

# NEW BOUNDS ON THE NUMBER OF REPRESENTATIONS OF $T$ AS A BINOMIAL COEFFICIENT

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## Abstract

For  $t > 1$ , let  $N(t)$  denote the number of ways that  $t$  can be written as a binomial coefficient. We prove that  $N(t) = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right)$ .

## 1. Introduction and statement of results

If  $t > 1$ , then let  $N(t)$  denote the number of ways that  $t$  can be written as a binomial coefficient. Abbot, Erdős, and Hanson show in [1] that

$$N(t) = O\left(\frac{\log t}{\log \log t}\right).$$

They also note that if Cramer's conjecture is true (i.e. if for some  $x_0$  and all  $x > x_0$  there is always a prime number between  $x$  and  $x + \log^2 x$ ), then this bound can be improved to

$$N(t) = O\left((\log t)^{(2/3+\epsilon)}\right)$$

for any  $\epsilon > 0$ . It has also been conjectured by D. Singrester that  $N(t) = O(1)$ .

We improve on the first of these bounds by proving the following theorem.

**Theorem 1.** *With  $N(t)$  defined above,*

$$N(t) = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).$$

## 2. Preliminary Lemmas

Here are several important facts that we shall be using. If  $\Gamma(z)$  denotes the Euler gamma-function, then

$$(2.1) \quad \log \Gamma(z+1) = \frac{1}{2} \log(2\pi) + \left(z + \frac{1}{2}\right) \log(z) - z + \frac{1}{12z} + O\left(\frac{1}{z^3}\right).$$

This holds uniformly in the region of the complex plane where  $\Re(z) > 1$ . This follows readily from the  $m = 2$  case of

(see [2])

$$\log \Gamma(z+1) = \frac{1}{2} \log(2\pi) + \left(z + \frac{1}{2}\right) \log(z) - z + \sum_{j=1}^m \frac{B_{2j}}{(2j-1)(2j)z^{2j-1}} - \frac{1}{2m} \int_0^\infty \frac{B_{2m}(x-[x])}{(x+z)^{2m}} dx.$$

Also note that since

$$N(t) = \left| \left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m} \right\} \right|,$$

we have by the symmetry of Pascal's triangle that

$$(2.2) \quad N(t) \leq 2 \left| \left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m}, 2m \leq n \right\} \right|.$$

**Lemma 2.1.** *If  $F(x) : \mathbb{R} \rightarrow \mathbb{R}$  is an infinitely differentiable function and if  $F(x) = 0$  for  $x = x_1, x_2, \dots, x_{n+1}$  (where  $x_1 < x_2 < \dots < x_{n+1}$ ), then  $F^{(n)}(y) = 0$  for some  $y \in (x_1, x_{n+1})$ .*

*Proof of Lemma 2.1.* We proceed by induction on  $n$ . The case of  $n = 1$  is Rolle's Theorem. Given the statement of Lemma 2.1 for  $n - 1$ , if there exists such an  $F$  with  $n + 1$  zeroes,  $x_1 < x_2 < \dots < x_{n+1}$ , then by Rolle's theorem, there exist points  $y_i \in (x_i, x_{i+1})$  ( $1 \leq i \leq n$ ) so that  $F'(y_i) = 0$ . Then since  $F'$  has at least  $n$  roots, by the induction hypothesis there exists a  $y$  with  $x_1 < y_1 < y < y_n < x_{n+1}$ , and  $F^{(n)}(y) = (F')^{(n-1)}(y) = 0$ .  $\square$

If we let  $F(x)$  equal  $f(x) - p(x)$  where  $p(x)$  is a degree  $n$  polynomial, we get that

**Corollary 2.1.** *If  $f(x)$  is an infinitely differentiable function and if  $p(x)$  is a polynomial of degree  $n$  so that  $f(x) = p(x)$  for  $x = x_1, x_2, \dots, x_{n+1}$  where  $x_1 < x_2 < \dots < x_{n+1}$ , then there exists a  $y \in (x_1, x_{n+1})$  so that  $f^{(n)}(y) = p^{(n)}(y)$ .*

