Cosmology

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1 Friedmann-Robertson-Walker Metric

On very large scales, the Universe appears homogeneous and isotropic. The density and pressure are hence uniform throughout the Universe, though they change in time. The only metrics which satisfy homogeneity and isotropy are

$$\boxed{\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]}$$

The metric inside the squackets uses *comoving coordinates*. Comoving observers have *fixed* comoving coordinates (r, θ, ϕ) but the actual physical distance between them is scaled over time by $R(t)$. An alternative radial coordinate is χ:

$$r = S_K(\chi) \equiv \left\{\begin{matrix} \sin\left(\sqrt{K}\chi\right)/\sqrt{K} & K > 0 \\ \chi & K = 0 \\ \sinh\left(\sqrt{-K}\chi\right)/\sqrt{-K} & K < 0 \end{matrix}\right\} \Rightarrow \mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R(t)^2\left[\mathrm{d}\chi^2 + S_K(\chi)^2\mathrm{d}\Omega\right]$$

A homogeneous ideal fluid has the following stress-energy tensor:

$$T^{\mu\nu} = \left(\rho c^2 + P\right)u^\mu u^\nu - Pg^{\mu\nu} \qquad \rho = \rho(t), \qquad P = P(t), \qquad u^\mu = \delta^\mu_0$$

Substituting the FRW metric and this $T^{\mu\nu}$ into $\mathfrak{EFE}$ gives the Friedmann Equations:

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}} \qquad \boxed{\left(\frac{\dot{R}}{R}\right)^2 + \frac{Kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}} \qquad (\mathfrak{F}1, 2)$$

1.0.1 Density Parameters

Defining the *Hubble parameter* $\mathbf{H} = \dot{R}/R$ and the *Hubble constant* $\mathbf{H}_0$ as its current value, we can then write $\mathfrak{F}2$ at the current time as

$$\mathbf{H}_0^2 + \frac{Kc^2}{R_0^2} = \frac{8\pi G\rho_0}{3} + \frac{\Lambda c^2}{3} \qquad \Rightarrow \qquad 1 = \overbrace{\frac{\rho_0}{3\mathbf{H}_0^2/8\pi G}}^{\Omega_m} - \overbrace{\frac{Kc^2}{\mathbf{H}_0^2 R_0^2}}^{\Omega_K} + \overbrace{\frac{\Lambda c^2}{3\mathbf{H}_0^2}}^{\Omega_\Lambda}$$

The dimensionless $\Omega_m, \Omega_K, \Omega_\Lambda$ quantify the contributions of matter, curvature, and the cosmological constant to the energy of the Universe today. Ω_m defines a critical density $\rho_c = 3\mathbf{H}_0^2/8\pi G$, such that if $\Lambda = 0$, the balance of ρ_0 and ρ_c reveals the curvature of the Universe:

$$\rho_0 > \rho_c \Rightarrow K > 0 \qquad \rho_0 = \rho_c \Rightarrow K = 0 \qquad \rho_0 < \rho_c \Rightarrow K < 0$$

1.0.2 Conservation of Energy

Using $\mathrm{d}U = -P\mathrm{d}V$ (or else writing $\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2 R^3) = -P\frac{\mathrm{d}}{\mathrm{d}t}(R^3) \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}R}(\rho R^3) = -3\frac{P}{c^2}R^2} \text{ or } \boxed{\dot{\rho} + 3\mathbf{H}\left(\rho + \frac{P}{c^2}\right) = 0}$$

The evolution $\rho(R)$ thus depends on the *equation of state* $P = w\rho c^2$. Note that the speed of sound $c_s = \sqrt{|\,\mathrm{d}P/\mathrm{d}\rho\,|} = \sqrt{|w|}c$ requires $|w| \leq 1$. Letting $\rho \propto R^\alpha$, the above gives $3 + \alpha = -3w \Rightarrow \alpha = -3(w+1) \Rightarrow \rho \propto R^{-3(w+1)}$. For regular matter (*dust*) $w = 0$ and so $\rho_m \propto R^{-3}$; relativistic matter and light (*radiation*) has $w = 1/3$ so $\rho_\gamma \propto R^{-4}$.

1.1 Cosmological Redshift

Consider two comoving observers, i.e. with a constant χ_{AB} between them but a changing true, physiccal distance $R(t)\chi_{AB}$. Consider two wavecrests of a photon emitted from one observer (at $\chi = 0$) to the other ($\chi = \chi_{AB}$) at times t_e and $t_e + 1/\nu_e$. Photons travel along null paths, so

$$c^2\,\mathrm{d}t^2 = R^2\,\mathrm{d}\chi^2 \qquad \Rightarrow \qquad \frac{\mathrm{d}t}{R(t)} = \frac{1}{c}\mathrm{d}\chi$$

If the photons are received at t_0 and $t_0 + 1/\nu_r$ then we have both

$$\int_{t_e}^{t_0} \frac{\mathrm{d}t}{R(t)} = \frac{1}{c}\int_0^{\chi_{AB}} \mathrm{d}\chi, \qquad \int_{t_e+1/\nu_e}^{t_0+1/\nu_r} \frac{\mathrm{d}t}{R(t)} = \frac{1}{c}\int_0^{\chi_{AB}} \mathrm{d}\chi = \int_{t_e}^{t_0} \frac{\mathrm{d}t}{R(t)}$$

$$\Rightarrow \int_{t_e}^{t_e+1/\nu_e} \frac{\mathrm{d}t}{R(t)} = \int_{t_0}^{t_0+1/\nu_r} \frac{\mathrm{d}t}{R(t)} \quad \Rightarrow \frac{1}{\nu_e R(t_e)} \approx \frac{1}{\nu_r R_0} \quad \Rightarrow 1 + \mathbf{z} \equiv \frac{\lambda_r}{\lambda_e} = \frac{\nu_e}{\nu_r} = \frac{R_0}{R(t_e)}$$

So when we look at a distant galaxy with redshift $\mathbf{z}$, we can deduce the $R(t_e)$ from when that photon was emitted at t_e, compared to the current R_0 (whose value is irrelevant). This also explains why the energy density of radiation decreases as R^{-4} – the number density decreases by R^{-3} (as with dust), but also the photon energy $h\nu \propto R^{-1}$.

Using $\rho \propto R^{-3}$ ($w \approx 0$ since $t \sim 50$kyr) and $R = R_0(1+\mathbf{z})^{-1}$, $\mathfrak{F}2$ becomes

$$\frac{\dot{R}}{R} = \mathbf{H}_0\sqrt{\frac{\rho}{3\mathbf{H}_0^2/8\pi G} - \frac{K}{\mathbf{H}_0^2 R^2} + \frac{\Lambda}{3\mathbf{H}_0^2}} = \mathbf{H}_0\sqrt{\Omega_m\left(\frac{R_0}{R}\right)^3 + \Omega_K\left(\frac{R_0}{R}\right)^2 + \Omega_\Lambda}$$

$$\Rightarrow \mathrm{d}t = \frac{\mathrm{d}R}{R\mathbf{H}_0\sqrt{\Omega_m\left(\frac{R_0}{R}\right)^3 + \Omega_K\left(\frac{R_0}{R}\right)^2 + \Omega_\Lambda}} = -\frac{\mathrm{d}\mathbf{z}}{\mathbf{H}_0(1+\mathbf{z})\sqrt{\Omega_m(1+\mathbf{z})^3 + \Omega_K(1+\mathbf{z})^2 + \Omega_\Lambda}}$$

$$= -\frac{\mathrm{d}\mathbf{z}}{\mathbf{H}_0(1+\mathbf{z})E(\mathbf{z})} \text{ where } E(\mathbf{z}) \equiv \sqrt{\Omega_m(1+\mathbf{z})^3 + \Omega_K(1+\mathbf{z})^2 + \Omega_\Lambda}$$

1.2 Cosmic Microwave Background

The total energy density of the photons produced by a black body is given by

$$\rho_\gamma c^2 = \int_0^\infty \mathrm{d}\nu\, \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/k_B T_\gamma} - 1} = aT_\gamma^4$$

and thus $\rho_\gamma \propto T_\gamma^4$. Now $\rho_\gamma \propto R^{-4}$, so $T_\gamma \propto R^{-1} \propto \nu$.

