Stellar Dynamics

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1 Keplerian Orbits

The Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GMm}{r}$$

gives the equations of motion:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}, \qquad r^2\dot{\phi} = \text{const.} \equiv h \qquad \Rightarrow \qquad \ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}$$

We then change variables to $u = 1/r$ and eliminate the time-dependence of u and ϕ to obtain $u(\phi)$. Firstly, note $\dot{\phi} = hu^2$, and define $u' \equiv \mathrm{d}u/\mathrm{d}\phi$. Then

$$\dot{r} = -\frac{\dot{u}}{u^2} = -h\frac{\dot{u}}{\dot{\phi}} = -hu' \qquad\qquad \ddot{r} = -hu''\dot{\phi} = -h^2u^2u''$$

On substitution we then have

$$-h^2u^2u'' - h^2u^3 = -GMu^2 \qquad \Rightarrow \qquad u'' + u - \frac{GM}{h^2} = 0$$

which if we don't care about orientation, has solution

$$u(\phi) = \frac{GM}{h^2}(1 + e\cos\phi) \qquad \Rightarrow \qquad r(\phi) = \frac{\ell}{1 + e\cos\phi} \qquad\qquad (h^2 = GM\ell)$$

where e is an integration constant (> 0 wlog) which determines the shape of the orbit.

1.1 $e < 1$: Elliptical Orbits

Apsides. There are constant minimum (periapsis) and maximum (apoapsis) distances to the primary mass M. By inspection these are:

$$r_p = \frac{\ell}{1 + e} \qquad\qquad r_a = \frac{\ell}{1 - e}$$

Semi-Major Axis. The semi-major axis a of the ellipse is the average of the apsides:

$$a = \frac{1}{2}\frac{\ell(1 + e) + \ell(1 - e)}{1 - e^2} = \frac{\ell}{1 - e^2} \qquad \Rightarrow \qquad \boxed{r = \frac{a(1 - e^2)}{1 + e\cos\phi}}$$

Energy. At the pericentre $\dot{r} = 0$, so the energy per particle mass E is easy to calculate[1]:

$$E = \frac{1}{2} r_p^2 \dot{\phi}^2 - \frac{GM}{r_p} = \frac{h^2}{2r_p^2} - \frac{GM}{r_p} = \frac{GM\ell}{2\ell^2/(1+e)^2} - \frac{GM}{\ell/(1+e)} = \frac{GM(1+e)}{2\ell}(1+e-2)$$
$$= -\frac{GM}{2}\frac{1-e^2}{\ell} = -\frac{GM}{2a}$$

so we see that the geometric quantities (ℓ, a, e) depend on the orbital quantities (h, E, both h and E) respectively.

Semi-Minor Axis. The semi-minor axis of the orbit, b, is found using Figure 1. We have

$$b = -ae \tan \phi_b \qquad \sqrt{b^2 + a^2 e^2} = \frac{a(1-e^2)}{1 + e\cos\phi_b}$$

$$\Rightarrow \sqrt{b^2 + a^2 e^2} = \frac{a(1-e^2)}{1 - e/\sqrt{1 + b^2/a^2 e^2}} \qquad (\cos\phi_b < 0)$$
$$= \frac{a(1-e^2)}{1 - ae^2/\sqrt{b^2 + a^2 e^2}}$$
$$\Rightarrow \sqrt{b^2 + a^2 e^2} - ae^2 = a(1-e^2) \qquad b^2 + a^2 e^2 = a^2 \qquad \Rightarrow \qquad b = a\sqrt{1-e^2}$$

showing that e represents the eccentricity of the ellipse.

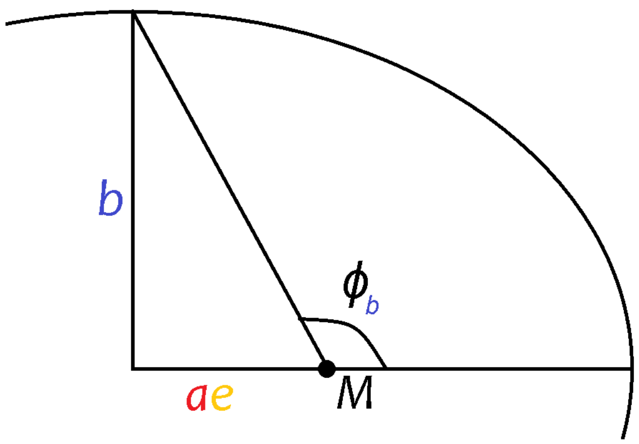

Figure 1 | Geometry of an Elliptical Orbit. A right-angled triangle is formed between the ellipse's centre, the primary, and the vertical, which forms an angle ϕ_b to the horizontal.

Period. The rate at which area of the ellipse is being swept out is constant ($\mathfrak{K}2\mathfrak{L}$):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \dot{\phi} = \frac{h}{2}$$

The area of an ellipse is $\pi ab = \pi a^2 \sqrt{1-e^2}$. The period of the orbit is thus

$$T = \frac{\pi a^2 \sqrt{1-e^2}}{h/2} = \frac{2\pi a^2 \sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} = \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

where we have used $h = \sqrt{GM\ell} = \sqrt{GMa(1-e^2)}$. The proportionality $T^2 \propto a^3$ is $\mathfrak{K}3\mathfrak{L}$.

[1]The calculation at apocentre gives the same result.

1.2 $e > 1$: Hyperbolic Orbits

Asymptotes. By inspection, $r \to \infty$ as $\phi \to \phi_\infty$, where

$$\phi_\infty = \arccos\left(-\frac{1}{e}\right)$$

Energy. Most of the above derivation for an elliptical orbit still applies, however we can't identify $a = \ell/(1-e^2)$ as a physical distance because it is negative for $e > 1$. If we just define a like that anyway, then as before

$$E = -\frac{GM}{2a} > 0$$

as now $a < 0$. At $r = \infty$, this energy will be equal to $\frac{1}{2}v_\infty^2$.

Impact Parameter. If gravity were switched off and the particle simply flew past the primary, the impact parameter b is what its closest approach would be. The angular momentum is thus $h = bv_\infty$. Using $E = \frac{1}{2}v_\infty^2$, we have

$$b^2 = \frac{h^2}{2E} = \frac{GM\ell}{-GM/a} = -\ell a = -a^2(1-e^2) \qquad \Rightarrow \qquad b = -a\sqrt{e^2-1}$$

which sort of justifies the use of the letter b for the impact parameter.

1.3 Binary Orbits

Suppose we no longer fix the larger mass in place and instead have two masses M_1 and M_2 at locations $\mathbf{r_1}$ and $\mathbf{r_2}$. Define the vector $\mathbf{d} = \mathbf{r_1} - \mathbf{r_2}$. Then

$$\ddot{\mathbf{d}} = \ddot{\mathbf{r}}_\mathbf{1} - \ddot{\mathbf{r}}_\mathbf{1} = -\frac{G(M_1+M_2)}{d^3}\mathbf{d}$$

which is the same equation for $\ddot{\mathbf{r}}$ in the fixed-mass potential above, but with a primary M_1+M_2. The vector $\mathbf{d}$ traces out an ellipse in the same way that $\mathbf{r}$ traces out an ellipse.

Period. The period is thus

$$T = \frac{2\pi}{\sqrt{G(M_1+M_2)}}a^{3/2}$$

where a is the semi-major axis of the ellipse traced out by $\mathbf{d}$; also $a = (d_{\text{min}} + d_{\text{max}})/2$.

In the centre-of-mass frame, $M_1\mathbf{r_1} + M_2\mathbf{r_2} = \mathbf{0}$. Together with $\mathbf{d} = \mathbf{r_1} - \mathbf{r_2}$ the positions of the two masses $\mathbf{r_1}$ and $\mathbf{r_2}$ can thus be recovered from $\mathbf{d}$:

$$\mathbf{r_1} = \frac{M_2}{M_1+M_2}\mathbf{d} \qquad\qquad \mathbf{r_2} = -\frac{M_1}{M_1+M_2}\mathbf{d}$$

Angular Momentum. About the origin, the angular momentum vector (not per unit mass) is

$$\begin{aligned}\mathbf{H} &= M_1\mathbf{r_1} \times \dot{\mathbf{r}}_\mathbf{1} + M_2\mathbf{r_2} \times \dot{\mathbf{r}}_\mathbf{2} = \left[M_1\left(\frac{M_2}{M_1+M_2}\right)^2 + M_2\left(-\frac{M_1}{M_1+M_2}\right)^2\right]\mathbf{d} \times \dot{\mathbf{d}} \\ &= \mu\mathbf{d} \times \dot{\mathbf{d}}\end{aligned}$$

$$\text{where } \mu = \frac{M_1M_2^2 + M_2M_1^2}{(M_1+M_2)^2} = \frac{M_1M_2}{M_1+M_2}$$

There is thus a dynamical equivalence between two masses M_1 and M_2 orbiting each other, and a mass μ orbiting a fixed mass $M_1 + M_2$.

