Date :	Oct 11 2024	**Board** :	CBSE
Class :	12	**Session # :**	6
Subject :	Mathematics	**Assignment # :**	VaA3
Topic :	Vector Algebra	**Subtopic(s) :**	Section Formula, Scalar (Dot) Product, Projection of a vector on a line, Projection of a vector on another vector,Cauchy-Schwarz Inequality, Triangle Inequality
Lecture #:	3		

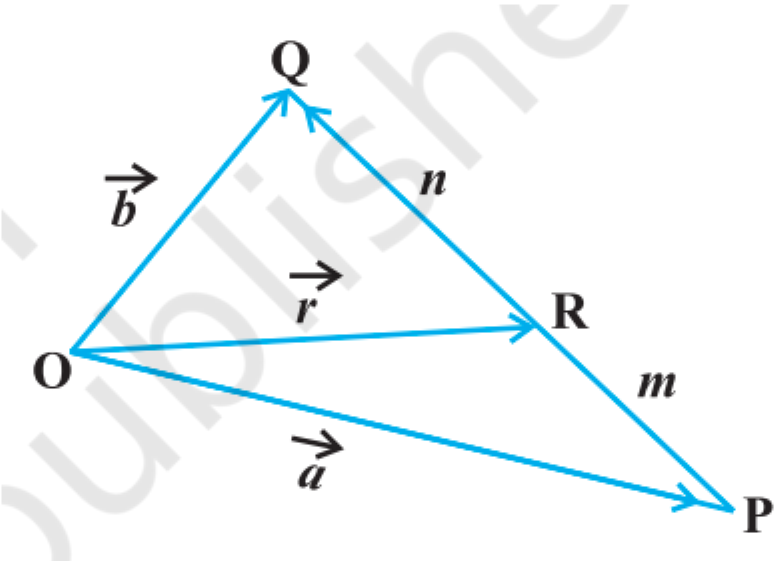
Section Formula:

The section formula can be applied for both internal and external division of a line segment. Here's how they differ:

Internal Division

When a point $\mathbf{R}$ divides the line segment $\mathbf{PQ}$ internally in the ratio $m : n$, it lies between points $\mathbf{P}$ and $\mathbf{Q}$. The formula for the position vector $\mathbf{r}$ of point $\mathbf{R}$ is:

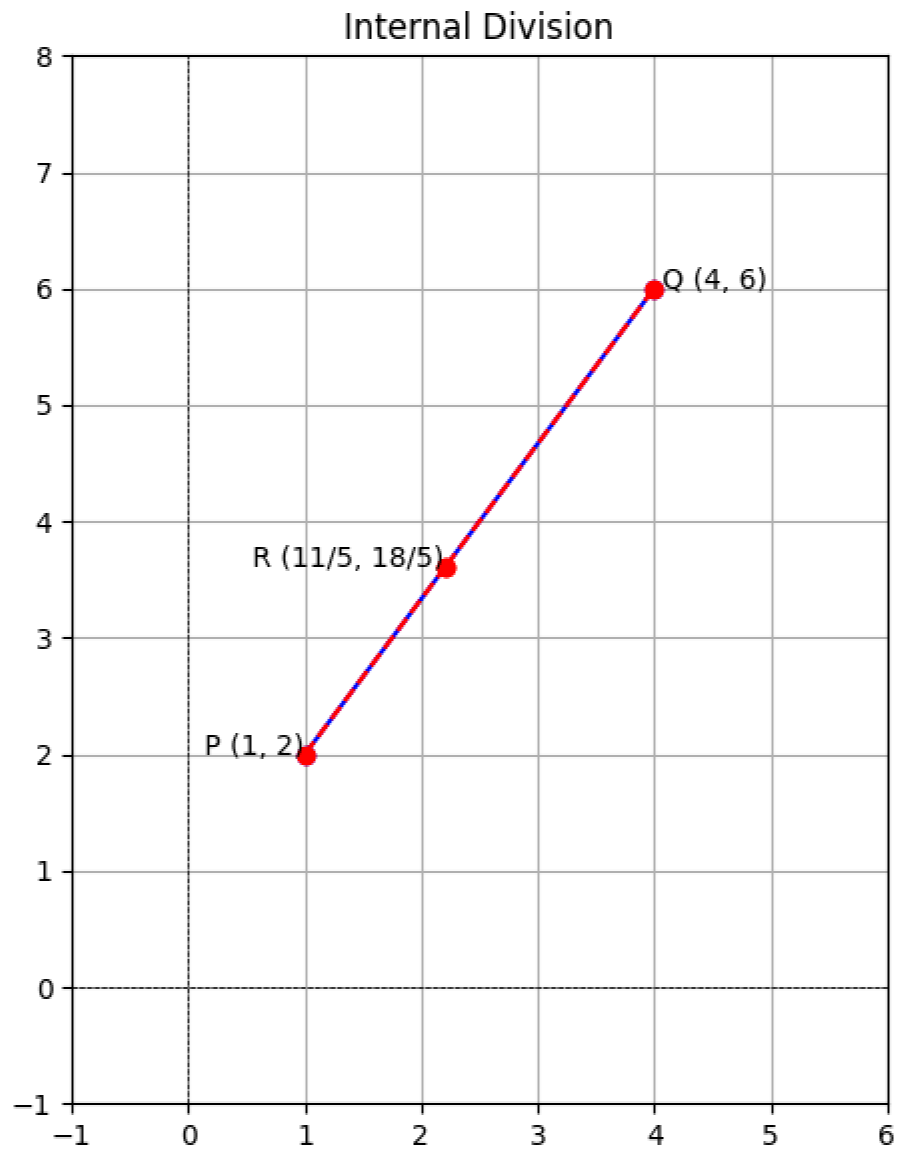
$$\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}$$



