Black Holes

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1 Basic Properties

A black hole (BH) is an object that creates strong enough spacetime curvature that it is out of causal contact with the rest of the Universe. The no-hair theorem states that fields external to BHs are characterised only by the mass, spin, and charge; charged BHs are unrealistic and so we do not consider the charged (Reissner-Nordström and Kerr-Newman) solutions.

1.1 Observations

The BH mass distribution function is bimodal. Stellar mass BHs have $M \in\sim [3, 10^2]M_\odot$ and are the remains of stars with $M \gtrsim 25M_\odot$. They are observed in X-ray binaries (XRBs) and the gravitational waves of merging BH binaries. Supermassive BHs (SMBHs) have $M \in\sim [10^5, 10^{10}]M_\odot$, and are at the centres of most massive galaxies. Rapidly accreting SMBH can outshine all of the stars in their galaxy, being then labelled *active galactic nuclei* (AGN), the most powerful of which being called *quasars*.

BH solutions of GR require large densities and hence compactness. This is evidenced by fast variability (short crossing times, hence small radii), large Doppler shifts in surrounding gas (hence fast orbits), or, in the case of Sgr A* in the Milky Way, resolved orbits of stars orbiting an point source that is invisible without the aid of an Earth-sized radio telescope.

1.2 Schwarzschild Black Holes

In Newtonian gravity, the energy equation is

$$\frac{1}{2}\dot{r}^2 + \Phi_{\text{eff}}(r) = E, \qquad \Phi_{\text{eff}}(r) = \frac{h^2}{2r^2} - \frac{GM}{r}$$

where h is the specific AM. Newtonian theory is inaccurate when $r \lesssim r_G \equiv GM/c^2$. Instead:

$$\frac{1}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V(r)^2 = E^2, \qquad V(r)^2 = \left(1 - \frac{2GM}{rc^2}\right)\left(1 + \frac{h^2}{r^2c^2}\right); \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = E\left(1 - \frac{2GM}{rc^2}\right)^{-1}$$

where τ is *proper* time and h is relativistic specific AM. As $r \to 2GM/c^2$, we see that $V \to 0$ and $\mathrm{d}t/\mathrm{d}\tau \to \infty$. The surface $r = r_h \equiv 2GM/c^2 = 2r_G$ is the event horizon[1]. Some curves for $V(r)$ are shown in Figure 1. Differentiating $V(r)$ shows[2] its stationary points to be:

[1] Mathematically, this surface is not a singularity of spacetime as the curvature tensor is finite; mass can freely fall through this surface (though only in one direction!).

[2] This is least painfully done by finding $\mathrm{d}V^2/\mathrm{d}(r_g/r)$, which the chain rule says is proportional to $\mathrm{d}V/\mathrm{d}r$.

$$\frac{r}{r_G} = \frac{\bar{h}^2}{2}\left(1 \pm \sqrt{\bar{h}^2 - 12}\right), \qquad \bar{h} \equiv \frac{hc}{GM}$$

Hence for $h \leq 2\sqrt{3}GM/c$, the potential is monotonically increasing, and for $h > 2\sqrt{3}GM/c$ there are two stationary points due to a centrifugal barrier emerging; it turns out one is stable and the other unstable. If $E = \max V$, then we will have $\mathrm{d}r/\mathrm{d}\tau = 0$ at the maximum of the potential, so the particle cannot fall into the BH. But if $E > \max V$, then the centrifugal barrier can be overcome, for any h.

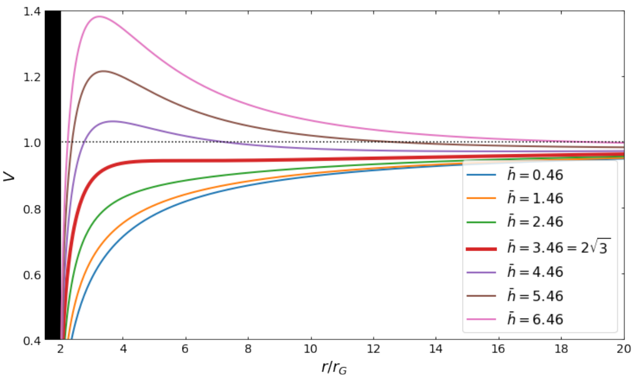

Figure 1 | Relativistic potentials for different AM. The critical curve, for $\bar{h} = 2\sqrt{3}$, is shown in red.

As h approaches the critical value from above, the location of the stable orbit (at which a particle can stably orbit the BH) moves inwards; at the critical h, we see that the circular orbit is at $r = 6r_G = 6GM/c^2$: this is the *innermost stable circular orbit*, or ISCO. It is here that accretion disks around BHs will have an inner edge, with anything within plummeting.

1.3 Kerr Black Holes

BH rotation affects the spacetime. Their rotation is encapsulated by the distance $a \equiv h_{\mathrm{BH}}/c$, where h_{BH} is the specific AM of the rotating BH. The event horizon is modified from $2r_G$ to $r_G + \sqrt{r_G^2 - a^2}$ in the equatorial plane, naturally tending to the Schwarzschild radius as $a \to 0$. The ISCO depends in a complicated way on a; for $a \to r_G$ (a *maximally spinning* BH), we find $r_{\mathrm{ISCO}} \to r_G$ and $r_h \to r_G$ so the orbits go right to the edge. Importantly, this allows more energy to be churned out of accreting matter before it is lost within the event horizon.

Matter orbiting in the opposite direction to a maximally spinning BH has $r_{\mathrm{ISCO}} = 9r_G$. Counter-rotating matter is dragged round onto a corotating orbit before crossing the horizon.

1.4 Entropy & Collisions

Stephen Hawking's epitaph says that the entropy of a BH is $S_{\mathrm{BH}} = (k_B c^3/4\hbar G)A$, where A is its surface area. Thus the sum of the surface areas of all the BHs in the Universe can never decrease unless something very entropic happens. The area turns out to be given by:

$$A = 4\pi\left[r_h^2 + a^2\right] = 4\pi\left[\left(r_G + \sqrt{r_G^2 - a^2}\right)^2 + a^2\right] \xrightarrow{a\to 0} 4\pi\left(\frac{2GM}{c^2}\right)^2$$

A rotating BH can therefore have mass (i.e. energy) extracted from it, so long as it is compensated by decreasing a and the mass does not fall below $M_{\mathrm{irr}} = \sqrt{A/16\pi}$; mass cannot be extracted from a Schwarzschild BH. Consider two Schwarzschild BHs, each with mass M_1, colliding to form a single BH with mass M_2. Let the colllision be head-on, so that the resulting BH will have no AM and thus also be Schwarzschild. We require

$$4\pi\left(\frac{2GM_2}{c^2}\right)^2 \geq 2 \times 4\pi\left(\frac{2GM_1}{c^2}\right)^2 \qquad \Rightarrow \qquad M_2 \geq \sqrt{2}M_1$$

which is less than the original amount of mass! The rest (at most $(2-\sqrt{2})M_1 \approx 0.59M_1$) is radiated away as gravitational waves.