At early times and small R, T_γ was high and everything was ionised; photons were constantly scattering off the free electrons and hence in thermal equilibrium with them. Eventually the photons became cool enough to allow hydrogen atoms to stay bound without ionising them again immediately. This lowered the number of free electrons and eventually the scattering rate slowed to a rate comparable with $\mathbf{H}$, making photons effectively free. This "freeze-out" occurred at $\mathbf{z} \sim 1100$: the "last scattering surface" of the Universe. Having been in thermal equilibrium prior to this, the Universe was very well-approximated by a black body, and because $T_\gamma \propto \nu$, as the Universe cooled each photon cooled in proportion, preserving the black-body spectrum. Today this spectrum peaks in the microwave and $T_{\gamma,0} = 2.726\text{K}$. The CMB is isotropic, but as we are not fundamental observers there is a dipole of about 0.003K.

1.3 Age

From the result at the end of §1.1, we can write

$$\mathrm{d}t = -\frac{\mathrm{d}\mathbf{z}}{\mathbf{H}_0(1+\mathbf{z})E(\mathbf{z})} \qquad \Rightarrow \qquad t(\mathbf{z}) = -\frac{1}{\mathbf{H}_0}\int_\infty^{\mathbf{z}} \frac{\mathrm{d}\mathbf{z}'}{(1+\mathbf{z}')E(\mathbf{z}')}$$

For $K = 0$, the (surprisingly satisfying) integral can be evaluated:

$$t(\mathbf{z}) = \frac{2}{3\mathbf{H}_0\sqrt{1-\Omega_m}} \sinh^{-1}\left(\sqrt{\frac{1-\Omega_m}{\Omega_m(1+\mathbf{z})^3}}\right)$$

For a redshift $\mathbf{z} = 0$ (i.e. here and now) and $\Omega_m = 0.31$, we find $t_0 = 13.84\text{Gyr}$.

1.4 Distances

As photons' worldlines are null, $c\,\mathrm{d}t = R\,\mathrm{d}\chi$. For light emitted at some t and arriving now (t_0),

$$\chi = c\int_t^{t_0} \frac{\mathrm{d}t'}{R(t')} = \frac{c}{\mathbf{H}_0}\int_{R(t)}^{R_0} \frac{\mathrm{d}R}{R^2 E(\mathbf{z})} = \frac{c}{\mathbf{H}_0 R_0}\int_0^{\mathbf{z}} \frac{\mathrm{d}\mathbf{z}'}{E(\mathbf{z}')}$$

where we used $\dot{R} = \mathbf{H}_0 E(\mathbf{z}) R$ to convert from t to R, and then $R = \frac{R_0}{1+\mathbf{z}}$ to convert from R to $\mathbf{z}$; these are both common conversions.

1.4.1 Angular Diameter Distance D_A

Say we put a standard ruler (e.g. a H atom) of length ℓ at a coordinate distance χ away. From the FRW metric we see that it will appear to subtend an angle θ such that $\ell = RS_K\theta$. If we define the angular diameter distance D_A by how far away an object would have to be in regular Euclidean space to have an angular size θ, that is, $\ell = D_A\theta$, we have

$$D_A = RS_K = \frac{R_0 S_K}{1+\mathbf{z}}$$

For the case $\Omega_K = \Omega_\Lambda = 0$, $S_K(\chi) = \chi$ and $E(\mathbf{z}) = (1+\mathbf{z})^{3/2}$, which gives

$$S_K = \frac{2c}{\mathbf{H}_0 R_0}\left[1 - \frac{1}{\sqrt{1+\mathbf{z}}}\right] \qquad \Rightarrow \qquad D_A = \frac{2c}{\mathbf{H}_0}\frac{1}{1+\mathbf{z}}\left[1 - \frac{1}{\sqrt{1+\mathbf{z}}}\right]$$

which tends to $c\mathbf{z}/\mathbf{H}_0 \approx v/\mathbf{H}_0$ (the Euclidean distance according to Hubble's Law) for low $\mathbf{z}$, peaks at $\mathbf{z} = 5/4$, and then decreases: things further away start to cover larger angles!

1.4.2 Luminosity Distance

Say we put a standard candle (e.g. a Type Ia SN) of luminosity L at χ away. The photons from this source will be spread out over an area $4\pi R_0^2 S_K^2$ by the time they reach Earth. As discussed in §1.1, frequencies are reduced by a factor of $1 + \mathbf{z}$: not only does this decrease the energy of each photon (and hence the energy flux) by this factor, it also reduces the rate at which these photons arrive. As such, the energy flux is in fact $L/4\pi R_0^2 S_K^2 (1+\mathbf{z})^2$. Defining the luminosity distance D_L by how far away such an object would have to be in regular Euclidean space to have this flux, that is, $L/4\pi D_L^2$, we have

$$D_L = R_0 S_K (1 + \mathbf{z}) = D_A (1 + \mathbf{z})^2$$

1.4.3 Particle Horizon

The particle horizon is the radius of the past light cone at a given time: if at $t = 0$, every point everywhere emitted a photon towards us, the photon from the point now at a proper distance d_{ph} would have only just reached us; photons from sources further away have not reached us yet, and hence those sources are not in causal contact with us.

As before, $\mathrm{d}\chi = c\mathrm{d}t/R(t)$, so

$$\chi_{ph} = c \int_0^{t_0} \frac{\mathrm{d}t'}{R(t')} \qquad \Rightarrow \qquad d_{ph} = R_0 c \int_0^{t_0} \frac{\mathrm{d}t'}{R(t')}$$

For any Universe model, this gives a distance of order $d_{ph} \sim ct_0$.

1.4.4 Event Horizon

The event horizon is the radius of the *future* light cone at a given time: suppose we emit some light isotropically at t_0; it will eventually reach things that are currently located a distance

$$d_{eh} = R_0 c \int_{t_0}^{\infty} \frac{\mathrm{d}t'}{R(t')}$$

away. If this converges, this is the radius of the event horizon: the furthest point away that we will ever be able to affect. Similarly, objects currently further away than this will never be able to affect us.

2 Thermal History of the Universe

2.1 Statistical Physics Tools

2.1.1 Distributions and Densities

The phase-space density distribution function $f(\mathbf{r}, \mathbf{p})$ is the probability per $\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{p}\,/h^3$ of a particle having a position $\mathbf{r}$ and momentum $\mathbf{p}$; the factor of h^3 comes from the volume of a phase space element or something. For homogeneity $f(\mathbf{r}, \mathbf{p})$ cannot depend on position, so $f = f(\mathbf{p})$; for isotropy it cannot depend on the direction so $f = f(p)$. The number density is $\frac{1}{h^3}\int_{\mathbb{R}^3} f(\mathbf{p})\mathrm{d}^3\mathbf{p} = \frac{4\pi}{h^3}\int_0^\infty p^2 f(p)\,\mathrm{d}p$.

The fs for bosons and fermions, in terms of energy $E = E(p)$, are respectively:

$$f_B(E) = \frac{1}{e^{(E-\mu)/k_B T} - 1} \qquad f_F(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1} \tag{$\mathfrak{BE}/\mathfrak{FD}$}$$

The number density and total energy density for a particular species is hence:

$$n_i = \frac{4\pi g_i}{h^3}\int_0^\infty f(E(p))p^2\,\mathrm{d}p \qquad \rho_i c^2 = \frac{4\pi g_i}{h^3}\int_0^\infty f(E(p))E(p)p^2\,\mathrm{d}p$$

where g_i is the number of the species which can fit in the same $\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{p}$ (e.g. $g_{e^-} = 2$ as the electron has two spin states).

