Proof of conjecture in A215068

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Theorem. *Suppose n has the property that for all divisors d of n, $d+1$ is either a prime or a perfect power. Then n is a divisor of 48 or a Mersenne prime.*

Proof. If odd prime p divides n, $p+1$ is not a prime, so it must be a perfect power (x^k where $x \geq 2$ and $k \geq 2$). But $x^k - 1$ is divisible by $x - 1$, and therefore can't be prime unless $x = 2$, i.e. $p = 2^k - 1$ is a Mersenne prime.

Every Mersenne prime $\equiv 3 \bmod 4$. So $pq + 1$, where p and q are (not necessarily distinct) Mersenne primes, is divisible by 2, but not by 4, and can't be a prime or a perfect power. Therefore n can't be divisible by the square of a Mersenne prime or by two distinct Mersenne primes.

Since $2^5 + 1 = 33$ is neither a prime nor a perfect power, n can't be divisible by 2^5.

At this point the only possibilities are $n = 2^j$ or $2^j(2^m - 1)$ where $0 \leq j \leq 4$, and $2^m - 1$ is a Mersenne prime. Now if $2^m - 1$ is a Mersenne prime > 3, m is odd so $2 \cdot (2^m - 1) + 1 = 2^{m+1} - 1$ is divisible by 3, and can't be a prime. But since it differs from the perfect power 2^{m+1} by 1, it can't be a perfect power by Mihăilescu's theorem. So n can't be divisible by $2 \cdot (2^m - 1)$. This leaves only 2^j or $2^j \cdot 3$ for $j \leq 4$ (which are the divisors of 48) and the Mersenne primes.

$\square$