1.5 Luminosity & Radiative Efficiency

AGN are sourced by extraction of potential energy from accreting gas onto a compact SMBH. In the Newtonian limit, the potential energy released per unit mass falling onto a body of mass M and radius R is GM/R; for a neutron star ($M = M_\odot$, $R = 10$km), this gives 10^{16}J/kg. Nuclear fusion gives $0.007c^2 = 6 \times 10^{14}$J/kg, which is much lower; clearly this will be *way* lower than the energy from BH accretion. Neutron stars have a hard surface, so accreting gas shocks near the surface, converting all its kinetic energy to thermal energy and then radiation, with a luminosity $L = GM\dot{M}/R$, where $\dot{M}$ is the rate at which mass is falling onto the star. However, BHs don't have a solid surface, so some energy is accreted through the event horizon, adding to the BH mass and never being radiated. We could parametrise this by a correcting factor to $GM\dot{M}/R$ but that stops being good relativistically, so we instead write

$$L = \varepsilon \dot{M} c^2$$

defining the important parameter ε. Note that $\dot{M}$ is not the rate of change of the mass M of the black hole, which we will write $\dot{M}$ in a different colour[3]; $\dot{M}$ is the rate at which matter is falling onto the BH, some of which may be radiated away as energy rather than be added to the BH mass. Indeed, the change in the mass of the BH will be the mass that is *not* being radiated out, that is, $\dot{M} = (1 - \varepsilon)\dot{M}$. Hence

$$L = \frac{\varepsilon}{1 - \varepsilon} \dot{M} c^2$$

Nuclear fusion gives $\varepsilon = 0.007$, which would require ridiculously large $\dot{M}$ to match observed quasar luminosities. Schwarzschild BHs' maximum ε can be estimated using a Newtonian formula for their released energy of $L = GM\dot{M}/2r_{\text{ISCO}}$, which implies $\varepsilon = 1/12 = 0.083$; a more accurate GR calculation gives $\varepsilon = 0.057$. Corotating Kerr BHs have ISCOs much closer to their horizons: maximally spinning BHs[4] have $\varepsilon = 0.42$. Observationally, $\varepsilon \sim 0.1$.

1.6 Eddington Luminosity

For a steady, spherical flow of fully ionised hydrogen accreting onto a BH of mass M, the Eddington luminosity L_{Edd} is that whereby the radiation pressure balances the gravitational force. Radiation pressure is mainly exerted on the electrons, as their Thomson cross section σ_T is $(m_p/m_e)^2 = 3 \times 10^6$ greater than protons', but the electrons will drag the protons electrostatically so that the radiation pressure on the electrons is essentially felt by the protons as well. Setting the rate at which momentum is being absorbed by protons (via electrons) equal to the gravitational force on them, we have

$$\frac{L_{\text{Edd}}}{c} \frac{\sigma_T}{4\pi r^2} = \frac{GMm_p}{r^2} \qquad \Rightarrow \qquad L_{\text{Edd}} = \frac{4\pi GMm_p c}{\sigma_T} = 1.3 \times 10^{31} \left(\frac{M}{M_\odot} \right) \text{W}$$

SMBH luminosities are observed up to 10^{41}W, suggesting masses of $M \sim 10^{10} M_\odot$. We can also define the Eddington accretion rate $\dot{M}_{\text{Edd}} \equiv L_{\text{Edd}}/\varepsilon c^2$. Now because $L \propto \dot{M}$, we have that $\dot{M} \propto M$, suggesting exponential growth for BHs accreting at the Eddington luminosity.

[3] It can be written $\dot{M}_{\text{BH}}$ if you don't have a coloured pencil handy.

[4] GR effects show that maximally spinning BHs are not possible: a can only actually get up to $0.998r_G$, which turns out to give the significantly lower $\varepsilon = 0.3$

2 Seed Formation

The SMBHs in the Universe today have grown from seeds by accretion and mergers. It is uncertain exactly how these seeds form: it may be from massive Population III stars, the runaway collapse of star clusters, or even directly through the gravitational collapse of gas.

2.1 Star Cluster Formation

Although a seemingly unrelated process, the process of star formation is important to understand all three seed formation processes: the formation of Pop. III stars, the formation of star clusters, and fundamentally this is all about the collapse of gas.

The virial theorem gives a bulk relationship between the different kinds of energy in a gravitating system: $2T + \Omega = 0$, where T is the total kinetic energy (both bulk and microscopic/internal, including e.g. gas pressure), and Ω is the potential energy. Consider a spherical cloud of uniform density ρ and temperature T. Its potential and kinetic energy will be

$$\Omega = \int_0^R -\frac{G}{r} \cdot \frac{4}{3}\pi r^3 \rho \cdot 4\pi r^2 \rho \, \mathrm{d}r = -\frac{16\pi^2 G\rho^2}{3}\frac{R^5}{5} = -\frac{3}{5}\frac{GM^2}{R} = -\frac{3}{5}\left(\frac{4}{3}\pi\rho\right)^{1/3} GM^{5/3}$$

$$T = \int \frac{1}{\gamma - 1} p \, \mathrm{d}V = \frac{3kT}{2\mu m_p} M$$

where we have taken $\gamma = 5/3$, assuming atomic H. For the cloud to collapse, we require $|\Omega| > 2T$, so that the cloud's gravity overcomes outward gas pressure. This condition is

$$\frac{3}{5}\left(\frac{4}{3}\pi\rho\right)^{1/3} GM^{5/3} > \frac{3kT}{\mu m_p} M \qquad \Rightarrow \qquad M > \sqrt{\frac{375}{4\pi\rho}\left(\frac{kT}{\mu m_p}\right)^3} \propto \sqrt{\frac{T^3}{\rho}}$$

where the critical mass M_J is the *Jeans mass*. If any volume exceeds this mass it will collapse.

Depending on the metallicity (and hence opacity) of the cloud, the gravitational energy released during collapse may or may not be able to be radiated away. As such the collapse may be isothermal or adiabatic.

- **Isothermal collapse.** The cloud must be metal-rich ($Z \gtrsim 10^{-3} Z_\odot = 10^{-5}$) to radiate away heat generated and maintain constant temperature. In this regime we have $M_J \propto \rho^{-1/2}$, so as the cloud contracts and its density increases, the threshold for collapse decreases, enabling smaller regions of the cloud to collapse. The cloud thus fragments.

- **Adiabatic collapse.** Eventually, the density increases enough that the cloud's opacity reaches the *opacity limit* and cooling becomes inefficient; the cooling time becomes longer than the collapse time. We then have $p \propto \rho^{5/3} \Rightarrow T \propto \rho^{2/3} \Rightarrow M_J \propto \rho^{1/2}$. The cloud is no longer susceptible to fragmentation, collapsing *monolithically* into a constant number of bodies. The density and temperature continue to rise until T reaches about 100K, at which point $M_J \sim 10^3 M_\odot$, and the transitions of any H_2 in the gas allow it to cool, ending adiabaticity. If the cloud is able to collapse further before the onset of e.g. nuclear fusion, its M_J will decrease and it may fragment. As such the maximum mass of a population III star is $\sim 10^3 M_\odot$.

In a low-Z cloud, the above process forms a cluster of enormous stars. These stars run out of fuel in a few Myr, forming large stellar mass BHs. Alternatively, they may collide with other stars, disrupting their fusion processes and forming a large BH.

2.2 Cluster Dynamics

2.2.1 Relaxation

Consider a cluster of radius R_C of N identical stars of mass M. Let one star pass by another in the x-direction at a large impact parameter b, such that its x-directed velocity v remains roughly constant. It will be deflected in the y-direction by the interaction. The y-directed force experienced will be

$$F_y = \frac{GM^2}{r^2}\frac{b}{r} = \frac{GM^2 b}{(b^2+x^2)^{3/2}} = \frac{GM^2 b}{(b^2+v^2t^2)^{3/2}}$$

$$\Rightarrow \delta v_y = \frac{1}{M}\int_{-\infty}^{\infty} GM^2 b \frac{\mathrm{d}t}{(b^2+v^2t^2)^{3/2}} = \frac{2GM}{bv}$$

