Structure & Evolution of Stars

Xander Byrne

Michaelmas 2021

1 Basic Properties

1.1 Motion

1.1.1 Radial

Radial velocities are measured using the Doppler effect. Spectral features appear at redder wavelengths for stars that are receding from us. The *redshift* z is defined by:

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \approx \beta$$

according to SR.

1.1.2 Transverse

The speed at which stars move across the sky year on year is known as the star's *proper motion* (measured in arcseconds / year). With parallax effects superimposed, stars appear to travel along helical paths across the sky.

1.2 Magnitude

The magnitude scale is reverse logarithmic with base $\sqrt[5]{100} = 10^{0.4} \approx 2.512$. A star 100 times brighter (in terms of flux received at Earth) thus has an apparent magnitude 5 lower; a star 100 times dimmer has an apparent magnitude of 5 greater:

$$\frac{F_2}{F_1} = 10^{-0.4(m_2-m_1)} \qquad \Rightarrow \qquad \boxed{m_2 - m_1 = -2.5\log_{10}\left(\frac{F_2}{F_1}\right)}$$

The *absolute magnitude*, M, is the apparent magnitude a particular star *would* have if it were 10pc away. If a star is d parsecs away and we receive a flux F_d, then because $F \propto L/d^2$, at 10pc we would receive a flux F_{10} where

$$F_{10}(10)^2 = F_d d^2 \qquad \Rightarrow \qquad \frac{F_d}{F_{10}} = 10^{-0.4(m-M)} = \left(\frac{10}{d}\right)^2$$

$$\Rightarrow \qquad m - M = -2.5\log\left(\frac{10}{d}\right)^2 = 5\log\left(\frac{d}{10}\right) = \boxed{5\log d - 5}$$

Magnitudes sometimes refer to a particular *filter* through which the flux is being measured. The filters are named: U (near-UV), B (blue), V ("visual" – green), R (red), I (near-IR), J, H, K, L (progressively far-IR).

If there is intervening dust blocking some starlight, the star may be assigned an unfairly high absolute magnitude. The *bolometric magnitude* M_{bol} accounts for this.

1.3 Mass and Binary Stars

A star's mass determines most of its other properties. It is easiest to measure the mass of distant stars by how their gravity affects another star in a binary system. In what follows (and in the notes, despite the misleading pictures), it is assumed for simplicity that the orbits are concentric circles with no eccentricity, as in Figure 1. Binary stars are classified based on how we know that they aren't just one really bright star.

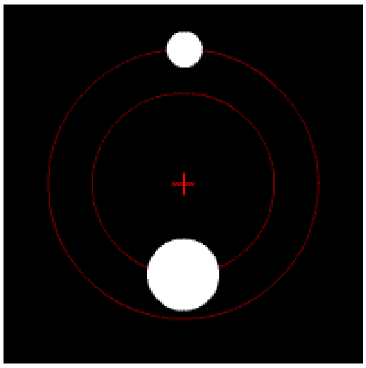

Figure 1 | Circular Binary Star System.

1.3.1 Visual Binaries

These can be individually resolved and the orbits tracked over time. The motion of a binary orbit about the centre of mass is such that:

$$\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{\theta_2}{\theta_1}$$

where a_k is the orbital radius, and $\theta_k = a_k/d$ the angle thereby subtended (assuming the system is face-on), of body k. Furthermore, from the two-body problem we also have:

$$G(M_1 + M_2)T^2 = 4\pi^2(a_1 + a_2)^3 = 4\pi^2 d^3(\theta_1 + \theta_2)^3$$

where T is the period of the orbit and d is the distance to the system. From this and the above ratio, one can deduce both M_1 and M_2 simply by measuring T, d, θ_1 and θ_2.

Binary orbits are rarely face-on; they are usually *inclined* by an angle i to the plane of the sky, making the orbital radii appear smaller by a factor of $\cos i$ than they actually are. If the inclination angle is known, the above formula can simply be adjusted:

$$G(M_1 + M_2)P^2 = 4\pi^2 d^3 \left(\frac{\theta_1 + \theta_2}{\cos i}\right)^3$$

where θ_k are the angles which are actually measured from our point of view, so $\theta_k/\cos i$ are the (larger) angles which we would see if we were face-on.

1.3.2 Spectroscopic Binaries

if $i \neq 0$, the stars have an oscillating radial velocity and hence an oscillating redshift, so we can measure $v_{k,r} = v_k \sin i$ at the peak redshift. For circular orbits, $v_{k,r} = (2\pi a_k/T)\sin i$, so:

$$\frac{M_1}{M_2} = \frac{v_{2,r}}{v_{1,r}}$$

However to use $\mathfrak{K}3\mathfrak{L}$ to find the *sum* of the masses, we need to know i:

$$G(M_1+M_2)T^2 = 4\pi^2\left(\frac{T}{2\pi}v_1 + \frac{T}{2\pi}v_2\right)^3 \qquad \Rightarrow \qquad G(M_1+M_2) = \frac{T}{2\pi}\left(\frac{v_{1,r}+v_{2,r}}{\sin i}\right)^3$$

In *single-lined spectroscopic binary* systems, only one set of oscillating spectral lines is seen; this may occur if the unseen body is a dim star, a planet, or a black hole. Using the ratio above,

$$G(M_1+M_2) = \frac{T}{2\pi}\left(v_{1,r}\frac{1+M_1/M_2}{\sin i}\right)^3 \qquad \Rightarrow \qquad \frac{M_2^3}{(M_1+M_2)^2}\sin^3 i = \frac{Tv_{1,r}^3}{2\pi G}$$

The left-hand side of the second equation is the *mass function* and depends on i which is unknown. It does at least give a lower bound of $M_2 > Tv_{1,r}^3/2\pi G$.

1.3.3 Eclipsing Binaries

If i is very close to $\pi/2$, the orbit will be edge-on as seen from Earth, and the stars eclipse each other at certain points in their orbits, causing periodic dimming. By recording the light curve of the system very accurately, one can obtain information about the radii and temperatures of the two stars.

1.4 Effective Temperature and Luminosity

The *specific intensity* $I(\lambda)$ is *the power flux per wavelength per solid angle.* That is, if $U(\lambda)\mathrm{d}\lambda$ is the energy coming through an area $\mathrm{d}A$ at an angle θ to the area's normal, in a time $\mathrm{d}t$, in a wavelength band of width $\mathrm{d}\lambda$ around λ and heading in the direction of the solid angle $\mathrm{d}\Omega$, then

$$U(\lambda)\,\mathrm{d}\lambda = I(\lambda)\,\mathrm{d}A\cos\theta\,\mathrm{d}t\,\mathrm{d}\lambda\,\mathrm{d}\Omega$$

Most stars' surfaces behave approximately as black bodies, which have a specific intensity given by stat thermo as:

$$I(\lambda) = B(\lambda;T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT}-1}$$

where T is the *effective temperature*: that of the surface layers. Integrating over wavelength:

$$\int_0^\infty B(\lambda;T)\,\mathrm{d}\lambda = 2hc^2\int_0^\infty \frac{1}{\lambda^5}\frac{\mathrm{d}\lambda}{e^{hc/\lambda kT}-1} = \frac{2k^4T^4}{h^3c^2}\underbrace{\int_0^\infty \frac{u^3\,\mathrm{d}u}{e^u-1}}_{\pi^4/15} = \frac{\sigma T^4}{\pi} \qquad \left(\sigma = \tfrac{2\pi^5k^4}{15h^3c^2}\right)$$

