Astrophysical Fluid Dynamics

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1 Fundamentals

A fluid is a medium with well-defined macroscopic properties (T, ρ etc.). A fluid element is a region of size ℓ with a large number of particles and all macroscopic variables roughly constant:

$$n^{-1/3} \ll \ell \ll \frac{q}{|\nabla q|}$$

where n is the fluid's number density and q is any macroscopic variable.

Collisional fluids have a mean free path $\ll \ell$, and hence bumble to equilibrium very quickly. Collisionless fluids have mean free paths $\gtrsim \ell$, so particles travel long distances and macroscopic properties depend on initial conditions.

The Eulerian description of a fluid assigns a field $q(\mathbf{r}, t)$ to every point in the fluid, relative to a fixed reference frame. The Lagrangian description instead looks at properties of each fluid element; because these move in space as well as time, we can define a Lagrangian derivative:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - q(\mathbf{r}, t)}{\delta t} = \lim \frac{\delta t\, \partial q/\partial t + \delta\mathbf{r} \cdot \nabla q}{\delta t} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q$$

There are three ways of lining a fluid, which all coincide if $\partial u/\partial t = \mathbf{0}$:

- **Streamlines**: Curves whose tangents are velocity fields $\mathbf{u}(\mathbf{r}, t)$. If $\mathbf{u}$ is known, then $\mathrm{d}y/\mathrm{d}x = \dot{y}/\dot{x}$ is solved by the streamlines
- **Streaklines**: a streakline of $\mathbf{r_0}$ is the locus of points that once passed through $\mathbf{r_0}$
- **Particle paths**: the paths taken by individual fluid elements over time

1.1 Conservation of mass

Consider an arbitrary fixed Eulerian volume V and its bounding surface ∂V. By conservation of mass and arbitrarity of V,

$$\frac{\partial}{\partial t} \int_V \rho \,\mathrm{d}V = -\int_{\partial V} \rho\mathbf{u} \cdot \mathrm{d}\mathbf{S} \Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\right] \mathrm{d}V = 0 \qquad \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0} \tag{$\mathfrak{M}$}$$

This can also be expressed in the Lagrangian form as $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla \cdot \mathbf{u} = 0$, so *incompressible* fluids, which by definition have $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, are also divergenceless $\nabla \cdot \mathbf{u} = 0$.

1.2 Conservation of Momentum

Consider the force on an arbitrary *Lagrangian* volume V in an arbitrary direction $\hat{\mathbf{n}}$:

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_{\partial V} p\hat{\mathbf{n}}\cdot \mathrm{d}\mathbf{S} + \int_V \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V = \hat{\mathbf{n}}\cdot\int_V [-\boldsymbol{\nabla}p + \rho\mathbf{g}]\,\mathrm{d}V$$

This will be equal to the Lagrangian rate of change of momentum in this direction:

$$\hat{\mathbf{n}}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = \hat{\mathbf{n}}\cdot\int_V [-\boldsymbol{\nabla}p + \rho\mathbf{g}]\,\mathrm{d}V \qquad \Rightarrow \qquad \frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = \int_V [-\boldsymbol{\nabla}p + \rho\mathbf{g}]\,\mathrm{d}V$$

because $\hat{\mathbf{n}}$ was arbitrary. Now we cannot take the derivative inside the integral as before, as it is Lagrangian and the volume may be changing. We therefore take the limit as $V \to \delta V$:

$$[-\boldsymbol{\nabla}p + \rho\mathbf{g}]\delta V = \frac{\mathrm{D}(\rho\mathbf{u}\delta V)}{\mathrm{D}t} = \mathbf{u}\underbrace{\frac{\mathrm{D}(\rho\delta V)}{\mathrm{D}t}}_{0} + \rho\delta V\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$$

$$\Rightarrow \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \boxed{\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = -\boldsymbol{\nabla}p + \rho\mathbf{g}} \qquad (\mathfrak{p})$$

Working our way back to Eulerian form:

$$\begin{aligned}
\rho\partial_t u_i + \rho u_j\partial_j u_i &= -\partial_i p + \rho g_i \\
\partial_t(\rho u_i) &= -\rho u_j\partial_j u_i + u_i\partial_t\rho - \partial_j(p\delta_{ij}) + \rho g_i = -\rho u_j\partial_j u_i - u_i\partial_j(\rho u_j) - \partial_j(p\delta_{ij}) + \rho g_i \\
&= -\partial_j\big(\underbrace{\rho u_i u_j}_{\text{Ram}} + \underbrace{p\delta_{ij}}_{\text{Thermal}}\big) + \rho g_i \equiv -\partial_j\sigma_{ij} + \rho g_i
\end{aligned}$$

The first term in the stress tensor, of the form $\rho\mathbf{u}\otimes\mathbf{u}$, arises from bulk motion that affects the flow; $p\delta_{ij}$ is just due to isotropic thermal pressure from microscopic motions.

1.3 Gravitation

The gravitational potential (Φ, no matter what the notes say) is given by $\mathbf{g} = -\boldsymbol{\nabla}\Phi$. The potential around a point mass is $\Phi = -GM/r$; for a system of point masses:

$$\Phi = -\sum_i \frac{GM_i}{|\mathbf{r}-\mathbf{r}_i|} \qquad \Rightarrow \qquad \mathbf{g} = -\boldsymbol{\nabla}\Phi = -\sum_i \frac{GM_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} \to -G\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V'$$

$$\Rightarrow \boldsymbol{\nabla}\cdot\mathbf{g} = -\boldsymbol{\nabla}^2\Phi = -G\int \rho(\mathbf{r}')4\pi\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' = -4\pi G\rho \qquad \Rightarrow \qquad \boxed{\boldsymbol{\nabla}^2\Phi = 4\pi G\rho} \qquad (\mathfrak{P})$$

$$\Rightarrow \int_V \boldsymbol{\nabla}\cdot\mathbf{g}\,\mathrm{d}V' = -4\pi G\int_V \rho\,\mathrm{d}V \qquad \Rightarrow \qquad \int_{\partial V}\mathbf{g}\cdot\mathrm{d}\mathbf{S} = -4\pi G M_V$$

which can be useful if the situation has a nice symmetry. For a spherical distribution radius r,

$$-4\pi r^2|\mathbf{g}| = -4\pi G\int_0^r 4\pi r'^2\rho(r')\,\mathrm{d}r' \qquad \Rightarrow \qquad |\mathbf{g}| = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2}\int_0^r 4\pi r'^2\rho(r')\,\mathrm{d}r'$$

$$\Rightarrow \Phi(r) = \int_\infty^r \frac{G}{s^2}\left(\int_0^s 4\pi r'^2 \rho(r')\,\mathrm{d}r'\right)\mathrm{d}s = -\frac{GM_r}{r} + 4\pi G\int_\infty^r s\rho(s)\,\mathrm{d}s$$

The gravitational potential energy of a pair of point masses is $-GM_1M_2/|\mathbf{r}_1 - \mathbf{r}_2|$. For a system of point masses, the total gravitational potential energy is

$$\Omega = -\frac{1}{2}\sum_i\sum_{j\neq i}\frac{GM_iM_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2}\sum_j M_j\Phi(\mathbf{r}_j) \to \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r})\,\mathrm{d}V$$

Consider the quantity[1] $I = \sum_i m_i r_i^2$. Half its second derivative is

$$\frac{1}{2}\ddot{I} = \sum_i m_i \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \cdot \dot{\mathbf{r}}_i) = \sum_i m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i + \sum_i m_i \dot{\mathbf{r}}_i^2 = \sum_i\sum_{j\neq i}\mathbf{r}_i \cdot \mathbf{F}_{ij} + 2T$$
$$= \sum_i\sum_{j<i}(\mathbf{r}_i - \mathbf{r}_j)\cdot\mathbf{F}_{ij} + 2T = \Omega + 2T$$

In a steady-state (AKA *relaxed* or *virialised*) system, $\dot{I} = 0$, so $\boxed{\Omega + 2T = 0}$ ($\mathfrak{V}$).

1.4 Energy

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0} \qquad \boxed{\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Phi} \qquad \boxed{\boldsymbol{\nabla}^2\Phi = 4\pi G\rho}$$

In order to close this system of equations in $\rho, \mathbf{u}, p$ and Φ, we need another equation relating p to the other three. This is fulfilled by the *equation of state* of the fluid, which relates p to ρ; because astrophysical fluids are quite dilute, this is just the ideal gas law:

$$\boxed{p = \frac{\rho k_B T}{\mu m_H}} = \frac{\mathcal{R}_*}{\mu}\rho k_B T \qquad \text{where} \qquad \mathcal{R}_* = \frac{k_B}{m_H} = 8314\ \mathrm{Jkg^{-1}K^{-1}}$$

where μ is the average particle mass – including the electrons; neutral hydrogen has $\mu = 1$; ionised hydrogen has $\mu = 0.5$. Unfortunately, a new variable, T has been introduced.

