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| **Date** : | Oct 14 2024 | **Board** : | CBSE |
| **Class** : | 12 | **Session # :** | 7 |
| **Subject** : | Mathematics | **Assignment # :** | VaA4 |
| **Topic** : | Vector Algebra | **Subtopic(s) :** | Vector (or cross) product of two vectors, Properties (Right-hand rule, Distributive Law, Multiplication by Scalar) |
| **Lecture #:** | 4 | | |

**Vector (or cross) product of two vectors:**

The **vector product** (or **cross product**) of two vectors is a mathematical operation that results in a third vector that is perpendicular to both of the original vectors. This operation is defined only in three-dimensional space.

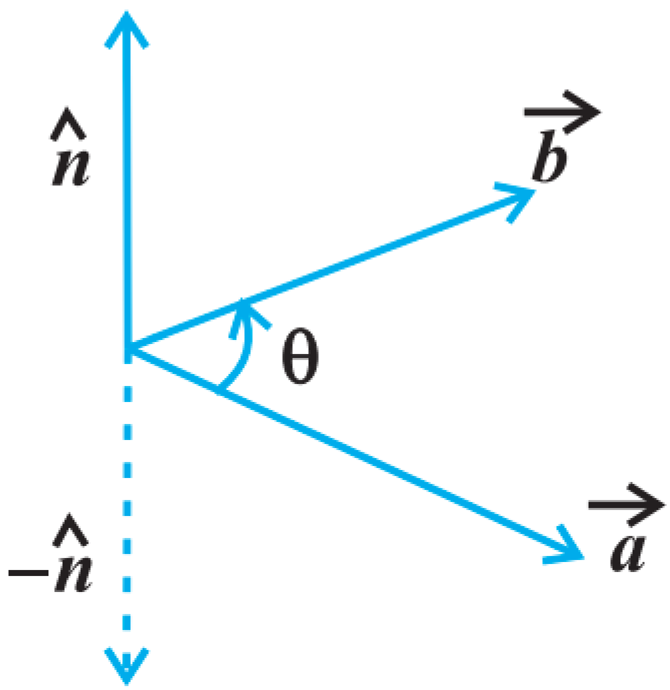
## Formula:

If you have two vectors **A** and **B** in 3D space:

- **A** = $\langle A_x, A_y, A_z \rangle$
- **B** = $\langle B_x, B_y, B_z \rangle$

The cross product **A × B** is given by:

$$A \times B = \langle A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x \rangle$$




$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

In determinant form, the cross product can be represented as:

$$A \times B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

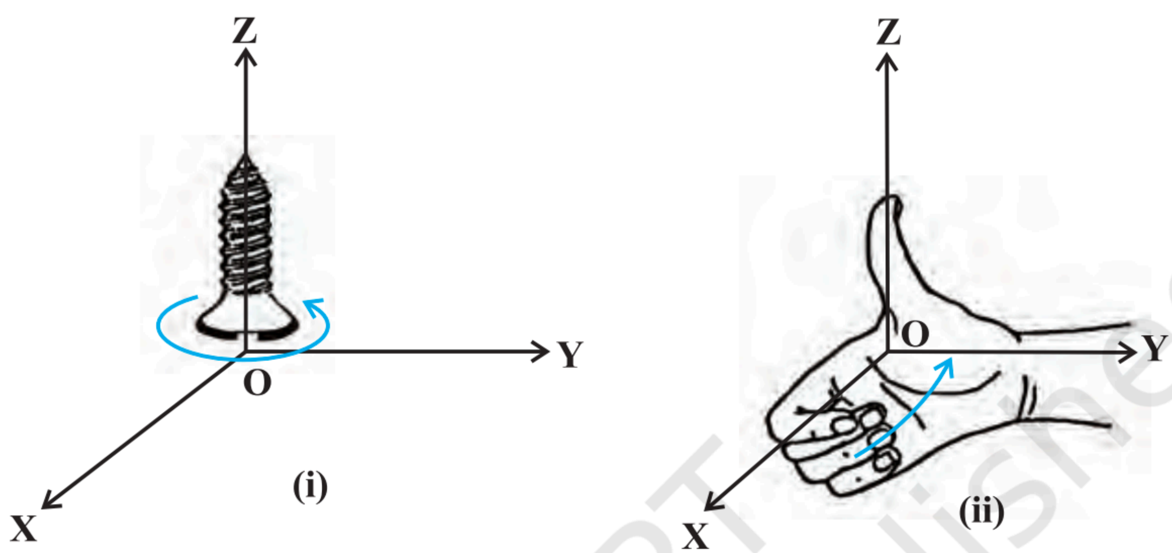
Where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the x, y, and z directions, respectively.