The *total* energy per unit time coming out of a star (its luminosity) is given by:

$$L = \int \mathrm{d}\Omega\int \mathrm{d}A\cos\theta\int_0^\infty \mathrm{d}\lambda\,B(\lambda;T) = \underbrace{\int_0^{\pi/2}(4\pi R^2)\cos\theta(2\pi\sin\theta\,\mathrm{d}\theta)}_{4\pi^2R^2}\underbrace{\int_0^\infty B(\lambda;T)\,\mathrm{d}\lambda}_{\sigma T^4/\pi} = \boxed{4\pi R^2\sigma T^4}$$

so σT^4 is the total flux from the surface of the star. L depends on both R and T; its brightness as seen from Earth depends on R, T, and d. Using the fact that black bodies are isotropic, we also have that $u(\lambda)$, the energy per unit volume, per wavelength, is given by:

$$u(\lambda) = \frac{1}{c}I(\lambda)\int \mathrm{d}\Omega = \frac{4\pi}{c}I(\lambda)$$

Integrating *this* over wavelength gives the *total* energy density:

$$u = \frac{4\pi}{c}\int_0^\infty I(\lambda)\,\mathrm{d}\lambda = \frac{4\pi}{c}\int_0^\infty B(\lambda;T)\,\mathrm{d}\lambda = \frac{4\sigma}{c}T^4 \equiv aT^4$$

Photons exert a *radiation pressure*. Each photon carries momentum E/c; if it is reflected it will transfer twice this, and hence the radiation pressure is given by

$$P = \int \mathrm{d}\Omega\cos\theta\int_0^\infty \mathrm{d}\lambda\,\frac{2I(\lambda)\cos\theta}{c} = \frac{2}{c}\underbrace{\int_0^{\pi/2}\cos^2\theta(2\pi\sin\theta\,\mathrm{d}\theta)}_{2\pi/3}\underbrace{\int_0^\infty B(\lambda;T)\,\mathrm{d}\lambda}_{\sigma T^4/\pi}$$

$$= \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4$$

Setting $\mathrm{d}B/\mathrm{d}\lambda = 0$ gives $\boxed{\lambda_{\max}T = 2.90\text{mm K}}$, so the colour of a star is a measure of its temperature: the bluer the hotter. Rather than measuring the entire spectrum, often one takes the magnitudes through two different filters and subtracts them, giving e.g. a "$B-V$ colour" $m_B - m_V$ $(= M_B - M_V)$. Lower $B-V$ values indicate a bluer, and thus hotter, star. A scatter plot (*Hertzsprung-Russell Diagram*) of M_V (a proxy for brightness/luminosity) against $B-V$ (a proxy for temperature) shows that most stars fall on a narrow strip, known as the *Main Sequence*, diagonal from dim red stars to bright blue stars. Other islands of stars also appear, in the bright red and dim white regions, home to red giants and white dwarfs.

1.4.1 Luminosity, Mass and Lifetime

From measuring the luminosities and masses of many stars, we find that luminosity is strongly related to the initial mass by the power law $L \propto M^{3.5}$. Luminosity is essentially how quickly a star is converting its mass into energy and shining it out into space, so $L \propto -\mathrm{d}M/\mathrm{d}t$. Integrating, we find:

$$\boxed{t \propto M_0^{-2.5}}$$

This is a strong dependence: a star of two solar masses will live only 18% as long.

2 Atmospheres

2.1 Opacity $\kappa(\lambda)$

When light of specific intensity I travels through a cloud of gas, some of it is absorbed:

$$\mathrm{d}I = -\kappa(\lambda)\rho I\,\mathrm{d}s$$

where κ is the wavelength dependent *opacity* of the gas. We see that the mean free path of photons through the gas is $1/\kappa\rho$. Thermodynamically, the mean free path is given by $1/\sigma n$ where $\sigma(\lambda)$ is the interaction cross-section. Thus

$$\kappa(\lambda)\rho = \sigma(\lambda)n$$

2.1.1 Sources of Opacity

- **Bound-bound**: very strong near atomic transition wavelengths and very weak elsewhere
- **Bound-free**: very strong below a certain value of λ at which common species can be ionised (to bring about ionisation or molecular photodissociation). This gives a mean opacity of the form $\kappa \propto \rho T^{-7/2}$.
- **Free-free**: also a continuum, arising from free electrons absorbing photons to accelerate. This also gives a mean opacity of the form $\kappa \propto \rho T^{-7/2}$.
- **Thomson scattering**: the non-relativistic limit of Compton scattering[1]. Only relevant in very dense or hot stars (high n), as the σ for this interaction (involving photons being scattered by electrons) is very small. This gives a mean opacity of the form $\kappa \propto 1 + X$, where X is the hydrogen mass fraction.

The relationship between κ and T is shown in Figure 2. We see that at low T, κ rises rapidly as easily-ionised elements like K and Na are ionised, forming both H^- ions (H has a higher ionisation potential), which have strong bound-free opacity, and free electrons for free-free absorption. As T increases H and He are then ionised, further increasing the electron density and κ. When T is high enough that most atoms are ionised, κ simply depends on the details of the free-free process, so $\kappa \propto \rho T^{-3.5}$. At yet higher T, κ levels out as temperature-independent electron scattering dominates. The bump at $\log T \approx 5.2$ is where $k_B T \approx 13.6\text{eV}$.

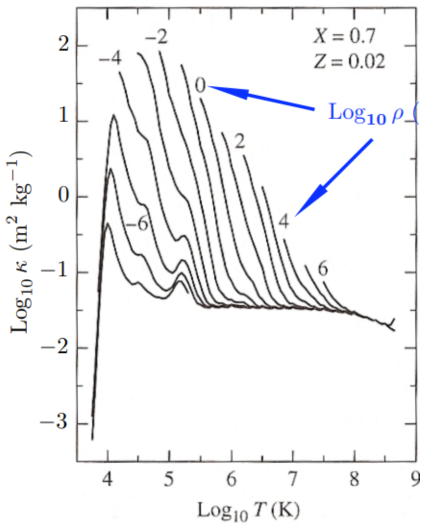

Figure 2 | Opacity κ and Temperature T.

- **O stars** are hot enough that Thomson scattering is relevant, as well as bound-free absorptions for H and He, and free-free absorption.
- **B and A stars** mostly just have bound-free absorptions for H, and free-free absorption.
- **F–M stars** (including the Sun) are cool enough to have lots of H^- for bound-free absorptions, and high electron density for free-free absorptions. Particularly cool stars also have molecules like TiO, leading to many bound-bound and bound-free contributions.

2.1.2 Optical Depth

The optical depth $\tau(\lambda)$ is a dimensionless property of a given path through some gas. Typically this path starts at some depth within a star and ends at its surface. Its definition is:

$$\boxed{\tau(\lambda) = \int_0^s \kappa(\lambda; s')\rho(s')\,\mathrm{d}s'} \qquad \Rightarrow \qquad I(s) = I_0 e^{-\tau}$$

[1]In this limit, the frequency of the photons are not changed, though their direction is

where I_0 is the incident specific intensity at the wavelength in question. τ can be thought of as the number of mean free paths contained from $s' = 0$ to $s' = s$.

The stellar photosphere is defined to be the depth from which $\boxed{\tau = 2/3}$ as viewed radially. If one looks at a star from a fixed viewpoint, looking at the centre of the star *is* looking radially, and so one sees right to the lower limits of the photosphere. However, looking at the sides of the star (the "limbs") does *not* see to the lower limits of the photosphere[2], but to a smaller radial depth of the star, at which the matter is cooler and less luminous. This leads to "limb darkening", whereby the limbs of stars appear progressively redder and dimmer.

