TESTING CHEBYSHEV'S BIAS FOR PRIME NUMBERS UP TO 10^{15}

Andrey Sergeevich SHCHEBETOV

Lomonosovskaya School & MIEM HSE

Moscow, Russia

Presentation for the 30th European Union Contest for Young Scientists

EUCYS 2018

Dublin, Ireland

September 14-18, 2018

GOALS AND TARGETS FOR THE PROJECT

- *To test Chebyshev's bias* **for 15 "most biased prime number races"**
- **To extend the range tested by mathematicians 1000 times to 10^{15}** ($10*10^{14}$ – *the upper bound and the last number of the tested range*)
- **To define exactly the main characteristics of all sign-changing zones** *(known, as well as newly found), including their beginning, end and number of terms*
- **To check newly discovered zones against predictions**
- **To test and confirm all previously known sign-changing zones for** $\Delta_{q,a,b}(x)$ **up to 10^{12}**
- **To make all primary data available** *to a wide group of mathematicians working in number theory* **field through OEIS publication and deposit in author's own repository**
- **To define all data in a uniform way and with unified format**

The main goal of the project was to test Chebyshev's bias for 15 selected modulus and pairs of residues for prime numbers up to 10^{15}.

LETTER FROM CHEBYSHEV TO FUSS (1853)
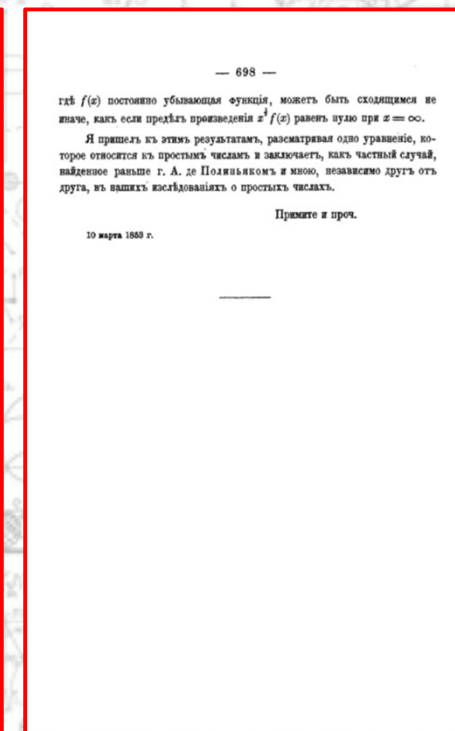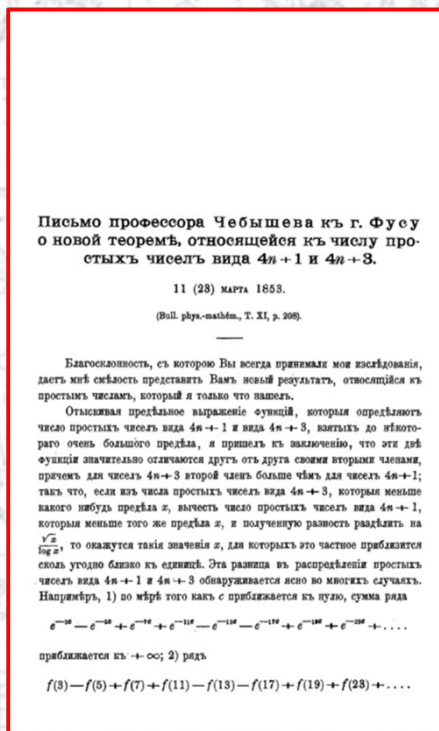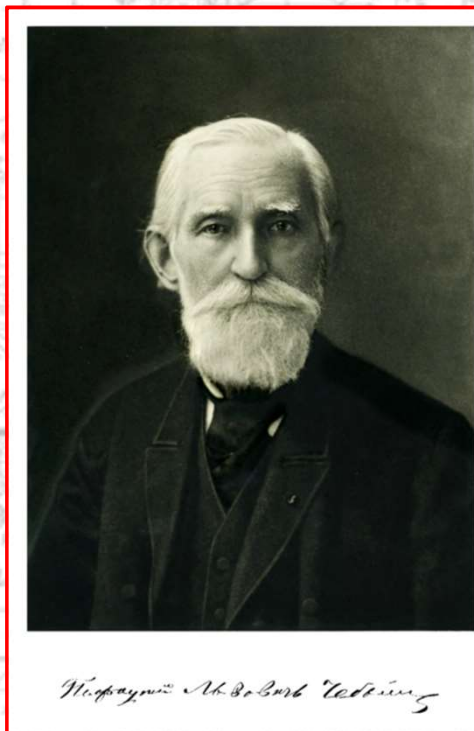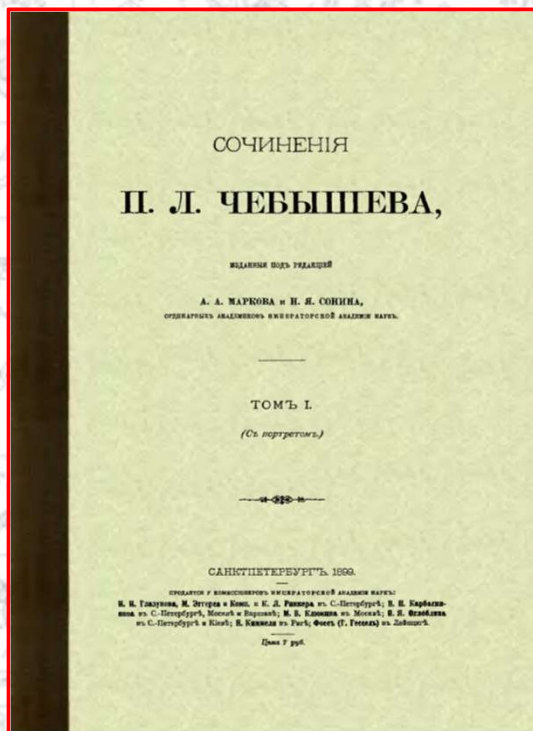
Chebyshev's Bias (Chebyshev, 1853). **"There is a notable difference in the splitting of the prime numbers between the two forms 4n + 3, 4n + 1: the first form contains a lot more than the second."**

In 1853 Chebyshev suggested that there are always more primes of the form 4n + 3 than primes of the form 4n + 1.