As the star passes through the cluster, it will see a surface density of stars of order $N/\pi R_C^2$. If passing through the centre, the number of interactions at impact parameters between b and $b+\mathrm{d}b$ will be $(N/\pi R_C^2)\cdot 2\pi b\,\mathrm{d}b$. The mean deflection δv_y will of course be 0 by symmetry, but the mean *square* deflection will not:

$$\delta v^2 = \int_{b_{\min}}^{b_{\max}} \delta v_y^2(b)\frac{N}{\pi R_C^2}\cdot 2\pi b\,\mathrm{d}b = \frac{8NG^2M^2}{R_C^2 v^2}\ln\Lambda$$

where the Coulomb parameter $\Lambda \equiv b_{\max}/b_{\min}$. We can approximate $b_{\max} \sim R_C$ and $b_{\min}$ as the b such that $\delta v_y = v \Rightarrow b_{\min} = 2GM/v^2$. This gives a value of $\Lambda = R_C v^2/2GM$. Now the typical speed of a particle in a virialised system is such that

$$2T + \Omega = 0 \;\Rightarrow\; NMv^2 = \frac{G(NM)^2}{R_C} \;\Rightarrow\; v^2 = \frac{GNM}{R_C} \qquad \Rightarrow \qquad \delta v^2 = \frac{8GM}{R_C}\ln\left(\frac{N}{2}\right)$$

This is the mean square change in velocity when the star crosses the cluster once. After a certain number of crossings, n_c, the change in the square velocity $n_c\delta v^2$ will become of order v^2, the initial velocity, at which point the cluster is said to become *relaxed*: information about the initial conditions is lost. We find

$$n_c = \frac{v^2}{\delta v^2} = \frac{GNM}{R_C}\frac{R_C}{8GM\ln(N/2)} = \frac{N}{8\ln(N/2)} \qquad \Rightarrow \qquad t_{\text{relax}} \sim n_c \cdot \frac{R_C}{v} = \frac{N}{8\ln(N/2)}\frac{R_C}{v}$$

2.2.2 Evaporation

In reality, stars in a cluster will have a distribution of speeds. Some stars will have a speed greater than the cluster's escape velocity, and will therefore be able to leave the cluster. The escape velocity is at a given point $v_{\text{esc}} = \sqrt{-2\Phi}$. The mass-averaged escape velocity is then

$$\left\langle v_{\text{esc}}^2 \right\rangle_\rho \equiv \frac{\int \rho v_{\text{esc}}^2\,\mathrm{d}^3\mathbf{r}}{\int \rho\,\mathrm{d}^3\mathbf{r}} = -\frac{2}{M_C}\int \rho\Phi\,\mathrm{d}^3\mathbf{r} = -\frac{4\Omega}{M_C}$$

where we have used $\Omega = \frac{1}{2}\int \rho\Phi\,\mathrm{d}^3\mathbf{r}$ and the cluster mass $M_C \equiv \int \rho\,\mathrm{d}^3\mathbf{r}$. Now the mean velocity is given by the virial theorem, $2T+\Omega = 0$:

$$T = \sum \frac{1}{2}M_i v_i^2 = \frac{1}{2}M_C\left\langle v^2\right\rangle_\rho \qquad \Rightarrow \qquad M_C\left\langle v^2\right\rangle = \frac{1}{4}M_C\left\langle v_{\text{esc}}^2\right\rangle \qquad \Rightarrow \qquad \left\langle v_{\text{esc}}^2\right\rangle = 4\left\langle v^2\right\rangle$$

Thus stars with speeds $> 2\times$ the rms velocity will be able to escape. For a Maxwell-Boltzmann distribution this escape fraction is approximately 7×10^{-3}. This fraction of stars escape on roughly a relaxation timescale. The evaporation timescale of the cluster would then be of order $t_{\text{relax}}/7 \times 10^{-3}$.

The total energy of the cluster can be written $E = T + \Omega$; using $\Omega = -2T$ we have $E = -T \Rightarrow \partial E/\partial T = -1$. As the kinetic energy T is a measure of the "dynamical temperature" of the system, the system effectively has a negative heat capacity. Therefore, as energy is carried away from the cluster by evaporating stars, the temperature and average KE of the cluster actually increases, meaning that *more* stars are then able to escape the cluster... a positive feedback loop is established, which formally would lead to the total evaporation of the cluster in a so-called *gravithermal catastrophe* in a finite amount of time. However, at a certain point several processes (including the formation of binary systems, three-body encounters, and stellar winds) save the cluster from total evaporation, and the cluster ends up with a small high-density and high-temperature core.

With high stellar densities and speeds, head-on stellar collisions would be relatively frequent, leading to very massive ($\sim 10^3 M_\odot$) stars which will form SMBH seeds. The rest of the cluster core is soon accreted onto the BH, allowing it to grow into an SMBH. N-body codes suggest this to be the most realistic scenario for SMBH seed formation.

2.2.3 Dynamical Friction & Mass Segregation

We have so far considered a cluster with stars of equal mass, which is not realistic. A next-order assumption would be a cluster with two populations, with stars of masses M and m.

If $M \gg m$, such as a BH moving through a sea of field stars (or even H atoms), the effect of the ms will be a drag force on M. Chandrasekhar derived the following formula:

$$\frac{\mathrm{d}\mathbf{v}_M}{\mathrm{d}t} = -16\pi^2 G^2 M m \ln \Lambda \left(\int_0^{v_M} f(v_m) v_m^2 \, \mathrm{d}v_m \right) \frac{\mathbf{v}_M}{v_M^3}$$

where $f(v_m)$ is the distribution of speeds of the ms. Via the upper bound of the integral, $\mathrm{d}\mathbf{v}_M/\mathrm{d}t$ depends in a complicated way on v_M, but becomes simpler in limiting cases. If $v_M \ll \langle v_m \rangle$, then $f(v_m) \approx f(0)$ for the entire integral, and so the integral becomes approximately proportional to v_M^3 and we have $\mathrm{d}\mathbf{v}_M/\mathrm{d}t \propto -\mathbf{v}_M$. In the fast limit $v_M \gg \langle v_m \rangle$, the upper bound can be approximated at infinity, and the integral becomes proportional to the number density $n = \int_0^\infty 4\pi v_m^2 f(v_m) \, \mathrm{d}v_m$; we then have $\mathrm{d}\mathbf{v}_M/\mathrm{d}t \propto -v_M^{-2}$.

Dynamical friction leads to a segregation of the masses M and m in a cluster. Again taking the analogy of stars and gas particles, a result of the equipartition theorem is that the *kinetic* energy (not the total energy) is shared between all objects. As such, larger masses must have a lower $\langle v^2 \rangle$ to have the same KE as the smaller stars; if this is not the case then dynamical friction will transfer energy from the large masses to the smaller masses. As the large masses lose energy, they will sink towards the centre of the cluster and fling the smaller masses to the outside. This segregation apparently occurs on a timescale $\sim t_{\text{relax}}(m/M) \ll t_{\text{relax}}$, which is quite fast, and clearly this segregation will accelerate the process of evaporation and SMBH seed formation, by helping the contraction of the core.

3 Accretion

3.1 Sołtan's Argument

In the high-redshift Universe we see accreting SMBHs as AGN. The *AGN luminosity function* $\Phi(L, z)$ is the number density of AGN per luminosity per redshift; observationally, biases and the obscuration of the AGN must be accounted for, giving an uncertainty of a factor of ~ 2.

Sołtan argued that the mass density of SMBHs today should be equal to the integrated mass accreted by all those SMBHs through time; that is

$$\rho_{\mathrm{SMBH}}(\mathrm{now}) = \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}L\, \Phi(L, z)\frac{\mathrm{d}M}{\mathrm{d}z} = \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}L\, \Phi(L, z)\frac{\mathrm{d}t}{\mathrm{d}z}\dot{M}$$

This can be simplified using the earlier result that $L = (\varepsilon/(1-\varepsilon))\dot{M}c^2$:

$$\Rightarrow \rho_{\mathrm{SMBH}}(\mathrm{now}) = \frac{1-\varepsilon}{\varepsilon c^2}\int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}L\, \Phi(L, z)L$$

– the luminosity integral now gives the total AGN luminosity per redshift.