Taking the ultrarelativistic limit $E = pc \gg \mu$, the energy densities are respectively:

$$\text{Bosons} \qquad \rho_i c^2 = \frac{4\pi g_i}{h^3}\int_0^\infty \frac{p^3 c}{e^{pc/k_B T_i} - 1}\,\mathrm{d}p = \frac{4\pi g_i c}{h^3}\left(\frac{k_B T_i}{c}\right)^4 \underbrace{\int_0^\infty \frac{u^3}{e^u - 1}\,\mathrm{d}u}_{\pi^4/15} = \frac{g_i}{2}aT_i^4$$

$$\text{Fermions} \qquad \rho_j c^2 = \frac{4\pi g_j}{h^3}\int_0^\infty \frac{p^3 c}{e^{pc/k_B T_i} + 1}\,\mathrm{d}p = \frac{4\pi g_j c}{h^3}\left(\frac{k_B T_j}{c}\right)^4 \underbrace{\int_0^\infty \frac{u^3}{e^u + 1}\,\mathrm{d}u}_{(7/8)\pi^4/15} = \frac{7}{8}\frac{g_j}{2}aT_j^4$$

where we have substituted the value of a.

2.1.2 Effective Statistical Weight

The total energy density of a relativistic soup containing many different kinds of particles will be the sum of all their ρ_i contributions. We can define an *effective statistical weight* g_{eff} by imagining that the mixture of particles consists purely of photons which proportionally represent the energy contributions of all the species present. That is:

$$\frac{g_{\text{eff}}}{2}aT_\gamma^4 = \sum_{\text{Bosons}} \frac{g_i}{2}aT_i^4 + \frac{7}{8}\sum_{\text{Fermions}}\frac{g_j}{2}aT_j^4 \qquad \Rightarrow \qquad g_{\text{eff}} = \sum_{\text{Bosons}} g_i\left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8}\sum_{\text{Fermions}} g_j\left(\frac{T_j}{T_\gamma}\right)^4$$

where we allow the $\mathfrak{BE}/\mathfrak{FD}$ distributions to have different temperature parameters in general, in case the different components in the soup are not in thermal equilibrium, perhaps after some interaction process has frozen out. If they *are* in thermal equilibrium, the expression simplifies:

$$g_{\text{eff}} = \sum_{\text{Bosons}} g_i + \frac{7}{8}\sum_{\text{Fermions}} g_j$$

2.1.3 Chemical Potentials

For the reaction $A + B \rightleftharpoons C + D$ to be in equilibrium, it is required that $\mu_A + \mu_B = \mu_C + \mu_D$. Photons have no chemical potential because their numbers are not conserved, hence by considering $X + \bar{X} \rightleftharpoons \gamma$ we have $\mu_X = -\mu_{\bar{X}}$.

2.2 Particles in the Early Universe

Non-relativistic particles have $E \approx mc^2 + p^2/2m$, and hence the number density is about

$$n_i \approx \frac{4\pi g_i}{h^3} \int_0^\infty \frac{p^2}{e^{(m_i c^2 - \mu_i + p^2/2m_i)/k_B T} \pm 1} \,\mathrm{d}p$$

If we assume that $e^{(m_i c^2 - \mu_i)/k_B T} \gg 1$ (justified shortly) then the integral simplifies and the difference between bosons and fermions disappears:

$$n_i \approx \frac{4\pi g_i}{h^3} e^{-(m_i c^2 - \mu_i)/k_B T} \int_0^\infty p^2 e^{-p^2/2m_i k_B T} \,\mathrm{d}p = g_i e^{-(m_i c^2 - \mu_i)/k_B T} \left[\frac{h}{\sqrt{2 m_i \pi k_B T}} \right]^{-3}$$

where the quantity in the squackets is the thermal de Broglie wavelength. Rearranging,

$$g_i e^{-(m_i c^2 - \mu_i)/k_B T} = n_i \lambda^3$$

So the condition that $e^{(m_i c^2 - \mu_i)/k_B T} \gg 1$ is to impose that the gas is not quantumly dense.

Note that the number density is proportional to $e^{-m_i c^2/k_B T}$. In the early Universe $k_B T \gg 2m_i c^2$ for all particles in the Standard Model and hence they are all being produced ultrarelativistically in particle-antiparticle pairs, and hence contribute to the sum for g_{eff}. As T cools off, progressively lighter particles cease to be relativistic and annihilate with their antiparticles, exponentially suppressing their numbers. Today the only particles which are still relativistic and can contribute to g_{eff} are the massless γ and the almost-massless ν.

2.2.1 $k_B T \gtrsim$ 1MeV

Initially, all particles in the Universe were relativistic and being created and annihilated all the time: $g_{\text{eff}} = 106.75$. At $k_B T \sim 150\text{GeV}$, the electromagnetic and weak forces split (prior to this is *electroweak unification*) causing $W^\pm$ and Z^0 to drop out and g_{eff} to fall to about 80.

2.2.2 1MeV $\lesssim k_B T \lesssim$ 150MeV

As the temperature drops below 150MeV, a "QCD phase transition" occurs, causing quarks and gluons to drop out, and shortly afterwards the muons drop out. The only particles remaining are γ, ν and $e^\pm$. Weak interactions cause ν to be in equilibrium with $e^\pm$, and electromagnetic interactions "couple" $e^\pm$ and γ, so everything is at the same temperature. Hence[1]

$$g_{\text{eff}} = \underbrace{2}_{\gamma} + \frac{7}{8}\Bigg[\overbrace{\underbrace{2}_{\uparrow/\downarrow} \times \underbrace{2}_{e^+/e^-}}^{e^\pm} + \overbrace{3 \times 2}^{\nu_e/\nu_\mu/\nu_\tau} \Bigg] = 2 + \frac{70}{8} = 10.75$$

[1] Neutrinos are weird. Either they are their own antiparticle and have e.g. $g_{\nu_e} = 2$, or they are not their own antiparticle and $g_{\bar{\nu}_e} = g_{\nu_e} = 1$. Either way, each flavour of neutrino contributes 2.

2.2.3 $k_B T \ll 1$MeV and Neutrino Temperature

As T cools below 3MeV, the ν decouple from the $e^\pm$, but initially remain at the same temperature despite no longer being in equilibrium – they just aren't interacting with anything; they are *relics* of the time when they were interacting with $e^\pm$. The decoupling occurs because the rate of weak interactions[2] is $\Gamma_W = n_e \sigma_W c \propto T^3 T^2 = T^5$, whereas the Hubble parameter in the radiation-dominated era (where $\rho \propto R^{-4}$) is $\mathbf{H} = \mathbf{H}_0 \sqrt{\Omega_m (R_0/R)^4} \propto R^{-2} \propto T^2$. Thus as the temperature cools, Γ decreases faster than $\mathbf{H}$, eventually $\Gamma \ll \mathbf{H}$ and the weak interactions which were keeping ν and $e^\pm$ in equilibrium cease to occur at a meaningful rate. We say that neutrinos and electrons have *decoupled*, and that neutrinos have *frozen out*.

As the temperature cools below 1MeV, most photons are no longer able to do $e^\pm$ pair production, so the equilibrium fails and $e^\pm$ annihilate into γ unopposed. This will raise T_γ but not change T_ν because neutrinos are no longer influenced by γ or $e^\pm$. To deduce the temperature discrepancy, we need to look at entropy in an FRW metric. It can be shown that the entropy is generally given by

$$S = \frac{R^3}{T}(\rho + P) \propto \frac{\rho}{T} \propto g_{\text{eff}} T^3 R^3$$

where we have used $P \propto \rho$. The total entropy of the $e^\pm$ and γ (the ν have stopped interacting with these) is the same before and after the annihilation. Before, we have an energy density of $\frac{2}{2} a T_1^4 = a T_1^4$ from the γ and $\frac{7}{8} \times \frac{4}{2} a T_1^4 = \frac{7}{4} a T_1^4$ from the $e^\pm$, where T_1 is the temperature of the γ, $e^\pm$, and ν before the annihilation. Afterwards, we simply have an energy density of $a T_2^4$ from the photons alone; the neutrinos don't know that any of this has happened and are still at T_1. Setting $S \propto g_{\text{eff}} T^3$ for the electrons and photons equal before and after,

$$\left(2 + \frac{7}{2}\right) T_1^3 = 2 T_2^3 \qquad \Rightarrow \qquad T_1 = \left(\frac{4}{11}\right)^{1/3} T_2$$

$\sqrt[3]{4/11} \approx 0.71$, so the cosmic neutrino background is today expected to have $T_\nu \approx 1.95$K; less than that of the photons which gained a boost from the electron annihilation. Substituting this into the expression for g_{eff}, we have the current value as

$$g_{\text{eff}} = 2 + \frac{7}{8} \times 6 \times \left(\frac{4}{11}\right)^{4/3} \approx 3.36$$

2.3 Recombination

Jumping forwards a bit, the formation of atoms occurred when the temperature was at eV levels, thousands of years after all that other stuff.

