On the remarkable properties of the pentagonal numbers [*]

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1. To the class of pentagonal numbers I refer not only to those which are generally and indeed appropriately called so by custom and comprised of the form $\frac{3nn-n}{2}$, but also those which are made in this form: $\frac{3nn+n}{2}$. Thus, the general form of all these numbers will be $\frac{3nn\pm n}{2}$, from which if in place of n is written the succession of numbers 0, 1, 2, 3, 4, etc., the following sequence of paired numbers arises:

n	0,	1,	2,	3,	4,	5,	6
pentagonal	0,	1,	5,	12,	22,	35,	51
numbers	0,	2,	7,	15,	26,	40,	57

That is, any number taken by n produces two numbers, which are written as one above the other, such that the top sequence contains the pentagonal numbers properly called, and the bottom indeed those which I refer to by the very same class, and which come forth after the above sequence.

2. But if we put these numbers in a single sequence in order of their

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magnitudes, this progression will unfold:

$$0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, \text{ etc.},$$

whose order is clearly broken up, seeing that the progression of differences is

$$1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, \text{ etc.},$$

which is made from the sequence of natural numbers with others mixed in. Moreover, it is possible for this sequence to be led into a continuation, if after every third term a certain fraction is interpolated. Namely, between the terms 2 and 5 is placed $\frac{10}{3}$, then indeed $\frac{28}{3}$ between 7 and 12, similarly $\frac{55}{3}$ between 15 and 22, such that the sequence will be consummated as:

$$1, 2, \frac{10}{3}, 5, 7, \frac{28}{3}, 12, 15, \frac{55}{3}, 22, 26, \text{ etc.};$$

it follows by continuing along with this rationale that the sequence of differences will then be:

$$1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 4 \text{ etc.}$$

Moreover, it is clear that the same sequence is seen if all the triangular numbers are divided by 3. From this therefore, a beautiful property of our pentagonal numbers shows itself, seeing that every third shows itself to be a triangular number.

3. So extraordinary are the properties which can be derived from the general formula and which moreover are able to be applied to other polygonal numbers, toward which consequently I do not gaze. At this moment, it is preferable to recall a proposition of mine on certain astonishing properties which the pentagonal numbers above all the other polygonal numbers are gifted with. Here, a significant property of these numbers comes to mind which I have already shown, for there to be a very close connection between this sequence of pentagonal numbers and the progression formed by the sums of the divisors of the natural numbers, so that by means of this, in fact, a rule for this most irregular sequence can be assigned, with respect to which it will be worthwhile to quickly repeat the work.

4. If for any number N its divisors are collected into a single sum, then we will signify this sum with the character $\int N$: then from the natural numbers is born a sequence which when first looked at is most irregular:

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N	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11
$\int N$	1,	3,	4,	7,	6,	12,	8,	15,	13,	18,	12

in which the terms progress so inordinately, at one time increasing and at another time decreasing, that any rule for them will be uncovered with difficulty, seeing that the sequence of prime numbers themselves is obviously enveloped in them.

5. I have nevertheless however demonstrated that this progression, although certainly irregular, can be considered in the class of recurrent sequences, where each term immediately next is able to be determined by a certain rule from the preceeding ones. For, if $\int N$ denotes the sum of all the divisors of the number N, with itself not excepted, it will always be found to be:

$$\int N = \int (N-1) + \int (N-2) - \int (N-5) - \int (N-7)+$$
$$\int (N-12) + \int (N-15) - \int (N-22) - \int (N-26) + \text{ etc.}$$

in which the numbers which are successively subtracted from N clearly make up our sequence of pentagonal numbers

$$1, 2, 5, 7, 12, 15, 22, 26, 35, 40, \text{ etc.}$$

where the terms that come forth for n taken as odd have the sign $+$, while indeed when it is even the sign $-$ is formed. Then indeed, after each case it is required for the expressions to be continued thusly, as long as the number written after the $\int$ sign does not come out to be negative; if the expression $\int (N-N)$ occurs, in place of it should be written the number N. Thus if we select $N = 12$, it will be:

$$\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0$$

and it will therefore be:

$$\int 12 = 12 + 18 - 8 - 6 + 12 = 28.$$

To be sure, if we take up $N = 13$, it will be:

$$\int 13 = \int 12 + \int 11 - \int 8 - \int 6 + \int 1,$$

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that is, it will be:

$$\int 13 = 28 + 12 - 15 - 12 + 1 = 14.$$

6. Seeing then that the order in which the sums of divisors procede, which appears most deservedly as irregular and which no one has been able to come to a definite conclusion about, has been able to be explored with the pentagonal numbers, from which this observation is by all means to be admired highly. I will set forth moreover another property of this which is indeed closely connected to what has been related, but which leads to several no less admirable properties, which will arise equally from the nature of our pentagonal numbers which has been established.

7. The beginning of all of these remarkable properties is encountered in the expansion of this infinite product:

$$S = (1 - x)(1 - xx)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7)(\text{ etc.},$$

for I have demonstrated that if each of the factors are multiplied by the others in order, then in the end this series results:

$$S = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{ etc.},$$

where the exponents for each x form our sequence of pentagonal numbers, with the rule for the signs $+$ and $-$ that they alternate in pairs of two, so that the exponents which procede with n taken as even have the sign $+$, and indeed the the others which come from it odd the sign $-$. This then deserves our admiration no less than the properties mentioned before, with no fixed rule apparant from which any connection can be understood between the expansion of this product and our pentagonal numbers.

8. Therefore with this series of powers of x equal to the above infinite product, if we take it equal to nothing, so that we would have the equation:

$$0 = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{ etc.}$$

it will involve each of the roots which the former product forces to be equal to nothing. According to the first factor $1 - x$ it will be of course $x = 1$; from

the second factor $1 - xx$ it will be both $x = +1$ and $x = -1$; from the third factor $1 - x^3$ originate these three roots:

$$1°) \, x = 1, \qquad 2°) \, x = -\frac{1 + \sqrt{-3}}{2}, \qquad 3°) \, x = -\frac{1 - \sqrt{-3}}{2};$$

then again, from the fourth factor $1 - x^4 = 0$ emerge these four roots:

$$1°) \, x = +1, \qquad 2°) \, x = -1, \qquad 3°) \, x = +\sqrt{-1} \qquad \text{and} \qquad 4°) \, x = -\sqrt{-1};$$

next the fifth factor $1 - x^5 = 0$ yields these five roots:

$$1°) \, x = 1, \qquad 2°) \, x = \frac{-1 - \sqrt{5} + \sqrt{(-10 + 2\sqrt{5})}}{4},$$

$$3°) \, x = \frac{-1 - \sqrt{5} - \sqrt{(-10 + 2\sqrt{5})}}{4}, \qquad 4°) \, x = \frac{-1 + \sqrt{5} + \sqrt{(-10 - 2\sqrt{5})}}{4},$$

$$5°) \, x = \frac{-1 + \sqrt{5} - \sqrt{(-10 - 2\sqrt{5})}}{4};$$

in turn the sixth factor produces these six roots:

$$1°) \, x = 1, \qquad 2°) \, x = -1, \qquad 3°) \, x = \frac{+1 + \sqrt{-3}}{2},$$

$$4°) \, x = \frac{+1 - \sqrt{-3}}{2}, \qquad 5°) \, x = \frac{-1 + \sqrt{-3}}{2},$$

$$6°) \, x = \frac{-1 - \sqrt{-3}}{2} \qquad \text{etc. etc.}$$

9. Consequently then, it is clear for all the roots of unity to simultaneously be roots of our equation, and if we consider the general problem in which it is to be put $1 - x^n = 0$, it is at once apparent for $x = 1$ to always be a root, and if n is an even number for another root to be $x = -1$. For the other roots, the trionomial factors of $1 - x^n$ ought to be considered, which, making use of what has already been set forth elsewhere, are contained in the general form:

$$1 - 2x \cos \frac{2i\pi}{n} + xx,$$

taking successively for i all the integral numbers no greater than $\frac{1}{2}n$. By equating this factor with nothing, these two roots are elicited:

$$x = \cos \frac{2i\pi}{n} + \sqrt{-1} \sin \frac{2i\pi}{n} \quad \text{and}$$

$$x = \cos \frac{2i\pi}{n} - \sqrt{-1} \sin \frac{2i\pi}{n}.$$

For then it will be in turn:

$$x^n = \cos 2i\pi \pm \sqrt{-1} \sin 2i\pi.$$

Moreover, when it is $\cos 2i\pi = 1$ and $\sin 2i\pi = 0$, then $x^n = 1$, from which, if for n and i are taken all the successive integral numbers, this formula:

$$x = \cos \frac{2i\pi}{n} \pm \sqrt{-1} \sin \frac{2i\pi}{n}$$

will produce all the roots of our equation

$$0 = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}$$

so that plainly we may have the power to assign all the roots of this equation.

10. But if we indicate all the roots of this equation by the letters $\alpha, \beta, \gamma \,\delta, \epsilon$, etc., the factors will be $\frac{1-x}{\alpha}, \frac{1-x}{\beta}, \frac{1-x}{\gamma}, \frac{1-x}{\delta}$, etc., from which we deduce from the nature of equations for the sum of all these fractions $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} +$ etc. $= 1$, then in fact for the sum of the products two apiece to be equal to -1, the sum of products three apiece to indeed be equal to 0, the sum of the products four apiece to be equal to 0, the sum of the products five apiece to be equal to -1, the sum of the products six apiece to be equal to -1, etc. Then on the other hand, we conclude as before for the sum of the squares of these fractions to be

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} + \text{ etc.} = 3,$$

the sum of the cubes

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} + \text{ etc.} = 4,$$

the sum of the biquadratics

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} + \text{ etc.} = 7,$$

and thusly as before; in this certainly no order is observed.

11. Insofar as we have the fractions $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, etc., moreover furthermore α, β, γ, etc. prevail as roots themselves. For if α is a root of our equation, we can express this root in the form:

$$\cos \frac{2i\pi}{n} \pm \sqrt{-1} \sin \frac{2i\pi}{n}.$$

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Then moreover it is

$$\frac{1}{\alpha} = \frac{1}{\cos\frac{2i\pi}{n} \pm \sqrt{-1}\sin\frac{2i\pi}{n}} = \cos\frac{2i\pi}{n} \mp \sqrt{-1}\sin\frac{2i\pi}{n},$$

which will likewise be a root of our equation; from this it is apparent that if $\frac{1}{\alpha}$ is a root of our equation, then α will also be a root.

12. Therefore, were α to denote any root of the equation $1 - x^n = 0$, seeing that then it will also be a root of our equation

$$1 - x - xx + x^5 + x^7 - x^{12} - x^{15} + \text{ etc.} = 0,$$

then therefore it will be $\alpha^n = 1$. In addition, in fact all powers of α will also be roots of the equation $1 - x^n = 0$. If for in place of x we write $\alpha\alpha$, it will be $1 - x^n = 1 - \alpha^{2n}$. With it moreover $\alpha^n = 1$, it is clear then for it to be $\alpha^{2n} = 1$, and therefore $1 - \alpha^{2n} = 0$, with respect to which the same is clear for the cube α^3 and all higher powers. Then therefore it follows that it is:

$$\alpha^{n+1} = \alpha \text{ and } \alpha^{n+2} = \alpha\alpha \text{ and } \alpha^{n+3} = \alpha^3.$$

It follows that in general it will be $\alpha^{in+\lambda} = \alpha^\lambda$.