Example (Internal Division):

Let $A(1,2)$ and $B(4,6)$ be two points. To find the point that divides AB in the ratio $2:3$:

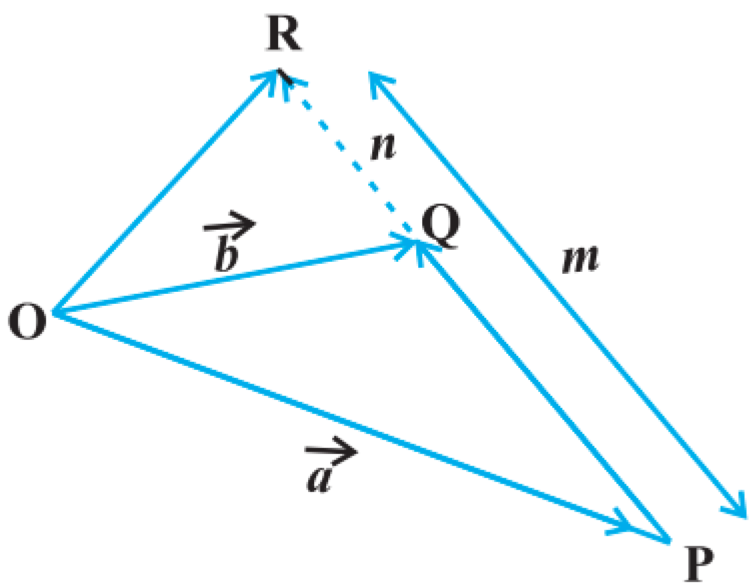
$$\mathbf{p} = \frac{3\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}4\\6\end{pmatrix}}{2+3} = \frac{\begin{pmatrix}3\\6\end{pmatrix} + \begin{pmatrix}8\\12\end{pmatrix}}{5} = \begin{pmatrix}\frac{11}{5}\\\frac{18}{5}\end{pmatrix}$$



External Division

When a point $\mathbf{R}$ divides the line segment $\mathbf{PQ}$ externally in the ratio $m : n$, it lies outside the segment. The formula for the position vector $\mathbf{r}$ of point $\mathbf{R}$ is:

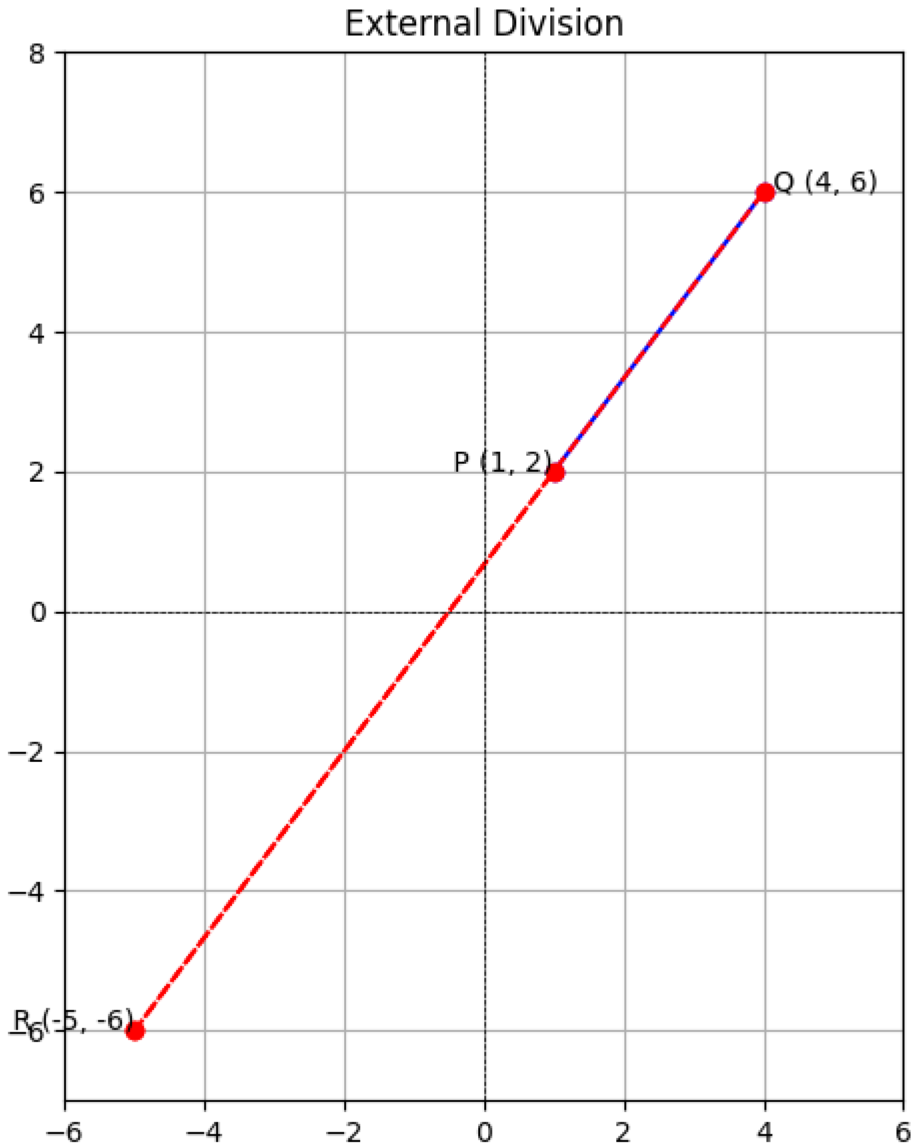
$$\mathbf{r} = \frac{n\mathbf{a} - m\mathbf{b}}{n - m}$$



Example of External Division:

Using the same points $\mathbf{P}(1, 2)$ and $\mathbf{Q}(4, 6)$, if $\mathbf{R}$ divides $\mathbf{PQ}$ externally in the ratio $2 : 3$:

$$\mathbf{r} = \frac{3\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}4\\6\end{pmatrix}}{3-2} = \frac{\begin{pmatrix}3\\6\end{pmatrix} - \begin{pmatrix}8\\12\end{pmatrix}}{1} = \begin{pmatrix}3-8\\6-12\end{pmatrix} = \begin{pmatrix}-5\\-6\end{pmatrix}$$



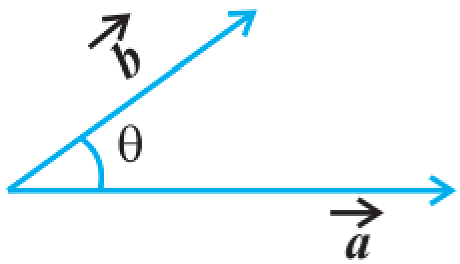
Scalar (Dot) Product:

The scalar (dot) product is a fundamental operation in vector mathematics. It combines two vectors to produce a scalar (a single number) and has several important applications, including calculating the angle between vectors and determining projections.

Definition of the Dot Product

For two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, the dot product is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$



Geometric Interpretation

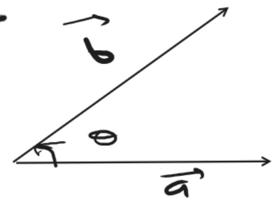
The dot product can also be expressed in terms of the magnitudes of the vectors and the cosine of the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Where:

- $|\mathbf{a}|$ is the magnitude of vector $\mathbf{a}$.
- $|\mathbf{b}|$ is the magnitude of vector $\mathbf{b}$.
- θ is the angle between the two vectors.