1.4.1 Barotropic Fluids $p = p(\rho)$

There are three examples of barotropic fluids, whose EoSs close the three equations above.

Isothermal. In an isothermal fluid, strong heating and cooling processes quickly flatten out any temperature inhomogeneities and T is constant throughout the fluid. Thus $p \propto \rho$.

Electron-degenerate. In electron-degenerate fluids such as those in white dwarves, quantum effects lead to $p \propto \rho^{5/3}$, or $p \propto \rho^{4/3}$ if the electrons become relativistic.

Adiabatic. Adiabatic gases move with no change in entropy or transfer of heat. Their barotropic relation is harder to derive. Per unit mass of fluid, $1\mathfrak{LT}$ with $\delta Q = 0$ gives

$$0 = \mathrm{d}e + p_M\,\mathrm{d}V \qquad p_M = \frac{k_B T}{V\mu m_H} \text{ as } \rho_M = \frac{1}{V} \qquad \frac{\mathrm{d}e}{\mathrm{d}T} = C_{V,M} \equiv f\frac{k_B}{2\mu m_H}$$

[1]This is a bit like the system's moment of inertia, but the r_i in the formula for moment of inertia are the distances to a particular axis; in the quantity I here we are looking at the distances to some origin.

where e is the internal energy per unit mass (dependent only on temperature) and $C_{V,M}$ is the constant volume heat capacity per mass, dependent on f, the number of degrees of freedom (just 3 for a monatomic, up to 7 for a diatomic). We can also write:

$$e = C_{V,M}T = f\frac{k_B}{2\mu m_H}T = \frac{f}{2}\frac{p}{\rho} = \frac{1}{\gamma-1}\frac{p}{\rho} \qquad \Rightarrow \qquad \boxed{e = \frac{1}{\gamma-1}\frac{p}{\rho}}$$

in accordance with equipartition. Substituting into $\mathbf{1}\mathfrak{LT}$,

$$0 = f\frac{k_B}{2\mu m_H}\,\mathrm{d}T + \frac{k_B T}{V\mu m_H}\,\mathrm{d}V \Rightarrow 0 = \frac{f}{2}\frac{\mathrm{d}T}{T} + \frac{\mathrm{d}V}{V} \Rightarrow V \propto T^{-f/2} \propto (pV)^{-f/2} \Rightarrow p \propto V^{-\frac{f+2}{f}} \propto \rho^{\frac{f+2}{f}}$$

The exponent $\frac{f+2}{f}$ is equal to γ, the specific heat ratio, so for an adiabatic fluid $p \propto \rho^\gamma$:

$$\delta Q = C_{V,M}\,\mathrm{d}T + \mathrm{d}(p_M V) - V\,\mathrm{d}p = \left(C_{V,M} + \frac{k_B}{\mu m_H}\right)\mathrm{d}T - V\,\mathrm{d}p$$

$$\Rightarrow C_{p,M} = C_{V,M} + \frac{k_B}{\mu m_H} = C_{V,M}\left(1 + \frac{2}{f}\right) = C_{V,M}\frac{f+2}{f} \qquad \Rightarrow \qquad \gamma \equiv \frac{C_p}{C_V} = \frac{f+2}{f}$$

1.4.2 General Fluids

Generally, $p \neq p(\rho)$. For a fluid element, per unit mass;

$$\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\delta W}{\delta t} + \frac{\delta Q}{\delta t} \qquad \text{where} \qquad \frac{\delta W}{\delta t} = -p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} \qquad \text{and} \qquad \frac{\delta Q}{\delta t} = -\dot{Q}_{\text{cool}}$$

$$\Rightarrow \frac{\mathrm{D}e}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\text{cool}} = -\frac{p}{\rho}\boldsymbol{\nabla}\cdot\mathbf{u} - \dot{Q}_{\text{cool}}$$

Now e is only the *internal* energy per unit mass; the fluid element is generally also moving and in a potential. The total energy per unit *volume* in a time-independent potential is:

$$\boxed{E = \rho\left(e + \frac{1}{2}u^2 + \Phi\right)} \qquad \Rightarrow \qquad \frac{\mathrm{D}E}{\mathrm{D}t} = \overbrace{\frac{\mathrm{D}\rho}{\mathrm{D}t}}^{\mathfrak{M}}\frac{E}{\rho} + \rho\left(\overbrace{\frac{\mathrm{D}e}{\mathrm{D}t}}^{\text{above}} + \mathbf{u}\cdot\overbrace{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}}^{\mathfrak{p}} + \frac{\mathrm{D}\Phi}{\mathrm{D}t}\right)$$

$$\Rightarrow \qquad \boxed{\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = -\rho\dot{Q}_{\text{cool}}} \qquad (\mathfrak{E})$$

If there are no cooling processes, we have a simple Eulerian equation relating the energy change to the *enthalpy* $(E+p)$ flux. If $\dot{Q}_{\text{cool}} = 0$, then $\mathfrak{E}$ relates $\rho, p, \mathbf{u}, \Phi$, hence closing $\mathfrak{M}, \mathfrak{p}, \mathfrak{P}$ and we have a fully determined set of equations.

However, $\dot{Q}_{\text{cool}} \neq 0$ generally. Here are some possibilities:

- **Radiative Cooling.** Light is radiated away by the fluid. This may occur from recombination, collisional excitation, and electron Bremsstrahlung, each of which involve two-body interactions and so the overall rate of cooling is $\propto \rho^2$; thus the rate of cooling per unit mass, $\dot{Q}_{\text{cool}} \propto \rho$. Thus $\dot{Q}_{\text{cool,rad}} = \rho f(T)$.

- **Cosmic Ray Heating.** External relativistic particles enter the fluid and create a cascade of particles as it blitzes through, heating the fluid. The heating rate is simply proportional to the density, so $\dot{Q}_{\text{cool},CR} \propto \rho^0 = -H$; negative because this *heats* the fluid.

Combining these two, $\dot{Q}_{\text{cool}} = \rho f(T) - H$. Energy can move through a fluid by various transport processes. These include thermal conduction (flux proportional to $-\boldsymbol{\nabla}T$), radiative transport (flux in optically thick media proportional to $-\boldsymbol{\nabla}E_{\text{rad}}$) and convection.

2 Hydrostatic Equilibrium

Hydrostatic equilibrium is defined by $\mathbf{u} = \mathbf{0}$, $\partial/\partial t = 0$. Hence $\mathfrak{M}$ is satisfied and $\mathfrak{p}$ becomes:

$$\boldsymbol{\nabla} p = \rho \mathbf{g} = -\rho \boldsymbol{\nabla} \Phi$$

If the fluid is barotropic ($p = p(\rho)$) then this relates ρ and Φ. In some fluids Φ will be externally imposed (e.g. the atmosphere in the Earth's gravitational field). In others, the fluid will be *self-gravitating*, and $\boldsymbol{\nabla}^2 \Phi = 4\pi G \rho$, a second equation for ρ and Φ.