## Properties:

1. **Perpendicular Vector**: The resulting vector is perpendicular to both **A** and **B**.
2. **Magnitude**: The magnitude of the cross product is:

$$|A \times B| = |A||B| \sin \theta$$

Where $\theta$ is the angle between the vectors **A** and **B**.



3. **Right-Hand Rule**: The direction of the resulting vector follows the right-hand rule. If you curl the fingers of your right hand from vector **A** to vector **B**, your thumb points in the direction of the cross product.
4. **Non-Commutativity**: The cross product is not commutative:

$$A \times B \neq B \times A$$

In fact, $A \times B = -(B \times A)$.

5. **Zero Vector**: If the vectors are parallel or anti-parallel (i.e., $\theta = 0^\circ$ or $180^\circ$), the cross product will be a zero vector.

**6. Distributive law:**

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

**Note:** Don't change the order of vectors in vector products.

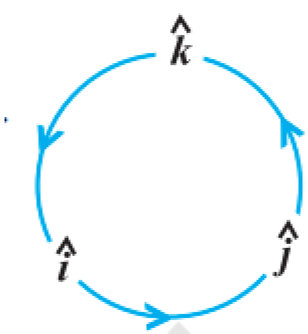
**Recall that,**

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

**and**

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}.$$




**7) Multiplication by Scalar:**

$$\lambda(\mathbf{A} \times \mathbf{B}) = (\lambda\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (\lambda\mathbf{B})$$

## Example:

Let $\mathbf{A} = \langle 1, 2, 3 \rangle$, $\mathbf{B} = \langle 4, 5, 6 \rangle$, and $\lambda = 3$.

1. First, compute the cross product $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \langle 2 \times 6 - 3 \times 5, 3 \times 4 - 1 \times 6, 1 \times 5 - 2 \times 4 \rangle$$

$$\mathbf{A} \times \mathbf{B} = \langle 12 - 15, 12 - 6, 5 - 8 \rangle = \langle -3, 6, -3 \rangle$$

2. Now multiply the result by the scalar $\lambda$:

$$\lambda(\mathbf{A} \times \mathbf{B}) = 3 \times \langle -3, 6, -3 \rangle = \langle -9, 18, -9 \rangle$$

Alternatively, you could first multiply one of the vectors by the scalar $\lambda$ and then compute the cross product, and you would obtain the same result:

$$(3\mathbf{A}) \times \mathbf{B} = \langle 3 \times 1, 3 \times 2, 3 \times 3 \rangle \times \mathbf{B} = \langle 3, 6, 9 \rangle \times \langle 4, 5, 6 \rangle$$

$$= \langle 6 \times 6 - 9 \times 5, 9 \times 4 - 3 \times 6, 3 \times 5 - 6 \times 4 \rangle = \langle 36 - 45, 36 - 18, 15 - 24 \rangle = \langle -9, 18, -9 \rangle$$

In both cases, you get the same final result.