13. If then α denotes some arbitrary root of our equation, so that it is thus $\alpha^n = 1$, if in place of each x we write α, clearly this series arises:

$$1 - \alpha^1 - \alpha^2 + \alpha^5 + \alpha^7 - \alpha^{12} - \alpha^{15} + \alpha^{22} + \text{ etc.} = 0.$$

Thereafter indeed, by putting $x = \alpha\alpha$ it will be:

$$1 - \alpha^2 - \alpha^4 + \alpha^{10} + \alpha^{14} - \alpha^{24} - \alpha^{30} + \alpha^{44} + \text{ etc.} = 0,$$

and in general if in place of x we write α^i, denoting with i any integral number, it will further be:

$$1 - \alpha^i - \alpha^{2i} + \alpha^{5i} + \alpha^{7i} - \alpha^{12i} - \alpha^{15i} + \alpha^{22i} + \text{ etc.} = 0.$$

Besides this it occurs on the other hand that when negative numbers are taken for i, as we have already shown, each of the roots will be $\frac{1}{\alpha^2}, \frac{1}{\alpha^3}, \frac{1}{\alpha^4}, \frac{1}{\alpha^5}$, etc.

14. Seeing here that we have assumed α to be a root of the equation $1 - x^n = 0$, we have run through the succession of cases in which n is either

1, or 2, or 3, or 4, etc; and indeed in the first, where $n = 1$, by necessity it is $\alpha = 1$. By substituting this value into our general equation, this form is induced:

$$1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + \text{ etc.};$$

this series is evidently composed from infinitely many periods of which each contains these terms: $1 - 1 - 1 + 1$, according to which the value of each of each of the periods is equal to 0, for which reason then the infinitely many periods likewise have their sum taken as equal to 0. Moreover, it should be understood that by continuing, if now the infinitely many periods above are run through after one [sic] term is constituted, the sum will be equal to 0; if three are constituted, the sum will be will -1, and if four are constituted it will be equal to 0, in which case an entire period has been added. Therefore, since this infinite sum of numbers never terminates, the sum of this infinite series will be kept between the four sums 1, 0, -1, 0 just considered. The middle is obtained if these four numbers are aggregated into one that is then divided by four; then moreover it is apparent to come out as 0, which therefore is rightly decreed the sum of our series.

15. One may know that a reckoning similar to this can be carried out, which is usually referred to by the sum of the Leibniz series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \text{ etc.}$ that is equal to $\frac{1}{2}$; with this being assumed, the truth of the present assertion can shine forth. Seeing that it is:

$$1 - 1 + 1 - 1 + 1 - 1 + \text{ etc.} = \frac{1}{2}, \text{ it will be}$$

$$-1 + 1 - 1 + 1 - 1 + 1 - \text{ etc.} = -\frac{1}{2},$$

hence by combining these two series it will be:

$$1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 - 1 - \text{ etc.} = 0.$$

16. We now consider the case in which $n = 2$ and $\alpha\alpha = 1$, in which α is either $+1$ or -1. However, we retain the letter α for designating either of these, and because it is

$$\alpha^3 = \alpha, \alpha^4 = 1, \alpha^5 = \alpha, \alpha^6 = 1, \text{ etc.},$$

substiting what has been made into our general equation induces this form:

$$1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha + 1| + 1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha + 1 + \text{ etc.};$$

this series is advanced by a certain period, which is continuosly repeated, and every one of which consists of these eight terms:

$$1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha + 1,$$

of which the sum is 0, and however large, the complete period is certain to disappear. If indeed the first one, two, three or all the way the eight terms are began with, the sums will be obtained in the following way:

given the first terms	the sum will be
one	1
two	$1 - \alpha$
three	$-\alpha$
four	0
five	α
six	$\alpha - 1$
seven	-1
eight	0

of which eight the sum of their aggregate is 0, from which we can safely conclude for the sum of the whole of this series that we have made, continued infinitely, to be equal to 0.