2.1 Polytropic Stars

Stars are spherically symmetric, self-gravitating, and can be approximated as in hydrostatic equilibrium and barotropic. Stars whose barotropic relation is a monomial of order > 1 are *polytropic*, and can be written $p = K\rho^{1+1/n}$. The equation for hydrostatic equilibrium is then:

$$\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}r} &= -\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(K\rho^{1+1/n}\right) = -(n+1)K\left[\frac{1}{n+1}\left(\rho^{\frac{n+1}{n}}\right)^{\left(\frac{1}{n+1}-1\right)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{\frac{n+1}{n}}\right)\right] \\
&= -(n+1)K\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\rho^{\frac{n+1}{n}}\right)^{\frac{1}{n+1}}\right] = -(n+1)K\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{1/n}\right) \\
\Rightarrow \Phi &= -(n+1)K\rho^{1/n} + \text{const.}
\end{aligned}$$

At the surface, $\rho = 0$ and we label $\Phi = \Phi_T$ here. At the core, $\rho = \rho_c$ and $\Phi = \Phi_c$, so:

$$\rho = \left(\frac{\Phi_T - \Phi}{(n+1)K}\right)^n \qquad \rho_c = \left(\frac{\Phi_T - \Phi_c}{(n+1)K}\right)^n \qquad \Rightarrow \qquad \rho = \rho_c\left(\frac{\Phi_T - \Phi}{\Phi_T - \Phi_c}\right)^n$$

We also have $\mathfrak{P}$ to relate ρ and Φ; substituting for Φ:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho_c\left(\frac{\Phi_T - \Phi}{\Phi_T - \Phi_c}\right)^n$$

Substituting θ below, where $\theta_c = 1$, $\theta_T = 0$, and $\mathrm{d}\theta/\mathrm{d}r\,\big|_c \propto g_c = 0$ at the centre:

$$\theta \equiv \frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \qquad \Rightarrow \qquad -(\Phi_T - \Phi_c)\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 4\pi G\rho_c\theta^n$$

and then rescaling the radial coordinate; we will have $\theta(\xi = 0) = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi\,\big|_0 = 0$

$$\xi = r\sqrt{\frac{4\pi G\rho_c}{\Phi_T - \Phi_c}} \qquad \Rightarrow \qquad \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \qquad \text{(Lane-Emden)}$$

Solutions of the Lane-Emden equation are known as polytropes, and are analytic for $n = 0, 1, 5$. The first zero of θ_n, at ξ_0, gives the value of ξ corresponding to the surface of the star.

2.1.1 Scaling

Consider a family of stars with the same polytropic index n (e.g. white dwarfs with $p = K\rho^{5/3} \Rightarrow n = 3/2$ where K is a bunch of quantum constants). All the stars in this family will have the same $\theta(\xi)$, but their $\xi(r)$ and $\theta(\Phi)$ will be rescaled according to Φ_c, or more physically, ρ_c. From the relation above between ρ_c and Φ_c, we have

$$\rho_c = \left(\frac{\Phi_T - \Phi_c}{(n+1)K}\right)^n \qquad \Rightarrow \qquad \Phi_T - \Phi_c \propto K\rho_c^{1/n} \qquad \Rightarrow \qquad \xi \propto K^{-1/2}\rho_c^{\frac{1}{2}(1-1/n)}r$$

$$M = \int_0^R 4\pi r^2 \rho(r)\,\mathrm{d}r \propto K^{3/2}\rho_c^{-\frac{3}{2}(1-1/n)}\int_0^{\xi_0}\xi^2\rho_c\theta^n\mathrm{d}\xi = K^{3/2}\rho_c^{\frac{1}{2}(3/n-1)}\int_0^{\xi_0}\theta^n\xi^2\mathrm{d}\xi \propto K^{3/2}\rho_c^{\frac{1}{2}(3/n-1)}$$

whereas $R \propto \xi_0 K^{1/2}\rho_c^{\frac{1}{2}(1/n-1)} \propto K^{1/2}\rho_c^{\frac{1}{2}(1/n-1)}$.

Treating K as a constant, we can eliminate ρ_c between M and R, giving $M \propto R^{\frac{3/n-1}{1/n-1}} = R^{\frac{3-n}{1-n}}$, which for white dwarfs with $n = 3/2$ gives $M \propto R^{-3}$, which works quite well.

However, for MS stars this is not very accurate, even for those with $n = 3/2$. This is because most MS stars have different values of K, but due to the universality of nuclear processes they do have the same T_c, enabling the following proportionality:

$$K = p_c\rho_c^{-1-1/n} = \frac{\rho_c k_B T_c}{\mu m_H}\rho_c^{-1-1/n} \propto \rho_c^{-1/n}$$

Hence $M \propto \rho_c^{-1/2}$ and $R \propto \rho_c^{-1/2}$, hence $M \propto R$, which is more accurate.

3 Sound & Shocks

3.1 Sound Waves

We do a first-order perturbation analysis of

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad\qquad \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Phi$$

Suppose the equilibrium situation is to have $\rho(\mathbf{r}) = \rho_0(\mathbf{r})$, $p(\mathbf{r}) = p_0(\mathbf{r})$ and $\mathbf{u}(\mathbf{r}) = \mathbf{0}$. Consider a *Lagrangian* perturbation about this equilibrium, such that the fluid element currently at location $\mathbf{r}$ has $\rho = \rho_0(\mathbf{r}) + \Delta\rho(\mathbf{r})$ and $p = p_0(\mathbf{r}) + \Delta p(\mathbf{r})$ and $\mathbf{u} = \Delta\mathbf{u}(\mathbf{r})$. The equations of fluid dynamics are in terms of Eulerian derivatives, in which picture the disturbances are instead for example $\rho_0 + \delta_e\rho$; the relationships between the Eulerian perturbation and the Lagrangian perturbations are, to first order,

$$\Delta\rho = \delta_e\rho + \Delta\mathbf{r}\cdot\boldsymbol{\nabla}\rho \approx \delta_e\rho + \Delta\mathbf{r}\cdot\boldsymbol{\nabla}\rho_0, \qquad \Delta p = \delta_e p + \Delta\mathbf{r}\cdot\boldsymbol{\nabla}p_0, \qquad \Delta\mathbf{u} = \delta_e\mathbf{u}$$

For a uniform medium $\boldsymbol{\nabla}\rho_0 = \boldsymbol{\nabla}p_0 = \mathbf{0}$ and in that case the distinction between Eulerian and Lagrangian pictures vanishes to first order.

$\mathfrak{M}$ becomes in general

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}(\rho_0 + \delta_e\rho) + \boldsymbol{\nabla}\cdot((\rho_0 + \delta_e\rho)\mathbf{u}) \\
&= \frac{\partial}{\partial t}(\rho_0 + \Delta\rho - \Delta\mathbf{r}\cdot\boldsymbol{\nabla}\rho_0) + \boldsymbol{\nabla}\cdot((\rho_0 + \Delta\rho - \Delta\mathbf{r}\cdot\boldsymbol{\nabla}p_0)\Delta\mathbf{u}) \\
&= \frac{\partial\Delta\rho}{\partial t} - \Delta\mathbf{u}\cdot\boldsymbol{\nabla}\rho_0 + \boldsymbol{\nabla}\cdot(\rho_0\Delta\mathbf{u}) = \frac{\partial\Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot\Delta\mathbf{u} \qquad \Rightarrow \qquad \frac{\partial^2\Delta\rho}{\partial t^2} = -\rho_0\boldsymbol{\nabla}\cdot\frac{\partial\Delta\mathbf{u}}{\partial t}
\end{aligned}$$

and $\mathfrak{p}$ becomes, neglecting gravity for some reason,

$$\frac{\partial \Delta \mathbf{u}}{\partial t} + (\Delta \mathbf{u} \cdot \boldsymbol{\nabla}) \Delta \mathbf{u} = -\frac{1}{\rho_0 + \Delta \rho - \Delta \mathbf{r} \cdot \boldsymbol{\nabla} \rho_0} \boldsymbol{\nabla}(p_0 + \Delta p - \Delta \mathbf{r} \cdot \boldsymbol{\nabla} p_0)$$

Before continuing the derivation, it is useful to consider the quantity $\boldsymbol{\nabla}(\Delta \mathbf{r} \cdot \boldsymbol{\nabla} p_0)$. This quantity's x-component is, by definition:

$$\frac{\partial(\Delta \mathbf{r} \cdot \boldsymbol{\nabla} p_0)}{\partial x} \equiv \lim_{\delta x \to 0} \frac{\delta x [\boldsymbol{\nabla} p_0]_x + \overbrace{\delta y}^{0} [\boldsymbol{\nabla} p_0]_y + \overbrace{\delta z}^{0} [\boldsymbol{\nabla} p_0]_z}{\delta x} = [\boldsymbol{\nabla} p_0]_x$$

from which we can conclude that $\boldsymbol{\nabla}(p_0 - \Delta \mathbf{r} \cdot \boldsymbol{\nabla} p_0) = \mathbf{0}$. To first order we then have

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \boldsymbol{\nabla}(\Delta p) = -\frac{1}{\rho_0} \left. \frac{\mathrm{d}p}{\mathrm{d}\rho} \right|_{\rho_0} \boldsymbol{\nabla}(\Delta \rho) = -\frac{c^2}{\rho_0} \boldsymbol{\nabla}(\Delta \rho)$$

where we assume a barotropic equation of state $p = p(\rho)$ and that c is independent of $\mathbf{r}$. The form of c depends on whether the wave passes through isothermally or adiabatically. In the isothermal case, $p = \rho k_B T / \mu m_H$ and so $c = \sqrt{k_B T / \mu m_H} = \sqrt{p/\rho}$. For the (more common) adiabatic case, $p = \rho^\gamma \Rightarrow c = \sqrt{\gamma p / \rho} = \sqrt{\gamma k_B T / \mu m_H}$, greater by a factor $\sqrt{\gamma}$.

