, 163, 173, 179, 211, 229, 239, 241, 257, 263, 269, 293, 347, 349, 359, 379, 397, 431, 433, 443, 461, 491, 499, 509, 521,	A019337
8	3, 5, 11, 29, 53, 59, 83, 101, 107, 131, 149, 173, 179, 197, 227, 269, 293, 317, 347, 389, 419, 443, 461, 467, 491, 509,	<u>A019338</u>
9	2 (this is all, since 9 is square number, thus 9 is <u>quadratic</u> residue mod all primes and cannot be <u>primitive root</u> mod any odd primes)	
<u>10</u>	7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, 223, 229, 233, 257, 263, 269, 313, 337, 367, 379, 383, 389, 419, 433, 461, 487, 491, 499, 503, 509, 541,	A001913
11	2, 3, 13, 17, 23, 29, 31, 41, 47, 59, 67, 71, 73, 101, 103, 109, 149, 163, 173, 179, 197, 223, 233, 251, 277, 281, 293, 331, 367, 373, 383, 419, 443, 461, 463, 467, 487, 499,	<u>A019339</u>
<u>12</u>	5, 7, 17, 31, 41, 43, 53, 67, 101, 103, 113, 127, 137, 139, 149, 151, 163, 173, 197, 223, 257, 269, 281, 283, 293, 317, 353, 367, 379, 389, 401, 449, 461, 509, 523,	<u>A019340</u>
<u>13</u>	2, 5, 11, 19, 31, 37, 41, 47, 59, 67, 71, 73, 83, 89, 97, 109, 137, 149, 151, 167, 197, 227, 239, 241, 281, 293, 307, 317, 349, 353, 359, 379, 383, 397, 401, 431, 449, 457, 479, 487, 509, 541,	<u>A019341</u>
<u>14</u>	3, 17, 19, 23, 29, 53, 59, 73, 83, 89, 97, 109, 127, 131, 149, 151, 227, 239, 241, 251, 257, 263, 277, 283, 307, 313, 317, 353, 359, 373, 389, 419, 421, 431, 433, 467, 487, 521, 541,	A019342
<u>15</u>	2, 13, 19, 23, 29, 37, 41, 47, 73, 83, 89, 97, 101, 107, 139, 149, 151, 157, 167, 193, 199, 227, 263, 269, 271, 281, 313, 337, 347, 373, 379, 383, 389, 401, 433, 439, 449, 457, 461, 467, 499, 503, 509, 521,	<u>A019343</u>
<u>16</u>	(not exist, since 16 is square number, thus 16 is <u>quadratic</u> residue mod all primes and cannot be <u>primitive root</u> mod any odd primes, this only remains to check the prime 2, but 2 divides 16)	A000000 (the empty sequence)
17	2, 3, 5, 7, 11, 23, 31, 37, 41, 61, 97, 107, 113, 131, 139, 167, 173, 193, 197, 211, 227, 233, 269, 277, 283, 311, 313, 317, 347, 367, 379, 401, 419, 431, 439, 449, 479, 487, 499, 503, 521,	<u>A019344</u>
<u>18</u>	5, 11, 29, 37, 43, 53, 59, 61, 67, 83, 101, 107, 109, 139, 149, 157, 163, 173, 179, 181, 197, 227, 251, 269, 277, 283, 293, 317, 347, 349, 379, 389, 397, 419, 421, 461, 467, 491, 509, 523, 541,	<u>A019345</u>
19	2, 7, 11, 13, 23, 29, 37, 41, 43, 47, 53, 83, 89, 113, 139, 163, 173, 191, 193, 239, 251, 257, 263, 269, 281, 293, 311, 317, 337, 347, 359, 367, 401, 419, 433, 443, 449, 463, 467, 479, 491, 499,	A019346

	503, 509, 521,	
<u>20</u>	3, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 103, 107, 113, 137, 157, 163, 167, 173, 223, 227, 233, 257, 263, 277, 283, 293, 313, 317, 337, 347, 353, 367, 383, 397, 433, 443, 463, 467, 487, 503,	<u>A019347</u>
21	2, 19, 23, 29, 31, 53, 71, 97, 103, 107, 113, 137, 139, 149, 157, 179, 181, 191, 197, 223, 233, 239, 263, 271, 281, 307, 313, 317, 347, 359, 389, 397, 401, 409, 431, 443, 449, 491, 523,	<u>A019348</u>
22	5, 17, 19, 31, 37, 41, 47, 53, 71, 83, 107, 131, 139, 191, 193, 199, 211, 223, 227, 233, 269, 281, 283, 307, 311, 317, 337, 347, 367, 383, 389, 397, 409, 421, 487, 491, 509, 523,	A019349
23	2, 3, 5, 17, 47, 59, 89, 97, 113, 127, 131, 137, 149, 167, 179, 181, 223, 229, 281, 293, 307, 311, 337, 347, 389, 401, 421, 433, 439, 443, 457, 487, 491, 499, 521,	<u>A019350</u>
<u>24</u>	7, 11, 13, 17, 31, 37, 41, 59, 83, 89, 107, 109, 113, 137, 157, 179, 181, 223, 227, 229, 233, 251, 257, 277, 281, 347, 353, 373, 397, 401, 419, 421, 443, 463, 467, 487, 491, 541,	<u>A019351</u>
25	2 (this is all, since 25 is square number, thus 25 is <u>quadratic</u> <u>residue</u> mod all primes and cannot be <u>primitive root</u> mod any odd primes)	
<u>26</u>	3, 7, 29, 41, 43, 47, 53, 61, 73, 89, 97, 101, 107, 131, 137, 139, 157, 167, 173, 179, 193, 239, 251, 269, 271, 281, 283, 347, 353, 359, 373, 383, 389, 409, 419, 443, 449, 457, 463, 467, 479, 491,	A019352
<u>27</u>	2, 5, 17, 29, 53, 89, 101, 113, 137, 149, 173, 197, 233, 257, 269, 281, 293, 317, 353, 389, 401, 449, 461, 509, 521,	<u>A019353</u>
28	5, 11, 13, 17, 23, 41, 43, 67, 71, 73, 79, 89, 101, 107, 173, 179, 181, 191, 229, 257, 263, 269, 293, 313, 331, 347, 353, 359, 379, 397, 409, 431, 433, 443, 461, 463, 487, 491, 499, 509, 521,	A019354
29	2, 3, 11, 17, 19, 41, 43, 47, 73, 79, 89, 97, 101, 113, 127, 131, 137, 163, 191, 211, 229, 251, 263, 269, 293, 307, 311, 317, 331, 337, 359, 389, 409, 433, 443, 449, 461, 467, 479, 491, 503,	A019355
30	11, 23, 41, 43, 47, 59, 61, 79, 89, 109, 131, 151, 167, 173, 179, 193, 197, 199, 251, 263, 281, 293, 307, 317, 349, 383, 419, 421, 433, 439, 449, 457, 491, 503, 521, 523, 541,	A019356
31	2, 7, 17, 29, 47, 53, 59, 61, 67, 71, 73, 89, 107, 131, 137, 197, 227, 229, 241, 269, 277, 283, 307, 311, 313, 337, 353, 359, 379, 389, 401, 419, 431, 433, 439, 443, 449, 457, 461, 467, 479, 503, 509,	A019357
<u>32</u>	3, 5, 13, 19, 29, 37, 53, 59, 67, 83, 107, 139, 149, 163, 173, 179, 197, 227, 269, 293, 317, 347, 349, 373, 379, 389, 419, 443, 467, 509, 523,	<u>A019358</u>
33	2, 5, 7, 13, 19, 23, 43, 47, 53, 59, 71, 73, 89, 113, 137, 179, 191,	<u>A019359</u>

	251, 257, 269, 311, 317, 337, 349, 353, 383, 389, 409, 419, 439, 443, 449, 457, 467, 509,	
34	19, 23, 31, 41, 43, 53, 59, 67, 73, 79, 83, 101, 113, 149, 157, 167, 179, 193, 199, 233, 241, 251, 293, 311, 313, 337, 349, 367, 373, 389, 401, 431, 439, 449, 461, 467, 479, 491, 503, 509, 523,	A019360
35	2, 3, 11, 37, 41, 47, 53, 61, 71, 79, 83, 89, 101, 103, 137, 151, 167, 179, 191, 197, 211, 223, 227, 229, 233, 239, 241, 269, 283, 317, 331, 359, 373, 379, 383, 409, 431, 457, 461, 467, 499, 503, 509, 521,	<u>A019361</u>
<u>36</u>	(not exist, since 36 is square number, thus 36 is <u>quadratic</u> residue mod all primes and cannot be <u>primitive root</u> mod any odd primes, this only remains to check the prime 2, but 2 divides 36)	A000000 (the empty sequence)

Another example is in base 10 there are <u>unique primes</u>, the primes p such that there is no other prime q such that the period length of the decimal expansion of its <u>reciprocal</u>, 1/p, is equal to the period length of the reciprocal of q, 1/q, a number n is a unique period (i.e. there is only one prime p such that the decimal expansion of 1/p has period length n) if and only if the <u>Zsigmondy number</u> Zs(n,10,1) (see <u>Zsigmondy's theorem</u>) is a prime power p', and hence p is the unique prime with period length n, and this can be generalized to other <u>bases</u> b, a number n is a unique period (i.e. there is only one prime p such that the decimal expansion of 1/p has period length n) if and only if the <u>Zsigmondy number</u> Zs(n,b,1) (see <u>Zsigmondy's theorem</u>) is a prime power p', and hence p is the unique prime with period length p in base p, if p is a true power of a prime (i.e. p' with p in the prime p is a generalized <u>Wieferich prime</u> base p (reference: <u>list of generalized Wieferich primes</u> p is a generalized <u>Wieferich prime</u> base p (reference: <u>list of generalized Wieferich primes</u> p is a generalized unique primes base p (list for bases p is not a perfect power). All generalized repunit primes base p (list for bases p is not all generalized Fermat primes (list for bases p is 1000) are generalized unique primes base p, and there is a list of top 20 known generalized unique primes (with period length p is an order trivial).

b	unique periods in base <i>b</i> (≤ 4096) (written in base 10)
2	2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 30, 31, 32, 33, 34, 38, 40, 42, 46, 49, 54, 56, 61, 62, 65, 69, 77, 78, 80, 85, 86, 89, 90, 93, 98, 107, 120, 122, 126, 127, 129, 133, 145, 147, 150, 158, 165, 170, 174, 184, 192, 195, 202, 208, 234, 254, 261, 280, 296, 312, 322, 334, 342, 345, 366, 374, 382, 398, 410, 414, 425, 447, 471, 507, 521, 550, 567, 579, 590, 600, 602, 607, 626, 690, 694, 712, 745, 795, 816, 889, 897, 909, 954, 990, 1106, 1192, 1224, 1230, 1279, 1384, 1386, 1402, 1464, 1512, 1554, 1562, 1600, 1670, 1683, 1727, 1781, 1834, 1904, 1990, 1992, 2008, 2037, 2203, 2281, 2298, 2353, 2406, 2456, 2499, 2536, 2838, 3006, 3074, 3217, 3415, 3418, 3481, 3766, 3817, 3927,
3	1, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 20, 21, 24, 26, 32, 33, 36, 40, 46, 60, 63, 64, 70, 71, 72, 86, 103, 108, 128, 130, 132, 143, 145, 154, 161, 236, 255, 261, 276, 279, 287, 304, 364, 430, 464, 513, 528, 541, 562, 665, 672, 680, 707, 718, 747, 760, 782, 828, 875, 892, 974, 984, 987, 1037, 1058, 1070, 1073, 1080, 1091, 1154, 1248, 1367, 1426, 1440, 1462, 1524, 1598, 1623, 1627, 1863, 1985, 2132, 2188, 2196, 2340, 2460, 2508, 2626, 2640, 2739, 2856, 3092, 3158, 3262, 3315,

	3326, 3482, 3638, 3982, 4018, 4036,
<u>4</u>	1, 2, 3, 4, 6, 8, 10, 12, 16, 20, 28, 40, 60, 92, 96, 104, 140, 148, 156, 300, 356, 408, 596, 612, 692, 732, 756, 800, 952, 996, 1004, 1228, 1268, 2240, 2532, 3060, 3796, 3824, 3944,
<u>5</u>	1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 18, 24, 28, 47, 48, 49, 56, 57, 88, 90, 92, 108, 110, 116, 120, 127, 134, 141, 149, 161, 171, 181, 198, 202, 206, 236, 248, 288, 357, 384, 420, 458, 500, 530, 536, 619, 620, 694, 798, 897, 929, 981, 992, 1064, 1134, 1230, 1670, 1807, 2094, 2162, 2369, 2516, 2649, 2988, 3407, 3888,
<u>6</u>	1, 2, 3, 4, 5, 6, 7, 8, 18, 21, 22, 24, 29, 30, 42, 50, 62, 71, 86, 90, 94, 118, 124, 127, 129, 144, 154, 186, 192, 214, 271, 354, 360, 411, 480, 509, 558, 575, 663, 764, 814, 825, 874, 1028, 1049, 1050, 1102, 1113, 1131, 1158, 1376, 1464, 1468, 1535, 1622, 1782, 1834, 1924, 2096, 2176, 2409, 2464, 2816, 3013, 3438, 3453, 3663,
7	3, 5, 6, 8, 13, 18, 21, 28, 30, 34, 36, 46, 48, 50, 54, 55, 58, 63, 76, 84, 94, 105, 122, 131, 148, 149, 224, 280, 288, 296, 332, 352, 456, 528, 531, 581, 650, 654, 730, 740, 759, 1026, 1047, 1065, 1460, 1660, 1699, 1959, 2067, 2260, 2380, 2665, 2890, 3238, 4020,
8	1, 2, 3, 6, 9, 18, 30, 42, 78, 87, 114, 138, 189, 303, 318, 330, 408, 462, 504, 561, 1002, 1389, 1746, 1794, 2040, 2418, 2790, 3894, 4077,
9	1, 2, 4, 6, 10, 12, 16, 18, 20, 30, 32, 36, 54, 64, 66, 118, 138, 152, 182, 232, 264, 336, 340, 380, 414, 446, 492, 540, 624, 720, 762, 1066, 1094, 1098, 1170, 1230, 1254, 1320, 1428, 1546, 2018, 2574, 2724, 2804, 2920, 3074, 3316, 3646,
<u>10</u>	1, 2, 3, 4, 9, 10, 12, 14, 19, 23, 24, 36, 38, 39, 48, 62, 93, 106, 120, 134, 150, 196, 294, 317, 320, 385, 586, 597, 654, 738, 945, 1031, 1172, 1282, 1404, 1426, 1452, 1521, 1752, 1812, 1836, 1844, 1862, 2134, 2232, 2264, 2667, 3750, 3903, 3927,
11	2, 4, 5, 6, 8, 9, 10, 14, 15, 17, 18, 19, 20, 27, 36, 42, 45, 52, 60, 73, 91, 104, 139, 205, 234, 246, 318, 358, 388, 403, 458, 552, 810, 855, 878, 907, 1114, 1131, 1220, 1272, 1431, 1470, 1568, 1614, 1688, 1696, 1907, 2029, 2136, 2288, 2535, 2577,
<u>12</u>	1, 2, 3, 5, 10, 12, 19, 20, 21, 22, 56, 60, 63, 70, 80, 84, 92, 97, 109, 111, 123, 164, 189, 218, 276, 317, 353, 364, 386, 405, 456, 511, 636, 675, 701, 793, 945, 1090, 1268, 1272, 1971, 2088, 2368, 2482, 2893, 2966, 3290,
<u>13</u>	2, 3, 5, 6, 7, 8, 9, 12, 16, 22, 24, 28, 33, 34, 38, 78, 80, 102, 137, 140, 147, 224, 230, 283, 304, 341, 360, 372, 384, 418, 420, 436, 483, 568, 570, 594, 737, 744, 855, 883, 991, 1021, 1193, 1222, 1615, 1628, 1838, 2032, 2146, 2302, 2530, 2830, 2958, 3030, 3528, 3671, 3885,
14	1, 3, 4, 6, 7, 14, 19, 24, 31, 33, 35, 36, 41, 55, 60, 106, 114, 129, 152, 153, 172, 222, 265, 286, 400, 448, 560, 584, 864, 1006, 1335, 1363, 1520, 1536, 1659, 1862, 1925, 2332, 2458, 2687, 3381, 3512, 3870, 3976,
<u>15</u>	3, 4, 6, 7, 14, 24, 43, 54, 58, 73, 85, 93, 102, 184, 220, 221, 228, 232, 247, 291, 305, 486, 487, 505, 551, 552, 590, 1029, 1194, 1274, 1406, 1444, 1532, 1548, 1748, 1986, 2093, 2182, 2202, 2579, 2781, 3054, 3239, 3696,

<u>16</u>	2, 4, 6, 8, 10, 14, 20, 30, 46, 48, 52, 70, 74, 78, 150, 178, 204, 298, 306, 346, 366, 378, 400, 476, 498, 502, 614, 634, 1120, 1266, 1530, 1898, 1912, 1972, 2548, 2770, 3738, 3850,
17	1, 2, 3, 5, 7, 8, 11, 12, 14, 15, 34, 42, 46, 47, 48, 50, 71, 77, 94, 110, 114, 147, 154, 176, 228, 235, 258, 275, 338, 350, 419, 450, 480, 515, 589, 624, 666, 716, 724, 810, 815, 1232, 1490, 1934, 2106, 2391, 2732, 2904, 3462, 3912, 4053,
<u>18</u>	1, 2, 3, 6, 14, 17, 21, 24, 30, 33, 38, 45, 46, 72, 78, 114, 146, 168, 288, 414, 440, 448, 665, 792, 801, 816, 975, 1165, 1176, 1267, 1466, 1513, 1882, 1920, 1998, 2194, 2272, 2643, 2800, 2946, 3434, 3504, 3813, 3866, 3957,
19	2, 3, 4, 6, 19, 20, 31, 34, 47, 56, 59, 61, 70, 74, 91, 92, 96, 98, 107, 120, 145, 156, 168, 242, 276, 314, 326, 337, 387, 565, 602, 612, 892, 984, 1061, 1067, 1079, 1262, 1328, 2356, 3033, 3419, 3501, 3963,
<u>20</u>	1, 3, 4, 6, 8, 9, 10, 11, 17, 30, 98, 100, 110, 126, 154, 158, 160, 168, 178, 182, 228, 266, 270, 280, 340, 416, 480, 574, 774, 980, 1052, 1139, 1338, 1418, 1474, 1487, 1594, 1902, 2326, 3112, 3520, 3808, 3830,
21	2, 3, 5, 6, 8, 9, 10, 11, 14, 17, 26, 43, 64, 74, 81, 104, 192, 271, 321, 335, 348, 404, 437, 445, 516, 671, 694, 788, 1788, 1943, 2343, 2742, 3031, 3135,
22	2, 5, 6, 7, 10, 21, 25, 26, 69, 79, 86, 93, 100, 101, 154, 158, 161, 171, 202, 214, 294, 354, 359, 424, 454, 602, 687, 706, 744, 857, 1028, 1074, 1136, 1150, 1345, 1408, 1525, 1572, 1578, 1988, 2142, 2665,
23	2, 5, 8, 11, 15, 22, 26, 39, 42, 45, 54, 56, 132, 134, 145, 147, 196, 212, 218, 252, 343, 580, 662, 816, 820, 846, 1078, 1092, 1174, 1189, 1548, 1632, 2040, 2180, 2348, 2732, 3100, 3181, 4010,
<u>24</u>	1, 2, 3, 4, 5, 8, 14, 19, 22, 38, 45, 53, 54, 70, 71, 117, 140, 144, 169, 186, 192, 195, 196, 430, 653, 661, 744, 834, 855, 870, 927, 1128, 1158, 1390, 1516, 1555, 1617, 1844, 2022, 2060, 2208, 2812, 3153, 3952,
25	2, 4, 6, 12, 14, 24, 28, 44, 46, 54, 58, 60, 118, 124, 144, 192, 210, 250, 268, 310, 496, 532, 1258, 1494, 1944, 2050, 2498, 2728, 3324, 3418, 3646, 3862, 4014,
<u>26</u>	1, 2, 4, 7, 9, 18, 20, 22, 24, 30, 43, 69, 132, 140, 186, 200, 210, 218, 267, 347, 454, 495, 554, 585, 645, 694, 980, 1028, 1060, 1098, 1432, 1714, 1828, 3513, 3786,
<u>27</u>	2, 3, 12, 21, 24, 36, 87, 93, 171, 249, 276, 360, 480, 621, 732, 780, 1716, 3843,
28	1, 2, 3, 5, 6, 8, 17, 21, 38, 81, 91, 96, 102, 132, 148, 156, 240, 258, 260, 276, 457, 464, 465, 500, 506, 535, 684, 746, 838, 930, 982, 1015, 1189, 1296, 1335, 1345, 1390, 1423, 2062, 2723, 2893, 3078,
29	4, 5, 6, 7, 8, 14, 30, 32, 39, 45, 50, 76, 116, 151, 222, 357, 402, 462, 570, 588, 636, 671, 695, 844, 1498, 1650, 1770, 3175, 3195, 3312, 3538, 3719,
30	1, 2, 5, 9, 11, 12, 21, 36, 51, 64, 91, 163, 174, 195, 230, 278, 318, 342, 346, 424, 530, 569, 578, 795, 984, 1094, 1167, 1335, 1564, 1605, 1658, 1789, 2159, 2204, 2225, 3366, 3458, 3615,
31	3, 7, 12, 17, 24, 30, 31, 33, 40, 176, 218, 308, 404, 420, 630, 693, 890, 915, 922,

	1475, 2122, 2185, 2487, 2541, 2907, 3387, 4055,	
<u>32</u>	1, 6, 30, 85, 110, 120, 320, 1050, 1065, 1385, 2490, 3080, 3920,	
33	1, 2, 3, 10, 16, 25, 28, 30, 35, 36, 45, 56, 76, 87, 110, 134, 135, 197, 200, 220, 228, 314, 324, 330, 396, 498, 583, 624, 725, 806, 940, 1145, 1240, 1644, 1750, 2171, 2268, 2675, 2781, 2790, 2808, 3581,	
34	3, 6, 8, 10, 13, 20, 24, 56, 87, 154, 164, 196, 282, 363, 428, 652, 744, 780, 860, 902, 952, 1178, 1493, 1540, 1643, 1904, 2184, 2277, 2468, 2943,	
35	2, 4, 6, 8, 18, 21, 22, 26, 42, 128, 154, 158, 170, 180, 184, 192, 254, 313, 450, 624, 737, 762, 798, 874, 912, 1002, 1006, 1098, 1234, 1297, 1418, 1714, 1926, 2325, 2343, 2368, 2998, 3567, 4064,	
<u>36</u>	2, 4, 12, 62, 72, 96, 180, 240, 382, 514, 688, 732, 734, 962, 1048, 1088, 1232, 1408, 2088, 2176, 2248, 2724, 3180,	

Another example is in base 10 there are automorphic numbers, the natural numbers n whose square "ends" in the same digits as n itself, and this can be generalized to other bases b. Given a number base b, a natural number n with k digits is an automorphic number if n is a fixed point of the polynomial function $f(x) = x^2$ over $Z/b^k Z$, the ring of integers modulo b^k . As the inverse limit of Z/b^kZ is Z_b , the ring of b-adic integers, automorphic numbers are used to find the numerical representations of the fixed points of $f(x) = x^2$ over Z_b . A fixed point of f(x) is a zero of the function g(x) = f(x) - x. In the ring of integers modulo b, there are $2^{\frac{omega(b)}{2}}$ zeroes to $g(x) = x^2 - x$, where omega(b) is the number of distinct prime factors in b. An element x in Z/bZ is a zero of $g(x) = x^2 - x$ if and only if $x == 0 \mod p^{\frac{valuation(b,p)}{2}}$ or x == 1mod $p^{valuation(b,p)}$ for all primes p dividing b (for the examples of valuation(b,p) for primes p = 2, 3, 5, 7, see the OEIS sequences A007814, A007949, A112765, A214411, respectively). Since there are two possible values in the <u>set</u> $\{0,1\}$, and there are <u>omega(b)</u> such p <u>dividing</u> b, there are $2^{\frac{omega(b)}{2}}$ zeroes of $g(x) = x^2 - x$, and thus there are $2^{\frac{omega(b)}{2}}$ fixed points of $f(x) = x^2 - x$ x^2 . According to Hensel's lemma, if there are k zeroes or fixed points of a polynomial function modulo b, then there are k corresponding zeroes or fixed points of the same function modulo any power of b, and this remains true in the inverse limit. Thus, in any given base b there are $2^{\frac{\text{omega}(b)}{b}}$ b-adic fixed points of $f(x) = x^2$.

As 0 is always a <u>zero divisor</u>, 0 and 1 are always fixed points of $f(x) = x^2$, and 0 and 1 are automorphic numbers in every base. These solutions are called trivial automorphic numbers. If b is a <u>prime power</u>, then the ring of b-adic numbers has no <u>zero divisors</u> other than 0, so the only fixed points of $f(x) = x^2$ are 0 and 1. As a result, nontrivial automorphic numbers, those other than 0 and 1, only exist when the base b has at least two distinct <u>prime factors</u>.

b	nontrivial automorphic numbers in base b
<u>6</u>	4155152221350213,1400403334205344
<u>10</u>	6259918212890625,3740081787109376
<u>12</u>	B3452B21B61B3854,0876909A05A08369

<u>14</u>	A8CBA57337AA0C37,3512386AA633D1A8
<u>15</u>	CE8570624D4BDA86,20697E8CA1A3146A
<u>18</u>	01G4C968DA4E1249,HG1D58B947D3GFDA
<u>20</u>	9F1B657D121AB6B5,A4I8DEC6IHI98D8G
21	J03D7HID8J86H7G7,1KH7D327C1CE3D4F
22	A1F0E7IGDI8D185B,BK6L7E3583D8KDGC
<u>24</u>	KK4L76I751E4D0L9,33J2GH5GIM9JAN2G
<u>26</u>	NODPBN31MM3H1G6D,21C0E2MO33M8O9JE
28	E2ILKR7QB6IDAAQ8,DP9670K1GL9EHH1L
30	GQ881C8LBQ6LB2J6,R2230RO2307OH13A,G1JIRJR6F3FE1Q7F,DSAB2A2NEQEFS3MG,2RRQT25RQTM5CSQL,D3LLSHL8I3N8IRAP
33	BE9LG6LOKN0BVC7C,LINBGQB8C9WL1KPM
34	HVLAS5K7H4HI248H,G2CN5SDQGTGFVTPI
35	S7AV6H8SIPXWTC1F,6RO3SHQ6G9125MXL
<u>36</u>	PNZH5ZDJPZBEDN29,AC0IU0MGA0OLMCXS

Thus, we had better study about the base b digits of primes for other bases b. For the repunit primes, there are <u>a list</u> of repunit primes or <u>PRPs</u> in all bases $2 \le b \le 160$ and length \leq 32803, and <u>a list</u> of repunit primes or <u>PRPs</u> in all bases $2 \leq b \leq 999$ and length \leq 3571, also see OEIS sequences A084740 and A084738 for the smallest repunit (probable) primes in base b; for the near-repdigit primes, there was no list of the smallest such primes (only a list of factorization of such numbers in decimal (base 10)), but recently I built a list of the smallest primes or PRPs (searched to length 5000, lists 0 if no primes or PRPs in this form with length \leq 5000) in given near-repdigit form $x\{y\}$ (i.e. xyyy...yyy) or $\{x\}y$ (i.e. xxx...xxxy) (where x and y are digits in base b) in bases $2 \le b \le 36$ (I stop at base 36 since this base is a maximum base for which it is possible to write the numbers with the symbols 0, 1, ..., 9 (the 10 Arabic numerals) and A, B, ..., Z (the 26 Latin letters) of the Latin alphabet, references: http://www.tonymarston.net/php-mysql/converter.html https://www.dcode.fr/base-36-cipher https://docs.python.org/3/library/functions.html#int https://reference.wolfram.com/language/ref/BaseForm.html https://baseconvert.com/ https://www.calculand.com/unit-converter/zahlen.php?og=Base+2-36&ug=1 http://factordb.com/index.php?showid=100000000000000127 https://linesegment.web.fc2.com/application/math/numbers/RadixConversion.html (in Japanese)); for the left-truncatable primes, there is a list for primes $\leq 10^6$ for bases $2 \leq b \leq 10^6$ 20, and there is a graph of the actual values and estimation formulas for bases $3 \le b \le 120$ (no such prime exists for b = 2), also there is a page for find largest such prime in a given base b, also see OEIS sequences A103443 and A103463 and A076623 for the largest lefttruncatable primes in base b and the total number of left-truncatable primes in base b; for the

right-truncatable primes, there is a <u>list</u> for bases $2 \le b \le 20$, and there is <u>data</u> for bases $3 \le b$

 \leq 90 (no such prime exists for b=2), also see OEIS sequences A023107 and A103483 and A076586 for the largest right-truncatable primes in base b and the total number of righttruncatable primes in base b. Thus, this new problem on the digits of primes (i.e. the problem on the digits of primes inspired from a classical theorem in formal language theory) should also be generalized to other bases, and this problem in various bases is exactly the target of this article (in this article we aim to solve this problem in bases $2 \le b \le 36$ (I stop at base 36) since this base is a maximum base for which it is possible to write the numbers with the symbols 0, 1, ..., 9 (the 10 Arabic numerals) and A, B, ..., Z (the 26 Latin letters) of the Latin alphabet, references: http://www.tonymarston.net/php-mysgl/converter.html https://www.dcode.fr/base-36-cipher https://docs.python.org/3/library/functions.html#int https://reference.wolfram.com/language/ref/BaseForm.html https://baseconvert.com/ https://www.calculand.com/unit-converter/zahlen.php?og=Base+2-36&ug=1 http://factordb.com/index.php?showid=100000000000000127 https://linesegment.web.fc2.com/application/math/numbers/RadixConversion.html (in Japanese)), but since this problem (finding all minimal primes) is much harder than finding all left-truncatable primes or all right-truncatable primes for the same base, in this article we only solve this problem in bases $2 \le b \le 16$, and I left as a challenge to readers the task of solving this problem in bases $17 \le b \le 36$, of course, you can also try to solve this problem in bases $2 \le b \le 120$ as the same problem for the left-truncatable primes, but this will be extremely difficult).

There is a <u>conjecture</u> that there are <u>infinitely many</u> repunit primes in all bases b which are not <u>perfect powers</u> (if b is a perfect power, then it can be shown that there is at most one repunit prime in base b, since the numbers with all digits 1 in these bases can be factored algebraically (this strategy (algebraic factorization) will also be used in our problem, to show that some families (such as 10^n1 in base 8 and 38^n in base 9) contains no primes > base)), and it is also conjectured that there are also <u>infinitely many</u> primes in any given near-repdigit form $x\{y\}$ (i.e. xyyy...yyy) or $\{x\}y$ (i.e. xxx...xxxy) (where x and y are digits in base b) if this form cannot be proven as only contain composites or only contain finitely many primes, also, it is conjectured that there are finitely many left-truncatable primes and finitely many right-truncatable primes in any given base b, however, unlike minimal primes (which can be proven to be finite in any given base b by using the theorem that there are no <u>infinite</u> <u>antichains</u> for the <u>subsequence ordering</u>), none of these conjectures are proven.

Problems about the digits of prime numbers have a long history, and many of them are still unsolved. For example, are there infinitely many primes, all of whose base-10 digits are 1? Currently, there are only five such "repunits" known, corresponding to $(10^p - 1)/9$ for $p \in \{2,$ 19, 23, 317, 1031}. It seems likely that four more are given by $p \in \{49081, 86453, 109297, 109277, 109277, 109277, 109277, 109277, 109277, 109277, 109277, 1092777, 109277, 109277, 109277,$ 270343, 5794777, 8177207, but this has not yet been rigorously proven. This problem also exists for other bases, e.g. for base 12, there are only nine proven such numbers, corresponding to $(12^p - 1)/11$ for $p \in \{2, 3, 5, 19, 97, 109, 317, 353, 701\}$. It seems likely that five more are given by $p \in \{9739, 14951, 37573, 46889, 769543\}$, but this has not yet been rigorously proven. However, for some bases there exists no such primes, these bases are 9, 25, 32, 49, 64, 81, 121, 125, 144, ..., (https://oeis.org/A096059) this is because the numbers with all digits 1 in these bases can be factored algebraically (this strategy (algebraic factorization) will also be used in our problem, to show that some families (such as 10ⁿ1 in base 8 and 38ⁿ in base 9) contains no primes > base). Some positive integers n are repunit in some base $2 \le b \le n-2$ (every integer $n \ge 3$ are trivially repunit in base b = n-1 since n is written "11" in base b = n-1, but every integer $n \ge 2$ are not repunit in any base $b \ge n$ since n is written "10" in base b = n and n is single-digit number (and this digit is not 1) in any base b> n), they are called Brazilian numbers, all integers >6 which are neither primes nor squares of primes are Brazilian numbers, but it is unknown whether there are infinitely many primes which are also Brazilian numbers (however, it is known that every squares of primes except 121 = "11111" in base 3 are not Brazilian numbers).

