What makes a good prediction interval or probabilistic forecast?

A thesis submitted for the degree of

Masters of Applied Econometrics

by

Beinan Xu

26401746

Department of Econometrics and Business Statistics

Monash University

Australia

June 2018

Contents

Abstract

Statistical forecasting models usually provide an estimate of the forecast distribution, or at least a prediction interval, for each forecast horizon. This thesis introduces scoring rules and using them to evaluate interval forecasts and probabilistic forecasts. The measures will be evaluated empirically, by comparing the forecast distributions obtained from a range of statistical models applied to some large collections of time series. In the past few decades, the interval forecasts and probabilistic forecasts have a very important development and are attracting more and more attention. More and more organizations and individuals begin to use probability prediction instead of point prediction to carry out the future. However, the traditional evaluation methods of point prediction cannot effectively evaluate the results of probabilistic prediction. Because if we want to evaluate the probability prediction effectively, we should not only evaluate the sharpness of the prediction distribution but also evaluate its calibration. For evaluating the result of interval forecasts and probabilistic forecasts, scoring rules is a very effective method. It can evaluate the sharpness of the prediction of distribution while assessing calibration. In this article, we have used different scoring rules to evaluate the different forecasting result base on different models at the index of ASX 200 and M3 datasets.

Chapter 1

Introduction

In this world, people wish to understand the future development of different events. For example, residents want to know tomorrow's weather and temperatures to decide what kind of clothes they need to choose. Securities investors want to know the future price trend of securities in order to formulate a suitable investment portfolio. Unfortunately, it is very difficult for humans to predict the future because uncertainty is a universal feature of this world. Although we have a variety of ways to predict future events, such as building suitable time series models based on past information, then predicting future trends over time, none of these methods provide absolutely accurate future predictions. The limitations are that, first of all, various activities and phenomena in the real world are difficult to be perfectly represented by mathematical models, especially humanistic phenomena such as the purchase of lottery tickets (the outcome of the lottery is random). Although there are some natural phenomena, they have certain rules to follow, such as temperature have seasonal changes, so it is very difficult to establish prediction models for them. Second, human cognition is limited. For the event, people cannot collect and obtain all of the information for the relevant factors. Because of these limitations, using these methods to make predictions must not be absolutely accurate. For now, to accurately forecast future is still a difficult task, but as more and more prediction methods and models are developed, forecasters can use them to get what they want. The following problem is how to evaluate these models because choosing the correct assessment method can

effectively compare the accuracy of forecast results by using different models, then the forecaster can obtain the most suitable prediction model.

In choosing the forecast method, point forecast is the most commonly used. But forecasts should be probabilistic (Gneiting and Katzfuss (2014)) and point estimation gradually transform to distribution estimation (Stigler (1975)). Therefore, interval forecasts and probabilistic are being used more and more frequently. Probabilistic prediction is a method to forecast future uncertain events and development by generating probability prediction distribution. Base on the available information set, to maximize the sharpness of prediction distribution and subject to calibrate (Gneiting and Katzfuss (2014)). Comparing the point forecasts can produce a single point result, such as predicted a stock price in the next day, probabilistic prediction can supply more information to the forecaster by assigning a probability distribution to each future possible outcome as supplying the probabilistic distribution on different prices on the second day. Obviously, probabilistic forecasting has more obvious advantages than point forecasting, so people begin to use probabilistic forecasts to predict activities rather than using point forecasts in many fields, such as finance, weather, medicine etc. In the Raftery (2016) paper, it discussed five potential predictors who have a need for probability forecasts.

For evaluating the accuracy of point forecast results, the common ways are to calculate the forecast errors, scale-dependent errors (as Mean absolute error, Root mean squared error), percentage errors (as Mean absolute percentage error) or scale errors (as the mean absolute scaled error) (Hyndman and Athanasopoulos (2018)). But for interval forecasts and probabilistic forecasts, the scoring rules are used more generally. At present, scoring rules are mainly used in weather forecasting system. They provide such an assessment by giving a numerical score based on probability and actual observation (Winkler (1996)) and proper scoring rules encourage the forecaster to conduct careful assessment and honesty (Gneiting and Raftery (2007)).

This thesis introduces the scoring rules in Chapter 2. In this Chapter, probability forecasts, sharpness and calibration is reviewed. Also, interval scoring rules and distribution scoring rules are reviewed separately, and they are applied to evaluate interval forecasts and probability forecasts respectively. Meanwhile, the corresponding formula and derived

formula are shown in Chapter 2.1 and 2.2. In the third chapter, after using two different models to fit the financial data, interval forecast and probabilistic forecast are performed separately. Then the prediction results are evaluated by using suitable interval scoring rules and distribute scoring rules. In the same way, in Chapter 4, we select M3 data sets that have more abundant data types, and choose more different models to make forecasts, then to evaluate.

Chapter 2

Scoring rules

The traditional prediction method is mainly based on point forecasts, which can provide forecasters with future development trend information under given predictive interval level. But the future is extremely uncertain. It's hard to predict an accurate future through the past information. For example, when watching a football match, if the level of the two teams is very different, we can easily judge that the team is more likely to win, but how many goals is hard to know. At this point, the limitations of point forecasts are reflected. The prediction should be probabilistic. The preditive interval or probabilistic forecasts can be given an interval or probability distribution for all possible future results so that more information can be obtained to predict the uncertain future. If we can assign a different probability to different results in the game, the fans will be able to judge the result of the match.

Scoring rules are usually used to evaluate the results of interval prediction and probability prediction by assessing calibration and sharpness of interval or probabilistic forecast at same time. The meaning of sharpness refers to the centralization of the predicted distribution and the calibration refers to the statistical consistency between the predicted distribution and the observed value. (Gneiting, Balabdaoui, and Raftery (2007)) They affect the quality of probabilistic forecast. Therefore, to evaluate the calibration and sharpness of probability prediction is an important means to evaluate probability prediction results. In this article, we mainly use two kinds of score, interval score and distribution score.

And four scoring rules are used: one is interval scoring rule, three are distribution scoring rules.

2.1 Difference between confidence interval and prediction interval

Before discussing scoring rules, we must first understand the difference between confidence interval and prediction interval. These two kinds of intervals are often considered to be the same, but this view is wrong, so they need to be very careful when used. For their differences, Hyndman (2013) gave a detailed introduction on his blog. The prediction interval is an interval related to the random variable, and all of the random variable are located in the interval. In contrast, confidence interval is a concept of frequency, which is related to parameters. The prediction interval is used in the interval forecasts and probabilistic forecasts.

2.2 Interval scoring rules

The interval score is used to evaluate the results of interval prediction. For interval forecasts, sometimes full predictive distributions are difficult to specify and the forecaster might quote predictive quantiles, such as value at risk in financial applications (Gneiting and Raftery (2007)). So the interval prediction is used, it is a special case of quantile prediction. The $(1-\alpha)\times 100\%$ is represent the central prediction interval. $\frac{\alpha}{2}$ and $\ell - \frac{\alpha}{2}$ quantiles are upper and lower endpoints (Gneiting, Balabdaoui, and Raftery (2007)).

2.2.1 Property of interval scoring rules

Suppose that the quantiles at the levels $\alpha_1, \cdots, \alpha_k \in (0,1)$ are sought. If the forecaster quotes $r_1, \cdots, r_k$ and x materializes, then the scores $S(r_1, \cdots, r_k; P)$ will be rewarded.

$$S(r_1, \cdots, r_k; P) = \int S(r_1, \cdots, r_k; x) dP(x)$$

as the expected score under the probability meansure P when the forecaster quotes the quantiles $r_1, \cdots, r_k$.

Following Cervera and Muñoz (1996), the scoring rule S is proper if

$$S(q_1, \cdots, q_k; P) \geq S(r_1, \cdots, r_k; P)$$

for all real numbers $r_1, \cdots, r_k$ and for all probability meansures $P \in \mathcal{P}$

2.2.2 Winkler loss scoring rule

We have chosen Winkler loss scoring rules to assessing interval forecast. The most commonly used interval scoring rules is Winkler Loss scoring rules, it was proposed by Winkler (1972). Predictors are rewarded with narrow prediction intervals, and they will be punished. If the missed intervals are observed, the size depends on α.

$$S_\alpha^{int}(l, u; x) = (u - l) + \frac{2}{\alpha}(l - x)1\{x < l\} + \frac{2}{\alpha}(x - u)1\{x > u\}$$

where l and u represent for the quoted $\frac{\alpha}{2}$ and $\ell - \frac{\alpha}{2}$ quantiles.