CHEBYSHEV'S BIAS FOR TWO RESIDUES

$$\pi(x) = \pi_{4,3}(x) + \pi_{4,1}(x) + 1$$
$$\Delta_{4,3,1}(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$$

$\pi(x)$ – prime counting function
$q = 4$ – modulus
$a = 3$ and $b = 1$ – residues
$(a, q) = 1$, $(b, q) = 1$
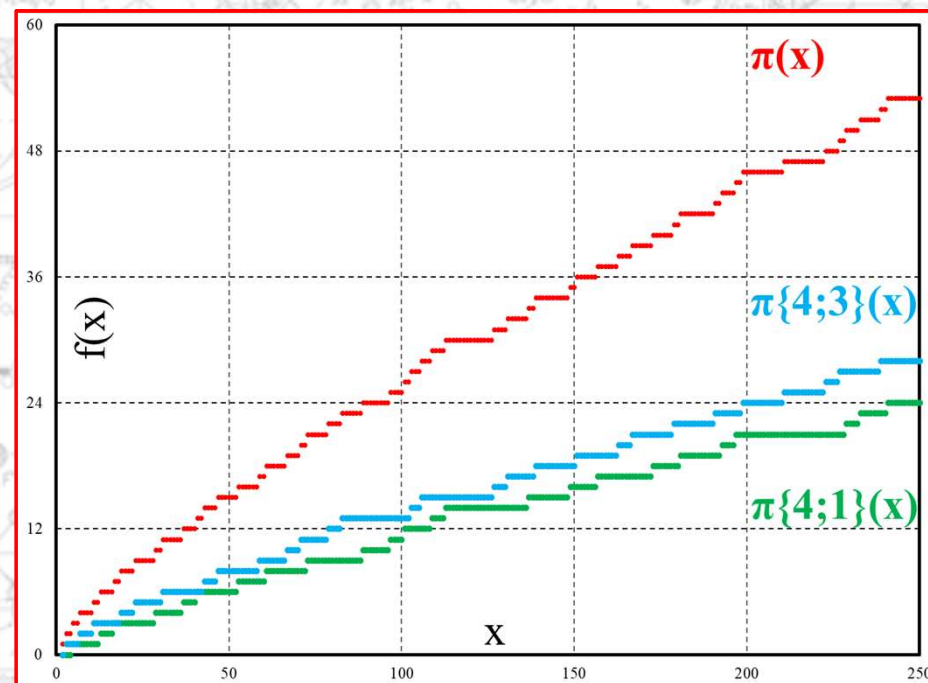
Chebyshev's Bias(1853): $\Delta_{4,3,1}(x) > 0$ *for all x*

- *Initial conjecture for q = 4, a = 3, b = 1*
- *Similar situation for q = 3, a = 2, b = 1*
- *«Prime number races»*

Chebyshev's Bias is easily formulated through prime counting function for two residues a and b modulo q.

CHEBYSHEV'S BIAS EXAMPLE FOR TWO RESIDUES MOD 4

x	$\pi(x)$	#{4n + 2}	#{4n + 3}	#{4n + 1}	Δ{4, 3, 1}	%
100	25	1	13	11	2	2.000%
200	46	1	24	21	3	1.500%
300	62	1	32	29	3	1.000%
400	78	1	40	37	3	0.750%
500	95	1	50	44	6	1.200%
600	109	1	57	51	6	1.000%
700	125	1	65	59	6	0.857%
800	139	1	71	67	4	0.500%
900	154	1	79	74	5	0.556%
1000	168	1	87	80	7	0.700%
2000	303	1	155	147	8	0.400%
3000	430	1	218	211	7	0.233%
4000	550	1	280	269	11	0.275%
5000	669	1	339	329	10	0.200%
6000	783	1	399	383	16	0.267%
7000	900	1	457	442	15	0.214%
8000	1007	1	507	499	8	0.100%
9000	1117	1	562	554	8	0.089%
10,000	1229	1	619	609	10	0.100%
20,000	2262	1	1136	1125	11	0.055%

- *The phenomena is **small, but permanent***
- *Effective percentage has **tendency to decrease***
- *At Chebyshev's times and 100 years after **no negative zones for** $\Delta_{4,3,1}$ **were known***
- *Only in 1957 **the first and the second zones were discovered***

The first and second zones where Chebyshev's Bias was violated were discovered only in 1957 – more than 100 years after the letter to Fuss.

MAIN WORKS IN CHEBYSHEV'S BIAS AREA

1853 *Letter from P.L. Chebyshev to P.N. Fuss*

1914 *J. E. Littlewood, «Sur la distribution des nombres premiers»*

1957 *J. Leech, «Note on the distribution of prime numbers»*

1959 *D. Shanks «Quadratic Residues and the Distribution of Primes»*

1962 *S. Knapowski and P. Turán, «Comparative Prime-Number Theory»*

1978 *C. Bays u R. Hudson, «Details of the first region of integers x with $\pi\{3,2\}(x) < \pi\{3,1\}(x)$»*

1978 *R. H. Hudson u C. Bays, «The appearance of tens of billion of integers x with $\pi\{24, 13\}(x) < \pi\{24, 1\}(x)$ in the vicinity of 10^{12}»*

1979 *C. Bays u R. H. Hudson, «Numerical and graphical description of all axis crossing regions for the moduli 4 and 8 which occur before 10^{12}»*

1994 *M. Rubinstein u P. Sarnak, «Chebyshev's Bias»*

2001 *C. Bays, K. Ford, R. H. Hudson u M. Rubinstein, «Zeros of Dirichlet L-functions near the Real Axis and Chebyshev's Bias»*

2001 *K. Ford u R. H. Hudson, «Sign changes in pi{q;a}(x) - pi{q;b}(x)»*

2006 *A. Granville u G. Martin, «Prime Number Races»*

2012 *G. Martin u J. Scarfy, «Comparative Prime Number Theory»*

2013 *D. Fiorilli u G. Martin, «Inequities in the Shanks-Renyi prime number race: an asymptotic formula for the densities»*

«Chebyshev's conjecture was the origin for a big branch of modern Number Theory, namely, comparative prime-number theory» as was written by S.V. Konyagin (Russia) and K. Ford (USA) in a joint paper.

CHEBYSHEV'S BIAS AND OTHER THEOREMS

Dirichlet prime number theorem for arithmetic progression *(Dirichlet, 1837). Let a, q $\in Z^+$ be such that gcd(a, q) = 1. Then there are infinitely many prime numbers p such that p $\equiv$ a (mod q). Therefore, as a result:*

$$\frac{\#\{\text{primes } qn + a \leq x\}}{\#\{\text{primes } qn + b \leq x\}} \to 1 \ (as \ x \to \infty)$$

Theorem (Littlewood, 1914). *There are arbitrarily large values of x for which there are more primes of the form 4n + 1 up to x than primes of the form 4n + 3. In fact, there are arbitrarily large values of x for which:*

$$\#\{primes \ 4n + 1 \leq x\} - \#\{\text{primes } 4n + 3 \leq x\} \geq \frac{1}{2}\frac{\sqrt{x}}{\ln x} \ln \ln \ln x$$

Conjecture (Knapowski and Turán, 1962). *As X $\to \infty$, the percentage of integers x $\leq$ X for which there are more primes of the form 4n + 3 up to x than of the form 4n + 1 goes to 100%.*

Theorem (Kaczorowski, Rubinstein-Sarnak, 1994). *If the Generalized Riemann Hypothesis GRH is true, then the Knapowski-Turán Conjecture is false.*

The connection between Chebyshev's Bias and Generalized Riemann Hypothesis (GRH) was proven in 1994.

CHEBYSHEV'S BIAS AND OTHER THEOREMS

Generalized Riemann Hypothesis (GRH) (Piltz, 1884): *For any χ mod q and all complex s = σ + it such as 0 ≤ σ ≤ 1 and L(σ + it, χ) = 0, all the non-trivial zeroes of the Dirichlet L-function L(s, χ) (Re(s) > 1) lie on the straight line Re(s) = 1/2.*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\ primes} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Dirichlet L-function for "race of primes 4n + 3 vs. primes 4n + 1" (Re(s) > 1):

$$L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \cdots$$

Therefore, the sum over primes in arithmetic progression is equivalent to the sum over zeros of Dirichlet L-function.