2 Poisson's Equation

$$\nabla^2\Phi = +4\pi G\rho$$

2.1 Spherical Symmetry

For $\Phi(\mathbf{r}) = \Phi(r)$, Poisson's Equation simplifies to

$$\frac{1}{r}(r\Phi)'' = 4\pi G\rho$$

which is simply integrated twice for $\Phi(r)$ if $\rho(r)$ is known. One boundary condition is that $\Phi(\infty) = 0$ conventionally; another is that $\Phi(0)$ is finite, unless there is a point mass there. Similarly, when solving in multiple regions, Φ and $\nabla\Phi$ should be continuous except in areas of infinite spatial density (e.g. a line or surface density), in which case Gauss' Law can be used.

Suppose $\rho(r)$ follows a power law[2]:

$$\rho(r) = \rho_0\left(\frac{a}{r}\right)^\alpha \qquad \Rightarrow \qquad \begin{aligned} (r\Phi)'' &= \rho_0 a^\alpha r^{1-\alpha} \\ r\Phi &= \frac{\rho_0 a^\alpha}{(2-\alpha)(3-\alpha)} r^{3-\alpha} + Ar + B \\ \Phi(r) &= -\frac{\rho_0 a^\alpha}{(\alpha-2)(3-\alpha)} r^{2-\alpha} \end{aligned}$$

where A and B have been set to 0. For the total mass within small r to be finite, we require $\alpha < 3$, and for $\lim_{r\to\infty}\Phi = 0$ we require $\alpha > 2$, so $2 < \alpha < 3$.

The potential due to a spherical shell, of radius r' and mass δm, is

$$\begin{aligned} \Phi(r) &= -G\int \frac{\delta m\,\mathrm{d}\Omega}{4\pi}\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \\ &= -\frac{G\delta m}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} \\ &= -\frac{G\delta m}{2}\int_{|r-r'|}^{r+r'}\frac{\mathrm{d}R}{rr'} \qquad\qquad (R = \sqrt{r^2+r'^2-2rr'\cos\theta}) \\ &= \begin{cases} -G\delta m/r & \text{if } r > r' \\ -G\delta m/r' & \text{if } r < r' \end{cases} \end{aligned}$$

From this we see that within a spherical shell the potential is non-zero but constant, so there is no force. Also, if we are outside a spherically-symmetric mass, the mass can be though of as lots of shells going all the way down to the origin. The potential is additive, so the shells will all give a $1/r$ dependence, and the overall potential will simply be $-GM/r$, the same as if it were all concentrated at the centre.

2.2 Axisymmetric Distributions

If there is now θ-dependence, so $\Phi(\mathbf{r}) = \Phi(r,\theta)$, Poisson's Equation is instead:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 4\pi G\rho$$

[2]The usefulness of such models is always limited by the resulting divergence as $r \to 0$.

Outside the matter distribution, where $\rho = 0$, we have then Laplace's Equation. Setting $\Phi(r,\theta) = \mathcal{R}(r)\Theta(\theta)$, and separating, we obtain

$$\underbrace{\frac{r}{\mathcal{R}}(r\mathcal{R})''}_{n(n+1)} + \underbrace{\frac{1}{\Theta\sin\theta}(\sin\theta\,\Theta')'}_{-n(n+1)} = 0$$

where $n(n+1)$ is a separation constant. The most general finite solution to the equation for $\Theta(\theta)$ is

$$\Theta(\theta) = A_n P_n(\cos\theta)$$

where n is an integer and P_n is the nth Legendre Polynomial. The most general solution to the equation for $\mathcal{R}(r)$ is

$$\mathcal{R}(r) = B_n r^n + \frac{C_n}{r^{n+1}}$$

so most generally

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1}\right) P_n(\cos\theta)$$

Depending on whether we are considering the region including $r = 0$ or $r = \infty$, one of these terms can be thrown away. Furthermore, density distributions will almost always be symmetrical about $\theta = \pi/2$, so only even n terms are non-zero by symmetry. To find the remaining constants, often there is a special radius or angle (such as the vertical axis $\theta = 0$) where the potential can be deduced by other reasons; the potential at all points in consideration can then be deduced, though often only the first couple of terms are important.

2.3 Cylindrical Distributions – Thin Disks

We now change variables from (r,θ) to $(R = r\cos\theta, z = r\sin\theta)$. In these coordinates, Laplace's Equation becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0$$

This again separates $\Phi(R,z) = \mathfrak{R}(R)Z(z)$ into

$$\underbrace{\frac{1}{R\mathfrak{R}}(R\mathfrak{R}')'}_{-k^2} + \underbrace{\frac{1}{Z}Z''}_{k^2} = 0$$

For an infinitely thin disk, the solution for $Z(z)$ is therefore $Z(z) = A_k \exp(-k|z|)$ to ensure Φ's finity at large z; the discontinuity is allowed by the infinite density in the disk. Meanwhile, the equation for $\mathfrak{R}(R)$ is Bessel's equation order $\nu = 0$. The finite solution is then $\mathfrak{R}(R) = B_k J_0(kR)$. Overall then,

$$\Phi(R,z) = \int_0^\infty \mathrm{d}k\, \tilde{A}(k) J_0(kR) \exp(-|k|z)$$

for some coefficients $\tilde{A}(k)\,\mathrm{d}k$. To find out what $\tilde{A}(k)$ is, apply Gauss' Theorem to a pillbox area S enclosing a portion of the disk, with surface density $\Sigma(R,z)$:

$$4\pi G\Sigma S = S\left(\left.\frac{\partial\Phi}{\partial z}\right|_{z=0+} - \left.\frac{\partial\Phi}{\partial z}\right|_{z=0-}\right) = 2S\left.\frac{\partial\Phi}{\partial z}\right|_{0+} \qquad \text{(by symmetry)}$$

$$= -2\int_0^\infty \mathrm{d}k\,\tilde{A}(k)J_0(kR)k \equiv -2A(R)$$

where $A(R)$ is the inverse Hankel transform of $\tilde{A}(k)$, order $\nu = 0$. Taking the Hankel transform of both sides:

$$\tilde{A}(k) = -2\pi G\tilde{\Sigma}(k) = -2\pi G\int_0^\infty \Sigma(R)J_0(kR)R\,\mathrm{d}R$$

giving finally

$$\Phi(R,z) = -2\pi G\int_0^\infty \left(\int_0^\infty \Sigma(s)J_0(ks)s\,\mathrm{d}s\right)J_0(kR)e^{-|k|z}\,\mathrm{d}k$$

The circular velocity $v_c(R)$ is given by

$$\frac{v_c^2}{R} = \left.\frac{\partial\Phi}{\partial R}\right|_{z=0} \qquad \Rightarrow \qquad v_c = \left[2\pi GR\underbrace{\int_0^\infty \underbrace{\left(\int_0^\infty \Sigma(s)J_0(ks)s\,\mathrm{d}s\right)}_{\nu=0\text{ transform}} J_1(kR)k\,\mathrm{d}k}_{\nu=1\text{ inverse transform}}\right]^{1/2}$$

2.3.1 Mestel Disk: $\Sigma = \Sigma_0 R_0/R$

The total mass contained within such a disk is

$$M(R) = \int_0^R \frac{\Sigma_0 R_0}{R'}2\pi R'\,\mathrm{d}R' = 2\pi\Sigma_0 R_0 R$$

which diverges. We further find

$$\tilde{A}(k) = -2\pi G\Sigma_0 R_0\int_0^\infty J_0(kR)\,\mathrm{d}R = -\frac{2\pi G\Sigma_0 R_0}{k} \qquad \left(\int_0^\infty J_\nu(x)\,\mathrm{d}x = 1\right)$$

$$\Phi(R,z) = -2\pi G\Sigma_0 R_0\int_0^\infty \frac{1}{k}J_0(kR)e^{-k|z|}\,\mathrm{d}k$$

$$v_c = \sqrt{2\pi G\Sigma_0 R_0 R\int_0^\infty J_1(kR)\,\mathrm{d}k} = \sqrt{2\pi G\Sigma_0 R_0} = \sqrt{\frac{GM(R)}{R}}$$

so the radial velocity is actually constant, and it has a Keplerian form despite the non-sphericity of the potential.