3.2 Bondi Accretion

Bondi accretion is the accretion of gas of ambient density ρ_∞ onto a mass M, assuming:

- Steady State
- Negligible magnetic field
- Negligible radiation field
- Spherical symmetry
- Negligible viscosity
- Point mass potential
- Negligible AM
- Negligible turbulence
- No self-gravity or DM

We seek the mass accretion rate $\dot{M} = -4\pi r^2 \rho u$ (remember, $\dot{M} \neq \dot{M}$), which is constant in the steady state. Thus:

$$\frac{\mathrm{d}\ln \dot{M}}{\mathrm{d}r} = 0 \qquad \Rightarrow \qquad \frac{2}{r} + \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\mathrm{d}\ln u}{\mathrm{d}r} = 0 \tag{1}$$

The momentum equation gives

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} \qquad \Rightarrow \qquad u^2\frac{\mathrm{d}\ln u}{\mathrm{d}r} = -c_s^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2} \tag{2}$$

$$\text{or,} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}r}\underbrace{\left[\frac{1}{2}u^2 + \int\frac{\mathrm{d}p}{\rho} - \frac{GM}{r}\right]}_{\text{Bernoulli constant}} = 0 \tag{3}$$

Using (1) to eliminate $\mathrm{d}\ln\rho/\mathrm{d}r$ from (2), we find

$$(u^2 - c_s^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

We see that there may be a sonic radius $r_s = GM/2c_s^2$ where $u = -c_s$. Thus we can write

$$\dot{M} = 4\pi\left(\frac{GM}{2c_s^2}\right)^2\rho_s c_s = \pi\frac{G^2M^2}{c_s^3}\rho_s$$

The only remaining unknowns are c_s and ρ_s, which can both be found using the Bernoulli constant, in a way depending on the equation of state. In the general polytropic case, we have $p = K\rho^{1+1/n}$, and so $c_s^2 = \frac{n+1}{n}\rho^{1/n}$ and $\int \mathrm{d}p/\rho = (n+1)K\rho^{1/n} = nc_s^2$. Note that c_s is a function of radius: we write $c_s = c_{s\infty}$ and c_{ss} at infinity and the sonic point. The Bernoulli constant, compared at $r = \infty$ and $r = r_s$, gives

$$\frac{1}{2}\underbrace{u_\infty^2}_{0} + nc_{s\infty}^2 = \frac{1}{2}c_s^2 + nc_{ss}^2 - \underbrace{\frac{GM}{r_s}}_{2c_{ss}^2} \quad\Rightarrow\quad c_{ss}^2 = \frac{n}{n-3/2}c_{s\infty}^2 \quad\Rightarrow\quad \rho_s = \left(\frac{n}{n-3/2}\right)^n \rho_\infty$$

$$\Rightarrow \dot{M} = \pi\frac{G^2M^2}{c_{ss}^3}\rho_s = \pi\left(\frac{n}{n-3/2}\right)^{n-3/2}\frac{G^2M^2}{c_{s\infty}^3}\rho_\infty \xrightarrow{n\to\infty} \pi e^{3/2}\frac{G^2M^2}{c_s^3}\rho_\infty$$

where the $n \to \infty$ limit corresponds to the limit of isothermal accretion at constant c_s. The important $n \to 3/2$ limit, corresponding to $\gamma = 5/3$, causes the bracket to tend to 1.

The more general Bondi-Hoyle-Lyttleton[5] accretion model assumes that M moves through the ambient gas with a speed v_∞. This replaces $c_{s\infty}$ in the above equations with $\sqrt{c_{s\infty}^2 + v_\infty^2}$.

3.2.1 Radiative Efficiency

Define the "accretion radius" $r_{\mathrm{acc}} \equiv 2GM/c_{s\infty}^2$. For $r \ll r_{\mathrm{acc}}$, we expect gravity to dominate the flow, and so $u(r) \approx -\sqrt{2GM/r} \propto r^{-1/2}$. Now $\dot{M} \propto r^2\rho u$ is a constant, so $\rho \propto r^{-3/2}$ for small r. For an adiabatic flow with $\gamma = 5/3$, $T \propto \rho^{\gamma-1} = \rho^{2/3} \propto r^{-1}$, so the gas will heat up as $r \to 0$. Due to these high temperatures, the gas will likely become a plasma and emit radiation by Bremsstrahlung. The luminosity per unit volume of Bremsstrahlung is given by $4\pi j_0 n_e^2 T^{1/2}$, so the total Bremsstrahlische luminosity will be

$$\begin{aligned} L &= \int_{r_h}^{r_{\mathrm{acc}}} 4\pi j_0 n_e^2 T^{1/2} \cdot 4\pi r^2\,\mathrm{d}r = 16\pi^2 j_0 \int_{r_h}^{r_{\mathrm{acc}}} \left[n_{\mathrm{acc}}\left(\frac{r}{r_{\mathrm{acc}}}\right)^{-3/2}\right]^2 \left[T_{\mathrm{acc}}\left(\frac{r}{r_{\mathrm{acc}}}\right)^{-1}\right]^{1/2} r^2\,\mathrm{d}r \\ &= 16\pi^2 j_0 n_{\mathrm{acc}}^2 T_{\mathrm{acc}}^{1/2} r_{\mathrm{acc}}^{7/2} \int_{r_h}^{r_{\mathrm{acc}}} r^{-3/2}\,\mathrm{d}r \approx 32\pi^2 j_0 \left[n^2 T^{1/2} r^{7/2}\right]_{\mathrm{acc}} r_h^{-1/2} \end{aligned}$$

where in the final approximation we have assumed that $r_h \ll r_{\mathrm{acc}}$, i.e. $c_{s\infty} \ll c$. Now $r_{\mathrm{acc}} = 2GM/c_{s\infty}^2 \propto MT_\infty^{-1}$; approximating $n_{\mathrm{acc}} \approx n_\infty$ etc., we can then write $L \propto n_\infty^2 M^3 T_\infty^{-3}$. We can also find that the radiative efficiency ε is

$$\varepsilon = \frac{L}{\dot{M}c^2} = \frac{L}{c^2}\frac{c_{s\infty}^3}{\pi G^2 M^2 \rho_\infty} \propto n_\infty^2 M^3 T_\infty^{-3}\frac{T_\infty^{3/2}}{M^2 n_\infty} \propto n_\infty M T_\infty^{-3/2} \propto \frac{L^{1/2}}{M^{1/2}} \propto \sqrt{\frac{L}{L_{\mathrm{Edd}}}}$$

where we have used the fact that L_{Edd} is proportional to M alone. The proportionality constant between ε and $\sqrt{L/L_{\mathrm{Edd}}}$ is apparently 0.009, so even if $L \sim L_{\mathrm{Edd}}$, the radiative efficiency of Bondi accretion is much lower than the observed values of ~ 0.1.

In fact, we can show that $L \ll L_{\mathrm{Edd}}$, making things worse. The optical depth to radius r is

$$\tau = \sigma_T \int_r^\infty n_e\,\mathrm{d}r \approx \sigma_T n_{\mathrm{acc}} r_{\mathrm{acc}}^{3/2} \int_r^\infty r^{-3/2}\,\mathrm{d}r = 2\sigma_T n_{\mathrm{acc}} r_{\mathrm{acc}}^{3/2} r^{-1/2} = 2\sigma_T \frac{\rho_\infty}{m_p}\left(\frac{GM}{c_{s\infty}^2}\right)^{3/2} r^{-1/2}$$

[5] For some reason loads of people spell his name Littleton, which is literally incorrect. Also, turns out his secondary school was the rival of my secondary school. Mad.

$$= \frac{8\pi GMc}{L_{\rm Edd}} \frac{\dot{M}}{\pi\sqrt{GM}} r^{-1/2} = \frac{8\dot{M}c^2}{L_{\rm Edd}} \sqrt{\frac{GM}{c^2}} r^{-1/2} = \frac{8}{\varepsilon} \frac{L}{L_{\rm Edd}} \left(\frac{r}{r_h}\right)^{-1/2} \approx \frac{8}{0.009} \frac{L}{L_{\rm Edd}} \sqrt{\frac{r_h}{r}}$$

So the optical depth to the horizon will be $\sim \varepsilon^{-1} \sim 10^2$. The radiation produced by the Bremsstrahlung will therefore be trapped in the "radiatively inefficient" flow and dragged into the BH. We will only be able to see the radiation produced down to some radius $r \gg r_h$.