Rubinstein and Sarnak (1994): *The sum over primes in arithmetic progressions results into:*

$$\sum_{\substack{k\geq 1 \\ p^k \equiv a\ mod\ q}} \sum_{k\leq x} \frac{1}{k} = \pi(x; q, a) + \frac{1}{2}\left|\{p \leq \sqrt{x}: p^2 \equiv a\ mod\ q\}\right| + error$$

The second term in the formula is the source of Chebyshev's Bias.

Chebyshev's Bias (modern formulation): *There are more primes of the form **qn + a** than of the form **qn + b** if **a** is non-square and **b** is a square residue modulo **q**.*

The source and the origin of Chebyshev's Bias is the presence of the square residue b among residues modulo q.

DISPROVAL OF KNAPOWSKI-TURÁN CONJECTURE

Maximum percentage of values of $x \leq X$ for which $\pi_{4,1}(x) > \pi_{4,3}(x)$

Range	Max %
0-10^7	2.6%
10^7-10^8	0.6%
10^8-10^9	0.1%
10^9-10^{10}	1.6%
10^{10}-10^{11}	2.8%

⬇ Leech: 1957

⬇ Lehmer: 1969

⬇ Lehmer: 1969

⬆ Bays & Hudson: 1979

⬆ Bays & Hudson: 1979-1996

- *With exact formulation of Knapowski-Turán conjecture in 1962 **the extensive search for $\Delta_{q,a,b}$ sign-changing zones started** for various moduli and residues*
- ***It became clear that Knapowski-Turán conjecture was fa**lse after a number breakthrough works and papers of C. Bays and R.H. Hudson (USA) who discovered several new sign-changing zones for $\Delta_{4,3,1}$ between 1979 and 1996*

Empirical data supported Knapowski-Turán conjecture up to 10^9 only. After Bays and Hudson research it became clear that it was wrong.

EMPIRICAL RESULTS: 1957-1996 (q = 3, 4 & 8)

Status of $\Delta_{q,a,b}(x)$ sign-changing zones search from 1957 to 1996

q	#	b	a	Beginning	Discovered	
3	1	1	2	608,981,813,029	Bays & Hudson, 1978	🟢
4	1	1	3	26,861	Leech, 1957	🟢
4	2	1	3	616,841	Leech, 1957	🟢
4	3	1	3	12,306,137	Lehmer, 1969	🟢
4	4	1	3	951,784,481	Lehmer, 1969	🟢 *7 known zones*
4	5	1	3	6,309,280,709	Bays & Hudson, 1979	🟢
4	6	1	3	18,465,126,293	Bays & Hudson, 1979	🟢
4	7	1	3	1,488,478,427,089	Bays & Hudson, 1996	🟢
8	1	1	3	Not known up to 10^{12}	Not discovered	🔴
8	1	1	5	588,067,889	Bays & Hudson, 1979	🟢
8	2	1	5	35,615,130,497	Bays & Hudson, 1979	🟢
8	1	1	7	Not known up to 10^{12}	Not discovered	🔴

- *The search for new zones had been very slow - **sometimes decades** passed between the discoveries*
- *Several outstanding mathematicians such as J. Leech ("Leech lattice"), D.H. Lehmer ("Lucas-Lehmer primality test") and C. Bays & R.H. Hudson (prime number research and estimates for "Skewes number") **contributed greatly to the search***
- *Most sign-changing zones (7) were found for $\Delta_{4,3,1}$*

There had been extensive search for Δ sign-changing zones up to 10^{12} from 1957 to 1996.

EMPIRICAL RESULTS: 1957-1996 (q = 12 & 24)

Status of $\Delta_{q,a,b}(x)$ sign-changing zones search from 1957 to 1996

q	#	b	a	Beginning	Discovered	
12	1	1	5	Not known up to 10^{12}	Not discovered	🔴
12	1	1	7	Not known up to 10^{12}	Not discovered	🔴
12	1	1	11	Not known up to 10^{12}	Not discovered	🔴
24	1	1	5	Not known up to 10^{12}	Not discovered	🔴
24	1	1	7	Not known up to 10^{12}	Not discovered	🔴
24	1	1	11	Not known up to 10^{12}	Not discovered	🔴
24	1	1	13	«Around 10^{12}»	Bays & Hudson, 1978	🟡 *Not defined exactly*
24	1	1	17	Not known up to 10^{12}	Not discovered	🔴
24	1	1	19	Not known up to 10^{12}	Not discovered	🔴
24	1	1	23	Not known up to 10^{12}	Not discovered	🔴

- *Apart from $\Delta_{24,13,1}$ there had been no other found Δ sign-changing zones for q = 12 & 24*

- *For $\Delta_{24,13,1}$ the first zone was defined only approximately without exact boundaries and number of terms*

Mod 12 and 24 presented a major problem as there had been almost nothing discovered and known about them.

EMPIRICAL RESULTS: 1996-2016

Status of △q,a,b(x) sign-changing zones search from 1996 to 2016

q	#	b	a	Beginning	Discovered
3	2	1	2	6,148,171,711,663	Johnson, 2011
8	1	1	7	192,252,423,729,713	Martin, 2016

🟡 *Found with mistakes*

🟡 *Only first point found*

- *New zones were discovered quite rarely*

- *The range beyond 10^{12} was beyond the technical capabilities for a long time*

- *Both Johnson and Martin were programmers, not mathematicians*

- *"Practice Is the Sole Criterion of Truth": no theoretical model would ever disprove the numerically confirmed $\Delta_{q,a,b}$ sign-changing zones*

- *Direct numerical calculations for $\Delta_{q,a,b}$ sign-changing zones **have absolute accuracy***

In 20+ years since 1996 there have been only two sign-changing zones found, although with incomplete or inaccurate information.

LOGARITHMIC DENSITY/PROBABILITY OF $\pi_{4,3}(x) > \pi_{4,1}(x)$

Theorem (Rubinstein and Sarnak, 1994). *As $X \to \infty$,*

$$\frac{1}{\log X} \sum_{\substack{x \leq X \\ \pi_{4,3}(x) > \pi_{4,1}(x)}} \frac{1}{x} \to 0.9959 \ldots$$

In other words, Chebyshev was right 99.59% of the time!

Theorem (Rubinstein and Sarnak, 1994) *Let $(a;q) = (b;q) = 1$ such that $a \neq b \mod q$. The logarithmic density*

$$\delta(q; a, b) := \lim_{X \to \infty} \frac{1}{\log X} \int_{\substack{t \in [2,X] \\ \pi(t;q,a) > \pi(t;q,b)}} \frac{dt}{t}$$

exists and is positive.

In 1994 the existence of positive logarithmic density, for Δ, meaning "the probability that $\pi_{q,a}(x) > \pi_{q,b}(x)$" was proved.