Substituting the above into $\mathfrak{M}_S$, we have

$$\frac{\partial^2 \Delta \rho}{\partial t^2} = c^2 \rho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{\nabla}(\Delta \rho) \right) \tag{$*$}$$

3.1.1 Uniform Media

In this case, $(*)$ simplifies to $\partial_t^2(\Delta \rho) = c^2 \boldsymbol{\nabla}^2 (\Delta \rho)$, which is just the wave equation.

3.1.2 Stratified Media

In this case there is no simplification, but we can rewrite $(*)$ as

$$\frac{\partial^2 \Delta \rho}{\partial t^2} = c^2 \boldsymbol{\nabla}^2 (\Delta \rho) - c^2 \frac{\boldsymbol{\nabla} \rho_0}{\rho_0} \cdot \boldsymbol{\nabla}(\Delta \rho)$$

which has a damping term. On substituting a disturbance of the form $\Delta \rho \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ and the form of ρ_0, one can find $\Delta \mathbf{u}$ from $\mathfrak{M}_S$, and also the dispersion relation. Often there is an acoustic cutoff frequency below which $\mathbf{k}$ becomes non-real. There may also be regions where $\Delta \rho / \rho_0$ ceases to be small as assumed, and the first-order analysis fails. A shock forms.

3.2 Shocks

Shocks are surfaces where the properties of a fluid are discontinuous on either side, as a result of being forced above its sound speed. To deduce the discontinuity conditions (*Rankine-Hugoniot* relations, $\mathfrak{RH}$1-3) consider Figure 1, in the frame of the shock. We integrate $\mathfrak{M}$, $\mathfrak{p}$, and $\mathfrak{E}$ across the (very thin) boundary to obtain the discontinuities, with the assumptions that mass, momentum, and energy do not pile up at the shock front.

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} = 0 \qquad \Rightarrow \qquad \boxed{\left. \rho u_x \right|_1 = \left. \rho u_x \right|_2} \tag{$\mathfrak{RH}1$}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p \qquad \Rightarrow \qquad \boxed{\rho u_x^2+p\Big|_1=\rho u_x^2+p\Big|_2}\,; \qquad u_y\Big|_1=u_y\Big|_2 \text{ etc.} \qquad (\mathfrak{RH}2)$$

The discontinuity condition for $\mathfrak{E}$ depends on whether the shock is isothermal or adiabatic.

3.2.1 Isothermal Shocks

$T\big|_1=T\big|_2$. As $c=\sqrt{k_BT/\mu m_H}$, this means the sound speeds are also equal in this case. We also have $p=c^2\rho$. Writing $u_x=Mc$ where M is the *Mach number*, dividing $\mathfrak{RH}2$ by ρu_x gives

$$M_1c+\frac{c^2}{M_1c}=M_2c+\frac{c^2}{M_2c} \qquad \Rightarrow \qquad M_1M_2=1$$

Because $M_1>1$ (as $u_1>c$ to create the shock), the fluid slows from supersonic to subsonic, the bulk KE being dissipated as heat. $\mathfrak{RH}1$ then gives

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{u_1}{c^2/u_1}=\left(\frac{u_1}{c}\right)^2=M_1^2$$

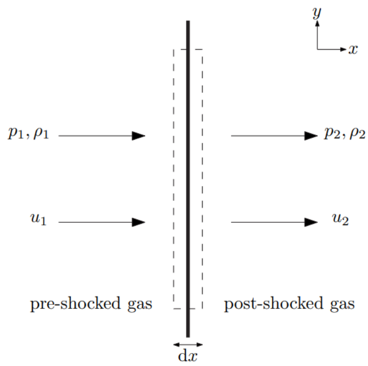

Figure 1 | A Shock Front

3.2.2 Adiabatic Shocks

Instead, we have $\dot{Q}_{\text{cool}}=0$, and so $\mathfrak{E}$ becomes:

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}[(E+p)u_x]=0 \qquad \Rightarrow \qquad \left[\rho\left(\frac{1}{2}u^2+e\right)+p\right]u_x\Big|_1=\left[\rho\left(\frac{1}{2}u^2+e\right)+p\right]u_x\Big|_2$$

$$\Rightarrow \qquad \boxed{\frac{1}{2}u^2+e+\frac{p}{\rho}\Big|_1=\frac{1}{2}u^2+e+\frac{p}{\rho}\Big|_2} \qquad (\mathfrak{RH}3)$$

where we have used $\mathfrak{RH}1$ to divide through by ρu_x on both sides. Using $e=\frac{1}{\gamma-1}\frac{p}{\rho}$ and $c^2=\frac{\gamma p}{\rho}$:

$$e+\frac{p}{\rho}=\left(\frac{1}{\gamma-1}+1\right)\frac{c^2}{\gamma}=\frac{c^2}{\gamma-1} \qquad \Rightarrow \qquad \boxed{\frac{u^2}{2}+\frac{c^2}{\gamma-1}\Big|_1=\frac{u^2}{2}+\frac{c^2}{\gamma-1}\Big|_2}$$

which is more commonly used. We note that $c^2=\frac{\gamma p}{\rho}=\frac{\gamma k_BT}{\mu m_H}$ is not the same on both sides of an adiabatic shock, as $T_1\neq T_2$.

Using $\mathfrak{RH}1-3$, an annoying amount of algebra[2] eventually gives

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{(\gamma-1)p_1+(\gamma+1)p_2}{(\gamma+1)p_1+(\gamma-1)p_2}$$

For a "strong" shock, $p_2\gg p_1$, and so $\frac{\rho_2}{\rho_1}\to\frac{\gamma+1}{\gamma-1}$; this could also be derived from the start by assuming $p_1\approx 0$. There is hence an upper limit to the density and velocity discontinuities across a shock front (for a monatomic gas, this is a factor of 4).

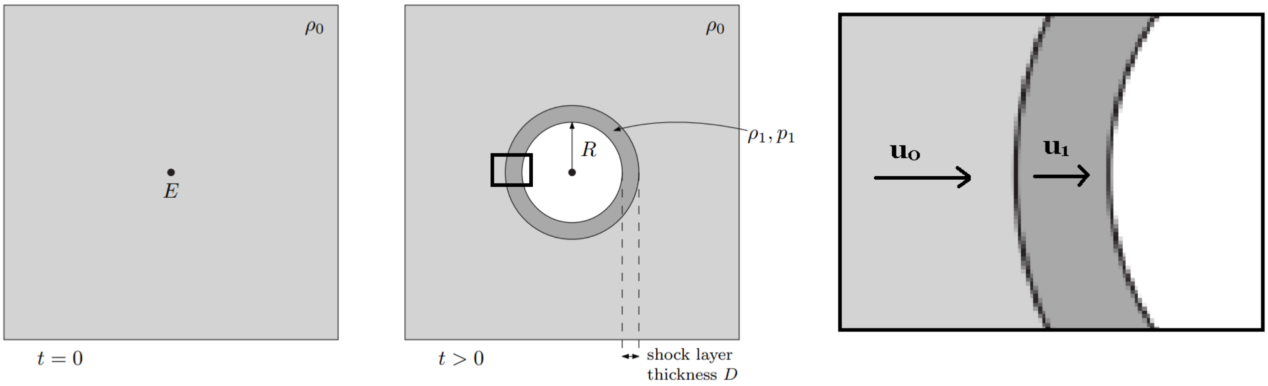

Figure 2 | A Supernova Blast Wave. In the high-quality final frame the velocities are given relative to the shock front.