Suppose a photon has a mean free path λ and is trying to diffuse outwards. We define a characteristic trapping radius r_t, at which the outward diffusion velocity is approximately equal to the infalling flow: within this radius, the gas will be infalling faster than the photon is able to diffuse outwards and the photon will be dragged inwards. The speed of the infalling gas at r_t will approximately be $\sqrt{2GM/r_t}$; we now derive the outward diffusion speed. The distance it travels after N scatterings will be $D = \sqrt{N}\lambda$, according to standard diffusion theory. The optical depth, qualitatively defined as the number of mean free paths that fit into the path in question, is therefore $\tau = D/\lambda = \sqrt{N}$. The diffusion velocity is then

$$v_d \sim \frac{D}{N\lambda/c} = \frac{c\sqrt{N}\lambda}{N\lambda} = \frac{c}{\sqrt{N}} = \frac{c}{\tau} \qquad \Rightarrow \qquad \sqrt{\frac{2GM}{r_t}} = \frac{c}{\tau(r_t)} \sim \frac{c\varepsilon}{8} \sqrt{\frac{L_{\rm Edd}}{L} \frac{r_t}{r_h}}$$

$$\Rightarrow r_t \sim \frac{8}{\varepsilon} \frac{L_{\rm Edd}}{L} r_h \sim 10^2 \frac{L_{\rm Edd}}{L} r_h$$

So if $L \sim L_{\rm Edd}$, any radiation generated within $\lesssim 10^2 r_h$ (which is where a lot of the radiation is generated) is trapped and dragged into the BH. This suggests that Bondi accretion is not a very good model for the accretion processes occurring in AGN, because we can see them.

3.3 Disks

In this section we look at axisymmetric planar inflows, where the material forms a disk and inspirals as its energy is dissipated by viscosity.

3.3.1 Steady-State Thin Disk Solution

In steady state, the mass conservation equation becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma u_r) = 0 \qquad \Rightarrow \qquad \dot{M} = -2\pi r \Sigma u_r = \text{const.}$$

as we are now working in 2D[6]. The radial and azimuthal Navier-Stokes equations are

$$\underbrace{\frac{\partial u_r}{\partial t}}_{0} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}; \qquad \Sigma\left(\underbrace{\frac{\partial u_\phi}{\partial t}}_{0} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)$$

where as usual $\Omega = u_\phi/r$ and $\Sigma = \int \rho \, \mathrm{d}z$. Rearranging for u_r in the second equation gives

$$u_r = \frac{\frac{\partial}{\partial r}(\bar{\nu}\Sigma r^3 \, \mathrm{d}\Omega/\mathrm{d}r)}{r^2 \Sigma(\partial u_\phi/\partial r + u_\phi/r)} = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^{1/2}\right)$$

But we also have $u_r = -\dot{M}/2\pi r \Sigma$, so we can write

$$-\frac{\dot{M}}{2\pi r \Sigma} = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^{1/2}\right) \qquad \Rightarrow \qquad \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^{1/2}\right) = \frac{\dot{M}}{6\pi r^{1/2}}$$

[6] We keep r as the coordinate in consistency with the Disks course

This can be integrated from some inner radius r_* at which $\bar{\nu}\Sigma = 0$ (such as r_{ISCO} where $\Sigma \to 0$):

$$(\bar{\nu}\Sigma)(r) = \frac{1}{r^{1/2}} \frac{\dot{M}}{3\pi} \left(r^{1/2} - r_*^{1/2}\right) = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_*}{r}}\right)$$

$$\Rightarrow u_r(r) = -\frac{3\bar{\nu}}{\bar{\nu}\Sigma r^{1/2}} \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^{1/2}\right) = -3\bar{\nu} \frac{3\pi}{\dot{M}\left(1 - \sqrt{r_*/r}\right) r^{1/2}} \frac{\dot{M}}{6\pi r^{1/2}} = -\frac{3\bar{\nu}(r)}{2r}\left(1 - \sqrt{\frac{r_*}{r}}\right)^{-1}$$

We assume that T is independent of z. In hydrostatic equilibrium we then have:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{GM}{r^2}\frac{z}{r} \qquad \Rightarrow \qquad c_s^2 \frac{\partial \rho}{\partial z} = \Omega^2 z \rho \qquad \Rightarrow \qquad \rho(r,z) = \rho(r,0)\exp\left(-\frac{z^2}{2H^2}\right), \quad H \equiv \frac{c_s}{\Omega}$$

defining the characteristic *scale height* of the disk. We can then estimate $\rho \sim \Sigma/H$.

The rate of energy dissipation per unit area is given (over a single face of the disk) by

$$D(r) = \bar{\nu}\Sigma\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right)$$

This needs to match with how the temperature structure $T(z)$ changes in the disk. The flux through a surface is related to the temperature gradient by

$$F(r,z) = -\frac{16\sigma T^3}{3\bar{\kappa}\rho}\frac{\partial T}{\partial z} = -\frac{4\sigma}{3\bar{\kappa}\rho}\frac{\partial T^4}{\partial z}$$

$$\Rightarrow D(r) \approx F(r,H) - F(r,0) \approx -F(r,0) = \frac{4\sigma}{3\bar{\kappa}\rho|_0}\left.\frac{\partial T^4}{\partial z}\right|_0 \sim \frac{4\sigma}{3\tau}T(r,0)^4$$

3.3.2 Shakura-Sunyaev Solution

We now assess the importance of the viscosity, through the *Reynolds number*, the approximate ratio of the inertial terms in the Navier-Stokes equation to the viscous terms. It is found experimentally that if Re is above about 2 000 then turbulent flows set in. The *least* turbulent accretion disk conditions give Reynolds numbers of $\sim 10^{14}$, so viscosity is definitely important.

Turbulence is a series of eddies, wherein the fluid elements move chaotically at speeds $\sim v_e$ around length scales $\sim \lambda_e$; the viscosity is then $\bar{\nu} \sim \lambda_e v_e$. Now the eddies have to fit into the disk, so we expect $\lambda_e \lesssim H$. We also expect $v_e \lesssim c_s$, else it would quickly shock until it was. Shakura and Sunyaev therefore prescribed $\bar{\nu} = \alpha c_s H$ to collect the uncertainties on how turbulence works into the parameter α, which we expect to be < 1.

We can use this parameter to make order-of-magnitude estimates of important parameters:

- u_r. The work in the previous subsection suggests $u_r \sim \bar{\nu}/r \sim \alpha(H/r)c_s \lll c_s$. We thus expect the radial motion to be highly subsonic and quite slow.

- u_ϕ. Vertical hydrostatic equilibrium gives

$$\frac{1}{\rho}\left|\frac{\partial p}{\partial z}\right| \sim \frac{GMz}{r^3} \qquad \Rightarrow \qquad \frac{c_s^2}{H} \sim \left(\frac{u_\phi}{r}\right)^2 H \qquad \Rightarrow \qquad u_\phi \sim \frac{r}{H}c_s$$

so we expect aximuthal motion to be highly *super*sonic.

- Departures from Keplerian motion. The Mach number $\mathcal{M} \equiv u_\phi / c_s$; we have $\mathcal{M} \sim r/H$ from the above. The centripetal force is mostly gravitational but offset by the radial pressure gradient. Compared to the gravitational force, this offset will be of order

$$\frac{1}{\rho}\frac{\partial p}{\partial r}\frac{r^2}{GM} \sim \frac{c_s^2}{u_\phi^2} \sim \mathcal{M}^{-2}$$

Thus $u_\phi = \sqrt{GM/r}[1 + \mathcal{O}(\mathcal{M}^{-2})]$, so we can reasonably assume Keplerian motion.