The lengths of the smallest repunit primes in base b for b = 2, 3, 4, ... 36 are (0 if no repunit primes exist for this base b) 2, 3, 2, 3, 2, 5, 3, 0, 2, 17, 2, 5, 3, 3, 2, 3, 2, 19, 3, 3, 2, 5, 3, 0, 7, 3, 2, 5, 2, 7, 0, 3, 13, 313, 2 (*OEIS* sequence A084740)

The smallest base such that the repunit with length p is prime for the first 100 primes p (i.e. p = 2, 3, 5, 7, ..., 541) are 2, 2, 2, 5, 2, 2, 10, 6, 2, 61, 14, 15, 5, 24, 19, 2, 46, 3, 11, 22, 41, 2, 12, 22, 3, 2, 12, 86, 2, 7, 13, 11, 5, 29, 56, 30, 44, 60, 304, 5, 74, 118, 33, 156, 46, 183, 72, 606, 602, 223, 115, 37, 52, 104, 41, 6, 338, 217, 13, 136, 220, 162, 35, 10, 218, 19, 26, 39, 12, 22, 67, 120, 195, 48, 54, 463, 38, 41, 17, 808, 404, 46, 76, 793, 38, 28, 215, 37, 236, 59, 15, 514, 260, 498, 6, 2, 95, 3 (*OEIS* sequence A066180)

b	lengths of repunit primes in base b (written in base 10) (such lengths must be primes, since if m divides n , then the repunit with length m divides the repunit with length n , in the same base b) ($ltalic$ for unproven $probable$ $primes$) (with link of the $factorization$ ($\geq 33.3333\%$ factored) of $factorization$ ($\geq 35.3333\%$ factored) of $factorization$ ($\geq 10^{299}$)	OEIS sequence
2	2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161,, 74207281,, 77232917,, 82589933, (the Mersenne primes, all	<u>A000043</u>

	are proven primes, since these primes can be proven prime using the <u>Lucas–Lehmer primality test</u>)	
3	3, 7, 13, 71, 103, 541, <u>1091</u> , <u>1367</u> , <u>1627</u> , <u>4177</u> , <u>9011</u> , <u>9551</u> , 36913, 43063, 49681, 57917, 483611, 877843, <u>2215303</u> , 2704981, 3598867,	A028491
4	2 (this is all, since $(4^n-1)/3 = (2^n-1)/3 * (2^n+1)$ for even n (and both factors are > 1 if $n > 2$), and $(4^n-1)/3 = (2^n-1) * (2^n+1)/3$ for odd n (and both factors are > 1 if $n > 1$, and for $n = 1$, $(4^n-1)/3 = 1$, which is not prime))	
<u>5</u>	3, 7, 11, 13, 47, 127, 149, 181, <u>619, 929, 3407, 10949,</u> 13241, 13873, 16519, 201359, 396413, 1888279, 3300593,	A004061
<u>6</u>	2, 3, 7, 29, 71, 127, 271, <u>509</u> , <u>1049</u> , <u>6389</u> , <u>6883</u> , <i>10613</i> , 19889, 79987, 608099, 1365019,	<u>A004062</u>
7	5, 13, 131, 149, <u>1699</u> , <i>14221</i> , <i>35201</i> , <i>126037</i> , <i>371669</i> , <i>1264699</i> ,	<u>A004063</u>
8	3 (this is all, since $(8^n-1)/7 = (2^n-1)/7 * (4^n+2^n+1)$ for n divisible by 3 (and both factors are > 1 if $n > 3$), and $(8^n-1)/7 = (2^n-1) * (4^n+2^n+1)/7$ for n not divisible by 3 (and both factors are > 1 if $n > 1$, and for $n = 1$, $(8^n-1)/7 = 1$, which is not prime))	
9	not exist (since $(9^n-1)/8 = (3^n-1)/4 * (3^n+1)/2$ for even n (and both factors are > 1), and $(9^n-1)/8 = (3^n-1)/2 * (3^n+1)/4$ for odd n (and both factors are > 1 if $n > 1$, and for $n = 1$, $(9^n-1)/8 = 1$, which is not prime))	A000000 (the empty sequence)
<u>10</u>	2, 19, 23, <u>317</u> , <u>1031</u> , 49081, 86453, 109297, 270343, 5794777, 8177207,	A004023
<u>11</u>	17, 19, 73, 139, <u>907</u> , <u>1907</u> , <u>2029</u> , <u>4801</u> , <u>5153</u> , <i>10867</i> , 20161, 293831, 1868983,	A005808
<u>12</u>	2, 3, 5, 19, 97, 109, <u>317</u> , <u>353</u> , <u>701</u> , 9739, 14951, 37573, 46889, 769543,	A004064
<u>13</u>	5, 7, 137, <u>283</u> , <u>883</u> , <u>991</u> , <u>1021</u> , <u>1193</u> , <u>3671</u> , <i>18743</i> , <u>31751</u> , <i>101089</i> , <i>1503503</i> ,	<u>A016054</u>
<u>14</u>	3, 7, 19, 31, 41, <u>2687</u> , <i>19697</i> , <i>59693</i> , <i>67421</i> , <i>441697</i> ,	A006032
<u>15</u>	3, 43, 73, <u>487</u> , <u>2579</u> , <i>8741</i> , <i>37441</i> , <i>89009</i> , <i>505117</i> , 639833,	<u>A006033</u>
<u>16</u>	2 (this is all, since $(16^n-1)/15 = (4^n-1)/15 * (4^n+1)$ for even n (and both factors are > 1 if $n > 2$), and $(16^n-1)/15 = (4^n-1)/3 * (4^n+1)/5$ for odd n (and both factors are > 1 if $n > 1$, and for $n = 1$, $(16^n-1)/15 = 1$, which is not prime))	

17	3, 5, 7, 11, 47, 71, <u>419</u> , <u>4799</u> , 35149, 54919, 74509, 1990523,	A006034
<u>18</u>	2, 25667, 28807, 142031, 157051, 180181, 414269, 1270141,	A133857
19	19, 31, 47, 59, 61, 107, <u>337</u> , <u>1061</u> , <i>9511</i> , <i>22051</i> , 209359,	A006035
<u>20</u>	3, 11, 17, <u>1487</u> , <i>31013</i> , <i>48859</i> , <i>61403</i> , <i>472709</i> , <i>984349</i> ,	A127995
21	3, 11, 17, 43, <u>271</u> , <i>156217</i> , <i>328129</i> ,	A127996
22	2, 5, 79, 101, <u>359</u> , <u>857</u> , <u>4463</u> , <i>90</i> 29, 27823,	A127997
23	5, <u>3181</u> , 61441, 91943, 121949, 221411,	<u>A204940</u>
<u>24</u>	3, 5, 19, 53, 71, <u>653</u> , <u>661</u> , <i>10343</i> , <i>49307</i> , <i>115597</i> , <i>152783</i> ,	A127998
25	not exist (since $(25^n-1)/24 = (5^n-1)/6 * (5^n+1)/4$ for even n (and both factors are > 1), and $(25^n-1)/24 = (5^n-1)/4 * (5^n+1)/6$ for odd n (and both factors are > 1 if $n > 1$, and for $n = 1$, $(25^n-1)/24 = 1$, which is not prime))	A000000 (the empty sequence)
<u>26</u>	7, 43, <u>347</u> , 12421, 12473, 26717,	<u>A127999</u>
27	3 (this is all, since $(27^n-1)/26 = (3^n-1)/26 * (9^n+3^n+1)$ for n divisible by 3 (and both factors are > 1 if $n > 3$), and $(27^n-1)/26 = (3^n-1)/2 * (9^n+3^n+1)/13$ for n not divisible by 3 (and both factors are > 1 if $n > 1$, and for $n = 1$, $(27^n-1)/26 = 1$, which is not prime))	
28	2, 5, 17, <u>457</u> , <u>1423</u> , <i>115</i> 877,	<u>A128000</u>
29	5, 151, <u>3719</u> , <i>4</i> 92 <i>11</i> , <i>77</i> 2 <i>37</i> ,	<u>A181979</u>
30	2, 5, 11, 163, <u>569</u> , <u>1789</u> , <i>844</i> 7, <i>72871</i> , <i>78857</i> , <i>82883</i> ,	A098438
31	7, 17, 31, <u>5581</u> , 9973, 101111, 535571,	A128002
<u>32</u>	not exist (since $(32^n-1)/31 = (2^n-1)/31$ * $(16^n+8^n+4^n+2^n+1)$ for n divisible by 5 (and both factors are > 1 if $n > 5$, and for $n = 5$, $(32^n-1)/31 = 1082401 = 601$ * 2401, which is not prime), and $(32^n-1)/31 = (2^n-1)$ * $(16^n+8^n+4^n+2^n+1)/31$ for n not divisible by 5 (and both factors are > 1 if $n > 1$, and for $n = 1$, $(32^n-1)/31 = 1$, which is not prime))	A000000 (the empty sequence)
33	3, 197, <u>3581</u> , <i>6871</i> , <i>183661</i> ,	A209120
34	13, <u>1493</u> , <i>5851</i> , <i>6379</i> , <i>125101</i> ,	A185073
35	<u>313, 1297, 568453,</u>	<u>A348170</u>
	·	

<u>36</u>	2 (this is all, since $(36^n-1)/35 = (6^n-1)/35 * (6^n+1)$ for even n (and both factors are > 1 if $n > 2$), and $(36^n-1)/35 = (6^n-1)/5 * (6^n+1)/7$ for odd n (and both factors are > 1 if $n > 1$, and for $n = 1$, $(36^n-1)/35 = 1$, which is not prime))	
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Another unsolved problem about the digits of prime numbers is whether there are infinitely many palindromic primes (primes which remain the same when their digits are reversed, such as 151 and 94849) in base 10? So far, the largest known such prime is 101234567 -20342924302 * 10⁶¹⁷²⁷⁸ - 1, this number has 1234567 digits, can also be written as 9⁶¹⁷²⁷⁸796570756979⁶¹⁷²⁷⁸, and the largest 20 known such primes are listed in this page. Of course, this problem also exists for other bases, there is no single bases for which it is known whether there are infinitely many palindromic primes. Some positive integers n are not palindromic in any base $2 \le b \le n-2$ (every integer $n \ge 3$ are trivially palindromic in base b = n-1 since *n* is written "11" in base b = n-1, also every positive integer *n* are trivially palindromic in any base b > n since n is single-digit number in any base b > n, but every integer $n \ge 2$ are not palindromic in base b = n since n is written "10" in base b = n), they are called strictly non-palindromic numbers, all such integers > 6 are primes, since all composites n > 6 is either "product of two numbers k and m with $m-k \ge 2$ " (in this case, n is written "kk" in base b = m-1) or "square of prime p" (in this case, n is written "121" in base b= p-1 if p > 3, or written "1001" in base b = 2 if p = 3), it is also unknown whether there are infinitely many such integers, but it is known that in every base b, almost all palindromic numbers are composite (neither 1 nor prime), see this reference.

Table

|x| is the length of x, and in the " $max(x, x \in L_b)$ " column, xy^nz means xyyy...yyyz with n y's (the n-value is written in decimal), not y to the nth power.

b	$ M(L_b) $	$max(x, x \in M(L_b))$	$max(x , x \in M(L_b))$	Algebraic form of $max(x, x \in M(L_b))$
2	1	<u>11</u>	2	<u>3</u>
<u>3</u>	3	<u>111</u>	3	<u>13</u>
<u>4</u>	5	221	3	<u>41</u>
<u>5</u>	22	<u>10⁹³13</u>	96	<u>5⁹⁵+8</u>
<u>6</u>	11	40041	5	<u>5209</u>
<u>7</u>	71	<u>3¹⁶1</u>	17	<u>(7¹⁷-5)/2</u>
8	75	4 ²²⁰ 7	221	<u>(4*8²²¹+17)/7</u>
<u>9</u> 1	≥149	30115811	1161	<u>3*9¹¹⁶⁰+10</u>
<u>10</u>	77	<u>50²⁸27</u>	31	<u>5*10³⁰+27</u>

<u>11</u> ^①	≥914	<u>557¹⁰¹¹</u> or 57 ⁿ with <i>n</i> >50000	1013	(607*11 ¹⁰¹¹ -17)/10
<u>12</u>	106	40 ³⁹ 77	42	<u>4*12⁴¹+91</u>
<u>13</u> ¹⁾²	≥2497	80 ³²⁰¹⁷ 111 or 95 ⁿ with n>50000 or A3 ⁿ A with n>50000	32021	<u>8*13³²⁰²⁰+183</u>
<u>14</u> ^①	≥606	<u>4D¹⁹⁶⁹⁸</u>	19699	<u>5*14¹⁹⁶⁹⁸-1</u>
<u>15</u> ^①	≥1212	715597	157	(15 ¹⁵⁷ +59)/2
<u>16</u> ¹⁾²	≥2045	<u>DB³²²³⁴</u>	32235	(206*16 ³²²³⁴ -11)/15

Notes:

- ① I have not proved these bases, these are the largest elements in $M(L_b)$ known to me, and they are just the <u>lower bounds</u>.
- ② Data based on results of strong <u>probable primality tests</u>, i.e. at least one element in the set $M(L_b)$ is only <u>strong probable prime</u> (i.e. numbers which passed the <u>Miller-Rabin primality tests</u> to first few prime bases, for the smallest *composite* number which passed the Miller-Rabin primality test to first n prime bases, see https://oeis.org/A014233) and not definitely prime, thus we cannot definitely say that the corresponding families can be removed from the list of unsolved families, and we cannot definitely <u>compute</u> this part of the <u>set $M(L_b)$ </u>, e.g. since $80^{32017}111$ (base 13) is only strong probable prime and it is the smallest (probable) prime in family $8\{0\}111$ (base 13) can be removed from the list of unsolved families, and since DB^{32234} (base 16) is only strong probable prime and it is the smallest (probable) prime in family $D\{B\}$ in base 16, we cannot definitely say that the family $D\{B\}$ (base 16) can be removed from the list of unsolved families.

It is found that both $|M(L_b)|$ and $max(|x|, x \in M(L_b))$ are roughly $e^{y^*(b-1)^*\underline{eulerphi}(b)}$, the value (b-1)*eulerphi(b) is the number of possible (first digit, last digit) combos (ordered pair) of a minimal prime in base b (these (first digit, last digit) combos are also all possible (first digit, last digit) combos (ordered pair) of a prime > b in base b) (these (first digit, last digit) combos for decimal (base b = 10) are listed in A085820, except the single-digit numbers (i.e. 1, 3, 7, 9) (i.e. first digit is 0, and hence the number has leading zeros) in this sequence, the smallest primes with these (first digit, last digit) combos listed in A085820 (except the singledigit numbers (i.e. 1, 3, 7, 9) in this sequence) are (italic for primes which are not minimal primes): 11, 13, 17, 19, 211, 23, 227, 29, 31, 313, 37, 349, 41, 43, 47, 409, 521, 53, 547, 59, 61, 613, 67, 619, 71, 73, 727, 79, 811, 83, 827, 89, 911, 953, 97, 919, and the smallest minimal primes with these (first digit, last digit) combos listed in A085820 (except the singledigit numbers (i.e. 1, 3, 7, 9) in this sequence) are (0 if no such minimal prime exists): 11, 13, 17, 19, 251, 23, 227, 29, 31, 0, 37, 349, 41, 43, 47, 409, 521, 53, 557, 59, 61, 0, 67, 6469, 71, 73, 727, 79, 821, 83, 827, 89, 991, 0, 97, 9049) (they are only all "possible" (first digit,last digit) combos (ordered pair) of a minimal prime in base b, this does not mean that they must be realized, e.g. there are no minimal primes with (first digit, last digit) = (2,2) in base 3, and there are no minimal primes with (first digit, last digit) = (3,3), (6,3), or (9,3) in base 10, but it is conjectured that there are only finitely many such examples (i.e. for every enough large base b, for any given such (first digit, last digit) combo, there is a minimal prime with this (first

digit, last digit) combo), also, it is conjectured that all such examples have qcd(first digit, last digit, b-1) > 1 (i.e. there is a prime number which divides first digit, last digit, and b-1simultaneously), since the first digit has b-1 choices (all digits except 0 can be the first digit), and the last digit has eulerphi(b) choices (only digits coprime to b (i.e. the digits in the reduced residue system mod b) can be the last digit), by the rule of product, there are (b-1)*eulerphi(b) choices of the (first digit, last digit) combo. (the set of these (first digit, last digit) combos is exactly the Cartesian product of the set of the possible first digits of a prime number > b in base b and the set of the possible last digits of a prime number > b in base b, i.e. $\{d \mid d \text{ is integer}, 1 \le d \le b-1\} \times \{d \mid d \text{ is integer}, 1 \le d \le b-1, \gcd(d,b) = 1\}, \text{ or } (Z/bZ - \{0\})$ \times (($\mathbb{Z}/b\mathbb{Z}$) $^{\times}$)) Thus, (b-1)*eulerphi(b) is also the relative hardness for (finding and proving the set $M(L_b)$ in) base b, there is exactly a sequence of $(b-1)^*\underline{eulerphi}(b)$ in OEIS: A062955, for these (b-1)*eulerphi(b) possible (first digit, last digit) combos, we want to find all minimal primes with such (first digit, last digit) combo, if the string "first digit, last digit" represents a prime in base b, then this prime will be the only minimal prime with this (first digit, last digit) combo (since the string "first digit, last digit" is a subsequence of all numbers with this (first digit, last digit) combo), otherwise, we should find all digits which can be inserted to this (first digit, last digit) combo, i.e. the string "first digit, such digit, last digit" is neither prime nor have a subsequence which represents a prime, then do this repeatedly (find the possible (first digit, last digit) combos for the string which inserted to the starting (first digit, last digit) combo, etc.), then do program loops, these program loops must be finite by the theorem that there are no infinite antichains for the subsequence ordering, see the "proof" section and this forum post and this article.

base (b)	number of possible first digits of a prime > b in base b (equal b-1, since all digits except 0 can be the first digit)	number of possible last digits of a prime > b in base b (equal eulerphi(b), since only digits coprime to b (i.e. the digits in the reduced residue system mod b) can be the last digit)	number of possible (first digit,last digit) combos of a prime > b in base b (equal (b-1)*eulerphi(b), by the rule of product), also the relative hardness for base b
2	1	1	1
<u>3</u>	2	2	4
4	3	2	6
<u>5</u>	4	4	16
<u>6</u>	5	2	10
<u>7</u>	6	6	36
<u>8</u>	7	4	28
9	8	6	48
<u>10</u>	9	4	36
<u>11</u>	10	10	100

12 11 4 44 13 12 12 144 14 13 6 78 15 14 8 112 16 15 8 120 17 16 16 256 18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 3				
14 13 6 78 15 14 8 112 16 15 8 120 17 16 16 256 18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816 <td><u>12</u></td> <td>11</td> <td>4</td> <td>44</td>	<u>12</u>	11	4	44
15 14 8 112 16 15 8 120 17 16 16 256 18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>13</u>	12	12	144
16 15 8 120 17 16 16 256 18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	14	13	6	78
17 16 16 256 18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>15</u>	14	8	112
18 17 6 102 19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>16</u>	15	8	120
19 18 18 324 20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	17	16	16	256
20 19 8 152 21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>18</u>	17	6	102
21 20 12 240 22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	19	18	18	324
22 21 10 210 23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>20</u>	19	8	152
23 22 22 484 24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	21	20	12	240
24 23 8 184 25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	22	21	10	210
25 24 20 480 26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	23	22	22	484
26 25 12 300 27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>24</u>	23	8	184
27 26 18 468 28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	25	24	20	480
28 27 12 324 29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>26</u>	25	12	300
29 28 28 784 30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	<u>27</u>	26	18	468
30 29 8 232 31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	28	27	12	324
31 30 30 900 32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	29	28	28	784
32 31 16 496 33 32 20 640 34 33 16 528 35 34 24 816	30	29	8	232
33 32 20 640 34 33 16 528 35 34 24 816	31	30	30	900
34 33 16 528 35 34 24 816	<u>32</u>	31	16	496
35 34 24 816	33	32	20	640
	34	33	16	528
<u>36</u> 35 12 420	35	34	24	816
	<u>36</u>	35	12	420

(Note: not all (first digit,last digit) combos must be realized for a minimal prime base b, e.g. there are no minimal primes with (first digit,last digit) = (2,2) in base 3, and there are no minimal primes with (first digit,last digit) = (3,3), (6,3), or (9,3) in base 10, for more such examples, see this post)

The probability for a random prime to have a given (first digit, last digit) combo (ordered pair) which is a possible (first digit, last digit) combo (ordered pair) of a prime > b in base b (i.e. "first digit" is not 0, and "last digit" is coprime to b) are all the same, i.e. they are all $1/((b-1)^*eulerphi(b))$ no matter which (first digit, last digit) combo (ordered pair) is given, the only condition is that "first digit" is not 0, and "last digit" is coprime to b, for the first digit, there is a reference about this, the primes do not follow the Benford's law (reference of Benford's law to other bases) (only the prime factors of the numbers with exponential growth (such as the repunits and the Fibonacci numbers) follow), instead, all nonzero digits have the same probability (i.e. probability 1/(b-1)) for a random prime in base b, just like a positive integer in base b, for the last digit, by the prime number theorem (extended to arithmetic progression), all digits coprime to b have the same probability (i.e. probability $1/\underline{eulerphi}(b)$) for a random prime in base b, however, according to Chebyshev's bias, if d_1 is a quadratic residue mod b_1 , d_2 is a quadratic nonresidue mod b_1 (i.e. d_1 can be the last digit of a square number, while d_2 cannot be), then for the primes $\leq N$ for a random positive integer N, the probability for the number of primes end with d_2 in base b is more than the number of primes end with d_1 in base b is larger than 50%, e.g. the smallest N such that the number of primes end with 1 in base 4 is more than the number of primes end with 3 in base 4 is 12203231 (26861 in decimal), and the smallest N such that the number of primes end with 1 in base 3 is more than the number of primes end with 2 in base 3 is 2011012212222201102200001 (608981813029 in decimal), references: https://oeis.org/A007350 https://oeis.org/A007352 https://oeis.org/A199547 https://oeis.org/A306891 https://oeis.org/A038698 https://oeis.org/A112632 https://oeis.org/A275939 https://oeis.org/A306499 https://oeis.org/A306500, this is a classic example of the strong law of small numbers (Prime Glossary page), another classic example is it appears that the sum of the Liouville function (which is an important function in number theory, defined as $(-1)^{\underline{bigomega}(n)}$, which is $\underline{A008836}(n)$) of the positive integers $\leq N$ is ≤ 0 if N > 1, is it always true? (the \underline{Polya} conjecture), the smallest N such that this conjecture is false is 906150257 (this conjecture is important in number theory since if this conjecture is true, then the Riemann hypothesis can be proved, and hence many conjectures in number theory can also be proved, e.g. Mills' primes will be known to be 2, 11, 1361, 2521008887, 16022236204009818131831320183, 4113101149215104800030529537915953170486139623539759933135949994882770404 074832568499, ... https://oeis.org/A051254, and the Mills' constant will be known to be 1.30637788386308069046861449260260571291678... https://oeis.org/A051021, which (let this constant be A) floor (A^{3^n}) are primes for all positive integers n, and this formula will be the first known formula for primes which only use exponential functions and floor functions (and not use factorial), thus can be easily to calculate, and there will not be "the largest known prime number"! (since floor(A^{3^n}) contains infinitely many numbers), currently, the largest known Mills' prime is (((((((((1361^3+6)^3+80)^3+12)^3+450)^3+894)^3+3636)^3+70756)^3+97220)^3+66768)^3 +300840)^3+1623568, which has 555154 digits, see PRP top), for more examples of the strong law of small numbers, see https://primes.utm.edu/glossary/xpage/LawOfSmall.html and https://oeis.org/A005165/a005165.pdf, and there are also examples of the strong law of small numbers which are related to the research in this article: Are the base 10 numbers 527, 5027, 50027, 500027, 5000027, 50000027, ..., all composite? (which is corresponding to the largest minimal prime in base 10: 50²⁸27) Are the base 8 numbers 47, 447, 4447, 44447, 4444447, ..., all composite? (which is corresponding to the largest minimal prime in base 8: 4²²⁰7) Are the base 16 numbers DB, DBB, DBBBB, DBBBBB,

2187001477972027873637433214911446252018853474384761589836346227953714449 2484599310778624146468224150373895489844303219383829573677353011540369291 867378470695590964880740521967077028064041941947533607 is the largest minimal prime in base 8, 705490352625161496279722666407220454094798939 is the largest minimal prime in base 12, etc. and there are also other paradoxes related to this paradox: the Berry paradox, the Richard's paradox, they are related to Cantor's diagonal argument to prove that the set of the real numbers is uncountable (this is also related to Gödel's incompleteness theorems, these theorems are widely, but not universally, interpreted as showing that Hilbert's program to find a complete and consistent set of axioms for all mathematics is impossible, we can use a simple proposition to show this: This proposition has no formal proof, and consider whether this proposition is true or not), but it can be proved that the set of the rational numbers, the set of the algebraic numbers, the set of the computable numbers, the set of the definable numbers, are all countable, i.e. card(these sets) are all equal to card(N), where N is the set of the natural numbers, but card(R) (R is the set of the real numbers) is larger than card(N), and the continuum hypothesis is that card(R)= 2^{card(N)}, references: Prime Curios! (the smallest number with no curios is 326) What's Special About This Number? (the smallest number not has property in this page is 391) Properties of the First 5000 Integers (the smallest number not in this page is 291), also, currently the smallest number without its own article is Wikipedia is 262, also, currently the smallest number not in OEIS is 20067.

Excluding the primes $\leq b$ (i.e. only counting the primes > b) makes the <u>formula</u> of the number of possible (first digit,last digit) combo of a minimal prime in base b more simple and <u>smooth number</u>, since if only excluding the primes < b (i.e. counting the primes $\geq b$), then when b is prime, there is an additional possible (first digit,last digit) combo: (1,0), and hence the formula will be $(b-1)^*\underline{eulerphi}(b)+1$ if b is prime, or $(b-1)^*\underline{eulerphi}(b)$ if b is composite (the fully formula will be $(b-1)^*\underline{eulerphi}(b)+1$ if b is prime (b) or $(b-1)^*\underline{eulerphi}(b)+1$ if (b) if (b) if (b) if (b) if (b) if (b) is more complex, and if start with 1 (i.e. the <u>original minimal prime problem</u>), the formula is much more complex, since the prime digits (i.e. the single-digit primes) should be excluded (thus, e.g. for decimal (base b=10), the primes are limited in (b)0 and (for such prime (b)1 the first digit has (b)2 the first digit has (b)3 the first digit has (b)4 there are $(b-1-\underline{bi}(b))^*(b)^*(b)$ 6 choices of the (first digit,last digit) combo (if for such prime (b)5 instead of (b)6, then the formula will be $(b-1-\underline{bi}(b))^*(b)^*(b)^*(b)$ 6 if (b)7 is prime, or $(b-1-\underline{bi}(b))^*(b)^*(b)$ 8 if (b)8 is composite),

which is much more complex, (also, the possible (first digit, last digit) combo for a prime > bin base b are exactly the (first digit, last digit) combos which there are infinitely many primes have, while this is not true when the requiring of the prime is $\geq b$ or ≥ 2 instead of > b, since this will contain the prime factors of b, which are not coprime to b and hence there is only this prime (and not infinitely many primes) have this (first digit, last digit) combo), thus this problem is much better than the original minimal prime problem (another reason is that this problem is regardless whether 1 is considered as prime or not, i.e. no matter 1 is considered as prime or not prime (in the beginning of the 20th century, 1 is regarded as prime) (reference of why 1 is not prime), the sets $M(L_b)$ in this problem are the same, while the sets $M(L_b)$ in the original minimal prime problem are different, e.g. in base 10, if 1 is considered 449, 499, 6469, 6949, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049}, while if 1 is not considered as prime, then the set $M(L_b)$ in the original minimal prime problem is {2, 3, 5, 7, 11, 19, 41, 61, 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049}, however, in base 10, the set $M(L_b)$ in this problem is always {11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 227, 251, 257, 277, 281, 349, 409, 449, 499, 521, 557, 577, 587, 727, 757, 787, 821, 827, 857, 877, 881, 887, 991, 2087, 2221, 5051, 5081, 5501, 5581, 5801, 5851, 6469, 6949, 8501, 9001, 9049, 9221, 9551, 9649, 9851, 9949, 20021, 20201, 50207, 60649, 80051, 666649, 946669, 5200007, 22000001, 60000049, 66000049, is considered as prime or not prime) (another reason is that if we include the prime = b (i.e. the prime "10") or the primes < b (i.e. the single-digit primes), then some properties in this post will be incorrect), thus, start with b+1 (instead of b, 2, 1, b^2 , b^2+1 , b+2, 2*b, 2*b+1, ...) makes this minimal prime problem most beautiful (prime = b (i.e. the prime "10") and primes < b (i.e. single-digit primes) need to be excluded, while the prime = b+1 (i.e. the prime "11") and other two-digit primes and other repunit primes do not need).

), reference: https://mersenneforum.org/showpost.php?p=562832&postcount=52.

Data

The <u>data</u> of <u>bases</u> 14, 16, and the odd bases >8 are possibly not complete, only tested to the test limit in the discussion of these bases and found the smallest (probable) prime in some unsolved <u>families</u> of these bases, but there may be more unsolved families not found by me.

Our results assume that a number which has passed Miller–Rabin primality tests to all prime bases $p \le 64$ (i.e. the first 18 prime bases, bases 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, and 61, for the composite numbers which pass this test to the first n prime bases (i.e. numbers which are strong pseudoprimes to the first n prime bases), see https://oeis.org/A014233, we use n = 18 for the primality tests) and the strong Lucas primality test with parameters (P, Q) defined by Selfridge's Method A (for the composite numbers which pass this test (i.e. numbers which are strong Lucas pseudoprimes with parameters (P, Q) defined by Selfridge's Method A), see https://oeis.org/A217255) is in fact prime, since in some cases (e.g. b = 13 and b = 16) some candidate elements of $M(L_b)$ are too long to be proven prime rigorously (and neither N-1 nor N+1 can be $\geq 33.3333\%$ factored), and the probability that such a number is in fact composite is very low, e.g. for

such a number with 5000 decimal digits, the probability is less than 7.6*10⁻⁶⁸⁰, and for such a number with 100000 decimal digits, the probability is less than 1.3*10⁻¹⁰⁵⁸⁴, both of them are "almost" zero, i.e. we can "almost surely" (99.9999...% (with more than 10000 9's) surely, but not 100% surely) that they are primes, and the numbers which currently cannot be proven prime rigorously are usually very large (usually > 10⁵⁰⁰⁰, see top 20 ECPP proving page and top 20 Primo proving page, the largest prime which is proven by ECPP is p(1289844341), where p(n) is the integer partition function, this number has 40000 decimal digits, and this number is the largest known ordinary prime, i.e. none of $p^n \pm 1$ (for small n) factor enough to make the number easily provable using the classical methods of primality proof), and if such a number is larger, then probability that this number is in fact composite is lower, thus the probability is much less than 7.6*10⁻⁶⁸⁰, see this page, also, our tests (combine of the Miller-Rabin primality tests to the first 13 prime bases and the strong Lucas primality test with parameters (P, Q) defined by Selfridge's Method A) cover the Baillie-PSW primality test (which is only combine of the Miller-Rabin primality tests to base 2 and the strong Lucas primality test with parameters (P, Q) defined by Selfridge's Method A, i.e. (let D be the first number in the sequence 5, -7, 9, -11, 13, -15 ... such that $\left(\frac{D}{N}\right) = -1$ (N is the number which we want to test primality), where $\left(\frac{m}{n}\right)$ is the <u>Jacobi symbol</u>), set P=1 and Q=1(1-D)/4), and no known composites which pass the Baillie-PSW test, and no composites < 264 pass the Baillie-PSW test (reference and reference), although it is still conjectured that there exist infinitely many "Baillie-PSW pseudoprimes", i.e. composites which pass the Baillie-PSW test, thus if a such number is in fact composite, it will be a pseudoprime to the Baillie-PSW test, which currently no single example is known!

There are five unsolved families (families for which not even a probable prime is known nor can be ruled out as only contain composites (only count the numbers > b)) for bases $2 \le b \le 16$ found by me and searched to length 50000 with no (probable) prime found.