3.3 Supernova Blast Waves

When a supernova releases an energy E in a point-like explosion, material is swept outwards at supersonic speeds by a strong adiabatic shock. The shocked material has a thickness $D \ll R$ and contains all the material within a radius R at a density $\rho_1 = \frac{\gamma+1}{\gamma-1}\rho_0$, where ρ_0 is the density of the ISM. D must hence be given by

$$\frac{4}{3}\pi R^3 \rho_0 = 4\pi R^2 D \rho_1 \qquad \Rightarrow \qquad D = \frac{\rho_0}{\rho_1}\frac{R}{3} = \frac{1}{3}\frac{\gamma-1}{\gamma+1}R$$

In the frame of the shock front, the velocity of the shock is u_1 and that of the ISM is u_0, as shown in Figure 2. The velocity of the frame itself (relative to the ISM, or the point of the explosion) is the negative of the velocity of the ISM, $-u_0$. $\mathfrak{RH}1$ gives $u_1 = \frac{\gamma-1}{\gamma+1}u_0$, so the velocity of the shocked gas in the rest frame of the explosion is:

$$U = -u_1 + u_0 = \frac{2}{\gamma+1}u_0$$

Consider the pressure in the shocked region. We suppose that the ISM has negligible pressure; $\mathfrak{RH}2$ then gives

$$p_1 \approx \rho_0 u_0^2 - \rho_1 u_1^2 = \rho_0 u_0^2 \left[1 - \frac{\rho_1}{\rho_0}\frac{u_1^2}{u_0^2}\right] = \rho_0 u_0^2 \left[1 - \frac{\gamma-1}{\gamma+1}\right] = \frac{2}{\gamma+1}\rho_0 u_0^2$$

Consider the momentum transported by the shocked region, which we suppose must be sourced by the pressure on the *inside* of the shell, p_{in}. We assume that this inner pressure scales with the pressure within the shell, so that $p_{\text{in}} = \alpha p_1$ (it can be shown that conservation of energy requires $\alpha = 1/2$). The force on the inside of the shell is $4\pi R^2 p_{\text{in}}$; the integrated magnitude of momentum carried out is $\frac{4}{3}\pi R^3 \rho_0 U$, thus

$$\alpha \underbrace{\frac{2}{\gamma+1}\rho_0 u_0^2}_{p_1} 4\pi R^2 = \frac{4}{3}\pi\rho_0 \frac{2}{\gamma+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^3 u_0\right) \qquad \Rightarrow \qquad \alpha u_0^2 R^2 = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^3 u_0\right)$$

Now the frame is moving at a speed u_0, which is also the rate of increase of the shell $\dot{R}$, so $\alpha \dot{R}^2 R^2 = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^3 \dot{R}\right)$. The solutions of this are of the form $R \propto t^b$, where $b = (4 - 3\alpha)^{-1}$. By dimensional analysis, the energy released must be $E \propto \rho_0 u_0^2 R^3 \propto t^{5b-2}$, but as energy is conserved we require $E \propto t^0$. Thus $b = 2/5$ and $\alpha = 1/2$.

[2]Write $\alpha = \rho_1 u_1 = \rho_2 u_2$, substitute $u_i = \alpha/\rho_i$ into $\mathfrak{RH}2$ and 3, and then solve for ρ_2/ρ_1.

4 Transonic Flows

4.1 Bernoulli's Constant and Vorticity

Assuming a barotropic flow, $\dfrac{\boldsymbol{\nabla} p}{\rho} = \boldsymbol{\nabla} p \left(\dfrac{\mathrm{d}}{\mathrm{d}p} \displaystyle\int \dfrac{\mathrm{d}p}{\rho} \right) = \boldsymbol{\nabla} \left(\displaystyle\int \dfrac{\mathrm{d}p}{\rho} \right)$

Using $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u} \times \boldsymbol{\nabla} \times \mathbf{u}$ and assuming steady state $\partial/\partial t = 0$, $\mathfrak{p}$ is

$$\boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u} \times \boldsymbol{\nabla} \times \mathbf{u} = -\frac{\boldsymbol{\nabla} p}{\rho} - \boldsymbol{\nabla}\Phi \qquad \Rightarrow \qquad \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2 + \int \frac{\mathrm{d}p}{\rho} + \Phi\right) = \mathbf{u} \times \mathbf{w}$$

where the vorticity $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$. The quantity in brackets is Bernoulli's constant:

$$\boxed{H = \frac{1}{2}\mathbf{u}^2 + \int \frac{\mathrm{d}p}{\rho} + \Phi}$$

If $\mathbf{w} = \mathbf{0}$ everywhere, then $\boldsymbol{\nabla} H = \mathbf{0}$ and H takes the same value everywhere. If $\mathbf{w} \neq \mathbf{0}$, we can instead take $\mathbf{u}\cdot$ both sides, giving $\mathbf{u} \cdot \boldsymbol{\nabla} H = 0$, so H is only conserved along a streamline; different streamlines can have different H.

Relaxing the steady state requirement and taking the curl, gradients disappear and we instead obtain Helmholtz's Equation $\partial \mathbf{w}/\partial t = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$. Hence if $\mathbf{w} = \mathbf{0}$ everywhere initially then it will remain $\mathbf{0}$ thereafter. Also, Kelvin's circulation theorem gives that the flux of vorticity through a surface moving with the fluid is 0:

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \frac{\mathrm{D}}{\mathrm{D}t} \oint_{\partial S} \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} &= \oint_{\partial S} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{\ell} + \oint_{\partial S} \mathbf{u} \cdot \underbrace{\frac{\mathrm{D}\mathrm{d}\boldsymbol{\ell}}{\mathrm{D}t}}_{(\mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{\nabla})\mathbf{u}} \\
&= \int_S \boldsymbol{\nabla} \times \left(-\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Phi\right) \cdot \mathrm{d}\boldsymbol{\ell} + \oint_{\partial S} u_i \partial_j(u_i)\, \mathrm{d}\ell_j \\
&= -\int_S \frac{1}{\rho^2} \underbrace{(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p)}_{\mathbf{0}\because\|\because\text{barotropic}} \cdot \mathrm{d}\mathbf{S} + \underbrace{\oint_{\partial S} \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2\right) \cdot \mathrm{d}\boldsymbol{\ell}}_{0} = 0
\end{aligned}$$

An alternate proof is given in the notes but it doesn't sit right with me somehow.

The integral $\int \mathrm{d}p/\rho$ depends on the barotropic EoS. For example:

$$\begin{aligned}
&\text{Isothermal case:} && \int \frac{\mathrm{d}p}{\rho} = \frac{k_B T}{\mu m_H} \ln \rho = c^2 \ln \rho \\
&\text{Polytropic case:} && \int \frac{\mathrm{d}p}{\rho} = K\left(1 + \frac{1}{n}\right) \int \rho^{-1+1/n} = K\left(1 + \frac{1}{n}\right) n\rho^{1/n} = nc^2
\end{aligned}$$

where we have used $c^2 = \mathrm{d}p/\mathrm{d}\rho = K(1 + \frac{1}{n})\rho^{1/n}$ (not a constant) for the polytropic case.

4.2 De Laval Nozzle

Consider a compressible, barotropic, irrotational fluid moving through a z-oriented nozzle of slowly-varying area $A(z)$. The momentum equation gives

$$\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p \qquad \Rightarrow \qquad \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) = -c^2 \frac{\boldsymbol{\nabla}\rho}{\rho} \qquad \Rightarrow \qquad u^2 \boldsymbol{\nabla} \ln u = -c^2 \boldsymbol{\nabla} \ln \rho$$

Continuity is expressed simply by $\rho u A$ being a constant, $\dot{M}$. Taking $c^2\nabla \ln$ of continuity,

$$c^2\nabla \ln \rho + c^2\nabla \ln u + c^2\nabla \ln A = 0 \qquad \Rightarrow \qquad (u^2 - c^2)\nabla \ln u = c^2\nabla \ln A$$

Hence at the points in the nozzle where A is at a minimum u is either stationary or passing through $u = c$: undergoing a *transonic transition*. Such points are called *sonic points*.

4.2.1 Isothermal: $p = \rho k_B T/\mu m_H$

Equating Bernoulli's constant at a general point to its value at the sonic point:

$$\frac{1}{2}u^2 + c^2 \ln \rho = \frac{1}{2}c^2 + c^2 \ln \rho_m \qquad \Rightarrow \qquad u^2 = c^2\left[1 + 2\ln\left(\frac{\rho_m}{\rho}\right)\right] = c^2\left[1 + 2\ln\left(\frac{uA}{cA_m}\right)\right]$$

where $_m$ refers to quantities at the extremum of the nozzle and we have used conservation of mass ($\rho u A = \rho_m c A_m$). Anyway this is an implicit equation for $u(z)$ in terms of $A(z)$; we can then obtain $\rho(z)$ through $\dot{M} = \rho u A$.

