An observation on the sums of divisors*

Leonhard Euler

Summarium

The divisors of a number are the numbers by which it can be divided without a remainder; and among the divisors of any number are first unity and then the number itself, since every number can be divided at least by unity and itself. Now, those numbers which besides unity and themselves have no other divisors we call prime, while the others, which permit division without remainder by some number other than themselves, we call composite; and in common Arithmetic a method is taught for finding all the divisors of any number. The Author in this dissertation considers the sum of all divisors of any given number, not to the ends that others typically pursue, for the investigation of perfect or amicable numbers and other questions of this kind, but to explore the order and as it were the law by which the sums of the divisors of each number proceed conveniently. This should certainly be seen as most concealed, since for prime numbers the sum of divisors will exceed them by unity, but for composites it is the greater the more prime factors that comprise them. Therefore, since the rule for the progression of the prime numbers is so far a great mystery, into which not even Fermat was able to penetrate, since the rule for these is clearly involved in the sums of divisors, who should doubt that these too are not subject to any law? So this dissertation merits all the more attention, because such a law is brought here to light, even if it has not yet been demonstrated with complete rigor. The same thing happens for the Author as before with the theorem of Fermat, that soon after the defect in the demonstration has been corrected.¹ For what is still desired in the demonstration given here will be at once supplied in the following dissertation.

In order to clearly explain this, the Author uses the sign \int to indicate the sum of the divisors of any number. Thus $\int n$ indicates the sum of all the divisors of the

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¹Translator: Right now I presume this refers to Euler's E228 and E241, both on representing integers as sums of two square. However I have not read these papers.

number n. One can see that:

These sums can be easily defined from the known principle² that the sum of the divisors of the product mnpq, where the factors m, n, p, q are taken to be prime to each other, is equal to the product of each of the sums, or

$$\int mnpq = \int m \cdot \int n \cdot \int p \cdot \int q.$$

Thus

$$\int 20 = \int 4.5 = \int 4. \int 5 = 7.6 = 42$$
 and $\int 360 = \int 8.9.5 = \int 8. \int 9. \int 5 = 15.13.6 = 1170.$

The Author also considers the correspondence between the sums of divisors and the numbers in their usual order, which proceed as follows

in which progression certainly no law is seen, since now it is greater and now it is less, now even and now odd, and especially, the order of the prime numbers is clearly involved; since this is so impenetrable, who would suspect a law in this series? But nevertheless, the Author shows that these numbers constitute a series of the kind that is usually called recurrent, such that each term can be determined from some of the preceding according to a certain law. As $\int n$ denotes the sum of the divisors of the number n, the expression $\int (n-a)$ will denote the sum of the divisors of the number n-a. With this notation established, the law found by the Author is that

$$\int n = \int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \int (n-15) - \int (n-22) - \int (n-26) + \int (n-35) + \int (n-40) - \text{etc.},$$

in which rule the signs are taken two + followed by two -; and the numbers 1, 2, 5, 7, 12, 15 which are continually subtracted from n can be easily found from their differences:

$$\begin{array}{lll} \text{numbers} & 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57 & \text{etc.,} \\ \text{differences} & 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6 & \text{etc.,} \end{array}$$

which come alternately from two sequences. The relation is conveniently represented in the following way:

$$\int n = \begin{cases} +\int (n-1) - \int (n-5) + \int (n-12) - \int (n-22) + \int (n-35) - \text{etc.} \\ +\int (n-2) - \int (n-7) + \int (n-15) - \int (n-26) + \int (n-40) - \text{etc.} \end{cases}$$

 $^{^2}$ Translator: cf. pp. 53–54 of André Weil, Number theory: an approach through history.

For the application of this formula to any number, one should understand that these terms are taken until the number after the \int sign is negative, which is omitted together with all further terms. As well then, if the term $\int (n-n)$ or $\int 0$ occurs, since this by itself is not determinate, in this case the number n itself is written in place of the term. Thus, according to this law it will be

$$\int 21 = \int 20 + \int 19 - \int 16 - \int 14 + \int 9 + \int 6$$

or

$$\int 21 = 42 + 20 - 31 - 24 + 13 + 12 = 87 - 55 = 32;$$

and

$$\int 22 = \int 21 + \int 20 - \int 17 - \int 15 + \int 10 + \int 7 - \int 0$$

or

$$\int 22 = 32 + 42 - 18 - 24 + 18 + 8 - 22 = 100 - 64 = 36.$$

The Author has deduced the marvelous law for this progression from the consideration of the product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$$
 etc.