Ь	Unsolved family	Algebraic form
11	57 ⁿ	(57*11 ⁿ -7)/10
13	95 ⁿ	<u>(113*13*-5)/12</u>
13	A3 ⁿ A	(41*13 ⁿ⁺¹ +27)/4
16	3 ⁿ AF	<u>(16ⁿ⁺²+619)/5</u>
16	4 ⁿ DD	(4*16 ⁿ⁺² +2291)/15

base 2

11

base 3

12, 21, 111

base 4

11, 13, 23, 31, 221

base 5

12, 21, 23, 32, 34, 43, 104, 111, 131, 133, 313, 401, 414, 3101, 10103, 14444, 30301, 33001, 33331, 44441, 300031,

base 6

11, 15, 21, 25, 31, 35, 45, 51, 4401, 4441, 40041

base 7

base 8

base 9 (not proved, only checked to the prime 8333333333)

12, 14, 18, 21, 25, 32, 34, 41, 45, 47, 52, 58, 65, 67, 74, 78, 81, 87, 117, 131, 135, 151, 155, 175, 177, 238, 272, 308, 315, 331, 337, 355, 371, 375, 377, 438, 504, 515, 517, 531, 537, 557, 564, 601, 638, 661, 702, 711, 722, 735, 737, 751, 755, 757, 771, 805, 838, 1011, 1015, 1101, 1701, 2027, 2207, 3017, 3057, 3101, 3501, 3561, 3611, 3688, 3868, 5035, 5051, 5071, 5101, 5501, 5554, 5705, 5707, 7017, 7075, 7105, 7301, 8535, 8544, 8555, 8854, 20777, 22227, 22777, 30161, 33388, 50161, 50611, 53335, 55111, 55535, 55551, 57061, 57775, 70631, 71007, 77207, 100037, 100071, 100761, 105007, 270707, 301111, 305111,

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7777777777777777777777777777777777
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base 10

base 11 (not proved, only checked to the prime 1500000001)

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base 12

11, 15, 17, 1B, 25, 27, 31, 35, 37, 3B, 45, 4B, 51, 57, 5B, 61, 67, 6B, 75, 81, 85, 87, 8B, 91, 95, A7, AB, B5, B7, 221, 241, 2A1, 2B1, 2BB, 401, 421, 447, 471, 497, 565, 655, 665, 701,

base 13 (not proved, only checked to the prime 1010008001, also the numbers B0⁶⁵⁴⁰BBA and 80³²⁰¹⁷111 are only probable primes, i.e. not definitely primes)

14, 16, 1A, 23, 25, 2B, 32, 34, 38, 41, 47, 49, 52, 56, 58, 61, 65, 6B, 76, 7A, 7C, 83, 85, 89, 9A, A1, A7, A9, B6, B8, C1, C7, CB, 10C, 119, 11B, 122, 133, 155, 157, 173, 179, 17B, 188, 197, 1B1, 1B5, 1CC, 209, 212, 218, 229, 272, 274, 281, 287, 292, 296, 298, 29C, 2C9, 311, 313, 331, 33B, 355, 371, 373, 379, 397, 3A6, 3AA, 3B3, 3B9, 3BB, 3CA, 43C, 445, 44B, 45A, 463, 4A3, 4A5, 4B2, 4B4, 4BA, 50C, 511, 515, 533, 54A, 551, 559, 571, 575, 57B, 595, 599, 5B3, 5B9, 5CC, 607, 629, 63A, 643, 674, 704, 715, 724, 728, 731, 737, 739, 742, 751, 75B, 773, 775, 779, 782, 784, 791, 793, 797, 7B1, 812, 818, 874, 878, 8AB, 8B4, 902, 919, 922, 926, 92C, 937, 93B, 946, 95B, 962, 968, 971, 977, 979, 982, 98C, 9B3, 9B5, A03, A3C, A45, A4B, A54, AA3, AAB, B02, B0C, B11, B15, B17, B24, B33, B39, B42, B57, B59, B71, B93, B9B, BA4, BAA, BB1, BB9, BC2, BCC, C29, C43, C98, CA3, 1013, 1031, 1037, 105B, 1075, 10B7, 10BB, 1105, 1112, 1121, 1127, 113C, 1172, 1187, 1208, 1211, 1277, 12C8, 1307, 1309, 131C, 139C, 151C, 1721, 1727, 1787, 1901, 1909, 1912, 1918, 193C, 1981, 198B, 199C, 19B2, 19C3, 1B29, 1BB2, 1BBC, 1C28, 1C39, 2021, 2078, 2117, 2201, 2221, 2267, 2278, 2627, 2678, 2711, 2771, 2788, 3037, 3053, 306A, 3077, 3091, 309B, 30AC, 3305, 353C, 35AB, 35BA, 35BC, 3677, 3905, 390B, 39C5, 3A0C, 3AB5, 3B5C, 3C35, 3C59, 3C95, 403A, 40AB, 4333, 435B, 4403, 44C3, 4535, 4544, 454C, 45B5, 45BB, 480B, 4B35, 4B5B, 4C36, 5057, 5077, 509B, 50A4, 5107, 5305, 530B, 539C, 53AB, 53C9, 5444, 5455, 54C4, 5503, 5545, 55AB, 5774, 5794, 590B, 594B, 5974, 59B4, 5A4C, 5A53, 5AA4, 5AB5, 5ABB, 5ACA, 5B4B, 5B5A, 5BA5, 5CA4, 6227, 6278, 6667, 6698, 6733, 6872, 6928, 6944, 694C, 6973, 6986, 6997, 69C8, 6AC3, 6C92, 6C94, 7019, 7057, 70B5, 7103, 710B, 7118, 7127, 7129, 7172, 7178, 7192, 7211, 7217, 7219, 7271, 7303, 7408, 7433, 7444, 7505, 7507, 7574, 770B, 7774, 7778, 7787, 7871, 7877, 7888, 794B, 7994, 79B4, 7B43, 7B74, 7B94, 7BB2, 8027, 8072, 8081, 80BA, 8171, 8207, 821C, 848B, 8687, 8711, 8722, 87BB, 8867, 88B2, 88BA, 8B22, 8B2A, 8BAC, 9004, 9017, 9031, 9053, 9055, 9073, 9091, 90BB, 90C8, 9107, 9118, 913C, 9181, 91C3, 9284, 935C, 93C5, 9424, 9428, 9448, 9509, 959C, 96C4, 9703, 9743, 9745, 974B, 97B2, 9811, 981B, 987B, 98B1, 991C, 9967, 9998, 9B12, 9B74, 9B92, 9BBC, 9C55, 9C86, 9CC4, A0BA, A306, A436, A535, A5B5, A636, A6C3, A80B, AB04, AB22, AB35, AB3B, AB4C, AB55, ABAC, ABB5, AC36, ACA5, B044, B04A, B0B7, B129, B1B2, B219, B222, B291, B299, B2CA, B35A, B3A5, B404, B44C, B45B, B4B3, B501, B51C, B55A, B5A5, B5AB, B5C3, B707, B792, B794, B905, B912, B9C5, BA5B, BAB3, BB03, BB45, BB72, BBA5, BBB2, BC44, BC53, BC95, BC99, C30A, C36A, C395, C454, C535, C553, C593, C944, C953, C964, CC94, 10015, 10051, 10099, 10118, 10291, 10712, 10772, 10811, 10877, 10921, 10B92, 11111, 11135, 11171, 111C8, 11531, 11C03, 13001, 13177, 13777, 13915, 13951, 13991, 159BB, 17018, 17102, 17111, 17117, 17171, 17177, 17708, 17711, 17801, 18071, 18101, 18271, 18B27, 19003, 19153, 19315,

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455C3, 45C04, 488BC, 4B03B, 4B0B5, 4B55C, 4BB0B, 4C003, 4C054, 4C5C4, 50053,
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4A0000000000000B, ..., 7000000000000013, ..., 8000000000000111C, ...,
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COAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA, ...,
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777777777777. ....
99999999999991. ....
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base 14 (not proved, only checked to the prime 108000000D)

13, 15, 19, 21, 23, 29, 2D, 31, 35, 3B, 43, 45, 4B, 51, 53, 59, 5D, 65, 6D, 73, 75, 79, 7B, 81, 91, 95, 9B, 9D, A9, AB, B3, B9, BD, C5, CB, CD, D9, DB, 101, 111, 11D, 161, 17D, 1A1, 1AD, 1D1, 205, 22B, 255, 26B, 285, 2BB, 30D, 33D, 349, 389, 3D3, 40D, 41D, 44D, 469, 471, 499, 4AD, 4C1, 4D1, 50B, 525, 52B, 55B, 58B, 60B, 61B, 683, 689, 6A3, 701, 71D, 741, 771, 77D, 7DD, 803, 80B, 825, 82B, 833, 839, 84D, 86B, 88D, 893, 8AD, 8BB, 8D3, 983, 9A3, A0D, A1D, A25, A41, A4D, AAD, AC1, AC3, AD1, B05, B41, B5B, B6B, B85, BA1, BB1, C49, C61, C83, C89, CC1, D01, D03, D33, D4D, D61, D71, D7D, D83, DA1, DA5, DC3, DD1, 10BB, 10DD, 128B, 18DD, 1B71, 1B8B, 1C41, 1D8D, 2BA5, 33A3, 347D, 3863, 3A7D, 40A1, 478D, 4809, 48C9, 48DD, 4C09, 4D8D, 56BB, 6049, 60C1, 6171, 61C1, 628B, 6409, 6461, 64A1, 6711, 6761, 67A1, 6A11, 6A71, 6B71, 6B8B, 708D, 748D, 7611, 780D, 7CA1, 8009, 8055, 807D, 8089, 80C9, 80DD, 837D, 8505, 88A3, 89C9, 8A05, 8A85, 8C63, 8C99, 8CC9, 9489, 94C9, 9869, 9899, A063, A071, A0A1, A0A3, A303, A603, A611, A633, A663, A83D, A883, A8A5, AA01, AD8D, B02B, B061, B08B, B10B, BC01, C0A3, C141, C171, C401, C441, CCA3, D005, D055, D08D, D18D, D1C1, D225, D80D, D885, DC11, 1062B, 11BBB, 1668B, 1B00B, 1BBBB, 1D00D, 1DD0D, 1DDDD, 2B225, 30083, 308A3, 33383, 338C3, 37A8D, 38883, 38AA3, 38DDD, 3A033, 3A8DD, 3AA83, 3AAA3, 3CA63, 40061, 400C9, 40601, 40641, 44141, 444C9, 44601, 44661, 44849, 44A01, 44AA1, 46061, 46411, 48489, 5B555, 5BA55, 5BBB5, 60A01, 60AA1, 64401, 66411, 66601, 66649, 6666B, 666B1, 66949, 66B11, 6BC11, 766C1, 7A661, 7AA11, 80649, 80669, 80699, 80885, 80949, 80AA5, 84409, 84849, 84889, 85A55, 86099, 86449, 86609, 86999, 86C09, 8700D, 884C9, 88805, 88809, 88899, 88B55, 89069, 89099, 89449, 89609, 89889, 89999, 8A5A5, 8AA55, 8AAA3, 8B555, 8BAA5, 8CAA3, 908C9, 90989, 94449, 98C09, 99089, 99409, 99949, A0085, A0A85, A7A11, A7A61, A8005, AA383, AA711, AA7A1, AA855, ADDD5, B011B, B07C1, B0C71, B11BB, B2225, B5555, B5AA5, B67C1, B76C1, B7C11, BB2B5, BB88B, BBB55, C04A1, C0A01, C0AA1, C3A03, D0ADD, D3DDD, DA8DD, DD38D, DDA63, DDD25, DDD55, DDDAD, 10006B, 11088B, 116B2B, 166B2B, 20008B, 300A33, 30A363, 3CA003, 400041, 400489, 401441, 404001, 404089, 404411, 404441, 404CC9, 406611, 40CCC9, 440001, 440409, 444041, 444611, 444641, 460011, 460041, 466401, 4A0001, 4A6AA1, 5BAAA5, 600411, 604041, 640011, 660441, 6666C1, 666A61, 6A0061, 6A0601, 6A6061, 6AAA61, 76A6A1, 8000A5, 85B5A5, 869669, 884049, 8885A5, 888669, 8886C3, 888BA5, 888C69, 889849, 896669, 898049, 900049, 900649, 908449, 940009, 969649, 988849, 990649, A08555, A33333, A3A333, A3A363, A6A6A1, A6AAA1, A88855, AAA085, AAA3A3, ADAAA3, ADD085, B0001B, B000C1, B00711, B2000B, B2AAA5, B60071, B66011, B66071, B666C1, B66C11, BA5A55, BAA5A5, BAAA55, C00A11, C00A71, C3A333, CA0333, CA3AA3, CAAA03, CAAA11, CAAAA1, D1000D, D3DA8D, DDAAA3. 100008B. 100020B. 3000A03. 3000CA3. 308CCC3. 38CCCC3. 4000011.

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base 15 (not proved, only checked to the prime 555555557)

12, 14, 18, 1E, 21, 27, 2B, 2D, 32, 38, 3E, 41, 47, 4B, 4D, 54, 58, 5E, 67, 6B, 6D, 72, 74, 78, 87, 8B, 92, 94, 9E, A1, A7, AD, B2, B8, BE, C1, CB, CD, D2, D4, E1, ED, 111, 11B, 131, 137, 13B, 13D, 157, 15B, 15D, 171, 177, 197, 19D, 1B7, 1BB, 1D1, 1DB, 1DD, 234, 298, 311, 31B, 337, 33D, 344, 351, 357, 35B, 364, 377, 391, 39B, 39D, 3A4, 3BD, 3C4, 3D7, 3DB, 3DD, 452, 51B, 51D, 531, 53B, 551, 55D, 562, 571, 577, 5A2, 5B1, 5B7, 5BB, 5BD, 5C2, 5D1, 5D7, 634, 652, 681, 698, 717, 71B, 731, 737, 757, 75D, 77D, 79B, 79D, 7B1, 7B7, 7BD, 7D7, 7DD, 801, 852, 88D, 8D8, 91D, 93B, 93D, 95B, 95D, 971, 977, 97B, 97D, 988, 991, 9BD, 9C8, 9D1, A98, AAB, B1D, B31, B3B, B44, B51, B57, B7B, B7D, B97, B9B, BB7, BC4, BD1, BD7, BDD, C07, C34, C52, C7E, C98, CC7, CE7, D0E, D1D, D31, D51, D5B, D68, D77, D7B, D91, D97, DA8, DAE, DCE, DD1, EB4, EEB, 107B, 1091, 10B1, 1107, 110D, 1561, 1651, 1691, 1B01, 2052, 2502, 2522, 303B, 307D, 3097, 30BB, 30D1, 3107, 3361, 3701, 3907, 3B01, 3B0B, 3C97, 4434, 4498, 4834, 4898, 49A8, 4E34, 5037, 507D. 5091, 509B, 5107, 5161, 5202, 53C7, 5552, 570B, 590B, 590D, 59C7, 5A5B, 5C97, 5D0D, 5DAB, 6061, 6151, 6191, 6511, 6601, 6911, 707B, 7091, 7097, 70AE, 70BB, 70CE, 70DB, 7561, 760E, 7691, 76CE, 7907, 7961, 7A0E, 7A3B, 7AEE, 7B0B, 7BAB, 7C0E, 7C77, 7CAE, 7D0B, 7D61, 7DAB, 7E5B, 7E6E, 7E7B, 7EBB, 8098, 811D, 8191, 835D, 853D, 8881, 8908, 8951, 8968, 899D, 8D3D, 8D5D, 8D6E, 8DDD, 8E98, 9011, 9037, 9097, 90D7, 9301, 93C7, 95C7, 9611, 9631, 96A8, 9811, 9851, 989D, 990B, 990D, 998D, 99AB, 99C7, 99D8, 9A08, 9A9B, 9AA8, 9ABB, 9B61, 9BC7, 9D0B, 9DAB, 9DC7, 9DD8, A052, A304, A502, A55B, A9BB, AB04, AB64, B09D, B107, B10B, B161, B1AB, B1C7, B30D, B3C7, B50B, B664, B691, B6A4, B707, B761, B90D, B961, BA5B, BABB, BBAB, BBB4, BC37, BC77, C777, C937, C997, D011, D03D, D05D, D09B, D0B1, D0BD, D101, D10B, D30D,

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E7777797, E9066668, EE00E397, EE077797, EE0E0397, EEE00797, EEE07E97,
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base 16 (not proved, only checked to 100000000, also the number DB³²²³⁴ is only a probable prime, i.e. not definitely prime)

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BB00000BD, BB0C0000D, BBBBBA00B, BBBBBBABB, BE0EEEE0B, BE7777777,
C00000CAF, C00006AAF, C000082CD, C00063AFF, C000820CD, C00F00023,
C0444444D, C66666AFF, CCCD99999, CF0000023, CF66666AF, D00000009,
D0000044D, D0044000D, D040E000D, D0440000D, D0DD000D9, DAAAAAA45,
E004044DD, E004444DD, E044400DD, E0C00008D, E0C08000D, E0EAAAAA1,
E2000000D, E400044DD, EAAA4AAA1, EAAAAEAA1, EAAAEA041, EBBBBC00D,
EEEE00CCB, F00000545, F02600003, F066AAAAF, F0FF5666F, F3FFF3F23,
F60AAAA0F, F77777777, FFEEEEEET, FFFF33323, FFFF5666F, FFFFF2CC3,
FFFF7777, FFFFFEEE7, FFFFFF77, ..., 2666666663, ..., 400000000D, ..., 500000006F,
...., 700000077B, ...., 8000000AA1, ...., 800AAAAA01, ...., 8886888AAF, ...., 88888888AF, ....,
888888A8F, ..., 888AAFFFFF, ..., 9000000019, ..., 9000000109, ..., 908AAAAA01, ...,
AAAAAAAA1, ..., AAAAAAAE41, ..., C000CC866F, ..., C00CCCCAF, ..., C6666666AF,
..., CCCCCCAAF, ..., CFFFFFAAF, ..., E44444441, ..., E4444444DD, ..., EAAAAAA4A1,
..., F260000003, ..., FEEEEEEEE7, ..., FFFFFF56F, ..., 22000000007, ..., 4000000004B,
..., 400000000A5, ..., 52CCCCCCCD, ..., 80AAAAAAA01, ..., 87000000007, ...,
A044444441. .... A0AAAAAEA41. .... BEEEEEEEEEB. .... C0006666AFF. ....
C000CCC6AF, ..., C0AF000000F, ..., EAAAEAAAA1, ..., FAAAAAAAA8F, ...,
5888888887, ..., 800AAAAAAAA1, ..., 888888AFFFFF, ..., 88AFFFFFFFF, ...,
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D1000000005, ..., E0A04AAAAAA1, ..., 1A000000000B, ..., 5BBBBBBBBBBBB, ...,
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88888F8888F, ..., 88F88888888F, ..., A00000000A8F, ..., A0FFFFFFFF45, ...,
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C0A00000000F, ..., C4444444444D, ..., CFF0A0000000F, ..., D00000000007B, ...,
6866666666666F, ..., 68CCCCCCCCCCF, ..., 77700000000007D, ...,
8000000000001, ..., 888888AAAAAAAF, ..., 9B000000000009, ...,
AAAAAAAAAAAA45, .... CFFFFFFFFA000F, .... DDDDDDDDDDDDDDD. ....
58CCCCCCCCCCD, ..., 866666666666666, ..., 8ECCCCCCCCCCD, ...,
A00000000000000, ..., 8CFFFFFFFFFFFF, ..., 5C20000000000000, ...,
B0000000000000981, ..., CFFFFFFFFFFFA00F, ..., AAAAAAAAAAAAAAAAAAAA
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CFFA00000000000000F, ..., 70000000000000007D7, ..., E0000000000000000441, ...,
CFFFFFFA0000000000F, ..., 4000000000000000000085, ...,
8AAAAAAAAAAAAAAAAAAFF, ..., 8D0000000000000000000007, ...,
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8444444444444444444444AD. ....
8CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCFF. ....
A8AAAAAAAAAAAAAAAAAAAAAAAAAAAA.....
AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA
222222222222222222222222227, ..., CFA000000000000000000000000000, ...,
8AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA
EEEEEEEEEEEEEEEEEEEEEEEEEEEE, ...,
C000000000000000000000000000000000000DD, ....
CCCCCCCCF. ....
CCCCCCCCCCCAF, ...,
BBBBBBBBBBBBBBBBBBBBB. ....
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CCCCCD.

Proof

There are <u>lemmas</u>, <u>corollaries</u>, <u>theorems</u>, <u>proofs</u>, <u>conjectures</u>, <u>hypotheses</u>, <u>open problems</u>, <u>heuristic arguments</u>, for this <u>problem</u> about the <u>sets</u> of the <u>primes</u> with no <u>proper subsequence</u> whose <u>value</u> is also prime in the <u>positional numeral system</u> with <u>base (or radix)</u> b for $2 \le b \le 36$.

Proving $M(L_b)$ = the set S is equivalent to:

- * Prove that all elements in S, when read as base b representation, are primes > b.
- * Prove that all <u>proper subsequence</u> of all elements in *S*, when read as base *b* representation, which are > *b*, are composite.
- * Prove that all primes > b, when written in base b, contain at least one element in S as subsequence (equivalently, prove that all strings not containing any element in S as subsequence, when read as base b representation, which are > b, are composite).

 $(M(L_b) = S \text{ is proved if and only if all these three problems are proved, i.e. } M(L_b) = S \text{ is a theorem if and only if all these three "conjectures" are theorems)}$

- * Prove that all primes > 10 contain at least one element in {11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 227, 251, 257, 277, 281, 349, 409, 449, 499, 521, 557, 577, 587, 727, 757, 787, 821, 827, 857, 877, 881, 887, 991, 2087, 2221, 5051, 5081, 5501, 5581, 5801, 5851, 6469, 6949, 8501, 9001, 9049, 9221, 9551, 9649,

For the first part, since the numbers are clearly > b, thus we only need to prove that they are primes, we can use ECPP (such as Primo) to prove that these 77 numbers are definitely primes (i.e. not merely probable primes), in this case of base 10, the largest number has only 31 digits and can be proved primality in <1 second, but in other case, such as base 13, 14, and 16, there are numbers > 10^{10000} in the sets, thus ECPP (or N-1, N+1, if this prime -1 or +1 can be trivially factored, such as the case of base 14, the large prime 5*14¹⁹⁶⁹⁸–1 in this set) is need to prove their primality; the second part is the easiest part of these three parts, as we can use either trial division or Fermat test to prove their compositeness (if these numbers have small prime divisors, or these numbers fails the Fermat primality tests, then they are defined composite), unless the numbers are Fermat pseudoprimes to many bases (such as the Carmichael numbers and the numbers of the form p^*q with p, q primes and q =2*p-1 (https://oeis.org/A129521)) (reference of pseudoprimes) with no small prime factors (say < 2³²), in this case, we need to run either Miller-Rabin primality test or Lucas primality test to prove their compositeness (the worst case is that the number is a Carmichael number which is strong pseudoprime to several bases, see this article, this article gives a 397 digit such number, another example is this 23707 digit number), the combine of these two tests is Baillie-PSW primality test, and there is no known composites which pass this test, also it is known that no composites $\leq 2^{64}$ which pass this test, this is because strong Fermat pseudoprimes to base 2 (https://oeis.org/A001262) tend to fall into the residue class +1 (mod m) for many small m, whereas strong Lucas pseudoprimes (https://oeis.org/A217255) tend to fall into the residue class -1 (mod m) for many small m. As a result, a number which passes both a strong Fermat and a strong Lucas test is very likely to be prime.

Determining M(L) for arbitrary L is in general <u>unsolvable</u>, and can be difficult even when L is relatively simple, also, determining M(L) for arbitrary L may be an <u>open problem</u> or <u>NP-complete</u> or an <u>undecidable problem</u>, or an example of <u>Gödel's incompleteness theorems</u> (like the <u>continuum hypothesis</u> and the <u>halting problem</u>), or as hard as <u>the unsolved problems in mathematics</u>, such as the <u>Riemann hypothesis</u> and the <u>abc conjecture</u>, which are the two famous hard problems in <u>number theory</u>.

The following is a " $\underline{\text{semi-algorithm}}$ " that is guaranteed to produce M(L), but it is not so easy to implement:

```
(1) M = \emptyset
(2) while (L \neq \emptyset) do
(3) choose x, a shortest string in L
(4) M = M \cup \{x\}
```

(5) $L = L - \sup(\{x\})$

In practice, for arbitrary L, we cannot feasibly carry out step (5). Instead, we work with L', some regular overapproximation to L, until we can show $L' = \emptyset$ (which implies $L = \emptyset$). In practice, L' is usually chosen to be a finite <u>union</u> of sets of the form $L_1L_2*L_3$, where each of L_1 , L_2 , L_3 is finite. In the case we consider in this paper, we then have to determine whether such a language contains a prime or not.

However, it is not even known if the following simpler <u>decision problem</u> is recursively solvable:

Problem: Given strings x, y, z, and a base b, does there exist a prime number whose base-b expansion is of the form xy^nz for some $n \ge 0$? (If we say "yes", then we should find such a prime (the smallest such prime may be very large, e.g. > 2^{65536} , and if so, then we should use <u>primality testing programs</u> such as <u>PFGW</u> or <u>LLR</u> to find it, and before using these programs, we should use <u>sieving programs</u> such as <u>srsieve</u> (or sr1/2/5sieve) to remove the numbers either having small prime factors or having algebraic factors) and <u>prove its primality</u> (and if we want to solve the problem in this article, we should check whether this prime is the smallest such prime or not, i.e. prove all smaller numbers of the form xy^nz with $n \ge 0$ are composite, usually by <u>trial division</u> or <u>Fermat primality test</u>), and if we say "no", then we should prove that such prime does not exist, may by <u>covering congruence</u>, <u>algebraic factorization</u>, or combine of them)

An algorithm to solve this problem, for example, would allow us to decide if there are any additional <u>Fermat primes</u> (of the form $2^{2^n} + 1$) other than the known ones (corresponding to n = 0, 1, 2, 3, 4). To see this, take b = 2, x = 1, y = 0, and $z = 0^{16}1$, or take $b = 2, x = 10^{16}, y = 0$, and z = 1. Since if $2^n + 1$ is prime then n must be a power of two, a prime of the form $(xy^*z)_b$ must be a new Fermat prime. Besides, it would allow us to decide if there are infinitely many <u>Mersenne primes</u> (of the form $2^p - 1$ with prime p). To see this, take $b = 2, x = \lambda$ (the <u>empty string</u>), y = 1, and $z = 1^{n+1}$, or take $b = 2, x = 1^{n+1}, y = 1$, and $z = \lambda$ (the <u>empty string</u>), where n = 1 is the exponent of the Mersenne prime which we want to know whether it is the largest Mersenne prime or not. Since if $2^n - 1$ is prime then n = 1 must be a prime, a prime of the form $(xy^*z)_b$ must be a new Mersenne prime. Also, it would allow us to decide if 21181 is a

Sierpinski number (take b = 2, x = 1010010101111101, y = 0, and z = 1) and if 23669 is a Riesel number (take b = 2, x = 101110001110100, y = 1, and $z = \lambda$ (the empty string)). Also, it would allow us to solve the numbers n with unknown status (i.e. n = 603, 1244, 1861) in this page (take b = 10, x = 6031, 12441, 18611, respectively, y = 1, and $z = \lambda$ (the empty string), or take b = 10, x = 603, 1244, 1861, respectively, y = 1, and z = 1).

Therefore, in practice, we are forced to try to rule out prime representations based on <u>heuristics</u> such as <u>modular techniques</u> and <u>factorizations</u>.

It will be necessary for our algorithm to determine if families of the form $(xy^*z)_b$ contain a prime > b or not. We use two different heuristic strategies to show that such families contain no primes > b.

(Reference: the divisibility rule for base b:

- * For prime p dividing b, the number is divisible by p if and only if the last digit of this number is divisible by p.
- * For prime p dividing b-1, the number is divisible by p if and only if the sum of the digits of this number is divisible by p.
- * For prime p dividing b+1, the number is divisible by p if and only if the <u>alternating sum</u> of the <u>digits</u> of this number is divisible by p. (this can also show that all <u>palindromic primes</u> in any base b have an <u>odd</u> number of <u>digits</u>, the only possible exception is "11" in base b (i.e. b+1 itself))
- * The section "Divisibility Rules in Lotsa Various Bases" in its talk page
- * The divisibility rule of b^2-b+1 in base b

In the first strategy, we mimic the well-known technique of "covering congruences", by finding some finite set S of primes p such that every number in a given family is divisible by some element of S (examples: the conjectured smallest Sierpinski number 78557 and the conjectured smallest Riesel number 509203, which have covering sets $\{3, 5, 7, 13, 19, 37, 73\}$ and $\{3, 5, 7, 13, 17, 241\}$, respectively, and their periods are 36 and 24, respectively, see https://oeis.org/A244562, https://oeis.org/A244564, https://oeis.org/A244564, https://oeis.org/A244074, https://oeis.org/A244074, https://oeis.org/A244074, https://oeis.org/A258154, another examples are the families $9\{1\}3$ and $9\{4\}9$ and $9\{5\}9$ in base b = 10, see https://stdkmd.net/nrr/1/91113.htm#prime_period,

https://stdkmd.net/nrr/1/94449.htm#prime_period,

https://stdkmd.net/nrr/1/95559.htm#prime_period,

http://www.worldofnumbers.com/deplat.htm) (this is equivalent to finding an integer N such that all numbers in a given family are not <u>coprime</u> to N, e.g. all numbers in the family 2{5} in base 11 are not <u>coprime</u> to 6, $gcd((5*11^n-1)/2, 6)$ can only be 2 or 3, and cannot be 1). In the second strategy, we attempt to find an <u>algebraic factorization</u>, such as <u>difference-of-two-squares factorization</u>, <u>sum/difference-of-two-cubes factorization</u>, and <u>Aurifeuillian factorization</u> for x^4+4y^4 (<u>examples of Aurifeuillian factorizations</u>), if a, b, c are all r-th powers for some r > 1 (i.e. gcd(A052409(a), A052409(b), A052409(c)) > 1), then $\frac{a \cdot b}{gcd(a+c,b-1)}$ ($a \ge 1$,

 $b\ge 2$ (*b* is the base), $c\ne 0$, gcd(a,c)=1, gcd(b,c)=1) is always composite, with only a possible exception of very small *n*, the same holds for the situation when *b* and 4ac are both 4th

<u>powers</u>. For the examples of combine of the two strategies (i.e. combine of <u>covering</u> <u>congruences</u> and <u>algebraic factorization</u>), see <u>https://arxiv.org/pdf/1110.4671.pdf</u> and <u>https://www.fq.math.ca/Scanned/33-3/izotov.pdf</u> and <u>https://oeis.org/A213353</u>,

 $4008735125781478102999926000625*2^n+1 (=44745755^4*2^n+1)$ and

1518540332232392131536305922189449981332693305773307144086114457160111705 7698737700140317416496481*2ⁿ-1

(=3896845303873881175159314620808887046066972469809 $^{2*}2^n$ -1) are examples, and the family 38{1} in base b = 10 is also an example, see

http://www.worldofnumbers.com/em197.htm and

http://www.worldofnumbers.com/Appending%201s%20to%20n.txt and

<u>https://oeis.org/A069568</u> and <u>https://archive.fo/vKSJO</u>, also there are examples in the <u>Conjectures 'R Us</u> pages: <u>Sierpinski side</u> (k^*b^n +1) (see bases b = 55, 63, 200, 225, ...) and Riesel side (k^*b^n -1) (see bases b = 12, 19, 24, 28, 30, 33, ...).

Examples of the first strategy: (we can show that the corresponding numbers are > all elements in S, if n makes corresponding numbers > b (i.e. $n \ge 1$ for 51^n in base 9 and 25^n in base 11 and 4^n D in base 16 and 8^n F in base 16, $n \ge 0$ for other examples), thus these factorizations are nontrivial)

- * In base 10, all numbers of the form $46^{n}9$ (algebraic form: $(14*10^{n+1}+7)/3$) ($n \ge 0$) are divisible by 7, and no numbers of the form $46^{n}9$ (base 10) with $n \ge 0$ is equal to 7, thus no number of the form $46^{n}9$ (base 10) with $n \ge 0$ is prime (factordb)
- * In base 6, all numbers of the form 40^n1 (algebraic form: $4^*6^{n+1}+1$) ($n \ge 0$) are divisible by 5, and no numbers of the form 40^n1 (base 6) with $n \ge 0$ is equal to 5, thus no number of the form 40^n1 (base 6) with $n \ge 0$ is prime (factordb)
- * In base 15, all numbers of the form $96^{n}8$ (algebraic form: $(66*15^{n+1}+11)/7$) ($n \ge 0$) are divisible by 11, and no numbers of the form $96^{n}8$ (base 15) with $n \ge 0$ is equal to 11, thus no number of the form $96^{n}8$ (base 15) with $n \ge 0$ is prime (factordb)
- * In base 9, all numbers of the form 51^n (algebraic form: $(41*9^n-1)/8$) ($n\ge 1$) are divisible by some element of $\{2,5\}$, and no numbers of the form 51^n (base 9) with $n\ge 1$ is equal to 2 or 5, thus no number of the form 51^n (base 9) with $n\ge 1$ is prime (note: the prime 5 (i.e. n=0) is not allowed since the prime must be > base) (factordb)
- * In base 11, all numbers of the form 25^n (algebraic form: $(5*11^n-1)/2$) ($n\ge 1$) are divisible by some element of $\{2,3\}$, and no numbers of the form 25^n (base 11) with $n\ge 1$ is equal to 2 or 3, thus no number of the form 25^n (base 11) with $n\ge 1$ is prime (note: the prime 2 (i.e. n=0) is not allowed since the prime must be > base) (factordb)
- * In base 14, all numbers of the form $B0^n1$ (algebraic form: $11*14^{n+1}+1$) ($n \ge 0$) are divisible by some element of $\{3,5\}$, and no numbers of the form $B0^n1$ (base 14) with $n \ge 0$ is equal to 3 or 5, thus no number of the form $B0^n1$ (base 14) with $n \ge 0$ is prime (factordb)
- * In base 8, all numbers of the form 64^n 7 (algebraic form: $(46*8^{n+1}+17)/7$) ($n \ge 0$) are divisible by some element of $\{3,5,13\}$, and no numbers of the form 64^n 7 (base 8) with $n \ge 0$ is equal to 3, 5, or 13, thus no number of the form 64^n 7 (base 8) with $n \ge 0$ is prime (<u>factordb</u>)
- * In base 13, all numbers of the form 30^n95 (algebraic form: $3*13^{n+2}+122$) ($n\ge0$) are divisible by some element of $\{5,7,17\}$, and no numbers of the form 30^n95 (base 13) with $n\ge0$ is equal to 5, 7, or 17, thus no number of the form 30^n95 (base 13) with $n\ge0$ is prime (<u>factordb</u>)
- * In base 16, all numbers of the form 4^nD (algebraic form: $(4*16^{n+1}+131)/15$) ($n\ge 1$) are divisible by some element of $\{3,7,13\}$, and no numbers of the form 4^nD (base 16) with $n\ge 1$ is

equal to 3, 7, or 13, thus no number of the form 4^nD (base 16) with $n \ge 1$ is prime (note: the prime D (i.e. n = 0) is not allowed since the prime must be > base) (factordb)

* In base 16, all numbers of the form 8^nF (algebraic form: $(8*16^{n+1}+97)/15$) ($n \ge 1$) are divisible by some element of $\{3,7,13\}$, and no numbers of the form 8^nF (base 16) with $n \ge 1$ is equal to 3, 7, or 13, thus no number of the form 8^nF (base 16) with $n \ge 1$ is prime (factordb)

Examples of the second strategy: (we can show that both factors are > 1, if n makes corresponding numbers > b (i.e. $n \ge 2$ for 1^n in base 9, $n \ge 0$ for $10^n 1$ in base 8 and B4ⁿ1 in base 16, $n \ge 1$ for other examples), thus these factorizations are nontrivial)

- * In base 9, all numbers of the form 1^n (algebraic form: $(9^n-1)/8$) $(n \ge 2)$ factored as (3^n-1) * $(3^n+1)/8$, and since if $n \ge 3$, $3^n-1 \ge 3^3-1 = 26 > 8$, $3^n+1 \ge 3^3+1 = 28 > 8$, this factorization is nontrivial if $n \ge 3$, and this only remains to check the case n=2, but for n=2, $(9^n-1)/8 = 10$ and 10 is not prime, thus no number of the form 1^n (base 9) with $n \ge 2$ is prime (factordb) * In base 8, all numbers of the form 10^n1 (algebraic form: $8^{n+1}+1$) $(n \ge 0)$ factored as $(2^{n+1}+1)$ * $(4^{n+1}-2^{n+1}+1)$, and since if $n \ge 0$, $2^{n+1}+1 \ge 2^1+1 = 3 > 1$, $4^{n+1}-2^{n+1}+1 \ge 4^1-2^1+1 = 3 > 1$, this factorization is nontrivial, thus no number of the form 10^n1 (base 8) with $n \ge 0$ is prime (factordb)
- * In base 9, all numbers of the form 38^n (algebraic form: $4*9^n-1$) ($n \ge 1$) factored as $(2*3^n-1)*(2*3^n+1)$, and since if $n \ge 1$, $2*3^n-1 \ge 2*3^1-1 = 5 > 1$, $2*3^n+1 \ge 2*3^1+1 = 7 > 1$, this factorization is nontrivial, thus no number of the form 38^n (base 9) with $n \ge 1$ is prime (note: the prime 3 (i.e. n = 0) is not allowed since the prime must be > base) (factordb)

