Numerical Method Homework3

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1 Problem1

From the definition of norm of matrix, $\|A\|$ bounds the "amplifying power" of the matrix: $\|A\| \leq \|A\|\|x\|$. And since $b = Ax$ and $\delta x = A^{-1}\delta b$, we could obtain that

$$\|b\| \leq \|A\|\|x\| \tag{1}$$
$$\|\delta x\| \leq \|A^{-1}\|\|\delta b\| \tag{2}$$

Then from equation (1), we could obtain that

$$\frac{1}{\|x\|} \leq \|A\|\frac{1}{\|b\|} \tag{3}$$

Therefore from both equation (2) and equation (3), we get what we need

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|} \tag{4}$$

2 Problem2

2.1 Norm of an orthonormal matrix

The norm of matrix M could be obtained by following equation

$$\begin{aligned} \|M\| = \sqrt{\text{largest eigenvalue of } M^*M} \\ \textit{where } M^*M = M^TM \end{aligned} \tag{5}$$

Since matrix M is an orthonormal matrix, it has the properties of $M^T = M^{-1}$ and $M^{-1}M = I$. Then $M^*M = M^TM = M^{-1}M = I$. Because the eigenvalue of I is 1, therefore the norm of an orthonormal matrix M is 1 or -1.

2.2 Determinant of an orthonormal matrix

Let's assume that the orthogonal matrix is M. And its transpose is M^T.

According to the properties of transpose of a matrix, we obtain that $det(M) = det(M^T)$. Then from that equation, we get that $det(M)^2 = det(M)det(M) = det(M^T)det(M) = det(MM^T)$. As M is an orthogonal matrix, it has the property that $M^T = M^{-1}$ which entails that $MM^T = I$. Based on all we get, we finally obtain that $det(MM^T) = det(I) = 1$.

Therefore the determinant of an orthonormal matrix is 1 or -1.

3 Problem3

To show that any two eigenvectors of a symmetric matrix are orthogonal, we must show that the dot product of any two eigenvectors is zero. Let's assume that v_1 and v_2 are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 of matrix A. So we have $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Then we can obtain that

$$\lambda_1(v_1 \cdot v_2) = (\lambda_1 v_1) \cdot v_2 = (Av_1) \cdot v_2 \tag{6}$$

According to the properties of dot product of matrix, we have

$$(Av_1) \cdot v_2 = (Av_1)^T v_2 = (v_1^T A^T)v_2 = v_1^T(A^T v_2) = v_1 \cdot (A^T v_2) \tag{7}$$

Since A is a symmetric matrix, which has the property that $A = A^T$, then equation(2) will be

$$v_1 \cdot (A^T v_2) = v_1 \cdot (Av_2) = v_1 \cdot (\lambda_2 v_2) = \lambda_2(v_1 \cdot v_2) \tag{8}$$

Hence

$$\lambda_1(v_1 \cdot v_2) = \lambda_2(v_1 \cdot v_2) => (\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0 \tag{9}$$

Since λ_1 and λ_2 are distinct, we can conclude that $v_1 \cdot v_2 = 0$, which means v_1 and v_2 are orthogonal. Therefore the eigenvectors of a symmetric matrix are orthogonal.

4 Problem4

Let's assume there exits such y minimize the equation like

$$\frac{1}{2}y^T Ay - b^T y + c = a \textit{ where a is the minimum value} \tag{10}$$

And if $Ay = b$ holds, from equation (10) we should have

$$\frac{1}{2}y^T Ay - (Ay)^T y + c = \frac{1}{2}y^T Ay - y^T A^T y + c = a \tag{11}$$

Since A is symmetric, then it has the property of $A = A^T$. Then equation (11) will be

$$\frac{1}{2}y^T Ay - y^T Ay + c = -\frac{1}{2}y^T Ay + c = a \tag{12}$$

Since A is positive definite, $y^T Ay$ is always positive, which means $-\frac{1}{2}y^T Ay$ is always negative. Therefore equation (12) should hold for certain minimized a, and such y must exist which holds the two equations.

5 Problem5

I use *p5.cpp* to implement the solver. Basic idea is that do *Gaussian Elimination* on the matrix A first, then solve the problem according the eliminated matrix. Since we know that the matrix is a quint-diagonal matrix, I decrease the operations and memory usage by just operating on the non-zero elements.

ps. please use gcc to compile the file.

6 Problem6

I use *p6_pbc.m* and *p6_zbc.m* to implement *periodic boundary conditions* and *zero-padded boundary conditions* respectively. From the results I obtain, both methods work fine, but *pbc* is better.