## 3. Approximation of the terms in binomial coefficients equal to $t$

Suppose that for  $n \geq 2m$

$$\binom{n}{m} = t.$$

We can take logs of both sides, and then we have by (2.1) that

$$\begin{aligned}
 & \log t + \log(m!) = \\
 & \log(n!) - \log((n-m)!) \\
 & = \left(n + \frac{1}{2}\right) \log(n) - n + \frac{1}{12n} - \left(n-m + \frac{1}{2}\right) \log(n-m) + (n-m) - \frac{1}{12(n-m)} + O\left(\frac{1}{n^3}\right) \\
 & = m \log(n) - \left(n-m + \frac{1}{2}\right) \log\left(1 - \frac{m}{n}\right) - m + O\left(\frac{m}{n^2}\right) \\
 & = m \log(n) + \left(n-m + \frac{1}{2}\right) \left(\frac{m}{n} + \frac{m^2}{2n^2}\right) - m + O\left(\frac{m^3}{n^2}\right) \\
 & = m \log(n) + \frac{m}{n} \left(\frac{-m+1}{2}\right) + O\left(\frac{m^3}{n^2}\right) \\
 & = m \log(n - (m-1)/2) + O\left(\frac{m^3}{n^2}\right).
 \end{aligned}$$

Hence we have that

$$\log(n - (m-1)/2) = \frac{\log t + \log(m!)}{m} + O\left(\frac{m^2}{n^2}\right).$$

So,

$$\begin{aligned}
 (3.1) \quad n &= \exp\left(\frac{\log t + \log(m!)}{m}\right) \left(1 + O\left(\frac{m^2}{n^2}\right)\right) + \frac{m-1}{2} \\
 &= \exp\left(\frac{\log t + \log(m!)}{m}\right) + \frac{m-1}{2} + O\left(\frac{m^2}{n}\right).
 \end{aligned}$$

Notice that if we define

$$\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$$

we can use this to define an analytic function  $f(z)$  by

$$(3.2) \quad \binom{f(z)}{z} = t$$

that satisfies

$$f(z) = \exp\left(\frac{\log t + \log \Gamma(z+1)}{z}\right) + \frac{z-1}{2} + O\left(\frac{z^2}{f(z)}\right)$$

uniformly, so long as  $|f(z)| > |2z|$ . This will hold when

$$\left|\exp\left(\frac{\log t + \log \Gamma(z+1)}{z}\right)\right| > |6z|.$$

By (2.1), this holds when  $\Re(z) > 1$  and

$$\left|\exp\left(\frac{\log t}{z}\right)\right| > C,$$

for some constant  $C$ . This clearly holds if  $|\Im(\log z)| < \pi/4$  and if  $|z| < K \log t$  for some constant  $K$ .

Suppose that for  $\log \log t > \alpha > 1.2$

$$\binom{m^\alpha}{m} = t.$$

We have by (3.1) and (2.1) that

$$m^\alpha = \exp\left(\frac{\log t}{m}\right) \frac{m}{e} (1 + O(\log m/m)) + \frac{m-1}{2} + O(m^{2-\alpha}).$$

So, we get that

$$(\alpha - 1)m \log m = \log(t) + O(m).$$

Or that

$$m \log m = \frac{\log t}{\alpha - 1} + O(m).$$

Hence for sufficiently large  $t$

$$(3.3) \quad \frac{\log t}{\log \log t(\alpha - 1)} < m < \left( \frac{\log t}{\log \log t(\alpha - 1)} \right) \left( 1 + \frac{1}{\log \log t} \right).$$

#### 4. Approximations of the derivatives of $f(z)$

We wish to find bounds for

$$\frac{1}{k!} \frac{d^k}{dx^k} f(x)$$

where  $k \geq 2$  is an integer and for  $f(x)$  as defined in (3.2) and  $x$  real, less than  $K \log t/2$ , and more than 2. Notice that as a complex analytic function,

$$\frac{z^2}{f(z)} = O\left(z \exp\left(\frac{-\log t}{z}\right)\right).$$

Hence, by Cauchy's Integral Formula we have

$$\frac{1}{k!} \frac{d^k}{dx^k} \frac{x^2}{f(x)} = \int_C O\left(w \exp\left(\frac{-\log t}{w}\right) (w-x)^{-k-1}\right) dw.$$