 * In base 16, all numbers of the form $8F^n$ (algebraic form: $9*16^n-1$) ($n \ge 1$) factored as $(3*4^n-1)*(3*4^n+1)$, and since if $n \ge 1$, $3*4^n-1 \ge 3*4^1-1 = 11 > 1$, $3*4^n+1 \ge 3*4^1+1 = 13 > 1$, this factorization is nontrivial, thus no number of the form $8F^n$ (base 16) with $n \ge 1$ is prime (factordb)
- * In base 16, all numbers of the form F^n 7 (algebraic form: $16^{n+1}-9$) ($n \ge 1$) factored as $(4^{n+1}-3)$ * $(4^{n+1}+3)$, and since if $n \ge 1$, $4^{n+1}-3 \ge 4^2-3=13>1$, $4^{n+1}+3 \ge 4^2+3=19>1$, this factorization is nontrivial, thus no number of the form F^n 7 (base 16) with $n \ge 1$ is prime (note: the prime 7 (i.e. n = 0) is not allowed since the prime must be > base) (factordb)
- * In base 9, all numbers of the form 31^n (algebraic form: $(25^*9^n-1)/8$) ($n\ge 1$) factored as $(5^*3^n-1)^*$ (5^*3^n+1) / 8, and since if $n\ge 1$, $5^*3^n-1\ge 5^*3^1-1=14>8$, $5^*3^n+1\ge 5^*3^1+1=16>8$, this factorization is nontrivial, thus no number of the form 31^n (base 9) with $n\ge 1$ is prime (note: the prime 3 (i.e. n=0) is not allowed since the prime must be > base) (factordb)

 * In base 16, all numbers of the form 4^n1 (algebraic form: $(4^*16^{n+1}-49)/15$) ($n\ge 1$) factored as $(2^*4^{n+1}-7)^*$ ($(2^*4^{n+1}+7)^*$) / 15, and since if $n\ge 1$, $(2^*4^{n+1}-7)^*$ 2 2 4 2 7 = 25 > 15, $(2^*4^{n+1}+7)^*$ 2 2 4 4 7 = 25 > 15, this factorization is nontrivial, thus no number of the form $(4^n1)^n$ (base 16) with $(4^n1)^n$ 0 is prime (factordb)
- * In base 16, all numbers of the form 15^n (algebraic form: $(4*16^n-1)/3$) ($n\ge 1$) factored as $(2*4^n-1)*(2*4^n+1)/3$, and since if $n\ge 1$, $2*4^n-1\ge 2*4^1-1=7>3$, $2*4^n+1\ge 2*4^1+1=9>3$, this factorization is nontrivial, thus no number of the form 15^n (base 16) with $n\ge 1$ is prime (factordb)
- * In base 16, all numbers of the from C^nD (algebraic form: $(4*16^{n+1}+1)/5$) ($n\ge 1$) factored as $(2*4^{n+1}-2*2^{n+1}+1)*(2*4^{n+1}+2*2^{n+1}+1)/5$, and since if $n\ge 1$, $2*4^{n+1}-2*2^{n+1}+1\ge 2*4^2-2*2^2+1=25>5$, $2*4^{n+1}+2*2^{n+1}+1\ge 2*4^2+2*2^2+1=41>5$, this factorization is nontrivial, thus no number of the form C^nD (base 16) with $n\ge 1$ is prime (note: the prime D (i.e. n=0) is not allowed since the prime must be > base) (factordb)

* In base 16, all numbers of the form B4ⁿ1 (algebraic form: $(169*16^{n+1}-49)/15$) ($n \ge 0$) factored as $(13*4^{n+1}-7)*(13*4^{n+1}+7)/15$, and since if $n \ge 0$, $13*4^{n+1}-7 \ge 13*4^{1}-7 = 45 > 15$, $13*4^{n+1}+7 \ge 13*4^{1}+7 = 59 > 15$, this factorization is nontrivial, thus no number of the form B4ⁿ1 (base 16) with $n \ge 0$ is prime (factordb)

Examples of combine of the two strategies: (we can show that for the part of the first strategy, the corresponding numbers are > all elements in S, and for the part of the second strategy, both factors are > 1, if n makes corresponding numbers > b (i.e. $n \ge 0$ for $B^n \ni B$ in base 12, $n \ge 1$ for other examples), thus these factorizations are nontrivial)

- * In base 14, numbers of the form 8Dⁿ (algebraic form: 9*14ⁿ-1) ($n \ge 1$) are divisible by 5 if n is odd and factored as $(3*14^{n/2}-1)*(3*14^{n/2}+1)$ if n is even, and no numbers of the form 8Dⁿ (base 14) with $n \ge 1$ is equal to 5, and since if $n \ge 2$ (if $n \ge 1$ and n is even, then $n \ge 2$), $3*14^{n/2}-1 \ge 3*14^1-1 = 41 > 1$, $3*14^{n/2}+1 \ge 3*14^1+1 = 43 > 1$, this factorization is nontrivial, thus no number of the form 8Dⁿ (base 14) with $n \ge 1$ is prime (factordb)
- * In base 12, numbers of the form Bⁿ9B (algebraic form: $12^{n+2}-25$) ($n\ge 0$) are divisible by 13 if n is odd and factored as $(12^{(n+2)/2}-5)$ * $(12^{(n+2)/2}+5)$ if n is even, and no numbers of the form Bⁿ9B (base 12) with $n\ge 0$ is equal to 13, and since if $n\ge 0$, $12^{(n+2)/2}-5\ge 12^1-5=7>1$, $12^{(n+2)/2}+5\ge 12^1+5=17>1$, this factorization is nontrivial, thus no number of the form Bⁿ9B (base 12) with $n\ge 0$ is prime (factordb)
- * In base 14, numbers of the form $D^n 5$ (algebraic form: $14^{n+1} 9$) ($n \ge 1$) are divisible by 5 if n is even and factored as $(14^{(n+1)/2} 3)$ * $(14^{(n+1)/2} + 3)$ if n is odd, and no numbers of the form $D^n 5$ (base 14) with $n \ge 1$ is equal to 5, and since if $n \ge 1$, $14^{(n+1)/2} 3 \ge 14^1 3 = 11 > 1$, $14^{(n+1)/2} + 3 \ge 14^1 + 3 = 17 > 1$, this factorization is nontrivial, thus no number of the form $D^n 5$ (base 14) with $n \ge 1$ is prime (note: the prime 5 (i.e. n = 0) is not allowed since the prime must be > base) (factordb)
- * In base 17, numbers of the form 19^n (algebraic form: $(25*17^n-9)/16$) ($n\ge 1$) are divisible by 2 if n is odd and factored as $(5*17^{n/2}-3)*(5*17^{n/2}+3)/16$ if n is even, and no numbers of the form 19^n (base 17) with $n\ge 1$ is equal to 2, and since if $n\ge 2$ (if $n\ge 1$ and n is even, then $n\ge 2$), $5*17^{n/2}-3\ge 5*17^1-3=82>16$, $5*17^{n/2}+3\ge 5*17^1+3=88>16$, this factorization is nontrivial, thus no number of the form 19^n (base 17) with $n\ge 1$ is prime (factordb)
- * In base 19, numbers of the from 16^n (algebraic form: $(4*19^n-1)/3$) ($n\ge 1$) are divisible by 5 if n is odd and factored as $(2*19^{n/2}-1)*(2*19^{n/2}+1)/3$ if n is even, and no numbers of the form 16^n (base 19) with $n\ge 1$ is equal to 5, and since if $n\ge 2$ (if $n\ge 1$ and n is even, then $n\ge 2$), $2*19^{n/2}-1\ge 2*19^1-1=37>3$, $2*19^{n/2}+1\ge 2*19^1+1=39>3$, this factorization is nontrivial, thus no number of the form 16^n (base 19) with $n\ge 1$ is prime (factordb)

(for the base b forms xy^*z converted to the algebraic forms $\frac{a \cdot b}{gcd(a+c,b-1)}$ (b is the base, r is the length of z), using: https://stdkmd.net/nrr/exprgen.htm (only for base 10 forms) and https://www.numberempire.com/simplifyexpression.php (enter the obvious algebraic forms, e.g. for base 8 family 64^n 7, enter " $6*8^n$ (n+1)+ $4*8*(8^n$ -1)/7+7", this website will return " $(23*2^n(3*n+4)+17)$ 7", and this form can be easily converted to $(46*8^n(n+1)+17)$ 7) (b is given in its factorized form), also for the examples see page 16 of https://cs.uwaterloo.ca/~cbright/reports/mepn.pdf (all unsolved families in the original minimal prime problem (i.e. prime > base (b) is not required) for bases $2 \le b \le 30$) (a and b are given in their factorized form) and the excel file https://docs.google.com/spreadsheets/d/e/2PACX-

<u>1vRCn7Ytp1 Jbgi2b0MkjPxWE6yk3Eq81Wa3kWUUmRY8odQWJzGFBL1RZ4nqks3RJXuql UoWm37HO6pu/pubhtml</u> (all unsolved families in the original minimal prime problem (i.e. prime > base (*b*) is not required) for bases $2 \le b \le 50$ except b = 43, 47, 49) (there is also a zipped file <u>https://mersenneforum.org/attachment.php?attachmentid=25078&d=1623428406</u> for them))

(Note: the factors only shown the algebraic forms, if you want the base *b* forms, see this post)

As previously mentioned, in practice to <u>compute</u> $M(L_b)$ one works with an underapproximation M of $M(L_b)$ and an overapproximation L of $L_b - sup(M)$. One then refines such approximations until $L = \emptyset$ from which it follows that $M = M(L_b)$.

For the initial approximation, note that every minimal prime in base b with at least 4 digits is of the form xY^*z , where $x \in \{x \mid x \text{ is base-}b \text{ digit}, x \neq 0\}$, $z \in \{z \mid z \text{ is base-}b \text{ digit}, gcd(z,b) = 1\}$, and Y^* (for this (x,z) pair) = $\{y \mid xy, xz, yz, xyz \text{ are all composites}\}$. (Of course, if xz is prime, then the Y^* set for this (x,z) pair is \emptyset)

Making use of this, our algorithm sets M to be the set of base-b representations of the minimal primes with at most 3 digits (which can be found simply by brute force) and L to be $\bigcup_{x,z} (xY^*z)$ as described above.

All remaining minimal primes are members of L, so to find them we explore the families in L. During this process, each family will be decomposed into possibly multiple other families. For example, a simple way of exploring the family xY^*z where $Y = \{y_1, ..., y_n\}$ is to decompose it into the families xY^*y_1z , ..., xY^*y_nz . If the smallest member (say xy_iz) of any such family happens to be prime, it can be added to M and the family xY^*y_iz removed from consideration. Furthermore, once M has been updated it may be possible to simplify some families in L. In this case, xY^*y_jz (for $j \neq i$) can be simplified to $x(Y-y_i)^*y_jz$ since no minimal prime contains xy_iz as a proper subsequence.

We call families of the form xy^*z (where $x, z \in \Sigma_b^*$ and $y \in \Sigma_b$) simple families. Our algorithm then proceeds as follows:

1. Let

 $M := \{ \text{minimal primes in base } b \text{ of length } \leq 3 \}$

$$L:=\cup_{x,z\in \Sigma b} \quad (xY\quad ^*z)$$

where $x \neq 0$ and Y is the set of digits y such that xyz has no subword in M.

- 2. While *L* contains non-simple families:
- (a) Explore each family of *L*, and update *L*.
- (b) Examine each family of L:
- i. Let w be the shortest string in the family. If w has a subword in M, then remove the family from L. If w represents a prime, then add w to M and remove the family from L.

- ii. If possible, simplify the family.
- iii. Check if the family can be proven to contain no primes > base, and if so then remove the family from L.
- (c) As much as possible and update L; after each split examine the new families as in (b).

At the conclusion of the algorithm described, L will consist of simple families (of the form xy*z) which have not yet yielded a prime, but for which there is no obvious reason why there can't be a prime of such a form. In such a case, the only way to proceed is to test the primality of larger and larger numbers of such form and hope a prime is eventually discovered (we usually conjecture that there must be a prime at some point if it cannot be proved that there can't be a prime, by covering congruence, algebra factorization, or combine of them, since there is a heuristic argument that there are infinitely many such primes (reference), since by the prime number theorem, the chance that a random n-digit base b number is prime is approximately 1/n (reference reference) (also see this page and this page, the chance is approximately $\frac{b-1}{\ln(b)} \cdot \frac{b^{n-1}}{n}$, where $\ln n$ is the <u>natural logarithm</u>). If one conjectures the numbers xy*z behave similarly (i.e. "N of the form xy*z" and "N is prime" are independent events) you would expect $\sum_{n=2}^{\infty} \frac{1}{n} = \infty$ (harmonic series is divergent) primes of the form xy^*z , of course, this does not always happen, since some xy^*z families can be proven to contain no primes > base, and every xy*z family has its own Nash weight (or difficulty), xy*z families which can be proven to contain no primes > base have Nash weight (or difficulty) 0, thus xy*z families are not "completely" random (but we still conjectured that for a xy*z families which cannot be proven to contain no primes or only finitely primes, using covering congruence, algebra factorization, or combine of them, the number of primes with ≤ n digit is roughly $c^* \ln(n)$ for some positive constant c, the constant c varies with family xy^*z). They are random enough that the prime number theorem can be used to predict their primality, but divisibility by small primes is not as random and can easily be predicted: Once one candidate is found to be divisible by a prime p or to have an algebraic factorization (e.g. difference-of-two-squares factorization, sum/difference-of-two-cubes factorization, Aurifeuillian factorization for x^4+4y^4), another predictable candidate will also be divisible by p or also have the same algebraic factorization. This decreases the probability of expected primes. Sometimes though, the candidates will never be divisible by a prime p, which increases the probability of expected primes. However, it is at least a reasonable conjecture in the absence of evidence to the contrary, the numbers in simple families are of the form $\frac{a \cdot b \quad \because + c}{gcd(a+c,b-1)}$ for some fixed integer <u>triple</u> (a, b, c), where $a \ge 1$, $b \ge 2$ (b is the base), $c \ne 0$, gcd(a,c)=1, gcd(b,c)=1, this is an exponential sequence, there is also a similar conjecture for polynomial sequence: the Bunyakovsky conjecture, the condition is similar to our conjecture in this article, both are the small prime factors and the algebraic factors, the main difference is that polynomial sequence cannot have a covering set with >1 primes, however, unlike our conjecture (the analog of Bunyakovsky conjecture for exponential sequences), the analog of Dickson's conjecture and Schinzel's hypothesis H for exponential sequences is widely believed to be false, e.g. for all integer k divisible by 3, it is widely believed that there are only finitely many integers $n \ge 1$ such that $k^* 2^n \pm 1$ are twin primes (see this page and this page and this page, the conjecture that 237 is the smallest odd number k divisible by 3 such that $k^*2^n \pm 1$ are never twin primes will never be proven, the smaller odd numbers k divisible by 3 with no known such twin primes (and unlikely any exist) are {111, 123, 153, 159, 171, 183,

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189, 219, 225)), another example is that it is widely believed that 127 is the largest number n
such that the Mersenne number 2^n-1 and the Wagstaff number (2^n+1)/3 are both primes
(see New Mersenne Conjecture and its status page, the known such n are {3, 5, 7, 13, 17,
19, 31, 61, 127}, and they are listed in https://oeis.org/A107360) (in fact, if n is even number,
then (2^n+1)/3 is not integer, thus we only need to consider odd n, and for odd number n=1
2^*m+1, (2^n+1)/3 = (2^*4^m+1)/3, thus it can be written as the form \frac{a \cdot b^{-n}+c}{gcd(a+c,b-1)}, with (a, b, c) = (a \cdot b \cdot b)
(2, 4, 1), thus is included in this conjecture, also, if n is odd composite, then 2^n-1 and
(2^{n}+1)/3 are both composites, thus we only need to consider odd prime n), another example
is that it is widely believed that there are only finitely many integers n such that n and n±1 all
have primitive roots (the known such n are \{2, 3, 4, 5, 6, 10, 18, 26, 82, 242, 3^{541}-1\}), and
3^{541}–1 may be the largest such n, since it is widely believed that there are only finitely many
integers n≥1 such that the given pair of exponential sequences both produce primes:
(2*3^n-1, 2*3^n+1), ((3^n+1)/2, 3^n+2), ((3^n-1)/2, 3^n-2), \text{ see https://oeis.org/A305237}, \text{ also it is}
widely believed that for any polynomial sequence and any exponential sequence, there are
only finitely many n such that both sequences produce primes, e.g. it is widely believed that
only finitely many Mersenne exponents (i.e. primes p such that 2^{p}-1 is also prime) are
Sophie Germain primes (such primes p are listed in <a href="https://oeis.org/A065406">https://oeis.org/A065406</a>), i.e. the
number of primes p such that 2^{*}p+1 and 2^{p}-1 are both prime is expected to be finite, also it
is widely believed that only finitely many Mersenne exponents (i.e. primes p such that 2^p-1 is
also prime) are members of twin primes pair (such primes p are listed in
https://oeis.org/A346645), see this post and this thread). For example, the base 11 family
57^n, this family have already been searched to length 50000 with no prime or PRP found,
however (we use the sense of <a href="http://www.iakovlev.org/zip/riesel2.pdf">http://www.iakovlev.org/zip/riesel2.pdf</a>,
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https://stdkmd.net/nrr/1/10003.htm#prime_period.

https://stdkmd.net/nrr/3/30001.htm#prime_period,

https://stdkmd.net/nrr/1/13333.htm#prime_period.

https://stdkmd.net/nrr/3/33331.htm#prime_period,

https://stdkmd.net/nrr/1/11113.htm#prime_period,

https://stdkmd.net/nrr/3/31111.htm#prime_period,

https://mersenneforum.org/showpost.php?p=138737&postcount=24,

https://mersenneforum.org/showpost.php?p=153508&postcount=147, to show this, i.e. this family (the base 11 family 57^n) cannot be ruled out as contain no primes > base, by covering congruence, algebraic factorization, or combine of them) the algebraic form of this family is $(57*11^n-7)/10$, and there is no n satisfying that $57*11^n$ and 7 are both r-th powers for some r>1 to make this number have difference-of-two-r-th powers factorization (since 7 is not perfect power), nor there is n satisfying that $57*11^n$ and -7 are (one is 4th power, another is of the form $4*m^4$) to make this number have Aurifeuillian factorization for x^4+4y^4 (since -7 is neither 4th power nor of the form $4*m^4$), thus, base 11 family 57^n has no algebraic factorization for any n, thus 57^n eventually should yield a prime unless it can be proven to contain no primes > base using covering congruence, and we have:

```
57^n is divisible by 2 for n == 1 \mod 2

57^n is divisible by 13 for n == 2 \mod 12

57^n is divisible by 17 for n == 4 \mod 16

57^n is divisible by 5 for n == 0 \mod 5

57^n is divisible by 23 for n == 6 \mod 22

57^n is divisible by 601 for n == 8 \mod 600
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57^n is divisible by 97 for n == 12 \mod 48
57^n is divisible by 1279 for n == 16 \mod 426
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(for the factorization of 57ⁿ in base 11, see factordb page)

and it does not appear to be any covering set of primes (and its Nash weight (or difficulty) is positive, and it has prime candidate), so there must be a prime at some point.

The multiplicative order of b mod the primes is important in this problem, since if a prime p divides the number with *n* digits in a family in base *b*, then *p* also divides the number with k^*r+n digits in the same family in base b for all nonnegative integer k, where r is the multiplicative order of $b \mod p$ (unless the multiplicative order of $b \mod p$ is 1, i.e. p divides b-1, in this case p also divides the number with k^*p+n digits in the same family in base b for all nonnegative integer k), the primes p such that the multiplicative order of b mod p is n are exactly the primes p dividing Zs(n,b,1), where Zs is the Zsigmondy number, i.e. Zs(n,b,1) is the greatest divisor of b^n-1 that is coprime to b^m-1 for all positive integers m < n, with $b \ge 2$ and $n \ge 1$, if (and only if) there is only one such prime, then this prime is <u>unique prime</u> in base b, see list of the multiplicative order of b mod p for $b \le 128$ and primes $p \le 4096$, list of primes p such that the multiplicative order of b mod p is n for $2 \le b \le 64$ and $1 \le n \le 64$, smallest prime p such that znorder(Mod(m,p)) = (p-1)/n for $2 \le m \le 128$ and $1 \le n \le 128$, bases b such that Phi(n,b) (where Phi is cyclotomic polynomial) has algebra factors or small prime factors, bases b such that there is unique prime with period length n, unique period length in base b, also see factorization of $b^n\pm 1$ (which is equivalent to factorization of Zs(n,b,1)) with $b\le 12$ $13 \le b \le 99$ b=10 any b any b, also see this page and this page.

(these references only include the multiplicative order of the base (b) mod the primes (i.e. <u>znorder(Mod(b,p))</u> with prime p), if you want to calculate the <u>multiplicative order</u> of the base (b) mod a composite number c coprime to b, factor c to product of distinct prime powers, and calculate the <u>multiplicative order</u> of $b \mod p^e$ (i.e. <u>znorder</u>($Mod(b,p^e)$)) for all these prime powers p^e , and znorder(Mod(b, p^e)) = $p^{max(e-r(b,p),0)*}z$ norder(Mod(b,p)), where r(b,p) is the largest integer s such that p^s divides $p^{p-1}-1$, the primes p such that r(b,p) > 1 are called generalized Wieferich prime base b, and if r(p,q) and r(q,p) are both > 1 for primes p and q, then (p,q) are called Wieferich pair, there are currently only 7 known such (p,q) pairs: (2, 1093), (3, 1006003), (5, 1645333507), (5, 188748146801), (83, 4871), (911, 318917), (2903, 18787), primes p such that r(b,p) > 1 for b = the smallest primitive root mod p (A001918(n), if p is the n-th prime) are called non-generous primes (https://oeis.org/A055578), there are currently only 3 known such primes p: 2, 40487, 6692367337, generalized Wieferich primes and Wieferich pairs are related to Fermat Last Theorem and abc conjecture and Catalan <u>conjecture</u>, and for the values of r(b,p) see http://www.fermatquotient.com/FermatQuotienten/FermQ_Sort.txt and http://www.fermatguotient.com/FermatQuotienten/FermQ Sorg.txt and http://www.asahinet.or.jp/~KC2H-MSM/mathland/math11/fer quo.htm and https://archive.fo/Hd9Rr and http://www.sci.kobe-u.ac.jp/old/seminar/pdf/2008_yamazaki.pdf, data is available for primes $p \le \text{search limit in these pages, for a base } b$, if p is not list here then r(b,p) = 1, if p is list here with no exponent given then r(b,p) = 2, if p is list here with an exponent given then r(b,p) = 2this exponent, <u>perfect power</u> bases are not listed in these two pages, and $r(b^m, p) = p^{s*}r(b, p)$ if p is odd prime, where s is the largest nonnegative integer such that p^s divides m, $r(b^m, 2) =$

largest nonnegative integer s such that 2^s divides b^m-1 , finally, calculate the <u>least common multiple</u> of these multiplicative orders of $b \mod p^e$) (references: http://go.helms-net.de/math/expdioph/fermatquotients.pdf)

The smallest Wieferich primes in base b for b = 2, 3, 4, ... 36 are 1093, 11, 1093, 2, 66161, 5, 3, 2, 3, 71, 2693, 2, 29, 29131, 1093, 2, 5, 3, 281, 2, 13, 13, 5, 2, 3, 11, 3, 2, 7, 7, 5, 2, 46145917691, 3, 66161 (*OEIS* sequence A039951)

The smallest base such that p is a Wieferich prime for the first 100 primes p (i.e. p = 2, 3, 5, 7, ..., 541) are 5, 8, 7, 18, 3, 19, 38, 28, 28, 14, 115, 18, 51, 19, 53, 338, 53, 264, 143, 11, 306, 31, 99, 184, 53, 181, 43, 164, 96, 68, 38, 58, 19, 328, 313, 78, 226, 65, 253, 259, 532, 78, 176, 276, 143, 174, 165, 69, 330, 44, 33, 332, 94, 263, 48, 79, 171, 747, 731, 20, 147, 91, 40, 1260, 104, 707, 18, 476, 75, 223, 14, 257, 159, 242, 174, 1259, 632, 175, 280, 751, 369, 251, 867, 349, 194, 590, 210, 735, 52, 255, 863, 583, 10, 753, 346, 1449, 93, 308, 241, 555 (*OEIS* sequence A039678)

b	known generalized Wieferich primes base b (written in base 10) (search limit: $6*10^{17}$ for $b=2$ (and hence $b=4$, 8, 16, 32), $1.2*10^{15}$ for $b=3$, 5, 7 (and hence $b=9$, 25, 27), $1.479*10^{14}$ for other b)	OEIS sequence
2	1093, 3511	A001220
<u>3</u>	11, 1006003	A014127
4	1093, 3511	essentially the same as A001220, since $4 = 2^2$ (2 divides 2, thus no need to add this prime)
<u>5</u>	2, 20771, 40487, 53471161, 1645333507, 6692367337, 188748146801	A123692
<u>6</u>	66161, 534851, 3152573	A212583
<u>7</u>	5, 491531	<u>A123693</u>
8	3, 1093, 3511	essentially the same as A001220 plus the prime 3, since 8 = 2 ³
9	2 (order 2), 11, 1006003	essentially the same as $\frac{A014127}{2}$ plus the prime 2, since $9 = 3^2$

<u>10</u>	3, 487, 56598313	A045616
<u>11</u>	71	
<u>12</u>	2693, 123653	A111027
<u>13</u>	2, 863, 1747591	<u>A128667</u>
<u>14</u>	29, 353, 7596952219	A234810
<u>15</u>	29131, 119327070011	A242741
<u>16</u>	1093, 3511	essentially the same as A001220, since 16 = 2 ⁴ (2 divides 2, thus no need to add this prime)
17	2 (order 3), 3, 46021, 48947, 478225523351	A128668
<u>18</u>	5, 7 (order 2), 37, 331, 33923, 1284043	A244260
19	3, 7 (order 2), 13, 43, 137, 63061489	A090968
<u>20</u>	281, 46457, 9377747, 122959073	A242982
21	2	
22	13, 673, 1595813, 492366587, 9809862296159	A298951
23	13, 2481757, 13703077, 15546404183, 2549536629329	A128669
<u>24</u>	5, 25633	
25	2 (order 2), 20771, 40487, 53471161, 1645333507, 6692367337, 188748146801	essentially the same as A123692, since 25 = 5 ² (2 is already a Wieferich prime base 5)
<u>26</u>	3 (order 2), 5, 71, 486999673, 6695256707	<u>A306255</u>
<u>27</u>	11, 1006003	essentially the same as A014127, since 27 = 3³ (3 divides 3, thus no need to add this prime)
28	3 (order 2), 19, 23	

29	2	
30	7, 160541, 94727075783	<u>A306256</u>
31	7, 79, 6451, 2806861	<u>A331424</u>
<u>32</u>	5, 1093, 3511	essentially the same as A001220 plus the prime 5, since 32 = 2 ⁵
33	2 (order 4), 233, 47441, 9639595369	
34	46145917691	
35	3, 1613, 3571	
<u>36</u>	66161, 534851, 3152573	essentially the same as A212583, since $36 = 6^2$ (2 divides 6, thus no need to add this prime)

The numbers in simple families are of the form $\frac{a \cdot b^{-n} + c}{gcd(a+c,b-1)}$ for some fixed integers a, b, cwhere $a \ge 1$, $b \ge 2$ (b is the base), $c \ne 0$, gcd(a,c)=1, gcd(b,c)=1 (thus, all large minimal primes base b (but possible not all minimal primes base b if b is large, e.g. b = 25, 29, 31, 35) have a nice short algebraic description (see this page and this page, the prime numbers in these two pages do not have nice short algebraic descriptions, also see this page) and have simple expression (expression with ≤ 40 characters, all taken from "0" "1" "2" "3" "4" "5" "6" "7" "8" "9" "+" "-" "*" "/" "^" "(" ")"), factorial (!), double factorial (!!), and primorial (#) are not allowed since they can be used to ensure many small factors, see this page). Except in the special case $c = \pm 1$ and gcd(a+c,b-1) = 1, when n is large the known primality tests for such a number are too inefficient to run (since this special case $c = \pm 1$ and gcd(a+c,b-1) = 1 is the only case which N-1 and/or N+1 is smooth, i.e. the case c = 1 and gcd(a+c,b-1) = 1(corresponding to generalized Proth prime base b: a^*b^n+1 , they are related to generalized Sierpinski conjecture base b) can be easily proven prime using Pocklington N-1 method, and the case c = -1 and qcd(a+c,b-1) = 1 (corresponding to generalized Riesel prime base b: a^*b^n-1 , they are related to generalized Riesel conjecture base b) can be easily proven prime using Morrison N+1 method) (reference of primality tests). In this case one must resort to a probable primality test such as a Miller-Rabin primality test or a Baillie-PSW primality test, unless a divisor of the number can be found, and thus these numbers cannot be definitely primes and can only be probable primes, and we cannot definitely say that the corresponding families can be removed from the list of unsolved families, and we cannot definitely compute this part of the set $M(L_b)$. Since we are testing many numbers in an exponential sequence, it is possible to use a sieving process (such as srsieve software) to find divisors rather than using trial division, i.e. we will remove the integers n such that

 $\frac{a \cdot b^{-n} + c}{gcd(a+c,b-1)}$ either has a <u>prime factor</u> less than certain limit (say 2³²) or has algebraic factorization, and <u>test the primality</u> of $\frac{a \cdot b^{-n} + c}{gcd(a+c,b-1)}$ for other integers n.

To do this, we made use of Geoffrey Reynolds' <u>srsieve</u> software. This program uses the <u>baby-step giant-step algorithm</u> to find all primes p which divide a^*b^n+c where p and n lie in a specified <u>range</u>. Since this program cannot handle the <u>general case</u> $\frac{a \cdot b^{-n}+c}{gcd(a+c,b-1)}$ when gcd(a+c,b-1) > 1 we only used it to sieve the sequence a^*b^n+c for primes p not dividing gcd(a+c,b-1), and initialized the list of candidates to not include p for which there is some prime p dividing gcd(a+c,b-1) for which p divides $\frac{a \cdot b^{-n}+c}{gcd(a+c,b-1)}$. The program had to be modified slightly to remove a check which would prevent it from running in the case when p, and p were all p odd (since then 2 divides p out 2 may not divide p odd p odd p.

Once the numbers with small divisors had been removed, it remained to test the remaining numbers using a probable primality test. For this we used the software LLR by Jean Penné. Although undocumented, it is possible to run this program on numbers of the form $\frac{a \cdot b - n + c}{gcd(a + c, b - 1)}$ when gcd(a + c, b - 1) > 1, so this program required no modifications (also, LLR) can do a proven primality test (i.e. prove the primality) for numbers of the form $a^*b^n\pm 1$ (i.e. the special case $c = \pm 1$ and gcd(a+c,b-1) = 1) with $b^n > a$). A script was also written which allowed one to run srsieve while LLR was testing the remaining candidates, so that when a divisor was found by srsieve on a number which had not yet been tested by LLR it would be removed from the list of candidates. In the cases where the elements of $M(L_b)$ could be proven prime rigorously, we employed <u>PRIMO</u> by Marcel Martin, an <u>elliptic curve primality</u> <u>proving</u> implementation (for the primes of the form $\frac{a \cdot b^{-n} + c}{gcd(a+c,b-1)}$, with $c \neq \pm 1$ and/or $qcd(a+c,b-1) \neq 1$, we cannot use the classical tests (including the tests of N-1, N+1, N+1, N^2+N+1 , N^2-N+1 (all such polynomials are cyclotomic polynomials of N, and such tests are called cyclotomy proofs, see this page), and the combined tests), since for these primes, none of them is at least 1/3 factorable (Brillhart-Lehmer-Selfridge primality test) (see this page) (if we want to use the classical tests to prove the primality of N, then we must factor at least one of N-1, N+1, N+1, N+N+1, N-N+1 to the factored part \geq 33.3333% (i.e. product of known prime factors ≥ the cube root of N), and except trial division with the primes up to certain limit (say 2⁶⁴) and the algebra factors (e.g. difference-of-two-squares factorization, sum/difference-of-two-cubes factorization, Aurifeuillian factorization, and algebra factors of the Cunningham number $b^n \pm 1$ ($b^n - 1$ can be factored to product of $\Phi_d(b)$ with d dividing n. and b^n+1 can be factored to product of $\Phi_o(b)$ with d dividing 2n but not dividing n, where Φ is the cyclotomic polynomial), see this reference and this reference), we can use elliptic-curve factorization method (ECM), Pollard P-1 method, Williams P+1 method, special number field sieve (SNFS), general number field sieve (GNFS), etc. to factor the numbers (see this reference), however, all these factorization algorithms take long time, i.e. they cannot be done in polynomial time (unlike primality proving, when the numbers are sufficiently large, no efficient, non-quantum integer factorization algorithm is known, i.e. integer factorization may be NP-complete. However, it has not been proven that no efficient algorithm exists. Also, many OEIS sequences need factors, see https://oeis.org/wiki/OEIS_sequences_needing_factors. The presumed difficulty of this

problem is at the heart of widely used algorithms in cryptography such as RSA, there are many large semiprimes, called RSA numbers, which are very hard to factor and are part of the RSA Factoring Challenge. Besides, integer factorization can be used for public-key cryptography is because it has no known polynomial time algorithm. Many areas of mathematics and computer science have been brought to bear on the problem, including elliptic curves, algebraic number theory, and quantum computing), and hence to do this is impractically), i.e. they are ordinary primes, and if the prime is not large (say less than 10²⁵⁰⁰⁰), we can use elliptic curve primality proving (ECPP) to proof (see *PRIMO* top 20 records and elliptic curve primality proving top 20 records and top primes proven by Francois Morain's programs) and make primality certificate, but if the prime is very large (say > 10²⁵⁰⁰⁰), the known <u>primality tests</u> for such a number are too inefficient to run (although there is AKS primality test, which can prove the primality of any general prime in polynomial time, but since its time complexity is $O(In(N)^{12})$, and if we can do 10^9 bitwise operations per second, use this test to prove the primality of a 5000-digit (in decimal) prime need 5.422859049×10³⁹ seconds, or 1.719577324×10³² years, much longer than the age of the universe, thus to do this test is still impractically), thus we can only resort to a probable primality test such as Miller-Rabin primality test and Baillie-PSW primality test, unless a divisor of the number can be found, and hence we cannot prove the primality of this number, thus we cannot definitely say that the corresponding families can be removed from the list of unsolved families, and we cannot definitely <u>compute</u> this part of the <u>sets</u> $M(L_b)$.

Fermat pseudoprime (to base <i>b</i> = 2: https://oeis.org/A001567 , and see this data)	Lucas pseudoprime (to parameters (<i>P</i> , <i>Q</i>) = (1, -1): https://oeis.org/A081264 union https://oeis.org/A141137 , and see this data) (to parameters (<i>P</i> , <i>Q</i>) defined by Selfridge's Method <i>A</i> : https://oeis.org/A217120 , and see this data)
Strong Fermat pseudoprime (to base $b = 2$: https://oeis.org/A001262 , and see this data)	Strong Lucas pseudoprime (to parameters (<i>P</i> , <i>Q</i>) defined by Selfridge's Method <i>A</i> : https://oeis.org/A217255 , and see this data)
Over Fermat pseudoprime (to base $b = 2$: composite factors of $\underbrace{A019320(n)}_{gcd}(\underline{A019320(n)}, n) = \underbrace{A064078(n)}_{n}$ for some n , there is an OEIS sequence: $\underline{https://oeis.org/A141232}$	Over Lucas pseudoprime (to parameters (P , Q) = (1, -1): composite factors of $A061446(n)$ / $gcd(A061446(n), n)$ = $A178763(n)$ for some n)
Smallest <i>n</i> such that a given prime <i>p</i> divides 2 ⁿ -1: https://oeis.org/A014664	Smallest <i>n</i> such that a given prime <i>p</i> divides <i>Fibonacci</i> (<i>n</i>): https://oeis.org/A001177
Numbers <i>n</i> such that 2 ⁿ -1 is prime: https://oeis.org/A000043	Numbers <i>n</i> such that <i>Fibonacci</i> (<i>n</i>) is prime: https://oeis.org/A001605
Numbers <i>n</i> such that (2 ⁿ +1)/3 is prime: https://oeis.org/A000978	Numbers <i>n</i> such that <i>Lucas</i> (<i>n</i>) is prime: https://oeis.org/A001606
Numbers <i>n</i> such that 2 ⁿ -1 and (2 ⁿ +1)/3 are both primes: https://oeis.org/A107360	Numbers <i>n</i> such that <i>Fibonacci</i> (<i>n</i>) and <i>Lucas</i> (<i>n</i>) are both primes: https://oeis.org/A080327

Numbers n such that $A019320(n)$ / $gcd(A019320(n), n) = A064078(n)$ is prime: https://oeis.org/A161508	Numbers n such that $\underline{A061446}(n)$ / $gcd(\underline{A061446}(n), n) = \underline{A178763}(n)$ is prime: https://oeis.org/A152012	
<u>Unique primes</u> in base 2: <u>https://oeis.org/A144755</u> (exactly the primes dividing no over Fermat pseudoprime (to base $b = 2$)	Prime Fibonacci integers: https://oeis.org/A178762 (exactly the primes dividing no over Lucas pseudoprime (to parameters (<i>P</i> , <i>Q</i>) = (1, -1)	
Primes with primitive root 2: https://oeis.org/A001122	Primes with Fibonacci primitive root: https://oeis.org/A214029	
$\frac{\text{Cyclotomic polynomial }(\text{A019320}(n) = Phi(n,2), \ \underline{\text{A019321}}(n) = Phi(n,3), \ \ldots)$	Fibcyclotomic polynomial (A061446(n) = FibPhi(n ,1), A008555(n) = FibPhi(n ,2),)	
Baillie–PSW pseudoprime (none are known, none < 2 ⁶⁴ exist)		
Carmichael number (https://oeis.org/A002997)	Lucas-Carmichael number (https://oeis.org/A006972)	
Euler's totient function (https://oeis.org/A000010)	Dedekind psi function (https://oeis.org/A001615)	
Range of <u>Euler's totient function</u> (https://oeis.org/A002202), also even nontotient numbers (https://oeis.org/A005277)	Range of <u>Dedekind psi function</u> (https://oeis.org/A203444), also even non-Dedekind numbers (https://oeis.org/A307055)	
Pépin primality test (for Fermat numbers, i.e. numbers of the form 2 ⁿ +1 (https://oeis.org/A000051), if 2 ⁿ +1 is prime, then <i>n</i> must be power of 2, such primes are https://oeis.org/A019434)	Lucas—Lehmer primality test (for Mersenne numbers, i.e. numbers of the form 2 ⁿ -1 (https://oeis.org/A000225), if 2 ⁿ -1 is prime, then <i>n</i> must be prime, such primes are https://oeis.org/A000668)	
https://oeis.org/A060377 (Pépin primality test numbers)	https://oeis.org/A003010 (Lucas–Lehmer primality test numbers)	
https://oeis.org/A152153 (Residues of Pépin primality test for Fermat numbers)	https://oeis.org/A095847 (Residues of Lucas—Lehmer primality test for Mersenne numbers)	
https://oeis.org/A129802 (Possible bases for Pépin primality test for Fermat numbers, the original base for Pépin primality test is 3)	https://oeis.org/A018844 (Possible starting values for <u>Lucas—Lehmer primality test</u> for Mersenne numbers, the original starting value for <u>Lucas—Lehmer primality test</u> is 4)	
Proth primality test (for numbers of the form $k*2^n+1$ with k odd and $k<2^n$, i.e. Proth numbers, https://oeis.org/A080075, such primes are https://oeis.org/A080076, also there is a list of such primes sorted by k)	Lucas—Lehmer—Riesel primality test (for numbers of the form k^*2^n –1 with k odd and $k<2^n$, i.e. Proth numbers of the second kind, https://oeis.org/A112714 , such primes are https://oeis.org/A112715 , also there is a list of such primes sorted by k)	
Sierpiński problem (finding and proving the smallest odd <i>k</i> such that <i>k</i> *2 ⁿ +1 is	Riesel problem (finding and proving the smallest odd <i>k</i> such that <i>k</i> *2 ^{<i>n</i>} -1 is	

composite for all $n \ge 1$, the smallest such k is conjectured to be 78557, such k are called Sierpiński numbers, see https://oeis.org/A076336 , also there is a list of primes of the form $k \ge n + 1$ for odd k)	composite for all $n \ge 1$, the smallest such k is conjectured to be 509203, such k are called Riesel numbers, see https://oeis.org/A101036 , also there is a list of primes of the form $k \ge n-1$ for odd k)	
Pocklington <u>N-1 primality test</u> (for numbers <i>n</i> such that <i>n</i> -1 can be trivially fully factored)	Morrison <u>N+1 primality test</u> (for numbers <i>n</i> such that <i>n</i> +1 can be trivially fully factored)	
Generalized Sierpiński problems to bases $b > 2$ (finding and proving the smallest k such that k^*b^n+1 is composite for all $n \ge 1$)	Generalized Riesel problems to bases $b > 2$ (finding and proving the smallest k such that k * b ^{n} -1 is composite for all n \ge 1)	
Combined $N-1$ / $N+1$ primality test (and other cyclotomy tests, i.e. $\Phi_r(N)$ for small r (where Φ is the cyclotomic polynomial), including N^2+1 , N^2+N+1 , N^2-N+1)		
Pollard P-1 integer factorization method	Williams P+1 integer factorization method	

Some families xy^*z could not be ruled out as containing no primes > base, but no primes > base could be found in the family, even after searching through numbers with over 50000 digits. Many xy^*z families contain no small primes even though they do contain very large primes, for example: (show the factordb link for the list of the factors of numbers in these families, like https://stdkmd.net/nrr/1/10003.htm#prime_period,

https://stdkmd.net/nrr/3/30001.htm#prime_period,

https://stdkmd.net/nrr/1/13333.htm#prime_period,

https://stdkmd.net/nrr/3/33331.htm#prime_period,

https://stdkmd.net/nrr/1/11113.htm#prime_period,

https://stdkmd.net/nrr/3/31111.htm#prime_period)

^{*} In base 5, the smallest prime in the family $10^n 13$ (algebraic form: $5^{n+2} + 8$) ($n \ge 0$) is $10^{93} 13$ (algebraic form: $5^{95} + 8$) (factordb)

^{*} In base 8, the smallest prime in the family 4^n 7 (algebraic form: $(4*8^{n+1}+17)/7$) ($n \ge 1$) is 4^{220} 7 (algebraic form: $(4*8^{221}+17)/7$) (the prime 7 (i.e. n = 0) is not counted since the prime must be > base) (<u>factordb</u>)

^{*} In base 9, the smallest prime in the family 30^n11 (algebraic form: $3*9^{n+2}+10$) ($n \ge 0$) is $30^{1158}11$ (algebraic form: $3*9^{1160}+10$) (<u>factordb</u>) (<u>certificate of this prime</u>)

^{*} In base 9, the smallest prime in the family 27^n07 (algebraic form: $(23*9^{n+2}-511)/8$) ($n \ge 0$) is $27^{686}07$ (algebraic form: $(23*9^{688}-511)/8$) ($\frac{1}{2}$) (certificate of this prime)

^{*} In base 9, the smallest prime in the family $76^{n}2$ (algebraic form: $(31*9^{n+1}-19)/4$) ($n \ge 0$) is $76^{329}2$ (algebraic form: $(31*9^{330}-19)/4$ (<u>factordb</u>) (<u>certificate of this prime</u>)

^{*} In base 11, family 57^n (algebraic form: $(57*11^n-7)/10$) ($n\ge 1$) cannot be ruled out as containing no primes > base (using covering congruence, algebra factorization, or combine of them) but no primes > base found in the family after searching to length 50000 (the prime 5 (i.e. n = 0) is not counted since the prime must be > base) (<u>factordb</u>)

^{*} In base 13, the smallest prime in the family 80^n 111 (algebraic form: $8*13^{n+3}+183$) ($n \ge 0$) is 80^{32017} 111 (algebraic form: $8*13^{32020}+183$) (this prime is only a probable prime, i.e. not definitely prime) (factordb)

- * In base 13, the smallest prime in the family $2B30^n1$ (algebraic form: $484*13^{n+1}+1$) ($n \ge 0$) is $2B30^{15197}1$ (algebraic form: $484*13^{15198}+1$) (<u>factordb</u>) (this prime can be easily proven prime using the n-1 test, since $n-1=2^2*13^{15198}$ is trivially 100% factored)
- * In base 13, the smallest prime in the family B0ⁿBBA (algebraic form: $11*13^{n+3}+2012$) ($n \ge 0$) is B0⁶⁵⁴⁰BBA (algebraic form: $11*13^{6543}+2012$) (this prime is only a probable prime, i.e. not definitely prime) (factordb)
- * In base 13, the smallest prime in the family 390^n 1 (algebraic form: $48*13^{n+1}+1$) ($n\ge 0$) is 390^{6266} 1 (algebraic form: $48*13^{6267}+1$) (factordb) (this prime can be easily proven prime using the n-1 test, since $n-1=2^4*3*13^{6267}$ is trivially 100% factored)
- * In base 13, the smallest prime in the family 720^n2 (algebraic form: $93*13^{n+1}+2$) ($n \ge 0$) is $720^{2297}2$ (algebraic form: $93*13^{2298}+2$) (factordb)
- * In base 13, family 95^n (algebraic form: $(113*13^n-5)/12$) ($n \ge 1$) cannot be ruled out as containing no primes > base (using covering congruence, algebra factorization, or combine of them) but no primes > base found in the family after searching to length 50000 (<u>factordb</u>)
- * In base 13, family A3ⁿA (algebraic form: $(41*13^{n+1}+27)/4$) ($n \ge 0$) cannot be ruled out as containing no primes > base but no primes > base found in the family after searching to length 50000 (factordb)
- * In base 14, the smallest prime in the family $4D^n$ (algebraic form: $5*14^n-1$) ($n \ge 1$) is $4D^{19698}$ (algebraic form: $5*14^{19698}-1$) (factordb) (this prime can be easily proven prime using the n+1 test, since $n+1 = 2^{19698} * 5 * 7^{19698}$ is trivially 100% factored)
- * In base 16, family 3^n AF (algebraic form: $(16^{n+2}+619)/5$) ($n \ge 0$) cannot be ruled out as containing no primes > base (using covering congruence, algebra factorization, or combine of them) but no primes > base found in the family after searching to length 50000 (<u>factordb</u>)
- * In base 16, family 4^nDD (algebraic form: $(4*16^{n+2}+2291)/15$) ($n \ge 0$) cannot be ruled out as containing no primes > base (using covering congruence, algebra factorization, or combine of them) but no primes > base found in the family after searching to length 50000 (factordb)
- * In base 16, the smallest prime in the family DBⁿ (algebraic form: $(206*16^n-11)/15$) ($n\ge1$) is DB³²²³⁴ (algebraic form: $(206*16^{32234}-11)/15$) (this prime is only a probable prime, i.e. not definitely prime) (the prime D (i.e. n=0) is not counted since the prime must be > base) (factordb)
- * In base 16, the smallest prime in the family 5BC n D (algebraic form: (459*16 $^{n+1}$ +1)/5) (n≥0) is 5BC 3700 D (algebraic form: (459*16 3701 +1)/5) (factordb) (certificate of this prime)
- * In base 16, the smallest prime in the family 90^n91 (algebraic form: $9*16^{n+2}+145$) ($n \ge 0$) is $90^{3542}91$ (algebraic form: $9*16^{3544}+145$) (factordb) (certificate of this prime)
- * In base 16, the smallest prime in the family F8ⁿF (algebraic form: $(233*16^{n+1}+97)/15$) ($n \ge 0$) is F8¹⁵¹⁷F (algebraic form: $(233*16^{1518}+97)/15$) (factordb) (certificate of this prime)
- * In base 16, the smallest prime in the family D9ⁿ (algebraic form: $(68*16^n-3)/5$) ($n\ge 1$) is D9¹⁰⁵² (algebraic form: $(68*16^{1052}-3)/5$) (<u>factordb</u>) (<u>certificate of this prime</u>)
- * In base 16, the smallest prime in the family $88F^n$ (algebraic form: $137*16^n-1$) ($n \ge 0$) is $88F^{545}$ (algebraic form: $137*16^{545}-1$) (<u>factordb</u>) (this prime can be easily proven prime using the n+1 test, since $n+1 = 2^{2180} * 137$ is trivially 100% factored)
- * In base 16, the smallest prime in the family $5F^n6F$ (algebraic form: $6*16^{n+2}-145$) ($n \ge 0$) is $5F^{544}6F$ (algebraic form: $6*16^{546}-145$) (<u>factordb</u>) (<u>certificate of this prime</u>)

For any given base b, we find all (x,z) digits-pair such that $x \ne 0$ and gcd(z,b) = 1, and find the corresponding sets Y^* , see below.

Bold for minimal primes in base b, i.e. elements of the set $M(L_b)$

base 2

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1)
```