17. From this it is clear for the sum of this periodic series to likewise be equal to nothing, for any value the letter α may have; truly then, consideration is led to of the periods which arise under the value of α for which $\alpha\alpha = 1$, wherefore this series is able to be separated into two parts, of which one contains only unities, and the other indeed only the letters α. It is necessary that the sum of both be equal to nothing, so that it is:

$$1 - 1 - 1 + 1, +1 - 1 - 1 + 1, +1 - 1 - 1 + 1, \text{ etc.} = 0$$
$$-\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, \text{ etc.} = 0,$$

for which the truth of both is apparent from what has been established earlier.

18. In a similar way, for the event of having the cube root of 1 by placing $\alpha^3 = 1$, and seeing that several terms of the period will extend, we will refer to the general series by two terms written one below the other, so that in general:

$$\left.\begin{array}{l} 1 - \alpha + \alpha^5 - \alpha^{12} + \alpha^{22} - \alpha^{35} \text{ etc.} \\ -\alpha^2 + \alpha^7 - \alpha^{15} + \alpha^{26} - \alpha^{40} \text{ etc.} \end{array}\right\} = 0.$$

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Seeing that if it is now supposed $\alpha^3 = 1$, such that it is

$$\alpha^4 = \alpha, \alpha^5 = \alpha^2, \alpha^6 = 1, \alpha^7 = \alpha, \text{ etc.}$$

the follows periodic progression will extend:

$$\begin{array}{l} 1 - \alpha + \alpha^2 - 1 + \alpha - \alpha^2 + 1| - \alpha + \alpha^2 - 1 + \alpha - \alpha^2 + 1| \\ -\alpha^2 + \alpha - 1 + \alpha^2 - \alpha| + 1 - \alpha^2 + \alpha - 1 + \alpha^2 - \alpha| + 1 \end{array} \quad \text{etc.}$$

equal to nothing, in which each period is constituted by twelve terms of the three general ones, namely $1, \alpha, \alpha^2$. It is noticed without difficulty for the general terms taken for this series to yield nothing, for the unities constitute this series:

$$1 - 1 - 1 + 1, +1 - 1 - 1 + 1, +1 - 1 - 1 + 1, \text{ etc.} = 0,$$

and indeed the letters α and $\alpha\alpha$ constitute the following series:

$$-\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, \text{ etc.} = 0$$
$$-\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, \text{ etc.} = 0.$$

It is clear for the sums of both of these to be equal to nothing.

19. We will further now consider the biquadratic roots of unity, for which it is $\alpha^4 - 1$, and the following periodic series is produced:

$$\begin{array}{l} 1 - \alpha + \alpha - 1 + \alpha^2 - \alpha^3 + \alpha^3 - \alpha^2 + 1| - \alpha + \alpha - 1 + \text{ etc.} \\ -\alpha^2 + \alpha^3 - \alpha^3 + \alpha^2 - 1 + \alpha - \alpha| + 1 - \alpha^2 + \alpha^3 - \alpha^3 + \text{ etc.} \end{array}$$

for which each period is constituted from 16 terms. The general four which have been related are shown in the following four series each to be equal to nothing:

$$1 - 1 - 1 + 1, +1 - 1 - 1 + 1, +1 - 1 - 1 + 1, \text{ etc.} = 0$$
$$-\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, \text{ etc.} = 0$$
$$-\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, \text{ etc.} = 0$$
$$+\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, +\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, +\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, \text{ etc.} = 0.$$