Following the formula of interval score, the score mainly based on the results of interval forecasts at different predictive interval level. If the forecast is in the prediction interval, the score will be equal to upper minus lower endpoints. if not the score will equal to the difference between x and upper/lower endpoint by $\frac{2}{\alpha}$. This scoring rule is very easy to understand and apply, so it has a wide range of practicality and can be used to evaluate interval forecasts by various models. However, its formula also shows that, in the case of simultaneous assessment of different time series, each result needs to be standardized if the results are compared, since the data is not transformed in the formula to reduce the impact of the unit on the final comparison results.

2.3 Distribution scoring rules

The distributed scoring rule is usually used to assess probabilistic forecasts, which represents the estimation of the respective probabilities for all possible future outcomes of a random variable. Compared with single value prediction, probability prediction represents probability density function. Distribution scoring rules supply the summary

measures to evaluate probabilistic forecasts, it assigns a numerical score under the predictive distribution and the events that need to be predicted. (Gneiting, Balabdaoui, and Raftery (2007)) The function of scoring rules is to evaluate the calibration and the sharpness of the forecast distribution results at the same time, then evaluating the quality of probabilistic forecasts. For the results of produced scores, forecasters wish it can be minimized.

2.3.1 Property of scoring rules

Assume the result of probabilistic forecasts is F, $F \in \mathcal{F}$ where $\mathcal{F}$ is a suitable class of CDFs, and $G : \mathcal{F} \times \cdots \times \mathcal{F} \to \mathcal{F}$. Then the scoring rule will be $S(F, y)$, where $y \in R$ is the realized outcome.

The scoring rule S is proper relative to the class $\mathcal{F}$ if

$$S(F, G) \geq S(G, G)$$

for all $F, G \in \mathcal{F}$. Also when $F = G$, the two sides of equation are equal, then it meanings the scoring rules is strictly proper.

For evaluating probabilistic forecasts, we prefer to choose three scoring rules, logarithmic scoring rule (LogS), continuous ranked Probability scoring rule (CRPS) and Dawid-Sebastiani scoring rule (DDS). They are chosen to evaluate probabilistic forecasts all under Gaussian predictive distribution.

2.3.2 Logarithmic score

For the scoring rules for evaluating probabilistic forecasts, the of the most commonly used rules is the Logarithmic score (logS). It was first proposed by Good (1952). It is a modified version of relative entropy and can be calculated for real forecasts and realizations. (Roulston and Smith (2002)) It is a strictly proper scoring rule. But if the prediction is continuous, using ignorance is troublesome (Peirolo (2010)). Despite its shortcomings, it can directly evaluate the results through the forecast model. Therefore,

the logarithmic scoring rule can be used in many scenarios and is not limited to specific models.

The formula is:

$$LogS(F, y) = logF(y)$$

For this report, we use the scoring rules to evaluation the probabilistic forecasts under Gaussian predictive distribution. Then the formula of the logarithmic score can be rewritten as below.

$$LogS(N(\mu, \sigma^2), y) = \frac{(y-\mu)^2}{2\sigma^2} + log\sigma + \frac{1}{2}log2\pi$$

According to the formula of the logarithmic scoring rule, we can see that it can directly standardize the evaluating results. Therefore, when using this scoring rule, there is no need to standardize the score results.

2.3.3 Continuous Ranked Probability Score

It is generally considered that it is unrealistic to limit the density forecasts. In the absence of restriction on density forecasts, the CRPS can define scoring rules directly in terms of predictive cumulative distribution functions. It focuses on observing the whole of forecast distributions rather than the special points in these distributions. It can use deterministic values to evaluate the results of probabilistic forecasts. Also, comparing with the CRPS, logarithmic score is a local strictly proper scoring rule. Therefore, there are not many restrictions on its use.

The formula of continuous ranked probability Score:

$$CRPS(F, y) = \int_{-\infty}^{\infty} (F(x) - 1\{y \leq x\})^2 dx$$

$$= E_F|Y - y| - \frac{1}{2}E_F|Y - Y'|$$

where Y and Y' are independent random variables with CDF F and finite first moment (Gneiting and Raftery (2007)). The CPRS can compare the probabilistic forecasts and point

forecasts because when the CRPS drop to the absolute error, the probabilistic forecast is a point forecast. (Gneiting and Katzfuss (2014))

Also, when evaluating probabilistic forecasts under Gaussian predictive distribution the form will re-write:

$$CRPS(N(\mu,\sigma^2),y) = \sigma\left(\frac{y-\mu}{\sigma}\left(2\Phi\left(\frac{y-\mu}{\sigma}\right)-1\right)+2\varphi\left(\frac{y-\mu}{\sigma}\right)-\frac{1}{\sqrt{\pi}}\right)$$

Unlike the logarithmic score, according to the formula continuous ranked probability scoring rule, no transformation of the data is made, so when using this scoring rule, we need to pay attention to the standardization of the scoring results.

2.3.4 Dawid-Sebastianti score

Althouth CRPS scoring rule can be easy to understand and convenient to use, but it has a limitation. It can be hard to compute for complex forecast distributions (Gneiting and Katzfuss (2014)). Therefore, when we need to evaluate the probabilistic forecasts under the complex distribution, choosing Dawid-Sebastiani score is a viable alternative. Dawid and Sebastiani (1999) described this proper scoring rule depend on first and second moments only.

$$S(F,y) = -logdet\Sigma_F - (x-\mu_F)'\Sigma_P^{-}1(x-\mu_P) \qquad (1)$$

which is relected the generalized entropy function $G(F) = -logdet\Sigma_F - m$. This scoring rule is strictly proper relative to any convex class of probability measures characterized by the first two moments, then it is equivalent to the logarithmic score.

According to Gneiting and Raftery (2007) using the predictive model choice criterion (PMCC by Laud and Ibrahim (1995) and Gelfand and Ghosh (1998)) $PMCC = \sum_{i=1}^{n}(y_i - \mu_i)^2 + \sum_{i=1}^{n}\sigma_i^2$ can get the scoring rule formula as $S(F,y) = -(y-\mu_F)^2 - \sigma_F^2$, where F has mean μ_F and variance σ_F^2. But it is improper.

When the ture belief of forecasters is F and they want to maximize the expected score, they will use the point measure at μ_F, rather than the forecast distribution F. So, the predictive

model choice criterion should be replaced by a criterion based on the scoring rule formula (1). Then the DSS formula will be obtained as

$$DSS(F,y) = -\frac{(y-\mu_F)^2}{\sigma_F^2} - 2log\sigma_F$$

and the $m = 1$ and the observations are real-valued. Since Dawid-Sebastianti score has the same characteristics as logarithmic score, which can transform data directly. Therefore, there is no need to consider the standardization problem when use this scoring rule.

Chapter 3

Time series models

Time series models are used to analyze and predict time series data. And different time series models are applicable to different types of time series, such as the generalized autoregressive conditional heteroskedasticity model is widely used to analyze and forecast volatility in the financial time series. In this thesis, four commonly used time series models are selected to use in both case study, autoregressive integrated moving average model (ARIMA), generalized autoregressive conditional heteroskedasticity model (GARCH), exponential smoothing model (ETS) and random walk model (RW). The four models have different characteristics, and the ways of selection are different. According to their characteristics, ARIMA and GARCH model are used in case study one, to fit the financial data. In case two, we use ARIMA and ETS model. And the RW model is selected to remove the units effect to standardize the results by using some scoring rules, because the datasets in case two have over 3000 time series from different fields.

3.1 Autoregressive integrated moving average model

Autoregressive integrated moving average model aims to describe the autocorrelations in the data. The non-stationary ARIMA model is obtained by combine differencing with autoregression and a moving average model. The full model of ARIMA(p,d,q) model is written as

$$y'_t = c + \phi_1 y'_{t-1} + \cdots + \phi_p y'_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

where y'_t is the differented series, p is the order of the autoregressive part, d is the degree of first differencing involved and q is the order of the moving average part.

After using the backshift notation, Non-seasonal ARIMA models can be rewritten as below. This formula is much easier to use.

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)^d y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

where $y'_t = (1 - B)^d y_t$ is the mean of y'_t.