THE MOST "UNFAIR PRIME NUMBER RACES"
The "most unfair prime number races" (Fiorilli & Martin) & status (2013)

#	q	b	a	δ(q;a,1)	Status (2013)	
1	24	1	5	0.999988	Not found up to 10^{12}	🔴
2	24	1	11	0.999983	Not found up to 10^{12}	🔴
3	12	1	11	0.999977	Not found up to 10^{12}	🔴
4	24	1	23	0.999889	Not found up to 10^{12}	🔴
5	24	1	7	0.999834	Not found up to 10^{12}	🔴
6	24	1	19	0.999719	Not found up to 10^{12}	🔴
7	8	1	3	0.999569	Not found up to 10^{12}	🔴
8	12	1	5	0.999206	Not found up to 10^{12}	🔴
9	24	1	17	0.999125	Not found up to 10^{12}	🔴
10	3	1	2	0.999063	Known up to 10^{12}	🟢
11	8	1	7	0.998939	Not found up to 10^{12}	🔴
12	24	1	13	0.998722	Known up to 10^{12}	🟡
13	12	1	7	0.998606	Not found up to 10^{12}	🔴
14	8	1	5	0.997395	Known up to 10^{12}	🟢
15	4	1	3	0.995928	Known up to 10^{12}	🟢

- *Fundamental 2013 research* by Fiorilli and Martin on logarithmic densities
- *Logarithmic densities were calculated and ranked* **for 120 top "prime number races"**
- **Top 15 were selected for test** within the scope of this project

Not defined exactly

In 2013 the most "unfair prime number races" were theoretically defined, 15 of which were selected for this project up to 10^{15}.

PREDICTIONS OF NEW ZONES: q = 3, 4 & 8

Predictions of possible $\Delta_{q,a,b}(x)$ sign-changing zones up to 10^{20}

q	#	b	a	Beginning	Made by	
q=3	2	1	2	$6.15*10^{12}$	Bays & Hudson, 2001	**CHECK!**
q=3	3	1	2	$3.97*10^{19}$	Bays & Hudson, 2001	
q=3	3	1	2	$3.97*10^{19}$	Ford & Hudson, 2001	
q=4	8	1	3	$9.32*10^{12}$	Bays & Hudson, 2001	**CHECK!**
q=4	9	1	3	$9.97*10^{17}$	Deléglise, Dusart & Roblot, 2004	
q=8	1	1	3	$6.82*10^{18}$	Ford & Hudson, 2001	
q=8	1	1	5	$1.93*10^{14}$	Ford & Hudson, 2001	**CHECK!**
q=8	2	1	5	$9.32*10^{14}$	Ford & Hudson, 2001	**CHECK!**
q=8	1	1	7	$1.93*10^{14}$	Bays & Hudson, 2001	**CHECK!**
q=8	1	1	7	$1.93*10^{14}$	Ford & Hudson, 2001	**CHECK!**

- *One of the main goals of the project was **to check the predictions for new sign-changing zones made in the beginning of 2000s***
- *Some predictions $(>10^{18})$ **were located far beyond the technical capabilities of that time***
- *Even today working above 10^{18} requires **the use of supercomputers with many cores and efficient multi-threading***

For q = 3, 4 and 8 the existence of six Δ sign-changing zones were predicted up to the 10^{15} – the upper boundary of the project.

PREDICTIONS OF NEW ZONES: q = 12 & 24

Predictions of possible $\Delta_{q,a,b}(x)$ sign-changing zones up to 10^{20}

q	#	b	a	Beginning	Predicted by	
q=12	1	1	5	$9.84*10^{16}$	Ford & Hudson, 2001	
q=12	1	1	7	$9.78*10^{16}$	Ford & Hudson, 2001	
q=12	1	1	11	None $< 10^{20}$	Ford & Hudson, 2001	
q=24	1	1	5	None $< 10^{20}$	Ford & Hudson, 2001	
q=24	1	1	7	None $< 10^{20}$	Ford & Hudson, 2001	
q=24	1	1	11	None $< 10^{20}$	Ford & Hudson, 2001	
q=24	1	1	13	$6.74*10^{14}$	Ford & Hudson, 2001	CHECK!
q=24	1	1	17	$6.18*10^{14}$	Ford & Hudson, 2001	CHECK!
q=24	2	1	17	$7.11*10^{14}$	Ford & Hudson, 2001	CHECK!
q=24	1	1	19	$7.15*10^{14}$	Ford & Hudson, 2001	CHECK!
q=24	1	1	23	$7.44*10^{18}$	Ford & Hudson, 2001	

- *One of the main goals of the project was **to check the predictions for new sign-changing zones made in the beginning of 2000s***
- *The situation with q = 12 and 24 was similar: some predictions ($>10^{18}$) **were located far beyond the technical capabilities of that time***

For q = 12 and 24 the existence of four Δ sign-changing zones were predicted up to the 10^{15} – the upper boundary of the project.
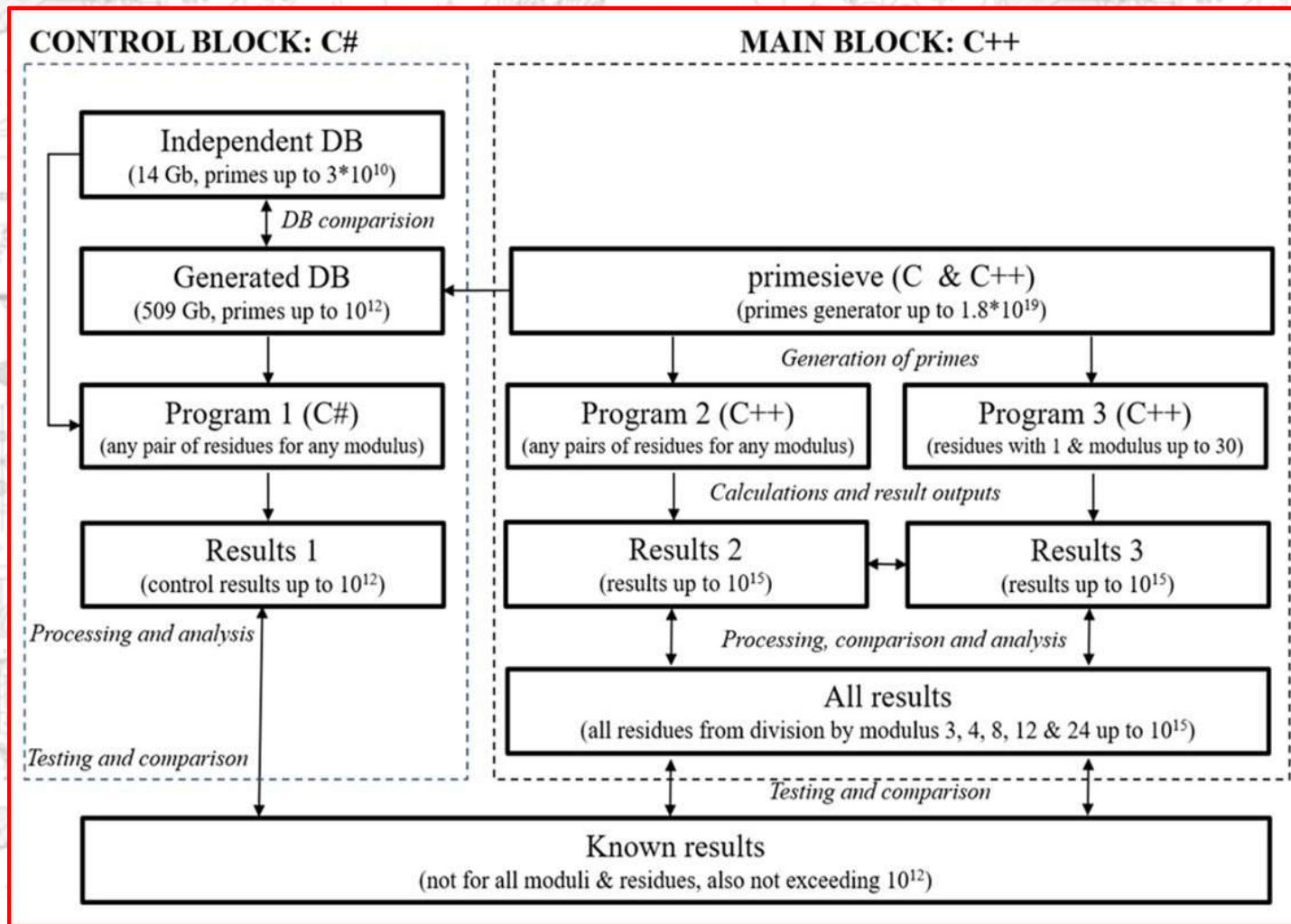
TECHNICAL DIFFICULTIES