When the reaction $p^+ + e^- \rightleftharpoons \mathrm{H} + k\gamma$ is in equilibrium, we have $\mu_p + \mu_e = \mu_H$. The number densities of the three species are

$$n_p = g_p e^{-(m_p c^2 - \mu_p)/k_B T} \lambda_p^{-3} \qquad n_e = g_e e^{-(m_e c^2 - \mu_e)/k_B T} \lambda_e^{-3} \qquad n_H = g_H e^{-(m_H c^2 - \mu_H)/k_B T} \lambda_H^{-3}$$

And hence the chemical potentials can be cancelled by taking interest in the fraction

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e} e^{-(m_H - m_p - m_e)c^2/k_B T} \left(\sqrt{\frac{m_H}{m_p m_e}} \frac{h}{\sqrt{2\pi k_B T}}\right)^3 \quad \Rightarrow \quad \frac{n_H}{n_e^2} \approx \left(\frac{h^2}{2 m_e \pi k_B T}\right)^{3/2} e^{Q/k_B T}$$

[2]The dependence of σ_W on T is apparently $\sigma_W \propto T^2$.

(the Saha equation) where we have used $g_H = 4$, $g_p = g_e = 2$, $m_H \approx m_p$, $n_e = n_p$ for neutrality, and defined $Q \equiv (m_p + m_e - m_H)c^2 = 13.6\text{eV}$.

As the temperature falls, this fraction will increase ($e^{Q/k_B T}$ beats $T^{-3/2}$). We are more interested in the ratio of electrons to baryons, so we define[3] the *ionisation fraction*:

$$x = \frac{n_e}{n_B} \qquad \Rightarrow \qquad 1 - x \approx 1 - \frac{n_e}{n_p + n_H} = \frac{n_p + n_H - n_E}{n_p + n_H} \approx \frac{n_H}{n_B} \qquad \Rightarrow \qquad \frac{n_H}{n_e^2} \approx \frac{1}{n_B}\frac{1-x}{x^2}$$

Hence Saha gives:

$$\frac{1-x}{x^2} = n_B \left(\frac{h^2}{2m_e \pi k_B T}\right)^{3/2} e^{Q/k_B T}$$

The number density of baryons is experimentally known to be $n_B = \eta n_\gamma$ where $\eta \approx 10^{-9}$, and n_γ can be derived as a function of temperature as

$$n_\gamma = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2}{e^{pc/k_B T} - 1}\,\mathrm{d}p = \frac{8\pi}{h^3}\left(\frac{k_B T}{c}\right)^3 \underbrace{\int_0^\infty \frac{x^2}{e^x - 1}\,\mathrm{d}x}_{\Gamma(3)\zeta(3) = 2\zeta(3)} = \frac{16\pi\zeta(3)}{h^3}\left(\frac{k_B T}{c}\right)^3$$

Hence the Saha equation becomes

$$\frac{1-x}{x^2} = \frac{16\pi\zeta(3)}{h^3}\left(\frac{k_B T}{c}\right)^3 \eta \left(\frac{h^2}{2m_e\pi k_B T}\right)^{3/2} e^{Q/k_B T} = \underbrace{\left(\frac{16\pi\zeta(3)}{(2\pi)^{3/2}}\right)}_{\approx 3.8} \eta \left(\frac{k_B T}{m_e c^2}\right)^{3/2} e^{Q/k_B T}$$

Thus we have a (quadratic) expression for the ionisation fraction as a function of temperature. Although $Q/k_B = 1.6 \times 10^5\text{K}$, we have $m_e c^2/k_B = 5.9 \times 10^9\text{K}$, so although the exponent starts to pick up around 10^5K, the small factors η and $(10^5/5.9 \times 10^9)^{3/2}$ mean that the temperature must reduce significantly further than this for the exponent to outweigh them[4].

The relationship between ionisation fraction and temperature is given in Figure 1. We see that x only begins to deviate from 1 (i.e. atoms only start forming) at around 7000K; this is because whenever an atom forms it is reionised almost immediately by one of the many billions of photons surrounding it.

It turns out that the analysis is more complicated, as other energy levels besides the ground state get involved. The ionisation fraction actually drops off a little later at about $T = 3800\text{K}$, which corresponds to a redshift of about 1100.

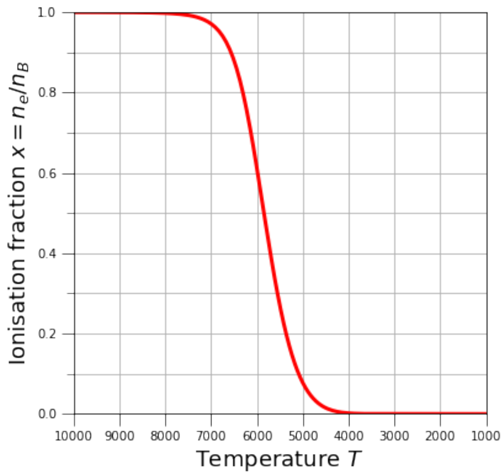

Figure 1 | Saha Model of Ionisation Fraction as the Universe Cools. Note that the fraction only starts to fall from 1 at around $T = 7000\text{K}$.

2.4 Nucleosynthesis

Primordial nucleosynthesis proceeds in several stages, overlapping with the stages of the previous sections. We begin by rewinding back to the early Universe when electrons were relativistic.

[3]Ignoring the contribution of He and other light elements to baryonic matter – in the next section we will see that there are about 0.08 as many He nuclei as H nuclei at this point

[4]Of course, the factor $\frac{k_B T}{m_e c^2}$ will also go down, but not as quickly as $e^{Q/k_B T}$ goes up

2.4.1 $k_B T \gtrsim 1$MeV: Equilibrium

At these temperatures p, n, γ, ν and $e^{\pm}$ are all kept in thermal equilibrium, by weak interactions like $n+\nu_e \rightleftharpoons p^+ + e^-$ and $e^{\pm}$ pair production. Observations are consistent with $\mu_{\nu_e} = \mu_{e^-} = 0$, so we must have $\mu_n = \mu_p$. Being very massive particles, protons and neutrons are non-relativistic at these temperatures; both are spin-1/2 so the ratio of their abundances (in equilibrium) is:

$$\frac{n_n}{n_p} = e^{-(m_n - m_p)c^2/k_B T} \left(\frac{m_n}{m_p}\right)^{3/2} \approx e^{-1.29\text{MeV}/k_B T}$$

2.4.2 $k_B T \approx 0.8$MeV: Freeze-out

$\Gamma_W \propto T^5$ and $\mathbf{H} \propto T^2$, so as T decreases eventually the weak interactions failed to maintain equilibrium between neutrons and protons. This happened when the Universe was about 1s old and $k_B T \approx 0.8$MeV, and hence when the neutron-to-proton ratio was

$$\frac{n_n}{n_p} = e^{-1.29/0.8} \approx \frac{1}{5}$$

It is fortunate that the details of the weak force ensure $\Gamma = \mathbf{H}$ at around $k_B T \sim (m_n - m_p)c^2$, as otherwise n_n/n_p would be nowhere near 1 and atomic nuclei would be difficult.