4.2.2 Polytropic: $p = K\rho^{1+1/n}$

The speed of sound is no longer constant, and Bernoulli's constant is also different:

$$H = \frac{1}{2}u^2 + nc^2$$

Mass conservation then gives an ugly expression for ρ_m:

$$\rho u A = \dot{M} = \rho_m c_m A_m = \rho_m\left(1 + \frac{1}{n}\right)^{1/2} K^{1/2}\rho_m^{1/2n} A_m$$

$$\Rightarrow \dot{M}^2 = \left(1 + \frac{1}{n}\right)K\rho_m^{(2n+1)/n} A_m^2 \qquad \Rightarrow \qquad \rho_m = \left[\left(\frac{\dot{M}}{A_m}\right)^2 \frac{n}{(n+1)K}\right]^{n/(2n+1)}$$

Then, using the Bernoulli constant and seeking ρ this time cause it's easier:

$$\frac{1}{2}u^2 + (n+1)K\rho^{1/n} = \left(n + \frac{1}{2}\right)c_m^2$$

$$\frac{1}{2}\left(\frac{\dot{M}}{\rho A}\right)^2 + (n+1)K\rho^{1/n} = \left(\frac{1}{2} + n\right)\left(1 + \frac{1}{n}\right)K\left[\left(\frac{\dot{M}}{A_m}\right)^2 \frac{n}{(n+1)K}\right]^{1/(2n+1)}$$

This time giving an implicit expression for $\rho(z)$ in terms of $A(z)$; we can find $u(z)$ using continuity as above. Note that the RHS is just a constant.

4.3 Spherical Flows

Consider steady-state spherically symmetric accretion of barotropic gas onto a point mass, such that $\lim_{r\to\infty} u = 0$. Continuity gives that $\rho u(4\pi r^2)$ is a constant, $-\dot{M}$, and so

$$\frac{\mathrm{d}\ln \rho}{\mathrm{d}r} = -\frac{\mathrm{d}\ln u}{\mathrm{d}r} - \frac{2}{r}$$

The momentum equation with $\dot{u} = 0$ and this time including gravity gives

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} \qquad \Rightarrow \qquad u^2\frac{\mathrm{d}\ln u}{\mathrm{d}r} = -c^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}$$

$$\text{Substituting from continuity} \qquad \Rightarrow (u^2 - c^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c^2}{r}\left(1 - \frac{GM}{2c^2 r}\right)$$

We see again that there is a sonic point, this time at $r_s = GM/2c^2$.

4.3.1 Isothermal

Bernoulli's constant now includes $\Phi = -GM/r$. Hence equating H at a general point to at a sonic point, and to the point at ∞ where $u = \Phi = 0$,

$$\begin{aligned} \frac{1}{2}u^2 + c^2\ln\rho - \frac{GM}{r} &= \frac{1}{2}c^2 + c^2\ln\rho_s - \frac{GM}{r_s} \\ &= c^2\left(\ln\rho_s - \frac{3}{2}\right) \\ \Rightarrow u^2 &= 2c^2\left[\ln\left(\frac{\rho_s}{\rho}\right) - \frac{3}{2}\right] + \frac{2GM}{r} \end{aligned} \qquad \begin{aligned} \frac{1}{2}c^2 + c^2\ln\rho_s - \frac{GM}{r_s} &= c^2\ln\rho_\infty \\ c^2\left(\ln\rho_s - \frac{3}{2}\right) &= c^2\ln\rho_\infty \\ \Rightarrow \rho_s &= \rho_\infty e^{3/2} \end{aligned}$$

which together with $\dot{M} = 4\pi r^2\rho u$ gives parametric equations for ρ and u. $\dot{M}$ is then

$$\dot{M} = 4\pi r_s^2\rho_s c = 4\pi\frac{G^2M^2}{4c^4}\rho_\infty e^{3/2}c = \pi e^{3/2}\frac{G^2M^2\rho_\infty}{c^3}$$

purely in terms of the parameters of the problem. Note $\dot{M} \propto M^2\rho_\infty T^{-3/2}$.

4.3.2 Polytropic

Bernoulli now gives:

$$\frac{1}{2}u^2 + (n+1)K\rho^{1/n} - \frac{GM}{r} = \overbrace{\frac{1}{2}c_s^2 + nc_s^2 - \frac{GM}{r_s}}^{(n-3/2)c_s^2} = \overbrace{(n+1)K\rho_\infty^{1/n}}^{nc_\infty^2}$$

where c_s is the sound speed *at the sonic point*. On substituting for r_s, the central quantity becomes $(n-3/2)c_s^2$, which is coincidentally 0 for a monatomic adiabatic gas. But this would mean that $\rho_\infty = 0$, which is one of the parameters of the problem so should be able to have any value we choose! The resolution of this is that $c_s^2 = (n+1)K\rho_\infty^{1/n}/(n-3/2)$ is in fact infinite (and hence the sonic point $r_s = 0$), which allows for a finite ρ_∞. The $n = 3/2$ case looks doable, but quite fiddly. Forgetting this edge case, using $\rho \propto c^{2n}$ we have:

$$\dot{M} = 4\pi r_s^2\rho_s c_s = 4\pi\left(\frac{GM}{2c_s^2}\right)^2\rho_\infty\left(\frac{c_s}{c_\infty}\right)^{2n}c_s = \pi\frac{G^2M^2\rho_\infty}{c_\infty^3}\left(\frac{c_s}{c_\infty}\right)^{2n-3} = \pi\left(\frac{n}{n-\frac{3}{2}}\right)^{n-\frac{3}{2}}\frac{G^2M^2\rho_\infty}{c_\infty^3}$$

which is very similar to the isothermal formula but with a different prefactor instead of $e^{3/2}$.

5 Instabilities

5.1 Convective Instability

If a fluid element is nudged upwards, it may be buoyantly forced further upwards. Suppose such a fluid element initially has (p, ρ), equal to its surroundings, and moves upwards a distance $\mathrm{d}z$ where the surroundings change to (p', ρ'). Typically the pressure of the fluid element will follow the surroundings, but its density evolves adiabatically: the fluid element then has (p', ρ^*). If $\rho^* > \rho'$, the fluid element will simply fall back down, but if $\rho^* < \rho'$, the element will be buoyant and float further upwards. We have $p = K\rho^\gamma$ and $p' = K\rho^{*\gamma}$, so

$$\rho^* = \rho(p'/p)^{1/\gamma} = \rho\left(1 + \frac{1}{p}\,\mathrm{d}p/\mathrm{d}r\;\delta r\right)^{1/\gamma} \approx \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}r}\delta r$$

whereas in the surrounding medium, $\rho' = \rho + \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r$. The equation of motion for a fluid element will be, per unit volume:

$$\rho\frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} = -(\rho^* - \rho')g \qquad \Rightarrow \qquad \frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} = -\frac{g}{\rho}\left(\frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right)\delta r$$

If the brackets are positive, the system oscillates. The stability criterion $\rho^* > \rho'$ is then:

$$\frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}r}\delta r > \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r \qquad \Rightarrow \qquad \frac{\mathrm{d}\ln p}{\mathrm{d}r} > \gamma\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} \qquad \Rightarrow \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p}{\rho^\gamma}\right) > 0}$$

that is, if elements higher up have a higher K. Substituting T for ρ gives:

$$\frac{\mathrm{d}T}{\mathrm{d}r} > \frac{1-\gamma}{\gamma}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}r}$$

Now $\mathrm{d}p/\mathrm{d}r < 0$ always, so positive T gradients are always stable; negative T gradients are stable provided they aren't *too* negative.

5.2 Gravitational Instability

A perturbation may cause an initially self-gravitating system to overcome gas pressure and collapse. Consider a barotropic, static system which is initially uniform and self-gravitating, and then ***introduce a perturbation***: $p = p_0 + \Delta p, \rho = \rho_0 + \Delta\rho_0, \Phi = \Phi_0 + \Delta\Phi, \mathbf{u} = \Delta\mathbf{u}$. Then, ***substitute***[3] these into $\mathfrak{M}, \mathfrak{p}, \mathfrak{P}$, ***linearise***, and move to ***Fourier*** space (that is, assume each perturbation is of the form $\Delta\Box = \Box_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$). This eventually gives:

$$-k^2\Phi_1 = 4\pi G\rho_1 \qquad \rho_0\mathbf{k}\cdot\mathbf{u}_1 = \omega\rho_1 \qquad \omega\rho_0\mathbf{u}_1 = \mathbf{k}c^2\rho_1 + \mathbf{k}\rho_0\Phi_1$$

$$\Rightarrow \qquad \omega^2 = k^2c^2 + 4\pi G\rho_0 = c^2\left(k^2 - k_J^2\right) \qquad \text{where} \qquad k_J^2 = \frac{4\pi G\rho_0}{c^2}$$

We see that high-k perturbations are non-dispersive ($\omega \to c^2k^2$), those with $k \gtrsim k_J$ are dispersive, and those with $k < k_J$, or a wavelength $\lambda > \lambda_J = \frac{2\pi}{k_J} = \sqrt{\pi c^2/G\rho_0}$ lead to gravitational instability. If the system is smaller than this, such perturbations cannot be supported, so gravitational instability requires sizes larger than λ_J, or equivalently a mass of $M > M_J \sim \rho_0\lambda_J^3 \propto c^3\rho_0^{-1/2} \propto T^3\rho_0^{-1/2}$. Thus if a $1M_J$ system isothermally collapses ($\rho_0 \uparrow$), the system will contain more M_J and fragment.