For seasonal data, seasonal ARIMA model can be used instead of non-seasonal ARIMA model for analysis and prediction. It includes a non-seasomal part and a seasonal part, it can be represented as $ARIMA(p, d, q)(P, D, Q)_m$, where m =number of observations per year, P is the order of the seasonal autoregressive part, D is the degree of seasonal first differencing involved and Q is the order of the seasonal moving average part. Seasonal ARIMA models in backshift notation as

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - \Phi_1 B^m - \cdots - \Phi_P B^{mP})(1 - B)^d (1 - B^m)^D y_t$$

$$= (1 + \theta_1 B + \cdots + \theta_q B^q)(1 + \Theta_1 B^m + \cdots + \Theta_Q B^{mQ})\varepsilon_t$$

For the selection of the ARIMA model, the way is that, plot the ACF and PACF of the data (if necessary, difference the data until stationary) to determine the corresponding models. Then choose the model with the smallest AIC or AICc to be the most suitable model. After checking the residuals from this model by plotting the ACF of the residuals and doing a portmanteau test of the residuals, to see whether the residuals look like white noise. If yes, this model can be use, if not, we should find the model again. In this process, AIC or AICc play a very important roles. Akaike's Information Criterion (AIC) was described by Akaike (1974), which useful in selecting predictors for regression. For ARIMA model the AIC is defined as

$$AIC = -2log(L) + 2(p + q + k + 1)$$

where L is the likelihood of the data, $k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

Also, the corrected AIC can be written as

$$AICc = AIC + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}$$

In this article, we choose to use 'auto.arima' function to select model. This function is from R package "forecast" by Hyndman (2018b). It uses a variation of the Hyndman-Khandakar algorithm (Hyndman and Khandakar (2008)), which combines unit root tests, minimization of the AICc and MLE to obtain an ARIMA model. It can automatically complete the selection process as above. Using this function, we can get the suitable model quickly and accurately.

3.2 Generalized autoregressive conditional heteroskedasticity model

GARCH model is an econometric model developed by Robert (1982), it is impartant way to analyze and forecast the volatility of financial data. It is actually formed on the basis of ARCH by increasing the p order autoregressiveness considering the heteroscedasticity function, which can effectively fit the heteroscedasticity function with long-term memory. In this article, we selection the GARCH models with the ARIMA model by using auto.arima function in R. Although the process of selecting the GARCH model cannot be carried out automatically, we can estimate some alternative models by using the garchFit function (from r package "fGarch" by Wuertz (2017)). Then choosing the model with minimum AIC is the most suitable model by comparing the results of the AIC.

ARMA-GARCH Model can be written as

$$y_t = \mu + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_r \varepsilon_{t-r}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_s \sigma_{t-s}^2 = \omega + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{s} \beta_i \sigma_{t-i}^2$$

where $\varepsilon_t \sim N(0, \sigma_t^2)$

According the formula of GARCH model, it has some important features. First, the big α_{t-1}^2 will follow a big α_t^2, which will generate a well-known phenomenon of volatility clusters in financial time series. Secondly, compared with the ARCH model, the tail of the GARCH model is thicker than that of the normal distribution. The third GARCH model can describe the evolution of volatility through a simple parameter function.

3.3 Exponential smoothing model

ETS model is a technique to make forecasts by using a weighted mean of past values, wherein more recent values are given higher weights. It was described by Brown (1959), Holt (1957) and Winters (1960). The types of ETS models can be divided into two categories: a model with additive errors and one with multiplicative errors. And they can continue to be subdivided by trends and seasonality. Each ETS model can be defined by these three factors to obtain ETS(Error, Trend, Seasonal). The possibilities for each component are: Error $= \{A, M\}$, Trend $= \{N, A, A_d\}$ and Seasonal $= \{N, A, M\}$, where N is none, A is additive, A_d is additive damped and M is multiplicative.

For example, according to the selecting rule as above, a ETS(A,A,A) model can be written as below. The one-step-ahead training errors are assumed as $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1} - s_{t-m} \sim NID(0, \sigma^2)$.

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$$

$$b_t = b_{t-1} + \beta \varepsilon_t$$

$$s_t = s_{t-m} + \gamma \varepsilon_t$$

Like 'aoto.arima', "forecast" package in R provides an automatic method ('ets') to select ETS model. Although 'ets' also chooses the most suitable model according to the AICc value, the formula is somewhat adjusted. For ETS model, the AIC is defined as

$$AIC = -2log(L) + 2k$$

where L is the likelihood of the model and k is the total number of parameters and initial states that have been estimated (including the residual variance).

And the AICc is be written as

$$AICc = AIC + \frac{k(k+1)}{T-k-1}$$

Although the ETS model is widely used, it is important to note that all ETS models are non-stationary. So they cannot be used to analyze the predicted the stationary time series data, such as financial data.

3.4 Random walk model

In case study two, the M3 datasets contain the time series from different fields, so they have units are also different. In order to remove the effect of unit before we compare the scoring results by using different scoring rules, standardization of results is needed. The method is that we use the time series models to get the scoring results divided the results obtained by our chosen model, which is used for standardization. The model is random walk model, it also be called naive model, because of two main features of random walk, long periods of apparent trends up or down and sudden and unpredictable changes in direction. Select it because it has some important features, one is the forecasts from a random walk model are equal to the last observation. And it can widely used for non-stationary data.

$$y_t = y_{t-1} + \varepsilon_t$$

where ε_t denotes white noise.

Similarly, the random walk model can also be modeled by an automatic program ('rwf' function from "forecast" package), and its modeling approach is more simple than the previous time series models, because there is no need to use information criteria as the the AIC to select the optimal model. The all forecasts to be the value of the last observation: $y_{T+h|\hat{T}} = y_T$ where h is the forecast horizon.

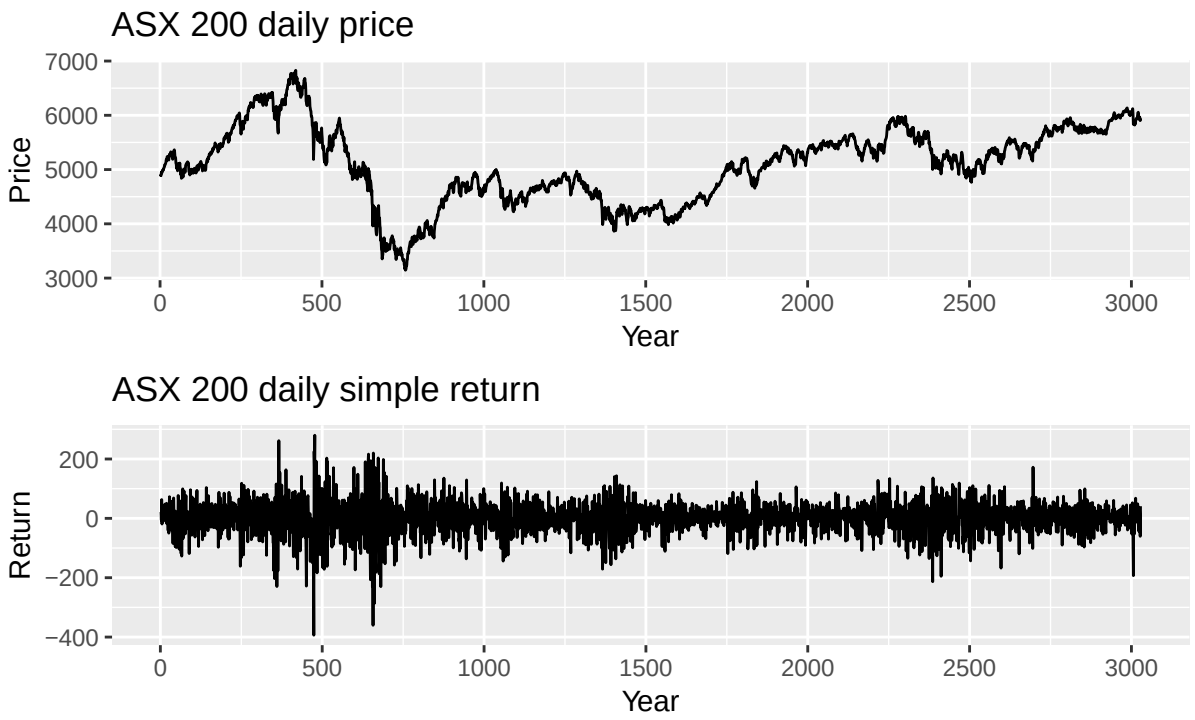
Chapter 4

Case study one: ASX 200

The ASX 200 is an index on the Australian Securities Exchange officially released on 31st March 2000. It uses market-weighted average calculations based on the 200 largest listed stocks in Australia. These stocks currently account for the Australian stock market value of 82%. It is considered to be the most important index to measure the operation of the Australian stock market.