20. Although now our conclusion about the higher roots has been well confirmed, it is however necessary to investigate the additional case of $\alpha^5 = 1$,

seeing that here not all the powers less than five occur. If then it is $\alpha^5 = 1$, this periodic series will advance:

$$\begin{array}{c} 1 - \alpha + 1 - \alpha^2 + \alpha^2 - 1 + \alpha - 1 + \alpha^2 - \alpha^2 + 1| - \alpha + 1 \\ -\alpha^2 + \alpha^2 - 1 + \alpha - 1 + \alpha^2 - \alpha^2 + 1 - \alpha| + 1 - \alpha^2 + \alpha^2 \end{array} \quad \text{etc.}$$

in which the powers α^3 and α^4 are thoroughly excluded. Since each of the periods is constituted by exactly 20 terms, it is necessary for the other powers to occur frequently; the separate three that are taken occur in the following periodic series:

$$1 + 1 - 1 - 1 - 1 - 1 + 1 + 1, 1 + 1 - 1 - 1 - 1 - 1 + 1 + 1, \text{ etc.} = 0$$
$$-\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, \text{ etc.} = 0$$
$$-\alpha^2 + \alpha^2 - \alpha^2 + \alpha^2 - \alpha^2 + \alpha^2 - \alpha^2 + \alpha^2, -\alpha^2 + \alpha^2 - \alpha^2 + \alpha^2, \text{ etc.} = 0.$$

By now the truth of the series of α themselves is clear from the preceding; for the remaining two, of which the period consists of eight terms, if the following which has been established earlier is examined, even now they will be discovered to be equal to nothing. For, not only do the terms alone in the periods eliminate each other, but also the terms of the basis from which the series is formed. Then from the series of unities is derived this base series:

$$1, 2, 1, 0, -1, -2, -1, 0,$$

of which the sum again vanishes; by the same experience the series of squares is disposed of.

21. From this it is now abundantly clear that these same properties will keep their place for higher roots, however many terms each period is composed of. This is certainly most amazing, with these properties able to occur in no other series of powers, and is a deep property of the series of pentagonal numbers.

22. So that moreover we can put eyes on the problem in general, it will be $\alpha^n = 1$, from which are born periods consisting of $4n$ terms, which will be either 1, or α, or α^2, or α^3, etc. While for the most part not all the power less than α^n will occur, the periods of each power of α are typically constituted from more than four terms. However, not only do the terms themselves of the period always destroy each other but also terms of the basis. Therefore if we

consider the powers α^r, with r arising as a number less than n, according to all the terms taken from our sequence of pentagonal numbers which divided by n leave behind the residue r. If these terms are given with the sign put in front, such a series comes forth:

$$\pm\alpha^r \pm \alpha^r \pm \alpha^r \pm \alpha^r \pm \alpha^r \pm \alpha^r \pm \alpha^r \pm \alpha^r \pm \text{ etc.,}$$

which always consists of a certain period with a rule for the signs $+$ and $-$, so that of each period all the terms that are taken at once destroy each other, and moreover the same for the series that occurs of the basis.

23. Truly, along with these properties that have been related so far there are countless others no less extraordinary that are drawn forth after these. For if α is a root of an power n of unity, so that $1 - \frac{x}{\alpha}$ is a factor of the form $1 - x^n$, it is evident for it to be a factor of the forms $1 - x^{2n}, 1 - x^{3n}, 1 - x^{4n}$, etc. into infinity. From this, because these forms are all factors of our progression:

$$1 - x - xx + x^5 + x^7 - x^{12} - \text{ etc.,}$$

each root α occurs not only once but indeed infinitely in this equation, so that this equation has infinitely many roots equal to α.

24. We know on the other hand from the nature of equations, that if some equation

$$1 + Ax + Bxx + Cx^3 + Dx^4 + \text{ etc.} = 0$$

has two root equal to α then in fact α is a root of the equation made by differentiating, namely:

$$A + 2Bx + 3Cxx + 4Dx^3 + \text{ etc.} = 0,$$

and if it has three roots equal to α then in addition α will likewise be a root of the equation that is made by differentiating, after we have multiplied this differentiated equation by x:

$$1^2 \cdot A + 2^2 \cdot Bx + 3^2 \cdot Cxx + 4^2 \cdot Dx^3 + \text{ etc.} = 0,$$

from which if these equation has λ equal roots, each of which are equal to α, then it will always be:

$$1^\lambda \cdot A + 2^\lambda \cdot B\alpha + 3^\lambda \cdot C\alpha\alpha + 4^\lambda \cdot D\alpha^3 + \text{ etc.} = 0,$$