We have now found 9 equations (some approximate...) in 9 unknowns $(\rho, \Sigma, H, c_s, p, \tau, \bar{\kappa}, T_c, \bar{\nu})$

$$\rho = \frac{\Sigma}{H}, \qquad H = \frac{c_s}{\sqrt{GM/r^3}}, \qquad c_s^2 = p/\rho, \qquad \tau = \Sigma\bar{\kappa}(\rho, T_c), \qquad \bar{\nu} = \bar{\nu}(\rho, T_c, \Sigma, \alpha),$$

$$p = \frac{\rho k T_c}{\mu m_p}, \qquad \bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_*}{r}}\right), \qquad u_r = -\frac{3\bar{\nu}}{2r}\left(1 - \sqrt{\frac{r_*}{r}}\right)^{-1},$$

$$\frac{3GM\dot{M}}{8\pi r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right) = \frac{4\sigma}{3\tau}T_c^4$$

where $T_c \equiv T(r, 0)$, and the variables $r, M, \dot{M}, \alpha$ are treated as parameters. Note also that we have neglected radiation pressure in favour of gas pressure – this is revisited in the next subsection. To solve this set of equations, the functional forms of $\bar{\nu}$ and $\bar{\kappa}$ need to be specified: a standard choice is $\bar{\nu} = \alpha c_s H$ and $\bar{\kappa} \propto \rho T_c^{-7/2}$ (Kramers' opacity law).

The analytic solution is complicated, but important results include:

- The solution is not very sensitive to α, which enters with |powers| of at most 1. This is nice, because we don't really know α.
- $H \sim r^{9/8} \Rightarrow H/r \sim r^{1/8}$. The disk therefore flares outwards. Also the proportionality constant is $\sim 10^{-2}$, so the disk is indeed thin, and stays thin out to large r.
- $\Sigma \sim r^{-3/4}$. Integrating $M_{\text{disk}} = \int_{r_*}^{r} 2\pi r \Sigma \, \mathrm{d}r$ out to some reasonable cutoff, apparently the disk has a total mass $M_{\text{disk}} \sim 10^{-10} M_\odot \ll M$, justifying our neglect of self-gravity.

3.3.3 Radiation-Pressure-Dominated Regions

The Shakura-Sunyaev solution also shows that radiation pressure does in fact dominate over gas pressure below a certain cutoff radius $r_{\text{rad}} \sim M^{11/21}\dot{M}^{16/21}$: that is, the larger M or $\dot{M}$, the larger the radiation-dominated region. Thus the inner regions of disks around NSs and BHs are expected to be dominated by radiation pressure. This turns out to be very important: we now show that this leads to the disk no longer being thin. We have $H = c_s/\Omega$, where

$$c_s^2 = \frac{p}{\rho} = \frac{1}{\rho}\frac{4\sigma T_c^4}{3c} = \frac{1}{\rho c}\frac{3GM\dot{M}}{8\pi r^3}\underbrace{\left(1 - \sqrt{\frac{r_*}{r}}\right)}_{g(r)}\tau = \frac{3GM\dot{M}}{8\pi\rho c r^3}g(r)\underbrace{\Sigma}_{\sim\rho H}\underbrace{\bar{\kappa}}_{\sim\sigma_T/m_p} \approx \frac{3GM\dot{M}H\sigma_T}{8\pi c m_p}\frac{g(r)}{r^3}$$

$$\Rightarrow H^2 = \frac{c_s^2}{\Omega^2} \approx \frac{3\dot{M}H\sigma_T}{8\pi c m_p}g(r) \qquad \Rightarrow \qquad H \approx \frac{3\dot{M}\sigma_T}{8\pi c m_p}g(r)$$

This can be written in terms of a critical accretion rate, achieved when accreting at around the Eddington rate:

$$L_{\text{Edd}} \equiv \varepsilon \frac{GM\dot{M}_{\text{crit}}}{r_*} \qquad \Rightarrow \qquad \dot{M}_{\text{crit}} = \frac{L_{\text{Edd}}}{\varepsilon}\frac{r_*}{GM} = \frac{r_*}{\varepsilon GM}\frac{4\pi GMm_pc}{\sigma_T} = \frac{4\pi m_pc}{\varepsilon\sigma_T}r_*$$

$$\Rightarrow H \approx \frac{3}{2\varepsilon}\frac{\dot{M}}{\dot{M}_{\text{crit}}}g(r)r_*$$

Thus if $\dot{M}$ is significantly greater than $\varepsilon\dot{M}_{\text{crit}}$ then H will be $\sim r$ for small r. As such the disk is no longer thin: radiation pressure puffs out the inner disk into almost even a spherical shape. This may obscure the inner regions of the disk.

3.3.4 Thick & Slim Disks

The most basic possible non-thin disk model has no accretion and is purely rotational: $u_r = u_z = 0, u_\phi = r\Omega$ for some non-necessarily-Keplerian Ω; mass conservation is automatically satisfied. The components of the momentum equation become:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial \Phi}{\partial r} + \Omega^2 r \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{\partial \Phi}{\partial z} \qquad \Rightarrow \qquad \frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Phi + \Omega^2\mathbf{R} \equiv \mathbf{g}_{\text{eff}}$$

where $\mathbf{R} = r\hat{\mathbf{e}}_r$, defining the effective gravity vector, which we see must be perpendicular to isobaric surfaces. Let's assume that the pressure is dominated by radiation pressure:

$$p \approx p_{\text{rad}} = \frac{4\sigma}{3c}T^4, \qquad \mathbf{F} = -\frac{16\sigma T^3}{3\bar{\kappa}\rho}\boldsymbol{\nabla} T = -\frac{c}{\bar{\kappa}\rho}\boldsymbol{\nabla} p = -\frac{c}{\bar{\kappa}}\mathbf{g}_{\text{eff}} = \frac{c}{\bar{\kappa}}\boldsymbol{\nabla}\Phi - \frac{c}{\bar{\kappa}}\Omega^2\mathbf{R}$$

One can then calculate the total luminosity $L \equiv \int_{\mathcal{D}} \mathbf{F}\cdot \mathrm{d}\mathbf{S}$, where $\mathcal{D}$ is the surface of the disk.

Continuing with steady axisymmetric flows, we now allow for some small non-**zero** poloidal velocity $\mathbf{u}_P = (u_R, u_z)$, with a magnitude much smaller than u_ϕ. Note that axisymmetry means $\boldsymbol{\nabla} = \boldsymbol{\nabla}_P$, so e.g. $\mathbf{u}\cdot\boldsymbol{\nabla} = \mathbf{u}_P\cdot\boldsymbol{\nabla} = \mathbf{u}_P\cdot\boldsymbol{\nabla}_P$. Continuity, and the toroidal and poloidal components of the Navier-Stokes equation are now

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0,$$

$$(\mathbf{u}_P\cdot\boldsymbol{\nabla}_P)\mathbf{u}_\phi + \frac{u_r}{r}\mathbf{u}_\phi = (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})_\phi; \qquad (\mathbf{u}_P\cdot\boldsymbol{\nabla}_P)\mathbf{u}_P = -\frac{1}{\rho}\boldsymbol{\nabla}_P p \overbrace{-\boldsymbol{\nabla}_P\Phi + \Omega^2\mathbf{R}}^{\mathbf{g}_{\text{eff}}}$$

where we have neglected the radial component of the viscous stress because radial velocities are small. The "slim disk" solution involves integrating all the terms in the z-directions out to infinity. If $\lim_{z\to\pm\infty}\rho u_z = 0$, then the continuity equation gives $r\Sigma u_r = \text{const.} = -\dot{M}/2\pi$ as familiar[7]. Doing the same averaging to the toroidal Navier-Stokes gives:

$$u_r\frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r} = \frac{1}{\Sigma r^2}\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) \qquad \Rightarrow \qquad \Sigma r u_r\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)$$

