PRIMES IN A FIXED INTERVAL

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ABSTRACT. The Hardy-Littlewood k-tuples conjecture is investigated using the Reimann prime counting function. Equations similar to the General Boson Ordering equations are created and interpreted in regard to the twin prime conjecture.

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The number of primes in an interval of consecutive integers $[x+1, x+c]$ can be calculated as $\pi(x+c) - \pi(x)$. Using the Riemann formula for $\pi()$, the number of primes in the interval is

$$\begin{aligned}\pi(x+c) - \pi(x) = &\sum_{N=1}^{\lfloor \frac{\ln(x+c)}{\ln 2} \rfloor} \frac{\mu(N)}{N} \left(\mathrm{Li}(x+c)^{\frac{1}{N}} - \sum_{\rho} \mathrm{Li}(x+c)^{\frac{\rho}{N}} - \ln 2 + \int_{(x+c)^{\frac{1}{N}}}^{\infty} \frac{\mathrm{d}t}{t(t^2-1)\ln(t)} \right) \\ &- \sum_{N=1}^{\lfloor \frac{\ln x}{\ln 2} \rfloor} \frac{\mu(N)}{N} \left(\mathrm{Li}(x^{\frac{1}{N}}) - \sum_{\rho} \mathrm{Li}(x^{\frac{\rho}{N}}) - \ln 2 + \int_{x^{\frac{1}{N}}}^{\infty} \frac{\mathrm{d}t}{t(t^2-1)\ln(t)} \right)\end{aligned}$$

If a restriction is applied to x and $x+c$, the upper bounds of the Mobius summations can be identical. The restriction is to prevent x and $x+c$ from spanning a power of 2. The restriction is

$$2^n \leq x < x + c < 2^{n+1}$$

This restriction is only to simplify further calculations. If x and $x+c$ did span a power of 2, the original difference could be shown as the sum of each difference that does not span a power of 2.

$$\begin{aligned}\pi(x+c) - \pi(x) = &(\pi(x+c) - \pi(2^{n+m})) + (\pi(2^{n+m}-1) - \pi(2^{n+m-1})) + \dots \\ &\dots + (\pi(2^{n+2}-1) - \pi(2^{n+1})) + (\pi(2^{n+1}-1) - \pi(x))\end{aligned}$$

When the grouping parentheses are dropped, all adjacent terms cancel because $\pi(2^n) = \pi(2^n - 1)$ for $n > 1$.

The logarithmic integral $\mathrm{Li}(x)$ is to be evaluated for each pair of numbers, and can be expanded as

$$\mathrm{Li}(x^{\frac{1}{N}}) = \gamma + \ln \ln x^{\frac{1}{N}} + \sum_{k=1}^{\infty} \frac{\ln^k x^{\frac{1}{N}}}{k\, k!}$$

The use of Li() in the Riemann formula requires calculating $\ln\ln x$. The following substitution eliminates this calculation.

$$\begin{aligned} x &= \mathrm{e}^{y} & y &= \ln x & \ln\ln x &= \ln y \\ x+c &= \mathrm{e}^{y+d} & y+d &= \ln(x+c) & \ln\ln(x+c) &= \ln(y+d) \end{aligned}$$

In the summations over the complex roots of the zeta function, each root and its conjugate are to be evaluated.

$$\sum_{\rho} \mathrm{Li}(v^{\frac{\rho}{N}}) = \sum_{\rho_\pm} \Big(\mathrm{Li}(v^{\frac{\rho_+}{N}}) + \mathrm{Li}(v^{\frac{\rho_-}{N}}) \Big)$$

Using the above restriction and substitutions, the number of primes in the interval $[x+1, x+c]$ can now be expressed as

$$\begin{aligned} \pi(x+c)-\pi(x) = \sum_{N=1}^{\lfloor \frac{\ln(x+c)}{\ln x} \rfloor} \frac{\mu(N)}{N} \Bigg[& \ln(1+\tfrac{d}{y}) + \sum_{k=1}^{\infty} \frac{((y+d)^k - y^k)}{N^k k k!} \\ & - \sum_{\rho} \Bigg(2\ln(1+\tfrac{d}{y}) + \sum_{k=1}^{\infty} \frac{(\rho_+^k + \rho_-^k)((y+d)^k - y^k)}{N^k k k!} \Bigg) \\ & - \int_{x^{\frac{1}{N}}}^{(x+c)^{\frac{1}{N}}} \frac{\mathrm{d}t}{t(t^2-1)\ln(t)} \Bigg] \end{aligned}$$

When c is held constant and the value of x increases, the integral term in the above equation decreases. As the value of x goes to ∞, the value of the integral goes to zero and vanishes. Therefore, the integral term will not be considered.

The focus is now on the summation over the complex roots of the zeta function. Each root can be expressed with a real and imaginary part.

$$\rho_+ = a + b\mathrm{i} \qquad\qquad \rho_- = a + b\mathrm{i}$$

By extracting a single term for a root of the zeta function from the summation, the term can be written as

$$2\ln(1+\tfrac{d}{y}) + \sum_{k=1}^{\infty} \frac{((a+b\mathrm{i})^k + (a-b\mathrm{i})^k)((y+d)^k - y^k)}{N^k k k!}$$

The numerator of each term in the summation of the complex roots can be expanded, thereby allowing the terms ro be reordered. The expansion is as follows

$$\begin{array}{cc}
k & ((a+b\mathrm{i})^k + (a-b\mathrm{i})^k)((y+d)^k) - y^k)) \\
1 & 2(a)(d) \\
2 & 2(a^2-b^2)(\binom{2}{1}yd + d^2) \\
3 & 2(a^3 - \binom{3}{2}ab^2)(\binom{3}{1}y^2d + \binom{3}{2}yd^2 + d^3) \\
4 & 2(a^4 - \binom{4}{2}a^2b^2 + b^4)(\binom{4}{1}y^3d + \binom{4}{2}))y^2d^2 + \binom{4}{3}yd^3 + d^4) \\
5 & 2(a^5 - \binom{5}{2}a^3b^2 + \binom{5}{4}ab^4)(\binom{5}{1}y^4d + \binom{5}{2}y^3d^2 + \binom{5}{3}y^2d^3 + \binom{5}{4}yd^4 + d^5) \\
\vdots & \vdots\vdots
\end{array}$$

$$2\tfrac{d}{y}\sum \tfrac{a^k y^k}{N^k k k!}\tbinom{k}{1} - 2\tfrac{b^2 d}{a^2 y}\sum \tfrac{a^k y^k}{N^k k k!}\tbinom{k}{1}\tbinom{k}{2}$$

This expansion has been chosen because it generates a $\frac{d}{y}$ term that is < 1, causing convergence as the term is raised to increasing powers. Each summation in the first column can be represented by a closed form. For clarity the falling factorials are shown in full text.

matrix style formula

Another pass of summing the closed forms, this time by columns in the order above.

matrix style formula

The infinite series in the first row is the identity for a logarithm of $1+v$ where $v \leq 1$, the terms of $\frac{d}{y}$ are less than 1 due to the initial restriction placed on x and $x+c$. The first row in the expansion can be expressed as

$$-2\ln(1+\tfrac{d}{y}) + 2\mathrm{e}^{\frac{ay}{N}}\mathrm{e}^{\frac{-N}{ay}}\ln(1+\tfrac{d}{y}) + 2\mathrm{e}^{\frac{ay}{N}}\left(\right)$$

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