whose factors continue to infinity. But if this is expanded by actual multiplication it will be found to form the following series

$$1 - x - x^{2} + x^{5} + x^{7} - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{etc.},$$

and it indeed has been possible to continue this operation further, whence the law of this series and the progression of the exponents has been concluded at least by induction, which seems perhaps to suffice for many. Truly the Author candidly admits that this observed agreement to be by no means demonstrated, but that its demonstration is still desired, which however he has communicated hardly much later to the Academy. With the equality of the product and the expanded series, the mentioned theorem about the order in the sums of divisors is thence clearly demonstrated, such that there cannot be any more doubt, even if it is accomplished by logarithms and differentiation, which seem to have little to do with the nature of divisors. From this, one may see how closely and wonderfully infinitesimal Analysis is connected not only with usual Analysis, but even with the theory of numbers, which seems remote from that higher kind of calculus.

- 1. For a given number n, let the formula $\int n$ denote the sum of all the divisors of the number n. Then since unity has no other divisors besides itself, $\int 1 = 1$; and since a prime number has exactly two factors, unity and itself, if n is a prime number then $\int n = 1 + n$. Next, since a perfect number is equal to the sum of its aliquot parts, where aliquot parts are the divisors other than itself, it is immediate that the sum of the divisors of a perfect number will be twice the number itself; that is, if n is a perfect number then $\int n = 2n$. Further, usually a number is called abundant if the sum of its aliquot parts is greater than it, so if n is an abundant number, then $\int n > 2n$; and if n is a deficient number, that is such that the sum of its aliquot parts is less than it, then $\int n < 2n$.
- 2. In this manner now, one can easily express with signs the essences of numbers, as far as this is contained in the sum of the aliquot parts or divisors.

For if $\int n = 1 + n$, then n will be a prime number, if $\int n = 2n$, then n will be a perfect number, and if $\int n > 2n$ or $\int n < 2n$, then n will be an abundant or deficient number respectively. Here the question can also be dealt with of those numbers which are usually called amicable, that is, for which the sum of the aliquot parts of each is equal to the other. For if m and n are amicable numbers, since the sum of the aliquot parts of the number m is $= \int m - m$ and of the number n is $= \int n - n$, by the nature of these numbers it will be $n = \int m - m$ and $m = \int n - n$, and thus $\int m = \int n = m + n$. Hence two amicable numbers have the same sum of divisors, which is simultaneously equal to the sum of both the numbers.

3. To easily find the sum of the divisors of any given number, it is most convenient to resolve the number into two factors which are prime to each other. For if p and q are numbers that are prime to each other, or which besides unity have no common divisor, then the sum of the divisors of the product pq will be equal to the product of the sums of the divisors of both, or

$$\int pq = \int p \cdot \int q.$$

Thence, with the sums of the divisors of smaller numbers found, it will not be difficult to extend the discovery of the sum of divisors to greater numbers.

4. If a, b, c, d etc. are the prime numbers, then every number, no matter what size, can always be reduced to the form $a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$ etc.; having gotten this form, the sum of the divisors of this number or $\int a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$ etc. will be

$$= \int a^{\alpha} \cdot \int b^{\beta} \cdot \int c^{\gamma} \cdot \int d^{\delta} \cdot \text{ etc.}$$

But because a, b, c, d etc. are prime numbers,

$$\int a^{\alpha} = 1 + a + a^2 + \dots + a^{\alpha} = \frac{a^{\alpha+1} - 1}{a - 1}$$

and so

$$\int\! a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}\,\mathrm{etc.} = \frac{a^{\alpha+1}-1}{a-1}\cdot\frac{b^{\beta+1}-1}{b-1}\cdot\frac{c^{\gamma+1}-1}{c-1}\cdot\frac{d^{\delta+1}-1}{d-1}\cdot\,\mathrm{etc.}$$

It will therefore suffice to have found only the sums of the divisors of all the powers of prime numbers. 3

5. However, I shall not pursue this inquiry further. To come nearer to what I have set to treat, let me write out for later use the sums of the divisors of numbers proceeding according to the natural order.

 $^{^3}$ Translator: The *Opera omnia* cites several sources on the multiplicativity of the sum of divisors. In particular, it mentions Euler's paper E152, on amicable numbers.