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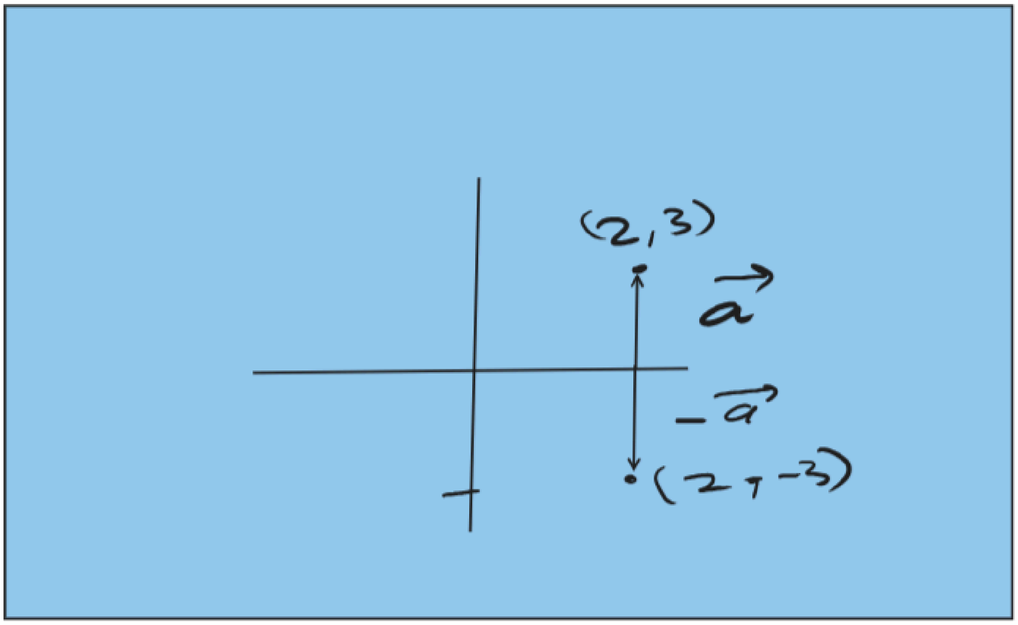
**Vector Algebra - Lecture board 4:**




$$\hat{i} \times \hat{i} = 0$$

$$\theta = 0$$

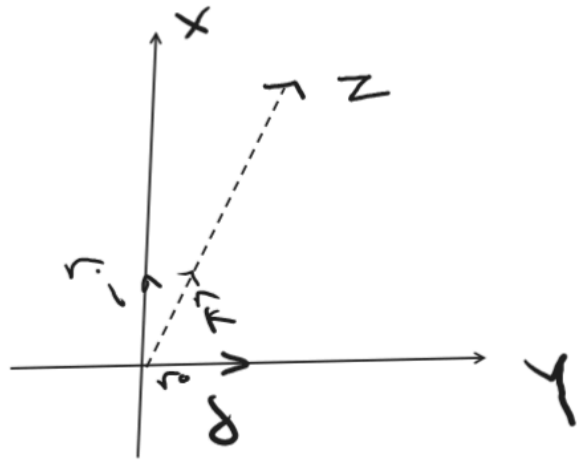
$$\hat{j} \times \hat{j} = 0$$

$$\hat{k} \times \hat{k} = 0$$

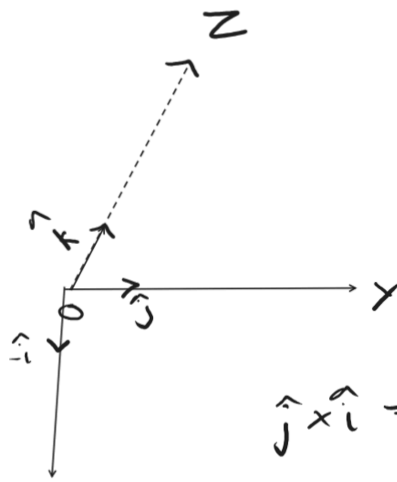
$$\hat{i} \times \hat{j} = \hat{k}$$

$\hat{k} \times \hat{j} = i$

$\hat{j} \times i = -\hat{k}$




$\hat{k} \times \hat{j} = -\hat{i}$




$\hat{j} \times \hat{i} = -\hat{k}$

$$\vec{a} = a_1\hat{i} + b_1\hat{j}$$

$$\vec{b} = a_2\hat{i} + b_2\hat{j}$$

$$\vec{a}\times\vec{b} = (a_1\hat{i} + b_1\hat{j})\times(a_2\hat{i} + b_2\hat{j})$$

