Notations:

- $\nu_p(N)$: the p-adic valuation of N, i.e., the highest power of p dividing N;
- $\Phi_k(n)$: the k-th cyclotomic polynomial evaluated at x=n;
- μ (): the Möbius function;
- $\operatorname{ord}_s(n)$: the multiplicative order of n modulo s.

Let p be a prime, then it is natural to ask: when does $\Phi_k(n)$ has p as a prime factor? Since $\Phi_k(n)|(n^k-1)$, there's no chance if p|n. Therefore, we can suppose $p \nmid n$.

Theorem. Let p be a prime. Suppose that $p \nmid n$, write $\lambda = \operatorname{ord}_p(n)$, then $p | \Phi_k(n)$ if and only if $k = p^e \lambda$. Moreover, if e > 0 and $k \neq 2$, then $p^2 \nmid \Phi_k(n)$.

Proof. Recall that for p not dividing d, if p > 2, then

$$\nu_p(n^{p^ed} - 1) = \begin{cases} \nu_p(n^{\lambda} - 1) + e, & \lambda | d, \\ 0, & \text{otherwise;} \end{cases}$$

if $p = 2, e \ge 1$,

$$\nu_2(n^{2^e d} - 1) = \nu_2(n - 1) + \nu_2(n + 1) + e - 1.$$

Write $k = p^e m$ with p not dividing m. Then:

$$\begin{split} \nu_p(\Phi_k(n)) &= \nu_p \Big(\prod_{d \mid k} (n^d - 1)^{\mu(\frac{k}{d})} \Big) \\ &= \sum_{d \mid m} \mu(\frac{k}{d}) \nu_p(n^d - 1) \\ &= \sum_{d \mid m} \Big(\mu(\frac{p^e m}{d}) \nu_p(n^d - 1) + \mu(\frac{p^e m}{p^d}) \nu_p(n^{p^d} - 1) + \dots \\ &\quad + \mu(\frac{p^e m}{p^{e-1}d}) \nu_p(n^{p^{e-1}d} - 1) + \mu(\frac{p^e m}{p^e d}) \nu_p(n^{p^e d} - 1) \Big) \\ &= \sum_{d \mid m} \Big(\mu(p^e) \mu(\frac{m}{d}) \nu_p(n^d - 1) + \mu(p^{e-1}) \mu(\frac{m}{d}) \nu_p(n^{p^d} - 1) + \dots \\ &\quad + \mu(p) \mu(\frac{m}{d}) \nu_p(n^{p^{e-1}d} - 1) + \mu(1) \mu(\frac{m}{d}) \nu_p(n^{p^e d} - 1) \Big) \\ &= \sum_{d \mid m, \lambda \mid d} \mu(\frac{m}{d}) \Big(\mu(p^e) \nu_p(n^d - 1) + \mu(p^{e-1}) \nu_p(n^{p^d} - 1) + \dots + \mu(p) \nu_p(n^{p^{e-1}d} - 1) + \mu(1) \nu_p(n^{p^e d} - 1) \Big) \\ &= \sum_{d \mid m, \lambda \mid d} \mu(\frac{m}{d}) \Big(\mu(p^e) \nu_p(n^{\lambda} - 1) + \mu(p^{e-1}) \nu_p(n^{p^{\lambda}} - 1) + \dots + \mu(p) \nu_p(n^{p^{e-1}\lambda} - 1) + \mu(1) \nu_p(n^{p^e\lambda} - 1) \Big) \\ &= \begin{cases} \mu(p^e) \nu_p(n^{\lambda} - 1) + \mu(p^{e-1}) \nu_p(n^{p^{\lambda}} - 1) + \dots + \mu(p) \nu_p(n^{p^{e-1}\lambda} - 1) + \mu(1) \nu_p(n^{p^e\lambda} - 1), & m = \lambda, \\ 0, & \text{otherwise.} \end{cases} \end{split}$$

It is easy to see $\nu_p(\Phi_k(n)) > 0$ if and only if $k = p^e \lambda$. In fact, if e > 0, then

$$\mu(p^e)\nu_p(n^{\lambda}-1) + \mu(p^{e-1})\nu_p(n^{p\lambda}-1) + \dots + \mu(p)\nu_p(n^{p^{e-1}\lambda}-1) + \mu(1)\nu_p(n^{p^e\lambda}-1)$$

$$=\nu_p(n^{p^e\lambda}-1)-\nu_p(n^{p^{e-1}\lambda}-1)$$

$$=\begin{cases} 1, & (p,e)\neq(2,1), \\ \nu_2(n+1), & (p,e)=(2,1), \end{cases}$$

which means that $p^2 \nmid \Phi_k(n)$ if e > 0 and $k \neq 2$.

Corollary. (i) Let p be a prime factor of $\Phi_k(n)$, then either $p \equiv 1 \pmod{k}$ or p is the largest prime factor of k.

(ii) Given k > 1, let p be the largest prime factor of k, then $gcd(k, \Phi_k(n)) = p$ if $p \nmid n$ and $k/ord_p(n)$ is a p-power, 1 otherwise.

Proof. (i) Write $\lambda = \operatorname{ord}_p(n)$. If $p \nmid k$, then $k = \lambda | (p-1) \Longrightarrow p \equiv 1 \pmod{k}$; if p | k, then $k = p^e \lambda \Longrightarrow p$ is the largest prime factor of k.

(ii)

$$\begin{split} &\gcd(k,\Phi_k(n))>1\\ \Rightarrow &\text{there exists prime } p \text{ such that } p|k,\ p|\Phi_k(n)\\ \Rightarrow &p \text{ is the largest prime factor of } k,p\nmid n \text{ and } k/\mathrm{ord}_p(n) \text{ is a } p\text{-power}\\ \Rightarrow &p|\gcd(k,\Phi_k(n))\\ \Rightarrow &\gcd(k,\Phi_k(n))=p \text{ since } p^2\nmid \Phi_k(n) \text{ unless } k=p=2, \text{ in which case } p^2\nmid k. \end{split}$$