Here  $C$  denotes the contour consisting of the circle of radius  $x/3$  centered at  $x$  traversed once in the counterclockwise direction. Notice that on this contour,

$$\Re\left(\frac{1}{w}\right) \geq \frac{3}{4x}.$$

Therefore, the integral is

$$O(x^{1-k} 3^k t^{-3/4x}) = O(x^{2-k} 3^k f(x)^{-3/4}).$$

It is clear that

$$\frac{d^k}{dx^k} \frac{x-1}{2} = 0.$$

Notice that by (2.1)

$$\frac{\log \Gamma(z+1)}{z} = \log z - 1 + O\left(\frac{\log z}{z}\right)$$

holds for complex  $z$ . This means that, by Cauchy's Integral Formula

$$\frac{1}{k!} \frac{d^k}{dx^k} \frac{\log \Gamma(x+1)}{x} = \frac{(-1)^{k+1}}{kx^k} + \int_C O\left(\frac{\log w}{w(w-x)^{k+1}}\right) dw,$$

where  $C$  is the circle about  $x$  through 1. This is then

$$\frac{(-1)^{k+1}}{kx^k} + O((\log^2 x)(x^{-k-1})).$$

Therefore,

$$\frac{1}{k!} \frac{d^k}{dx^k} \frac{\log t + \log \Gamma(x+1)}{x} = \frac{(-1)^k}{x^{k+1}} (\log t + O(\log^2 x + x/k)).$$

This has the same sign as  $(-1)^k$  as long as  $x < k \log t$ , which will hold in the case we are considering where  $x < K \log t/2$  and  $k \geq 1$ . We now wish to analyze the  $k$ th derivative with respect to  $x$  of

$$g(x) = \exp\left(\frac{\log t + \log \Gamma(x+1)}{x}\right).$$

Using our previous result, and expanding a Taylor series we find that

$$\begin{aligned} g(y) &= \\ \exp\left(\frac{\log t + \log \Gamma(y+1)}{y}\right) &= \\ \exp\left(\frac{\log t + \log \Gamma(x+1)}{x}\right) \exp(a_1(y-x) - a_2(y-x)^2 + \dots + O((y-x)^{k+1})) \end{aligned}$$

where  $a_i = \frac{-1}{x^{k+1}} (\log t + O(\log^2 x + x/k)) < 0$ . This means that the coefficients of the Taylor series about 0 of  $g(x-y)$  are all positive. Therefore,  $\frac{d^k}{dy^k} g(y)$  has the same sign as  $(-1)^k$ . Furthermore, the absolute values of these coefficients is at least

$$\begin{aligned} \exp\left(\frac{\log t + \log \Gamma(x+1)}{x}\right) \frac{|a_1|^k}{k!} &= \\ (f(x) + O(x)) \left(\frac{\log t + O(x/k)}{x^2}\right)^k \frac{1}{k!} &> \\ \frac{f(x) + O(x)}{x^k k!}. \end{aligned}$$

To find an upper bound on the absolute value of the  $k$ th coefficient, we note that if we write

$$\exp\left(\frac{\log t + \log \Gamma(x+1)}{y}\right) = \exp\left(\frac{\log t + \log \Gamma(x+1)}{x}\right) \exp(b_1(y-x) - b_2(y-x)^2 + \dots + O((y-x)^{k+1}))$$

then  $b_i$  and  $a_i$  will have the same sign, but  $|a_i| < |b_i|$ . Therefore, we know that the  $k$ th coefficient of the Taylor series for  $g(y)$  at  $x$  is at most

$$\left| \frac{1}{k!} \frac{d^k}{dy^k} \exp\left(\frac{\log t + \log \Gamma(x+1)}{y}\right) \right|_{y=x}.$$

Using Cauchy's Integral Formula, we find that

$$\left| \frac{1}{k!} \frac{d^k}{dx^k} \exp\left(\frac{c}{x}\right) \right| = \left| \frac{1}{2\pi i} \int_C \exp\left(\frac{c}{w}\right) (w-x)^{-k-1} dw \right|,$$

where  $C$  is the contour that traverses the circle about  $x$  with radius  $\frac{x}{\log(x)}$  once counter clockwise.