2.4.3 Deuterium Bottleneck and Helium Formation

Free neutrons decay over time as $e^{-t/880s}$, so time is running out for nucleosynthesis. Annoyingly, the only thing that can be made out of just proton-neutron collisions is D, which takes a nail-bitingly slow 300s to appreciably form (this is the "deuterium bottleneck"), for the same reasons that it took so long for recombination (low baryon-to-photon ratio and high $m_p c^2/k_B$). The neutron-to-proton ratio in the nuclei that form to rescue the neutrons from impending decay is then

$$\frac{n_n}{n_p} = \frac{1}{5} e^{-300/880} \approx 0.14$$

Those are the neutrons which go into forming D, which is stable, so is no longer under time pressure and can casually form He. The number density n_D before forming He is the same as that of neutrons n_n, but after two Ds make a He we have $n_{\text{He}} = n_n/2$. The number of hydrogen nuclei is the original number of protons minus the number that pair up with neutrons to make D, so $n_H = n_p - n_n$. We therefore have

$$\frac{n_{\text{He}}}{n_{\text{H}}} = \frac{n_n/2}{n_p - n_n} = \frac{0.07}{1 - 0.14} = 0.08$$

This is the *number* ratio of He to H. But He is more massive than H, so the mass ratio will be:

$$Y = \frac{4n_{\text{He}}}{n_H + 4n_{\text{He}}} = \frac{0.32}{1.32} \approx 0.25$$

which is pretty accurate, and far more than stellar nucleosynthesis could have achieved by now.

2.5 Timings

This has all been in no particular order, so we should get a handle on some timings.

2.5.1 Matter-Radiation Equality

Today, surveys suggest $\Omega_m \approx 0.31$, mostly dark matter, whereas Ω_r:

$$\Omega_r = \frac{\rho_\gamma c^2/c^2}{3\mathbf{H}_0^2/8\pi G} = \frac{8\pi G a T_{\gamma,0}^4}{3c^2\mathbf{H}_0^2} \approx 5\times 10^{-5}$$

which is orders of magnitude lower. However, radiation density is proportional to R^{-4}, whereas matter density is proportional to R^{-3}, so there was once a time when the two were equal:

$$\Omega_m\left(\frac{R_0}{R}\right)^3 = \Omega_r\left(\frac{R_0}{R}\right)^4 \qquad \Rightarrow \qquad \frac{R_0}{R} = 1+\mathbf{z}_{mr} = \frac{\Omega_m}{\Omega_r} \approx 6100$$

which is way off the true value of 3400 because we have forgotten to include the neutrinos, which bump Ω_r up to more like 9×10^{-5}. It is important to pin this redshift down because $R(t)$ evolves differently depending on whether matter or radiation are dominant at a given time.

2.5.2 Evolution and Composition

Neglecting curvature and the cosmological constant, $\mathfrak{F}2$ becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \qquad \Rightarrow \qquad \dot{R} \propto R\rho^{1/2}$$

Depending on the proportionality between ρ and R, this differential equation has different solutions. Since matter-radiation equality, matter has been dominant and $\rho \propto R^{-3}$; hence

$$\dot{R} \propto R^{-1/2} \qquad \Rightarrow \qquad R(t) \propto t^{2/3} \qquad \Rightarrow \qquad 1+\mathbf{z} \propto t^{-2/3}$$

We can use this to calculate the time of matter-radiation equality, as since then the Universe has been matter-dominated and we know what the redshift was then. We have

$$1+\mathbf{z}_{mr} = \left(\frac{t_{mr}}{t_0}\right)^{-2/3} \qquad \Rightarrow \qquad t_{mr} = (3400)^{-3/2} t_0 \approx 70,000 \text{ years}$$

a more accurate calculation gives 50,000 years but oh well.

Before matter-radiation equality, radiation was dominant and so $\rho \propto R^{-4}$; hence

$$\dot{R} \propto R^{-1} \qquad \Rightarrow \qquad R(t) \propto t^{1/2} \qquad \Rightarrow \qquad 1+\mathbf{z} \propto t^{-1/2}$$

During this period the temperature is also a measure of the time: $T \propto R^{-1} \propto t^{-1/2} \propto 1+\mathbf{z}$. Subtly though, because the entropy $g_{\text{eff}}T^3R^3$ is conserved, the proportionality constants change whenever g_{eff} does.

2.5.3 Everything Else

Everything else is a bit harder to calculate timings for, but the main events of the hot early Universe are summarised in Figure 2 overleaf.

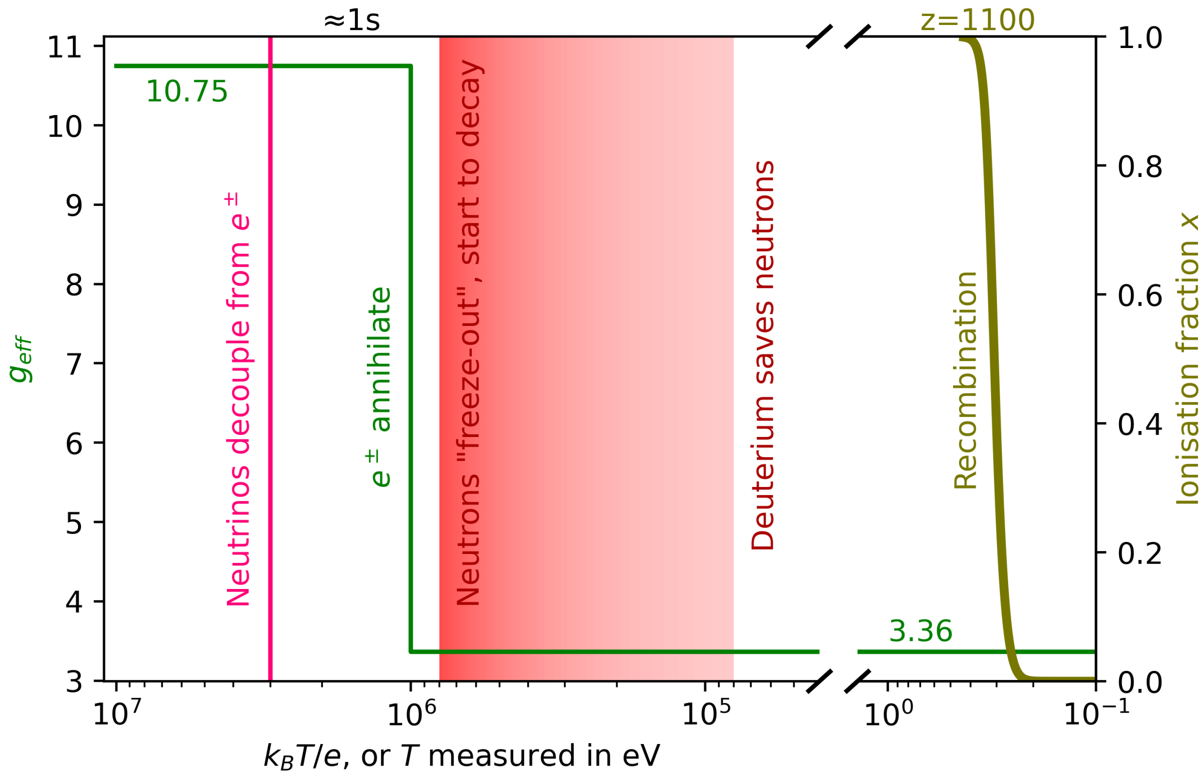

Figure 2 | Timeline of events in the hot early Universe.

