Homework 3

CSPP58001

April 28, 2011

1 Questions

1. Show for any matrix A that $\frac{||\delta x||}{||x||} \leq ||A||\,||A^{-1}||\,\frac{||\delta b||}{||b||}$ where $||.||$represents matrix norm.

2. What are the norm and determinant of an orthonormal matrix? Explain your result intuitively.

3. Show that the eigenvectors of a symmetric matrix are orthogonal.

4. Show that the vector y that minimizes the quadratic form $f(x) = \frac{1}{2}x^T Ax - b^T x + c$ gives a solution to the linear system $Ax = b$ for any symmetric, positive definite matrix A (positive definite means that $x^T Ax > 0$ for all vectors x).

5. Write a solver for a sparse system $Ax = b$ where A is quint-diagonal:

$$\begin{pmatrix}
c & d & e & 0 & 0 & 0 & 0 & 0 & 0 \\
b & c & d & e & 0 & 0 & 0 & 0 & 0 \\
a & b & c & d & e & 0 & 0 & 0 & 0 \\
0 & a & b & c & d & e & 0 & 0 & 0 \\
0 & 0 & a & b & c & d & e & 0 & 0 \\
0 & 0 & 0 & a & b & c & d & e & 0 \\
0 & 0 & 0 & 0 & a & b & c & d & e \\
0 & 0 & 0 & 0 & 0 & a & b & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & a & b & c
\end{pmatrix}$$

   You will be graded on the efficiency of your algorithm, both in terms of operations and memory usage. Include with your code an analysis of the required memory usage and number of floating point operations for an arbtirary matrix of size mxm.

6. Express the running average of homework 1 as a linear transformation (ie a smoothing matrix on an input vector).

   Consider two cases for handling the ends of the input vector

   (a) periodic boundary conditions

   (b) zero-padded boundary conditions For each case, assume a smoothing width of 3 and calculate the determinant, eigenvalues, eigenvectors, and condition number of the matrix (ok to use matlab). Comment intuitively on what each tells you about the linear transformation.

7. Repeat above for a differencing filter – ie one that takes the difference rather than the average of adjacent pairs of points.

8. Consider the following matrix A:

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

   For each of the following be sure to show how you arrived at your answer:

   (a) Write A as the product LU where L is a lower-diagonal and U an upper diagonal matrix.

   (b) Is A symmetric?

   (c) Is A positive definite?

   (d) Is A invertible?

   (e) Use the power iteration method to find the largest eigenvalue of A

   (f) Use any technique to find all of the eigenvalues of A

9. Consider the following nonlinear system of equations:

$$\begin{aligned} 2\sqrt{u_1}u_1 - \sqrt{u_1}u_2 &= 100; \\ -\sqrt{u_2}u_1 + 2\sqrt{u_2}u_2 - \sqrt{u_2}u_3 &= 100; \\ &\vdots \\ -\sqrt{u_n - 1}u_n - 2 + 2\sqrt{u_{n-1}}u_{n-1} - \sqrt{u_{n-1}}u_n &= 100; \\ -\sqrt{u_n}u_{n-1} + 2\sqrt{u_n}u_n &= 100; \end{aligned}$$

with n specified by the user and U is the solution vector $[u_1, u_2, ...u_n]$. Use Newton's method to solve the equation system to obtain 'U', starting with an initial guess of all ones (1,1,..).

In order to solve using Newton method, use a numerical Jacobian $(\partial F/\partial U)$ computed using $(\partial F/\partial U)_j = (F(U + \epsilon e_j) - F(U))/\epsilon$ where j is the column index [1 n], and ϵ is a small perturbation parameter (10^{-6}). Hence, to compute the entire Jacobian matrix, you need to call 'F(U)' n-times and use the above formula to assemble column-by-column. Next, solve the resulting linear system

Jacobian $\delta U = -$residua$l$

using the Gaussian elimination algorithm with partial pivoting implemented in the previous homework.

Tabulate for different values of n (the number of iterations) function evaluations to compute the solution to an accuracy of 1e-8. Also, suggest a more suitable starting guess profile for Newton method based on your final converged solutions.