2.3.2 Exponential Disk: $\Sigma = \Sigma_0 e^{-R/R_d}$

This is a better fit to what is observed. The disk has a finite total mass:

$$M = \int_0^\infty 2\pi R\Sigma_0 e^{-R/R_d}\,\mathrm{d}R = 2\pi\Sigma_0 R_d^2\Gamma(2) = 2\pi\Sigma_0 R_d^2$$

using the Γ function. Furthermore,

$$\tilde{A}(k) = -2\pi G\Sigma_0\int_0^\infty e^{-R/R_d}J_0(kR)R\,\mathrm{d}R = -2\pi G\Sigma_0\frac{R_d^2}{(1+k^2R_d^2)^{3/2}}$$

apparently. $\Phi(R,z)$ and $v_c(R)$ are then in unfortunate terms of modified Bessel functions.

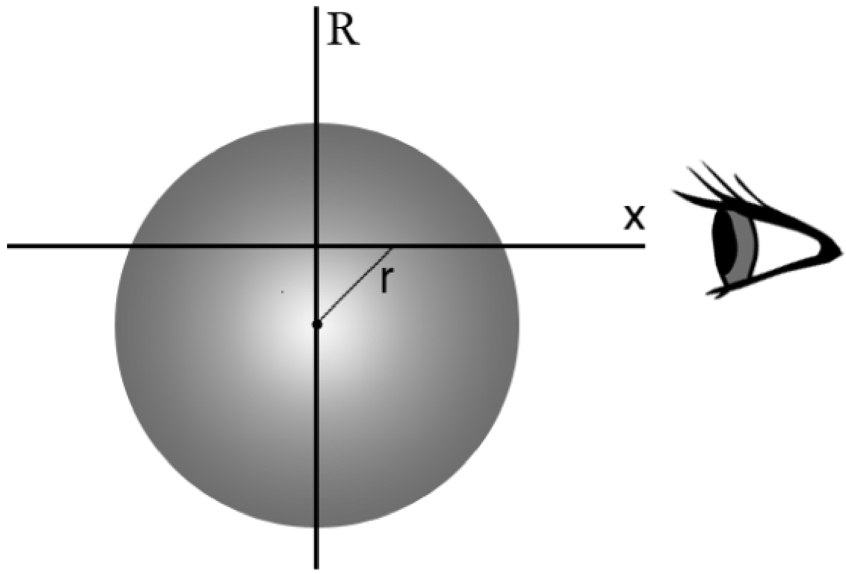

Figure 2 | **Surface Brightness Parameters.** R is the 2D distance on the night sky from the centre as observed by an observer at $x = \infty$; r is the physical 3D distance from the centre.

2.4 Surface Brightness Profiles

For a spherically symmetric star cluster or galaxy, the varying concentration of stars (assumed spherically symmetric) will correspond to a surface brightness profile $I(R)$ – a variation in brightness on the sky as one looks a distance R away from the centre. Assuming the luminosity density $j(r)$ is proportional to the mass density, we can find a $j(r)$ which corresponds to the observed $I(R)$, multiply by a constant to give $\rho(r)$, and then solve $\boldsymbol{\nabla}^2\Phi = 4\pi G\rho$ to find $\Phi(r)$.

We first find the relation between $I(R)$ and $j(r)$, using Figure 2:

$$I(R) = 2\int_0^\infty j(x(r,R))\,\mathrm{d}x = 2\int_R^\infty j(r)\frac{r\,\mathrm{d}r}{\sqrt{r^2 - R^2}}$$

The inverse of this is something like[3]

$$j(r) = -\frac{2}{\pi r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty I(R)\frac{R\,\mathrm{d}R}{\sqrt{R^2 - r^2}}$$

but somehow I doubt it needs memorising.

A profile with an analytic inversion is the modified Hubble profile:

$$I(R) = 2j_0 a\left(1 + \frac{R^2}{a^2}\right)^{-1} \quad\Rightarrow\quad j(r) = 2j_0\left[1 + \frac{r^2}{a^2}\right]^{-3/2} \quad\Rightarrow\quad \rho(r) = \rho_0\left[1 + \frac{r^2}{a^2}\right]^{-3/2}$$

In this model the total mass diverges logarithmically. Setting this formula for $\rho(r)$ equal to $\boldsymbol{\nabla}^2\Phi/4\pi G$ gives, on integrating twice,

$$\Phi(r) = 4\pi G\rho_0 a^3\frac{\ln\left(\sqrt{r^2 + a^2} + r\right)}{r}$$

[3]I've seen some wayward factors of 2 around here; they don't really matter in deriving the density profile as there is an unknown proportionality between luminosity density and mass density anyway.

3 Motion in Non-Keplerian Potentials

3.1 Radial Potentials

For a non-inverse-square-but-still-radial force law $\Phi = \Phi(r) \not\propto -1/r$, orbits generally precess.

$$E = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{2r^2} + \Phi(r) \qquad \Rightarrow \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2[E - \Phi(r)] - \frac{h^2}{r^2}}$$

The time period for one oscillation will be double the time taken to go from pericentre to apocentre, wherever they are on a given orbit. At these points, $\dot{r} = 0$ by definition, so they will be the two[4] radii r_p and r_a at which the above radicand is 0.

$$T_r = 2\int_{r_p}^{r_a} \frac{\mathrm{d}r}{\sqrt{2[E - \Phi(r)] - \frac{h^2}{r^2}}}$$

During this time, ϕ does not increase by 2π for general $\Phi(r)$, but by

$$\Delta\phi = \oint \mathrm{d}\phi = \oint \frac{\mathrm{d}\phi/\mathrm{d}t}{\mathrm{d}r/\mathrm{d}t}\,\mathrm{d}r = 2h\int_{r_p}^{r_a} \frac{\mathrm{d}r}{r^2\sqrt{2[E - \Phi(r)] - \frac{h^2}{r^2}}} \qquad \left(\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{h}{r^2}\right)$$

If $\Delta\phi/2\pi \in \mathbb{Q}$, a whole number of radial oscillations will fit into a (different) whole number of orbits of the primary, and the path will be closed. The rate of precession of the orbit is thus

$$\Omega_p \equiv \frac{\Delta\phi - 2\pi}{T_r}$$

If positive, the precession is prograde; negative, retrograde.

3.2 Axisymmetric Potentials

Consider now $\Phi = \Phi(R, z)$. $\mathfrak{N2L}$ is then

$$\ddot{R} - R\dot{\phi}^2 = \ddot{R} - \frac{h_z^2}{R^3} = -\frac{\partial\Phi}{\partial R} \qquad \Rightarrow \ddot{R} = -\frac{\partial\Phi_{\text{eff}}}{\partial R} \qquad \ddot{z} = -\frac{\partial\Phi}{\partial z}$$

$$\Phi_{\text{eff}} \equiv \Phi + \frac{h_z^2}{2R^2}$$

where we define the effective potential Φ_{eff} in terms of the z-component of the angular momentum vector h_z. We assume that this potential is symmetric about $z = 0$. Expanding about the stable point $C : (R, z) = (R_c, 0)$ where R_c is the circular radius ($\partial\Phi_{\text{eff}}/\partial R\,\big|_C = 0$), we obtain

$$\begin{aligned}\Phi_{\text{eff}}(R_c + \epsilon, z) &\approx \Phi_{\text{eff}}(C) + \epsilon \underbrace{\left.\frac{\partial\Phi_{\text{eff}}}{\partial R}\right|_C}_{0} + z \underbrace{\left.\frac{\partial\Phi_{\text{eff}}}{\partial z}\right|_C}_{0} + \frac{1}{2}\epsilon^2 \left.\frac{\partial^2\Phi_{\text{eff}}}{\partial R^2}\right|_C + \epsilon z \underbrace{\left.\frac{\partial^2\Phi_{\text{eff}}}{\partial R\partial z}\right|_C}_{0} + \frac{1}{2}z^2 \left.\frac{\partial^2\Phi_{\text{eff}}}{\partial z^2}\right|_C \\ &= \Phi_{\text{eff}}(C) + \frac{1}{2}\epsilon^2 \left.\frac{\partial^2\Phi_{\text{eff}}}{\partial R^2}\right|_C + \frac{1}{2}z^2 \left.\frac{\partial^2\Phi_{\text{eff}}}{\partial z^2}\right|_C\end{aligned}$$

[4] See Example Sheet Question 2.4

Differentiating and letting $R = R_c + \epsilon$, $\mathfrak{N}2\mathfrak{L}$ gives

$$\ddot{\epsilon} = -\epsilon \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right|_C = -\kappa^2 \epsilon \qquad\qquad \ddot{z} = -z \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right|_C = -\nu^2 z$$

We see that the particle makes both radial and vertical oscillations with frequencies κ and ν. If $\kappa \neq \Omega$, the orbit exhibits radial precession with a frequency $\Omega_p = \Omega - \kappa$; if $\nu \neq \Omega$, the orbit exhibits *nodal* precession[5] with a frequency $\Omega_z = \Omega - \nu$.