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from which if we multiply uniformly this equation by α, it will then be:

$$1^\lambda \cdot A\alpha + 2^\lambda \cdot B\alpha^2 + 3^\lambda \cdot C\alpha^3 + 4^\lambda \cdot D\alpha^4 + \text{ etc.} = 0.$$

25. Therefore with it having been put $\alpha^n = 1$, our equation formed from the pentagonal numbers:

$$1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{ etc.} = 0$$

has infinitely many roots equal to α, and it will actually be that α is a root of all equations contained in this general form:

$$-1^\lambda x - 2^\lambda x^2 + 5^\lambda x^5 + 7^\lambda x^7 - 12^\lambda x^{12} - \text{ etc.} = 0,$$

where λ is taken as any integral number. Therefore it will always be:

$$-1^\lambda \alpha - 2^\lambda \alpha^2 + 5^\lambda \alpha^5 + 7^\lambda \alpha^7 - 12^\lambda \alpha^{12} - \text{ etc.} = 0.$$

26. In order to clearly illustrate this we take $\alpha = 1$, and it will always be:

$$-1^\lambda - 2^\lambda + 5^\lambda + 7^\lambda - 12^\lambda - 15^\lambda + \text{ etc.} = 0,$$

and for the case of $\lambda = 0$, we have already examined the truth of this equation. Therefore if it were $\lambda = 1$, it is to be shown for the sum of this sucession which diverges to infinity to be equal to 0:

$$-1 - 2 + 5 + 7 - 12 - 15 + 22 + 26 - \text{ etc.}$$

Because this series is interrupted, that is, it has been mixed from two series, we may consider it separated into two parts by putting:

$$s = -1 + 5 - 12 + 22 - 35 + \text{ etc., and}$$
$$t = -2 + 7 - 15 + 26 - 40 + \text{ etc.}$$

and it ought to be shown to be $s + t = 0$.

27. From the doctrine of series which proceed with their signs alternating, just as $A - B + C - D + $ etc. continues, the sum of this series progressing to infinity will be equal to:

$$\frac{1}{2}A - \frac{1}{4}(B - A) + \frac{1}{8}(C - 2B + A) - \frac{1}{16}(D - 3C + 3B - A) \text{ etc.},$$

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This useful rule is set forth by differences, namely by a rule based on the signs. From the series of numbers A, B, C, D, E, etc. is formed the series of differences, so that each term of this series is subtracted from the following one: it will be a, b, c, d, etc. Again, in the very same, from this series of differences is formed a series of second differences, which will be a', b', c', d', etc., from which likewise is the series of third differences, which will be a'', b'', c'', d'', e'', etc., and in this way further differences are reached for until they are unchanging. Then moreover from the first terms of all of these series the sum of the proposed series may thus be determined, such that it will be:

$$\frac{1}{2}A - \frac{1}{4}a + \frac{1}{8}a' - \frac{1}{16}a'' + \frac{1}{32}a''' - \frac{1}{64}a'''' + \text{ etc.}$$

28. By this rule it is established, with the signs changed it will be:

$$-s = 1 - 5 + 12 - 22 + 35 - 51 + 70 - \text{ etc., and}$$
$$-t = 2 - 7 + 15 - 26 + 40 - 57 + 77 - \text{ etc.,}$$

whose terms are arranged in the following way with their differences written underneath them:

1,5,12,22,35,51,70 etc.	2,7,15,26,40,57,77 etc.
4,7,10,13,16,19	5,8,11,14,17,20
3,3,3,3,3	3,3,3,3,3
0,0,0,0	0,0,0,0

Therefore from this it is gathered to be:

$$-s = \frac{1}{2} - \frac{4}{4} + \frac{3}{8} = -\frac{1}{8}, \text{ that is, } s = \frac{1}{8}, \text{ and moreover,}$$
$$-t = \frac{2}{2} - \frac{5}{4} + \frac{3}{8} = \frac{1}{8}, \text{ that is, } t = -\frac{1}{8},$$

from which it is clearly combined to be $s + t = 0$.