[3] This technically requires in this case setting $\nabla^2\Phi = 4\pi G\rho_0$, which cannot be true as ρ_0 is assumed to be infinite and uniform. However, relativistically this swindle turns out to be fine.

5.3 Interface Instabilities

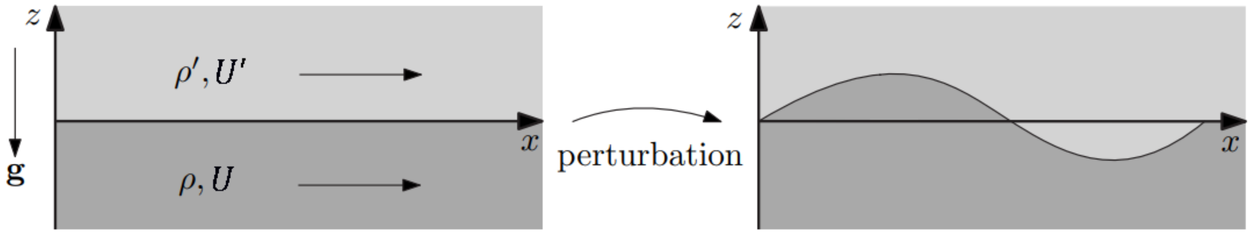

Figure 3 | Rayleigh-Taylor and Kelvin-Helmholtz instabilities

Consider two incompressible fluids in a uniform gravitational field, where the lower fluid has ρ and horizontal velocity U, while the upper has ρ' and U', as in Figure 3. A lengthy derivation gives that instabilities along the surface have the following dispersion relation:

$$\rho(kU-\omega)^2 + \rho'(kU'-\omega)^2 = kg(\rho-\rho')$$

Suppose first that the two fluids are initially at rest. Setting $U = U' = 0$ gives $\omega^2 = g\frac{\rho-\rho'}{\rho+\rho'}k$. For $\rho' < \rho$, such as air over oceans, this gives stable dispersive surface waves. For $\rho' > \rho$, this gives a **Rayleigh-Taylor instability** as the heavier fluid sinks into the lighter one.

Suppose the fluid is stable to RT ($\rho > \rho'$) but not at rest. The phase speed is then:

$$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{\rho-\rho'}{\rho+\rho'}\frac{g}{k} - \frac{\rho\rho'}{(\rho+\rho')^2}(U-U')^2}$$

Kelvin-Helmholtz instability arises if $\omega \notin \mathbb{R}$, that is, if

$$\frac{\rho\rho'}{(\rho+\rho')^2}(U-U')^2 > \frac{\rho-\rho'}{\rho+\rho'}\frac{g}{k} \qquad \Rightarrow \qquad k > g\frac{(\rho^2-\rho'^2)}{\rho\rho'(U-U')^2}$$

Thus if $g = 0$, instability arises unless $U = U'$ (surface waves). If $g \neq 0$ then low k are stable.

5.4 Thermal Instability

A long series of derivations shows that a system is unstable to thermal instabilities if

$$\boxed{\left.\frac{\partial \dot{Q}_{\text{cool}}}{\partial T}\right|_p < 0} \qquad \text{(Field Instability)}$$

This is fairly intuitive: if T being increased causes $\dot{Q}_{\text{cool}}$ to decrease, i.e. cooling becomes less efficient, T will then increase further and run away. The full analysis shows that even Field-stable fluids are unstable for large wavelengths if $\left.\partial \dot{Q}_{\text{cool}}\middle/\partial T\right|_\rho < 0$.

6 Viscosity

All the previous sections assume the mean free path $\lambda = 0$, so every particle in a given fluid element is moving with the same (bulk) velocity $\mathbf{u}$. For finite λ, thermal velocity means that particles can be transferred between fluid elements, the effect of which is that momentum can be transferred perpendicular to the bulk velocity. It turns out that this will reduce the stress tensor σ_{ij} (see §1.2) by σ'_{ij}, the *viscous stress tensor.*

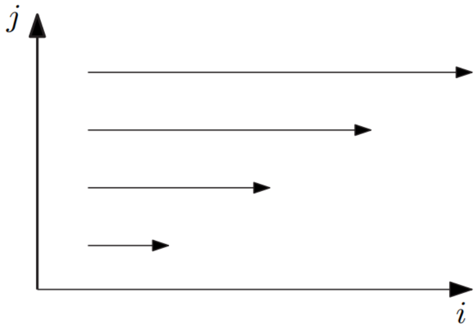

Figure 4 | A Linear Shear Flow

6.1 Linear Shear Flow

Consider the "linear shear flow" shown in Figure 4, where the bulk flow $\mathbf{u}$ is in the i-direction and depends on j, but there are also thermal velocities like $|v_j| = \alpha\sqrt{k_B T/m}$ where $\alpha \sim 1$, in the other directions, which can carry molecules between the layers, which we can think of as being separated by a distance $\delta x_j = \lambda = (n\sigma)^{-1}$. Now consider the *momentum flux in the i-direction carried across a surface area δA normal to the j-direction.* With a density ρ, the total momentum in the i-direction at a height x_j is $[\rho\delta A\delta x_j]u_i(x_j)$. The upward flux of i-momentum is then the i-momentum transferred per area per time, equal to $[\rho\delta A\delta x_j]u_i(x_j)/[\delta A(\delta x_j/v_j)] = \rho u_i(x_j)v_j$. There will also be a *downward* flux of i-momentum coming from the layer above; this will be $-\rho u_i(x_j + \delta x_j)v_j$. The net flux of i-momentum across this surface is then:

$$-\frac{\rho}{n\sigma}\frac{\partial u_i}{\partial x_j}v_j = -\frac{m}{\pi a^2}\frac{\partial u_i}{\partial x_j}\alpha\sqrt{\frac{k_B T}{m}} = -\frac{\alpha}{\pi a^2}\sqrt{mk_B T}\partial_j u_i \equiv -\eta\partial_j u_i$$

where η, the *shear viscosity* is temperature-dependent ($\propto T^{1/2}$ here; $\propto T^{5/2}$ for plasmas) but independent of ρ – denser gases have shorter λ but proportionally more particles to transport the momentum. Hence η is constant for an isothermal fluid. The total force on the volume bounded by δA and δx_j will be the net force (flux $\times$ area) on its upper surface minus the net force on its lower surface, which is

$$F \equiv \frac{\partial}{\partial t}([\rho\delta A\delta x_j]u_i) = -\left(-\eta\partial_j u_i\Big|_{x_j-\delta x_j} + +\eta\partial_j u_i\Big|_{x_j}\right)\delta A \qquad \Rightarrow \qquad \frac{\partial}{\partial t}(\rho u_i) = \partial_j(\eta\partial_j u_i)$$

6.2 The Navier-Stokes Equation

By comparison with the definition of the stress tensor in §1.2, this would suggest that $\sigma'_{ij} = \eta\partial_j u_i$. However, because σ'_{ij} has to be symmetric (to avoid unbalanced torques), isotropic, Galilean invariant, and because it depends linearly on the $\partial_m u_n$, the most general form is[4]:

$$\sigma'_{ij} = \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\partial_k u_k\delta_{ij}\right)$$

with the full stress tensor $\sigma_{ij} = \rho u_i u_j + p\delta_{ij} - \sigma'_{ij}$, we can then generalise $\mathfrak{p}$:

$$\Rightarrow \boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Phi + \nu\left(\boldsymbol{\nabla}^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right)} \qquad (\mathfrak{NS})$$

[4]We have neglected bulk viscosity ζ which is only relevant in shocks.

where we have assumed η is constant (isothermal) and defined the *kinematic viscosity* $\nu \equiv \eta/\rho$. The relevance of viscosity is encapsulated by the *Reynolds number* $\mathrm{Re} = UL/\nu$, where U and L are velocity and length scales of the system. This is the ratio of inertial to viscous forces: if Re$\gg 1$, viscosity is irrelevant and the flow is turbulent. Introducing viscosity tends to stabilise fluid instabilities.