The data from ASX 200 is a kind of financial time series, it has the following characteristics. Firstly, The unconditional distribution is leptokurtic, it means that comparing with Gaussian distribution it has a high peak and heavy tails. Also, the return series appears to have a constant unconditional mean, so the time series of return might be stationary, some trend models are not suitable to use for financial times series as ETS model. Because of the volatility of return changes over time and volatility tends to arrive in clusters, the GARCH model is very suitable for use. We choose to use ARIMA model and ARIMA-GARCH model to fit model. Also, using these two models can be very intuitively and clearly to compare the results of forecasting and scoring.

The raw data comes from YahooFinance (2018), it is the daily data over 10 years period until the beginning of 2018. Because of the features of financial time series, we processed data and obtain its simple return.



In order to facilitate the final assessment, we have set the data before 2017 as train data,the data from 2017 until early 2018 as test data. Then using train data to select suitable ARIMA model and GARCH model to make both interval forecasts and probabilistic forecasts. For the forecasting results, different scoring rules are used to evaluate them respectively. Finally compare the results of the score to evaluate the quality of the forecast results. Because in this case study, only single one time series is used, there is no the impact of the unit in the comparison. Therefore, we do not have to consider any standardization problem.

4.1 Select suitable models

4.1.1 ARIMA model

For the selection of models, we first use the 'auto.arima' function to find the most suitable model. Based on train set, the ARIMA model is selected to be shown below.

According to table 4.1, it shows that ARIMA model does not have the autoregressive part, and the order of the moving average part is equal to 3. So, MA(3) model is the model what we need to use. This result is used to continue selecting the GARCH model.

Table 4.1: *ARIMA model select*

	x
ma1	-0.0398745
ma2	0.0063880
ma3	-0.0504566

Table 4.2: *Garch model select*

	AIC	BIC	SIC	HQIC
garch11	10.608	10.623	10.608	10.614
garch12	10.609	10.626	10.609	10.615
garch21	10.609	10.626	10.609	10.616
garch22	10.610	10.629	10.610	10.617
arch1	10.779	10.791	10.779	10.783
arch2	10.729	10.744	10.729	10.734

4.1.2 GARCH model

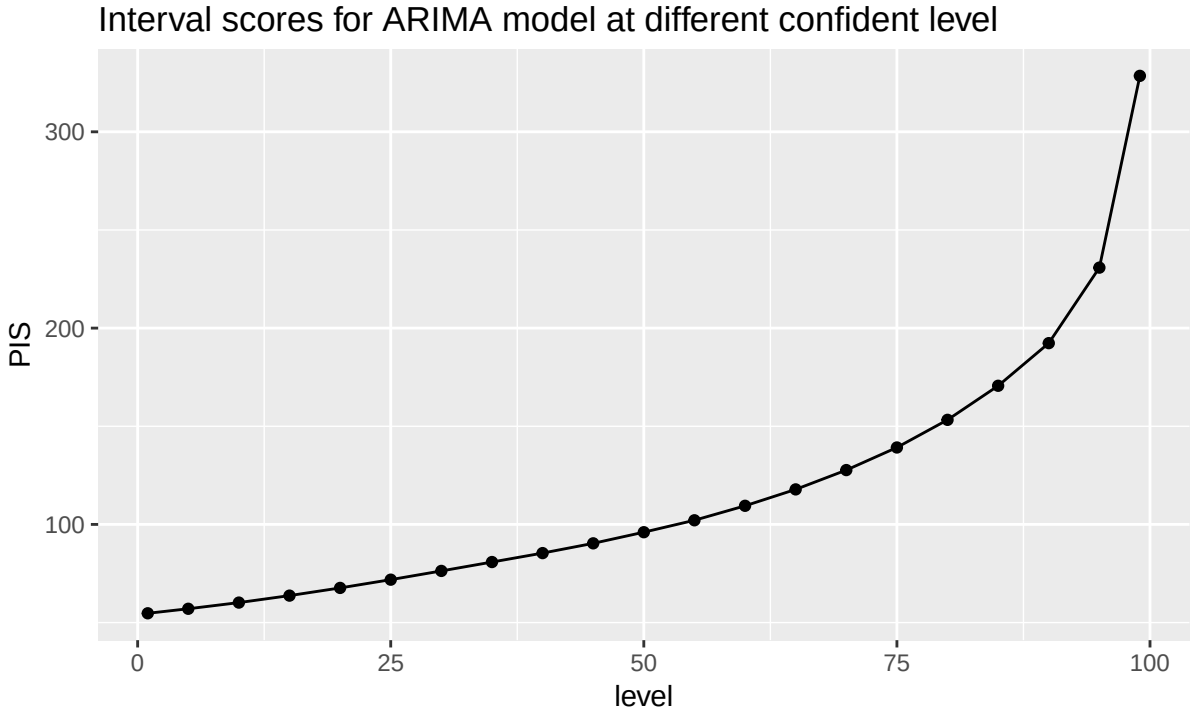
Unlike the selection of ARIMA models, there is no automatic program to help us select the most suitable GARCH model. Therefore, we define 6 different GARCH models based on the ARIMA model,which be seclected before, and then use 'garchFit' function from fGarch package to estimate them separately. According to the result, the model with minimum AIC is chosen as the optimal model.

According to the result as table 4.2, the AIC of the MA(3)-GARCH(1,1) is 10.608, it is the smaller than other models. Therefore, The MA(3)-Garch(1,1) is considered the most suitable model for the train set.

4.2 Interval forecast for the ASX 200 index

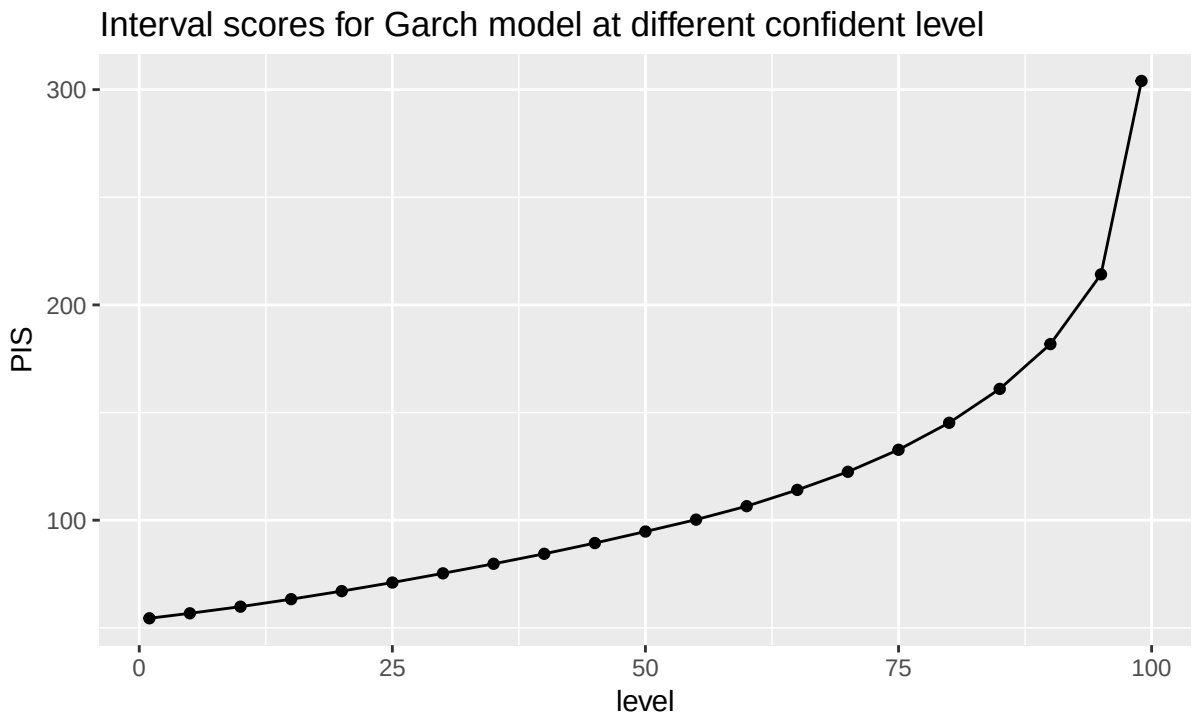
For interval prediction, Winkler loss scoring rule are used. According to its formula, the score is affected by the size of the predictive interval. Therefore, we want to know what changes will be made in the interval score under different predictive interval levels. First we use the previous MA(3) model to make forecast. By setting 21 different prediction intervals $(1-\alpha)\times 100\%$ from 1% to 99%, the results of interval forecasts at different prediction interval level are obtained.