3 Dark Matter

3.1 Ways of Measuring Mass

- **Virial Theorem.** In virialised systems, $\langle K \rangle = -\frac{1}{2} \langle U \rangle \Rightarrow \frac{1}{2}v^2 = GM_r/2r$. By looking measuring velocity dispersions (v) and the distances to the centre of the relevant system (galaxy, galaxy cluster, ...) one can estimate the mass enclosed at a given radius.
- **X-Ray Measurements.** For a system in hydrostatic equilibrium, its gas satisfies $\mathrm{d}P/\mathrm{d}r = -GM(r)\rho/r^2$ and $P = nk_BT = \rho k_B T/\mu m_H$. X-ray spectra determine $T(r)$ and $n(r)$, so substituting and solving for $M(r)$ gives[5]

$$M(r) = -\frac{k_B T r}{G\mu m_H}\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln n}{\mathrm{d}\ln r}\right)$$

- **Galaxy Luminosity Function.** Surveys give this as a Schechter function:

$$\phi(L)\,\mathrm{d}L = \frac{\phi^*}{L^*}\left(\frac{L}{L^*}\right)^{\alpha} e^{-L/L^*}\,\mathrm{d}L \qquad (\alpha \approx -1)$$

  Integrating[6], we obtain the total luminosity per unit volume. Suppose the Universe has the critical density ρ_c: this would give an average mass-to-luminosity ratio for the Universe. It is found to be 10^3 larger than that of stars.

[5] TPoC is missing lns in the derivatives and hence is dimensionally incorrect.

[6] The integral doesn't seem to converge for $\alpha = -1$ but oh well

- **Gravitational Lensing.** GR says that a point mass M deflects a light ray at a large impact parameter b by an angle

$$\alpha = \frac{4GM}{bc^2}$$

By considering the source plane in Figure 3, we require that $\alpha D_{LS} + \theta_S D_S = \theta D_S$ (angular diameter distances). Substituting α from above and $b = \theta D_L$, we obtain

$$\theta(\theta - \theta_S) = \frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S} \equiv \theta_E^2$$

– a quadratic in θ. The product of the two solutions $|\theta^+\theta^-| = \theta_E^2$. if the source is directly behind the lens ($\theta_S = 0$), then we have $\theta = \pm\theta_E$; in 2D this will appear as a ring around the lensing mass. Lensing also achromatically magnifies sources, giving a characteristic light curve as a lens passes in front of a source.

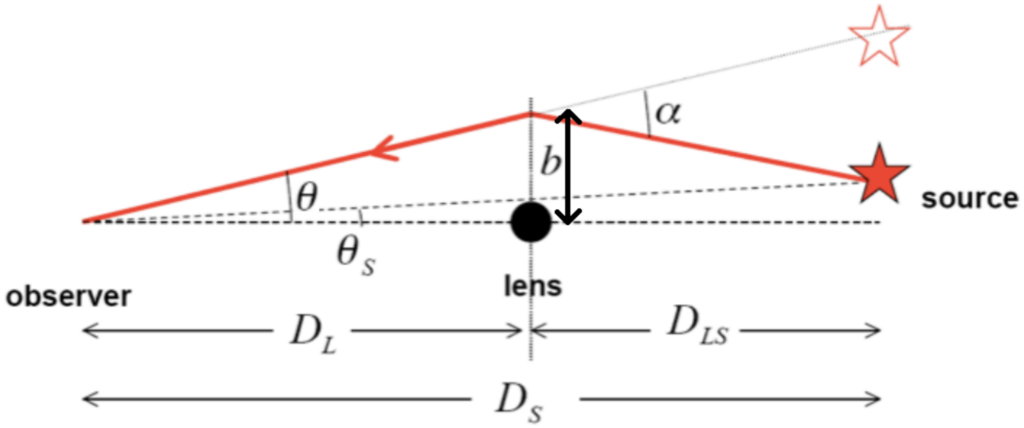

Figure 3 | Diagram of Gravitational Lens. The distances shown are angular diameter distances.

3.2 Dark Matter Candidates

- **Massive astrophysical compact halo objects (MACHOs).** These may consist of Jupiter-sized objects, or primordial black holes (those with $M \gtrsim 3M_\odot$ would not yet have decayed). However, microlensing surveys show that if MACHOs are between $10^{-7}M_\odot$ and $10M_\odot$, they cannot contribute very much to the Milky Way's halo mass. Also, no gravitational wave signals from substellar black holes have ever been detected.

- **Neutrinos.** These are neutral and don't interact much with light, so were compelling candidates. Calculating the fermion integrals for neutrinos and comparing to photons, we find

$$\rho_\nu = \frac{3}{11} m_\nu n_\gamma \qquad \Rightarrow \qquad \Omega_\nu \sim 10^{-3}$$

which is still nowhere near enough (and also weirdly similar to stars' contribution)

4 Dark Energy and Inflation

4.1 Cosmological Constant

Using $\mathfrak{F}2$, we can think of Λ as being due to a fluid with density $\rho_\Lambda = \Lambda c^2/8\pi G$. However, Λ (and so ρ_Λ) is a constant according to $\mathfrak{EFE}$. Using conservation, $\rho \propto R^{-3(w+1)}$, so for ρ_Λ to be constant we require its density to have $w_\Lambda = -1$. Very weird fluid.

According to $\mathfrak{F}1$, ρ and P cause the Universe's expansion to decelerate, and Λ causes it to accelerate; at late times, ρ and P will dilute away and Λ will cause *late-time acceleration*. Alternatively, some obscure fluid with $w < -1/3$ would also make the first term positive and help the Universe accelerate.

A suggestion was that Λ might originate from the QFT vacuum energy: each mode of the electromagnetic field has a ground state energy $\hbar\omega/2 = \hbar ck/2$. Integrating over all wavevectors up to $1/\ell_P$, we would have a vacuum energy of order $\hbar c\ell_P^{-4} \sim 10^{113}\mathrm{Jm}^{-3}$. However, according to measurements, $\rho_\Lambda c^2 = \Lambda c^4/8\pi G \sim 10^{-9}\mathrm{Jm}^{-3}$, wrong by of order 10^2 orders of magnitude!

4.2 Problems with the Early Universe

4.2.1 Horizon Problem

From §1.4.3, the particle horizon at a time t is $R(t)c\int_0^t \mathrm{d}t'\,/R(t')$. Now if a single fluid with some $w \geq 0$ is dominant in the Universe, $\rho \propto R^{-3(w+1)}$, for which $\mathfrak{F}2$ yields $\dot{R} \propto R^{-(3w+1)/2} \Rightarrow R \propto t^\alpha$ with $\alpha = 2/(3(w+1))$. For such a Universe, $d_{ph} = ct/(1-\alpha)$.

Now consider the almost totally uniform CMB. By setting $d_{ph,\mathrm{rec}} = D_{A,\mathrm{rec}}\theta_{\mathrm{rec}}$ and calculating d_{ph} and D_A at recombination ($z \approx 1100$), we find that $\theta_{\mathrm{rec}} \approx 1100^{-1/2} = 1.7°$(about 3.5 full Moons), so any correlations in the CMB over larger scales than this would be between regions not in causal contact. Yet the CMB is almost uniform across the sky, suggesting the whole thing was in thermal equilibrium. The Universe therefore cannot have had $R \propto t^\alpha$ as from a conventional fluid.

4.2.2 Flatness Problem

It turns out that $K = 0$ is an unstable equilibrium – if there were any curvature in the early Universe it would have become more important over time, as the curvature term goes as R^{-2}, rather than matter (R^{-3}) or radiation (R^{-4}). We know today that $|\Omega_K| < 0.01$, so the Universe is probably just flat, but if not then some very fine tuning would have been required.