$\int 1 = 1$	$\int 26 = 42$	$\int 51 = 72$	$\int 76 = 140$
$\int 2 = 3$	$\int 27 = 40$	$\int 52 = 98$	$\int 77 = 96$
$\int 3 = 4$	$\int 28 = 56$	$\int 53 = 54$	$\int 78 = 168$
$\int 4 = 7$	$\int 29 = 30$	$\int 54 = 120$	$\int 79 = 80$
$\int 5 = 6$	$\int 30 = 72$	$\int 55 = 72$	$\int 80 = 186$
$\int 6 = 12$	$\int 31 = 32$	$\int 56 = 120$	$\int 81 = 121$
$\int 7 = 8$	$\int 32 = 63$	$\int 57 = 80$	$\int 82 = 126$
$\int 8 = 15$	$\int 33 = 48$	$\int 58 = 90$	$\int 83 = 84$
$\int 9 = 13$	$\int 34 = 54$	$\int 59 = 60$	$\int 84 = 224$
$\int 10 = 18$	$\int 35 = 48$	$\int 60 = 168$	$\int 85 = 108$
$\int 11 = 12$	$\int 36 = 91$	$\int 61 = 62$	$\int 86 = 132$
$\int 12 = 28$	$\int 37 = 38$	$\int 62 = 96$	$\int 87 = 120$
$\int 13 = 14$	$\int 38 = 60$	$\int 63 = 104$	$\int 88 = 180$
$\int 14 = 24$	$\int 39 = 56$	$\int 64 = 127$	$\int 89 = 90$
$\int 15 = 24$	$\int 40 = 90$	$\int 65 = 84$	$\int 90 = 234$
$\int 16 = 31$	$\int 41 = 42$	$\int 66 = 144$	$\int 91 = 112$
$\int 17 = 18$	$\int 42 = 96$	$\int 67 = 68$	$\int 92 = 168$
$\int 18 = 39$	$\int 43 = 44$	$\int 68 = 126$	$\int 93 = 128$
$\int 19 = 20$	$\int 44 = 84$	$\int 69 = 96$	$\int 94 = 144$
$\int 20 = 42$	$\int 45 = 78$	$\int 70 = 144$	$\int 95 = 120$
$\int 21 = 32$	$\int 46 = 72$	$\int 71 = 72$	$\int 96 = 252$
$\int 22 = 36$	$\int 47 = 48$	$\int 72 = 195$	$\int 97 = 98$
	$\int 48 = 124$	$\int 73 = 74$	$\int 98 = 171$
$\int 24 = 60$	$\int 49 = 57$	$\int 74 = 114$	$\int 99 = 156$
$\int 25 = 31$	$\int 50 = 93$	$\int 75 = 124$	$\int 100 = 217$

- 6. If we contemplate now the series of the numbers 1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28 etc. which the sums of divisors corresponding to the numbers proceeding in their natural order constitute, not only is there no apparent law for the progression, but the order of these numbers seems so disturbed that they seem to be bound by no law whatsoever. For this series is clearly mixed up with the order of prime numbers, since the term of index n, or $\int n$, will always be = n + 1 exactly when n is a prime number; but it is well known that thus far it has not been possible to refer the prime numbers to any certain law of progression. And since our series involves the rule not only of the prime numbers but also all the other numbers, insofar as they are composed from primes, its law would seem more difficult to find than that of the series of prime numbers alone.
- 7. Since this is the case, I seem to have advanced the science of numbers by not a small amount when I found a certain fixed law according to which the terms of the given series 1, 3, 4, 7, 6 etc. progress, such that by this law each term of the series can be defined from the preceding; for I have found, which seems rather wonderful, that this series belongs to the kind of progression which are usually called recurrent and whose nature is such that each term is determined from the preceding according to some certain rule of relation. And who would

have ever believed that this series which is so disturbed and which seems to have nothing in common with recurrent series would nevertheless be included in this type of series, and that it would be possible to assign a scale of relation for it?⁴

8. Since the term of this series corresponding to the index n, which indicates the sum of the divisors of the number n, is $= \int n$, the prior terms in descending order are $\int (n-1), \int (n-2), \int (n-3), \int (n-4), \int (n-5)$ etc. And any term of this series, namely $\int n$, is conflated from some of the prior terms, as

$$\int n = \int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \int (n-15) - \int (n-22) - \int (n-26) + \int (n-35) + \int (n-40) - \int (n-51) - \int (n-57) + \int (n-70) + \int (n-77) - \int (n-92) - \int (n-100) + \int (n-117) + \int (n-126) - \text{etc.}$$