Because the scoring rule is to score every prediction result, to take the average of scoring results can be easy to observe the changes as the size of predictive interval increasing. Then the 21 scores under 21 predictive interval levels are obtained. Using them to make a curve as blow.

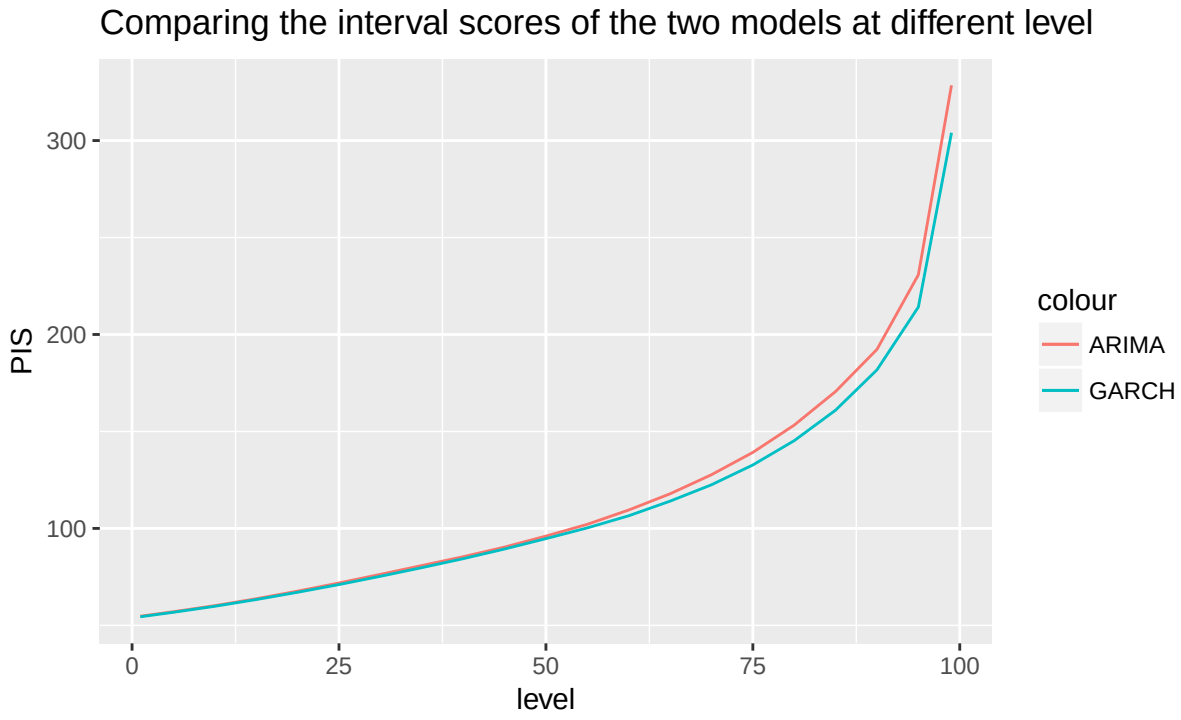


This graph shows that as the predictive interval increases, the interval score shows a trend of accelerating and increasing, and the score reaches the highest at 99%. The lower the score represents the better result of the interval forecasts, so the information from this curve shows that the interval forecasts for the simple return of the ASX200 by using ARIMA model are better when the prediction interval level is smaller.

Then use the same way to set prediction interval levals, and use MA(3)-GARCH(1,1) model to forecast the intervals again. The scores of forecast results under same predictive interval are also taken average respectively. After that, a score change curve for GARCH model is obtained, which shows the result very similar to that obtained by using the ARIMA model before.



This graph also shows that as the prediction interval expands, the score increases. It means that the smaller prediction intervals show better score results by using MA(3)-GRACH(1,1) model. Because the images produced by using the two different models are extremely similar, we put them together for comparison.



By comparing these two curves, their overall characteristics are extremely similar. Their scores are not very different at smaller predictive intervals level, but in the larger prediction interval level, the scores are slightly different by using these two different models. The interval scores of interval forecasts by using MA(3)-GRACH(1,1) model is becoming smaller than that by using MA(3) model as the prediction interval expands. So, at high predictive interval level, interval forecasts by using GARCH model has a relatively good performance. This result illustrates, for the interval forecast of financial return time series, GARCH model can provide the more efficient result to forecasters. And it is also proving that it is more suitable for fitting financial data.

4.3 Probabilistic forecasts for the ASX 200 index

For probabilistic forecasts, we still using the ARIMA(0,0,3) model and MA(3)-GARCH(1,1) model to fit data and make a forecast, which was produced at section 4.1. However, unlike the result obtained in section 4.2, we do not need to set the predictive interval. Instead, we use the functions of 'logs_norm', 'crps_norm' and 'dss_norm' from "scoringRules" package (Jordan, Krueger, and Lerch (2017)) in R to directly calculate the scores, and these three functions represent these three distribution scoring rules under Gaussian predictive distribution: Logarithmic score, Continuous Ranked Probability Score and Dawid-Sebastianti score.

For the scores by using the same model and the same score rule, we also get their mean. Then the table 4.3 is generated. According to this table, the scores of three type scoring rules of MA(3)-garch(1,1) model are all smaller than the result of MA(3) Model. This result shows that for the selected financial time series (ASX 200), the GARCH model can get better prediction results than the ARIMA model.

Table 4.3: *Scoring Rules for MA model and GARCH model*

	CRPS	LogS	DSS
GARCH	20.70	5.10	8.36
ARIMA	21.13	5.14	8.45

By observing all the previous results, no matter for interval prediction or probability prediction, the Garch model all can have a better performance than the ARIMA model, although it prediction effect under the high predictive interval level is much worse than that in the low predictive interval. This result also shows that compared with the ARIMA model, the GARCH model can analyze and predict the financial data more accurately.

Chapter 5

Case study two: M3 datasets

The M3 dataset includes 3003 different type time series, it is from R packages "Mcomp" (Hyndman (2018a)). These time series are from different fields, so their data units are different. Base on the time type, they can be divided into three large classes, yearly data monthly data and quarterly data. For each time series, there are a train set and a test set, which can be easily used to build each forecast models, then predicting and scoring. Different from previous financial data, M3 datasets can use different models for predictive analysis at the same time. Therefore, it can provide more information for evaluating forecasts by using scoring rule.

As in the previous chapter, before we start forecasting and evaluating, the suitable models should be selected. In this case study, three prediction models are chosen, ARIMA model, ETS model, and Random walk model. However, for the M3 datasets, there are more than 3000 different time series, so we have modeled all the time series separately by using these three models.

Before the analysis of the M3 dataset, there is a problem that needs to be noticed. These different time sequences come from different fields and their units are different. As we discussed in the second chapter, it is necessary to standardize each scoring result by using Winkler loss scoring rule and continuous ranked probability Scoring rule. If data are not standardized, the final result will be the mistake.

5.1 Model selection

As in the previous chapter, before we start forecasting and evaluating, the suitable models should be selected. In this case study, three prediction models are chosen, ARIMA model, ETS model, and Random walk model. However, for the M3 datasets, there are more than 3000 different time series, so we have modeled all the time series separately by using these three models. The three models can be selected by the automatic program, which are introduced in the third chapter. Then they can be used for interval prediction or probability prediction. So there is no more detailed discussion here.