2.2 Absorption Lines

Stellar spectra feature absorption lines from atoms and molecules contained within the atmosphere. For example, the Hα line is due to an absorption to promote an electron in a H atom from the $n = 2$ to $n = 3$ state, corresponding to a photon of energy $13.6\left(\frac{1}{4} - \frac{1}{9}\right) = 1.89$eV and wavelength 656nm – deep red.

2.2.1 Line Strength

The Hα absorption line requires $n = 2$ hydrogen atoms, and there are two competing temperature-dependent factors determining the population of this species.

Thermal Excitation. The probability of an atom being in a state with energy E_n and degeneracy g_n is given by Boltzmann's distribution:

$$P_n = \frac{1}{Z} g_n e^{-E_n/kT}$$

where the partition function $Z = \sum_1^\infty g_n e^{-E_n/kT}$ is a normalisation factor. In the limit $T \to \infty$, the exponential tends to 1 and the occupation of each energy level depends solely on the degeneracy of that level, rather than on its energy.

Collisional Ionisation. At higher temperatures, ionising interatomic collisions become more frequent, and so atoms are bonked into higher ionisation states. The Saha equation relates the densities of atoms (in units of m^{-3}) in ionisation state i relative to state $i+1$ (e.g. the densities of Fe II relative to Fe III):

$$\frac{n_{i+1}}{n_i} = \frac{2}{n_e \lambda_e^3} \frac{Z_{i+1}}{Z_i} \exp\left(-\frac{\chi_i}{k_B T}\right)$$

where χ_i is the energy needed to convert an atom of state i to state $i+1$, Z_i is the partition function of an atom in state i, and λ_e is the *thermal de Broglie wavelength*, that of an electron with kinetic energy $\pi k_B T$: $\lambda_e = h/\sqrt{2 m_e \pi k_B T}$.

The strength of the Hα line peaks at around $T = 10^4$K. It is weaker (i.e. there is not much $n = 2$ H I) in stars on either side of this temperature because:

- Cooler stars have most of their hydrogen in the $n = 1$ state
- Hotter stars have ionised most of their hydrogen to H II

[2] if one looks at the very sides of the star, one can see all the way through it!

For the same reasons all spectral lines have a peaked absorption strength profile, as shown in Figure 3. The strengths of different absorption lines can identify temperature to an accuracy of order 10K, which is much more accurate than comparison to blackbody spectra. Spectral features were use to classify stars in the system OBAFGKM, which is in decreasing order of temperature. This sequence is then subdivided from 0 to 9, O9 is just hotter than B0, etc. Hotter stars are also sometimes referred to as "early-type", and cooler stars as "late-type".

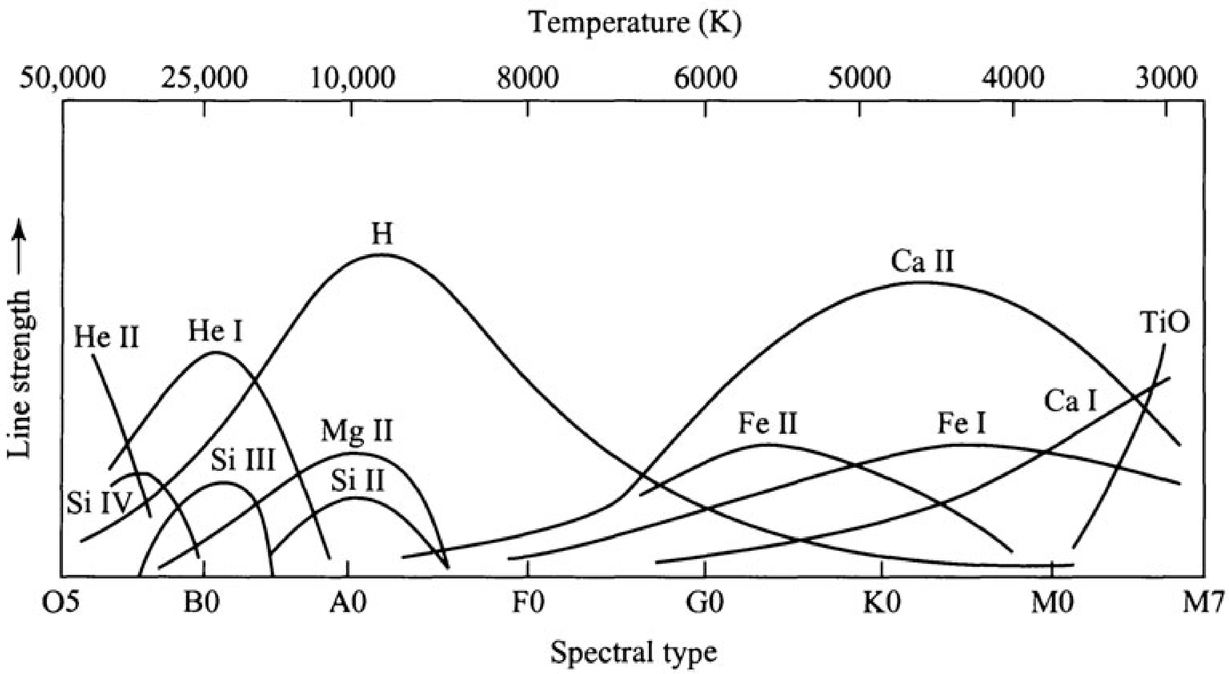

Figure 3 | Strengths of Spectral Lines at Various Temperatures.

2.2.2 Spectral Line Width

The width of a spectral line is quantified by the *equivalent width*, given by:

$$W = \int \frac{I_0 - I(\lambda)}{I_0} \, \mathrm{d}\lambda = \int (1 - e^{-\tau}) \, \mathrm{d}\lambda$$

where I_0 is the specific intensity on either side of the line. For no absorption, $I(\lambda) = I_0 \Rightarrow W = 0$. For total absorption within a band $\Delta\lambda$, $I(\lambda) = 0$ in this range $\Rightarrow W = \Delta\lambda$. A given absorption line does not just have $I(\lambda) \neq I_0$ at only one particular λ, but over a range and hence $W \neq 0$; there are three reasons why.

Natural Broadening. As a result of the uncertainty in the energies of higher energy states with finite lifetimes, spectral lines naturally have a Lorentzian profile centred on the redshifted wavelength:

$$\phi(\lambda; v) = \frac{1}{\pi} \frac{\delta_k}{\delta_k^2 + (\lambda - \lambda_0(v))^2}$$

where k is the upper level of the transition, and δ_k is called the *radiation damping constant*, which encodes the inverse of the excited state's lifetime.

Pressure broadening effectively increases δ_k by $\delta_p \propto T^{1/2} n$, as collisions decrease the lifetimes of upper levels. At a given temperature, more luminous stars are larger, have lower surface gravity and atmospheric pressure, so have thinner peaks; dimmer stars are smaller, higher pressure, and have wider peaks. The *luminosity class* scale from I (most luminous) to VII. Stars of class V are those on the Main Sequence; I through IV are various levels of giant star; class VI is a bit dimmer than most Main Sequence star; class VII are the white dwarfs.

Doppler Broadening. Due to thermal motion of individual atoms, the wavelengths of absorption can be redshifted or blueshifted if the atoms are moving on our line of sight. The radial velocity distribution is Gaussian, centred on the velocity of the star v_0:

$$\Psi(v) = \frac{1}{\sqrt{\pi}b} \exp\left(-\frac{(v-v_0)^2}{b^2}\right)$$

where b is the *Doppler width* of the distribution. If only thermal effects were involved, b would simply be $\sqrt{2kT/m}$ as predicted by statistical mechanics, but stellar atmospheric turbulence causes an increase in the width: $b^2 = b^2_{\text{thermal}} + b^2_{\text{turb}}$.