More can be deduced about κ:

$$\begin{aligned} \kappa^2 &= \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right|_C = \left. \frac{\partial^2 \Phi}{\partial R^2} \right|_C + \left. \frac{\partial^2}{\partial R^2} \left(\frac{h_z^2}{2R^2} \right) \right|_C = -f'(R_c) + 3\frac{h_z^2}{R_c^4} \\ &= 3\Omega^2 - f'(R_c) \end{aligned}$$

where $f(R)$ is the radial force and Ω is the frequency of a circular orbit. The stability of these oscillations is discussed in the next section.

3.3 Excursions and Epicycles

We now reconsider the same problem in a different way, with $z = 0$. Consider a circular orbit with radius $r(t) = r_c$ and angular velocity $\dot{\phi}(t) = \Omega$, and perturb it slightly while conserving angular momentum, so that now $r(t) = r_c + \epsilon(t)$ and $\dot{\phi}(t) = \Omega + \omega(t)$:

$$h = r_c^2 \Omega = (r_c + \epsilon)^2 (\Omega + \omega) \approx r_c^2 \Omega + 2 r_c \epsilon \Omega + r_c^2 \omega \qquad \Rightarrow \qquad r_c \omega = -2\epsilon\Omega$$

The general[6] radial force law $\ddot{r} - r\dot{\phi}^2 = f(r)$ becomes:

$$\begin{aligned} \ddot{r_c} + \ddot{\epsilon} - (r_c + \epsilon)(\Omega + \omega)^2 &= f(r_c + \epsilon) \\ \ddot{\epsilon} - r_c \Omega^2 - \epsilon \Omega^2 - 2 \underbrace{r_c \omega}_{-2\epsilon\Omega} \Omega &\approx \underbrace{f(r_c)}_{-r_c\Omega^2} + \epsilon f'(r_c) \\ \ddot{\epsilon} + \big(3\Omega^2 - f'(r_c)\big)\epsilon &= 0 \end{aligned}$$

So the radial excursions $\epsilon(t)$ are simple-harmonic[7], with frequency $\kappa = \sqrt{3\Omega^2 - f'(r_c)}$, as before. The condition for stability is that $\kappa^2 > 0$:

$$0 < 3\Omega^2 - f'(r_c) = -\frac{3f(r_c)}{r} - f'(r_c) = -\frac{1}{r_c^3}\big(3r_c^2 f(r_c) + r_c^3 f'(r_c)\big) = -\frac{1}{r_c^3} \frac{\mathrm{d}}{\mathrm{d}r}\left[r^3 f(r)\right]\Big|_{r_c}$$

so we require $r^3 f(r)$ to have a negative slope at $r = r_c$.

3.3.1 Epicycles

Relative to a frame which orbits the primary on the ordinary circular path $r = r_c$, $\dot{\phi} = \Omega$, the particle has a radial coordinate $\epsilon(t)$ and a tangential coordinate y whose velocity is $\dot{y} = r_c\omega = -2\epsilon\Omega$. We have seen that $\epsilon(t)$ obeys $\ddot{\epsilon} + \kappa^2 \epsilon = 0$, so we can write

$$\epsilon(t) = A \cos \kappa t \qquad \Rightarrow \qquad \dot{y} = -2\Omega A \cos \kappa t \qquad \Rightarrow \qquad y(t) = -\frac{2\Omega A}{\kappa} \sin \kappa t$$

The coordinates $(\epsilon(t), y(t))$ thus trace out a retrograde ellipse with frequency $-\kappa$, which in general is elongated tangentially[8].

[5] An orbital node being a point where the orbit intersects the $z = 0$ plane

[6] e.g. for Kepler, $f(r) = -GM/r^2$

[7] Provided the force law isn't too steep

[8] For Kepler, $\kappa = \Omega$ so the semi-major axis is simply double the semi-minor.

4 Dynamics in the Milky Way

4.1 Velocity Profile

We now wish to know the circular velocity profile $v(R)$ within our own Galaxy, from redshift observations of gas, as this enables estimates of $\Phi(r)$ and $M(r)$, which ultimately are the justification for the existence of dark matter.

Unfortunately, when observing things in our Galaxy, the distance d to the source is generally unknown, as is its distance to the Galactic centre R. Also, rather than the orbital velocity v we only know the line-of-sight recessional velocity v_r. We do know, however, the velocity v_0, and distance to the Galactic centre R_0, of the Sun, and also the Galactic longitude l of the object we are looking at. The situation is summarised in Figure 3 below.

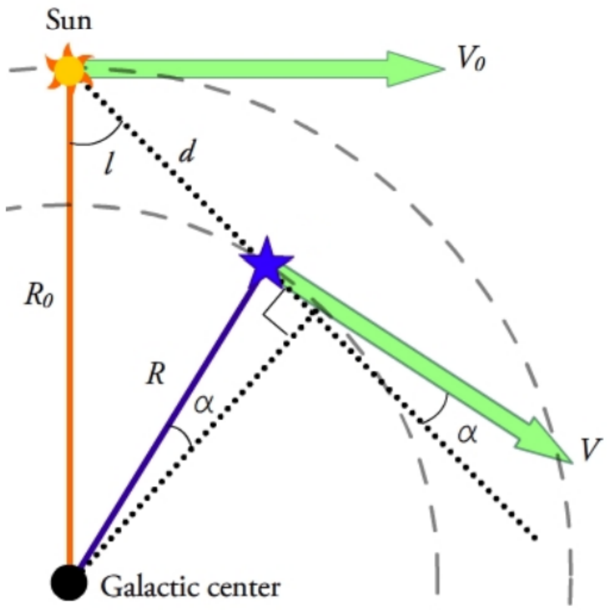

Figure 3 | Geometry of Milky Way Observations.
Known variables: v_0, R_0, l, d (for nearby stars), v_r (not shown)
Unknown variables: v, R, α

For a ray at a given angle l, there will be a range of v_r observed in the gas, as there are a range of orbital radii R that the line of sight will intersect. The largest v_r will be where the line of sight is tangent to the orbit – that is, with $\alpha = 0$ – as this is the lowest-radius orbit touched by the ray at l. For such orbits, $v_{r\max} = v - v_0 \sin l$ (noting that v_r is measured relative to the motion of the sun), and $\sin l = R/R_0$, so their orbital velocity is

$$v = v_{r\max} + v_0 \sin l = v_{r\max} + v_0 \frac{R}{R_0}$$

However, this will only work for $|l| < \pi/2$, because for $|l| > \pi/2$, the line of sight is never tangent to an orbit.