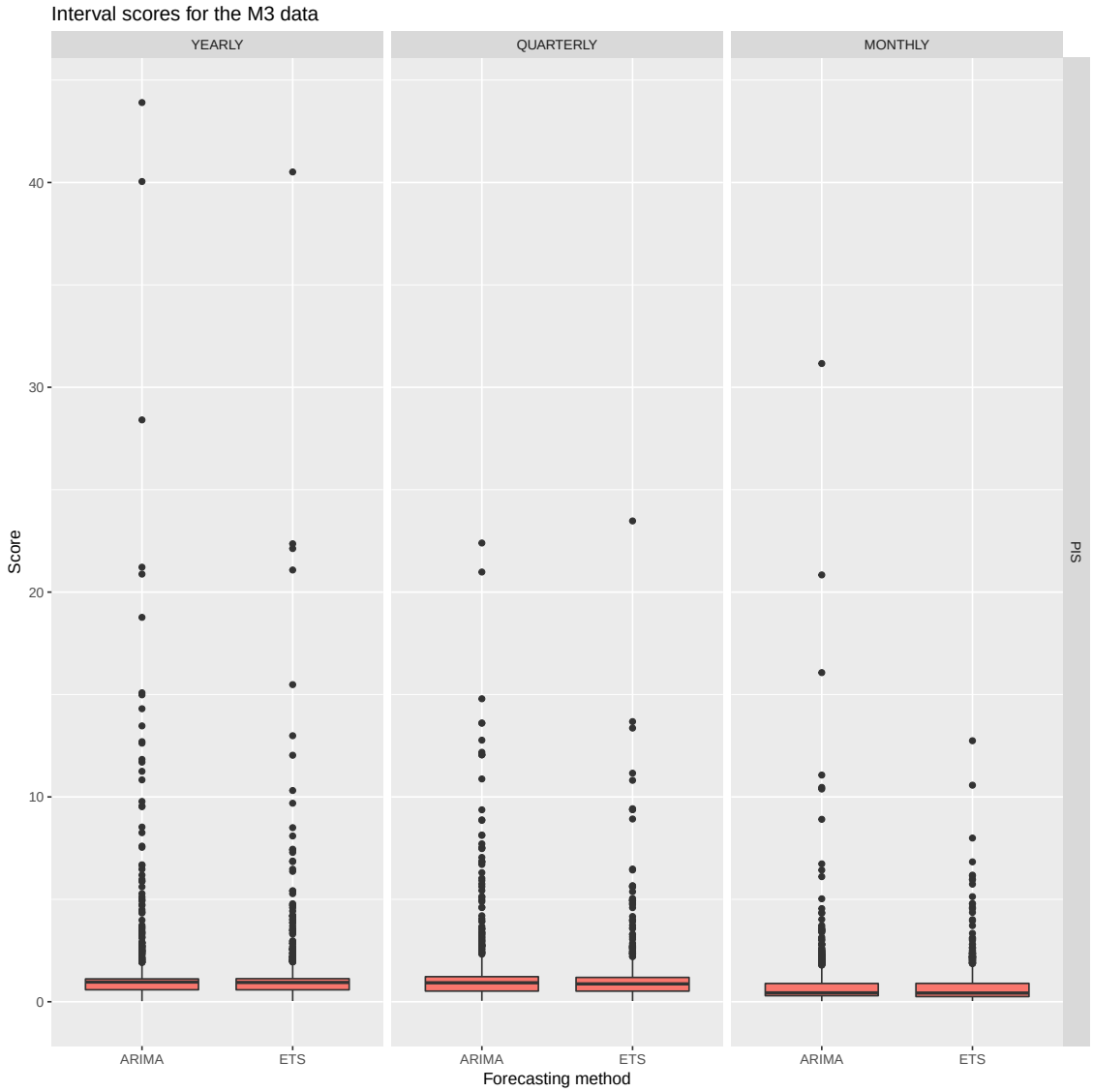
5.2 Interval forecast for the M3 competition data

Like case study one, firstly, we need to use the optimality model selected by automatic programs to predict interval for all time series from M3 datasets. Then use Winkler loss scoring rule to score each prediction result. The difference is that we no longer need to observe the impact of the different prediction interval level on the interval prediction results, so we did not set different predictive interval levels. Instead, all of the interval forecasts are under the 95% prediction interval. Each scoring results for every time series by evaluating forecasts, we also take their mean as the final result.

It should be noted that after getting the interval score, we need to use the scores by assessing the forecasts of random walk model to standardize the results from ARIMA and ETS model, to remove the impact of units from different time series.

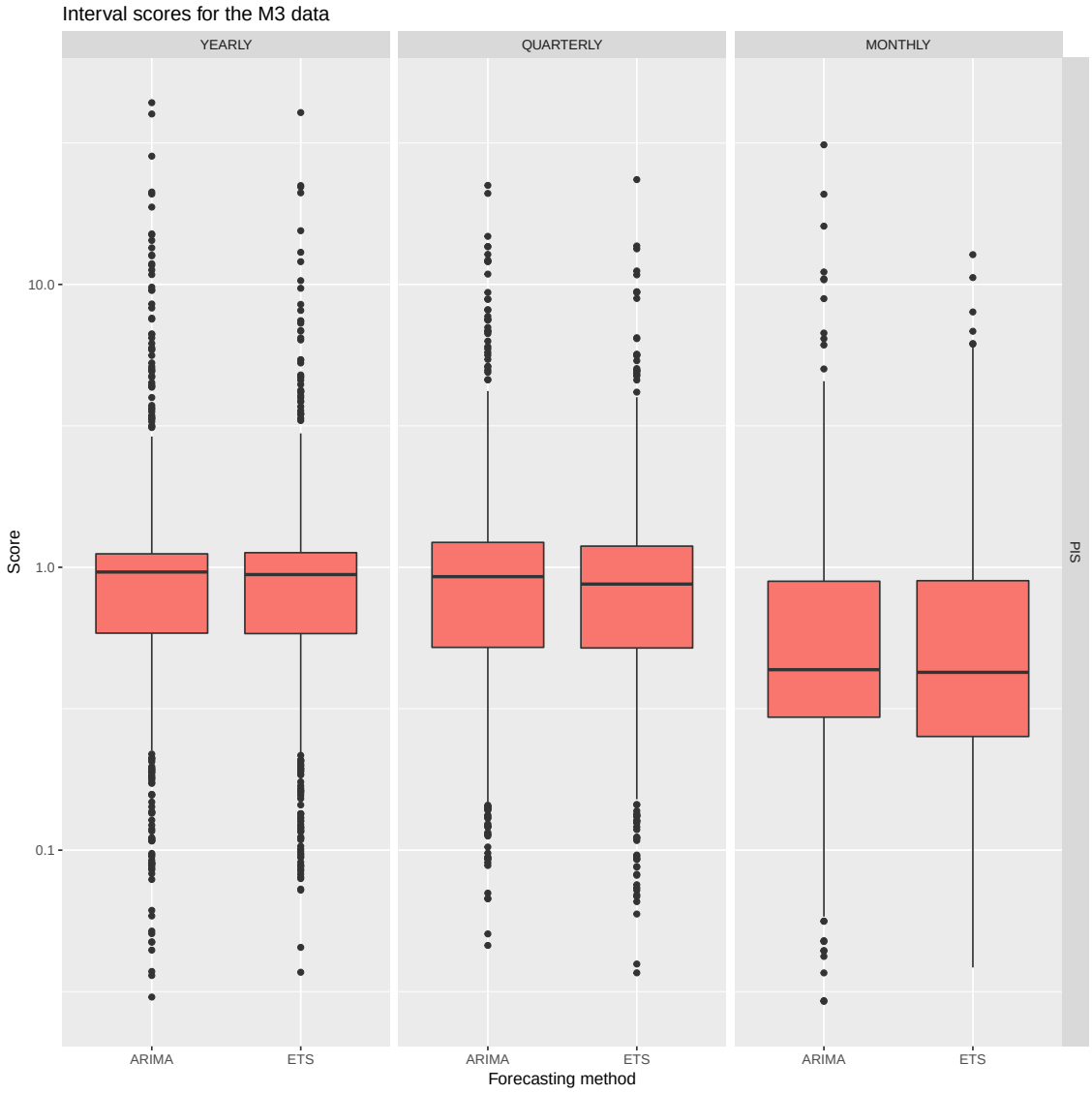
$$Standardized\ interval\ score = \frac{Interval\ scores\ for\ ARIMA\ model\ or\ ETS\ model}{Interval\ scores\ for\ RW\ model}$$

Then use the standardized scores to produce graph to compare the results. Here we use box plot, which an intuitively identify outliers in data sets and determine the degree of dispersion and bias in data sets. Also, use the different box plot to compare the scoring results from time series under different time types. Then the box plot are generated.