4.3 Inflaton Field ϕ

As mentioned, some fluid with $w < -1/3$ could also cause inflation. This is presented as a scalar "inflaton" field $\phi(\mathbf{x}, t)$, whose kinetic energy is $\frac{1}{2}\dot{\phi}^2$ and potential energy is some $V(\phi)$. If the inflaton fluid is isotropic ($\phi = \phi(t)$ alone), it turns out that

$$\boxed{\rho_\phi c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)} \qquad \boxed{P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)}$$

so depending on the exact functions $\phi(t)$ and $V(\phi)$, the equation of state parameter $w_\phi \equiv P_\phi/\rho_\phi c^2$ may (unlike Λ) change over time[7]. Furthermore, it turns out that the Euler-Lagrange

[7] in these theories, ϕ is called *quintessence*; this is not considered further

equations in GR give:

$$\boxed{\ddot{\phi} + 3\mathbf{H}\dot{\phi} + V'(\phi) = 0}$$

It helps intuition to think of the above as equivalent to the equation of motion for a particle moving in a potential $V(\phi)$, with a drag term proportional to $\mathbf{H}$ (known as *Hubble friction*). In general, this equation must be solved together with $\mathfrak{F}2$, which in this case is

$$\mathbf{H}^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

If the inflaton field changes only slowly, so that $\dot{\phi}^2 \ll V(\phi)$ (known as the *slow-roll condition*), then $\rho_\phi c^2 \approx V(\phi)$, and because $\dot{\phi}$ is small $V(\phi)$ barely changes, so ρ_ϕ is roughly constant[8]. Substituting $\rho_\phi \approx V(\phi)/c^2$ into $\mathfrak{F}1$ or $\mathfrak{F}2$ (with $\Lambda = K = 0$) gives

$$R(t) \propto e^{\mathbf{H}t} \qquad \text{where} \qquad \mathbf{H} = \sqrt{\frac{8\pi G}{3c^2}V(\phi)}$$

This exponential dependence does not have the same problems as the power-law R that a fluid has. If this exponential phase (or, if you will, *inflation*) lasts long enough that R increases by a factor of over about e^{60} (i.e. 60 "e-foldings") then the horizon and flatness problems disappear.

Inflation due to a scalar field can be made temporary. The equation of motion for ϕ shows that if $V(\phi)$ begins high on a slope, it will accelerate down said slope, increasing $|\dot{\phi}|$. As $V(\phi)$ approaches a minimum, the Hubble friction will slow it down, dissipating energy away from the scalar field into the creation of other components (like ordinary matter and radiation) which take over proceedings as ϕ fades away. This process is called *reheating*.

5 Fluctuations

5.1 Newtonian Approach

Writing the equations of fluid dynamics, including Newtonian gravity,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \qquad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla}\phi - \frac{1}{\rho}\boldsymbol{\nabla}P \qquad \boldsymbol{\nabla}^2\phi = 4\pi G\rho$$

where $\mathbf{v} = \mathrm{d}(R(t)\mathbf{x})/\mathrm{d}t$ (i.e. $\mathbf{v}$ is the physical velocity and $\mathbf{u}$ is the comoving velocity). Transforming the above equations to comoving coordinates, writing $\rho = \bar{\rho}(1 + \delta(\mathbf{x}, t))$, Fourier transforming, and solving,

$$\boxed{\ddot{\delta}_\mathbf{k} + 2\mathbf{H}\dot{\delta}_\mathbf{k} - c_s^2\left(k_J^2 - \frac{k^2}{R^2}\right)\delta_\mathbf{k} = 0}$$

where $c_s^2 \equiv \partial P/\partial\rho$ and $k_J = \sqrt{4\pi G\bar{\rho}}/c_s$. There is again a Hubble friction term.

[8]Note that the slow-roll condition also gives $P_\phi \approx -V(\phi) \approx \rho_\phi$, so $w \approx -1$. Together with a constant ρ, this mimics a cosmological constant $\Lambda_\phi = 8\pi G\rho_\phi/c^2 = 8\pi GV(\phi)/c^4$.

5.1.1 Large-Scale Perturbations

If the perturbations take place on a large scale, then the k^2/R^2 term will be negligible; this term is descended from the $\boldsymbol{\nabla}P$ term in the fluid equations so such perturbations are interpreted as those of a pressureless fluid. We then have $\ddot{\delta}_{\mathbf{k}} + 2\mathbf{H}\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho} = 0$. This will have different solutions depending on the Universe model used, as the forms of $\mathbf{H}$ and $\bar{\rho}$ will depend on the Ω_i. It is usually found that there is a growing mode and a decaying mode.

5.1.2 Small-Scale Perturbations

Smaller perturbations have $k^2/R^2 \gg k_J^2$, so we can then neglect the k_J term, equivalent to switching gravity off (setting $G = 0$). For very high-frequency perturbations we can also neglect $\mathbf{H}$, so we get simply $\ddot{\delta}_{\mathbf{k}} = -(c_s k/R)^2 \delta_{\mathbf{k}}$ and hence $\delta_{\mathbf{k}} \propto \exp(ic_s kt/R)$.

5.2 Correlation Function

By definition of $\bar{\rho}$, we must have $\langle \delta(\mathbf{x}) \rangle = 0$. The first non-trivial statistic we might look for would be the (two-point) correlation function, written

$$\xi(|\mathbf{x} - \mathbf{x}'|) = \langle \delta(\mathbf{x})\delta(\mathbf{x}') \rangle$$

where due to isotropy ξ can only depend on the distance between two points, not their relative orientation. We can therefore rewrite this as $\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x} + \mathbf{r}) \rangle$.

We now wish to convert this to Fourier space, using the convention

$$\delta(\mathbf{x}) = \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \delta_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{x}}$$

where we have used the fact that $\delta(\mathbf{x}) \in \mathbb{R}$ by its definition, so taking the complex conjugate does nothing. This fact makes it easier to the correlation function to $\mathbf{k}$-space:

$$\begin{aligned}
\xi(r) &= \frac{1}{V} \int \mathrm{d}^3\mathbf{x}\, \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}'\, \delta_{\mathbf{k}'}^* e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} \\
&= \frac{V}{(2\pi)^6} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{k}'\, \delta_{\mathbf{k}} \delta_{\mathbf{k}'}^* e^{-i\mathbf{k}'\cdot\mathbf{r}} \underbrace{\int \mathrm{d}^3\mathbf{x}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')} \\
&= \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, |\delta_{\mathbf{k}}|^2 e^{-i\mathbf{k}\cdot\mathbf{r}}
\end{aligned}$$

We see that $\xi(r)$ is the Fourier transform of the *power spectrum* $P(k) = |\delta_{\mathbf{k}}|^2$. If $P(k) \propto k^{n_s}$ in some model, then n_s is the *spectral index*.

5.3 Anisotropies in the CMB

Fluctuations occurring before recombination imprint anisotropies $\delta T(\theta, \phi)$ on the CMB. As we observe them on the inside of a sphere, they should be expanded as a spherical harmonic series:

$$\frac{\delta T}{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

We can get an idea of the angular scales which are most prominent in the CMB anisotropies by considering the CMB power spectrum. ℓ is the most important parameter for the angular scales, which are about $180°/\ell$; m is mostly about orientation. The CMB power spectrum is:

$$C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$$

It is conventional to show the CMB spectrum as a plot of $\ell(\ell+1)C_\ell/2\pi$ against ℓ, because apparently if $P(k) \propto k$ (that is, a spectral index of $n_s = 1$), then $C_\ell \propto [\ell(\ell+1)]^{-1}$. As the spectrum (Figure 4) shows, it is more complicated than this.