Similar considerations can be used to deduce the orbital parameters of the Sun, using observations of stars in the Solar neighbourhood. The recessional velocity is more generally:

$$v_r = v \cos \alpha - v_0 \sin l$$

and simple geometry gives

$$\frac{\sin l}{R} = \frac{\sin\left(\frac{\pi}{2}+\alpha\right)}{R_0} = \frac{\cos\alpha}{R_0}$$

so eliminating α:

$$\begin{aligned} v_r &= \left(\frac{v}{R}-\frac{v_0}{R_0}\right)R_0\sin l \approx \left((R-R_0)\left.\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{v}{R}\right)\right|_{R_0}\right)R_0\sin l \\ &\approx -d\cos l\left.\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{v}{R}\right)\right|_{R_0}R_0\sin l = Ad\sin 2l \end{aligned}$$

where we have restricted ourselves to nearby stars, made a small-angle approximation $R-R_0 \approx -d\cos l$, and defined Oort's first constant A

$$A \equiv -\frac{R_0}{2}\left.\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{v}{R}\right)\right|_{R_0} = \frac{1}{2}\left[\frac{v_0}{R_0}-\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}\right] = -\frac{R_0}{2}\left.\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right|_{R_0}$$

For stars in the Solar neighbourhood, the proper motion is often measurable, so we know its tangential velocity v_t. Theoretically this is given by $v_t = v\sin\alpha - v_0\cos l$. Now $R\sin\alpha = R_0\cos l - d$, so eliminating α:

$$\begin{aligned} v_t &= \left(\frac{v}{R}-\frac{v_0}{R_0}\right)R_0\cos l - \frac{v}{R}d \approx -d\cos l\left.\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{v}{R}\right)\right|_{R_0}R_0\cos l - \frac{v_0}{R_0}d \qquad (+\mathcal{O}(d^2/R_0^2)) \\ &= -\frac{d}{2}\left(\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}-\frac{v_0}{R_0}\right)(1+\cos 2l) - \frac{v_0}{R_0}d \\ &= Ad\cos 2l - \frac{1}{2}\left[\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}+\frac{v_0}{R_0}\right]d \equiv d(A\cos 2l + B) \qquad B \equiv -\frac{1}{2}\left[\frac{v_0}{R_0}+\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}\right] \end{aligned}$$

B is Oort's second constant. To summarise,

$$v_r = Ad\sin 2l \qquad \boxed{A \equiv \frac{1}{2}\left[\frac{v_0}{R_0}-\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}\right]}$$

$$v_t = d(A\cos 2l + B) \qquad \boxed{B \equiv -\frac{1}{2}\left[\frac{v_0}{R_0}+\left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0}\right]}$$

Thus plots of v_r/d and v_t/d against l as we scan around the sky will be sinusoidal, and the values of A and B can be read off the graphs: their values are about $15\mathrm{kms}^{-1}\mathrm{kpc}^{-1}$ and $-12\mathrm{kms}^{-1}\mathrm{kpc}^{-1}$. From their definitions,

$$\frac{v_0}{R_0} = \Omega_0 = A - B \qquad \left.\frac{\mathrm{d}v}{\mathrm{d}R}\right|_{R_0} = -A - B$$

from which we can find the Sun's orbital velocity ($230\mathrm{kms}^{-1}$) and period around the Galactic centre (230Myr); these suggest an enclosed mass of about $10^{11}M_\odot$.

The Oort constants also reveal the Sun's epicyclic frequency κ_0. We have

$$\kappa_0^2 = 3\Omega_0^2 - f'(R_0) = 3\Omega_0^2 + \left.\frac{\mathrm{d}}{\mathrm{d}R}(R\Omega^2)\right|_{R_0} = 4\Omega_0^2 + 2R_0\Omega_0\left.\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right|_{R_0}$$

$$= 4(A - B)^2 - 4(A - B)A = 4(A - B)(A - B - A) = -4B(A - B)$$

$$\boxed{\kappa_0 = 2\sqrt{-B(A - B)}} \quad \Rightarrow \quad \boxed{\frac{\kappa_0}{\Omega_0} = 2\sqrt{\frac{-B}{A - B}}}$$

For the Sun, this ratio is approximately 1.3, so the Sun's orbit precesses retrograde.

5 Stellar Interactions

We aim now to describe the dynamics of a self-gravitating many-body system, such as a globular cluster. Such systems are described on short timescales as *collisionless*, not (only) because the constituent stars collide incredibly rarely, but because they are all so immensely far away from each other relative to their size that interactions make only very small changes to their trajectories and the potential is pretty smooth and constant.

5.1 Weak Encounters

Consider two stars of mass m, one of which ("the primary") is "nailed down" at the point $(0, b)$, and the other ("the particle") moves past at speed v and a large impact parameter b (along the x-axis). To first order, this will not reduce the particle's speed in the direction of travel, so this will be constant, but it *will* impart a perpendicular component $\Delta v_\perp$ to its velocity. The force in the y-direction is given by:

$$m\dot{v}_\perp = F_y = \frac{Gm^2}{r^2}\frac{b}{r} = \frac{Gm^2 b}{[b^2 + x^2]^{3/2}} = \frac{Gm^2 b}{b^3}\left[1 + \left(\frac{x}{b}\right)^2\right]^{-3/2} = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

$$\Rightarrow \Delta v_\perp = \frac{Gm}{b^2}\int_{-\infty}^{\infty}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2} \mathrm{d}t = \frac{2Gm}{bv}$$

By assumption, $\Delta v_\perp \ll v \Rightarrow \frac{1}{2}v^2 \gg Gm/b$, so we are looking at a situation where the stars are very far apart (or moving very fast relative to each other).

Suppose a star makes a journey from one side of a cluster with N stars and radius R, to another, through the centre. We wish to deduce the overall effect of the weak encounters with all of the stars in the cluster on its journey. The number of stars which the star passes within between b and $b + \mathrm{d}b$ of is approximately $(2\pi b\,\mathrm{d}b)(N/\pi R^2)$, where we have assumed a constant surface density (perhaps using a surface density that is a function of R doesn't change things much). The total $v_\perp$ will probably be 0, because equally many cluster stars will tug this star of interest in one direction as in the opposite direction. However, $\Delta v_\perp^2$s are expected to accumulate in a random walk sort of way. On a single crossing, then,

$$v_\perp^2 \approx \int_{b_{\min}}^{R} 2\pi b\,\mathrm{d}b\,\frac{N}{\pi R^2} N\left(\frac{2Gm}{bv}\right)^2 = 8N\left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{R}{b_{\min}}\right) \equiv 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$$

where $b_{\min}$ is the closest impact parameter; that is, $N\pi b_{\min}^2 \approx \pi R^2$, and so $\ln\Lambda \approx \ln\sqrt{N} \sim \ln N$.

5.2 Relaxation Time

The system can only be described as collisionless on short timescales, as eventually the trajectories of stars become scrambled and almost impossible to trace back. This will occur when $v_\perp^2$ reaches order v^2, which will be after a number $n_{\text{relax}} = v^2/v_\perp^2$ crossings. If we write $v^2 \approx NGm/R$, we have

$$n_{\text{relax}} \sim \frac{1}{8N\ln N}\left(\frac{v^2 R}{Gm}\right)^2 \approx \frac{0.1N}{\ln N}$$

and the *relaxation time* is simply $t_{\text{relax}} = n_{\text{relax}} t_{\text{cross}}$ where

$$t_{\text{cross}} \sim \frac{R}{v} \approx \sqrt{\frac{R^3}{GNm}}$$

It is found that for galaxies $t_{\text{relax}} \sim 10^{16}$yr, whereas for globular clusters it is more like 10^5yr. As such the former are not (yet!) relaxed but the latter are.

5.3 Gravitational Drag

We now model a large mass M (such as a globular cluster, black hole, or big star) sweeping through a field of (initially stationary) regular stars of mass $m \ll M$. As M slingshots lots of ms, each of which robs it of a little momentum, M experiences a drag force (this effect is better known as *dynamical friction*). We assume that all the stars whose trajectories are deflected by an angle $\pi/2$ or greater in the frame of M lose all their momentum to M. The deflection angle depends on b and v, which in the M-frame is equal to the velocity of M. Thus there is some critical impact parameter $b_\perp$ within which all stars give up their momentum to M. Deflection by $\pi/2$ corresponds to $\phi_\infty = 3\pi/4 = \arccos(-1/e)$, so for this orbit $e = \sqrt{2}$. Now $b_\perp = -a\sqrt{e^2-1} = -a$, and $E = \frac{1}{2}v^2 = -GM/2a = GM/2b_\perp$, where we recall that $a < 0$ for hyperbolic orbits. We therefore find that $b_\perp = \frac{GM}{v^2}$.