Or since the signs + and - occur alternately in pairs, this series can be separated easily into two like this:

$$\int n = \begin{cases} \int (n-1) - \int (n-5) + \int (n-12) - \int (n-22) + \int (n-35) - \int (n-51) + \text{etc.} \\ \int (n-2) - \int (n-7) + \int (n-15) - \int (n-26) + \int (n-40) - \int (n-57) + \text{etc.} \end{cases}$$

9. From the above form, the order of the numbers which are successively subtracted from n in each series is easily seen; for each series is of the second order, having constant second differences.⁵ In fact, the numbers of the first series together with both their first and second differences are

Whence the general term of this series⁶ is $=\frac{3xx-x}{2}$ and thus contains exactly the pentagonal numbers. The other series is

and hence the general term is $\frac{3xx+x}{2}$ and contains the series of pentagonal numbers continued backwards.⁷

$$f(x+a) = \sum_{k=0}^{\infty} {x \choose k} \Delta^k f(a).$$

This is sometimes called "Newton's series" for f.

⁴Translator: scalam relationis=scale of relation=recurrence relation. Euler uses this term in his November 10, 1742 letter to Nicolaus I Bernoulli.

 $^{^5{\}mbox{Translator:}}$ Namely a second order arithmetic progression.

⁶Translator: Let $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^{k+1} f(x) = \Delta^k f(x+1) - \Delta^k f(x)$. If f(x) is a polynomial, then

⁷Translator: Continued to negative indices? viz. $\frac{-x(-3x-1)}{2} = \frac{3x+1}{2}$.

10. It is highly noteworthy here that the series of pentagonal numbers, itself and continued backwards, is applied with great effect to the order of the series of sums of divisors, since certainly one would not at all suspect there to be a connection between the pentagonal numbers and sums of divisors. For if one writes the series of pentagonal numbers forwards and continued backwards in this way

our formula enclosing the order of the sums of divisors can be presented with alternating signs ordered in this way

etc.
$$-\int (n-15) + \int (n-7) - \int (n-2) + \int (n-0) - \int (n-1) + \int (n-5) - \int (n-12) + \int (n-22) - \text{etc.} = 0,$$

in which the series on both sides continue to infinity, but in every case if it is correctly applied to our use a determinate numbers of terms will arise.

11. For if we want to find the sum of the divisors of the number n by means of our first exhibited formula

with the sums of the divisors of smaller numbers known, then we only need to take the terms in this formula until we reach sums of the divisors of negative numbers. Namely, all the terms which contain negative numbers after the \int sign are rejected; whence it is clear that if n is a small number just a few terms suffice, while if n is a larger number then it will be necessary to take more terms from our general formula.

- 12. Therefore, the sum of the divisors of a given number n is composed from the sum of divisors of some smaller numbers which I assume to be known, since in each case the sums for negative numbers are rejected. This is an easy provision, because one cannot even take the sum of the divisors of negative numbers; but it should be explained how this operation is done in those cases in which our formula yields the term $\int (n-n)$ or $\int 0$, which, since zero is divisible by all numbers, seems either infinite or indeterminate. This case will occur exactly when n is a number from either the series of pentagonal numbers or the series continued backwards; then in these cases the number n itself should be taken in place of the term $\int (n-n)$ or $\int 0$, and should be written with the sign which the term $\int (n-n)$ is affixed with in our formula.
- 13. With these precepts for the use of our formula explained, to begin with I shall give examples with small numbers which can be easily examined by means of our formula, and simultaneously the truth of the formula will be recognized.

$$\int 1 = \int 0$$
or
$$\int 1 = 1 = 1$$

$$\int 2 = \int 1 + \int 0$$
or
$$\int 2 = 1 + 2 = 3$$

$$\int 3 = \int 2 + \int 1$$
or
$$\int 3 = 3 + 1 = 4$$

$$\int 4 = \int 3 + \int 2$$
or
$$\int 4 = 4 + 3 = 7$$

$$\int 5 = \int 4 + \int 3 - \int 0$$
or
$$\int 5 = 7 + 4 - 5 = 6$$

$$\int 6 = \int 5 + \int 4 - \int 1$$
or
$$\int 6 = 6 + 7 - 1 = 12$$

$$\int 7 = \int 6 + \int 5 - \int 2 - \int 0$$
or
$$\int 7 = 12 + 6 - 3 - 7 = 8$$

$$\int 8 = \int 7 + \int 6 - \int 3 - \int 1$$
or
$$\int 8 = 8 + 12 - 4 - 1 = 15$$

$$\int 9 = \int 8 + \int 7 - \int 4 - \int 2$$
or
$$\int 9 = 15 + 8 - 7 - 3 = 13$$

$$\int 10 = \int 9 + \int 8 - \int 5 - \int 3$$
or
$$\int 10 = 13 + 15 - 6 - 4 = 18$$

$$\int 11 = \int 10 + \int 9 - \int 6 - \int 4$$
or
$$\int 11 = 18 + 13 - 12 - 7 = 12$$

$$\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0$$
or
$$\int 12 = 12 + 18 - 8 - 6 + 12 = 28.$$