29. Although the rules which these properties are supported by clearly leave no doubt, it will be by no means useless to exhibit further the truth for the case $\lambda = 2$, for it to be:

$$-1^2 - 2^2 + 5^2 + 7^2 - 12^2 - 15^2 + 22^2 + \text{ etc.} = 0.$$

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In the same way as before this series is divided into two, which with their signs changed are:

$$s = +1^2 - 5^2 + 12^2 - 22^2 + 35^2 - 51^2 + \text{ etc.}$$
$$t = 2^2 - 7^2 + 15^2 - 26^2 + 40^2 - 57^2 + \text{ etc.,}$$

and in order to find the sum of the first, the following procedure is carried out:

Series	1, 25, 144, 484, 1225, 2601, 4900
Diff. I	24, 119, 340, 741, 1376, 2299
Diff. II	95, 221, 401, 635, 923
Diff. III	126, 180, 234, 288
Diff. IV	54, 54, 54
Diff. V	0, 0

Hence it will then be:

$$s = \frac{1}{2} - \frac{24}{4} + \frac{95}{8} - \frac{126}{16} + \frac{54}{32} = +\frac{3}{16}.$$

In the same way for the other series:

Series	4, 49, 225, 676, 1600, 3249, 5929
Diff. I	45, 176, 451, 924, 1649, 2680
Diff. II	131, 275, 473, 725, 1031
Diff. III	144, 198, 252, 306
Diff. IV	54, 54, 54
Diff. V	0, 0

Then it is concluded:

$$t = \frac{4}{2} - \frac{45}{4} + \frac{131}{8} - \frac{144}{16} + \frac{54}{32} = -\frac{3}{16}.$$

On account of this, it prevails for the total sum to be $s + t = 0$.

30. Furthermore, we now consider the square root, i.e. when it is $\alpha^2 = 1$, and hence this series arises:

$$-1^\lambda \cdot \alpha - 2^\lambda + 5^\lambda \cdot \alpha + 7^\lambda \alpha - 12^\lambda - 15^\lambda \cdot \alpha + 22^\lambda + 26^\lambda - \text{ etc.} = 0,$$

from which if we separate the terms comprised of unity and α from each other, we will obtain two series both equal to nothing, namely:

$$-2^\lambda - 12^\lambda + 22^\lambda + 26^\lambda - 40^\lambda - 70^\lambda + 92^\lambda + \text{ etc.} = 0$$

and

$$-1^\lambda \cdot \alpha + 5^\lambda \cdot \alpha + 7^\lambda \cdot \alpha - 15^\lambda \cdot \alpha - 35^\lambda \cdot \alpha + 51^\lambda \cdot \alpha + 57^\lambda \cdot \alpha - \text{ etc.} = 0.$$

Insofar as we may want to display the truth of these series in the same manner which we have made use of before, each one ought to be broken into four others, until in the end we reach unchanging differences. Truly, if this work were to have been taken up, it will be certain for the aggregate of the sums of all the parts to be equal to 0.

31. Now most generally the whole matter is embraced, where it will be $\alpha^n = 1$, and we search for a series which contains all powers α^r. We select to this end from all our pentagonal numbers which when divided by n leave a residue the same as r. Then if the pentagonal numbers are A, B, C, D, E, etc., all of course of the form $\gamma n + r$ and of which the sign $\pm$, which is appropriate for them, is observed carefully. Then indeed it will always be:

$$\pm A^\lambda \pm B^\lambda \pm C^\lambda \pm D^\lambda \pm \text{ etc.} = 0,$$

for which the exponent λ may take any integral value. Then in this most general form, all series which we have elicited so far, and of which we have shown for the sums to be equal to nothing, are contained.