As the large mass passes through the field, in a time Δt it will pass $(\rho/m)(\pi b_\perp^2)(v\Delta t)$ stars, where ρ is the mass density and ρ/m the number density of stars. Each of these stars will extract a momentum mv from the large mass, so the drag force will be

$$\begin{aligned} F = M\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{1}{\Delta t}\left(\frac{\rho}{m}\pi b_\perp^2 v\Delta t\right)mv = -\pi b_\perp^2 \rho v^2 = -\pi\left(\frac{GM}{v^2}\right)^2 \rho v^2 \\ \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\pi G^2 M\rho}{v^2} \end{aligned}$$

5.4 Gravitational Focusing

For a given "target radius" r_t (perhaps the radius of a stellar atmosphere), there is a b_0 so that all bodies with impact parameters $b < b_0$ come within r_t of a mass M. This b_0 is the impact parameter for a hyperbolic orbit whose periapsis is the target radius. We have

$$E = \frac{1}{2}v_\infty^2 = -\frac{GM}{r_t} + \frac{1}{2}v_p^2 \qquad\qquad h = b_0 v_\infty = r_t v_p$$

On eliminating v_p,

$$b_0^2 = r_t^2\left[1 + \frac{2GM}{r_t v_\infty^2}\right] = r_t^2\left[1 + \left(\frac{v_{\text{esc}}}{v_\infty}\right)^2\right]$$

where v_{esc} is the escape velocity at r_t. Often one of the terms in the squackets is much larger than the other and the other can be approximated out.

5.5 Dynamics in Clusters

Clusters actually aren't generally collisionless. For globulars, $t_{\text{relax}} \sim$ age; for open clusters, $t_{\text{relax}} \ll$ age, so interactions must be important to the shape of clusters today. There are several important processes governing the evolution of clusters.

In analogy with the behaviour of gases, we can use the virial theorem $2T + \Phi = 0$, $E = T + \Phi \Rightarrow E = -T$ to show that $\partial E/\partial T = -1$. As the kinetic energy per mass T is a measure of dynamical temperature, self-gravitating systems like clusters thus have negative heat capacities. Gases have positive heat capacity, so removing some E decreases T, which makes further transfer of energy slower as the T gradient is then smaller. For negative heat capacities, removing E causes an *increase* in T, meaning the T gradient becomes larger and *more* E gets transferred away faster – the system is unstable. Clusters thence end up with low-energy, high-T regions (in the core) and high-energy, low-T regions (in the halo). Interactions between stars continually move energy from the core into the halo as stars get flung outwards. Over time, the core shrinks and the halo evaporates, leading to an "implosion" scenario known as a "gravithermal catastrophe".

Relaxation can give some stars $v > v_{\text{esc}}$, leading to their escape. This can be imagined as a fraction (deduced below) of a Maxwell-Boltzmann distribution having sufficient energy to escape and being lost, and the remaining stars relaxing back to an MB distribution in a time t_{relax}, and then some more escaping. . . this process is analogous to evaporation. The virial theorem gives $\overline{v^2} = 2T = -\Phi$. The escape velocity is $v_e^2 = 2T_e = -2\Phi$. The mean-square escape speed is then

$$\overline{v_e^2} = \frac{\int \rho v_e^2 \, \mathrm{d}^3\mathbf{x}}{\int \rho \, \mathrm{d}^3\mathbf{x}} = -\frac{2}{M}\int \rho\Phi \, \mathrm{d}^3\mathbf{x} = -\frac{4V}{M} = -4\Phi = 4\overline{v^2}$$

Hence the rms escape velocity is double the rms velocity. In an MB distribution, a fraction 7.4×10^{-3} of the stars have velocity greater than this, so would be expected to escape every relaxation time. The evaporation time is expected then to be of order $t_{\text{relax}}/7.4 \times 10^{-3}$, which is trillions of years.

Alternatively, stars from the halo may be *tidally stripped* by a larger body (such as the galaxy the cluster orbits).

Cluster cores often contain binary star systems; as the core becomes denser, more stars end up in $N > 2$-body systems. Often being unstable, these can fling stars back out of the cluster, "reinflating" it.

By equipartition, interactions tend to equilibrate the kinetic energies of stars. As such, $\overline{v^2} \propto m^{-1}$, so massive stars with lower v sink to the core; lighter stars get flung out to the halo.

A final important process in clusters is the mass loss due to stellar winds and supernovae.

6 Phase Space Analysis

Consider a large, collisionless system, such as a galaxy on timescales less than 10Gyr; this will have a nice smooth $\Phi(\mathbf{x}, t)$. There are far too many (N) objects to individually model the dynamics, so instead we use the function $f(\mathbf{x}, \mathbf{v}, t)$ as a *phase space density distribution function* in the 6-dimensional $\mathbf{x}$-$\mathbf{v}$ phase space. f is normalised so that

$$\iint f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}^3\mathbf{x} \, \mathrm{d}^3\mathbf{v} = 1$$

for all t. In words, $f\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}$ is the probability that a star will be found with position in the box $[\mathbf{x}, \mathbf{x}+\mathrm{d}\mathbf{x}]$ *and* velocity in the $\mathbf{v}$-box $[\mathbf{v}, \mathbf{v}+\mathrm{d}\mathbf{v}]$. f is also differentiable, as for collisionless processes, particles will not be rapidly jumping around in velocity space.

A given star has $\dot{\mathbf{v}} = -\boldsymbol{\nabla}\Phi(\mathbf{x}, t)$. As such, writing the 6D vector $\mathbf{w} = (\mathbf{x}, \mathbf{v})$, we have $\dot{\mathbf{w}} \equiv (\mathbf{v}, \dot{\mathbf{v}}) = (\mathbf{v}, -\boldsymbol{\nabla}\Phi(\mathbf{x}, t))$.

6.1 Collisionless Boltzmann Equation

In fluid mechanics, the conservation of mass density gives

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$

expressing that an increase in density at a point must be matched by an inflow. Similarly, stars cannot be lost from phase space[9], so an increase in star density in phase space must be matched by a 6-dimensional inflow. In equations,

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\mathbf{w}} \cdot (f \dot{\mathbf{w}}) = 0$$

The second term may be simplified

$$\boldsymbol{\nabla}_{\mathbf{w}} \cdot (f\dot{\mathbf{w}}) = \frac{\partial}{\partial w_i}(f\dot{w}_i) = \frac{\partial}{\partial x_i}(f v_i) + \frac{\partial}{\partial v_i}\left(-f\frac{\partial \Phi}{\partial x_i}\right) = v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}} f - \boldsymbol{\nabla}_{\mathbf{x}}\Phi \cdot \boldsymbol{\nabla}_{\mathbf{v}} f$$

which on substitution gives the Collisionless Boltzmann Equation:

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}} f - \boldsymbol{\nabla}_{\mathbf{x}}\Phi\cdot\boldsymbol{\nabla}_{\mathbf{v}} f = \frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0} \tag{CBE}$$

If $f(\mathbf{x}, \mathbf{v}, 0)$ is known, the CBE can be used to integrate it for all time.

Going back to the conservation equation of f, we can use a divergence identity $\boldsymbol{\nabla}\cdot(\phi\mathbf{a}) = \mathbf{a}\cdot\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\cdot\mathbf{a}$ and the fact that $\boldsymbol{\nabla}_{\mathbf{w}}\cdot\dot{\mathbf{w}} = \partial\dot{w}_i/\partial w_i = \partial v_i/\partial x_i + \partial\dot{v}_i/\partial v_i = 0$ (as v_i is independent of x_i and $\dot{v}_i$ depends only on x_i), to write

$$\frac{\partial f}{\partial t} + \dot{\mathbf{w}}\cdot\boldsymbol{\nabla}_{\mathbf{w}} f = 0 \tag{$*$}$$

Then, defining the *material derivative*[10] in the 6D phase space by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\mathbf{w}}\cdot\boldsymbol{\nabla}_{\mathbf{w}}$$

like the derivative moving along with the phase flow, we can instead write $\boxed{Df/Dt = 0}$, which is Liouville's Theorem.

[9]... at least on small timescales

[10]Be aware that this has many different names, including Lagrangian derivative (from the notes of this course) and convective derivative (from IB Classical Dynamics). Wikipedia lists 8 further names. What's wrong with people?

The form $(*)$ is the easiest to adapt to other coordinate systems. For example, in cylindrical coordinates (R, ϕ, z),

$$\begin{aligned}
0 &= \frac{\partial f}{\partial t} + \dot{w}_i \frac{\partial f}{\partial w_i} \\
&= \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R \frac{\partial f}{\partial v_R} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} + \dot{v}_z \frac{\partial}{\partial f}_{v_z} \\
&= \frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z}
\end{aligned}$$

where we have used

$$v_R = \dot{R} \qquad v_\phi = R\dot{\phi} \qquad v_z = \dot{z}$$

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R} \qquad \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{\phi}\right) = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} \qquad \ddot{z} = -\frac{\partial \Phi}{\partial z}$$

where the equations of motion on the second row eventually lead to

$$\dot{v}_R = \frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \qquad \dot{v}_\phi = -\frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)$$

Throughout one should remember that the $v_\square$ must have units of velocity, but that the coordinates themselves (like ϕ) need not have units of length.