2.2.3 Curves of Growth

The alternative definition of $\tau(\lambda)$ is

$$\tau(\lambda) = \int_0^s \sigma(\lambda) n(s')\,\mathrm{d}s' \approx N\sigma(\lambda)$$

where $N \equiv \int_0^s n(s')\,\mathrm{d}s'$ is the *column density* of the atom which causes the absorption (in m^{-2}). The relative atmospheric abundances of atoms can be deduced from their relative N. However, to find N from τ (in turn from W), we first need to know $\sigma(\lambda)$. It can be written as $\sigma(\lambda) = \sigma_0\Phi(\lambda)$, where σ_0 involves atomic parameters and $\Phi(\lambda) = \phi(\lambda; v) * \Psi(v)$ is the *broadening function*, accounting for all possible atomic radial velocities. The overall cross section is therefore $\sigma = \sigma_0\phi(\lambda; v) * \Psi(v)$, and the overall optical depth is then $\tau(\lambda) = N\sigma_0\phi(\lambda; v) * \Psi(v)$: a *Voigt function*.

2.3 Hydrodynamics

In any atmosphere, the equation for hydrostatic equilibrium is

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho}{r^2}} = -\rho g$$

A negative pressure gradient is intuitive as we expect there to be lower pressures higher up as there is less gas to support. It is often found that the density is proportional to the pressure (at least locally), and so the pressure decays as $\exp\left(-\frac{r}{H_P}\right)$, where H_P depends on the proportionality between ρ and P. H_P is a length scale known as the *pressure scale height*.

If some event takes place in a star (like a small explosion), its effects will not be felt instantly throughout the star. In the absence of pressure, a particle falling through a star would take a time of order

$$\sqrt{\frac{2R}{g}} = \sqrt{2R\frac{R^2}{GM}} = \sqrt{\frac{3}{2\pi}\frac{1}{G\langle\rho\rangle}}$$

We therefore take as a dynamical timescale for a given star the quantity $\boxed{t_{\text{dyn}} = (G\rho)^{-1/2}}$.

This is essentially the maximum timescale over which gas can move from one equilibrium state to another after a change, and applies to stars, interstellar clouds, and even whole galaxies.

3 Energy

3.1 Sources

3.1.1 Gravitational Potential

When a cloud collapses into a star, gravitational energy is converted into kinetic/thermal energy. According to the Virial Theorem:

$$\boxed{\langle K \rangle = -\frac{1}{2} \langle U \rangle} \qquad (\mathfrak{VT})$$

so half of the lost potential energy when a cloud is converted into thermal energy on gravitational collapse. Assuming uniform density, the total gravitational potential energy of a star is:

$$U = \int_0^R -\frac{G\frac{4}{3}\pi r^3 \rho}{r} 4\pi r^2 \rho \, \mathrm{d}r = -\frac{16 G\pi^2 R^5 \rho^2}{15} = \boxed{-\frac{3}{5}\frac{GM^2}{R}}$$

$3GM^2/5R$ is freed up; according to $\mathfrak{VT}$ half of that goes into heating up the new star; the other half is radiated away. For the Sun, this radiation is $3GM^2/10R = 10^{48}$erg, sufficient only to power the Sun at its current luminosity for 10Myr, so this is not what powers the Sun.

3.1.2 Fusion

On fusing four ^{1}H into ^{4}He, converts $\boxed{0.007}$ of the hydrogen mass into energy. If 10% of the Sun's mass were converted from ^{1}H to ^{4}He, that would provide $(0.007)(0.1)M_\odot c^2 = 1 \times 10^{51}$ergs, enough to power the Sun at its current luminosity for 10^{10} years.

High temperatures are required to fuse 4 ^{1}H into ^{4}He. Setting $\frac{3}{2}kT$ equal to the potential energy between two protons at the strong force radius ($r_f \approx 1$fm) gives $T = 10^{10}$K, way hotter than the Sun's core. The Sun is actually able to fuse because protons can quantum tunnel through the Coulomb potential barrier, so they don't need all that energy. Rather than getting right up to r_f, the protons only need to be within about 1 de Broglie wavelength before the fusion can take place. We have a double equation to solve for T:

$$\frac{(h/\lambda)^2}{2\mu} = \frac{3}{2}kT = \frac{e^2}{4\pi\epsilon_0\lambda}$$

using the reduced mass μ to calculate the de Broglie wavelength to account for the relative motion of the two particles. Eliminating λ gives $T = \mu e^4/12\pi^2\epsilon_0^2 h^2 k$, which for the Sun is a much more realistic 10^7K.

ϵ, the energy released per unit time per unit mass, is naturally proportional to the mass fractions of the two fusing species, proportional to the density for a two-body collision, and proportional to T^β where β depends on the fusion reaction in question.

Proton-Proton Chain. The main process for the Main Sequence. Overall,

$$4\,^1\mathrm{H} \to {}^4\mathrm{He} + 2\,\mathrm{e}^- + 2\nu_e$$

There are three pathways via which the proton-proton chain can occur, PPI through PPIII, though PPI occurs about 70% of the time. $\beta = 4$.

CNO Cycle. Also converts ^{1}H to ^{4}He , by a process catalytic in CNO. It is only more efficient than PP if the temperature is above about 10^7K (for a star of solar composition) as $\beta = 17$, so requires stars of $M \gtrsim 1.3M_\odot$. Because stars have a negative temperature gradient, CNO fusion occurs within a much smaller radius than PP.

Triple-Alpha Process. Three ^{4}He fuse to form ^{12}C. Of course, the probability of three meeting together simultaneously is very low; there is an intermediate ^{8}Be nucleus, but it is so short-lived that $\epsilon_{3\alpha}$ is proportional to ρ^2 instead of ρ. Requiring high temperatures ($\gtrsim 10^8$K) and pressures, the triple-alpha process has $\beta = 40$, being incredibly temperature-sensitive, and thus centrally-localised.

Alpha Ladder. Once there's enough ^{12}C around, they can fuse with ^{4}He to form O, Ne, Mg, Si, S (α*-capture nuclei*) with progressively less energy being released at each step.

Burning of Heavier Elements. For massive stars, core temperatures can reach 6×10^8K, high enough for carbon nuclei to fuse together into ^{24}Mg. At 10^9K oxygen nuclei can fuse into ^{32}S. At 1.5×10^9K, Wien's law gives a hard X-ray wavelength, photons of which can photofission the nuclei present, giving a messy chaos of nucleons flying around. At 3×10^9K, silicon and sulfur continue the alpha ladder up to Fe and Ni for a couple of days, but beyond that any fusion that occurs is endothermic and won't help hold the star up any longer.

If the core is hot enough to burn ^{4}He, there will be a shell further out which is not and regular PP fusion still occurs. For stars whose cores are hot enough to fuse the heaviest of nuclei, there is an onion structure around the core where progressively lighter nuclei are fused; there will also be a non-burning shell of hydrogen around the outside.

Neutron Capture. At photofissile temperatures, the orphaned neutrons are readily absorbed by any remaining nuclei as they have no Coulomb barrier. Nuclei thus progressively absorb neutrons until an unstable isotope is reached, which undergoes β^- decay into a more stable element one further along the Periodic Table. This then continues to absorb neutrons until an unstable isotope of *this* element is reached, and so on. This is the *s-process*, and it occurs a lot in stars on the asymptotic giant branch; typical resulting elements are Cu and Pb.