According to the whole box plots, we can not see clearly which model has better interval prediction results. However, it can be seen that quite a number of outliers are displayed over 95th percentile line. This shows that comparing the standard normal distribution, the score distribution shows that the tail is too heavy, and the degree of freedom is small. Because the outliers are concentrated on one side of the larger value, the distribution appears right-biased. The reason for this result is that the scores based on Winkler loss scoring rule formula are always greater than or equal to the difference of upper and lower endpoints. ALsoThe scores of the ETS model are lower than those of the ARIMA model at 5th, 25th, 50th, 75th and 95th percentiles, whether they are from monthly data or quarterly data for yearly data.

In order to observe the results more clearly, we produce the box plots on the log scale.



This is the result boxplot of interval scores on the log scale. The boxplots can show 5th, 25th, 50th, 75th and 95th percentiles of central prediction interval width. And for the yearly and quarterly time series the interval scores are similar by different model, only the line at the 50th percentile for ETS model shows a bit lower than that for ARIMA model. So we consider that ETS model has better prediction proformance to make interval forecasts for the yearly and quarterly time series. For monthly data, the score results show a great difference. The line at 25th percentile of ETS model is clearly lower than that of ARIMA model. And the distance between the upper and lower limits of ETS model is obviously greater than ARIMA. This shows that for the monthly time series ETS score has great

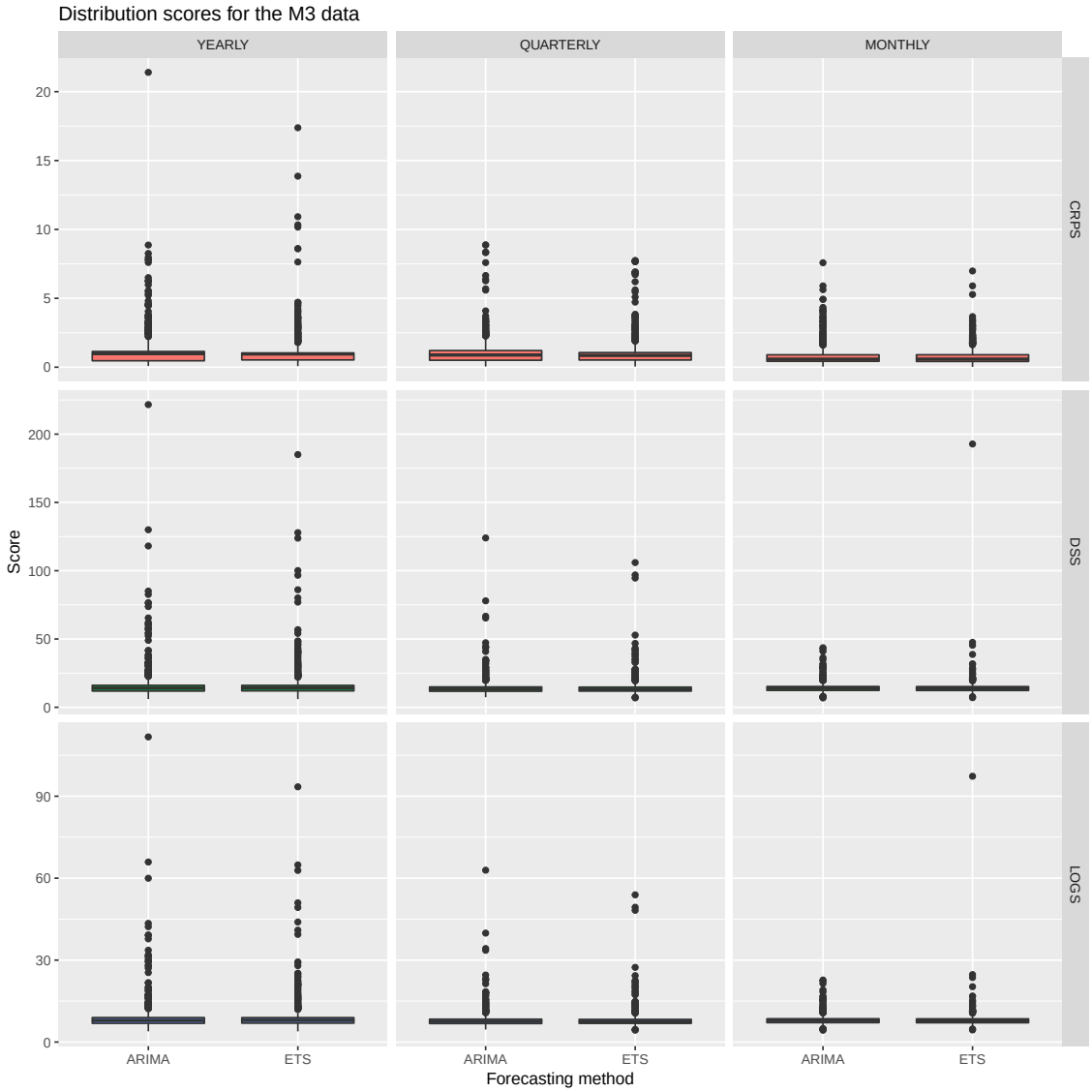
volatility and is not very concentrated, so the prediction result of ETS model is not as good as ARIMA model. The forecasts by ARIMA model have sharper sharpness and the calibration is more accurate,

5.3 Probabilistic forecasts for the M3 competition data

In this part, we make the probabilistic forecasts of each time series by using the same time series models as before. However, the scoring rules will use the distribution scoring rules. It is important to note that, as previously discussed in the second chapter, although the logarithmic score and Dawid-Sebastianti score will be directly transfrom the data so that the results should not standardized. But, when the continuous ranked probability score is used, the scoring results need to be standardized. The standardization method is the same as the method to deal with the interval prediction.

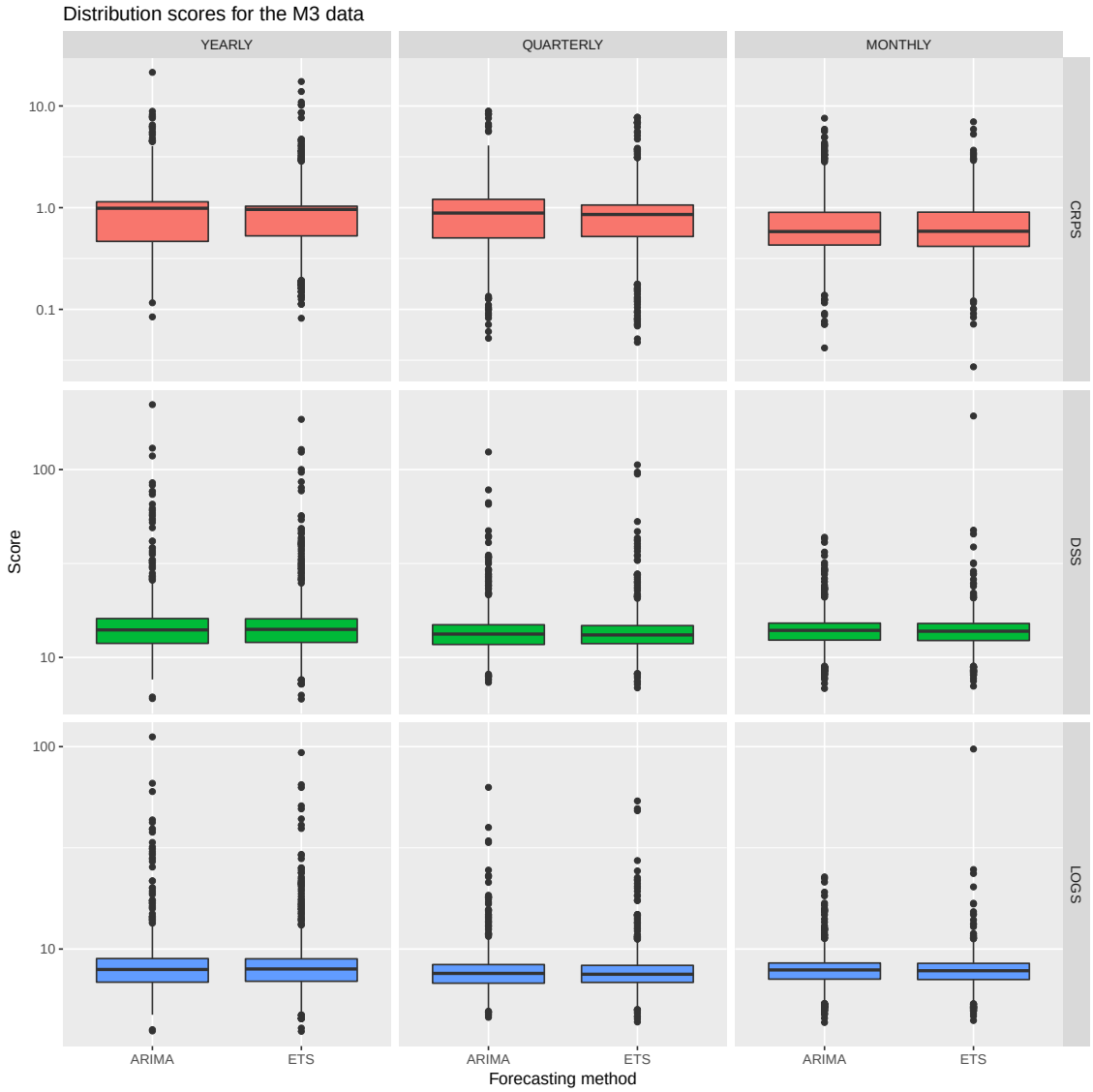
$$Standardized\ CRPS\ score = \frac{CRPS\ scores\ for\ ARIMA\ model\ or\ ETS\ model}{CRPS\ scores\ for\ RW\ model}$$