6.2 Jeans' Equations

f is very abstract; usually we simply know the number density of stars ν:

$$\boxed{\nu(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{v}}$$

and some velocity means (though usually just in the radial direction):

$$\boxed{\overline{v_i}(\mathbf{x}, t) = \frac{1}{\nu}\int v_i f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{v}} \qquad \boxed{\overline{v_i v_j}(\mathbf{x}, t) = \frac{1}{\nu}\int v_i v_j f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{v}}$$

The evolution of these quantities is derived by taking moments of the CBE.

6.2.1 0th Moment

Integrating CBE over velocity,

$$0 = \int \frac{\partial f}{\partial t}\, \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i}\, \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\underbrace{\int \frac{\partial f}{\partial v_i}\, \mathrm{d}^3\mathbf{v}}_{0} = \frac{\partial}{\partial t}\underbrace{\int f\, \mathrm{d}^3\mathbf{v}}_{\nu} + \frac{\partial}{\partial x_i}\underbrace{\int v_i f\, \mathrm{d}^3\mathbf{v}}_{\nu\overline{v_i}}$$

$$\Rightarrow \boxed{\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu\overline{v_i}) = 0} \qquad \Rightarrow \qquad \frac{\partial \nu}{\partial t} + \boldsymbol{\nabla}\cdot(\nu\mathbf{v}) \tag{$\mathfrak{J}0$}$$

which looks kinda like the fluid dynamics conservation equation; the vector form is more easily used for alternate coordinate systems. For instance, for axisymmetric systems $\mathfrak{J}0$ is

$$\frac{\partial \nu}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\nu v_R) + \frac{\partial}{\partial z}(\nu v_z) = 0$$

6.2.2 1st Moment

Multiplying the CBE by v_j and then integrating,

$$0 = \int v_j \frac{\partial f}{\partial t}\,\mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i}\,\mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i}\,\mathrm{d}^3\mathbf{v}$$

$$= \frac{\partial}{\partial t}\underbrace{\int v_j f\,\mathrm{d}^3\mathbf{v}}_{\nu\overline{v_j}} + \frac{\partial}{\partial x_i}\underbrace{\int v_i v_j f\,\mathrm{d}^3\mathbf{v}}_{\nu\overline{v_i v_j}} - \frac{\partial \Phi}{\partial x_i}\left(\underbrace{\int \frac{\partial}{\partial v_i}(v_j f)\,\mathrm{d}^3\mathbf{v}}_{0} - \underbrace{\int \frac{\partial v_j}{\partial v_i} f\,\mathrm{d}^3\mathbf{v})}_{\nu\delta_{ij}}\right)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\nu\overline{v_j}) + \frac{\partial}{\partial x_i}(\nu\overline{v_i v_j}) + \nu\frac{\partial \Phi}{\partial x_j} = 0} \tag{$\mathfrak{J}1$}$$

giving one form of the first moment equation. Ideally we would want to know $\partial\overline{v_j}/\partial t$, for which we need to subtract $\overline{v_j}\,\partial\nu/\partial t$, using $\partial\nu/\partial t = -\,\partial(\nu\overline{v_i})/\partial x_i$ from $\mathfrak{J}0$:

$$\nu\frac{\partial\overline{v_j}}{\partial t} - \overline{v_j}\frac{\partial}{\partial x_i}(\nu\overline{v_i}) + \frac{\partial}{\partial x_i}(\nu\overline{v_i v_j}) + \nu\frac{\partial \Phi}{\partial x_j} = 0 \tag{$\mathfrak{J}1$}$$

Defining a symmetric *velocity covariance matrix* $\sigma_{ij}^2 = \overline{(v_i - \overline{v_i})(v_j - \overline{v_j})}$:

$$\boxed{\sigma_{ij}^2 = \frac{1}{\nu}\int (v_i - \overline{v_i})(v_j - \overline{v_j}) f\,\mathrm{d}^3\mathbf{v}}$$

and rewriting as $\sigma_{ij}^2 = \overline{(v_i v_j - \overline{v_i}v_j - v_i\overline{v_j} + \overline{v_i v_j})} = \overline{v_i v_j} - \overline{v_i}\;\overline{v_j}$, we can write $\mathfrak{J}1$ again as:

$$\boxed{\nu\frac{\partial\overline{v_j}}{\partial t} + \frac{\partial}{\partial x_i}\left(\nu\sigma_{ij}^2\right) + \nu\overline{v_i}\frac{\partial\overline{v_j}}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0} \tag{$\mathfrak{J}1$}$$

reminiscent of one of Euler's equations of fluid dynamics

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\nabla}p + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \rho\boldsymbol{\nabla}\Phi = 0$$

6.2.3 Applications of Jeans' Equations

It is often useful to assume steady state, purely dispersive motion, and isotropic motion:

$$\frac{\partial}{\partial t} = 0 \qquad\qquad \overline{v_i} = 0 \qquad\qquad \sigma_{ij}^2 = \sigma^2\delta_{ij}$$

with which $\mathfrak{J}1$ becomes simply $\boldsymbol{\nabla}(\nu\sigma^2) + \nu\boldsymbol{\nabla}\Phi = 0$. For a spherically symmetric cluster, if $\nu(r)$ is known, then $\rho(r) = m\nu(r)$, then $\boldsymbol{\nabla}^2\Phi(r) = 4\pi G\rho$ so Φ can be found. Then $\sigma(r)$ can be found from the equation at the start of this paragraph. With the assumptions made, σ is related to the rms speed:

$$\overline{v_i v_j} - \underbrace{\overline{v_i}\;\overline{v_j}}_{0} = \sigma_{ij} = \sigma^2\delta_{ij} \qquad \Rightarrow \qquad \sigma^2 = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\overline{v_i v_i} = \frac{1}{3}\left\langle v^2\right\rangle \qquad \Rightarrow \qquad \sigma = \sqrt{\frac{\langle v^2\rangle}{3}}$$

The simplified first form of $\mathfrak{J}1$ can be applied to the Milky Way. MW is much thinner than it is wide, so the characteristic length scales are very different in the z and x-y directions. Choosing $j = z$ and using isotropy and steady state, and simplifying Poisson's equation,

$$\frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right) + \nu\frac{\partial\Phi}{\partial z} = 0 \qquad\qquad \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho$$

$$\rho = -\frac{1}{4\pi G}\frac{\partial}{\partial z}\left(\frac{1}{\nu}\frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right)\right)$$

A particular advantage of this is that although ρ here refers to all mass, ν could just be that of, say, G-type stars, making an analysis easy if rather noisy due to approximating two derivatives over large distances. This can be reduced by integrating $\rho(R, z)$ once over a region of z to give $\Sigma(R, < |z|)$:

$$\Sigma(R, < |z|) = \int_{-z}^{z} \rho(R, z')\,\mathrm{d}z' = -\frac{1}{2\pi G\nu}\frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right)\mathrm{d}z$$

the factor of 2 coming from the integration in both $+z$ and $-z$, assumed even. Observations show that at the Sun's R, $\Sigma(R, < |1.1\text{kpc}|) \approx 70M_\odot\ \text{pc}^{-2}$, but just adding up the stars and gas gives more like $40M_\odot\ \text{pc}^{-2}$, suggesting some dark matter.

Furthermore, the rotation curve observed is flat, i.e. v_c is constant. As $v_c^2/R \approx GM(R)/R^2$, we thus have that $M(R) \sim R$, or that the halo in which we reckon dark matter is in the shape of has a density profile $\rho \sim R^{-2}$. This ends up suggesting a dark matter halo contribution of about $\Sigma_D(R, < |1.1\text{kpc}|) \approx 30M_\odot\text{pc}^{-2}$ which adds up.

6.3 Jeans' Theorem and Spherical Isotropic Systems

Jeans Theorem concerns *constants of motion*: quantities which are conserved throughout any path which is a solution to the equations of motion. Example: for a free particle moving in 1D at speed v, the quantity $x - vt$ is a constant of motion. Another example: an orbiting particle's energy, or angular momentum.