As $\Sigma r u_r$ is a constant, this equation can be integrated:

$$\Sigma r u_r[ru_r]_{r_*}^{r} = \bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r} \qquad \Rightarrow \qquad u_r(\ell - \ell_*) = \bar{\nu}r^2\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

where we have defined $\ell \equiv ru_r = r^2\Omega$ and used the fact that there is no torque at the inner boundary. Substituting out u_r with $\dot{M}$, we find the angular momentum as a function of radius:

$$-\frac{\dot{M}}{2\pi r\Sigma}(\ell - \ell_*) = \bar{\nu}r^2\frac{\mathrm{d}\Omega}{\mathrm{d}r} \qquad \Rightarrow \qquad \ell = \ell_* - \frac{2\pi r^3\bar{\nu}\Sigma}{\dot{M}}\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

[7] Technically, u_r is now the average quantity $\frac{1}{\rho}\int \rho u_r \,\mathrm{d}z$

4 Winds

AGN can drive strong ionised winds into their host galaxies. Consider a steady spherically-symmetric flow, with outflow speed v; the rate of mass transfer of the wind will be $\dot{M} = 4\pi r^2 \rho v$ (written as $\dot{M}_W$ if you're boring). The momentum equation is then simply:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = a_{\mathrm{rad}} - \frac{GM}{r^2}$$

where a_{rad} is the acceleration due to radiation pressure; we neglect gas pressure. For an individual ion X in the wind, this is given by

$$a_{\mathrm{rad}} = \frac{n_{\mathrm{X}}}{4\pi r^2 \rho c}\int_{\nu_{\mathrm{X}}}^{\infty} L_\nu \kappa_\nu e^{-\tau_\nu}$$

where ν_{X} is the ionisation frequency of X (such that we only consider ionising photons, not sure why) and κ_ν is the total absorption cross section / unit mass (i.e. including bound-bound, bound-free, free-free, and Compton processes). Because that's a lot of opacities to deal with, we simplify things using a quantity called the dimensionless *force multiplier* $\mathcal{M}$:

$$\mathcal{M}(r) \equiv \frac{a_{\mathrm{rad,\ all}}(r)}{a_{\mathrm{rad,\ Compton}}(r)}; \qquad a_{\mathrm{rad,\ Compton}}(r) = \frac{n_e(r)\sigma_T L}{4\pi r^2 \rho(r) c}$$

so we only have to worry about the easy Compton effects and a factor $\mathcal{M} \geq 1$, with equality if the gas is fully ionised. We can then simplify:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = \underbrace{\frac{n_e}{\rho}}_{1/\mu m_p}\frac{\sigma_T L}{4\pi r^2 c}\mathcal{M} - \frac{GM}{r^2} = \frac{\sigma_T L}{4\pi r^2 \mu m_p c}\left(\mathcal{M} - \frac{1}{L}\frac{4\pi G M \mu m_p c}{\sigma_T}\right) = \frac{\sigma_T L}{4\pi r^2 \mu m_p c}\left(\mathcal{M} - \frac{L_{\mathrm{Edd}}}{L}\right)$$

Now to accelerate a wind, $\mathrm{d}v/\mathrm{d}r$ should be positive, so for sub-Eddington luminosities $L < L_{\mathrm{Edd}}$, we require a large $\mathcal{M}$ to outweigh L_{Edd}/L; physically, we require a powerful radiation flux ($\mathcal{M}$) to overcome gravity ($L_{\mathrm{Edd}} \propto GM$).

Suppose an AGN has an "Eddington wind": $L = L_{\mathrm{Edd}}$ and the wind has $\dot{M} = \dot{M}_{\mathrm{Edd}}$. Suppose also that the inner wind has $\tau = 1$, so that the photons emitted by the AGN scatter once on average before escaping. This means that the average photon will transfer 100% of its momentum to the wind, and the rate of momentum transfer to the wind is $\dot{p}_W = L_{\mathrm{Edd}}/c$. But $\dot{p}_W \equiv \dot{M}v = \dot{M}_{\mathrm{Edd}}v$, so

$$v = \frac{L_{\mathrm{Edd}}}{\dot{M}_{\mathrm{Edd}} c} = \frac{\varepsilon \dot{M}_{\mathrm{Edd}} c^2}{c} = \varepsilon c \sim 0.1c$$

These not-too-unreasonable assumptions lead to very fast winds!

4.1 Energy- and Momentum-Driven Winds

Clearly AGN winds can be highly supersonic. As they clatter into the ISM, a *discontinuity* will form as the ISM (made of gas) can't react quickly enough to communicate to gas further out: a *forward shock* then sweeps the gas outwards into the ISM. Also, news of the collision will propagate backwards through the wind, in a *reverse shock*. This is all shown in Figure 2. It is at the reverse shock that most of the KE of the wind is dissipated, and hence this interface is most important. The mechanism behind the propagation of the shock depends on how fast the shocked wind is able to cool, by radiation.

If heating is inefficient (the cooling time is longer than the "flow time" $\sim r/v$), then the energy of the wind gets transferred to the shocked material, which then expands (adiabatically) on either side of the discontinuity; the reverse shock makes it most of the way back to the BH. Such winds are driven by the adiabatic expansion of the shocked shell of ISM, and are hence called *energy-driven winds*. By contrast, if cooling is efficient and the cooling time is short, then the KE is pretty much all dissipated as soon as it hits the reverse shock. As such, the wind material inside the shock doesn't gain much energy, and remains very thin. The shock is then powered simply by the momentum of the wind slamming into it.

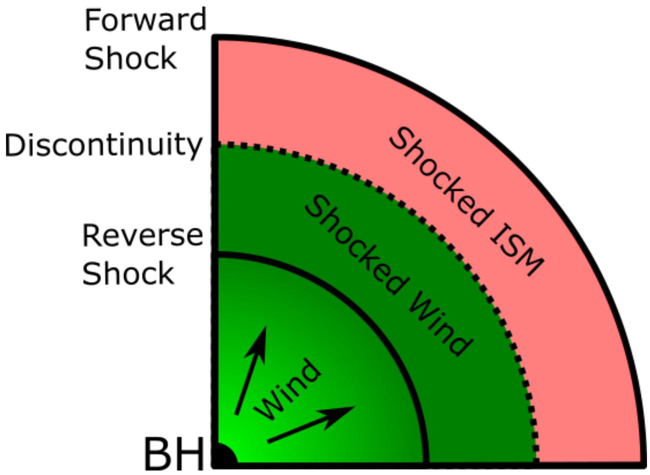

Figure 2 | Geometry of an AGN wind.

We now develop mathematical models of the propagation of these winds. I don't think they're very rigorous, and it's not always clear what radius in Figure 2 we're talking about, but oh well. In each case we will model the galaxy in which the AGN is embedded as a singular isothermal sphere, of total density $\rho(r) = \sigma^2/2\pi G r^2$. This includes dark matter, so the gas density, which we will assume to be a constant fraction ($f_g \lesssim \Omega_{\text{baryon}}/\Omega_{\text{DM}} \approx 0.16$) of the total density, will be $f_g \rho(r)$. The total mass (inc. DM) contained within a radius r will be:

$$M_t(<r) = \int_0^r 4\pi r^2 \frac{\sigma^2}{2\pi G r^2}\,\mathrm{d}r = \frac{2\sigma^2}{G} r$$

whereas the mass of the shocked shell, swept up entirely from the interior of the reverse shock, is $M_s(r) = 2 f_g \sigma^2 r/G$. Note that we have neglected the mass of the black hole itself, as we will generally be looking at large radii within which the enclosed DM mass $\gg M$.