- * Case (1,1):
- ** 11 is prime, and thus the only minimal prime in this family.

base 3

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,2), (2,1), (2,2)
```

- * Case (1,1):
- ** Since 12, 21, **111** are primes, we only need to consider the family 1{0}1 (since any digits 1, 2 between them will produce smaller primes)
- *** All numbers of the form 1{0}1 are divisible by 2, thus cannot be prime.
- * Case (1,2):
- ** 12 is prime, and thus the only minimal prime in this family.
- * Case (2,1):
- ** 21 is prime, and thus the only minimal prime in this family.
- * Case (2,2):
- ** Since 21, 12 are primes, we only need to consider the family 2{0,2}2 (since any digits 1 between them will produce smaller primes)
- *** All numbers of the form 2{0,2}2 are divisible by 2, thus cannot be prime.

base 4

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,3), (2,1), (2,3), (3,1), (3,3)
```

* Case (1,1):

** 11 is prime, and thus the only minimal prime in this family. * Case (1,3): ** **13** is prime, and thus the only minimal prime in this family. * Case (2,1): ** Since 23, 11, 31, 221 are primes, we only need to consider the family 2{0}1 (since any digits 1, 2, 3 between them will produce smaller primes) *** All numbers of the form 2{0}1 are divisible by 3, thus cannot be prime. * Case (2,3): ** 23 is prime, and thus the only minimal prime in this family. * Case (3,1): ** 31 is prime, and thus the only minimal prime in this family. * Case (3,3): ** Since 31, 13, 23 are primes, we only need to consider the family 3(0,3)3 (since any digits 1, 2 between them will produce smaller primes) *** All numbers of the form 3{0,3}3 are divisible by 3, thus cannot be prime. base 5 The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are: (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)* Case (1,1): ** Since 12, 21, 111, 131 are primes, we only need to consider the family 1{0,4}1 (since any digits 1, 2, 3 between them will produce smaller primes) *** All numbers of the form 1{0,4}1 are divisible by 2, thus cannot be prime. * Case (1,2):

** 12 is prime, and thus the only minimal prime in this family.

* Case (1,3):

- ** Since 12, 23, 43, **133** are primes, we only need to consider the family 1{0,1}3 (since any digits 2, 3, 4 between them will produce smaller primes)
- *** Since 111 is prime, we only need to consider the families 1{0}3 and 1{0}1{0}3 (since any digit combo 11 between (1,3) will produce smaller primes)
- **** All numbers of the form 1{0}3 are divisible by 2, thus cannot be prime.
- **** For the 1{0}1{0}3 family, since **10103** is prime, we only need to consider the families 1{0}13 and 11{0}3 (since any digit combo 010 between (1,3) will produce smaller primes)
- ***** The smallest prime of the form 1{0}13 is

- ***** All numbers of the form 11{0}3 are divisible by 3, thus cannot be prime.
- * Case (1,4):
- ** Since 12, 34, **104** are primes, we only need to consider the family 1{1,4}4 (since any digits 0, 2, 3 between them will produce smaller primes)
- *** Since 111, 414 are primes, we only need to consider the families 1{4}4 and 11{4}4 (since any digit combo 11 or 41 between them will produce smaller primes)
- **** The smallest prime of the form 1{4}4 is 14444.
- **** All numbers of the form 11{4}4 are divisible by 2, thus cannot be prime.
- * Case (2,1):
- ** 21 is prime, and thus the only minimal prime in this family.
- * Case (2,2):
- ** Since 21, 23, 12, 32 are primes, we only need to consider the family 2{0,2,4}2 (since any digits 1, 3 between them will produce smaller primes)
- *** All numbers of the form 2{0,2,4}2 are divisible by 2, thus cannot be prime.
- * Case (2,3):
- ** 23 is prime, and thus the only minimal prime in this family.
- * Case (2,4):
- ** Since 21, 23, 34 are primes, we only need to consider the family 2{0,2,4}4 (since any digits 1, 3 between them will produce smaller primes)
- *** All numbers of the form 2{0,2,4}4 are divisible by 2, thus cannot be prime.

- * Case (3,1):
- ** Since 32, 34, 21 are primes, we only need to consider the family 3{0,1,3}1 (since any digits 2, 4 between them will produce smaller primes)
- *** Since 313, 111, 131, **3101** are primes, we only need to consider the families 3{0,3}1 and 3{0,3}11 (since any digit combo 10, 11, 13 between (3,1) will produce smaller primes)
- **** For the 3{0,3}1 family, we can separate this family to four families:
- ***** For the 30{0,3}01 family, we have the prime **30301**, and the remain case is the family 30{0}01.
- ****** All numbers of the form 30{0}01 are divisible by 2, thus cannot be prime.
- ***** For the 30{0,3}31 family, note that there must be an even number of 3's between (30,31), or the result number will be divisible by 2 and cannot be prime.
- ****** Since 33331 is prime, any digit combo 33 between (30,31) will produce smaller primes.
- ******* Thus, the only possible prime is the smallest prime in the family 30{0}31, and this prime is **300031**.
- ***** For the 33{0,3}01 family, note that there must be an even number of 3's between (33,01), or the result number will be divisible by 2 and cannot be prime.
- ****** Since 33331 is prime, any digit combo 33 between (33,01) will produce smaller primes.
- ******* Thus, the only possible prime is the smallest prime in the family 33{0}01, and this prime is **33001**.
- ***** For the 33(0,3)31 family, we have the prime 33331, and the remain case is the family 33(0)31.
- ****** All numbers of the form 33{0}31 are divisible by 2, thus cannot be prime.
- **** All numbers of the form 3{0,3}11 are divisible by 3, thus cannot be prime.
- * Case (3,2):
- ** 32 is prime, and thus the only minimal prime in this family.
- * Case (3,3):
- ** Since 32, 34, 23, 43, **313** are primes, we only need to consider the family 3{0,3}3 (since any digits 1, 2, 4 between them will produce smaller primes)
- *** All numbers of the form 3{0,3}3 are divisible by 3, thus cannot be prime.
- * Case (3,4):
- ** **34** is prime, and thus the only minimal prime in this family.

- * Case (4,1):
- ** Since 43, 21, **401** are primes, we only need to consider the family 4{1,4}1 (since any digits 0, 2, 3 between them will produce smaller primes)
- *** Since 414, 111 are primes, we only need to consider the families 4{4}1 and 4{4}11 (since any digit combo 14 or 11 between them will produce smaller primes)
- **** The smallest prime of the form 4{4}1 is 44441.
- **** All numbers of the form 4{4}11 are divisible by 2, thus cannot be prime.
- * Case (4,2):
- ** Since 43, 12, 32 are primes, we only need to consider the family 4{0,2,4}2 (since any digits 1, 3 between them will produce smaller primes)
- *** All numbers of the form 4{0,2,4}2 are divisible by 2, thus cannot be prime.
- * Case (4,3):
- ** 43 is prime, and thus the only minimal prime in this family.
- * Case (4,4):
- ** Since 43, 34, **414** are primes, we only need to consider the family 4{0,2,4}4 (since any digits 1, 3 between them will produce smaller primes)
- *** All numbers of the form 4{0,2,4}4 are divisible by 2, thus cannot be prime.

base 6

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

$$(1,1), (1,5), (2,1), (2,5), (3,1), (3,5), (4,1), (4,5), (5,1), (5,5)$$

- * Case (1,1):
- ** 11 is prime, and thus the only minimal prime in this family.
- * Case (1,5):
- ** 15 is prime, and thus the only minimal prime in this family.
- * Case (2,1):
- ** **21** is prime, and thus the only minimal prime in this family.
- * Case (2,5):

- ** 25 is prime, and thus the only minimal prime in this family.
- * Case (3,1):
- ** 31 is prime, and thus the only minimal prime in this family.
- * Case (3,5):
- ** 35 is prime, and thus the only minimal prime in this family.
- * Case (4,1):
- ** Since 45, 11, 21, 31, 51 are primes, we only need to consider the family 4{0,4}1 (since any digits 1, 2, 3, 5 between them will produce smaller primes)
- *** Since **4401** and **4441** are primes, we only need to consider the families 4{0}1 and 4{0}41 (since any digits combo 40 and 44 between them will produce smaller primes)
- **** All numbers of the form 4{0}1 are divisible by 5, thus cannot be prime.
- **** The smallest prime of the form 4{0}41 is 40041
- * Case (4,5):
- ** 45 is prime, and thus the only minimal prime in this family.
- * Case (5,1):
- ** **51** is prime, and thus the only minimal prime in this family.
- * Case (5,5):
- ** Since 51, 15, 25, 35, 45 are primes, we only need to consider the family 5{0,5}5 (since any digits 1, 2, 3, 4 between them will produce smaller primes)
- *** All numbers of the form 5{0,5}5 are divisible by 5, thus cannot be prime.

base 7

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)
```

- * Case (1,1):
- ** Since 14, 16, 41, 61, **131** are primes, we only need to consider the family 1{0,1,2,5}1 (since any digits 3, 4, 6 between them will produce smaller primes)

```
*** Since the digit sum of primes must be odd (otherwise the number will be divisible by 2, thus cannot
be prime), there is an odd total number of 1 and 5 in the {}
**** If there are >=3 number of 1 and 5 in the {}:
***** If there is 111 in the {}, then we have the prime 11111
***** If there is 115 in the {}, then the prime 115 is a subsequence
***** If there is 151 in the {}, then the prime 115 is a subsequence
***** If there is 155 in the {}, then the prime 155 is a subsequence
***** If there is 511 in the {}, then the current number is 15111, which has digit sum = 12, but digit sum
divisible by 3 will cause the number divisible by 3 and cannot be prime, and we cannot add more 1 or
5 to this number (to avoid 11111, 155, 515, 551 as subsequence), thus we must add at least one 2 to
this number, but then the number has both 2 and 5, and will have either 25 or 52 as subsequence,
thus cannot be minimal prime
***** If there is 515 in the {}, then the prime 515 is a subsequence
***** If there is 551 in the {}, then the prime 551 is a subsequence
***** If there is 555 in the {}, then the prime 551 is a subsequence
**** Thus there is only one 1 (and no 5) or only one 5 (and no 1) in the {}, i.e. we only need to consider
the families 1{0,2}1{0,2}1 and 1{0,2}5{0,2}1
***** For the 1{0,2}1{0,2}1 family, since 1211 is prime, we only need to consider the family 1{0}1{0,2}1
****** Since all numbers of the form 1{0}1{0}1 are divisible by 3 and cannot be prime, we only need to
consider the family 1{0}1{0}2{0}1
******* Since 11201 is prime, we only need to consider the family 1{0}1{0}21
******* The smallest prime of the form 11{0}21 is 1100021
******* All numbers of the form 101{0}21 are divisible by 5, thus cannot be prime
******* The smallest prime of the form 1001{0}21 is 100121
******** Since this prime has no 0 between 1{0}1 and 21, we do not need to consider more families
***** For the 1{0,2}5{0,2}1 family, since 25 and 52 are primes, we only need to consider the family
1{0}5{0}1
****** Since 1051 is prime, we only need to consider the family 15{0}1
****** The smallest prime of the form 15{0}1 is 150001
* Case (1,2):
```

- ** Since 14, 16, 32, 52 are primes, we only need to consider the family 1{0,1,2}2 (since any digits 3, 4, 5, 6 between them will produce smaller primes)
- *** Since 1112 and 1222 are primes, there is at most one 1 and at most one 2 in {}
- **** If there are one 1 and one 2 in {}, then the digit sum is 6, and the number will be divisible by 6 and cannot be prime.
- **** If there is one 1 but no 2 in {}, then the digit sum is 4, and the number will be divisible by 2 and cannot be prime.
- **** If there is no 1 but one 2 in $\{\}$, then the form is $1\{0\}2\{0\}2$
- ***** Since 1022 and 1202 are primes, we only need to consider the number 122
- ***** 122 is not prime.
- **** If there is no 1 and no 2 in {}, then the digit sum is 3, and the number will be divisible by 3 and cannot be prime.
- * Case (1,3):
- ** Since 14, 16, 23, 43, **113**, **133** are primes, we only need to consider the family 1{0,5}3 (since any digits 1, 2, 3, 4, 6 between them will produce smaller primes)
- *** Since 155 is prime, we only need to consider the family 1{0}3 and 1{0}5{0}3
- **** All numbers of the form 1{0}3 are divisible by 2, thus cannot be prime.
- **** All numbers of the form 1{0}5{0}3 are divisible by 3, thus cannot be prime.
- * Case (1,4):
- ** 14 is prime, and thus the only minimal prime in this family.
- * Case (1,5):
- ** Since 14, 16, 25, 65, **115**, **155** are primes, we only need to consider the family 1{0,3}5 (since any digits 1, 2, 4, 5, 6 between them will produce smaller primes)
- *** All numbers of the form 1{0,3}5 are divisible by 3, thus cannot be prime.
- * Case (1,6):
- ** 16 is prime, and thus the only minimal prime in this family.
- * Case (2,1):
- ** Since 23, 25, 41, 61, **221** are primes, we only need to consider the family 2{0,1}1 (since any digits 2, 3, 4, 5, 6 between them will produce smaller primes)

- *** Since 2111 is prime, we only need to consider the families 2{0}1 and 2{0}1{0}1
- **** All numbers of the form 2{0}1 are divisible by 3, thus cannot be prime.
- **** All numbers of the form 2{0}1{0}1 are divisible by 2, thus cannot be prime.
- * Case (2,2):
- ** Since 23, 25, 32, 52, **212** are primes, we only need to consider the family 2{0,2,4,6}2 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 2{0,2,4,6}2 are divisible by 2, thus cannot be prime.
- * Case (2,3):
- ** 23 is prime, and thus the only minimal prime in this family.
- * Case (2,4):
- ** Since 23, 25, 14 are primes, we only need to consider the family 2{0,2,4,6}4 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 2{0,2,4,6}4 are divisible by 2, thus cannot be prime.
- * Case (2,5):
- ** 25 is prime, and thus the only minimal prime in this family.
- * Case (2,6):
- ** Since 23, 25, 16, 56 are primes, we only need to consider the family 2{0,2,4,6}6 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 2{0,2,4,6}6 are divisible by 2, thus cannot be prime.
- * Case (3,1):
- ** Since 32, 41, 61 are primes, we only need to consider the family 3{0,1,3,5}1 (since any digits 2, 4, 6 between them will produce smaller primes)
- *** Since 551 is prime, we only need to consider the family $3\{0,1,3\}1$ and $3\{0,1,3\}5\{0,1,3\}1$ (since any digits combo 55 between (3,1) will produce smaller primes)
- **** For the 3{0,1,3}1 family, since **3031** and 131 are primes, we only need to consider the families 3{0,1}1 and 3{3}3{0,1}1 (since any digits combo 03, 13 between (3,1) will produce smaller primes, thus for the digits between (3,1), all 3's must be before all 0's and 1's, and thus we can let the red 3 in 3{3}3{0,1}1 be the rightmost 3 between (3,1), all digits before this 3 must be 3's, and all digits after this 3 must be either 0's or 1's)
- ***** For the 3{0,1}1 family:

```
****** If there are >=2 0's and >=1 1's between (3,1), then at least one of 30011, 30101, 31001 will be
a subsequence.
****** If there are no 1's between (3,1), then the form will be 3{0}1
****** All numbers of the form 3{0}1 are divisible by 2, thus cannot be prime.
****** If there are no 0's between (3,1), then the form will be 3{1}1
****** The smallest prime of the form 3{1}1 is 31111
****** If there are exactly 1 0's between (3,1), then there must be <3 1's between (3,1), or 31111 will
be a subsequence.
******* If there are 2 1's between (3,1), then the digit sum is 6, thus the number is divisible by 6 and
cannot be prime.
******* If there are 1 1's between (3,1), then the number can only be either 3101 or 3011
****** Neither 3101 nor 3011 is prime.
******* If there are no 1's between (3,1), then the number must be 301
****** 301 is not prime.
***** For the 3{3}3{0,1}1 family:
****** If there are at least one 3 between (3,3{0,1}1) and at least one 1 between (3{3}3,1), then 33311
will be a subsequence.
****** If there are no 3 between (3,3{0,1}1), then the form will be 33{0,1}1
******* If there are at least 3 1's between (33,1), then 31111 will be a subsequence.
******* If there are exactly 2 1's between (33,1), then the digit sum is 12, thus the number is divisible
by 3 and cannot be prime.
******* If there are exactly 1 1's between (33,1), then the digit sum is 11, thus the number is divisible
by 2 and cannot be prime.
****** If there are no 1's between (33,1), then the form will be 33{0}1
******* The smallest prime of the form 33{0}1 is 33001
****** If there are no 1 between (3{3}3,1), then the form will be 3{3}3{0}1
******* If there are at least 2 0's between (3{3}3,1), then 33001 will be a subsequence.
******* If there are exactly 1 0's between (3{3}3,1), then the form is 3{3}301
******* The smallest prime of the form 3{3}301 is 33333301
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******* If there are no 0's between (3{3}3,1), then the form is 3{3}31
**** For the 3{0,1,3}5{0,1,3}1 family, since 335 is prime, we only need to consider the family
3{0,1}5{0,1,3}1
***** Numbers containing 3 between (3{0,1}5,1):
****** The form is 3{0,1}5{0,1,3}3{0,1,3}1
******* Since 3031 and 131 are primes, we only need to consider the family 35{3}3{0,1,3}1 (since any
digits combo 03, 13 between (3,1) will produce smaller primes)
******** Since 533 is prime, we only need to consider the family 353{0,1}1 (since any digits combo 33
between (35,1) will produce smaller primes)
******** Since 5011 is prime, we only need to consider the family 353{1}{0}1 (since any digits combo
01 between (353,1) will produce smaller primes)
******** If there are at least 3 1's between (353,{0}1), then 31111 will be a subsequence.
******* If there are exactly 2 1's between (353,{0}1), then the digit sum is 20, thus the number is
divisible by 2 and cannot be prime.
******* If there are exactly 1 1's between (353,{0}1), then the form is 3531{0}1
******** The smallest prime of the form 3531{0}1 is 3531001, but it is not minimal prime since 31001
is prime.
******* If there are no 1's between (353,{0}1), then the digit sum is 15, thus the number is divisible
by 6 and cannot be prime.
***** Numbers not containing 3 between (3{0,1}5,1):
***** The form is 3{0,1}5{0,1}1
******* If there are >=2 0's and >=1 1's between (3,1), then at least one of 30011, 30101, 31001 will be
a subsequence.
******* If there are no 1's between (3,1), then the form will be 3{0}5{0}1
******* All numbers of the form 3{0}5{0}1 are divisible by 3, thus cannot be prime.
******* If there are no 0's between (3,1), then the form will be 3{1}5{1}1
******* If there are >=3 1's between (3,1), then 31111 will be a subsequence.
******** If there are exactly 2 1's between (3,1), then the number can only be 31151, 31511, 35111
********* None of 31151, 31511, 35111 are primes.
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******** If there are exactly 1 1's between (3,1), then the digit sum is 13, thus the number is divisible
by 2 and cannot be prime.
******* If there are no 1's between (3.1), then the number is 351
****** 351 is not prime.
******* If there are exactly 1 0's between (3,1), then the form will be 3{1}0{1}5{1}1 or 3{1}5{1}0{1}1
******** No matter 3\{1\}0\{1\}5\{1\}1 or 3\{1\}5\{1\}0\{1\}1, if there are >=3 1's between (3,1), then 31111 will
be a subsequence.
******** If there are exactly 2 1's between (3,1), then the number can only be 311051, 310151,
310511, 301151, 301511, 305111, 311501, 315101, 315011, 351101, 351011, 350111
*********** Of these numbers, 311051, 301151, 311501, 351101, 350111 are primes.
********* However, 311051, 301151, 311501 have 115 as subsequence, and 350111 has 5011 as
subsequence, thus only 351101 is minimal prime.
******** No matter 3{1}0{1}5{1}1 or 3{1}5{1}0{1}1, if there are exactly 1 1's between (3,1), then the digit
sum is 13, thus the number is divisible by 2 and cannot be prime.
******** If there are no 1's between (3,1), then the number is 3051 for 3{1}0{1}5{1}1 or 3501 for
3{1}5{1}0{1}1
****** Neither 3051 nor 3501 is prime.
* Case (3,2):
** 32 is prime, and thus the only minimal prime in this family.
* Case (3,3):
** Since 32, 23, 43, 313 are primes, we only need to consider the family 3{0,3,5,6}3 (since any digits
1, 2, 4 between them will produce smaller primes)
*** If there are >=2 5's in {}, then 553 will be a subsequence.
*** If there are no 5's in {}, then the family will be 3{0,3,6}3
**** All numbers of the form 3{0,3,6}3 are divisible by 3, thus cannot be prime.
*** If there are exactly 1 5's in {}, then the family will be 3{0,3,6}5{0,3,6}3
**** Since 335, 65, 3503, 533, 56 are primes, we only need to consider the family 3{0}53 (since any
digit 3, 6 between (3,5{0,3,6}3) and any digit 0, 3, 6 between (3{0,3,6}5,3) will produce smaller primes)
***** The smallest prime of the form 3{0}53 is 300053
* Case (3,4):
```

- ** Since 32, 14, **304**, **344**, **364** are primes, we only need to consider the family 3{3,5}4 (since any digits 0, 1, 2, 4, 6 between them will produce smaller primes)
- *** Since **3334** and 335 are primes, we only need to consider the family 3{5}4 and 3{5}34 (since any digits combo 33, 35 between them will produce smaller primes)

**** The smallest prime of the form 3{5}4 is

54 with 9234 5's, which can be written as 359234 and equal the prime (23*79235-11)/6 (factordb entry) (shown in base 7) (factorization of the numbers of this form) (this number is only probable prime, i.e. not definitely prime) (not minimal prime, since 35555 and 5554 are primes)

- * Case (3,5):
- ** Since 32, 25, 65, **335** are primes, we only need to consider the family 3{0,1,4,5}5 (since any digits 2, 3, 6 between them will produce smaller primes)
- *** If there are at least one 1's and at least one 5's in {}, then either 155 or 515 will be a subsequence.
- *** If there are at least one 1's and at least one 4's in {}, then either 14 or 41 will be a subsequence.
- *** If there are at least two 1's in {}, then 115 will be a subsequence.
- *** If there are exactly one 1's and no 4's or 5's in {}, then the family will be 3{0}1{0}5
- **** All numbers of the form 3{0}1{0}5 are divisible by 3, thus cannot be prime.
- *** If there is no 1's in {}, then the family will be 3{0,4,5}5
- **** If there are at least to 4's in {}, then 344 and 445 will be subsequences.
- **** If there is no 4's in {}, then the family will be 3{0,5}5
- ***** Since 3055 and 3505 are primes, we only need to consider the families 3{0}5 and 3{5}5
- ****** All numbers of the form 3{0}5 are divisible by 2, thus cannot be prime.
- ****** The smallest prime of the form 3{5}5 is 35555

- **** If there is exactly one 4's in {}, then the family will be 3{0,5}4{0,5}5
- ***** Since 304, **3545** are primes, we only need to consider the families 34{0,5}5 (since any digits 0 or 5 between (3,4{0,5}5) will produce small primes)
- ****** All numbers of the form 34{0,5}5 are divisible by 5, thus cannot be prime.
- * Case (3,6):
- ** Since 32, 16, 56, **346** are primes, we only need to consider the family 3{0,3,6}6 (since any digits 1, 2, 4, 5 between them will produce smaller primes)
- *** All numbers of the form 3{0,3,6}6 are divisible by 3, thus cannot be prime.
- * Case (4,1):
- ** 41 is prime, and thus the only minimal prime in this family.
- * Case (4,2):
- ** Since 41, 43, 32, 52 are primes, we only need to consider the family 4{0,2,4,6}2 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 4{0,2,4,6}2 are divisible by 2, thus cannot be prime.
- * Case (4,3):
- ** 43 is prime, and thus the only minimal prime in this family.
- * Case (4,4):
- ** Since 41, 43, 14 are primes, we only need to consider the family 4{0,2,4,5,6}4 (since any digits 1, 3 between them will produce smaller primes)
- *** If there is no 5's in {}, then the family will be 4{0,2,4,6}4
- **** All numbers of the form 4{0,2,4,6}4 are divisible by 2, thus cannot be prime.
- *** If there is at least one 5's in {}, then there cannot be 2 in {} (since if so, then either 25 or 52 will be a subsequence) and there cannot be 6 in {} (since if so, then either 65 or 56 will be a subsequence), thus the family is 4{0,4,5}5{0,4,5}4
- **** Since 445, **4504**, 544 are primes, we only need to consider the family 4{0,5}5{5}4 (since any digit 4 between (4,5{0,4,5}4) and any digit 0, 4 between (4{0,4,5}5,4) will produce smaller primes)
- ***** If there are at least two 0's between (4,5{0,4,5}4), then **40054** will be a subsequence.
- ***** If there is no 0's between $(4,5\{0,4,5\}4)$, then the family will be $4\{5\}5\{5\}4$, which is equivalent to $4\{5\}4$

- ****** The smallest prime of the form 4{5}4 is 45555555555555554 (not minimal prime, since 4555 and 5554 are primes)
- ***** If there is exactly one 0's between (4,5{0,4,5}4), then the family will be 4{5}0{5}5{5}4
- ****** Since 4504 is prime, we only need to consider the family 40{5}5{5}4 (since any digit 5 between (4,0{5}5{5}4) will produce small primes), which is equivalent to 40{5}4
- ******* The smallest prime of the form $40{5}4$ is 40555555555555555555 (not minimal prime, since 4555 and 5554 are primes)
- * Case (4,5):
- ** Since 41, 43, 25, 65, **445** are primes, we only need to consider the family 4{0,5}5 (since any digits 1, 2, 3, 4, 6 between them will produce smaller primes)
- *** If there are at least two 5's in {}, then 4555 will be a subsequence.
- *** If there is exactly one 5's in {}, then the digit sum is 20, and the number will be divisible by 2 and cannot be prime.
- *** If there is no 5's in {}, then the family will be 4{0}5
- **** All numbers of the form 4{0}5 are divisible by 3, thus cannot be prime.
- * Case (4,6):
- ** Since 41, 43, 16, 56 are primes, we only need to consider the family 4{0,2,4,6}6 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 4{0,2,4,6}6 are divisible by 2, thus cannot be prime.
- * Case (5,1):
- ** Since 52, 56, 41, 61, **551** are primes, we only need to consider the family 5{0,1,3}1 (since any digits 2, 4, 5, 6 between them will produce smaller primes)
- *** If there are at least two 3's in {}, then 533 will be a subsequence.
- *** If there is no 3's in {}, then the family will be 5{0,1}1
- **** Since **5011** is prime, we only need to consider the family 5{1}{0}1
- ***** Since 11111 is prime, we only need to consider the families 5{0}1, 51{0}1, 511{0}1, 5111{0}1 (since any digits combo 1111 between (5,1) will produce small primes)
- ****** All numbers of the form 5{0}1 are divisible by 6, thus cannot be prime.
- ****** The smallest prime of the form 51{0}1 is 5100000001
- ****** All numbers of the form 511{0}1 are divisible by 2, thus cannot be prime.

- ****** All numbers of the form 5111{0}1 are divisible by 3, thus cannot be prime.
- *** If there is exactly one 3's in {}, then the family will be 5{0,1}3{0,1}1
- **** If there is at least one 1's between (5,3{0,1}1), then 131 will be a subsequence.
- ***** Thus we only need to consider the family 5{0}3{0,1}1
- ****** If there are no 1's between (5{0}3,1), then the digit sum is 12, and the number will be divisible by 3 and cannot be prime.
- ****** If there are exactly one 1's between (5{0}3,1), then the digit sum is 13, and the number will be divisible by 2 and cannot be prime.
- ****** If there are exactly three 1's between (5{0}3,1), then the digit sum is 15, and the number will be divisible by 6 and cannot be prime.
- ****** If there are at least four 1's between (5{0}3,1), then 11111 will be a subsequence.
- ****** If there are exactly two 1's between (5{0}3,1), then the family will be 5{0}3{0}1{0}1{0}1
- ******* Since 5011 is prime, we only need to consider the family 5311{0}1 (since any digit 0 between (5,1{0}1) will produce small primes, this includes the leftmost three {} in 5{0}3{0}1{0}1{0}1, and thus only the rightmost {} can contain 0)
- ******* The smallest prime of the form 5311{0}1 is 531101
- * Case (5,2):
- ** **52** is prime, and thus the only minimal prime in this family.
- * Case (5,3):
- ** Since 52, 56, 23, 43, **533**, **553** are primes, we only need to consider the family 5{0,1}3 (since any digits 2, 3, 4, 5, 6 between them will produce smaller primes)
- *** If there are at least two 1's in {}, then 113 will be a subsequence.
- *** If there is exactly one 1's in {}, then the digit sum is 12, and the number will be divisible by 3 and cannot be prime.
- *** If there is no 1's in {}, then the digit sum is 11, and the number will be divisible by 2 and cannot be prime.
- * Case (5,4):
- ** Since 52, 56, 14, **544** are primes, we only need to consider the family 5{0,3,5}4 (since any digits 1, 2, 4, 6 between them will produce smaller primes)
- *** If there are no 5's in {}, then the family will be 5{0,3}4
- **** All numbers of the form 5{0,3}4 are divisible by 3, thus cannot be prime.

- *** If there are at least one 5's and at least one 3's in {}, then either 535 or 553 will be a subsequence.
- *** If there are exactly one 5's and no 3's in {}, then the digit sum is 20, and the number will be divisible by 2 and cannot be prime.
- *** If there are at least two 5's in {}, then **5554** will be a subsequence.
- * Case (5,5):
- ** Since 52, 56, 25, 65, **515**, **535** are primes, we only need to consider the family 5{0,4,5}5 (since any digits 1, 2, 3, 6 between them will produce smaller primes)
- *** If there are no 4's in {}, then the family will be 5{0,5}5
- **** All numbers of the form 5{0,5}5 are divisible by 5, thus cannot be prime.
- *** If there are no 5's in {}, then the family will be 5{0,4}5
- **** All numbers of the form 5{0,4}5 are divisible by 2, thus cannot be prime.
- *** If there are at least one 4's and at least one 5's in {}, then either **5455** or **5545** will be a subsequence.
- * Case (5,6):
- ** **56** is prime, and thus the only minimal prime in this family.
- * Case (6,1):
- ** **61** is prime, and thus the only minimal prime in this family.
- * Case (6,2):
- ** Since 61, 65, 32, 52 are primes, we only need to consider the family 6{0,2,4,6}2 (since any digits 1, 3, 5 between them will produce smaller primes)
- *** All numbers of the form 6{0,2,4,6}2 are divisible by 2, thus cannot be prime.
- * Case (6,3):
- ** Since 61, 65, 23, 43 are primes, we only need to consider the family $6\{0,3,6\}3$ (since any digits 1, 2,
- 4, 5 between them will produce smaller primes)
- *** All numbers of the form 6{0,3,6}3 are divisible by 3, thus cannot be prime.
- * Case (6,4):
- ** Since 61, 65, 14 are primes, we only need to consider the family 6{0,2,3,4,6}4 (since any digits 1, 5 between them will produce smaller primes)
- *** If there is no 3's in {}, then the family will be 6{0,2,4,6}4

- **** All numbers of the form 6{0,2,4,6}4 are divisible by 2, thus cannot be prime.
- *** If there are exactly two 3's in {}, then the family will be 6{0,2,4,6}3{0,2,4,6}4
- **** All numbers of the form 6{0,2,4,6}3{0,2,4,6}4 are divisible by 2, thus cannot be prime.
- *** If there are at least three 3's in {}, then 3334 will be a subsequence.
- *** If there is exactly one 3's in {}, then the family will be 6{0,2,4,6}3{0,2,4,6}4
- **** If there is 0 between (6,3{0,2,4,6}4), then **6034** will be a subsequence.
- **** If there is 2 between (6,3{0,2,4,6}4), then 23 will be a subsequence.
- **** If there is 4 between (6,3{0,2,4,6}4), then 43 will be a subsequence.
- **** If there is 6 between (6,3{0,2,4,6}4), then **6634** will be a subsequence.
- **** If there is 0 between (6{0,2,4,6}3,4), then 304 will be a subsequence.
- **** If there is 2 between $(6\{0,2,4,6\}3,4)$, then 32 will be a subsequence.
- **** If there is 4 between (6{0,2,4,6}3,4), then 344 will be a subsequence.
- **** If there is 6 between (6{0,2,4,6}3,4), then 364 will be a subsequence.
- **** Thus the number can only be 634
- ***** 634 is not prime.
- * Case (6.5):
- ** 65 is prime, and thus the only minimal prime in this family.
- * Case (6,6):
- ** Since 61, 65, 16, 56 are primes, we only need to consider the family 6{0,2,3,4,6}6 (since any digits 1, 5 between them will produce smaller primes)
- *** If there is no 3's in {}, then the family will be 6{0,2,4,6}6
- **** All numbers of the form 6{0,2,4,6}6 are divisible by 2, thus cannot be prime.
- *** If there is no 2's and no 4's in {}, then the family will be 6{0,3,6}6
- **** All numbers of the form 6{0,3,6}6 are divisible by 3, thus cannot be prime.
- *** If there is at least one 3's and at least one 2's in {}, then either 32 or 23 will be a subsequence.
- *** If there is at least one 3's and at least one 4's in {}, then either 346 or 43 will be a subsequence.

base 8

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,3), (1,5), (1,7), (2,1), (2,3), (2,5), (2,7), (3,1), (3,3), (3,5), (3,7), (4,1), (4,3), (4,5), (4,7), (5,1), (5,3), (5,5), (5,7), (6,1), (6,3), (6,5), (6,7), (7,1), (7,3), (7,5), (7,7)
```

- * Case (1,1):
- ** Since 13, 15, 21, 51, **111**, **141**, **161** are primes, we only need to consider the family 1{0,7}1 (since any digits 1, 2, 3, 4, 5, 6 between them will produce smaller primes)
- *** Since 107, 177, 701 are primes, we only need to consider the number 171 and the family 1{0}1 (since any digits combo 07, 70, 77 between them will produce smaller primes)
- **** 171 is not prime.
- **** All numbers of the form $1{0}1$ factored as $10^n+1=(2^n+1)*(4^n-2^n+1)$, thus cannot be prime.
- * Case (1,3):
- ** 13 is prime, and thus the only minimal prime in this family.
- * Case (1,5):
- ** 15 is prime, and thus the only minimal prime in this family.
- * Case (1,7):
- ** Since 13, 15, 27, 37, 57, **107**, **117**, **147**, **177** are primes, we only need to consider the family 1{6}7 (since any digits 0, 1, 2, 3, 4, 5, 7 between them will produce smaller primes)
- *** The smallest prime of the form 1{6}7 is 16667 (not minimal prime, since 667 is prime)
- * Case (2,1):
- ** 21 is prime, and thus the only minimal prime in this family.
- * Case (2,3):
- ** 23 is prime, and thus the only minimal prime in this family.
- * Case (2,5):
- ** Since 21, 23, 27, 15, 35, 45, 65, 75, **225**, **255** are primes, we only need to consider the family 2{0}5 (since any digits 1, 2, 3, 4, 5, 6, 7 between them will produce smaller primes)
- *** All numbers of the form 2{0}5 are divisible by 7, thus cannot be prime.
- * Case (2,7):

- ** 27 is prime, and thus the only minimal prime in this family.
- * Case (3,1):
- ** Since 35, 37, 21, 51, **301**, **361** are primes, we only need to consider the family 3{1,3,4}1 (since any digits 0, 2, 5, 6, 7 between them will produce smaller primes)
- *** Since 13, 343, 111, 131, 141, 431, **3331**, **3411** are primes, we only need to consider the families 3{3}11, 33{1,4}1, 3{3,4}4{4}1 (since any digits combo 11, 13, 14, 33, 41, 43 between them will produce smaller primes)
- **** All numbers of the form 3{3}11 are divisible by 3, thus cannot be prime.
- **** For the 33{1,4}1 family, since 111 and 141 are primes, we only need to consider the families 33{4}1 and 33{4}11 (since any digits combo 11, 14 between them will produce smaller primes)
- ***** The smallest prime of the form 33{4}1 is 3344441
- ***** All numbers of the form 33{4}11 are divisible by 301, thus cannot be prime.
- **** For the 3{3,4}4{4}1 family, since 3331 and 3344441 are primes, we only need to consider the families 3{4}1, 3{4}31, 3{4}341, 3{4}3441, 3{4}34441 (since any digits combo 33 or 34444 between (3,1) will produce smaller primes)
- ***** All numbers of the form 3{4}1 are divisible by 31, thus cannot be prime.
- ***** Since 4443 is prime, we only need to consider the numbers 3431, 34431, 34341, 343441, 343441, 3434441 (since any digit combo 444 between (3,3{4}1) will produce smaller primes)
- ****** None of 3431, 34431, 34341, 344341, 343441, 3443441, 3434441, 34434441 are primes.
- * Case (3,3):
- ** Since 35, 37, 13, 23, 53, 73, **343** are primes, we only need to consider the family 3{0,3,6}3 (since any digits 1, 2, 4, 5, 7 between them will produce smaller primes)
- *** All numbers of the form 3{0,3,6}3 are divisible by 3, thus cannot be prime.
- * Case (3,5):
- ** **35** is prime, and thus the only minimal prime in this family.
- * Case (3,7):
- ** 37 is prime, and thus the only minimal prime in this family.
- * Case (4,1):
- ** Since 45, 21, 51, **401**, **431**, **471** are primes, we only need to consider the family 4{1,4,6}1 (since any digits 0, 2, 3, 5, 7 between them will produce smaller primes)

- *** Since 111, 141, 161, 661, **4611** are primes, we only need to consider the families 4{4}11, 4{4,6}4{1,4,6}1, 4{4}6{4}1 (since any digits combo 11, 14, 16, 61, 66 between them will produce smaller primes)
- **** The smallest prime of the form 4{4}11 is 4444444444411 (not minimal prime, since 444444441 is prime)
- **** For the 4{4,6}4{1,4,6}1 family, we can separate this family to 4{4,6}41, 4{4,6}411, 4{4,6}461
- ****** For the 4{4,6}41 family, since 661 and 6441 are primes, we only need to consider the families 4{4}41 and 4{4}641 (since any digits combo 64 or 66 between (4,41) will produce smaller primes)
- ****** The smallest prime of the form 4{4}41 is 444444441
- ****** The smallest prime of the form 4{4}641 is 444641
- ***** For the 4{4,6}411 family, since 661 and 6441 are primes, we only need to consider the families 4{4}411 and 4{4}6411 (since any digits combo 64 or 66 between (4,411) will produce smaller primes)
- ****** The smallest prime of the form 4{4}411 is 44444441
- ****** The smallest prime of the form 4{4}6411 is 44444444444446411 (not minimal prime, since 444444441 and 444641 are primes)
- ***** For the 4{4,6}461 family, since 661 is prime, we only need to consider the family 4{4}461
- ****** The smallest prime of the form 4{4}461 is 4444444461 (not minimal prime, since 444444441 is prime)
- **** For the 4{4}6{4}1 family, since 6441 is prime, we only need to consider the families 4{4}61 and 4{4}641 (since any digits combo 44 between (4{4}6,1) will produce smaller primes)
- ***** The smallest prime of the form 4{4}61 is 4444444461 (not minimal prime, since 444444441 is prime)
- ***** The smallest prime of the form 4{4}641 is 444641
- * Case (4,3):
- ** Since 45, 13, 23, 53, 73, **433**, **463** are primes, we only need to consider the family 4{0,4}3 (since any digits 1, 2, 3, 5, 6, 7 between them will produce smaller primes)
- *** Since **4043** and **4443** are primes, we only need to consider the families 4{0}3 and 44{0}3 (since any digits combo 04, 44 between them will produce smaller primes)
- **** All numbers of the form 4{0}3 are divisible by 7, thus cannot be prime.
- **** All numbers of the form 44{0}3 are divisible by 3, thus cannot be prime.
- * Case (4,5):
- ** 45 is prime, and thus the only minimal prime in this family.

- * Case (4,7):
- ** Since 45, 27, 37, 57, **407**, **417**, **467** are primes, we only need to consider the family 4{4,7}7 (since any digits 0, 1, 2, 3, 5, 6 between them will produce smaller primes)
- *** Since 747 is prime, we only need to consider the families 4{4}7, 4{4}77, 4{7}7, 44{7}7 (since any digits combo 74 between (4,7) will produce smaller primes)
- **** The smallest prime of the form 4{4}7 is

- **** The smallest prime of the form 4{4}77 is 4444477
- **** The smallest prime of the form 4{7}7 is 47777
- * Case (5,1):
- ** 51 is prime, and thus the only minimal prime in this family.
- * Case (5,3):
- ** **53** is prime, and thus the only minimal prime in this family.
- * Case (5,5):
- ** Since 51, 53, 57, 15, 35, 45, 65, 75 are primes, we only need to consider the family 5{0,2,5}5 (since any digits 1, 3, 4, 6, 7 between them will produce smaller primes)
- *** Since 225, 255, **5205** are primes, we only need to consider the families 5{0,5}5 and 5{0,5}25 (since any digits combo 20, 22, 25 between them will produce smaller primes)
- **** All numbers of the form 5{0,5}5 are divisible by 5, thus cannot be prime.

**** For the 5{0,5}25 family, since 500025 and 505525 are primes, we only need to consider the number 500525 the families 5{5}25, 5{5}025, 5{5}0025, 5{5}0525, 5{5}0525, 5{5}050525 (since any digits combo 000, 055 between (5,25) will produce smaller primes) ***** 500525 is not prime. ***** The smallest prime of the form 5{5}025 is 55555025 ***** The smallest prime of the form 5{5}0025 is 5555555555555555555550025, with 184 5's, which can be written as 51830025 and equal the prime (5*8¹⁸⁷-20333)/7 (factordb entry) (shown in base 8) (factorization of the numbers of this form) (not minimal prime, since 55555025 and 55555555555525 are primes) ***** The smallest prime of the form 5{5}0525 is **5550525** ***** The smallest prime of the form 5{5}00525 is **5500525** ***** The smallest prime of the form 5{5}05025 is 555555555555555555555505025, with 25 5's, which can be written as 52305025 and equal the prime (5*828-145773)/7 (factordb entry) (shown in base 8) (factorization of the numbers of this form) (not minimal prime, since 5550525, 55555025, and 555555555555525 are primes) * Case (5,7): ** 57 is prime, and thus the only minimal prime in this family. * Case (6,1): ** Since 65, 21, 51, 631, 661 are primes, we only need to consider the family 6{0,1,4,7}1 (since any digits 2, 3, 5, 6 between them will produce smaller primes) *** Numbers containing 4: (note that the number cannot contain two or more 4's, or 6441 will be a subsequence) **** The form is 6{0,1,7}4{0,1,7}1 ***** Since 141, 401, 471 are primes, we only need to consider the family 6{0,7}4{1}1 ****** Since 111 is prime, we only need to consider the families 6(0,7)41 and 6(0,7)411 ******* For the 6{0,7}41 family, since 60741 is prime, we only need to consider the family 6{7}{0}41 ******* Since 6777 is prime, we only need to consider the families 6{0}41, 67{0}41, 677{0}41 ******** All numbers of the form 6{0}41 are divisible by 3, thus cannot be prime.

******** All numbers of the form 67{0}41 are divisible by 13, thus cannot be prime.

- ******* All numbers of the form 677{0}41 are divisible by 3, thus cannot be prime.
- ******* For the 6{0,7}411 family, since 60411 is prime, we only need to consider the family 6{7}411
- ******** The smallest prime of the form 6{7}411 is 67777411 (not minimal prime, since 6777 is prime)
- *** Numbers not containing 4:
- **** The form is 6{0,1,7}1
- ***** Since 111 is prime, we only need to consider the families 6{0,7}1 and 6{0,7}1{0,7}1
- ****** All numbers of the form 6{0,7}1 are divisible by 7, thus cannot be prime.
- ****** For the $6\{0,7\}1\{0,7\}1$ family, since 711 and **6101** are primes, we only need to consider the family $6\{0\}1\{7\}1$
- ******* Since **60171** is prime, we only need to consider the families 6{0}11 and 61{7}1
- ******* All numbers of the form 6{0}11 are divisible by 3, thus cannot be prime.
- ******* The smallest prime of the form 61{7}1 is 617771 (not minimal prime, since 6777 is prime)
- * Case (6,3):
- ** Since 65, 13, 23, 53, 73, **643** are primes, we only need to consider the family 6{0,3,6}3 (since any digits 1, 2, 4, 5, 7 between them will produce smaller primes)
- *** All numbers of the form 6{0,3,6}3 are divisible by 3, thus cannot be prime.
- * Case (6,5):
- ** 65 is prime, and thus the only minimal prime in this family.
- * Case (6,7):
- ** Since 65, 27, 37, 57, **667** are primes, we only need to consider the family 6{0,1,4,7}7 (since any digits 2, 3, 5, 6 between them will produce smaller primes)
- *** Since 107, 117, 147, 177, 407, 417, 717, 747, **6007**, **6477**, **6707**, **6777** are primes, we only need to consider the families 60{1,4,7}7, 6{0}17, 6{0,4}4{4}7, 6{0}77 (since any digits combo 00, 10, 11, 14, 17, 40, 41, 47, 70, 71, 74, 77 between them will produce smaller primes)
- **** All numbers of the form 6{0}17 or 6{0}77 are divisible by 3, thus cannot be prime.
- **** For the 60{1,4,7}7 family, since 117, 147, 177, 417, 6477, 717, 747, 6777 are primes, we only need to consider the numbers 6017, 6047, 6077 and the family 60{4}7 (since any digit combo 11, 14, 17, 41, 47, 71, 74, 77 between (60,7) will produce smaller primes)
- ***** None of 6017, 6047, 6077 are primes.

- **** All numbers of the form 60{4}7 are divisible by 21, thus cannot be prime.
- **** For the 6{0,4}4{4}7 family, since 6007 and 407 are primes, we only need to consider the families 6{4}7 and 60{4}7 (since any digit combo 00, 40 between (6,4{4}7) will produce smaller primes)
- ***** All numbers of the form 6{4}7 are divisible by 3, 5, or 15, thus cannot be prime.
- ***** All numbers of the form 60{4}7 are divisible by 21, thus cannot be prime.
- * Case (7,1):
- ** Since 73, 75, 21, 51, **701**, **711** are primes, we only need to consider the family 7{4,6,7}1 (since any digits 0, 1, 2, 3, 5 between them will produce smaller primes)
- *** Since 747, 767, 471, 661, **7461**, **7641** are primes, we only need to consider the families 7{4,7}4{4}1, 7{7}61, 7{7}7{4,6,7}1 (since any digits combo 46, 47, 64, 66, 67 between them will produce smaller primes)
- **** For the 7{4,7}4{4}1 family, since 747, 471 are primes, we only need to consider the family 7{7}{4}1 (since any digits combo 47 between (7,4{4}1) will produce smaller primes)
- ***** The smallest prime of the form 7{7}1 is 77777777771

- ***** All numbers of the form 7{7}444441 are divisible by 7, thus cannot be prime.
- ***** The smallest prime of the form 7{7}4444441 is 77774444441
- ****** Since this prime has just 4 7's, we only need to consider the families with <=3 7's

- ******* The smallest prime of the form 7{4}1 is **744444441**
- ******* All numbers of the form 77{4}1 are divisible by 5, thus cannot be prime.
- ******* The smallest prime of the form 777{4}1 is 77744444444441 (not minimal prime, since 444444441 and 744444441 are primes)
- * Case (7,3):
- ** 73 is prime, and thus the only minimal prime in this family.
- * Case (7,5):
- ** **75** is prime, and thus the only minimal prime in this family.
- * Case (7,7):
- ** Since 73, 75, 27, 37, 57, **717**, **747**, **767** are primes, we only need to consider the family 7{0,7}7 (since any digits 1, 2, 3, 4, 5, 6 between them will produce smaller primes)
- *** All numbers of the form 7{0,7}7 are divisible by 7, thus cannot be prime.

base 10

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,3), (1,7), (1,9), (2,1), (2,3), (2,7), (2,9), (3,1), (3,3), (3,7), (3,9), (4,1), (4,3), (4,7), (4,9), (5,1), (5,3), (5,7), (5,9), (6,1), (6,3), (6,7), (6,9), (7,1), (7,3), (7,7), (7,9), (8,1), (8,3), (8,7), (8,9), (9,1), (9,3), (9,7), (9,9)
```

- * Case (1,1):
- ** 11 is prime, and thus the only minimal prime in this family.
- * Case (1,3):
- ** 13 is prime, and thus the only minimal prime in this family.
- * Case (1,7):
- ** 17 is prime, and thus the only minimal prime in this family.
- * Case (1,9):
- ** 19 is prime, and thus the only minimal prime in this family.
- * Case (2,1):

- ** Since 23, 29, 11, 31, 41, 61, 71, **251**, **281** are primes, we only need to consider the family 2{0,2}1 (since any digits 1, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)
- *** Since **2221** and **20201** are primes, we only need to consider the families 2{0}1, 2{0}21, 22{0}1 (since any digits combo 22 or 020 between them will produce smaller primes)
- **** All numbers of the form 2{0}1 are divisible by 3, thus cannot be prime.
- **** The smallest prime of the form 2{0}21 is 20021
- **** The smallest prime of the form 22{0}1 is 22000001
- * Case (2,3):
- ** 23 is prime, and thus the only minimal prime in this family.
- * Case (2,7):
- ** Since 23, 29, 17, 37, 47, 67, 97, **227**, **257**, **277** are primes, we only need to consider the family 2{0,8}7 (since any digits 1, 2, 3, 4, 5, 6, 7, 9 between them will produce smaller primes)
- *** Since 887 and **2087** are primes, we only need to consider the families 2{0}7 and 28{0}7 (since any digit combo 08 or 88 between them will produce smaller primes)
- **** All numbers of the form 2{0}7 are divisible by 3, thus cannot be prime.
- **** All numbers of the form 28{0}7 are divisible by 7, thus cannot be prime.
- * Case (2,9):
- ** 29 is prime, and thus the only minimal prime in this family.
- * Case (3,1):
- ** **31** is prime, and thus the only minimal prime in this family.
- * Case (3,3):
- ** Since 31, 37, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 3{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)
- *** All numbers of the form 3{0,3,6,9}3 are divisible by 3, thus cannot be prime.
- * Case (3,7):
- ** 37 is prime, and thus the only minimal prime in this family.
- * Case (3,9):
- ** Since 31, 37, 19, 29, 59, 79, 89, **349** are primes, we only need to consider the family 3{0,3,6,9}9 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)

*** All numbers of the form 3{0,3,6,9}9 are divisible by 3, thus cannot be prime. * Case (4.1): ** 41 is prime, and thus the only minimal prime in this family. * Case (4,3): ** 43 is prime, and thus the only minimal prime in this family. * Case (4,7): ** 47 is prime, and thus the only minimal prime in this family. * Case (4,9): ** Since 41, 43, 47, 19, 29, 59, 79, 89, **409**, **449**, **499** are primes, we only need to consider the family 4{6}9 (since any digits 0, 1, 2, 3, 4, 5, 7, 8, 9 between them will produce smaller primes) *** All numbers of the form 4{6}9 are divisible by 7, thus cannot be prime. * Case (5,1): ** Since 53, 59, 11, 31, 41, 61, 71, **521** are primes, we only need to consider the family 5{0,5.8}1 (since any digits 1, 2, 3, 4, 6, 7, 9 between them will produce smaller primes) *** Since 881 is prime, we only need to consider the families 5{0,5}1 and 5{0,5}8{0,5}1 (since any digit combo 88 between them will produce smaller primes) **** For the 5{0,5}1 family, since **5051** and **5501** are primes, we only need to consider the families 5(0)1 and 5(5)1 (since any digit combo 05 or 50 between them will produce smaller primes) ***** All numbers of the form 5{0}1 are divisible by 3, thus cannot be prime. ***** The smallest prime of the form 5{5}1 is 555555555551 **** For the 5{0,5}8{0,5}1 family, since **5081**, **5581**, **5801**, **5851** are primes, we only need to consider the number 581 ***** 581 is not prime. * Case (5,3): ** **53** is prime, and thus the only minimal prime in this family. * Case (5,7):

** Since 53, 59, 17, 37, 47, 67, 97, 557, 577, 587 are primes, we only need to consider the family

5{0,2}7 (since any digits 1, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)

- *** Since 227 and **50207** are primes, we only need to consider the families 5{0}7, 5{0}27, 52{0}7 (since any digits combo 22 or 020 between them will produce smaller primes)
- **** All numbers of the form 5{0}7 are divisible by 3, thus cannot be prime.
- **** The smallest prime of the form 52{0}7 is 5200007
- * Case (5,9):
- ** **59** is prime, and thus the only minimal prime in this family.
- * Case (6,1):
- ** 61 is prime, and thus the only minimal prime in this family.
- * Case (6,3):
- ** Since 61, 67, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 6{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)
- *** All numbers of the form 6{0,3,6,9}3 are divisible by 3, thus cannot be prime.
- * Case (6,7):
- ** 67 is prime, and thus the only minimal prime in this family.
- * Case (6,9):
- ** Since 61, 67, 19, 29, 59, 79, 89 are primes, we only need to consider the family 6{0,3,4,6,9}9 (since any digits 1, 2, 5, 7, 8 between them will produce smaller primes)
- *** Since 449 is prime, we only need to consider the families 6{0,3,6,9}9 and 6{0,3,6,9}4{0,3,6,9}9 (since any digit combo 44 between them will produce smaller primes)
- **** All numbers of the form 6{0,3,6,9}9 are divisible by 3, thus cannot be prime.
- **** For the $6\{0,3,6,9\}4\{0,3,6,9\}9$ family, since 409, 43, **6469**, 499 are primes, we only need to consider the family $6\{0,3,6,9\}49$
- ***** Since 349, 6949 are primes, we only need to consider the family 6{0,6}49
- ****** Since **60649** is prime, we only need to consider the family 6{6}{0}49 (since any digits combo 06 between {6,49} will produce smaller primes)
- ****** The smallest prime of the form 6{6}49 is 666649
- ******** Since this prime has just 4 6's, we only need to consider the families with <=3 6's

- ******** The smallest prime of the form 6{0}49 is **60000049**
- ********* The smallest prime of the form 66{0}49 is 66000049
- ******* The smallest prime of the form 666{0}49 is 66600049
- * Case (7,1):
- ** 71 is prime, and thus the only minimal prime in this family.
- * Case (7,3):
- ** 73 is prime, and thus the only minimal prime in this family.
- * Case (7,7):
- ** Since 71, 73, 79, 17, 37, 47, 67, 97, **727**, **757**, **787** are primes, we only need to consider the family 7{0,7}7 (since any digits 1, 2, 3, 4, 5, 6, 8, 9 between them will produce smaller primes)
- *** All numbers of the form 7{0,7}7 are divisible by 7, thus cannot be prime.
- * Case (7,9):
- ** 79 is prime, and thus the only minimal prime in this family.
- * Case (8,1):
- ** Since 83, 89, 11, 31, 41, 61, 71, **821**, **881** are primes, we only need to consider the family 8{0,5}1 (since any digits 1, 2, 3, 4, 6, 7, 8, 9 between them will produce smaller primes)
- *** Since **8501** is prime, we only need to consider the family 8{0}{5}1 (since any digits combo 50 between them will produce smaller primes)
- **** Since **80051** is prime, we only need to consider the families 8{0}1, 8{5}1, 80{5}1 (since any digits combo 005 between them will produce smaller primes)
- ***** All numbers of the form 8{0}1 are divisible by 3, thus cannot be prime.
- ***** The smallest prime of the form 8{5}1 is 85555555555555555555 (not minimal prime, since 555555555555555 is prime)
- ***** The smallest prime of the form 80{5}1 is 80555551
- * Case (8,3):
- ** 83 is prime, and thus the only minimal prime in this family.
- * Case (8,7):
- ** Since 83, 89, 17, 37, 47, 67, 97, **827**, **857**, **877**, **887** are primes, we only need to consider the family 8{0}7 (since any digits 1, 2, 3, 4, 5, 6, 7, 8, 9 between them will produce smaller primes)

- *** All numbers of the form 8{0}7 are divisible by 3, thus cannot be prime.
- * Case (8,9):
- ** 89 is prime, and thus the only minimal prime in this family.
- * Case (9,1):
- ** Since 97, 11, 31, 41, 61, 71, **991** are primes, we only need to consider the family 9{0,2,5,8}1 (since any digits 1, 3, 4, 6, 7, 9 between them will produce smaller primes)
- *** Since 251, 281, 521, 821, 881, **9001**, **9221**, **9551**, **9851** are primes, we only need to consider the families 9{2,5,8}0{2,5,8}1, 9{0}2{0}1, 9{0}5{0,8}1, 9{0,5}8{0}1 (since any digits combo 00, 22, 25, 28, 52, 55, 82, 85, 88 between them will produce smaller primes)
- **** For the 9{2,5,8}0{2,5,8}1 family, since any digits combo 22, 25, 28, 52, 55, 82, 85, 88 between (9,1) will produce smaller primes, we only need to consider the numbers 901, 9021, 9051, 9081, 9201, 9501, 9801, 90581, 95081, 95801
- ***** 95801 is the only prime among 901, 9021, 9051, 9081, 9201, 9501, 9801, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.
- **** For the 9{0}2{0}1 family, since 9001 is prime, we only need to consider the numbers 921, 9201, 9021
- ***** None of 921, 9201, 9021 are primes.
- **** For the 9{0}5{0,8}1 family, since 9001 and 881 are primes, we only need to consider the numbers 951, 9051, 9501, 9581, 90581, 95081, 95801
- ***** 95801 is the only prime among 951, 9051, 9501, 9581, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.
- **** For the 9{0,5}8{0}1 family, since 9001 and 5581 are primes, we only need to consider the numbers 981, 9081, 9581, 9801, 90581, 95801
- ***** 95801 is the only prime among 981, 9081, 9581, 9801, 90581, 95081, 95801, but it is not minimal prime since 5801 is prime.
- * Case (9,3):
- ** Since 97, 13, 23, 43, 53, 73, 83 are primes, we only need to consider the family 9{0,3,6,9}3 (since any digits 1, 2, 4, 5, 7, 8 between them will produce smaller primes)
- *** All numbers of the form 9{0,3,6,9}3 are divisible by 3, thus cannot be prime.
- * Case (9,7):
- ** 97 is prime, and thus the only minimal prime in this family.
- * Case (9,9):

- ** Since 97, 19, 29, 59, 79, 89 are primes, we only need to consider the family 9{0,3,4,6,9}9 (since any digits 1, 2, 5, 7, 8 between them will produce smaller primes)
- *** Since 449 is prime, we only need to consider the families 9{0,3,6,9}9 and 9{0,3,6,9}4{0,3,6,9}9 (since any digit combo 44 between them will produce smaller primes)
- **** All numbers of the form 9{0,3,6,9}9 are divisible by 3, thus cannot be prime.
- **** For the $9\{0,3,6,9\}4\{0,3,6,9\}9$ family, since **9049**, 349, **9649**, **9949** are primes, we only need to consider the family $94\{0,3,6,9\}9$
- ***** Since 409, 43, 499 are primes, we only need to consider the family 94{6}9 (since any digits 0, 3, 9 between (94,9) will produce smaller primes)
- ****** The smallest prime of the form 94{6}9 is 946669

base 12

The possible (first digit, last digit) for an element with >=3 digits in the minimal set of the strings for primes with at least two digits are:

```
(1,1), (1,5), (1,7), (1,B), (2,1), (2,5), (2,7), (2,B), (3,1), (3,5), (3,7), (3,B), (4,1), (4,5), (4,7), (4,B), (5,1), (5,5), (5,7), (5,B), (6,1), (6,5), (6,7), (6,B), (7,1), (7,5), (7,7), (7,B), (8,1), (8,5), (8,7), (8,B), (9,1), (9,5), (9,7), (9,B), (A,1), (A,5), (A,7), (A,B), (B,1), (B,5), (B,7), (B,B)
```

- * Case (1,1):
- ** 11 is prime, and thus the only minimal prime in this family.
- * Case (1,5):
- ** 15 is prime, and thus the only minimal prime in this family.
- * Case (1,7):
- ** 17 is prime, and thus the only minimal prime in this family.
- * Case (1,B):
- ** **1B** is prime, and thus the only minimal prime in this family.
- * Case (2,1):
- ** Since 25, 27, 11, 31, 51, 61, 81, 91, **221**, **241**, **2A1**, **2B1** are primes, we only need to consider the family 2{0}1 (since any digits 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B between them will produce smaller primes)
- *** The smallest prime of the form 2{0}1 is 2001
- * Case (2,5):
- ** 25 is prime, and thus the only minimal prime in this family.

- * Case (2,7):
- ** 27 is prime, and thus the only minimal prime in this family.
- * Case (2,B):
- ** Since 25, 27, 1B, 3B, 4B, 5B, 6B, 8B, AB, **2BB** are primes, we only need to consider the family 2{0,2,9}B (since any digits 1, 3, 4, 5, 6, 7, 8, A, B between them will produce smaller primes)
- *** Since 90B, **200B**, **202B**, **222B**, **229B**, **299B** are primes, we only need to consider the numbers 20B, 22B, 29B, 209B, 220B (since any digits combo 00, 02, 22, 29, 90, 92, 99 between them will produce smaller primes)
- **** None of 20B, 22B, 29B, 209B, 220B are primes.
- * Case (3,1):
- ** 31 is prime, and thus the only minimal prime in this family.
- * Case (3,5):
- ** **35** is prime, and thus the only minimal prime in this family.
- * Case (3,7):
- ** 37 is prime, and thus the only minimal prime in this family.
- * Case (3.B):
- ** **3B** is prime, and thus the only minimal prime in this family.
- * Case (4,1):
- ** Since 45, 4B, 11, 31, 51, 61, 81, 91, **401**, **421**, **471** are primes, we only need to consider the family 4{4,A}1 (since any digit 0, 1, 2, 3, 5, 6, 7, 8, 9, B between them will produce smaller primes)
- *** Since A41 and **4441** are primes, we only need to consider the families 4{A}1 and 44{A}1 (since any digit combo 44, A4 between them will produce smaller primes)
- **** All numbers of the form 4{A}1 are divisible by 5, thus cannot be prime.
- **** The smallest prime of the form 44{A}1 is 44AAA1
- * Case (4,5):
- ** 45 is prime, and thus the only minimal prime in this family.
- * Case (4,7):
- ** Since 45, 4B, 17, 27, 37, 57, 67, 87, A7, B7, **447**, **497** are primes, we only need to consider the family 4{0,7}7 (since any digit 1, 2, 3, 4, 5, 6, 8, 9, A, B between them will produce smaller primes)

- *** Since **4707** and **4777** are primes, we only need to consider the families 4{0}7 and 4{0}77 (since any digit combo 70, 77 between them will produce smaller primes)
- **** All numbers of the form 4{0}7 are divisible by B, thus cannot be prime.
- * Case (4,B):
- ** **4B** is prime, and thus the only minimal prime in this family.
- * Case (5,1):
- ** **51** is prime, and thus the only minimal prime in this family.
- * Case (5,5):
- ** Since 51, 57, 5B, 15, 25, 35, 45, 75, 85, 95, B5, **565** are primes, we only need to consider the family 5{0,5,A}5 (since any digits 1, 2, 3, 4, 6, 7, 8, 9, B between them will produce smaller primes)
- *** All numbers of the form 5{0,5,A}5 are divisible by 5, thus cannot be prime.
- * Case (5,7):
- ** 57 is prime, and thus the only minimal prime in this family.
- * Case (5,B):
- ** **5B** is prime, and thus the only minimal prime in this family.
- * Case (6,1):
- ** **61** is prime, and thus the only minimal prime in this family.
- * Case (6,5):
- ** Since 61, 67, 6B, 15, 25, 35, 45, 75, 85, 95, B5, **655**, **665** are primes, we only need to consider the family 6{0,A}5 (since any digits 1, 2, 3, 4, 5, 6, 7, 8, 9, B between them will produce smaller primes)
- *** Since **6A05** and **6AA5** are primes, we only need to consider the families 6{0}5 and 6{0}A5 (since any digit combo A0, AA between them will produce smaller primes)
- **** All numbers of the form 6{0}5 are divisible by B, thus cannot be prime.
- **** The smallest prime of the form 6{0}A5 is 600A5
- * Case (6,7):
- ** 67 is prime, and thus the only minimal prime in this family.

- * Case (6,B):
- ** **6B** is prime, and thus the only minimal prime in this family.
- * Case (7,1):
- ** Since 75, 11, 31, 51, 61, 81, 91, **701**, **721**, **771**, **7A1** are primes, we only need to consider the family 7{4,B}1 (since any digits 0, 1, 2, 3, 5, 6, 7, 8, 9, A between them will produce smaller primes)
- *** Since 7BB, 7441 and 7B41 are primes, we only need to consider the numbers 741, 7B1, 74B1
- **** None of 741, 7B1, 74B1 are primes.
- * Case (7,5):
- ** **75** is prime, and thus the only minimal prime in this family.
- * Case (7,7):
- ** Since 75, 17, 27, 37, 57, 67, 87, A7, B7, **747**, **797** are primes, we only need to consider the family 7{0,7}7 (since any digits 1, 2, 3, 4, 5, 6, 8, 9, A, B between them will produce smaller primes)
- *** All numbers of the form 7{0,7}7 are divisible by 7, thus cannot be prime.
- * Case (7,B):
- ** Since 75, 1B, 3B, 4B, 5B, 6B, 8B, AB, **70B**, **77B**, **7BB** are primes, we only need to consider the family 7{2,9}B (since any digits 0, 1, 3, 4, 5, 6, 7, 8, A, B between them will produce smaller primes)
- *** Since 222B, 729B is prime, we only need to consider the families 7{9}B, 7{9}2B, 7{9}22B (since any digits combo 222, 29 between them will produce smaller primes)
- **** The smallest prime of the form 7{9}B is 7999B
- **** The smallest prime of the form 7{9}2B is 79992B (not minimal prime, since 992B and 7999B are primes)
- **** The smallest prime of the form 7{9}22B is 79922B (not minimal prime, since 992B is prime)
- * Case (8,1):
- ** **81** is prime, and thus the only minimal prime in this family.
- * Case (8,5):
- ** **85** is prime, and thus the only minimal prime in this family.
- * Case (8,7):
- ** 87 is prime, and thus the only minimal prime in this family.
- * Case (8,B):

- ** **8B** is prime, and thus the only minimal prime in this family.
- * Case (9,1):
- ** 91 is prime, and thus the only minimal prime in this family.
- * Case (9,5):
- ** 95 is prime, and thus the only minimal prime in this family.
- * Case (9,7):
- ** Since 91, 95, 17, 27, 37, 57, 67, 87, A7, B7, **907** are primes, we only need to consider the family 9{4,7,9}7 (since any digit 0, 1, 2, 3, 5, 6, 8, A, B between them will produce smaller primes)
- *** Since 447, 497, 747, 797, **9777**, **9947**, **9997** are primes, we only need to consider the numbers 947, 977, 997, 9477, 9977 (since any digits combo 44, 49, 74, 77, 79, 94, 99 between them will produce smaller primes)
- **** None of 947, 977, 997, 9477, 9977 are primes.
- * Case (9,B):
- ** Since 91, 95, 1B, 3B, 4B, 5B, 6B, 8B, AB, **90B**, **9BB** are primes, we only need to consider the family 9{2,7,9}B (since any digit 0, 1, 3, 4, 5, 6, 8, A, B between them will produce smaller primes)
- *** Since 27, 77B, **929B**, **992B**, **997B** are primes, we only need to consider the families 9{2,7}2{2}B, 97{2,9}B, 9{7,9}9{9}B (since any digits combo 27, 29, 77, 92, 97 between them will produce smaller primes)
- **** For the 9{2,7}2{2}B family, since 27 and 77B are primes, we only need to consider the families 9{2}2{2}B and 97{2}2{2}B (since any digits combo 27, 77 between (9,2{2}B) will produce smaller primes)
- ***** The smallest prime of the form 9{2}2{2}B is 9222B (not minimal prime, since 222B is prime)
- ***** The smallest prime of the form 97{2}2{2}B is 97222222222B (not minimal prime, since 222B is prime)
- **** For the 97{2,9}B family, since 729B and 929B are primes, we only need to consider the family 97{9}{2}B (since any digits combo 29 between (97,B) will produce smaller primes)
- ***** Since 222B is prime, we only need to consider the families 97{9}B, 97{9}2B, 97{9}22B (since any digit combo 222 between (97,B) will produce smaller primes)
- ****** All numbers of the form 97{9}B are divisible by 11, thus cannot be prime.
- ****** The smallest prime of the form 97{9}2B is 979999992B (not minimal prime, since 9999B is prime)
- ****** All numbers of the form 97{9}22B are divisible by 11, thus cannot be prime.

- **** For the 9{7,9}9{9}B family, since 77B and 9999B are primes, we only need to consider the numbers 99B, 999B, 979B, 9799B, 9979B
- ***** None of 99B, 999B, 979B, 9799B, 9979B are primes.
- * Case (A,1):
- ** Since A7, AB, 11, 31, 51, 61, 81, 91, **A41** are primes, we only need to consider the family A{0,2,A}1 (since any digits 1, 3, 4, 5, 6, 7, 8, 9, B between them will produce smaller primes)
- *** Since 221, 2A1, **A0A1**, **A201** are primes, we only need to consider the families A{A}{0}1 and A{A}{0}21 (since any digits combo 0A, 20, 22, 2A between them will produce smaller primes)
- **** For the A{A}{0}1 family:
- ***** All numbers of the form A{0}1 are divisible by B, thus cannot be prime.
- ***** The smallest prime of the form AA{0}1 is AA000001
- ***** The smallest prime of the form AAA{0}1 is AAA0001
- ***** The smallest prime of the form AAAA{0}1 is AAAA1
- ****** Since this prime has no 0's, we do not need to consider the families {A}1, {A}01, {A}001, etc.
- **** All numbers of the form A{A}{0}21 are divisible by 5, thus cannot be prime.
- * Case (A,5):
- ** Since A7, AB, 15, 25, 35, 45, 75, 85, 95, B5 are primes, we only need to consider the family A{0,5,6,A}5 (since any digits 1, 2, 3, 4, 7, 8, 9, B between them will produce smaller primes)
- *** Since 565, 665, **A605**, **A605**, **A6A5**, **AA65** are primes, we only need to consider the families A{0,5,A}5 and A{0}65 (since any digits combo 56, 60, 65, 66, 6A, A6 between them will produce smaller primes)
- **** All numbers of the form A{0,5,A}5 are divisible by 5, thus cannot be prime.
- **** The smallest prime of the form A{0}65 is A00065
- * Case (A,7):
- ** A7 is prime, and thus the only minimal prime in this family.
- * Case (A,B):
- ** **AB** is prime, and thus the only minimal prime in this family.
- * Case (B,1):

- ** Since B5, B7, 11, 31, 51, 61, 81, 91, **B21** are primes, we only need to consider the family B{0,4,A,B}1 (since any digits 1, 2, 3, 5, 6, 7, 8, 9 between them will produce smaller primes)
- *** Since 4B, AB, 401, A41, **B001**, **B0B1**, **BB01**, **BB41** are primes, we only need to consider the families B{A}0{4,A}1, B{0,4}4{4,A}1, B{0,4,A,B}A{0,A}1, B{B}B{A,B}1 (since any digits combo 00, 0B, 40, 4B, A4, AB, B0, B4 between them will produce smaller primes)
- **** For the B{A}0{4,A}1 family, since A41 is prime, we only need consider the families B0{4}{A}1 and B{A}0{A}1
- ***** For the B0{4}{A}1 family, since **B04A1** is prime, we only need to consider the families B0{4}1 and B0{A}1
- ****** The smallest prime of the form B0{4}1 is B04441 (not minimal prime, since 4441 is prime)
- ****** The smallest prime of the form B0{A}1 is B0AAAAA1 (not minimal prime, since AAAA1 is prime)
- ***** For the B{A}0{A}1 family, since A0A1 is prime, we only need to consider the families B{A}01 and B0{A}1
- ****** The smallest prime of the form B{A}01 is **BAA01**
- ****** The smallest prime of the form B0{A}1 is B0AAAAA1 (not minimal prime, since AAAA1 is prime)
- **** For the B $\{0,4\}$ 4 $\{4,A\}$ 1 family, since 4441 is prime, we only need to consider the families B $\{0,4\}$ 4 $\{4,A\}$ 1 and B $\{0,4\}$ 4 $\{A\}$ 1
- ***** For the B{0}4{4,A}1 family, since B001 is prime, we only need to consider the families B4{4,A}1 and B04{4,A}1
- ****** For the B4{4,A}1 family, since A41 is prime, we only need to consider the family B4{4}{A}1
- ******* Since 4441 and BAAA1 are primes, we only need to consider the numbers B41, B441, B4A1, B4AA1, B4AA1, B4AAA1
- ******* None of B41, B441, B4A1, B44A1, B4AA1, B44AA1 are primes.
- ****** For the B04{4,A}1 family, since **B04A1** is prime, we only need to consider the family B04{4}1
- ******* The smallest prime of the form B04{4}1 is B04441 (not minimal prime, since 4441 is prime)
- ***** For the B{0,4}4{A}1 family, since 401, 4441, B001 are primes, we only need to consider the families B4{A}1, B04{A}1, B44{A}1, B044{A}1 (since any digits combo 00, 40, 44 between (B,4{A}1) will produce smaller primes)
- ****** The smallest prime of the form B4{A}1 is B4AAA1 (not minimal prime, since BAAA1 is prime)
- ***** The smallest prime of the form B04{A}1 is **B04A1**
- ****** The smallest prime of the form B44{A}1 is B44AAAAAAA1 (not minimal prime, since BAAA1 is prime)

- ****** The smallest prime of the form B044{A}1 is B044A1 (not minimal prime, since B04A1 is prime)
- **** For the $B\{0,4,A,B\}A\{0,A\}1$ family, since all numbers in this family with 0 between (B,1) are in the $B\{A\}0\{4,A\}1$ family, and all numbers in this family with 4 between (B,1) are in the $B\{0,4\}4\{4,A\}1$ family, we only need to consider the family $B\{A,B\}A\{A\}1$
- ***** Since **BAAA1** is prime, we only need to consider the families B{A,B}A1 and B{A,B}AA1
- ****** For the B{A,B}A1 family, since AB and **BAAA1** are primes, we only need to consider the families B{B}A1 and B{B}AA1
- ******* All numbers of the form B{B}A1 are divisible by B, thus cannot be prime.
- ****** The smallest prime of the form B{B}AA1 is BBBAA1
- ****** For the B{A,B}AA1 family, since **BAAA1** is prime, we only need to consider the families B{B}AA1
- ****** The smallest prime of the form B{B}AA1 is BBBAA1
- **** For the B{B}B{A,B}1 family, since AB and BAAA1 are primes, we only need to consider the families B{B}B1, B{B}BA1, B{B}BAA1 (since any digits combo AB or AAA between (B{B}B,1) will produce smaller primes)
- ***** The smallest prime of the form B{B}B1 is BBBB1
- ***** All numbers of the form B{B}BA1 are divisible by B, thus cannot be prime.
- ***** The smallest prime of the form B{B}BAA1 is BBBAA1
- * Case (B,5):
- ** **B5** is prime, and thus the only minimal prime in this family.
- * Case (B,7):
- ** **B7** is prime, and thus the only minimal prime in this family.
- * Case (B,B):
- ** Since B5, B7, 1B, 3B, 4B, 5B, 6B, 8B, AB, **B2B** are primes, we only need to consider the family B{0,9,B}B (since any digits 1, 2, 3, 4, 5, 6, 7, 8, A between them will produce smaller primes)
- *** Since 90B and 9BB are primes, we only need to consider the families B{0,B}{9}B
- **** Since 9999B is prime, we only need to consider the families $B\{0,B\}B$, $B\{0,B\}99B$, $B\{0,B\}999B$
- ***** All numbers of the form B{0,B}B are divisible by B, thus cannot be prime.
- ***** For the B{0,B}9B family:

****** Since **B0B9B** and **BB09B** are primes, we only need to consider the families B{0}9B and B{B}9B (since any digits combo 0B, B0 between (B,9B) will produce smaller primes)

******* All numbers of the from B{B}9B is either divisible by 11 (if totally number of B's is even) or factored as $10^{2*n}=(10^n-5)$ (if totally number of B's is odd number 2^n-1), thus cannot be prime.

***** For the B{0,B}99B family:

****** Since B0B9B and BB09B are primes, we only need to consider the families B{0}99B and B{B}99B (since any digits combo 0B, B0 between (B,99B) will produce smaller primes)

****** The smallest prime of the form B{0}99B is B00099B

****** The smallest prime of the form B{B}99B is BBBBBB99B

***** For the B{0,B}999B family:

****** Since B0B9B and BB09B are primes, we only need to consider the families B{0}999B and B{B}999B (since any digits combo 0B, B0 between (B,999B) will produce smaller primes)

****** The smallest prime of the form B{0}999B is be written as B01765999B and equal the prime 11*121769+16967 (factordb entry) (primality certificate of this prime) (shown in base 12) (factorization of the numbers of this form) (not minimal prime, since B00099B and B000000000000000000000000009B are primes)

****** The smallest prime of the form B{B}999B is

Conclusion and perspectives

References

Main reference for this article: The Mersenneforum thread https://mersenneforum.org/showthread.php?t=24972 (which is the entry of the researching in this article in Mersenneforum)

Other references:

- [1] http://primes.utm.edu/glossary/xpage/MinimalPrime.html (article "minimal prime" in The Prime Glossary)
- [2] https://en.wikipedia.org/wiki/Minimal_prime_(recreational_mathematics) (article "minimal prime" in Wikipedia)
- [3] https://www.primepuzzles.net/puzzles/puzz 178.htm (the puzzle of minimal primes (when the restriction of prime>base is not required) in The Prime Puzzles & Problems Connection, warning: the solutions for the minimal 4k+1 and 4k-1 primes given by Andrew Rupinsiki have errors, the list wrongly including many primes which are not minimal primes, for the correct solution see

https://raw.githubusercontent.com/curtisbright/mepn-

data/master/data/primes1mod4/minimal.10.txt and

https://raw.githubusercontent.com/curtisbright/mepn-

<u>data/master/data/primes3mod4/minimal.10.txt</u> (or <u>https://oeis.org/A111055/b111055.txt</u> and <u>https://oeis.org/A111056/b111056.txt</u>, note: since the limit of the numbers in OEIS b-file is 10^{1000} –1, the list <u>https://oeis.org/A111056/b111056.txt</u> does not include the large prime $2^{19151}99 = (2*10^{19153}+691)/9$, respectively)

- [4] https://www.primepuzzles.net/problems/prob 083.htm (the problem of minimal primes in The Prime Puzzles & Problems Connection)
- [5] <u>https://github.com/xayahrainie4793/non-single-digit-primes</u> (my data for these $M(L_b)$ sets for $2 \le b \le 16$)
- [6] http://recursed.blogspot.com/2006/12/prime-game.html (Shallit's The Prime Game page)

- [7] http://www.cs.uwaterloo.ca/~shallit/Papers/minimal5.pdf (Shallit's proof of base 10 minimal primes, when the restriction of prime>base is not required) (the same pdf files: http://www.wiskundemeisjes.nl/wp-content/uploads/2007/02/minimal.pdf and http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.7.5686&rep=rep1&type=pdf)
- [8] https://archive.ph/IGZE1 (proofs of minimal primes in bases $b \le 10$, when the restriction of prime>base is not required, warning: the sets of $M(L_b)$ have errors for b = 8 and b = 10, b = 8 misses the prime 6101 and b = 10 misses the primes 9001 and 9049 and instead wrongly including the primes 90001, 90469, and 9000049, thus the correct values of $|M(L_b)|$ for b = 8 and b = 10 are 15 and 26 (instead of 14 and 27), respectively) (this is the copy of the site to GitHub, original link is https://scholar.colorado.edu/downloads/hh63sw661)
- [9] https://cs.uwaterloo.ca/~cbright/reports/mepn.pdf (the article for this minimal prime problem in bases *b*≤30, when the restriction of prime>base is not required, **warning: this article incorrectly uses "subword" or "substring" for subsequence**) (a similar pdf file: https://cs.uwaterloo.ca/~shallit/Papers/br10.pdf)
- [10] <u>https://cs.uwaterloo.ca/~cbright/talks/minimal-slides.pdf</u> (the article for this minimal prime problem in bases $b \le 30$, when the restriction of prime>base is not required, **warning:** this article incorrectly uses "subword" for subsequence)
- [11] https://archive.ph/ci2yM (the article for this minimal prime problem in bases b≤30, when the restriction of prime>base is not required, warning: this article incorrectly uses "substring" for subsequence) (this is the copy of the site to GitHub, original links are https://doi.org/10.1080/10586458.2015.1064048 and https://raw.githubusercontent.com/curtisbright/mepn-data/master/report/report.tex)
- [12] https://github.com/curtisbright/mepn-data (data for these $M(L_b)$ sets and unsolved families for $2 \le b \le 30$, when the restriction of prime>base is not required, file "minimal.b.txt" is the data of all known minimal primes or PRPs in base b, and file "unsolved.b.txt" is the list of all unsolved families (families for which not even a probable prime is known nor can be ruled out as only contain composites) in base b, the format of the unsolved families is xy^*z for xyyy...yyyz, for bases $2 \le b \le 16$ and b = 18, 20, 22, 23, 24, 30 are completely solved, except the largest element in $M(L_{13})$ and largest 9 elements in $M(L_{23})$ (except the secondlargest element in $M(L_{23})$, it can be proven prime using N-1 primality test, since n-1 can be trivially fully factored for this number n) are only probable primes, i.e. not definitely primes, thus we cannot definitely say that the corresponding families can be removed from the list of unsolved families, and we cannot definitely compute this part of the sets $M(L_b)$, search limits of lengths for the unsolved families (families for which not even a probable prime is known nor can be ruled out as only contain composites) in base b: 1000000 for b = 17, 707000 for b = 19, 506000 for b = 21, 292000 for b = 25, 486000 for b = 26, 368000 for b = 27, 543000 for b = 28, 233000 for b = 29)
- [13] <u>https://github.com/RaymondDevillers/primes</u> (data for these $M(L_b)$ sets and unsolved families for $28 \le b \le 50$, when the restriction of prime>base is not required, using lowercase letters a-n to represent digit values 36 to 49 for bases b > 36, file "kernel b" is the data of all known minimal primes or PRPs in base b, and file "left b" is the list of all unsolved families

(families for which not even a probable prime is known nor can be ruled out as only contain composites) in base b, the format of the unsolved families is $x\{y\}z$ for xyyy...yyyz, only bases b = 30 and b = 42 are completely solved, search limits of lengths for the unsolved families (families for which not even a probable prime is known nor can be ruled out as only contain composites) in base b: 10000 for all b)

b	Number of known minimal primes or PRPs (when the restrictio n of prime>b ase is not required)	Num ber of unsol ved famili es when the restri ction of prim e>ba se is not requi red	Additional minimal primes or PRPs (when the restriction of prime>base is not required) not in the lists	Unneeded families when the restriction of prime>base is not required	Search limit higher then the lists
2	2	0			
<u>3</u>	3	0			
<u>4</u>	3	0			
<u>5</u>	8	0			
<u>6</u>	7	0			
<u>7</u>	9	0			
<u>8</u>	15	0			
9	12	0			
<u>10</u>	26	0			
<u>11</u>	152	0			
<u>12</u>	17	0			
<u>13</u>	228	0			
<u>14</u>	240	0			
<u>15</u>	100	0			
<u>16</u>	483	0			

	1				
17	1279	1			
<u>18</u>	50	0			
19	3462	1			
<u>20</u>	651	0			
21	2600	1			
22	1242	0			
23	6021	0			
<u>24</u>	306	0			
25	17597	12			
<u>26</u>	5662	2			
<u>27</u>	17210	5			
28	5783	1			
29	57283	14			
30	220	0			
31	79189	14	E8U ²¹⁸⁶⁶ P = 443*31 ²¹⁸⁶⁷ -6 IEL ²⁹⁷⁸⁷ = (5727*31 ²⁹⁷⁸⁷ -7)/10 LF ²¹⁰⁵² G = (43*31 ²¹⁰⁵³ +1)/2 MIO ¹⁰⁷⁴⁷ L = (3504*31 ¹⁰⁷⁴⁸ -19)/5 PEO0 ²²³⁶⁷ Q = 24483*31 ²²³⁶⁸ +26 L ¹⁰⁰¹² 9G = (7*31 ¹⁰⁰¹⁴ -3777)/10 R ²²¹³⁷ 1R = (9*31 ²²¹³⁹ -8069)/10	ILE{L} (no primes or PRPs up to ILEL30000, and IEL29787 is PRP) L0{F}G (no primes or PRPs up to L0F23000G, and LF21052G is PRP) {L}9IG (no primes or PRPs up to L130009IG, and L100129G is PRP)	M{P} (searched to length 41962) P{F}G (searched to length 37061) SP{0}K (searched to length 28000) {F}G (searched to length 4194303) {F}KO (searched to length 35000) {F}RA (searched to length 34000) {L}CE (searched to length 21000)

_	1	ı		1	
					{L}G (searched to length 30000) {L}IS (searched to length 25000) {L}SO (searched to length 22000) {P}I (searched to length 32000) {R}1 (searched to length 27000) {R}8 (searched to length 27000) {R}8 (searched to length 33000) {U}P8K (searched to length 30000)
32	45205	78			4{0}1 (searched to length 1717986918) G{0}1 (searched to length 3435973836) UG{0}1 (searched to length 560002)
33	57676	33			
34	56457	33			
35	182378	15			
<u>36</u>	6296	1	P ⁸¹⁹⁹³ SZ = (5*36 ⁸¹⁹⁹⁵ +821)/7		O{L}Z (searched to length 100000)

37	314988	275	FYa ²²⁰²¹ = 590*37 ²²⁰²¹ -1		
	31 1000	210	$R8a^{20895} = 1008*37^{20895}-1$		
38	106838	77			1{0}1 (searched to length 16777216)
39	230317	43			
40	37773	1	$QaU^{12380}X = (13998*40^{12381}+29)/13$		S{Q}d (searched to length 100000)
41	689061	335			
42	4551	0			
43	900795	536			
44	255911	103			
45	323437	47	O0 ¹⁸⁵²¹ 1 = 24*45 ¹⁸⁵²² +1	AO{0}1 (the smallest prime is AO0 ⁴⁴⁷⁹⁰ 1 = 474*45 ⁴⁴⁷⁹¹ +1, but O0 ¹⁸⁵²¹ 1 is prime) (Note: O{0}1F1 and O{0}ZZ1 are still needed, since they are only searched to length 10000)	9W1{0}1 (searched to length 100003)
46	399012	113			d4{0}1 (searched to length 500002)
47	1436289	994			
48	29103	6			a{0}1 (searched to length 500001)
49	4365269	1183	$11c0^{29736}1 = 2488*49^{29737}+1$ $Fd0^{18340}1 = 774*49^{18341}+1$ $SLm^{52698} = 1394*49^{52698}-1$ $Ydm^{16337} = 1706*49^{16337}-1$	(Note: S6L{m}, YUUd{m}, YUd{m} are	

			still needed, since they are only searched to length 10000)	
50	189914	62		1{0}1 (searched to length 16777216) a{n} (searched to length 121290)

[14] http://www.bitman.name/math/article/730 (article for minimal primes, when the restriction of prime>base is not required)

[15] <u>http://www.bitman.name/math/table/497</u> (data for minimal primes in bases $2 \le b \le 16$, when the restriction of prime>base is not required)

[16] https://oeis.org/A071071/a071071.pdf (research of minimal sets of powers of 2, when the restriction of >base is not required)

[17] http://nntdm.net/papers/nntdm-25/NNTDM-25-1-036-047.pdf (research of minimal set of totients+n in base b = 10 for $0 \le n \le 5$, when the restriction of >base is not required) (this is from the article: https://arxiv.org/pdf/1607.01548.pdf, which is research of minimal set of the range of Euler phi function and the range of Dedekind psi function, both in base b = 10)

S	the minimal set of S (in base $b = 10$) (unlike the research of the minimal primes in this article, the restriction of >base is not required)
primes	{2, 3, 5, 7, 11, 19, 41, 61, 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66600049}
composites	{4, 6, 8, 9, 10, 12, 15, 20, 21, 22, 25, 27, 30, 32, 33, 35, 50, 51, 52, 55, 57, 70, 72, 75, 77, 111, 117, 171, 371, 711, 713, 731}
squares	{1, 4, 9, 25, 36, 576, 676, 7056, 80656, 665856, 2027776, 2802276, 22282727076, 77770707876, 78807087076, 7888885568656, 8782782707776, 72822772707876, 555006880085056, 782280288087076, 827702888070276, 888288787822276, 2282820800707876, 7880082008070276, 80077778877070276, 88778000807227876, 7828288878078078276, 2707700770820007076, 7078287780880770276, 780828782727072820207276, 7808287827727072807876, 8008002202002207876, 2728277277702807876, 70880800720008787876, 72887222220777087876, 80028077888770207876, 80880700827207270276, 87078270070088278276, 88002002000028027076,

	2882278278888228807876, 8770777780888228887076, 777000272228288822007876, 702087807788807888287876, 788708087882007280808827876, 880070008077808877000002276, 888000227087070707880827076, 888588886555505085888555556, 777000080078008878828227776, 7782727788888878708800870276, 5000060065066660656065555556, 807000880082288008070880087876, 8078787080888888087277777227076, 80000808807082087087077778827876, 822822722220080888878078820887876,} (currently not known, and might be extremely difficult to found)