Supernovae and neutron star formation provide such a barrage of neutrons that unstable nuclei do not have time to undergo β^- decay before they absorb another neutron. As such, neutron capture continues until a silly neutron-saturated isotope like ^{11}Li or ^{22}C is produced. When the often brief neutron flux ends, the nuclei then undergo a series of β^- decays until they reach a stable nucleus. A typical indicator of this *r-process* is Eu.

3.2 Energy Transport

There are three processes by which the energy released by fusion reactions could be transported to the surface, in the form of photons and thermal energy of electrons and neutrinos:

- **Radiation** – photons being repeatedly absorbed and emitted until they reach the surface
- **Convection** – due to macroscopic motion of large buoyant cells of material
- **Conduction** – due to microscopic particle collisions (only important in white dwarfs)

3.2.1 Radiation

Eddington's Equation gives the temperature gradient necessary to carry away all the luminosity produced by a star:

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4}\frac{1}{ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2}}$$

where L_r is the total luminosity produced within a radius r.

If there is sufficient radiation and insufficient gravity, the outermost layers of the star will be blown away. The maximum luminosity is called the *Eddington luminosity* L_E. At the surface,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{R^2} \qquad P = \frac{1}{3}aT^4 \qquad \frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4}\frac{1}{ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi R^2}$$

Differentiating the second, substituting the third, and equating the first gives $L_\mathrm{E} = 4\pi cGM/\kappa$.

3.2.2 Convection

According to Kramers' Law, $\kappa \propto T^{-3.5}$, so on moving out from the star's core to cooler layers, κ rises, and thus so does the magnitude of the temperature gradient, as radiation is less able to pass through and equilibrate T. Large temperature gradients are unstable phenomena in any type of atmosphere, and lead to chunks of rising gas (with higher T than their surroundings) convecting up.

Assuming that rising gas parcels move adiabatically (i.e. too quickly to balance temperature with the surroundings), the Schwarzschild condition for convective stability is:

$$\frac{\mathrm{d}T}{\mathrm{d}r} > \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{k_B}{m_H}\frac{T}{C_P P}\frac{\mathrm{d}P}{\mathrm{d}r} \qquad \text{(Stable)}$$

Now $\mathrm{d}T/\mathrm{d}r$ and $\mathrm{d}P/\mathrm{d}r$ are both negative – the star is stable unless $\mathrm{d}T/\mathrm{d}r$ is *so* negative that it becomes more negative than the RHS. If that happens then the temperature gradients are too strong and convection occurs.

Stars are divided into convective layers and radiative layers, depending on which method of energy transport is dominant at different radii. If gas is only partially ionised, then not only is the opacity high (leading to large temperature gradients), but the specific heat is also high, moving the RHS above closer to 0 and leaving less room for $\mathrm{d}T/\mathrm{d}r$: both make convection more likely. As such, the coolest stars ($M < 0.4M_\odot$) are fully convective, as they mostly consist of partially ionised gas, which has high κ and C_P. Between $0.4M_\odot < M < 1.5M_\odot$, the core becomes hot enough for the gas to be fully ionised there, reducing κ and C_P and causing radiation to be more efficient; in the outer layers there is still only partial ionisation, so convection still occurs (the Sun's is radiative up to $0.7R_\odot$ and convective from there to the surface). By about $1.5M_\odot$, the star becomes hot/ionised/transparent/low-heat-capacity enough for the star to be fully radiative. For stars above $2M_\odot$, CNO and triple-alpha processes become dominant, leading to a massive increase in core luminosity and hence temperature gradients: hence convection begins again in the cores of the largest stars.

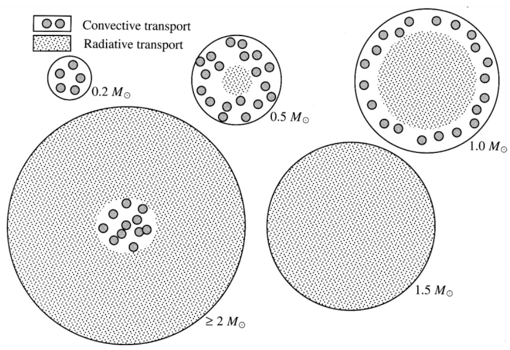

Figure 4 | Convective and Radiative Regions in Stars of Different Masses.

4 Models

4.1 Equations of Stellar Structure

1. **Mass**

$$\boxed{\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho} \tag{1}$$

2. **Hydrostatic**

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho}{r^2}} \tag{2}$$

3. **Luminosity**

$$\boxed{\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon} \tag{3}$$

4. **Transport**, radiative or convective

$$\boxed{\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} = -\frac{3}{4}\frac{1}{ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2}} \qquad \boxed{\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}} = -\frac{\gamma-1}{\gamma}\frac{\mu m_H}{k}\frac{Gm(r)}{r^2}} \tag{4r, 4c}$$

These are coupled to each other by equations of state for ϵ, κ, P and T_{eff}:

5. **Production**

$$\epsilon = \epsilon_0 \rho^\alpha T^\beta \tag{5}$$

6. **Opacity**

$$\kappa = \kappa_0 \rho T^{-3.5} \tag{6}$$

7. **Pressure**, gas or radiation

$$\boxed{P_{\mathrm{gas}} = \frac{\rho k T}{\mu m_H}} \qquad \boxed{P_{\mathrm{rad}} = \frac{1}{3}aT^4} \tag{7g, 7r}$$

μm_H is the average particle mass; μ depends on the mass fractions X and Y, in a way best understood from the following table. Neglecting the masses of electrons and assuming elements larger than He have equal numbers of protons and neutrons,

Quantity	^{1}H	^{4}He	$^{2Z}_{Z}$Z
Mass per avg m_H	Xm_H	Ym_H	$(1-X-Y)m_H$
Nuclei per m_H	X	$Y/4$	$(1-X-Y)/2Z$
Particles per m_H	$2X$	$3Y/4$	$(1-X-Y)/2$

$$\Rightarrow \mu = \frac{1(X) + 4(Y/4) + 2Z((1-X-Y)/2Z)}{2X + 3Y/4 + (1-X-Y)/2} = \frac{1}{3X/2 + Y/4 + 1/2} = \frac{4}{6X + Y + 2}$$

8. **Effective Temperature**

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4$$

T_{eff} is a different variable to T, unless one is explicitly evaluating T at the surface.

4.2 Homology

There is no analytic way to solve the above equations in general, so we restrict ourselves to *homologous* sets of stars. To be homologous, two stars must contain the same proportion of their mass within the same proportion of their radius. For instance, for the Sun, $m_\odot(0.25R_\odot) \approx 0.5M_\odot$, and many other stars have roughly $m(0.25R) \approx 0.5M$.

For a homologous set of stars, one can deduce proportionalities between various stellar properties by dimensional analysis. From equations (1)-(4), these are

$$\frac{M}{R} \propto R^2\rho \qquad \frac{P}{R} \propto \frac{M\rho}{R^2} \qquad \frac{L}{R} \propto R^2\rho\epsilon \propto R^2\rho^{1+\alpha}T^\beta \qquad \frac{T}{R} \propto \frac{\kappa\rho}{T^3}\frac{L}{R^2} \text{ or } \propto \frac{\mu G M}{R^2}$$

4.2.1 Luminosity-Mass Revisited

For a radiative star dominated by gas pressure, solving

$$\frac{M}{R} \propto R^2\rho \qquad \frac{P}{R} \propto \frac{M\rho}{R^2} \qquad \frac{T}{R} \propto \frac{\kappa\rho}{T^3}\frac{L}{R^2} \qquad P \propto \frac{\rho T}{\mu}$$

for L, and treating κ as a constant, we obtain $L \propto M^3$, not far off the observed $L \propto M^{3.5}$ considering the dodgy approximations.