The most interesting subset of the constants of motion are the *integrals of motion*, which are constants of motion which do not depend on time, such as an orbiting particle's energy or angular momentum. These are therefore a function of $\mathbf{w}$, i.e. of $\mathbf{x}$ and $\mathbf{v}$, so there is a value of E (say) associated with each point in phase space. Such integrals of motion almost always reduce the dimensionality of phase space that can be occupied by a particular star, which can only go to parts of phase space with the same value of E. Denoting an integral of motion by α, we have $\mathrm{d}\alpha/\mathrm{d}t = 0$ along an orbit, for instance $\mathrm{d}E/\mathrm{d}t = 0$. Then

$$0 = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \dot{\mathbf{w}}\cdot\boldsymbol{\nabla}_{\mathbf{w}}\alpha = \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\alpha + \dot{\mathbf{v}}\cdot\boldsymbol{\nabla}_{\mathbf{v}}\alpha = \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\alpha - \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}_{\mathbf{x}}\alpha$$

which is the same as the CBE if we take $\partial f/\partial t = 0$, the steady-state CBE. Jeans' Theorem states that if f is a steady-state ($\partial f/\partial t = 0$) function of the integrals of motion α_i, then it will satisfy the CBE:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial\alpha_i}\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}}f - \boldsymbol{\nabla}_{\mathbf{x}}\Phi\cdot\boldsymbol{\nabla}_{\mathbf{v}}f = 0$$

We now restrict our analysis to **Spherical Isotropic Systems**. In such systems, $f(\mathbf{x}, \mathbf{v}) = f(E(r, v)) = f(\Phi(r) + \frac{1}{2}v^2)$ is a function of E alone, (hence the distribution of $\hat{\mathbf{v}}$ is isotropic; this is not to say that everything is just moving radially, but everything is certainly not moving e.g. azimuthally). According to Jeans' Theorem, such f always satisfy the CBE, as E is a constant of motion. If f is known, we can find $\nu = \int f \, \mathrm{d}^3\mathbf{x}$ and hence ρ, and hence Φ, and hence $f = f(\Phi(r) + \frac{1}{2}v^2)$: this circular train enables us to find a model which is self-consistent.

To avoid lots of awkward $-$ signs, we define some silly new variables

- **Relative Energy** $\mathcal{E} = -E + \Phi_0$. This is also a constant of motion and so $f(\mathcal{E})$ also satisfies the CBE. Φ_0 is some convenient constant, sometimes set to give $f(\mathcal{E})$ particular properties.
- **Relative Potential** $\Psi = -\Phi(r) + \Phi_0$. We thus have $\nabla^2\Psi = -4\pi G\rho$, as well as $\lim_{r\to\infty}\Psi = \Phi_0$, and $\mathcal{E} = \Psi - \frac{1}{2}\mathbf{v}^2$.

A spherically symmetric system has $\nabla^2\Psi = \frac{1}{r^2}(r^2\Psi')' = -4\pi G\rho$. Writing[11] $\rho(r) = m\nu(r)$ (assuming stars have an average mass m), we have

$$\begin{aligned}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= -4\pi Gm\int f(r,v)\,\mathrm{d}^3\mathbf{v} = -16\pi^2 Gm\int_0^\infty f(\mathcal{E}(r,v))v^2\,\mathrm{d}v \\ &= -16\pi^2 Gm\int_0^\infty f\left(\Psi(r) - \frac{1}{2}v^2\right)v^2\,\mathrm{d}v\end{aligned}$$

If $f(\mathcal{E})$ is known (or angesetzt) and the integral can be done, the above is an ODE in $\Psi(r)$, which can be solved for Ψ and hence f and hence $\rho(r)$. We now give some Ansätze for f to see what they give.

6.3.1 Polytropic Models

$$f(\mathcal{E}) = \begin{Bmatrix} F\mathcal{E}^{n-3/2} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \le 0 \end{Bmatrix} = \begin{cases} F\left(\Psi - \frac{1}{2}v^2\right)^{n-3/2} & v < \sqrt{2\Psi} \\ 0 & v \ge \sqrt{2\Psi} \end{cases}$$

for constant F; note the discontinuity. We then have

$$\begin{aligned}\rho(r) &= m\int_0^\infty 4\pi v^2 f(\mathcal{E}(r,v))\,\mathrm{d}v = 4\pi mF\int_0^{\sqrt{2\Psi}}\left(\Psi - \frac{1}{2}v^2\right)^{n-3/2}v^2\,\mathrm{d}v \\ &= \frac{(2\pi)^{3/2}\Gamma(n-\frac{1}{2})mF}{\Gamma(n+1)}\Psi^n = C_n\Psi^n \qquad (\Psi > 0)\end{aligned}$$

where on going to the final line we have used some non-examinable maths about $\Gamma(x)$, and we rewrite the monstrous constant[12] as the more amicable C_n. This is only the formula for $\rho(r)$ where r is such that $\Psi(r) > 0$; for regions with $\Psi(r) < 0$ we have $\mathcal{E} < 0$ and hence $f = 0$ and $\rho = 0$. We then use Poisson to give:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = \begin{cases} -4\pi GC_n\Psi^n & r : \Psi(r) > 0 \\ 0 & r : \Psi(r) \le 0 \end{cases}$$

[11] Unlike the notes which write $\rho = m\rho$ (ok actually they redefine f but that's confusing); subsequent results may differ from the notes by a factor m cause I like being consistent

[12] This constant also restricts finite solutions to $n > 1/2$

Writing $\Psi = \Psi_0 \psi$ and $s = r\sqrt{4\pi G C_n \Psi_0^{n-1}}$, we obtain

$$\frac{1}{s^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2 \frac{\mathrm{d}\psi}{\mathrm{d}r}\right) = \begin{cases} -\psi^n & r : \psi(r) > 0 \\ 0 & r : \psi(r) \leq 0 \end{cases} \tag{Lane-Emden}$$

which is the Lane-Emden[13] Equation from fluid dynamics. The appropriate boundary conditions are that at $r = s = 0$: we can wlog set $\psi = 1$ and let Ψ_0 do the scaling of Ψ; $\mathrm{d}\psi/\mathrm{d}s \propto -\mathrm{d}\Phi/\mathrm{d}r = 0$ as there is no gravitational force at $r = 0$. The most appropriate analytic solution for this equation is for $n = 5$, and is known as the Plummer potential:

$$\psi(s) = \frac{1}{\sqrt{1 + s^2/3}} \qquad \Rightarrow \qquad \rho(s) \propto \left(1 + s^2/3\right)^{-5/2}$$

This density distribution is flat as $s \to 0$ as observed, and also decays as s^{-5} as $s \to \infty$ so contains a finite total mass. It describes globular clusters and dwarf galaxies well, but elliptical galaxies poorly.

6.3.2 Isothermal Solutions

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\mathcal{E}/\sigma^2}$$

This has the useful property that $f \propto e^{-v^2/2\sigma^2}$, yknow like a gas[14]. We find:

$$\rho(r) = \int_0^\infty 4\pi v^2 f(\mathcal{E}(r, v))\,\mathrm{d}v = \rho_1 e^{\Psi/\sigma^2} \frac{1}{(2\pi\sigma^2)^{3/2}} \int_0^\infty 4\pi v^2 e^{v^2/2\sigma^2}\,\mathrm{d}v = \rho_1 e^{\Psi/\sigma^2}$$

We could now substitute this ρ into Poisson to find $\Psi(r)$, and then find ρ; or we could substitute $\Psi = \sigma^2 \ln(\rho/\rho_1)$ into Poisson and solve for ρ directly. We write

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma^2 \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho\right) = -4\pi G\rho$$

One solution to this, called the *singular isothermal sphere* is $\rho(r) = Ar^{-2}$ with $A = \sigma^2/2\pi G$. Not only singular, but also having a diverging mass, this solution is awkward. The other solution has no analytic solution and the mass diverges anyway so it's not much better.

6.3.3 King Models

$$f(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2}\left(e^{\mathcal{E}/\sigma^2} - 1\right) & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

Like a truncated isothermal model. This cannot be solved analytically; at the first step $\rho(\Psi)$ involves an error function. Numerical integration shows that there is a *tidal radius* r_t at which $f(r > r_t, v) = 0$. These models are a good fit to globular clusters; it helps that there are some free parameters.

[13] More like *Lame*-Emden amirite

[14] One can calculate $\overline{v^2} = 3\sigma^2$