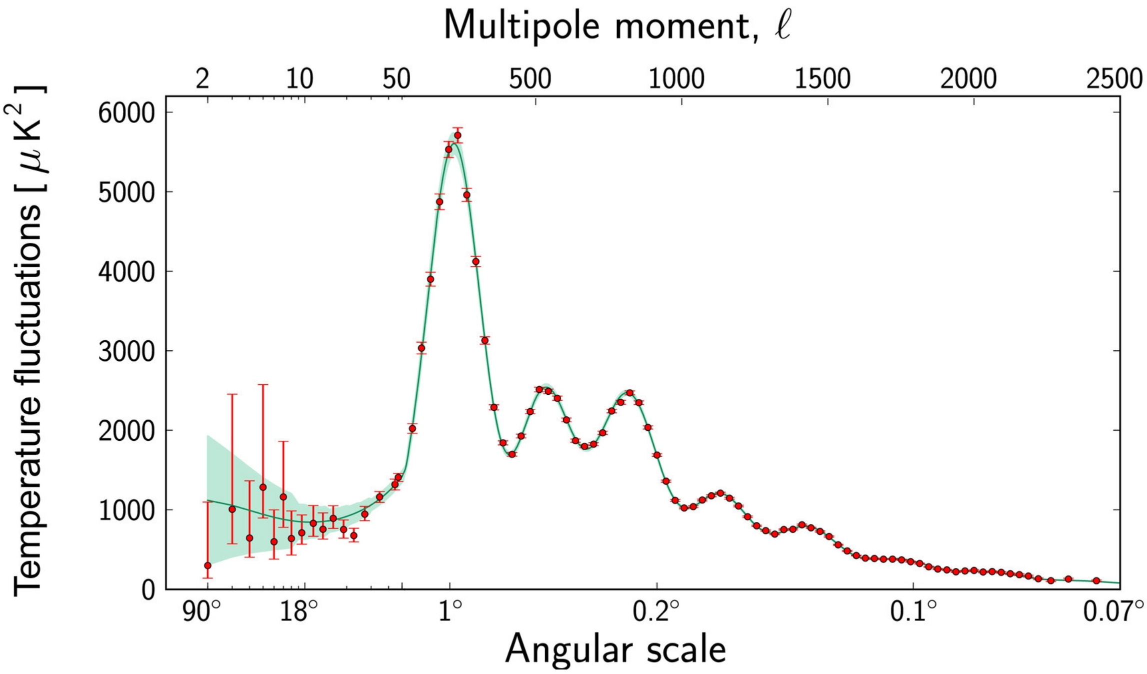

Figure 4 | **Power Spectrum of the CMB.** The red data points were measured by *Planck*; the green line is the cosmological model which best fits the data.

5.3.1 Features of the CMB Spectrum

- $\ell \lesssim 30$**, the Sachs-Wolfe Plateau.** These fluctuations are a result of random overdensities of size λ at the time of recombination, which would require photons to climb out of potential wells and thus be redshifted and appear colder. It turns out that if $P(k) \propto k^{n_s}$, then $\Delta T/T \propto \lambda^{(1-n_s)/2}$, so a spectral index of $n_s = 1$ would mean that $\Delta T/T$ is *scale-invariant*, and hence fluctuations of any (angular) size are equally likely. This plateau is quite flat so this is almost the case, but in fact the spectral index is slightly "tilted redward" to $n_s \approx 0.97$.

- $\ell \approx 200$**, First Acoustic Peak.** Before recombination, photons and baryons were constantly interacting ("strongly coupled") by Thomson scattering, and as such behave essentially as a single fluid. This radiation component of the fluid has a sound speed given by $c_s^2 = \mathrm{d}P/\mathrm{d}\rho = c^2/3$, so sound waves in this fluid (known as *baryon acoustic oscillations*) can travel at a maximum speed[9] of $c/\sqrt{3}$. Now recall from §4.2.1 that

[9]When doing this properly, the baryons do get involved and affect the sound speed; this analysis is an approximation to get the idea across.

the angular scale of the particle horizon at recombination is about 1.7°. The particle horizon involved seeing how far light could travel since the Big Bang; the baryon acoustic oscillations can only travel $1/\sqrt{3}$ as fast and hence would cover a size of only $1.7°/\sqrt{3} \approx 1°$– which is where the first peak in the CMB power spectrum is. This calculation was sensitive to the curvature of the Universe, and the result is consistent with $K = 0$ to a high degree of accuracy.

Further acoustic peaks correspond to sound waves that have integer numbers of oscillations within the sound horizon. Their relative amplitudes apparently give information about the amounts of baryons and dark matter.

- $\ell \gtrsim 1000$**, Silk Damping.** If the coupling between photons and baryons were perfect, the photon mean free path would be 0 and the two would form a perfect fluid. However the coupling was not perfect (even before recombination) so the mean free path would be finite, though not infinite. As such, temperature inhomogeneities get smoothed out, more so on smaller scales. Temperature fluctuations therefore show a damping tail.

5.4 Non-linear Fluctuations

5.4.1 Spherical Collapse Model

Consider a sphere of radius $r(t)$, somewhere within which is a spherically symmetric mass M. Then $r(t)$ will evolve according to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{r^2}$$

The solutions to this equation can be written parametrically (in the *development angle* θ) as:

$$r(\theta) = A(1 - \cos\theta) \qquad t(\theta) = B(\theta - \sin\theta) \qquad A^3 = GMB^2$$

We see that t is a monotonically increasing function of θ, and hence that r will increase from 0 up to $2A$ and then fall back to 0[10]. Expanding $t(\theta)$, inverting and substituting into $r(\theta)$,

$$r(\theta) = \left(\frac{9GMt^2}{2}\right)^{1/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3} + \dots\right]$$

The density of the perturbation can then be written at early times as:

$$\rho_p(t) \equiv \frac{M}{\frac{4}{3}\pi r^3} = \frac{1}{6\pi G t^2}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} + \dots\right]$$

Now for the model with $\Omega_K = \Omega_\Lambda = 0$ and $\Omega_m = 1$, solving $\mathfrak{F}2$ gives

$$R(t) = (6\pi G\bar{\rho}_0)^{1/3} R_0 t^{2/3} \qquad \Rightarrow \qquad \bar{\rho}(t) = \frac{1}{6\pi G t^2}$$

The circular mass described above can be thought of as a perturbation to this background overall density. The relative perturbation $\delta \equiv \rho_p/\bar{\rho} - 1$ is then given at small times by:

$$\delta = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

[10] This is what happens in a closed Universe; in an open Universe the solutions become hyperbolic.

In fact, if we used the linear theory in this Universe model, we would have also found a growing mode $\delta \propto t^{2/3}$.

Returning to the non-linear solution with sines and cosines, we see that r reaches a maximum at $r = 2A$ at $\theta = \pi$ and $t = B\pi$. At this time, we have

$$\rho_p = \frac{3M}{32\pi A^3}, \qquad \bar{\rho} = \frac{1}{6\pi^3 GB^2} \qquad \Rightarrow \qquad \delta = \frac{\rho_p}{\bar{\rho}} - 1 = \frac{9\pi^2}{16} - 1 \approx 4.55$$

whereas linear theory would at this point have given this ratio as $\frac{3}{20}(6\pi)^{2/3} \approx 1.06$, so the non-linear theory shows that perturbations can get much denser when they stop being linear.

5.4.2 Formation of Virialised Dark Matter Halos

At $\theta = 2\pi$, $r(\theta) \to 0$, which would suggest that $\rho_p \to \infty$. In fact the spherical collapse model becomes unrealistic here, and what actually happens is the perturbation settles into a virialised system, interpreted as a dark matter halo which will go on to host a galaxy. Being in virial equilibrium, we have $T = -\frac{1}{2}W$, and conservation of energy demands that $T + W = W_m$, where W_m denotes the potential when $r(\theta)$ is at a maximum (being stationary at that time, $T_m \ll W_m$). Hence we find that $W = 2W_m$. But $W_m \propto r^{-1}$; for a spherical system we have $W = -3GM^2/5r$. Hence rather than collapsing to $r = 0$, the system stabilises at $r = \frac{1}{2}r_m$. The density of the perturbation at $t = 2t_m = 2B\pi$ is thus $8\times$ the density at maximum expansion. Finally, using the time dependence of the background density $\bar{\rho}$, we find

$$\frac{\rho_p\big|_{t=2B\pi}}{\bar{\rho}\big|_{t=2B\pi}} = \frac{8\ \rho_p\big|_{t=B\pi}}{\frac{1}{4}\ \bar{\rho}\big|_{t=B\pi}} = 32\left(\frac{9\pi^2}{16}\right) = 18\pi^2 \qquad \Rightarrow \qquad \delta = 18\pi^2 - 1 \approx 177$$

This is the final density contrast for a virialised dark matter halo. More advanced models involving Λ give about 200; this is backed up by simulations. If we just used the linear theory up to $t = 2B\pi$, we would obtain just $\delta \approx 1.69$, so if we work within linear perturbation theory and δ reaches 1.69 then that should be interpreted as a collapse to virial equilibrium.