- 10^{15} **range seemed incredibly high 17 years ago** *(in 2001) when Bays & Hudson summarized their 25-year effort in Chebyshev's Bias area*
- **The direct brute-force method was extremely resource-consuming as well as sensitive to non-stop execution**
- **Fast and reliable prime number generators** *that were capable of working with large primes above 10^{12} and generate them without omissions and mistakes* **were absent**
- *The alternative way of getting primes –* **the preliminary generation with further database storage, required enormous memory size** *(hundreds of terabytes or even petabytes) and barely allowed to move above 10^{12} leaving alone 10^{15}*
- **Fast and affordable servers** *capable to work without mistakes and non-stop* **24 x 7** *for many weeks and months* **were required**
- **Many predicted points were located around 10^{18} – far above 10^{15},** *that also reduced substantially the desire for implementation*
- *To work above 10^{18}* **fast supercomputers with many cores and efficient multi-threading were required**

The direct brute force method to test Chebyshev's Bias even up to 10^{15} was difficult till recent advances in software and hardware development.

PROJECT TECHNICAL SET-UP

CONTROL BLOCK: C#

Independent DB
(14 Gb, primes up to $3*10^{10}$)

DB comparision

Generated DB
(509 Gb, primes up to 10^{12})

Program 1 (C#)
(any pair of residues for any modulus)

Results 1
(control results up to 10^{12})

Processing and analysis

Testing and comparison

MAIN BLOCK: C++

primesieve (C & C++)
(primes generator up to $1.8*10^{19}$)

Generation of primes

Program 2 (C++)
(any pairs of residues for any modulus)

Program 3 (C++)
(residues with 1 & modulus up to 30)

Calculations and result outputs

Results 2
(results up to 10^{15})

Results 3
(results up to 10^{15})

Processing, comparison and analysis

All results
(all residues from division by modulus 3, 4, 8, 12 & 24 up to 10^{15})

Testing and comparison

Known results
(not for all moduli & residues, also not exceeding 10^{12})

- ***2 main C++ programs***
- ***Primes up to $1.8*10^{19}$ (2^{64}) could be tested***
- ***Control C# program*** with 10^{12} database to check
- ***3 consecutive ranges to test:*** 10^{13}, 10^{14}, 10^{15}
- ***At least 2 passes*** for each range and "prime number race"
- *Initial numerical test was finished in the beginning of 2018*
- ***Project was extended to 10^{16}*** in May 2018

Several C++ & C# programs were written for the project. The fastest known prime number generator "primesieve" was used for tests.

RESULTS: q = 3 (primes and values of n for primes)

Sign-changing zones for q = 3: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS			
q = 3	1	1	2	608,981,813,029	610,968,213,787	20,590	A297006			
q = 3	2	1	2	6,148,171,711,663	6,156,051,951,677	63,733	A297006	NEW!	Ⓥ	*6.15*10^{12}*
Total	2	1	2			84,323	A297006			

Sign-changing zones for q = 3: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 3	1	1	2	23,338,590,792	23,411,791,034	20,590	A297005		
q = 3	2	1	2	216,415,270,060	216,682,882,512	63,733	A297005	NEW!	Ⓥ
Total	2	1	2			84,323	A297005		

- **Second zone matched exactly** with that predicted by Bays & Hudson (2001) at 6.15*10^{12}

- *New A297006 and A297005 sequences **were registered with OEIS***

For q = 3 the 2nd Δ sign-changing zone was found that almost exactly matched a zone predicted back in 2001.

RESULTS: q = 4 (primes)

Sign-changing zones for q = 4: primes

q	№	b	a	Beginning	End	# Δ= -1	OEIS			
q = 4	1	1	3	26,861	26,861	1	A051025			
q = 4	2	1	3	616,841	633,797	90	A051025			
q = 4	3	1	3	12,306,137	12,382,313	150	A051025			
q = 4	4	1	3	951,784,481	952,223,473	396	A051025			
q = 4	5	1	3	6,309,280,709	6,403,150,189	6,205	A051025			
q = 4	6	1	3	18,465,126,293	19,033,524,533	6,524	A051025			
q = 4	7	1	3	1,488,478,427,089	1,494,617,929,603	14,189	A051025			
q = 4	8	1	3	9,103,362,505,801	9,543,313,015,309	391,378	A051025	NEW!	Ⓥ	$9.32*10^{12}$
q = 4	9	1	3	64,083,080,712,569	64,084,318,523,021	13,370	A051025	NEW!	⚠	$9.97*10^{17}$
q = 4	10	1	3	715,725,135,905,981	732,156,384,107,921	481,194	A051025	NEW!	⚠	
Total	**10**	**1**	**3**			**913,497**	**A051025**			

- *The 8th zone happened lower than was predicted by Bays & Hudson (2001) at $9.32*10^{12}$*

- *The 9th & 10th zones were not expected up to 10^{18}*

- *OEIS sequence A051025 with only 30 terms was complemented and now includes 913,497 terms*

For q = 4 three new zones (8th, 9th & 10th) were discovered. According to the theoretical models the last two had not been expected below 10^{18}.