Both methods have a determinant of 0. Determinant shows the measure of volume is multiplied under the transformation, so the result shows that both of them work good. Also the eigenvectors/eigenvalues look good and similar between them. However, *zbc* has an Inf condition number while *pbc* always has a really small one. Condition numbers measure the asymptotically worst case of how much the function can change in proportion to small changes in the argument. And one with a low condition number is said to be well-conditioned, while one with a high condition number is said to be ill-conditioned. In conclusion, *pbc* works better than *zbc*.

ps. The input of both programs must follow the format as *[1 2 3 4 5 6]*

7 Problem7

I use *p7.m* to implement the *pair difference*. The results shows that the matrix has a determinant of 1 which is relevantly high compared with the previous two. Besides, it has a fair condition number which shows that it works fine.

ps. The input of both programs must follow the format as *[1 2 3 4 5 6]*

8 Problem8

8.1 LU

I use *Crout's Method* to solve this problem. Since the upper and lower triangle are all zeros, the L and U should be like

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ 0 & l_{32} & 1 & 0 & 0 \\ 0 & 0 & l_{43} & 1 & 0 \\ 0 & 0 & 0 & l_{54} & 1 \end{pmatrix} \tag{13}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & 0 & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 & 0 \\ 0 & 0 & u_{33} & u_{34} & 0 \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{pmatrix} \tag{14}$$

Then I multiple these two matrices and get the following equations

$$
\begin{aligned}
1 \times u_{11} &= 4 \\
1 \times u_{12} &= 1 \\
l_{21} \times u_{11} &= 1 \\
l_{21} \times u_{12} + 1 \times u_{22} &= 4 \\
1 \times u_{23} &= 1 \\
l_{32} \times u_{22} &= 1 \\
l_{32} \times u_{23} + 1 \times u_{33} &= 4 \\
1 \times u_{34} &= 1 \\
l_{43} \times u_{33} &= 1 \\
l_{43} \times u_{34} + 1 \times u_{44} &= 4 \\
1 \times u_{45} &= 1 \\
l_{54} \times u_{44} &= 1 \\
l_{54} \times u_{45} + 1 \times u_{55} &= 4
\end{aligned}
$$

From these equations, I obtain L and U which are

$$
L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 & 0 \\ 0 & 0 & \frac{15}{56} & 1 & 0 \\ 0 & 0 & 0 & \frac{56}{209} & 1 \end{pmatrix} \tag{15}
$$

$$
U = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 & 0 \\ 0 & 0 & \frac{56}{15} & 1 & 0 \\ 0 & 0 & 0 & \frac{209}{56} & 1 \\ 0 & 0 & 0 & 0 & \frac{780}{209} \end{pmatrix} \tag{16}
$$

8.2 Symmetric

A matrix is symmetric if its transpose and itself are the same ($M = M^T$). In this question, A's transpose A^T equals to A, which shows that A is symmetric.

8.3 Positive Definite

Since A is a real symmetric matrix, it is positive definite if $z^T M z > 0$ for all non-zero vectors z. Therefore I obtain the equation

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \tag{17}$$

After multiplying these three matrices, I obtain the following equation

$$\begin{aligned} 4z_1^2 + 2z_1z_2 + 4z_2^2 + 2z_2z_3 + 4z_3^2 + 2z_3z_4 + 4z_4^2 + 2z_4z_5 + 4z_5^2 \\ = 3z_1^2 + (z_1 + z_2)^2 + 2z_2^2 + (z_2 + z_3)^2 + 2z_3^2 + \\ (z_3 + z_4)^2 + 2z_4^2 + (z_4 + z_5)^2 + 3z_5^2 > 0 \end{aligned} \tag{18}$$

Thus A is positive definite.

8.4 Invertible

Matrix A is invertible if there exists a matrix B such that $AB = BA = I_n$. Then for the matrix in this question, I find a matrix B which meets that $AB = BA = I_n$.

$$B = \begin{pmatrix} 0.2679 & -0.0718 & 0.0192 & -0.0051 & 0.0013 \\ -0.0718 & 0.2872 & -0.0769 & 0.0205 & -0.0051 \\ 0.0192 & -0.0769 & 0.2885 & -0.0769 & 0.0192 \\ -0.0051 & 0.0205 & -0.0769 & 0.2872 & -0.0718 \\ 0.0013 & -0.0051 & 0.0192 & -0.0718 & 0.2679 \end{pmatrix} \tag{19}$$

Therefore A is invertible.

8.5 Largest Eigenvalue

I use *p8_e.m* computing the largest eigenvalue by *Power Iteration Method*. And the result is 5.7312.

8.6 All Eigenvalue

I use *p8_f.m* computing all of the eigenvalues of A by *QR Method*. And the five eigenvalues are $5.7321, 5, 4, 3, 2.2679$.

9 Problem9