$$a_1\hat{i}\times(a_2\hat{i} + b_2\hat{j}) + b_1\hat{j}\times(a_2\hat{i} + b_2\hat{j})$$

$$(a_1\hat{i}\times a_2\hat{i}) + (a_1\hat{i}\times b_2\hat{j}) + (b_1\hat{j}\times a_2\hat{i}) + (b_1\hat{j}\times b_2\hat{j})$$

$$\Rightarrow 0 + (a_1)(b_2)\hat{k} + (b_1)(a_2)(-\hat{k}) + 0$$

$$a_1 b_2\hat{k} - b_1 a_2\hat{k}$$

$$(a_1 b_2 - b_1 a_2)\hat{k}$$

$|x|$

Square matrix $2\times 2$, $3\times 3$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\underline{a_1 b_2} - \underline{b_1 a_2}$$

$$\begin{vmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{5} \\ \overset{-}{1} & \overset{+}{1} & \overset{-}{2} \\ \overset{+}{1} & \overset{-}{-1} & \overset{+}{3} \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$2 \times \left( \begin{vmatrix} \overset{+}{1} & \overset{-}{2} \\ -1 & 3 \end{vmatrix} \right) - 3 \left( \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \right)$$

$$+ 5 \times \left( \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right)$$

$$2 \times (1 \times 3 - 2 \times (-1)) - 3 \times (1 \times 3 - 2 \times 1)$$
$$+ 5 \times (1 \times (-1) - 1 \times 1)$$

$$10 - 3 - 10$$

$$-3$$

$$\vec{a} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$$
$$\vec{b} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$$

$$\begin{array}{l} \hat{i}\,\hat{j}\,\hat{k} \longrightarrow \\ \vec{a} \longrightarrow \\ \vec{b} \longrightarrow \end{array} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$+ \hat{i} \times (b_1 c_2 - c_1 b_2) - \hat{j} (a_1 c_2 - c_1 a_2)$$
$$+ \hat{k} (a_1 b_2 - b_1 a_2)$$

**Now complete this assignment #VaA4:**

# #VaA4:

**Example 22** Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

**Example 23** Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

**Example 24** Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

**Example 25** Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

**1.** Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

**2.** Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

**3.** If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle $\theta$ with $\hat{k}$, then find $\theta$ and hence, the components of $\vec{a}$.

**4.** Show that

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$$

**5.** Find $\lambda$ and $\mu$ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

**6.** Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

**7.** Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

**8.** If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

**9.** Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

**10.** Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

**11.** Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

**12.** Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

## *Miscellaneous Examples*

**Example 26** Write all the unit vectors in XY-plane.

**Example 27** If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

**Example 28** Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$.

**Example 29** Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Evaluate the quantity $\mu = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$, if $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$.

**Example 30** If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

## Miscellaneous Exercise

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of $x$-axis.
2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$ ? Justify your answer.
5. Find the value of $x$ for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.
6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.
7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.
8. Show that the points A (1, – 2, – 8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.
9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

**15.** Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a}\neq\vec{0}, \vec{b}\neq\vec{0}$.

Choose the correct answer in Exercises 16 to 19.

**16.** If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b}\geq 0$ only when

(A) $0<\theta<\frac{\pi}{2}$ (B) $0\leq\theta\leq\frac{\pi}{2}$

(C) $0<\theta<\pi$ (D) $0\leq\theta\leq\pi$

**17.** Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta=\frac{\pi}{4}$ (B) $\theta=\frac{\pi}{3}$ (C) $\theta=\frac{\pi}{2}$ (D) $\theta=\frac{2\pi}{3}$

**18.** The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is

(A) 0 (B) –1 (C) 1 (D) 3

**19.** If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

**10.** The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

**11.** Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

**12.** Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}\cdot\vec{d}=15$.

**13.** The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

**14.** If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{c}\cdot\vec{d}=15$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

End