RESULTS: q = 4 (values of n for primes)

Sign-changing zones for q = 4: values of n for primes ($\pi(x)$ function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 4	1	1	3	2,946	2,946	1	A051024		
q = 4	2	1	3	50,378	51,622	90	A051024		
q = 4	3	1	3	806,808	811,528	150	A051024		
q = 4	4	1	3	48,517,584	48,538,970	396	A051024		
q = 4	5	1	3	293,267,470	297,424,714	6,205	A051024		
q = 4	6	1	3	817,388,828	841,415,718	6,524	A051024		
q = 4	7	1	3	55,152,203,450	55,371,233,730	14,189	A051024		
q = 4	8	1	3	316,064,952,540	330,797,040,308	391,378	A051024	NEW!	Ⓥ
q = 4	9	1	3	2,083,576,475,506	2,083,615,410,040	13,370	A051024	NEW!	⊘
q = 4	10	1	3	21,576,098,946,648	22,056,324,317,296	481,194	A051024	NEW!	⊘
Total	**10**	**1**	**3**			**913,497**	**A051024**		

- ***The 8th zone happened lower** than was predicted by Bays & Hudson (2001)*

- ***The 9th and 10th zone were not expected so low***

- ***OEIS sequence** A051024 with only 33 terms was complemented and now includes 913,497 terms*

For q = 4 three new zones (8th, 9th & 10th) were discovered. According to the theoretical models the last two had not been expected below 10^{18}.

RESULTS: q = 8 (primes)

Sign-changing zones for q = 8: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 8	1	1	3	Not found up to 10^{15}					
q = 8	1	1	5	588,067,889	593,871,533	488	A297448		
q = 8	2	1	5	35,615,130,497	37,335,021,821	22,305	A297448		
q = 8	3	1	5	5,267,226,902,633	5,312,932,515,721	109,831	A297448	NEW!	⊙
q = 8	4	1	5	5,758,938,230,761	5,768,749,719,461	48,229	A297448	NEW!	⊙
q = 8	5	1	5	6,200,509,945,537	6,209,511,651,289	18,048	A297448	NEW!	⊙
q = 8	6	1	5	192,189,726,613,273	194,318,969,449,909	465,274	A297448	NEW! Ⓥ	*1.93*10^14*
q = 8	7	1	5	930,525,161,507,057	932,080,335,660,277	186,057	A297448	NEW! Ⓥ	*9.32*10^14*
Total	**7**	**1**	**5**			**850,232**	**A297448**		
q = 8	1	1	7	192,252,423,729,713	192,876,135,747,311	234,937	A295354	NEW! Ⓥ	*1.93*10^14*
Total	**1**	**1**	**7**			**234,937**	**A295354**		

- *Not a single zone discovered for $\Delta_{8,3,1}(x)$*
- *Out of 5 discovered zones for $\Delta_{8,5,1}(x)$ only the 6th and 7th (2 widest ones) were predicted correctly at 1.93*10^14 and 9.32*10^14 respectively*
- *The 1st zone for $\Delta_{8,7,1}(x)$ was also predicted correctly at 1.93*10^14*
- *4 new sequences were registered: A297448, A297447, A295354 and A295353*

Out of 5 new discovered zones for $\Delta_{8,5,1}(x)$ theoretical models correctly predicted only 2. The prediction for $\Delta_{8,7,1}(x)$ was also confirmed.

RESULTS: q = 8 (values of n for primes)

Sign-changing zones for q = 8: values of n for primes ($\pi(x)$ function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 8	1	1	3	Not found up to 10^{15}					
q = 8	1	1	5	30,733,704	31,021,248	488	A297447		
q = 8	2	1	5	1,531,917,197	1,602,638,725	22,305	A297447		
q = 8	3	1	5	186,422,420,112	187,982,502,637	109,831	A297447	NEW!	⚠
q = 8	4	1	5	203,182,722,672	203,516,651,165	48,229	A297447	NEW!	⚠
q = 8	5	1	5	218,192,372,353	218,497,974,121	18,048	A297447	NEW!	⚠
q = 8	6	1	5	6,033,099,205,868	6,097,827,689,926	465,274	A297447	NEW!	Ⓥ
q = 8	7	1	5	27,830,993,289,634	27,876,113,171,315	186,057	A297447	NEW!	Ⓥ
Total	7	1	5			850,232	A297447		
q = 8	1	1	7	6,035,005,477,560	6,053,968,231,350	234,937	A295353	NEW!	Ⓥ
Total	1	1	7			234,937	A295353		

- *Not a single zone discovered for $\Delta_{8,3,1}(x)$*
- *Out of 5 discovered zones for $\Delta_{8,5,1}(x)$ **only the 6th and 7th (2 widest ones) were predicted correctly***
- *The 1st zone for $\Delta_{8,7,1}(x)$ **was also predicted correctly***
- *4 new sequences were registered: A297448, A297447, A295354 and A295353*

Out of 5 new discovered zones for $\Delta_{8,5,1}(x)$ theoretical models correctly predicted only 2. The prediction for $\Delta_{8,7,1}(x)$ was also confirmed.

RESULTS: q = 12 (primes and values of n for primes)

Sign-changing zones for q = 12: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 12	1	1	5	25,726,067,172,577	25,727,487,045,613	8,399	A297355	NEW!	⚠ $9.84*10^{16}$
Total	1	1	5			8,399	A297355		
q = 12	1	1	7	27,489,101,529,529	27,555,497,263,753	55,596	A297357	NEW!	⚠ $9.78*10^{16}$
Total	1	1	7			55,596	A297357		
q = 12	1	1	11	Not found up to 10^{15}					

Sign-changing zones for q = 12: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 12	1	1	5	862,062,606,318	862,108,594,325	8,399	A297354	NEW!	⚠
Total	1	1	5			8,399	A297354		
q = 12	1	1	7	919,096,512,484	921,242,027,614	55,596	A297356	NEW!	⚠
Total	1	1	7			55,596	A297356		
q = 12	1	1	11	Not found up to 10^{15}					

- *Not a single zone discovered for $\Delta_{12,1,1}(x)$*
- *Discovered zone for $\Delta_{12,5,1}(x)$ happened to be narrow and lower than predicted at $9.84*10^{16}$*
- *Discovered zone for $\Delta_{12,7,1}(x)$ happened to be narrow and lower than predicted at $9.78*10^{16}$*
- *Four new OEIS sequences were registered A297355, A297354, A297357 and A297356*

In 10^{15} range theoretical models failed to predict both discovered zones unknown before. This requires explanation and change in the models!

RESULTS: q = 24 (primes)