The right hand side of the preceding equation is at most

$$\exp\left(\frac{c}{x} \left(1 + O\left(\frac{1}{\log(x)}\right)\right)\right) \left(\frac{x}{\log(x)}\right)^{-k}.$$

Hence we have that for large  $x$ ,

$$\begin{aligned} \left| \frac{1}{k!} \frac{d^k}{dx^k} g(x) \right| &< \\ \exp\left(\frac{\log t + \log \Gamma(x+1)}{x}\right)^{1+2/(\log x)} x^{-k} (\log x)^k &< \\ f(x)^{1+2/\log(x)} x^{-k} (\log x)^k. \end{aligned}$$

Hence, we have that if

$$f(x)^{7/4} > x^2 3^{k+1} k!,$$

then we have that

$$(4.1) \quad 0 < \left| \frac{1}{k!} \frac{d^k}{dx^k} f(x) \right| < 2f(x) e^{2\frac{\log f(x)}{\log x}} x^{-k} (\log x)^k$$

## 5. The Strategy

Let

$$\begin{aligned} A(t) &= \left| \left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m}, 2m < n < m^{6/5} \right\} \right| \\ B(t) &= \left| \left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m}, m^{6/5} < n < m^{\frac{\log \log(t)}{24 \log \log \log(t)}} \right\} \right| \\ C(t) &= \left| \left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m}, m^{\frac{\log \log(t)}{24 \log \log \log(t)}} < n \right\} \right|. \end{aligned}$$

It is clear that

$$(5.1) \quad N(t) = 2A(t) + 2B(t) + 2C(t) + O(1).$$

We now have to prove that

$$A(t), B(t), C(t) \leq O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).$$

## 6. Bounds on $A(t)$

It is clear that if  $2m < n < m^{6/5}$  and

$$t = \binom{n}{m},$$

then by (3.3)  $n < (\log(t))^{6/5}$  and from the proof of theorem 3 in [1] (pg. 258) ,

$$(6.1) \quad A(t) \leq (\log(t))^{3/4} = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).$$

## 7. Bounds on $B(t)$

Let

$$k = \frac{\log \log t}{12 \log \log \log t}.$$

Here we shall consider real  $x$  so that  $x^{\frac{\log \log t}{24 \log \log \log t}} > f(x) > x^{6/5}$ . Notice that  $\log_x f(x)$  is a decreasing function of  $x$  by (3.3). Notice also that  $f(x)^{7/4}x^{-2}$  is also a decreasing function of  $x$  by (3.2). Therefore, in this range,  $f(x)^{7/4}x^{-2}$  is minimal when  $f(x) = x^{6/5}$ . In this case, we have that  $f(x)^{7/4}x^{-2}$  is  $x^{1/10}$ , and by (3.3), this is at least

$$(\log t)^{1/10}(\log \log t)^{-1/10}.$$

Whereas we have that,

$$3^k k! < \exp(k \log k + k) < \exp\left(\frac{1}{12} (\log \log t) \left(1 + \frac{1}{\log \log \log t}\right)\right) < f(x)^{7/4}x^{-2}$$

for all sufficiently large  $t$ .

Additionally, we have by (3.3) that

$$x < \left(\frac{5 \log t}{\log \log t}\right) \left(1 + \frac{1}{\log \log t}\right) < \frac{K \log t}{2}$$

for sufficiently large  $t$ . Hence for sufficiently large  $t$ , and  $x$  in this range, the conditions of (4.1) are satisfied.

Therefore, by (4.1)

$$\begin{aligned}
 0 &< \left| \frac{1}{k!} \frac{d^k}{dx^k} f(x) \right| \\
 &< 2f(x) e^{2 \frac{\log f(x)}{\log x}} x^{-k} (\log x)^k \\
 &< 2x^{k/2} e^k x^{-k} (\log x)^k \\
 &< 2e^k x^{-k/2} (\log x)^k \\
 &< 2e^k \left( \frac{\log t}{k \log \log t} \right)^{-k/2} (\log \log t)^k \\
 &< 2(\log t)^{-k/2} (e \log \log t)^{2k} \\
 (7.1) \quad &< (\log t)^{-(k+1)/3}
 \end{aligned}$$

for all sufficiently large  $t$ .

