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Class :	12	**Session # :**	9
Subject :	Mathematics	**Assignment # :**	DeA2
Topic :	Differential Equations	**Subtopic(s) :**	Homogeneous Differential Equations (First-order)
Lecture #:	2		

Homogeneous Differential Equations

A **homogeneous differential equation** is a type of differential equation where the relationship between the variables can be expressed in a form where each term has the same degree when considered as a function of the independent variable and its derivatives.

There are two major contexts where "homogeneous" is used for differential equations:

1. **First-order homogeneous differential equations** (referring to functions of one variable).
2. **Higher-order homogeneous linear differential equations** (with constant or variable coefficients).

1. First-Order Homogeneous Differential Equations

A **first-order homogeneous differential equation** is typically of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

In this case, the function $f\left(\frac{y}{x}\right)$ is a homogeneous function of degree 0, meaning it can be expressed entirely in terms of the ratio $\frac{y}{x}$. These types of differential equations are solvable using a substitution technique that simplifies the equation into a separable form.

Substitution to Solve:

1. Set $v = \frac{y}{x}$, so that $y = vx$.
2. Differentiate $y = vx$ with respect to x to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
3. Substitute v and $\frac{dy}{dx}$ into the original equation, which reduces it to a separable form in terms of v and x.
4. Solve the resulting equation using standard techniques for separable differential equations.

Example:

$$\frac{dy}{dx} = \frac{y}{x}$$

Here, the right-hand side is a homogeneous function of degree 0, since $\frac{y}{x}$ depends only on the ratio of y and x.

Solution:

1. Set $v = \frac{y}{x}$, so that $y = vx$.
2. Differentiate to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
3. Substitute: $v + x\frac{dv}{dx} = v$, and simplify to $x\frac{dv}{dx} = 0$.
4. This implies $\frac{dv}{dx} = 0$, so v is constant, meaning $v = C$.
5. Therefore, $y = Cx$, which is the general solution.

Session Board:

Homogeneous diff equation

$$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

$f(x,y)$

$f(kx,ky)$

$$\frac{dy}{dx} = \frac{k(x)+2k(y)}{k(x)-k(y)}$$

$$= \frac{k'[x+2y)}{k'(x-y)}$$

$$= \underline{\underline{k^0}}\left(\frac{x+2y}{x-y}\right)$$

$$k^n\left(f(x,y)\right)$$

$$\frac{dy}{dx} = \boxed{\underline{\underline{x^2}}}$$

kx^2, 1

$$f(x,y) = x^2$$

(2).

k is a constant

$$f(kx, ky) = k^2(x^2)$$

$$= k^2 f(x,y)$$

$$= k^n f(x,y)$$

$$\frac{dy}{dx} = \frac{x^2 + y}{1 + y}$$

$$f(x,y) = \frac{x^2 + y}{1 + y}$$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda y}{1 + \lambda y}$$

$$= \frac{\lambda[\lambda x^2 + y)}{(1 + \lambda y)}$$

$$\underbrace{\frac{dy}{dx}} = \underbrace{\frac{3x-2y}{9x+5y}}$$

$$F(x,y) = \frac{3x-2y}{9x+5y}$$

$$x = \lambda x$$

$$y = \lambda y$$

$$F(\lambda x, \lambda y) = \frac{3(\lambda x) - 2(\lambda y)}{9(\lambda x) + 5\lambda y}$$

$$F(\lambda x, \lambda y) = \frac{\lambda(3x-2y)}{\lambda(9x+5y)}$$

$$F(\lambda x, \lambda y) = \lambda^0 \left(\frac{3x-2y}{9x+5y} \right)$$

$$= \boxed{\lambda^m \times (F(x,y))}$$

(Homogeneous).

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}$$

$$F(x, y) = \frac{\frac{x}{x} + \frac{2y}{x}}{\frac{x}{x} - \frac{y}{x}}$$

$$= \frac{1 + 2\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)} = g\left(\frac{y}{x}\right)$$

Homogeneous

$$(x-y)\frac{dy}{dx} = (x+2y)$$

$$\rightarrow \frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{(Homogeneous)}$$

Let $y = V \times x$ (v & x are variables)

$$\boxed{\frac{dy}{dx} = x\frac{dv}{dx} + v \times 1}$$

$$\frac{d(f_1 \times g_1)}{dx} = f_1 \times \frac{d(g_1)}{dx} + g_1 \times \frac{d(f_1)}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x + 2vx}{x - vx}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}}{\cancel{x}}\frac{(1+2v)}{(1-v)}$$

$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\int \frac{(1-v)}{1+v+v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{v-1}{v^2+v+1}\,dv = \int -\frac{1}{x}\,dx$$

$$= -\log(x) + c$$

v'

v^2+v+1

$\Rightarrow (2v+1)$

$$\int \frac{\frac{2v+1-1}{2} - 1}{v^2+v+1}\,dv$$

$$\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\,dv + \int \frac{-\frac{1}{2} - 1}{v^2+v+1}\,dv$$

$v^2+v+1 = t \Rightarrow \frac{1}{2}\int \frac{1}{t}\,dt$

$(2v+1)dv = dt \Rightarrow \frac{1}{2}\log(v^2+v+1)$

$$-\frac{3}{2}\int \frac{1}{\boxed{v^2+v+1}}\,dv$$

$$-\frac{3}{2}\int \frac{1}{v^2+v+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dv$$

$$-\frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dv \qquad -\frac{1}{4}+1$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$-\frac{3}{\cancel{2}}\,\frac{1\times\cancel{2}}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)$$

$$-\frac{3\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$-\sqrt{3}\,\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)$$

$$\underline{\qquad\qquad} = \underline{-\log x} + C$$

$$\frac{1}{2}\log\left[v^2+v+1\right) - \sqrt{3}\,\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log x + C$$

$$v = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

$$F(x,y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y \times \cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda x}{\lambda x \quad \cos\left(\frac{\lambda y}{\lambda x}\right)}$$

$$= \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\left(\frac{y}{x}\right)\right)}$$

$$= \lambda^0 \, F(x, y)$$

$$F(x,y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

$$= \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + 1}{\cos\left(\frac{y}{x}\right)}$$

$$= g\left(\frac{y}{x}\right)$$

Now complete this assignment #DeA2:

#DeA2:

Example 10 Show that the differential equation $(x-y)\dfrac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Example 11 Show that the differential equation $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y\cos\left(\dfrac{y}{x}\right) + x$ is homogeneous and solve it.

Example 12 Show that the differential equation $2y\,e^{\frac{x}{y}}dx + \left(y - 2x\,e^{\frac{x}{y}}\right)dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Example 13 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\dfrac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

2. $y' = \dfrac{x + y}{x}$

3. $(x - y)\, dy - (x + y)\, dx = 0$

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

5. $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

7. $\left\{x\cos\left(\dfrac{y}{x}\right) + y\sin\left(\dfrac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\dfrac{y}{x}\right) - x\cos\left(\dfrac{y}{x}\right)\right\} x\, dy$

8. $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

9. $y\, dx + x\log\left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

10. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0$; $y = 1$ when $x = 1$

12. $x^2\, dy + (xy + y^2)\, dx = 0$; $y = 1$ when $x = 1$

13. $\left[x\sin^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0$; $y = \dfrac{\pi}{4}$ when $x = 1$

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec}\left(\dfrac{y}{x}\right) = 0$; $y = 0$ when $x = 1$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$; $y = 2$ when $x = 1$

16. A homogeneous differential equation of the from $\dfrac{dx}{dy} = h\left(\dfrac{x}{y}\right)$ can be solved by making the substitution.

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

End