14. By inspecting these examples with attention and also by continuing to greater numbers, it will be apparent not without admiration how, as it were against expectation, that the true sum of divisors of the given number is obtained; and to make it easier to recognize this pattern, I have already given above the sums of the divisors of all numbers not greater than one hundred, whence the truth of our formula can be tested with greater numbers. In particular we will find not without delight that the given number is prime when the sum found from our formula for it is greater than the number by unity. Let us work out an example to this end, with the given number n = 101, and test it as if ignorant about whether or not this number is prime. The operation will

happen thus:

$$\int 101 = \int 100 + \int 99 - \int 96 - \int 94 + \int 89 + \int 86 - \int 79 - \int 75 - \int 75$$

Therefore by collecting the pairs of two terms together we will have

$$\int 101 = +373 - 396
+222 - 204
+206 - 177
+92 - 14$$

or

$$\int 101 = +893 - 791 = 102.$$

Therefore, we find that the sum of the divisors of the number 101 is greater than it by unity, namely 102, whence even if it were not otherwise known, it clearly follows that the number 101 is prime. This rightly seems miraculous, since no operation was done which referred in any way to the calculation of divisors; also, the divisors whose sum is found by this method remain themselves unknown, although they can frequently be figured out from the consideration of this sum.⁸

- 15. These special properties which the sums of divisors are gifted with would be no less memorable if their demonstration were obvious, and as it were exposed to the daylight. But the demonstration was in fact abstruse and depended on rather difficult properties of numbers, whence to no small degree the value of this law discovered for the progression is increased; for the investigation of truths is to be recommended the more the more hidden they are. Truly, I am compelled to admit that now not only have I not been able to find a demonstration of this truth, but that I have even nearly been brought to despair, and I do not know whether because of this the knowledge of a truth whose demonstration is hidden to us should be valued even more highly. And so this truth has been confirmed by a great many examples, since it has not been permitted that I exhibit a demonstration of it.
- 16. Thus here we have an extraordinary example of the kind of proposition whose truth we can in no way doubt, even if we have not achieved its demonstration. This will seem rather surprising to most, since in common mathematics no propositions are counted as true unless they can be derived from indubitable principles. Yet in the meanwhile, I have come to the knowledge of this truth not by chance and, as it were, by divination; for to whom would it have come to mind to try to elicit by conjecture alone an order that might perhaps occur in the sums of divisors, from the nature of recurrent series and of the pentagonal

⁸Translator: Is Euler saying that if we know $\int n$ for all n, then for any particular number n it is simple to find its divisors just using operations involving \int ?

numbers? For which reason I judge it not to be foreign from our purpose if I clearly explain the way by which came to the knowledge of this order, especially since it is very recondite and was discovered in a long roundabout way.

17. I was led to this observation by considering the infinite formula

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)$$
 etc.,

which if actually expanded by multiplication and then arranged according to the powers of x, I discovered to be transformed into the following series

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \text{etc.},$$

where exactly those numbers occur in the exponents of x which I described above, namely the pentagonal numbers themselves and them continued backwards. To more easily see this order, the series can be exhibited thus, going to infinity on each side

$$s = \text{etc.} + x^{26} - x^{15} + x^7 - x^2 + x^0 - x^1 + x^5 - x^{12} + x^{22} - x^{35} + x^{51} - \text{etc.}$$

18. The equality of these two formulas exhibiting s is now the very thing which I am not able to confirm with a solid demonstration; nevertheless undertaking to successively multiply out the factors of the first formula

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)$$
 etc.

leads to the initial terms of the other series

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{etc.}$$

and neither is it difficult to see that the two signs + and - occur alternately in pairs and that the exponents of the powers of x follow the same law that I explained enough already. But conceded the equality of these two infinite formulas, the properties of the sums of divisors which I indicated before can be rigidly demonstrated; and on the other hand, if these properties are admitted as true, the true agreement of our two formulas will follow.