Suppose that  $m_1 < m_2 < \dots < m_{k+1}$  are integers and that  $f(m_i)$  is also an integer for all  $1 \leq i \leq k+1$ . Then if we define the polynomial

$$P(x) = \sum_{i=1}^{k+1} \frac{f(m_i) \prod_{1 \leq j \leq k+1, j \neq i} (x - m_j)}{\prod_{1 \leq j \leq k+1, j \neq i} (m_i - m_j)},$$

Then  $P$  is of degree  $k$ , and  $P(m_i) = f(m_i)$  for all  $1 \leq i \leq k+1$ . We also have that

$$\frac{1}{k!} \frac{d^k}{dx^k} P(x) = \sum_{i=1}^{k+1} \frac{f(m_i)}{\prod_{1 \leq j \leq k+1, j \neq i} (m_i - m_j)}.$$

This is an integer multiple of

$$M = \left( \prod_{1 \leq i < j \leq k+1} (m_j - m_i) \right)^{-1} > (m_{k+1} - m_1)^{-k(k+1)/2}.$$

Therefore if

$$\frac{1}{k!} \frac{d^k}{dx^k} P(x) \neq 0,$$

then

$$(7.2) \quad \left| \frac{1}{k!} \frac{d^k}{dx^k} P(x) \right| > (m_{k+1} - m_1)^{-k(k+1)/2}.$$

Hence, if  $f(m_i) > m_i^{6/5}$  and  $f(m_i) < m_i^{k/2}$  for all  $i$ , we have by the corollary to Lemma 2.1, (7.1) and (7.2) that

$$(m_{k+1} - m_1)^{-k(k+1)/2} < (\log t)^{-(k+1)/3},$$



or that

$$\begin{aligned}
 m_{k+1} - m_1 &> (\log t)^{1/3k} \\
 &= (\log t)^{\frac{4 \log \log \log t}{\log \log t}} \\
 &= (\log \log t)^4.
 \end{aligned}
 \tag{7.3}$$

Let  $m_1 < m_2 < \dots < m_{B(t)}$  be all of the integers so that for all  $i$ ,  $f(m_i)$  is an integer where  $m_i^{k/2} > f(m_i) > m_i^{6/5}$ . It is clear that  $0 < m_1 < m_{B(t)} < \log t$ . Therefore,

$$\sum_{i=1}^{[B(t)/(k+1)]} (m_{(k+1)i} - m_{(k+1)(i-1)+1}) < \log(t).$$

Therefore, by (7.3)

$$\sum_{i=1}^{[B(t)/(k+1)]} (\log \log t)^4 < \log(t).$$

Or,  $[B(t)/(k+1)] < (\log t)(\log \log t)^{-4}$ . Therefore,

$$B(t) < k + (k+1)(\log t)(\log \log t)^{-4} < \frac{\log t}{(\log \log t)^3} = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).
 \tag{7.4}$$

## 8. Bounds on $C(t)$

We have by (3.3) that if  $f(x) > x^{\frac{\log \log t}{24 \log \log \log t}}$  that

$$x = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).$$

Therefore the largest  $m$  appearing in an element of

$$\left\{ (n, m) \in \mathbb{N}^2 : t = \binom{n}{m}, m^{\frac{\log \log(t)}{24 \log \log \log(t)}} < n \right\}$$

is

$$O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).$$

Which implies that

$$C(t) = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right).
 \tag{8.1}$$

Our result that

$$N(t) = O\left(\frac{\log t \cdot \log \log \log t}{(\log \log t)^2}\right)$$

now follows from (5.1), (6.1), (7.4) and (8.1).

**Remark.** This process can be done without the use of complex analysis except for the possible exception of the bounding of the derivatives of  $\log \Gamma(x+1)/x$ . This is done by claiming that solutions to  $t = \binom{n}{m}$  correspond to points near the curve  $f(x) = (t \cdot x!)^{1/x} + (x-1)/2$ . This method requires that there be better bounds on the number of points where  $n$  and  $m$  are closer to each other (we would need bounds for solutions where  $n < m^2$ ), but this can be provided by looking at the greatest common divisors of products of nearby sequences of integers.

## REFERENCES

- [1] Abbot, H. L.; Erdős, P.; Hanson, D., *On the Number of Times an Integer Occurs as a Binomial Coefficient*, Amer. Math. Monthly **81** (1974), 256-261.
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