4.1.1 Energy-Driven Winds

The energy produced by an Eddington wind is $\dot{E} = \frac{1}{2}\dot{M}_{\text{Edd}} v^2 \approx \frac{1}{2}\frac{L_{\text{Edd}}}{\varepsilon c^2}(\varepsilon c)^2 = \frac{\varepsilon}{2} L_{\text{Edd}}$. This energy will be converted into the internal and potential energy of the shocked material (wind and ISM), as well as the $p\,\mathrm{d}V$ work it does on the ISM:

$$\frac{\varepsilon}{2} L_{\text{Edd}} = \dot{U}_s + \frac{G M_t(<r) M_s(r)}{r^2}\dot{r} + p\dot{V}, \qquad U_s = \frac{pV}{\gamma - 1},$$

where r is the radius of the forward shock, $V = \frac{4}{3}\pi r^3$ as the reverse shock expands back almost to $r = 0$, and p can be found from force balance on the surface of the forward shock:

$$4\pi r^2 p = \frac{\mathrm{d}}{\mathrm{d}t}[M_s(r)\dot{r}] + \frac{G M_s(r) M_t(<r)}{r^2}$$

Substituting everything in, neglecting M in $M_t(<r)$, and taking $\gamma = 5/3$, one finds

$$\frac{\varepsilon}{2} L_{\text{Edd}} = 2 f_g \frac{\sigma^2}{G}\left[\frac{1}{2} r^2 \dddot{r} + 3 r \dot{r} \ddot{r} + \frac{3}{2}\dot{r}^3\right] + 10 f_g \frac{\sigma^4}{G}\dot{r}$$

It turns out this admits solutions where the shell coasts outwards at constant speed: $\ddot{r} = \dddot{r} = 0$. Consider the case where this constant speed $\dot{r}$ is the escape velocity of the galaxy, which is at every radius 2σ – that is, we consider *marginally outbound* solutions. This gives

$$\frac{\varepsilon}{2} L_{\mathrm{Edd}} = \frac{44 f_g \sigma^5}{G} \qquad \Rightarrow \qquad M \propto \sigma^5$$

– an example of an M-σ relation, which AGN scientists care about for some reason.

4.1.2 Momentum-Driven Winds

The equation of motion is, assuming $L = L_{\mathrm{Edd}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}[M_s(r)\dot{r}] = \frac{L_{\mathrm{Edd}}}{c} - \frac{GM_t(<r)M_s(r)}{r^2}$$

Substituting everything in as before, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}(r\dot{r}) = -2\sigma^2\left(1 - \frac{M}{M_\sigma}\right), \qquad M_\sigma \equiv \frac{f_g \sigma_T \sigma^4}{\pi G^2 m_p} \propto \sigma^4$$

$$(\times r\dot{r}) \qquad \Rightarrow r\dot{r}\frac{\mathrm{d}}{\mathrm{d}t}(r\dot{r}) = -2\sigma^2\left(1 - \frac{M}{M_\sigma}\right)r\dot{r} \qquad \Rightarrow \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{r}^2) = -\sigma^2\left(1 - \frac{M}{M_\sigma}\right)\frac{\mathrm{d}}{\mathrm{d}t}(r^2)$$

$$\Rightarrow r^2\dot{r}^2 = -2\sigma^2\left(1 - \frac{M}{M_\sigma}\right)r^2 + \text{const.} \qquad \Rightarrow \qquad \dot{r}^2 \approx -2\sigma^2\left(1 - \frac{M}{M_\sigma}\right)$$

for large r. Now this necessitates $M > M_\sigma$; physically this means that the BH luminosity must push the shocked material more strongly than it is attracted in by the galactic potential. If $M < M_\sigma$, a wind will not be supported and some sort of Bondi accretion process will ensue – M will rise until it reaches M_σ. At this stage, its winds will prevent further accretion and the mass will stabilise at $M \approx M_\sigma \propto \sigma^4$, another M-σ relation. Observationally, $\mathrm{d}\ln M/\mathrm{d}\ln\sigma = 4.4 \pm 0.3$, so we're on the right lines with these two solutions.

If M increases any further above M_σ, it turns out that the gas will be able to reach such a large radius that it will be able to cool efficiently – the energy-driven solution ensues, and the mass tends to $M \propto \sigma^5$. The story is therefore that an AGN initially performs momentum-driven winds (on ∼pc scales), before becoming massive enough to perform energy-driven winds (on ∼kpc scales), as shown in Figure 3.

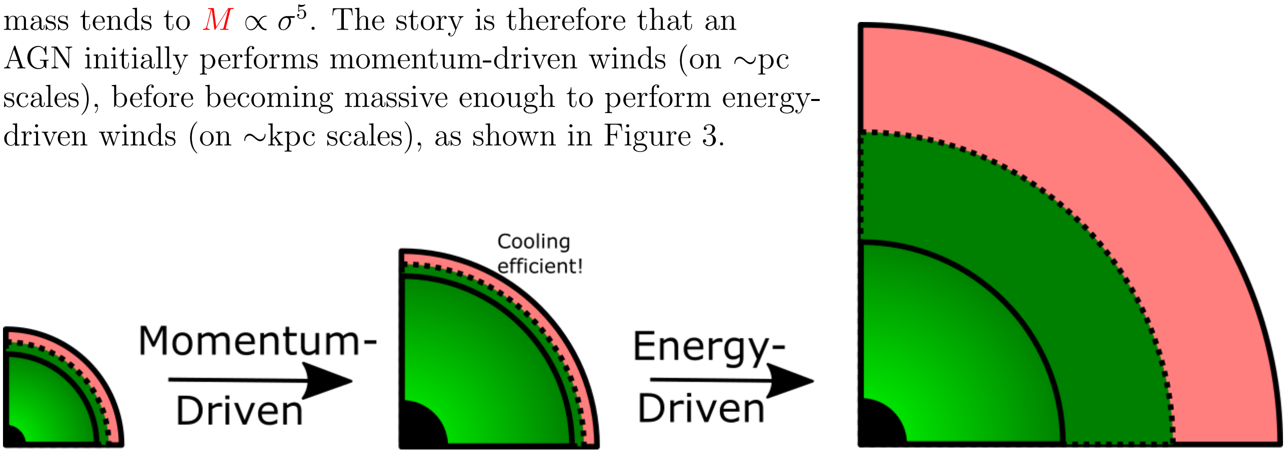

Figure 3 | Evolution of an AGN wind. Initially the wind is momentum-driven, and $M \approx M_\sigma$. If for whatever reason M increases above M_σ at the intermediate stage, the wind can reach such a radius that cooling becomes important, and energy-driven winds ensue.

4.2 Radiation-Driven Winds

A third possibility for the acceleration of winds is by radiation pressure. The force due to radiation pressure at a particular wavelength λ is:

$$f_\lambda = \int \frac{L_\lambda}{c} \frac{\kappa_\lambda \cdot 4\pi r^2 \rho \, \mathrm{d}r}{4\pi r^2} = \left(\int \kappa_\lambda \rho \, \mathrm{d}r \right) \frac{L_\lambda}{c} = \tau_\lambda \frac{L_\lambda}{c}$$

This multiplicative factor of τ arises due to there being several photon reflection events in an optically thick medium: each outgoing photon reflects off the inside of the shock, imparting its momentum, races to the other side of the sphere, reflects again, imparting its momentum again... this happens an average of τ times, so each photon imparts its momentum τ times. The total radiative pressure force, over all wavelengths is apparently $(1 - e^{-\tau_{\mathrm{UV}}} + \tau_{\mathrm{IR}})L/c$, so that for $\tau_{\mathrm{IR}}, \tau_{\mathrm{UV}} \ll 1$, this tends to $\tau_{\mathrm{UV}} L/c$ which is good for some reason.

An analogous calculation to the energy- and momentum-driven solutions apparently gives a lower limit to the UV luminosity required to launch an unbound wind:

$$\tau_{\mathrm{UV}} = 1, \tau_{\mathrm{IR}} = 0 \qquad \Rightarrow \qquad L_\sigma = \frac{4 f_g c \sigma^4}{G}$$

If one sets this equal to the Eddington luminosity, one finds that the critical mass M_σ is in fact the same as that found when considering momentum-driven winds, which makes sense as the equation at the start of that derivation assumed a radiation force of L_{Edd}/c, corresponding to a single scattering event of each photon ($\tau = 1$).