Sign-changing zones for q = 24: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 24	1	1	5	Not found up to 10^{15}					
q = 24	1	1	7	Not found up to 10^{15}					
q = 24	1	1	11	Not found up to 10^{15}					
q = 24	1	1	13	978,412,359,121	989,462,029,561	9,920	A295356		
q = 24	2	1	13	1,005,578,970,337	1,009,517,096,641	22,648	A295356	NEW!	⚠
q = 24	3	1	13	1,025,403,695,233	1,096,157,101,033	111,408	A295356	NEW!	⚠
q = 24	4	1	13	648,452,989,927,609	649,632,972,248,893	202,195	A295356	NEW!	⚠
q = 24	5	1	13	655,404,854,710,621	662,189,414,787,361	594,414	A295356	NEW!	Ⓥ $6.74*10^{14}$
q = 24	6	1	13	687,936,222,802,693	699,914,738,212,849	1,441,319	A295356	NEW!	Ⓥ
Total	6	1	13			2,381,904	A295356		
q = 24	1	1	17	617,139,273,158,713	618,051,990,355,993	73,201	A297450	NEW!	Ⓥ $6.18*10^{14}$
q = 24	2	1	17	709,763,768,223,841	714,186,411,923,009	773,982	A297450	NEW!	Ⓥ $7.11*10^{14}$
q = 24	3	1	17	772,451,788,864,537	772,739,867,710,897	116,739	A297450	NEW!	⚠
Total	3	1	17			963,922	A297450		
q = 24	1	1	19	706,866,045,116,113	709,591,447,226,587	260,586	A298821	NEW!	⚠
q = 24	2	1	19	716,328,072,795,619	725,993,117,452,657	833,790	A298821	NEW!	Ⓥ $7.15*10^{14}$
q = 24	3	1	19	731,496,205,367,611	733,085,386,984,849	306,557	A298821	NEW!	⚠
q = 24	4	1	19	739,965,838,936,153	756,906,118,578,763	1,586,533	A298821	NEW!	⚠
q = 24	5	1	19	761,403,326,459,539	766,164,822,666,883	449,524	A298821	NEW!	⚠
Total	5	1	19			3,436,990	A298821		
q = 24	1	1	23	Not found up to 10^{15}					

- **Six new OEIS sequences were registered A295356, A295355, A297450, A297449, A298821 and A298820**

For q = 24 13 new zones were discovered for $\Delta_{24,13,1}(x)$, $\Delta_{24,17,1}(x)$ and $\Delta_{24,19,1}(x)$. None were found for $\Delta_{24,5,1}(x)$, $\Delta_{24,7,1}(x)$, $\Delta_{24,11,1}(x)$ & $\Delta_{24,23,1}(x)$.

RESULTS: q = 24 (values of n for primes)

Sign-changing zones for q = 24: values of n (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 24	1	1	5	Not found up to 10^{15}					
q = 24	1	1	7	Not found up to 10^{15}					
q = 24	1	1	11	Not found up to 10^{15}					
q = 24	1	1	13	36,826,322,708	37,226,458,011	9,920	A295355		
q = 24	2	1	13	37,809,796,159	37,952,282,986	22,648	A295355	NEW!	ⓘ
q = 24	3	1	13	38,526,874,563	41,082,097,577	111,408	A295355	NEW!	ⓘ
q = 24	4	1	13	19,606,529,038,612	19,641,125,979,304	202,195	A295355	NEW!	ⓘ
q = 24	5	1	13	19,810,330,673,460	20,009,166,153,467	594,414	A295355	NEW!	Ⓥ
q = 24	6	1	13	20,763,192,869,094	21,113,714,560,133	1,441,319	A295355	NEW!	Ⓥ
Total	**6**	**1**	**13**			**2,381,904**	**A295355**		
q = 24	1	1	17	18,687,728,175,380	18,714,528,041,257	73,201	A297449	NEW!	Ⓥ
q = 24	2	1	17	21,401,790,499,965	21,531,111,289,460	773,982	A297449	NEW!	Ⓥ
q = 24	3	1	17	23,232,693,876,716	23,241,097,440,243	116,739	A297449	NEW!	ⓘ
Total	**3**	**1**	**17**			**963,922**	**A297449**		
q = 24	1	1	19	21,317,046,795,798	21,396,751,256,986	260,586	A298820	NEW!	ⓘ
q = 24	2	1	19	21,593,726,305,432	21,876,231,682,201	833,790	A298820	NEW!	Ⓥ
q = 24	3	1	19	22,037,035,819,978	22,083,466,138,743	306,557	A298820	NEW!	ⓘ
q = 24	4	1	19	22,284,455,265,595	22,779,076,769,443	1,586,533	A298820	NEW!	ⓘ
q = 24	5	1	19	22,910,331,360,479	23,049,274,819,456	449,524	A298820	NEW!	ⓘ
Total	**5**	**1**	**19**			**3,436,990**	**A298820**		
q = 24	1	1	23	Not found up to 10^{15}					

- *6 sequences registered A295356, A295355, A297450, A297449, A298821 & A298820*

For q = 24 13 new zones were discovered for $\Delta_{24,13,1}(x)$, $\Delta_{24,17,1}(x)$ and $\Delta_{24,19,1}(x)$. None were found for $\Delta_{24,5,1}(x)$, $\Delta_{24,7,1}(x)$, $\Delta_{24,11,1}(x)$ & $\Delta_{24,23,1}(x)$.

RESULTS

The most "unfair prime number races" – the largest δ(q;a,1) and status as of 2018

#	q	b	a	δ(q;a,1)	Status (2018)
1	24	1	5	0.999988	Not found up to 10^{15}
2	24	1	11	0.999983	Not found up to 10^{15}
3	12	1	11	0.999977	Not found up to 10^{15}
4	24	1	23	0.999889	Not found up to 10^{15}
5	24	1	7	0.999834	Not found up to 10^{15}
6	24	1	19	0.999719	Found up to 10^{15}
7	8	1	3	0.999569	Not found up to 10^{15}
8	12	1	5	0.999206	Found up to 10^{15}
9	24	1	17	0.999125	Found up to 10^{15}
10	3	1	2	0.999063	Known up to 10^{15}
11	8	1	7	0.998939	Found up to 10^{15}
12	24	1	13	0.998722	Known up to 10^{15}
13	12	1	7	0.998606	Found up to 10^{15}
14	8	1	5	0.997395	Known up to 10^{15}
15	4	1	3	0.995928	Known up to 10^{15}

2013   *2018*

5 NEW ZONES

1 NEW ZONE

3 NEW ZONES

1 NEW ZONE

1 NEW ZONE

5 NEW ZONES

1 NEW ZONE

5 NEW ZONES

3 NEW ZONES

25 NEW ZONES

- **Discovered 4 first ever zones ($\Delta_{12,5,1}(x)$, $\Delta_{12,7,1}(x)$, $\Delta_{24,17,1}(x)$, $\Delta_{24,19,1}(x)$) for 4 out of 15 most interesting and "unfair prime number races"**
- **In total 25 new $\Delta_{q,a,b}(x)$ sign-changing zones discovered**
- **In total 18 sequences were registered or substantially extended with OEIS**
- **Sign-changing zones for only 6 "most unfair prime number races" remain unknown**

Project implementation allowed to advance substantially in search for Δ sign-changing zones for the most interested "prime number races".

