Date :	Oct 4 2024	**Board** :	CBSE
Class :	12	**Session # :**	3
Subject :	Mathematics	**Assignment # :**	IA4
Topic :	Integration	**Subtopic(s) :**	Integration Using Trigonometric identities, Some particular functions

Important Trigonometric Identities:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

1. $sin(a) + sin(b) = 2sin(\frac{a+b}{2})cos(\frac{a-b}{2})$

2. $sin(a) - sin(b) = 2cos(\frac{a+b}{2})sin(\frac{a-b}{2})$

3. $cos(a) + cos(b) = 2cos(\frac{a+b}{2})cos(\frac{a-b}{2})$

4. $cos(a) - cos(b) = -2sin(\frac{a+b}{2})sin(\frac{a-b}{2})$

$\sin(a+b)+\sin(a-b)= 2\sin(a)\cos(b)$

$\sin(a+b)-\sin(a-b) = 2 \cos(a)\sin(b)$

$\cos(a+b)+\cos(a-b)=2\cos(a)\cos(b)$

$\cos(a+b)-\cos(a-b)= -2\sin(a)\sin(b)$

$$\sin A \cos B = \frac{1}{2}\,[\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2}\,[\sin(A+B) - \sin(A-B)]$$

$$\cos A \cos B = \frac{1}{2}\,[\cos(A+B) + \cos(A-B)]$$

$$\sin A \sin B = \frac{1}{2}\,[\cos(A-B) - \cos(A+B)]$$

$$\sin(a+b) = \sin a \cos b + \cancel{\cos a \sin b}$$

$$\sin(a-b) = \sin a \cos b - \cancel{\cos a \sin b}$$

$$+$$

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$$

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

Integration using Trigonometric Identities:

$$\int \cos^2 x\,dx$$

$$\frac{1}{2}\int (1+\cos 2x)\,dx$$

$$\frac{1}{2}\left[\int 1\,dx + \int \cos 2x\,dx\right]$$

$$\frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

$$\frac{1}{2}x + \frac{\sin 2x}{4} + C$$

$$\int \sin 2x \cos 3x\,dx$$

$$\frac{1}{2}\left[\int \sin(2x+3x)\,dx + \int \sin(2x-3x)\,dx\right]$$

$$\frac{1}{2}\left[\int \sin 5x\,dx - \int \sin x\,dx\right]$$

$$\frac{1}{2}\left[\frac{-1}{5}\cos 5x + \cos x\right] + C$$

Method 1:

$$\int \sin^3 x \, dx$$

$$\int \sin^2 x \times \sin x \, dx$$

$$\int \left(\frac{1 - \cos 2x}{2} \right) \times \sin x$$

$$\frac{1}{2} \left[\sin x - \sin x \cos 2x \right] dx$$

Method 2:

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\frac{1}{4} \int (3 \sin x - \sin 3x) \, dx$$

$$\frac{1}{4} \int 3 \sin x - \int \sin 3x \, dx$$

$$\boxed{-\frac{3}{4} \cos x + \frac{1}{3} \cos 3x + C}$$

Method 3:

$$\int \sin^3 x \, dx$$

$$\int \sin^2 x \times \sin x \, dx$$

$$\int (1 - \cos^2 x) \times \sin x \, dx$$

$$\cos x = t$$

$$-\sin x \, dx = dt$$

$1 - \cos^2 x$

$1 - \cos^2 x = t$

$\cancel{1} - 2\cos x$

$dx = d$

$\cos^2 x$

$-2\sin x \cos x$

$$\int -(1 - t^2) \, dt$$

$$\int (t^2 - 1) \, dt$$

$$\frac{t^3}{3} - t + C$$

$$\frac{\cos^3 x}{3} - \cos x + C$$

Method 1:

$$\int \sin^2(2x+5)\, dx$$

$$\sin^2(x) = \frac{1-\cos 2x}{2}$$

$$\int \frac{1-\cos(4x+10)}{2}$$

$$\frac{1}{2}\left[x - \frac{\sin(4x+10)}{4}\right] + C$$

Method 2:

$$\cos(4x+10)$$

$$\cos(4x)\,\underline{\cos(10)} - \sin(4x)\sin(10)$$

$$\frac{\cos(10)}{4}\sin 4x + \frac{\sin(10)}{4}\cos 4x + C$$

Now try these questions:

Example 7 Find (i) $\int \cos^2 x\, dx$ (ii) $\int \sin 2x \cos 3x\, dx$ (iii) $\int \sin^3 x\, dx$

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2(2x+5)$ **2.** $\sin 3x \cos 4x$ **3.** $\cos 2x \cos 4x \cos 6x$

4. $\sin^3(2x+1)$ **5.** $\sin^3 x \cos^3 x$ **6.** $\sin x \sin 2x \sin 3x$

7. $\sin 4x \sin 8x$ **8.** $\dfrac{1-\cos x}{1+\cos x}$ **9.** $\dfrac{\cos x}{1+\cos x}$

10. $\sin^4 x$ **11.** $\cos^4 2x$ **12.** $\dfrac{\sin^2 x}{1+\cos x}$

13. $\dfrac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$ **14.** $\dfrac{\cos x-\sin x}{1+\sin 2x}$ **15.** $\tan^3 2x \sec 2x$

16. $\tan^4 x$ **17.** $\dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$ **18.** $\dfrac{\cos 2x+2\sin^2 x}{\cos^2 x}$

19. $\dfrac{1}{\sin x\cos^3 x}$ **20.** $\dfrac{\cos 2x}{(\cos x+\sin x)^2}$ **21.** $\sin^{-1}(\cos x)$

22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}\,dx$ is equal to

(A) $\tan x+\cot x+C$ (B) $\tan x+\operatorname{cosec} x+C$

(C) $-\tan x+\cot x+C$ (D) $\tan x+\sec x+C$

24. $\displaystyle\int \frac{e^x(1+x)}{\cos^2(e^x x)}\,dx$ equals

(A) $-\cot(ex^x)+C$ (B) $\tan(xe^x)+C$

(C) $\tan(e^x)+C$ (D) $\cot(e^x)+C$

Integration of some particular functions:

(1) $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$$

(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C$

(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(4) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(5) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

(7) To find the integral $\int \frac{dx}{ax^2+bx+c}$, we write

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Now, put $x+\frac{b}{2a}=t$ so that $dx = dt$ and writing $\frac{c}{a}-\frac{b^2}{4a^2}=\pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int\frac{dt}{t^2\pm k^2}$ depending upon the sign of $\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int\frac{px+q}{ax^2+bx+c}dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px+q = \mathrm{A}\frac{d}{dx}(ax^2+bx+c)+\mathrm{B} = \mathrm{A}\,(2ax+b)+\mathrm{B}$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int\frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

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