After the probabilistic forecasts from each time series has been scored by three distribution scoring rules, and the scores has been taken the average, we get the final scoring results. Certainly, the results by using the CRPS scoring rules should be standardized. Using the same way of the interval score, we get the box plot with different scoring rules under different time types. The original image is shown next page.



According to the information of these box plots, we are hard to distinguish which model is better for making probabilistic forecasting under different time types of time series. But these plots are still clearly display that the most outliers are concentrated over the 95th percentile. This shows that the tail of the score distribution is heavy and the distribution is right-skewed.

In order to display the results more clearly, we have also processed this figure, to produce the boxplots on log scale.



According to this graphs, where CRPS scoring rules are used, the prediction scores for all time types of time series show that the ETS model has better prediction sharpness, because the distance between his upper and lower limits, and distance of the 25th percentile and 75th percentile lines are all small than the distances of the ARIMA model. Although the median lines from this two model are almost equal, we still think ETS models have better predictive performance by using CRPS scoring rules.

For score results by using LogS and DSS scoring rules, although the differences shown in the box plots are not particularly large, we can still see that the score distribution of the ETS model is more concentrated. It indicates that the sharpness of probabilistic prediction of ETS model is more sharp. Therefore, the score results by using LogS and DSS scoring

rules also show that ETS models have better probabilistic predictive performance for the time series from M3 datasets.

Chapter 6

Conclusion and future discussion

6.1 Conclusion

As a common means to evaluate interval prediction and probability prediction, scoring rules are widely used. The score of the prediction is scored by evaluating the sharpness of the prediction result and the calibration. This article presents two different types of scores, interval scores and distribution scores. And detailed introduction of four different scoring rules, Winkler loss score, Logarithmic score, Continuous Ranked Probability Score and Dawid-Sebastianti score. In the case study, the different time series models are used to make interval prediction and probabilistic prediction for different types of time series data. After scoring all the prediction results using the corresponding evaluation rules, the score results are compared and evaluated separately.

For financial data, because of its characteristics, the GARCH model can more accurately analyze and predict the financial time series, which has the phenomenon of volatility clusters. Therefore, the GARCH model has good performance both in interval prediction and probabilistic prediction. Of course, the prediction effect of the GARCH model is not very ideal under the high prediction interval level. For other types of time series data, different models have different predictive performance. In the case study, the higher-quality interval predictions can be given by ARIMA model, while ETS models perform better in probabilistic predictions. But no matter what kind of model you use to

make predictions, using appropriate scoring rules can intuitively evaluate the interval or probabilistic prediction results.

6.2 Future discussion

For probabilistic prediction and interval prediction, univariate time series are mainly used for analysis and prediction. However, in many cases, the development of an event is not only influenced by its own changes, but other events also affect the observed events. Therefore, it is important to study the multivariate time series model and use this model for interval and probability prediction. At the same time, we need to study new scoring rules for this situation to evaluate the accuracy of the prediction.

Bibliography

Akaike, H (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control* **19**(6), 716–723.

Brown, RG (1959). *Statistical Forecasting for Inventory Control*. McGraw/Hill.

Cervera, JL and J Muñoz (1996). Proper Scoring Rules for Fractiles. *Bayesian Statistics 5, eds*, 513–520.

Dawid, AP and P Sebastiani (1999). Coherent Dispersion Criteria for Optimal Experimental Design. *The Annals of Statistics* **27**, 253–263.

Gelfand, AE and SK Ghosh (1998). Model Choice: A Minimum Posterior Predictive Loss Approach. *Biometrika* **85**, 1–11.

Gneiting, T, F Balabdaoui, and AE Raftery (2007). Probabilistic forecasts, calibration and sharpness. *J. R. Stat. Soc. B* **69**(2), 243–268.

Gneiting, T and M Katzfuss (2014). Probabilistic forecasting. *Annual Review of Statistics and Its Application* **1**(1), 125–151.

Gneiting, T and AE Raftery (2007). Strictly proper scoring rules, prediction, and estimation. *J. Am. Stat. Assoc* **102**(477), 359–378.

Good, IJ (1952). Rational decisions. *J. R. Stat. Soc. B* **14**, 107–114.

Holt, CC (1957). *Forecasting Trends and Seasonals by Exponentially Weighted Averages*. Tech. rep. Carnegie Institute of Technology. Pittsburgh%20Office%20of%20Naval%20Research%20memorandum%20no.%2052.

Hyndman, RJ (2013). *The difference between prediction intervals and confidence intervals*. https://robjhyndman.com/hyndsight/intervals/.

Hyndman, RJ (2018a). *Data from the M-Competitions*. https://CRAN.R-project.org/package=Mcomp. R package version 2.7.

Hyndman, RJ (2018b). *forecast: Forecasting functions for time series and linear models*. https://CRAN.R-project.org/package=forecast. R package version 8.3.

Hyndman, RJ and G Athanasopoulos (2018). *Forecasting: principles and practice. 2nd ed*. https://OTexts.org/fpp2/. Melbourne, Australia: OTexts.

Hyndman, RJ and Y Khandakar (2008). Automatic Time Series Forecasting: The Forecast Package for R. *Decision Analysis* **27**(1), 1–22.

Jordan, A, F Krueger, and S Lerch (2017). *Scoring Rules for Parametric and Simulated Distribution Forecasts*. https://CRAN.R-project.org/package=scoringRules. R package version 0.9.4.

Laud, PW and JG Ibrahim (1995). Predictive Model Selection. *Journal of the Royal Statistical Society* **57**, 247–262.

Peirolo, R (2010). Information gain as a score for probabilistic forecasts. *Meteorological Applications* **18**(1), 9–17.

Raftery, AE (2016). Use and communication of probabilistic forecasts. *Statistical Analysis and Data Mining: The ASA Data Science Journal* **9**(6), 397–410.

Robert, E (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica* **50**(4), 987–1007.

Roulston, MS and LA Smith (2002). Evaluating Probabilistic Forecasts Using Information Theory. *Monthly Weather Review* **130**(6), 1653–1660.

Stigler, SM (1975). The transition from point to distribution estimation. *Bull. int. Stat. Inst.* **46**, 332–340.

Winkler, RL (1972). A Decision-Theoretic Approach to Interval Estimation. *Journal of the American Statistical Association* **67**, 187–191.

Winkler, RL (1996). Scoring rules and the evalution of probabilities. *Test* **5**(1), 1–60.

Winters, PR (1960). Forecasting Sales by Exponentially Weighted Moving Averages. *Management Science* **6**, 324–342.

Wuertz, D (2017). *Rmetrics - Autoregressive Conditional Heteroskedastic Modelling*. https://CRAN.R-project.org/package=fGarch. R package version 3042.83.

YahooFinance (2018). *Financial data*. https://au.finance.yahoo.com.