RESULTS: PUBLISHED DATA
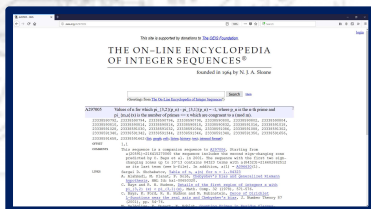
A297005: $\pi(x)\{\Delta_{3,2,1}(x)=-1\}$ **A051024:** $\pi(x)\{\Delta_{4,3,1}(x)=-1\}$ **A297447:** $\pi(x)\{\Delta_{8,5,1}(x)=-1\}$ **A295353:** $\pi(x)\{\Delta_{8,7,1}(x)=-1\}$ **A297354:** $\pi(x)\{\Delta_{12,5,1}(x)=-1\}$
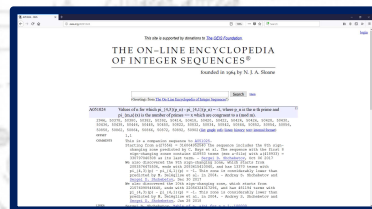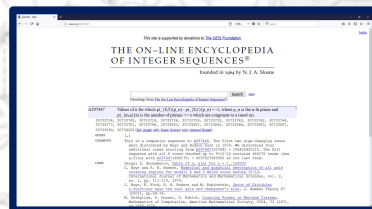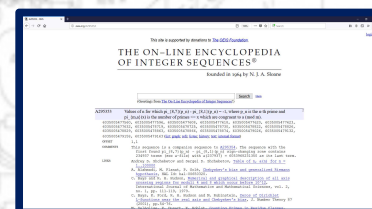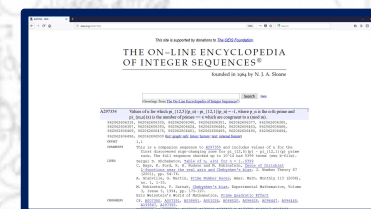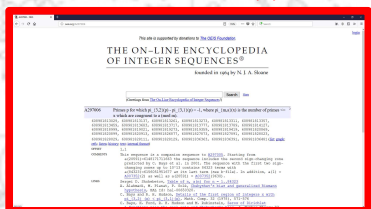
A297006: $p(x)\{\Delta_{3,2,1}(x)=-1\}$ **A051025:** $p(x)\{\Delta_{4,3,1}(x)=-1\}$ **A297448:** $p(x)\{\Delta_{8,5,1}(x)=-1\}$ **A295354:** $p(x)\{\Delta_{8,7,1}(x)=-1\}$ **A297355:** $p(x)\{\Delta_{12,5,1}(x)=-1\}$
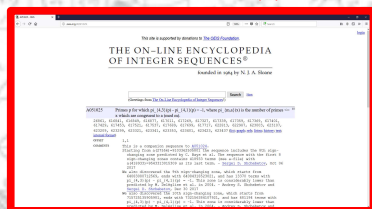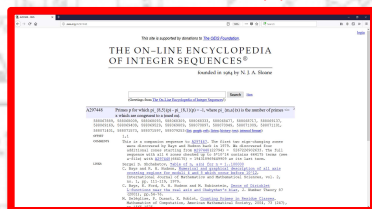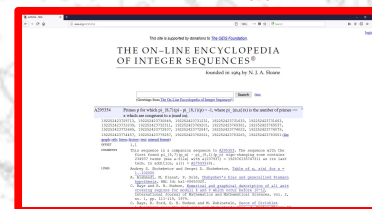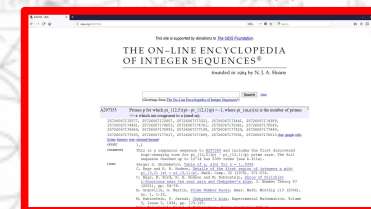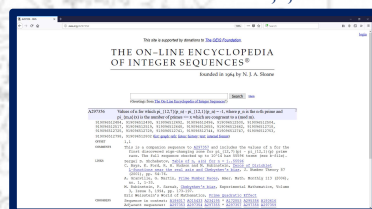
A297356: $\pi(x)\{\Delta_{12,7,1}(x)=-1\}$ **A295355:** $\pi(x)\{\Delta_{24,13,1}(x)=-1\}$ **A297449:** $\pi(x)\{\Delta_{24,17,1}(x)=-1\}$ **A298820:** $\pi(x)\{\Delta_{24,19,1}(x)=-1\}$
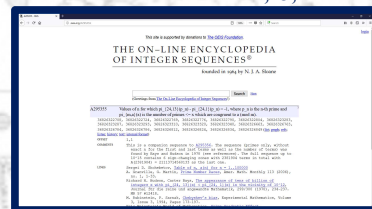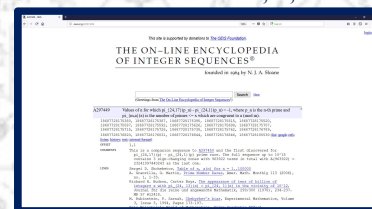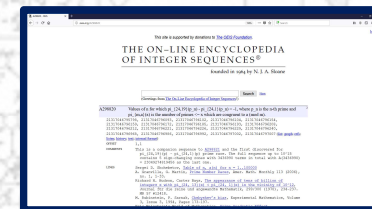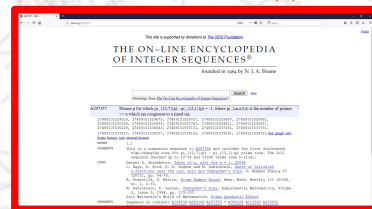
A297357: $p(x)\{\Delta_{12,7,1}(x)=-1\}$ **A295356:** $p(x)\{\Delta_{24,13,1}(x)=-1\}$ **A297450:** $p(x)\{\Delta_{24,17,1}(x)=-1\}$ **A298821:** $\pi(x)\{\Delta_{24,19,1}(x)=-1\}$
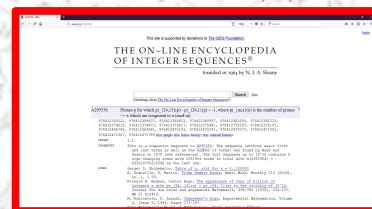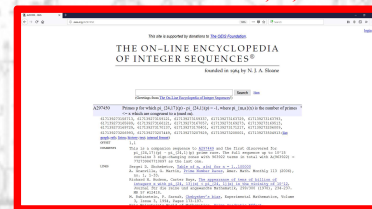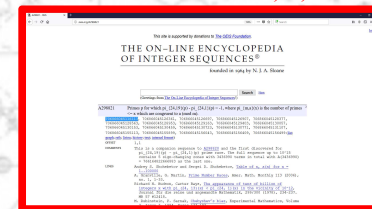
All data were published in The Online Encyclopedia of Integer Sequences (OEIS) as 18 separate sequences.

RESULTS: PUBLISHED DATA

All results are available at project repository at www.math101.guru (http://math101.guru/en/downloads-2/repository/).

RESULTS: CONCLUSIONS

- ***Chebyshev's Bias was tested up to 10^{15}*** *for selected 15 "most biased prime number races", established theoretically in 2013*
- ***First sign-changing zones were discovered*** *for 4 "most biased prime number races" out of selected 15 **(6 still remain unknown)***
- ***In total, 25 new sign-changing zones for delta were found***
- ***It was confirmed that theoretical models fail to predict small and narrow zones*** *that occur more frequently than assumed*
- *It was confirmed that **theoretical models predict big and wide zones relatively well***
- ***18 sequences were registered or substantially extended*** *with OEIS*
- ***All zones were accurately and exactly defined*** *(beginning, end, number of terms)*
- ***Full and complete data are available*** *to everybody*
- ***Created software allows to test Chebyshev's Bias up to 2^{64} $(1.8*10^{19})$***
- ***The article is under work for publication in*** *«Mathematics of Computation»*

Project implementation allowed to extend substantially our knowledge on Chebyshev's Bias and define the accuracy of theoretical models.