cubes	{1, 8, 27, 64, 343, 729, 3375, 4096, 35937, 39304, 46656, 50653, 79507, 97336, 300763, 405224, 456533, 474552, 493039, 636056, 704969, 3307949, 4330747, 5545233, 5639752, 5735339, 6539203, 9663597, 23393656, 23639903, 29503629, 37933056, 40353607, 45499293, 50243409, 54439939, 57066625, 57960603, 70444997, 70957944, 73560059, 76765625, 95443993, 202262003, 236029032, 350402625, 377933067, 379503424, 445943744, 454756609, 537367797, 549353259, 563559976, 567663552, 773620632, 907039232,} (currently not known, and might be extremely difficult to found)
primes == 1 mod 4	{5, 13, 17, 29, 37, 41, 61, 73, 89, 97, 101, 109, 149, 181, 233, 277, 281, 349, 409, 433, 449, 677, 701, 709, 769, 821, 877, 881, 1669, 2221, 3001, 3121, 3169, 3221, 3301, 3833, 4969, 4993, 6469, 6833, 6949, 7121, 7477, 7949, 9001, 9049, 9221, 9649, 9833, 9901, 9949, 11969, 19121, 20021, 20201, 21121, 23021, 23201, 43669, 44777, 47777, 60493, 60649, 66749, 80833, 90121, 91121, 91921, 91969, 94693, 111121, 112121, 119921, 199921, 220301, 466369, 470077, 666493, 666649, 772721, 777721, 777781, 779981, 799921, 800333, 803333, 806033, 833033, 833633, 860333, 863633, 901169, 946369, 946669, 999169, 1111169, 1999969, 4007077, 4044077, 4400477, 4666693, 8000033, 8006633, 8006633, 8600633, 8660633, 8860033, 8830033, 8863333, 8866633, 22000001, 40400077, 44040077, 60000049, 66000049, 66600049, 79999981, 80666633, 83333333, 86606633, 86666633, 8860033, 88888033, 88880033, 88880033, 88880333, 88880333, 88880333, 88880333, 888880333, 888880333, 888888333, 400404444444444477, 777777777777921, 888888888888888888888888888888888888
primes == 3 mod 4	{3, 7, 11, 19, 59, 251, 491, 499, 691, 991, 2099, 2699, 2999, 4051, 4451, 4651, 5051, 5651, 5851, 6299, 6451, 6551, 6899, 8291, 8699, 8951, 8999, 9551, 9851, 22091, 22291, 66851, 80051, 80651, 84551, 85451, 86851, 88651, 92899, 98299, 98899, 200891,

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208891, 228299, 282299, 545551, 608851, 686051, 822299,
828899, 848851, 866051, 880091, 885551, 888091, 888451,
902299, 909299, 909899, 2000291, 2888299, 2888891, 8000099,
8000891, 8000899, 8028299, 8808299, 8808551, 8880551,
8888851, 9000451, 9000899, 9908099, 9980099, 9990899,
9998099, 9999299, 60000851, 60008651, 60086651, 60866651,
68666651, 80088299, 80555551, 80888299, 88808099, 88808899,
88880899, 90000299, 90080099, 222222899, 800888899,
808802899, 808880099, 808888099, 888800299, 888822899,
992222299, 2222288899, 8808888899, 8888800099, 8888888299,
888888891, 48555555551, 55555555551, 999999999999999,
2288888888888888888888899.
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222222222222222222222222222299 (with 19151 2's, which can be written as 21915199 and equal the prime (2*10¹⁹¹⁵³+691)/9 (factordb entry) (shown in base 10) (primality certificate of this prime)}

palindromic primes

{2, 3, 5, 7, 11, 919, 94049, 94649, 94849, 94949, 96469, 98689, 9809089, 9888889, 9889889, 9908099, 9980899, 9989899, 900808009, 906686609, 906989609, 908000809, 908444809, 908808809, 909848909, 960898069, 968999869, 988000889.

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989040989, 996686699, 996989699, 999686999, 90689098609,
9089999809, 90999899909, 96099899069, 96600800669,
96609890669, 98000000089, 98844444889, 9009004009009,
9099094909909, 9600098900069, 9668000008669,
969998999969, 984444444489, 9899900099989,
9900004000099, 9900994990099, 900006898600009,
900904444409009, 966666989666669, 966668909866669.
96669989996669, 999090040090999, 999904444409999,
9000006860000009, 90000008480000009, 90000089998000009,
90999444444499909, 96000060806000069, 99900944444900999,
99990009490009999, 99999884448899999,
9000090994990900009, 9000094444444900009,
966666080806666669, 966666666866666669,
990999994999999999, 9999444444444449999.
999990994999099999, 9999990994990999999,
90000000080000000009, 900999994444499999009,
9899999444444499999989, 990444444444444444444099,
9094444444444444444444444909.
9000000099999994999999000000009.
9000000999999999994999999999990000009. ....
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	44444444444444444444444444444444444444
powers of 2	{1, 2, 4, 8, 65536} (conjectured by Jeffrey Shallit, not proven, however of course, if all powers of 2 except 65536 contain at least one of 1, 2, 4, 8, then this conjecture is true)
Fibonacci numbers	{1, 2, 3, 5, 8} (conjectured by Jeffrey Shallit, not proven, however of course, if all Fibonacci numbers contain at least one of 1, 2, 3, 5, 8, then this conjecture is true)
range of Euler phi function	{1, 2, 4, 6, 8, 30, 70, 500, 900, 990, 5590, 9550, 5555555555
range of Dedekind psi function	{1, 3, 4, 6, 8, 20, 72, 90, 222, 252, 500, 522, 552, 570, 592, 750, 770, 992, 7000, 55555555555555555555555555555555
totients+1	{2, 3, 5, 7, 9, 11, 41, 61, 81}
totients+2	{3, 4, 6, 8, 10, 12, 20, 22, 50, 72, 90, 770, 992, 5592, 9552, 55555555552} (conjectured, not proven, this conjecture is true if and only if there are no totients of the form 6{9}8, and such totients are conjectured not exist, since such totients are == 2 mod 12, thus must be of the form $(p-1)^*p^n$ with p prime and p odd)
totients+3	{4, 5, 7, 9, 11, 13, 21, 23, 31, 33, 61, 63, 81, 83}
totients+4	$\{5, 6, 8, 10, 12, 14, 20, 22, 24, 32, 34, 40, 44, 70, 74, 92, 300, 472, 772, 900, 904, 994\}$ (conjectured, not proven, this conjecture is true if and only if there are no totients of the form $\{3,9\}26$ or $\{3,9\}86$, and such totients are conjectured not exist, since such totients are == 2 mod 12, thus must be of the form $(p-1)^*p^n$ with p prime and p odd)
totients+5	$\{6, 7, 9, 11, 13, 15, 21, 23, 25, 33, 35, 41, 45, 51, 53, 83, 85, 301, 443, 505, 801, 881, 5555555555555555\}$ (conjectured, not proven, this conjecture is true if and only if there are no totients of the form $3\{9\}8$, and such totients are conjectured not exist, since such totients are == 2 mod 12, thus must be of the form $(p-1)^*p^n$ with p prime and p odd)

- [19] http://www.prothsearch.com/rieselprob.html (the Riesel problem)
- [20] http://www.primegrid.com/ (with projects for the Sierpinski problem, the Riesel problem, the Prime Sierpinski problem, the Extended Sierpinski problem, Sierpinski/Riesel base 5 problem, generalized Fermat prime search)
- [21] <u>http://www.prothsearch.com/</u> (lists for primes of the form k^*2^n+1 for odd k<1200, also factoring status of generalized Fermat numbers of the form $a^{2^n} + b^{2^n}$ for $1 \le b < a \le 12$)
- [22] https://archive.fo/VkelU (lists for primes of the form k^*2^n-1 for odd k<10000)
- [23] https://www.rieselprime.de/default.htm (lists for primes of the form $k*2^n\pm1$)
- [24] http://www.noprimeleftbehind.net/crus/Sierp-conjectures.htm (generalized Sierpinski conjectures in bases $b \le 1030$, some primes found in these conjectures are minimal primes in base b, especially, all primes for k < b (if exist for a (k,b) combo) are always minimal primes in the base b) (also some examples for simple families contain no primes > b)
- [25] http://www.noprimeleftbehind.net/crus/Riesel-conjectures.htm (generalized Riesel conjectures in bases $b \le 1030$, some primes found in these conjectures are minimal primes in base b, especially, all primes for k < b (if exist for a (k,b) combo) are always minimal primes in the base b) (also some examples for simple families contain no primes > b)
- [26] http://www.noprimeleftbehind.net/crus/tab/CRUS tab.htm (list for the status of the generalized Sierpinski conjectures and the generalized Riesel conjectures in bases b≤1030)
- [27] https://www.utm.edu/staff/caldwell/preprints/2to100.pdf (article for generalized Sierpinski conjectures in bases b≤100)
- [28] https://oeis.org/A076336/a076336c.html (the dual Sierpinski problem)
- [29] https://mersenneforum.org/showthread.php?t=10761 (list of large (probable) primes for the dual Sierpinski problem) (for the full list see https://www.mit.edu/~kenta/three/prime/dual-sierpinski/ezgxqqdm/dualsierp-excerpt.txt)
- [30] http://www.kurims.kyoto-u.ac.jp/EMIS/journals/INTEGERS/papers/i61/i61.pdf (article for the mixed (original+dual) Sierpinski problem)
- [31] https://mersenneforum.org/showthread.php?t=6545 (research for the mixed (original+dual) Riesel problem)
- [32] https://mersenneforum.org/showthread.php?t=26328 (research for the mixed (original+dual) Sierpinski base 5 problem)
- [33] <u>http://www.fermatquotient.com/</u> (generalized repunit primes (primes of the form $(b^n-1)/(b-1)$) in bases $b \le 160$, the smallest such prime for base b (if exists) is always minimal

prime in base *b*) and (generalized half Fermat primes (primes of the form $(b^{2^n} + 1)/2$) sorted by *n*, the smallest such prime for base *b* (if exists) is always minimal prime in base *b*)

- [34] <u>https://archive.ph/tf7jx</u> (generalized repunit primes (primes of the form $(b^n-1)/(b-1)$) in bases $b \le 1000$, the smallest such prime for base b (if exists) is always minimal prime in base b)
- [35] <u>http://jeppesn.dk/generalized-fermat.html</u> (generalized Fermat primes (primes of the form $b^{2^n} + 1$) in even bases $b \le 1000$, the smallest such prime for base b (if exists) is always minimal prime in base b)
- [36] <u>http://www.noprimeleftbehind.net/crus/GFN-primes.htm</u> (generalized Fermat primes (primes of the form $b^{2^n} + 1$) in even bases $b \le 1030$, the smallest such prime for base b (if exists) is always minimal prime in base b)
- [37] <u>https://harvey563.tripod.com/wills.txt</u> (primes of the form $(b-1)*b^n-1$ for bases $b \le 2049$, the smallest such prime for base b (if exists) is always minimal prime in base b)
- [38] <u>https://www.rieselprime.de/ziki/Williams_prime_MM_least</u> (the smallest primes of the form $(b-1)^*b^n-1$ for bases $b \le 2049$, these primes (if exists) is always minimal prime in base b)
- [39] <u>https://www.rieselprime.de/ziki/Williams_prime_MP_least</u> (the smallest primes of the form $(b-1)^*b^n+1$ for bases $b \le 1024$, these primes (if exists) is always minimal prime in base b)
- [40] <u>https://www.rieselprime.de/ziki/Riesel_prime_small_bases_least_n</u> (the smallest primes of the form k^*b^n -1 for $k \le 12$ and bases $b \le 1024$, these primes (if exists) is always minimal prime in base b if b > k)
- [41] <u>https://www.rieselprime.de/ziki/Proth_prime_small_bases_least_n</u> (the smallest primes of the form k^*b^n+1 for $k \le 12$ and bases $b \le 1024$, these primes (if exists) is always minimal prime in base b if b > k)
- [42] https://docs.google.com/spreadsheets/d/e/2PACX-1vTKkSNKGVQkUINlp1B3cXe90FWPwiegdA07EE7-U7sqXntKAEQrynol1sbFvvKriieda3LfkqRwmKME/pubhtml (my list for the smallest primes or PRPs (only primes (or PRPs) > base are considered) in given simple family in bases b ≤ 1024, including these families:
- * Repunit family $(b^n-1)/(b-1)$ (family $\{1\}$, $n \ge 2$ is needed, since n=1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 100000)
- * Fermat family b^n+1 (family 1{0}1, $n \ge 1$ is needed) (search limit of the length: ≥ 8388608)
- * Half Fermat family $(b^n+1)/2$ (family **{#}**\$, $n \ge 2$ is needed, since n = 1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 262143)

- * Wagstaff family $(b^n+1)/(b+1)$ (family $\{z0\}z1$, $n \ge 3$ is needed, since n must be odd, and n = 1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 17326)
- * Proth families k^*b^n+1 for $2 \le k \le 12$ (this includes families **2{0}1**, **3{0}1**, **4{0}1**, **5{0}1**, **6{0}1**, **7{0}1**, **8{0}1**, **9{0}1**, **A{0}1**, **B{0}1**, **C{0}1**, as in the Sierpinski conjectures, $n \ge 1$ is needed) (search limit of the length: ≥ 100000)
- * Riesel families k^*b^n-1 for $2 \le k \le 12$ (this includes families 1{z}, 2{z}, 3{z}, 4{z}, 5{z}, 6{z}, 7{z}, 8{z}, 9{z}, A{z}, B{z}, as in the Riesel conjectures, $n \ge 1$ is needed) (search limit of the length: ≥ 100000)
- * b^n+k for $2 \le k \le 4$ (this includes families **1{0}2**, **1{0}3**, **1{0}4**, $n \ge 1$ is needed) (search limit of the length: ≥ 5000)
- * $b^n k$ for $2 \le k \le 4$ (this includes families $\{z\}y$, $\{z\}x$, $\{z\}w$, $n \ge 2$ is needed, since n = 1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 5000)
- * Williams families $(b\pm 1)^*b^n\pm 1$ (this includes families 11{0}1 (case "++"), 10{z} (case "+-"), z{0}1 (case "-+"), y{z} (case "--"), $n \ge 1$ is needed) (search limit of the length: ≥ 100000)
- * Dual Williams families $b^n \pm (b \pm 1)$ (this includes families 1{0}11 (case "++", $n \ge 2$ is needed, since n = 1 will produce the number "21", which is not in the family), 1{0}z (case "+-", $n \ge 1$ is needed), {z}yz (case "-+", $n \ge 2$ is needed, since n = 1 will produce negative numbers), {z}1 (case "--", $n \ge 2$ is needed, since n = 1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 5000)
- * Families $x\{y\}$ and $\{x\}y$ with x, $y \le 4$ (not all done, currently only families $1\{y\}$ and $\{x\}1$ and $\{1\}2$ and $2\{1\}$ are in the list) (search limit of the length: ≥ 5000)
- * Families $x\{0\}y$ with $x, y \le 4$ (search limit of the length: ≥ 5000)
- * Family $((b-2)^*b^n+1)/(b-1)$ (family $\{y\}z$, $n \ge 2$ is needed, since n = 1 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 5000)
- * Family $(b^n-(2^*b-1))/(b-1)$ (family **{1}0z**, $n \ge 3$ is needed, since n = 1 will produce negative numbers, and n = 2 will produce single-digit numbers, which is not allowed in this research) (search limit of the length: ≥ 5000)

where z means b–1, y means b–2, x means b–3, w means b–4, # means (b–1)/2 (for odd b), \$ means (b+1)/2 (for odd b), the format of the families is $x\{y\}z$ for xyyy...yyyz, numbers in the list are the lengths of these primes or PRPs in base b, "RC" means this family can be ruled out as only contain composite numbers (only count numbers > base), "NB" means this family is not interpretable in this base (including the case which this family has either leading zeros (leading zeros do not count) or trailing zeros (numbers ending in zero cannot be prime > base) in this base), "unknown" means this family has no known primes or PRPs, the smallest primes in these families may not be minimal primes in the same base b:

family	the smallest prime in this family is a minimal prime if and only if there is no smaller prime of this family(s)
1{0}11	1{0}1
10{z}	1{z}
11{0}1	1{0}1

{1}0z	{1}, {1}z ({1}z is not in the list)
{1}2	{1}
2{1}	{1}
{z0}z1	(almost cannot be minimal prime, this family is of interest only because generalized Wagstaff primes)
{z}yz	{z}y

and the smallest primes in other families in the list (if exists) are always minimal primes in the same base b, and since only primes (or PRPs) > base are considered, the smallest allowed length is 2 (i.e. length 1 is not allowed).

Family	Algebraic form of the family (<i>n</i> is the length)	The smallest allowed base b (if the base b is not allowed, then listed as "NB" in the table)	The smallest allowed length	Bases such that this family can be ruled out as only contain composite numbers (only count numbers > base) (listed "RC" in the table)	Search limit of the lengths (<i>n</i>)
1{0}1	<i>b</i> ^{<i>n</i>-1} +1	2	2	$b == 1 \mod 2$ (trivial factor 2) $b = m^r$ with odd $r > 1$ (sum-of- r -th-powers factorization)	≥8388608
1{0}2	<i>b</i> ^{<i>n</i>-1} +2	3	2	b == 0 mod 2 (trivial factor 2) b == 1 mod 3 (trivial factor 3)	≥5000
1{0}3	<i>b</i> ⁿ⁻¹ +3	4	2	$b == 1 \mod 2$ (trivial factor 2) $b == 0 \mod 3$ (trivial factor 3)	≥5000
1{0}4	<i>b</i> ⁿ⁻¹ +4	5	2	$b == 0 \mod 2$	≥5000

				(trivial factor 2) $b == 1 \mod 5$ (trivial factor 5) $b == 14 \mod 15$ (covering set $\{3,5\}$) $b = m^4$ (Aurifeuillian factorization for $x^4 + 4y^4$)	
1{0}z	<i>b</i> ^{n−1} +(<i>b</i> −1)	2	2	(none)	≥5000
1{0}11	<i>b</i> ⁿ⁻¹ +(<i>b</i> +1)	2	3 (there is no number in this family with length 2 at all)	b == 1 mod 3 (trivial factor 3)	≥5000
10{z}	(<i>b</i> +1)* <i>b</i> ^{<i>n</i>−2} −1	2	3 (the number with length 2 is 10, whose value is <i>b</i> and not > <i>b</i> , thus not allowed)	(none)	≥100000
11{0}1	(<i>b</i> +1)* <i>b</i> ^{<i>n</i>-2} +1	2	3 (there is no number in this family with length 2 at all)	b == 1 mod 3 (trivial factor 3)	≥100000
{1}0z	(b ⁿ -(2*b-1))/ (b-1)	2	3 (the number with length 2 is z, whose value is <i>b</i> –1 and not > <i>b</i> , thus not allowed)	b such that b and $2*b-1$ both squares (difference-of-squares factorization) (this includes $b = 25, 841$)	≥5000
{1}	(b ⁿ -1)/(b-1)	2	2	b = m ^r with r>1 (difference-of-r-th-powers factorization)	≥100000
{1}2	(b ⁿ +(b-2))/(b -1)	3	2	b == 0 mod 2 (trivial factor 2)	≥5000

	1	I	T	T	
1{2}		3	2		≥2500
1{3}		4	2		≥2500
1{4}		5	2		≥2500
1{z}	2* <i>b</i> ⁿ⁻¹ -1	2	2	(none)	≥100000
2{0}1	2* <i>b</i> ^{<i>n</i>-1} +1	3	2	b == 1 mod 3 (trivial factor 3)	≥100000
2{0}3	2* <i>b</i> ^{<i>n</i>-1} +3	4	2		≥2500
2{1}	((2* <i>b</i> -1)* <i>b</i> ⁿ⁻¹ -1)/(<i>b</i> -1)	3	2	b such that b and $2*b-1$ both squares (difference- of-squares factorization) (this includes b = 25, 841)	≥5000
{2}1		3	2		≥2500
2{z}	3* <i>b</i> ^{<i>n</i>-1} -1	3	2	b == 1 mod 2 (trivial factor 2)	≥100000
3{0}1	3* <i>b</i> ^{<i>n</i>-1} +1	4	2	b == 1 mod 2 (trivial factor 2)	≥100000
3{0}2	3* <i>b</i> ^{<i>n</i>-1} +2	4	2		≥2500
3{0}4	3* <i>b</i> ^{<i>n</i>-1} +4	5	2		≥2500
{3}1		4	2		≥2500
3{z}	4* <i>b</i> ^{<i>n</i>-1} -1	4	2	$b == 1 \mod 3$ (trivial factor 3) $b == 14 \mod 15$ (covering set $\{3,5\}$) $b == 4 \mod 5$ (even length: factor 5, odd length: difference-of-squares factorization) $b = m^2$ (difference-of-squares	≥100000

				factorization)	
4{0}1	4* <i>b</i> ⁿ⁻¹ +1	5	2	$b == 1 \mod 5$ (trivial factor 5) $b == 14 \mod 15$ (covering set $\{3,5\}$) $b = m^4$ (Aurifeuillian factorization for x^4+4y^4)	≥100000
4{0}3	4*b ⁿ⁻¹ +3	5	2		≥2500
{4}1		5	2		≥2500
4{z}	5* <i>b</i> ^{<i>n</i>-1} -1	5	2	b == 1 mod 2 (trivial factor 2)	≥100000
5{0}1	5* <i>b</i> ⁿ⁻¹ +1	6	2	b == 1 mod 2 (trivial factor 2) b == 1 mod 3 (trivial factor 3)	≥100000
5{z}	6* <i>b</i> ^{<i>n</i>-1} -1	6	2	$b == 1 \mod 5$ (trivial factor 5) $b == 34 \mod 35$ (covering set $\{5,7\}$) $b = 6*m^2$ with $m == 2, 3 \mod 5$ (odd length: factor 5, even length: difference-of-squares factorization) (this includes $b = 24, 54, 294, 384, 864, 1014)$	≥100000
6{0}1	6* <i>b</i> ^{<i>n</i>-1} +1	7	2	b == 1 mod 7 (trivial factor 7) b == 34 mod 35 (covering set {5,7})	≥100000

6{z}	7* <i>b</i> ⁿ⁻¹ -1	7	2	b == 1 mod 2 (trivial factor 2) b == 1 mod 3 (trivial factor 3)	≥100000
7{0}1	7* <i>b</i> ⁿ⁻¹ +1	8	2	b == 1 mod 2 (trivial factor 2)	(no bases $b \le 1024$ have this family as unsolved family, base $b = 1004$ is the last to drop at length $n = 54849$)
7{z}	8* <i>b</i> ⁿ⁻¹ -1	8	2	$b == 1 \mod 7$ (trivial factor 7) $b == 20 \mod 21$ (covering set $\{3,7\}$) $b == 83, 307 \mod 455$ (covering set $\{5,7,13\}$) (this includes $b = 83, 307, 538, 762, 993)$ $b = m^3$ (difference-of-cubes factorization)	≥100000
8{0}1	8* <i>b</i> ^{<i>n</i>-1} +1	9	2	$b == 1 \mod 3$ (trivial factor 3) $b == 20 \mod 21$ (covering set $\{3,7\}$) $b == 47, 83 \mod 195$ (covering set $\{3,5,13\}$) b = 467 (covering set $\{3,5,7,19,37\}$) b = 722 (covering set $\{3,5,13,73,73,73,73,73,73,73,73,73,73,73,73,73$	≥100000

				109}) $b = m^3$ (sum- of-cubes factorization) $b = 128$ (no possible prime since $7*r+3$ cannot be power of 2)	
8{z}	9* <i>b</i> ^{<i>n</i>-1} -1	9	2	$b == 1 \mod 2$ (trivial factor 2) $b == 4 \mod 5$ (even length: factor 5, odd length: difference-of-squares factorization) $b = m^2$ (difference-of-squares factorization)	≥100000
9{0}1	9* <i>b</i> ⁿ⁻¹ +1	10	2	b == 1 mod 2 (trivial factor 2) b == 1 mod 5 (trivial factor 5)	≥100000
9{z}	10* <i>b</i> ′′ ⁻¹ -1	10	2	b == 1 mod 3 (trivial factor 3) b == 32 mod 33 (covering set {3,11})	≥100000
A{0}1	10* <i>b</i> ^{<i>n</i>-1} +1	11	2	b == 1 mod 11 (trivial factor 11) b == 32 mod 33 (covering set {3,11})	≥100000
A{z}	11* <i>b</i> ⁿ⁻¹ -1	11	2	$b == 1 \mod 2$ (trivial factor 2) $b == 1 \mod 5$ (trivial factor 5) $b = 11*m^2$	≥100000

				with $m == 2$, 3 mod 5 (odd length: factor 5, even length: difference-of-squares factorization) (this includes $b = 44$, 99, 539, 704)	
B{0}1	11* <i>b</i> ^{<i>n</i>-1} +1	12	2	$b == 1 \mod 2$ (trivial factor 2) $b == 1 \mod 3$ (trivial factor 3)	≥100000
B{z}	12* <i>b</i> ^{<i>n</i>-1} -1	12	2	$b == 1 \mod 11$ (trivial factor 11) $b == 142 \mod 143$ (covering set $\{11,13\}$) $b = 307$ (covering set $\{5, 11, 29\}$) $b = 901$ (covering set $\{7, 11, 13, 19\}$)	≥100000
C{0}1	12* <i>b</i> ^{<i>n</i>-1} +1	13	2	$b == 1 \mod 13$ (trivial factor 13) $b == 142 \mod 143$ (covering set $\{11,13\}$) $b = 296, 901$ (covering set $\{7, 11, 13, 19\}$) $b = 562, 828, 900$ (covering set $\{7, 13, 19\}$) $b = 563$ (covering set $\{5, 7, 13, 19, 29\}$) b = 597	≥100000

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				(covering set {5, 13, 29})	
{#}\$	(<i>b</i> ⁿ +1)/2	3 (only odd bases are allowed)	2	$b = m^r$ with odd $r > 1$ (sum-of- r -th- powers factorization)	≥524287
{y}z	((b-2)*b ⁿ +1)/ (b-1)	3	2	(none)	≥5000
y{z}	$(b-1)*b^{n-1}-1$	3	2	(none)	≥100000
z{0}1	$(b-1)^*b^{n-1}+1$	2	2	(none)	≥100000
{z0}z1	(<i>b</i> ⁿ⁺¹ +1)/(<i>b</i> +1	2	2 (only even lengths exist)	$b = m^r$ with odd $r > 1$ (sum-of- r -th- powers factorization) $b = 4*m^4$ (Aurifeuillian factorization for $x^4 + 4y^4$)	≥17326
{z}yz	<i>b</i> ⁿ -(<i>b</i> +1)	2	2	(none)	≥5000
{z}1	<i>b</i> ⁿ -(<i>b</i> -1)	2	2	(none)	≥5000
{z}w	<i>b</i> ⁿ -4	5	2	$b == 0 \mod 2$ (trivial factor 2) $b == 1 \mod 3$ (trivial factor 3) $b == 14 \mod 15$ (covering set $\{3,5\}$) $b == 4 \mod 5$ (odd length: factor 5, even length: difference-of-squares factorization) $b = m^2$ (difference-of-squares factorization)	≥5000
{z}x	<i>b</i> ⁿ -3	4	2	b == 1 mod 2 (trivial factor 2)	(no bases <i>b</i> ≤ 1024 have this family as unsolved

					family, base $b = 542$ is the last to drop at length $n = 1944$)
{z}y	<i>b</i> ⁿ –2	3	2	b == 0 mod 2 (trivial factor 2)	≥5000

- [43] https://www.rose-hulman.edu/~rickert/Compositeseg/ (a problem related to this project)
- [44] http://www.worldofnumbers.com/Appending%201s%20to%20n.txt (a problem related to this project)
- [45] http://www.worldofnumbers.com/deplat.htm (list of plateau and depression primes)
- [46] http://www.worldofnumbers.com/wing.htm (list of palindromic wing primes)
- [47] https://stdkmd.net/nrr/prime/primecount.txt (near- and quasi- repdigit (probable) primes sorted by count)
- [48] https://stdkmd.net/nrr/prime/primedifficulty.txt (near- and quasi- repdigit (probable) primes sorted by difficulty)
- [49] https://stdkmd.net/nrr/coveringset.htm (covering set of near-repdigit-related sequences)
- [50] http://www.numericana.com/answer/primes.htm (the article about the primes and the primality tests)
- [51] http://www.rieselprime.de/dl/CRUS pack.zip (srsieve, sr1sieve, sr2sieve, pfgw, and llr softwares)
- [52] https://www.bc-team.org/app.php/dlext/?cat=3 (srsieve, sr1sieve, sr2sieve, sr5sieve software)
- [53] https://sourceforge.net/projects/openpfgw/ (pfgw software)
- [54] http://jpenne.free.fr/index2.html (*IIr* software)
- [55] http://www.ellipsa.eu/public/primo/primo.html (PRIMO software)
- [56] https://primes.utm.edu/prove/index.html (website for primality proving)
- [57] https://primes.utm.edu/notes/prp_prob.html (the probability that a random probable prime is in fact composite)

- [58] https://oeis.org/wiki/User:Charles R Greathouse IV/Tables of special primes (expected number of primes in first *n* terms of a given sequence)
- [59] https://primes.utm.edu/curios/page.php?number_id=22380 (the largest base 10 minimal prime in Prime Curios!)
- [60] https://oeis.org/A347819 (OEIS sequence for base 10 minimal primes)
- [61] https://oeis.org/A326609 (OEIS sequence for the largest base *b* minimal prime, when the restriction of prime>base is not required)
- [62] https://primes.utm.edu/primes/lists/all.txt (top definitely primes)
- [63] http://www.primenumbers.net/prptop/prptop.php (top probable primes)
- [64] http://factordb.com (online factor database, including many primes which are minimal primes in a small base)

For list of more references, see

https://mersenneforum.org/showpost.php?p=571731&postcount=140 and https://mersenneforum.org/showpost.php?p=582061&postcount=154

Also see https://primes.utm.edu/curios/includes/primetest.php?file=primetest.html and https://www.numberempire.com/primenumbers.php and https://www.bigprimes.net/primalitytest and https://www.archimedes-lab.org/primOmatic.html and https://www.sonic.net/~undoc/java/PrimeCalc.html for links of prime checkers.

Also see https://www.numberempire.com/numberfactorizer.php and https://www.alpertron.com.ar/ECM.HTM and http://www.javascripter.net/math/calculators/primefactorscalculator.htm and https://www.se16.info/js/factor.htm and <a href="https://www.se16.info/js/factor.htm

Also see https://baseconvert.com/ and https://www.cut-the-knot.org/Curriculum/Algorithms/BaseConversion.shtml and https://linesegment.web.fc2.com/application/math/numbers/RadixConversion.html (in Japanese) for links of base converters.

Also see https://primes.utm.edu/lists/small/millions/ and https://oeis.org/A000040/b000040_1.txt and https://oeis.org/A000040/a000040_1B.7z and https://primefan.tripod.com/500Primes1.html (warning: this site incorrectly includes 1 as a prime and misses the primes 3229 and 3329) and https://www.primos.mat.br/indexen.html and https://www.rsok.com/~jrm/printprimes.html and https://en.wikipedia.org/wiki/List_of_prime_numbers#The_first_1000_prime_numbers_for_links of lists of small primes.

Also see http://primefan.tripod.com/500factored.html and https://en.wikipedia.org/wiki/Table_of_prime_factors for links of lists of factorizations of small integers.

Also see https://en.wikipedia.org/wiki/Table_of_bases for links of lists of small integers in various bases.

(In fact, you can use <u>Wolfram Alpha</u> for prime checker, integer factorizer, and base converter, besides, many <u>mathematical softwares</u> also already have prime checkers, integer factorizers, and base converters, including <u>Maple</u>, <u>wolfram Mathematica</u>, <u>PARI/GP</u>, <u>Python</u>, <u>GMP</u>, <u>Magma</u>, <u>SageMath</u>, see the table below, you can download these softwares by clicking the links)

software	<u>Maple</u>	Wolfram Mathema tica	PARI/GP	<u>Python</u>	<u>GMP</u>	<u>Magma</u>	<u>SageMat</u> <u>h</u>
check if a number is probable prime		PrimeQ[number]	ispseudo prime(<i>nu</i> <i>mber</i>)				
check if a number is definitely prime		Provable PrimeQ[number]	isprime(<i>n</i> umber)				
factor a number		FactorInt eger[nu mber]	factor(nu mber)				
convert a number to base b		BaseFor m[numbe r, base] IntegerDi gits[num ber,	digits(<i>nu</i> <i>mber</i> , <i>base</i>)	int(numb er, base)			

Finally, there is a <u>C</u> <u>code</u> for the problem in this article: (need run with <u>GMP</u>), see <u>this forum</u> <u>post</u>.