Dot Product



θ = angle

$\cos\theta$ = fraction

$-1 \le \cos\theta \le 1$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta .$$

Properties of the Dot Product

1. Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
3. Associative with Scalars: $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$

Example

Let's take two vectors:

- $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
- $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Calculating the Dot Product:

$$\mathbf{a} \cdot \mathbf{b} = 2 \cdot 4 + 3 \cdot 1 = 8 + 3 = 11$$

Magnitudes of Vectors:

- $|\mathbf{a}| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$
- $|\mathbf{b}| = \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}$

Angle Calculation:

Using the dot product formula with angle:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

$$11 = \sqrt{13} \cdot \sqrt{17} \cos(\theta)$$

From this, you can solve for $\cos(\theta)$ and subsequently find the angle θ.

Understanding using examples:

$\theta = 0^\circ$

$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}|$

$\vec{a} \cdot \vec{b} = 0$

(i) $\theta = 90^\circ$, $\left(\frac{3\pi}{2}\right)$ or (270°)

(ii) either $\vec{a}$ or $\vec{b}$ or both are zero vectors

$$\underline{\hat{i}} \cdot \underline{\hat{i}} = 0$$

$$\hat{i} \cdot \hat{i} = |\hat{i}| \cdot |\hat{i}| \cdot \cos(\theta)$$

$$= \underline{1} \cdot \underline{1} \cdot \underline{1}$$

$$= \boxed{1}$$

$$\hat{i} \cdot \hat{j} = |\hat{i}| \cdot |\hat{j}| \cdot \cos(90^\circ)$$

$$0$$

$$\lambda = 5$$

$$\vec{a} = 2\hat{i} - 5\hat{j}$$

$$\vec{b} = \hat{i} + 2\hat{j}$$

$$\lambda \vec{a} = 5(2\hat{i} - 5\hat{j})$$

$$10\hat{i} - 25\hat{j}$$

$$\lambda \vec{b} = 5\hat{i} + 10\hat{j}$$

(i) $(\lambda \vec{a}) \cdot \vec{b}$

$$(10\hat{i} - 25\hat{j}) \cdot (\hat{i} + 2\hat{j})$$

$$10\hat{i} \cdot (\hat{i} + 2\hat{j}) - 25\hat{j} \cdot (\hat{i} + 2\hat{j})$$

$$\underline{10\hat{i} \cdot \hat{i}} + 10\underline{\hat{i}} \cdot 2\underline{\hat{j}} - 25\underline{\hat{j}} \cdot \underline{\hat{i}} - 25\hat{j} \cdot 2\hat{j}$$

$$10 \times 1 + 20 \times 0 - 25 \times 0 - 50 \times 1$$

$$10 + 0 - 0 - 50$$

$$\boxed{-40}$$

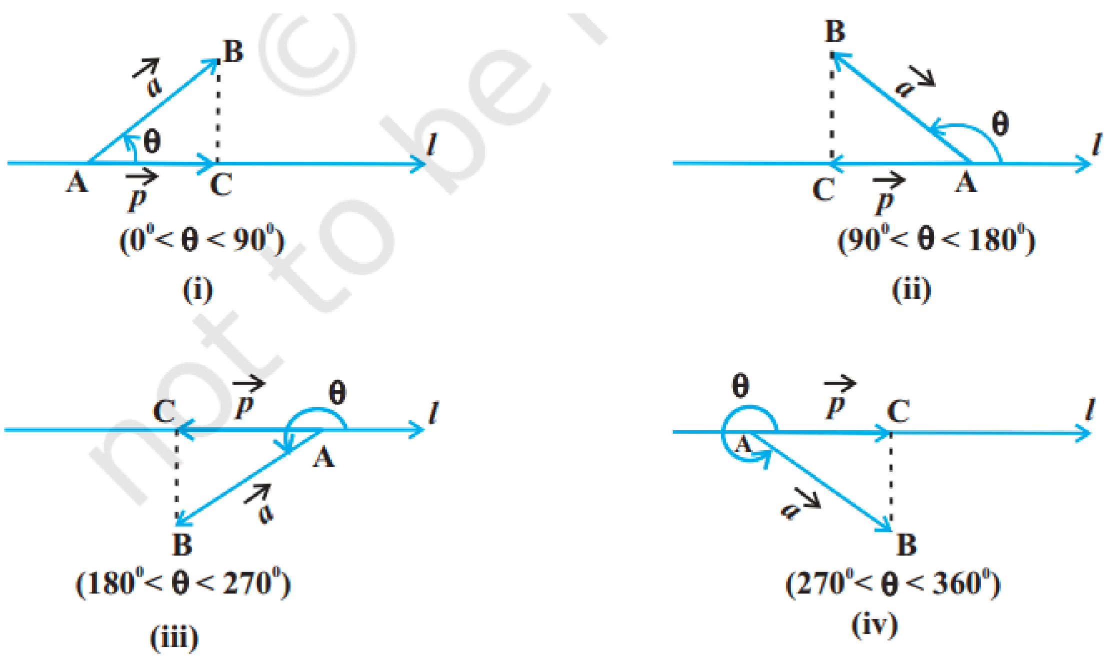
Projection of a vector on a line:

The projection of a vector onto a line is a geometric concept that describes how much of the vector lies in the direction of that line. This can be generalized to any vector and any line represented in a vector space.

General Definition of Projection

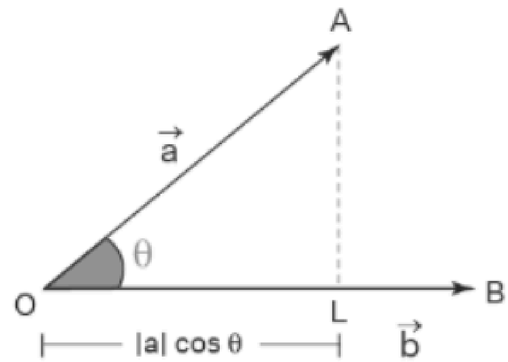
1. **Vector Representation**:
   - Let $\mathbf{a}$ be the vector you want to project.
   - Let $\mathbf{p}$ be the direction vector of the line onto which $\mathbf{a}$ is being projected.
2. **Projection Formula**: The projection of vector $\mathbf{a}$ onto the line defined by direction vector $\mathbf{p}$ is given by:

$$\mathrm{proj}_{\mathbf{p}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{p}}{|\mathbf{p}|^2}\mathbf{p}$$



Projection of Vector a on Vector b = $\dfrac{a.b}{|b|}$

Projection of Vector a on Vector b



Observations

1. If $\hat{p}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a}\cdot\hat{p}$.

2. Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}\cdot\hat{b}, \quad \text{or} \quad \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}\,|}\right), \quad \text{or} \quad \frac{1}{|\vec{b}\,|}(\vec{a}\cdot\vec{b})$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

4. If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

The Cauchy-Schwarz inequality is a fundamental result in linear algebra and analysis that provides an important relationship between vectors. It states that the dot product of two vectors is bounded by the product of their magnitudes.

Cauchy-Schwarz Inequality

For any vectors $\mathbf{a}$ and $\mathbf{b}$ in an inner product space, the Cauchy-Schwarz inequality can be expressed as:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

The triangle inequality is a fundamental concept in mathematics, particularly in geometry and vector spaces. It states that, for any two points (or vectors), the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides.

Triangle Inequality

For any vectors $\mathbf{a}$ and $\mathbf{b}$ in a vector space, the triangle inequality states:

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

Now complete this assignment #VaA3:

#VaA3:

Example 11 Consider two points P and Q with position vectors $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2:1, (i) internally, and (ii) externally.

15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

(i) internally (ii) externally

16. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

Example 13 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\vec{a} \cdot \vec{b} = 1$.

Example 14 Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Example 15 If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

Example 16 Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Example 17 Find $|\vec{a} - \vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$.

Example 18 If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, then find $|\vec{x}|$.

Example 19 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy-Schwartz inequality).

Example 20 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (triangle inequality).

Example 21 Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear.

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$.

2. Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

4. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

5. Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}),\ \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}),\ \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

7. Evaluate the product $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

10. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

11. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

12. If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

13. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

14. If either vector $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \cdot \vec{b} = 0$. But the converse need not be true. Justify your answer with an example.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overline{\text{BA}}$ and $\overline{\text{BC}}$].

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = 1/|\lambda|$

End