4.2.2 Minimum Stellar Mass

The minimum temperature for proton-proton core fusion is about 4×10^6K; the Sun's core temperature is about 1.5×10^7K. Taking $\alpha = 1$ and $\beta = 4$, we have $\frac{L}{R} \propto R^2\rho^2T^4$. In combination with the four above, one eventually obtains $M \propto T^{7/4}$. The star homologous to the Sun with the minimum mass so that its core temperature is high enough to initiate PP fusion thus has mass $M_{\text{min}} = (4 \times 10^6/1.5 \times 10^7)^{7/4}M_\odot = 0.1M_\odot$.

4.3 Degeneracy Pressure

White dwarfs and neutron stars are supported by degeneracy pressure; the derivation in the notes was so bad I've stolen the PQM one (without the stray factors of 2).

Consider a quantum gas of fermions confined to a cube of size L: each will have a wavevector $\mathbf{k} = (2\pi/L)(n_x, n_y, n_z), n_i \in \mathbb{Z}$, occupying a $\mathbf{k}$-space volume $(2\pi/L)^3 = 8\pi^3/V$. If there are N fermions, they will occupy a $\mathbf{k}$-space volume of

$$\frac{N}{2}\frac{8\pi^3}{V} = \frac{4}{3}\pi k_F^3 \qquad \Rightarrow \qquad k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = \left(\frac{3\pi^2\rho}{m}\right)^{1/3}$$

where k_F is the Fermi momentum, the maximum momentum magnitude (provided everything is in as low an energy state as possible) and the factor of $\frac{1}{2}$ at the start is due to the spins. The total energy is then

$$E = \int_0^{k_F} 2\underbrace{\frac{\hbar^2k^2}{2m}}_{\substack{\text{Energy}\\\text{per state}}}\underbrace{\frac{4\pi k^2\,\mathrm{d}k}{8\pi^3/V}}_{\substack{\text{Number}\\\text{of states}}} = \frac{\hbar^2 V}{10m\pi^2}k_F^5 = \frac{\hbar^2}{10m\pi^2}\left(3\pi^2N\right)^{5/3}V^{-2/3}$$

The degeneracy pressure is given by $P_d = -\,\partial E/\partial V$:

$$\Rightarrow P_d = \frac{\hbar^2}{15m\pi^2}\left(\frac{3\pi^2 N}{V}\right)^{5/3} = \frac{\hbar^2}{15m\pi^2}\left(\frac{3\pi^2 \rho}{m}\right)^{5/3} \boxed{\propto T^0 \rho^{5/3} m^{-8/3}}$$

so the pressure is independent of T and proportional to $N^{5/3}$. Also, $P_d \propto m^{-8/3}$, so neutron degeneracy pressure is much smaller than electron. In the limit of relativistically moving electrons, the energy per state is $pc = \hbar ck \propto k$ rather than $\propto k^2$, so $E \propto k_F^4$ and $P_d \propto \rho^{4/3}$ rather than $\rho^{5/3}$.

The gravitational potential energy is $U_g = -3GM^2/5R \propto -M^2V^{-1/3}$; hence the gravitational pressure $P_g \propto -M^2V^{-4/3}$; compare $P_d \propto N^{5/3}V^{-5/3} \propto M^{5/3}V^{-5/3}$. Setting the two pressures equal to (minus) each other for equilibrium, we find MV is constant; if more mass is added to a degenerate body it will shrink.

5 Evolution

Stellar evolution is complicated, and some of this section will be about observations where we don't quite understand the reasons behind them and so are justified poorly.

5.1 Formation

5.1.1 Collapse of Giant Molecular Cloud

An equilibrium self-gravitating system will have $2K+U=0$; if $2K+U>0$, there is too much kinetic/thermal energy and the cloud expands. If $2K+U<0$, perhaps because the cloud has been knocked out of equilibrium by shockwaves or collisions, then there is not enough K to support the cloud under gas pressure and it will collapse. The potential energy of a uniform sphere is

$$U = -\frac{3GM^2}{5R} = -\frac{3GM^2}{5}\left(\frac{4\pi\rho}{3M}\right)^{1/3} = -\frac{3G}{5}\left(\frac{4\pi\rho}{3}\right)^{1/3} M^{5/3}$$

whereas the kinetic energy is $\frac{3}{2}\frac{M}{\mu m_H}kT$. The instability criterion is then:

$$\frac{3MkT}{\mu m_H} < \frac{3G}{5}\left(\frac{4\pi\rho}{3}\right)^{1/3} M^{5/3} \qquad \Rightarrow \qquad M > \underbrace{\left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}}_{M_J}$$

If the mass exceeds the Jeans mass M_J for a given density, the cloud will collapse in a time of order $(G\rho)^{-1/2}$. We see that high densities and low temperatures are conducive to gravitational collapse. Alternate criteria can be derived for the Jeans radius and density for given masses, densities, or radii.

Stars form from the cores of **giant molecular clouds**, which are so cold (10K) that hydrogen exists mostly as molecular H_2, and relatively dense (10^{10}m^{-3}). As the GMC collapses (roughly isothermally), $\rho \uparrow$ and so $M_J \downarrow$, causing the cloud to fragment into many smaller collapsing clouds. As these clouds collapse, the lost potential energy is converted to equal parts thermal energy, and IR radiation which is lost to space.

5.1.2 Protostars

When ρ (and hence κ_{IR}) become high enough that the IR can no longer escape, the IR too is forced to convert to thermal energy and the gas pressure increases to the point of preventing further collapse – the quasistable result is a **protostar**. Gas continues to fall onto the protostar, releasing more GPE and heating it up.

When T reaches 2×10^3K, the thermal energy is sufficient to dissociate $H_2 \rightarrow 2H$. Thermal energy thus goes into dissociating H_2 rather than keeping the protostar up, and hydrostatic equilibrium is suspended. The star resumes collapse until enough thermal energy is released to support the now-atomised protostar, at which point hydrostatic equilibrium resumes and the protostar restabilises. T continues to rise due to accreting gas, eventually ionising the star.

When mostly ionised, $T \sim 5\times10^4$K, which is not only far too low for fusion, but also connotes high κ due to a decent remaining concentration of H^-. Radiative transport is slow and the protostar is fully convective and hence chemically homogeneous. Such stars are known as **T Tauri stars**, and are associated with sudden variability (as dust continues to fall onto them every now and then) and high activity, as their churning convection cells and often rapid rotation create strong magnetic fields, which generate jets that illuminate the ISM into emission nebulae called **Herbig-Haro objects**.

Convective stars of a given mass lie on the **Hayashi track**, the almost vertical boundary of the low-T (right) end of the HR diagram at about 4000K. The protostar continues to collapse (more slowly now) and as κ is so high the increase in core temperature doesn't increase the effective temperature much. The radius of the star decreases, but the surface T_{eff} increases only very slowly; the star thus moves almost downwards along the Hayashi track.

When T in the core is high enough, the $\kappa \propto T^{-3.5}$ regime is reached and the opacity starts to decrease; eventually the protostar develops a radiative core. Energy can then escape more easily and L increases; T continues to gradually increase on collapse. The star bounces up and left on the HR diagram.

Eventually, the core reaches temperatures high enough for fusion of ^{1}H into ^{4}He , and the protostar becomes a true star. It settles on the zero-age main sequence (ZAMS) at a position depending on its mass, and doesn't move far for potentially billions of years. The *initial mass function*, a mass frequency distribution, shows that small stars are much more common than large stars. The popular Salpeter IMF is $f(M) \propto M^{-2.35}$.