19. If for doing the demonstration we assume that both

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)$$
 etc.

and

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}$$

by taking logarithms we get

$$ls = l(1-x) + l(1-x^2) + l(1-x^3) + l(1-x^4) + l(1-x^5) + \text{etc.}$$

and

$$ls = l(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}).$$

Then taking the differentials of each formula we have

$$\frac{ds}{s} = -\frac{dx}{1-x} - \frac{2xdx}{1-x^2} - \frac{3x^2dx}{1-x^3} - \frac{4x^3dx}{1-x^4} - \frac{5x^4dx}{1-x^5} - \text{etc.}$$

and

$$\frac{ds}{s} = \frac{-dx - 2xdx + 5x^4dx + 7x^6dx - 12x^{11}dx - 15x^{14}dx + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}}$$

Let us multiply both of these by $\frac{-x}{dx}$, so that we have

$$\begin{split} &\text{I.} - \frac{xds}{dx} = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \text{etc.} \\ &\text{II.} - \frac{xds}{sdx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}} \end{split}$$

20. First, let's consider the first of the two equal expressions, and let us convert all the terms into geometric progressions in usual manner; with this done, arranging these infinitely many geometric progressions according to powers of x will yield:

21. Now if the coefficients of all the powers of x are collected, one will have

$$-\frac{xds}{sdx} = x^{1} + x^{2}(1+2) + x^{3}(1+3) + x^{4}(1+2+4) + x^{5}(1+5) + x^{6}(1+2+3+6) + \text{etc.},$$

where it is clear that the coefficient of each power of x is the sum of all the numbers by which the exponent of the power is divisible. Namely, the coefficient of the power x^n will be the sum of all the divisors of the number n; thus according to the manner of signification explained above it will be $= \int n$. Then the series found equal to $-\frac{xds}{sdx}$ can thus be exhibited as

$$-\frac{xds}{sdx} = x \int 1 + x^2 \int 2 + x^3 \int 3 + x^4 \int 4 + x^5 \int 5 + x^6 \int 6 + x^7 \int 7 + \text{etc.},$$

and by putting x = 1 this yields the progression of the sums of divisors, which assembles all all numbers proceeding in the natural order.

22. Let us now designate this series by t, so that

$$t = x^{1} \int 1 + x^{2} \int 2 + x^{3} \int 3 + x^{4} \int 4 + x^{5} \int 5 + x^{6} \int 6 + x^{7} \int 7 + \text{etc.},$$

and as $t = -\frac{xds}{sdx}$, it will also be that

$$t = \frac{x^1 + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}}$$

Then it is necessary that the series obtained for t from the expansion of this fraction be equal to that which the prior form has provided. From this it is apparent that that series found for t is recurrent: each of its terms is determined from the preceding by a certain scale of relation, which the denominator $1-x-x^2+x^5+x^7-x^7$ etc. indicates.

23. Now so that the character of this recurrent series can be easily understood, let us equate the two values found for t and in order to get rid of the fraction, let each be multiplied by the denominator $1-x-x^2+x^5+x^7-x^{12}-x^{15}+$ etc. This done, by arranging the terms according to powers of x this will arise

24. Since now the coefficients of each power of x need to destroy each other, we may elicit the following equalities

which clearly reduce to these

25. Here it is evident that the numbers which ought to be continually subtracted from the given number, the sum of whose divisors is sought, are the very numbers from the series 1, 2, 5, 7, 12, 15, 22, 26 etc.; in each case, they are to be taken as long as they do not exceed the given number. As well the signs follow the same rule which was described above. Therefore for any given number n it will clearly be

$$\int n = \int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \int (n-15) - \text{etc.};$$

the terms are to be continued until the numbers having the sign \int in front of them become negative. So from the origin of this recurrent series the rule is transparent why this progression is not continued any further.

26. Then, for what pertains to the actual numbers which are appended at the end of certain of the found formulas, it is clear that they arise from the numerator of the fraction whose value was found expressing t (§22), and interrupt the law of continuity for exactly those cases in which the number n is a term from the series 1, 2, 5, 7, 12, 15, 22, 26 etc., but even in this case the law of signs is not affected. In these cases the actual number which is to be added is equal to the given number itself, keeping the same sign; and if we consider the law described before, we see that this number corresponds to the term $\int (n-n)$ there. From this the rule is transparent why, whenever in applying the formula

$$\int n = \int (n-1) + \int (n-2) - \int (n-5) - \int (n-7) + \int (n-12) + \text{etc.}$$

the term $\int (n-n)$ is encountered, it is not omitted, but rather its the number n itself should be written for its value. Therefore the rule explained above is confirmed in all its parts.