5.1.3 OB Associations and Strömgren Spheres

Groups of stars containing luminous O and B stars are called OB associations. With short lifetimes, these must have recently formed; indeed they are found near molecular clouds.

Their temperatures are high enough to ionise H atoms in the surrounding ISM, creating a spherical HII region called a Strömgren sphere. In this region, the H are continuously ionised and recombining, the latter of which makes the region glow in Hα. If Q is the ionisation rate (i.e. the number of ionising photons emitted per unit time) and $\alpha n_H n_e \approx \alpha n_H^2$ is the recombination rate per unit volume, then in steady-state:

$$Q = \alpha n_H^2 \frac{4}{3}\pi r_S^3 \qquad \Rightarrow \qquad r_S = \left(\frac{3Q}{4\pi\alpha n_H^2}\right)^{1/3}$$

where r_S is the Strömgren radius of the sphere.

5.2 Main Sequence (MS)

A star is on MS while ^{1}H is burning in its core.

5.2.1 $M \leq M_\odot$

As H is slowly converted to He in the core, $\mu = 4/(6X + Y + 2)$ steadily increases and core pressure $P = \rho k T/\mu m_H$ decreases. As a result the core is gradually squished inwards by the weight of the outer layers, and ρ (and hence T, and also T_{eff} and R) increase to compensate. As the energy released in PP fusion is proportional to $X^2 \rho T^4$ and X doesn't change much compared to ρ and T, the luminosity L of the star increases. Solar mass stars therefore move *up and left* over most of their lifetimes; indeed the Sun is brighter and hotter than it once was.

By around 10Gyr, the star has a small ^{4}He core, which is inert (not hot enough for burning) and hence isothermal ($L = 0 \Rightarrow \mathrm{d}T/\mathrm{d}r = 0$). An isothermal core can only support about 0.1 of the mass of the star (the **Schönberg-Chandrasekhar Limit**), unless there is some degeneracy pressure to help it up to about 0.13. ^{1}H fusion still takes place in a shell surrounding the core, but because this shell is essentially as hot as the core ($\mathrm{d}T/\mathrm{d}r = 0$), the luminosity produced continues to increase. However it doesn't all go into the total luminosity of the star, some of it goes into slowly expanding the star's envelope/atmosphere, and hence the rate of increase of L slows down, R increases and T_{eff} actually decreases. The star moves *up and right*, landing on the nearly horizontal **Subgiant Branch**, where almost all surplus energy production goes into inflating the star and reducing T_{eff} for about 2Gyr.

As the envelope expands, the ^{4}He core contracts apparently. The **Mirror Principle** is the empirical observation that for stars with a burning shell at intermediate radii, the core within does the opposite of what the envelope without does, as regards expanding and contracting. If the core contracts, GPE is released, T in the shell increases, energy production increases, and the envelope can expand.

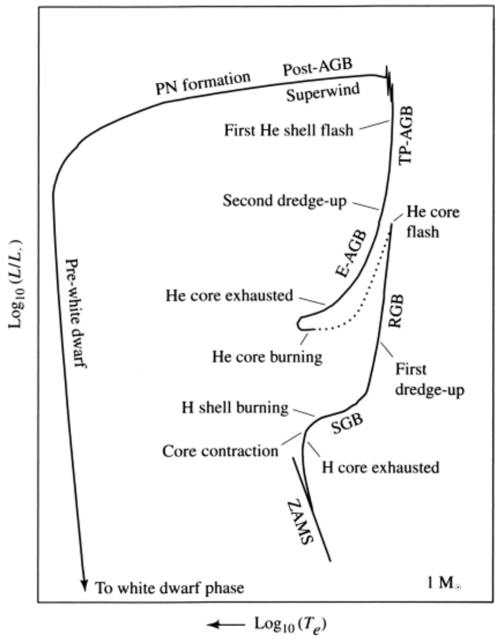

Figure 5 | HR Diagram Path of a $1M_\odot$ star.

5.2.2 $M > M_\odot$

In larger stars, the higher energy production rates cause the envelope expansion to occur throughout its MS life, and so these stars move up and right during their MS.

Large stars have convective cores, due to the strong T dependence of the CNO cycle, and the good mixing means that the core can be uniformly depleted of ^{1}H . As ^{1}H is fused, positrons are produced, which annihilate to reduce electron density and electron scattering, reduce κ, and enable radiative transport to take over in the outskirts of the once-fully convective core, which therefore shrinks. When ^{1}H runs out in the ever-shrinking core, the star collapses a bit until T in the shell is high enough to support it. This causes T_{eff} to lurch upwards but the energy production is basically being relocated so L doesn't change much – the star's path hooks to the left.

5.3 $M < 8M_\odot$ Post-MS

5.3.1 Red Giant Branch

^{1}H fusion continues in the shell of a subgiant star, dumping the ^{4}He on the core. Being almost degenerate at this point (see Section 4.3), this causes the core to contract, the shell temperature and density to increase, and hence the energy production rate to increase, faster ^{1}H fusion... the runaway process leads to accelerating increase in L. The shrinkage in the core leads to the envelope inflating, increasing R so quickly that T_{eff} in fact continues to decrease. These inflating stars are **Red Giants**, accelerating up the **Red Giant Branch** for about 0.5Gyr.

The position of the RGB depends slightly on metallicity: metal-rich stars have larger atmospheric κ, hence a thinner photosphere (recall the bottom of the photosphere is defined by $\tau = 2/3 \sim \int \kappa\rho\,\mathrm{d}s$), and hence lower T_{eff}.

The atmospheres are sometimes cool enough for grains such as silicates to condense. The large R, low g and high L are sufficient to blow away about $0.3M_\odot$ on the RGB.

For most of a star's life its nuclear products are stuck in the radiative zone, unable to mix with outer layers, but as the outer layers decrease in temperature the base of the convective zone eventually reaches enriched regions and **dredges up** the nuclear products. Atmospheric He abundance increases up the RGB.

5.3.2 Helium Flash

At the tip of the RGB, core P and T are sufficient (10^8K) to begin ^{4}He fusion by the triple-alpha process in the degenerate core. This proceeds differently to triple-alpha fusion in very big main sequence stars where the core is not degenerate, as P_d of a degenerate gas is independent of T. As such, the energy released from 3α fusion goes entirely to raising T, and because $\epsilon \propto T^{40}$, the fusion rate increases rapidly, leading to a runaway **Helium Flash** at the TRGB.

5.3.3 Horizontal Branch

It is difficult to tell what happens at this sudden juncture, but He flashes have never been observed; the energy released by the flash is probably absorbed to lift the degeneracy of the core (as well as raise the temperature of the rest of the star), at which point the core expands like a more regular gas; the envelope thus shrinks. Now, the star has a non-degenerate He core, in which ^{4}He fusion proceeds stably, surrounded by a ^{1}H burning shell. ^{4}He fusion is less efficient than ^{1}H fusion, and the ^{1}H fusion takes place in a shell which is less dense than it was on the SGB. L therefore decreases, but the energy released by the flash, combined with the decrease in R, increases T_{eff}, so the star moves *down and left* onto the **Horizontal Branch**.

For stars with initial $M \geq 2M_\odot$, their cores exceed the Schönberg-Chandrasekhar limit, collapse, and it gets hot enough for triple-alpha fusion before the He core ever becomes degenerate. They therefore skip the He flash stage, casually settling onto the HB.

The HB is similar to the SGB, but the He core is now burning rather than inert and so L is higher; an inert, isothermal, degenerate inner core consisting of ^{12}C and ^{16}O forms once the core ^{4}He fuel is exhausted. For $M < 8M_\odot$, the ^{12}C /^{16}O will never undergo fusion. Stars slowly move rightwards as their atmospheres expand and T_{eff} decreases. However, the HB lasts 0.01 as long as SGB, as there is less ^{4}He , ^{4}He fusion is quicker, and it releases less energy.

5.3.4 Asymptotic Giant Branch

Analogous to the SGB, there is now an inert, degenerate core and a ^{4}He burning shell around it. The ^{4}He burning dumps ^{12}C /^{16}O onto the core, causing it to contract, heat, increase burning speed, etc. The outer layers expand as L rises, reducing T_{eff} and forming the **Asymptotic Giant Branch**, analogous to the RGB. The reduced T_{eff} increases κ, increasing convection, and causing a second dredge-up, further increasing the ^{4}He content of the atmosphere, as well as ^{12}C ,^{16}O , and s-process elements such as ^{99}Tc created amidst ^{4}He fusion.

The fusion of ^{1}H and ^{4}He in two different shells is complicated; sometimes they alternate which is the most active. This leads to **Thermal Pulsation**; this region is the TP-AGB. As with the RGB, much mass loss occurs on this stage, and the pulsations compound this, leading to a **superwind** at 10-15km s^{-1}. The outer layers of the star are blown away, revealing inner layers progressively close to the core, with higher T_{eff} – the star moves *left*. The wind now becomes tenuous but much faster, crashing into the slower wind from earlier and creating **planetary nebulae**. Planetary nebulae are further lit up by light from the remainder of the star, which no longer does fusion and is held up by electron degeneracy – **White Dwarfs**.

5.4 White Dwarfs

White dwarfs consist mostly of their C/O core, but have helium and hydrogen blankets: due to high g, white dwarfs are highly stratified.

With no fusion, WDs are supported by electron[3] degeneracy pressure (see Section 4.3). If the mass of a WD increases (e.g. due to another star dumping matter onto it) above the Chandrasekhar limit of $1.44M_\odot$, electron degeneracy pressure is unable to support it and it collapses into a neutron star, releasing a Type Ia supernova.

As there is no fusion source remaining, WDs simply radiate away their thermal energy, continuously decreasing in T_{eff} and L, and hence moving *down and right* over a period at least as long as MS. Electrons can have very high momenta in the degenerate core, and conduction becomes such an efficient energy transport mechanism that the WD is almost isothermal, though it is covered by the non-degenerate H/He blankets, slowing the cooling process. WDs take about 10Gyr to cool down to 3kK, explaining why the WD luminosity distribution drops off suddenly at around $10^{-4}L_\odot$.

5.5 $M > 8M_\odot$ Post-MS

The differences at this boundary are that T becomes high enough for ^{12}C /^{16}O burning ($> 11M_\odot$ burn everything up to Fe), and the high ϵ causes strong winds and high mass loss even on the MS.

As the core repeatedly runs out of a fuel, collapses causing envelope expansion and $T_{\text{eff}} \downarrow$, increases its T, begins burning a new fuel, and increases $T_{\text{eff}} \uparrow$ again, the star's luminosity increases but not by much (there is a lot of envelope in the way!). The star thus simply zigzags bluewards and redwards on the HR diagram, leading to **blue supergiants** and **red supergiants**. More massive stars are able to fuse heavier elements and so zigzag more times.

5.5.1 Stellar Winds

Interactions with previously-blown off matter can cause wind-blown nebulae. These exhibit P Cygni line profiles for UV lines of highly ionised species like C IV. The profiles show maximum velocities[4] of up to 3×10^6m s^{-1}.

The winds are driven by line absorption and re-emission. The radial momentum transferred by a photon of frequency ν_a which is absorbed is $h\nu_a/c$. When the photon is re-emitted (potentially at a different frequency), the direction is random and so the average momentum transferred in the emission is $\mathbf{0}$.

The mass loss depends on the metallicity: higher metallicity winds can absorb photons at a more diverse range of wavelengths and hence more momentum gets transferred to them. Further, at lower T, κ is higher leading to greater mass loss from an A giant than an O giant of the same luminosity.

[3]Nuclei are still there of course, but as $P_d \propto m^{-8/3}$, they don't contribute much to the pressure

[4]The maximum velocity is calculated from a P Cygni profile based on the wavelength emitted by the gas flying directly towards us, λ_{min}, and the wavelength emitted by the gas flying tangentially (which gives the redshift of the star itself), λ_{star}. The redshifts of the approaching gas and of the star itself are calculated, and their velocities subtracted.

5.5.2 Wolf-Rayet and Luminous Blue Variable Stars

Stars with $25M_\odot < M < 40M_\odot$ are so luminous while burning ^{4}He that everything except the core is blown away. They are then surrounded by a tenuous region of gas which isn't quite part of the star, but is lit up by the bright core into an emission nebula.

Above $40M_\odot$, the mass is lost to stellar winds quicker than by nuclear fusion. These **luminous blue variable** stars are often found among OB associations (Section 5.1.3), and lose mass too quickly to ever zigzag and have RSGs phases.

5.6 Supernovae

5.6.1 Type Ia

When a CO white dwarf is in a binary system, it may accrete matter from its companion until it approaches the Chandrasekhar Limit. Before reaching this limit[5], the temperature is raised to ^{12}C fusion temperatures, and the WD effectively undergoes a "carbon flash", which causes the WD to violently tear apart (*deflagrate*), leaving nothing behind but a remnant. Ia supernovae are decent standard candles, producing light from the decay of ^{56}Ni and then ^{56}Co.

5.6.2 Types Ib, Ic, II (Core-Collapse)

Stars with $M > 11M_\odot$ develop an Fe core with a ^{28}Si-burning shell around it for about a day. The core is supported by electron degeneracy pressure; if any Fe does burn, it does not contribute to keeping the core up. When the Chandrasekhar mass is exceeded, the gravity becomes too strong and the core contracts. ^{28}Si-burning temperatures are high enough to photofission nuclei, leading to a soup of nucleons; as contraction continues and pressure rises, inverse β decay ($\mathrm{p}^+ + \mathrm{e}^- \longrightarrow \mathrm{n} + \nu_e$) occurs, releasing neutrons and neutrinos.[6]

The core continues to collapse until the density is high enough for the neutrons to become degenerate. Still having inward momentum, the neutrons fall a bit too far in before quickly bouncing back outwards. This generates a shockwave, which (after stalling slightly) shunts material outwards and generates the core-collapse supernova.

Just after the bounce, but before the shock fully gets going, matter accretes onto the core. If the ZAMS mass of the star is above a certain limit ($\sim 25M_\odot$ perhaps), even neutron degeneracy pressure is insufficient and the core collapses into a black hole.

The rapid neutron fluxes generated by photofission, inverse β decay, and the explosion itself, cause r-process nucleosynthesis within the inner layers of the star (up to about the O/Ne layers). The decay of ^{56}Ni (the most abundant r-process product) and ^{56}Co (the decay product) create the light curve of the core-collapse supernovae as well. Due to their different half-lives, the decay of different nuclei dominate the light curve at different times.

[5] The core mass reaches about 0.99 of the Chandrasekhar mass – the WD narrowly avoids collapsing into a neutron star

[6] A similar process occurs if $8M_\odot < M < 11M_\odot$, but with O/Ne cores supported by electron degeneracy pressure. The inverse β decay stuff occurs before the star has a chance to start fusing heavier elements.