Expanding the Lagrangian derivative, taking the curl, and assuming constant density, we obtain a modified version of the Helmholtz equation:

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \nu \boldsymbol{\nabla}^2 \mathbf{w}$$

The first term is carrying $\mathbf{w}$ with the flow; the second diffuses $\mathbf{w}$ through the fluid.

6.3 Viscous Dissipation

The rate of change of the kinetic energy density in a viscous incompressible fluid is

$$\partial_t \left(\frac{1}{2} \rho u^2 \right) = u_i \partial_t (\rho u_i) = -u_i \partial_i (\rho u_i u_j) - u_i \partial_i p + u_i \partial_j \sigma'_{ij} = -\partial_i \left(\frac{1}{2} \rho u_j u_j u_i + p u_i + \sigma'_{ij} u_j \right) - \sigma'_{ij} \partial_i u_j$$

The first term is the divergence of enthalpy and work done by viscous forces; the second is due to viscous dissipation of kinetic energy into heat. For an incompressible fluid, $\sigma'_{ij} = \eta(\partial_i u_j + \partial_j u_i)$, so this component of the kinetic energy change is equal to $-\frac{1}{2}\eta(\partial_i u_j + \partial_j u_i)^2$. $2\mathfrak{LT}$ decrees that microscopic processes cannot convert heat to kinetic energy, so $\eta \geq 0$.

7 Accretion Disks

- Axisymmetry: $\partial_\phi = 0$,
- Equilibrium in the z-direction: $u_z = 0$
- Keplerian angular velocity: $\Omega = \sqrt{GM/R^3} \Rightarrow u_\phi = \sqrt{GM/R}$

$\mathfrak{M}$ then gives:

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \rho u_R) + \frac{1}{R} \overbrace{\frac{\partial}{\partial \phi}}^{0} (\rho u_\phi) + \frac{\partial}{\partial z} \overbrace{(\rho u_z)}^{0} = 0 \qquad \Rightarrow \qquad \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) = 0$$

Then, consider conservation of angular momentum for an annulus width ΔR at R. The angular momentum in the annulus is $[2\pi R \Delta R \Sigma] R^2 \Omega = 2\pi R^3 \Delta R \Omega \Sigma$; the bracket is the mass of the annulus. The rate at which angular momentum is entering from inner R is $[2\pi R \Sigma u_R] R^2 \Omega = 2\pi R^3 \Omega \Sigma u_R$; the bracket is the mass entering per unit time. There are also viscous torques arising from neighbouring annuli, which are moving at different speeds to the annulus in question. Here we use an alternative and more conventional definition of η: the proportionality between shear stress and velocity gradient, that is, $\frac{F}{A} = \eta \frac{\mathrm{d}u}{\mathrm{d}R}$. In fact, in cylindrical systems, this is modified to $\frac{F}{A} = \eta R \frac{\mathrm{d}\Omega}{\mathrm{d}R}$. Now the area over which some shear is being done is $2\pi R \Delta z$, so $\Delta F = 2\pi R \eta \Delta z R \frac{\mathrm{d}\Omega}{\mathrm{d}R} = 2\pi R^2 \frac{\mathrm{d}\Omega}{\mathrm{d}R} \nu \rho \Delta z$; integrating over z and multiplying by R gives the torque $G = 2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R}$ exerted from within. Finally, the rate of change in angular momentum is equal to the rate of AM entering from within, minus the rate of AM leaving outside, plus the torque exerted from within minus the torque exerted outside, so cancelling factors of 2π:

$$\frac{\mathrm{d}}{\mathrm{d}t}(R^3 \Delta R \Omega \Sigma) = \left[R'^3 \Omega \Sigma u_R \right] \Big|_R - \left[R'^3 \Omega \Sigma u_R \right] \Big|_{R+\Delta R} + \left[R'^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R} \nu \Sigma \right] \Bigg|_{R+\Delta R} - \left[R'^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R} \nu \Sigma \right] \Bigg|_R$$

$$\Rightarrow R^3\Omega\frac{\partial\Sigma}{\partial t} = -\frac{\partial}{\partial R}(R^3\Omega\Sigma u_R) + \frac{\partial}{\partial R}\left(R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\nu\Sigma\right)$$

$$= -\frac{\partial}{\partial R}(R^2\Omega)R\Sigma u_R + R^2\Omega R\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial R}\left(R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\nu\Sigma\right)$$

$$\Rightarrow \frac{\partial}{\partial R}(R^2\Omega)R\Sigma u_R = \frac{\partial}{\partial R}\left(R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\nu\Sigma\right) \qquad \Rightarrow \qquad \frac{1}{2}R^{-1/2}R\Sigma u_R = \frac{\partial}{\partial R}\left(-\frac{3}{2}R^{1/2}\nu\Sigma\right)$$

$$\Rightarrow \boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\nu\Sigma)\right]}$$

where we have twice substituted from the conservation equation. Solving this equation shows that if initially $\Sigma(R,0) \propto \delta(R-R_0)$, then this ring will broaden and then relax to the centre.

Dimensional analysis gives a timescale of $t_v \sim R^2/\nu = (Ru_\phi/\nu)(R/u_\phi) = \mathrm{Re}/\Omega \sim \mathrm{Re}$ orbital periods. But for thermal viscosity, $\mathrm{Re} \sim 10^{14}$, which would give a viscous timescale $> t_H$. There is in fact extra “effective viscosity” from MHD turbulence, reducing Re.

In the steady state, it can be shown using intermediate results above that

$$\nu\Sigma = \frac{\dot{m}}{3\pi}\left(1 - \sqrt{\frac{R_*}{R}}\right)$$

where $\dot{m} = -2\pi R u_R \Sigma$ is an integration constant interpreted as the accretion rate, and R_* is a radius where $\nu\Sigma = 0$ (no viscous torque), (e.g. surface of a star, ISCO of a black hole).

Recall that the rate of viscous dissipation of kinetic energy is $\partial_t\left(\frac{1}{2}\rho u^2\right) = -\frac{1}{2}\eta(\partial_i u_j + \partial_j u_i)^2$, in this case $= -\eta R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$. Integrating this over z gives a dissipation rate per surface area of $\nu\Sigma R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$. Using the steady-state solution above, this is

$$F_{\mathrm{diss}} = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right)$$

$$\Rightarrow T_{\mathrm{eff}} = \left[\frac{F_{\mathrm{diss}}}{2\sigma}\right]^{1/4} = \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right)\right]^{1/4} \qquad L = \int_{R_*}^{\infty} 2\pi R\,\mathrm{d}R\,F_{\mathrm{diss}} = \frac{GM\dot{m}}{2R_*}$$

The factor of 2 in the T_{eff} is due to radiation from both top and bottom of the disk. The luminosity is as expected from the virial conversion of $\frac{1}{2}\times$ gravitational potential to radiation.

8 Magnetohydrodynamics

8.1 MHD Equations

Consider two overlapping fluids, such as of protons and electrons. Continuity requires both

$$\frac{\partial n^+}{\partial t} + \boldsymbol{\nabla}\cdot\left(n^+\mathbf{u}^+\right) = 0 \qquad \frac{\partial n^-}{\partial t} + \boldsymbol{\nabla}\cdot\left(n^-\mathbf{u}^-\right) = 0$$

where $\mathbf{u}^\pm$ is the bulk velocity of each fluid. The total density is $\rho = \rho^+ + \rho^- = m^+n^+ + m^-n^-$, and the centre-of-mass velocity is

$$\mathbf{u} = \frac{m^+n^+\mathbf{u}^+ + m^-n^-\mathbf{u}^-}{m^+n^+ + m^-n^-} \qquad \Rightarrow \qquad \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$

The *charge* density $q = q^+ + q^- = e^+n^+ + e^-n^-$, and the current density $\mathbf{j} = e^+n^+\mathbf{u}^+ + e^-n^-\mathbf{u}^-$;

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$$

For momentum, we include the Lorentz force:

$$\rho^+\left(\frac{\partial \mathbf{u}^+}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}^+\right) = -f^+\boldsymbol{\nabla}p - \rho^+\boldsymbol{\nabla}\Phi + q^+\left(\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}\right)$$

where f^+ is "the fraction of pressure gradient that accelerates positive charges", whatever that means; similarly for $\mathbf{u}^-$. Summing the two,

$$\boxed{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}\right) = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Phi + \mathbf{j} \times \mathbf{B}} + q\mathbf{E}$$

We also have Ohm's Law $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$, for conductivity σ; for high σ we require $\mathbf{E} \perp \mathbf{B}$. To close the system, require:

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0} \quad \boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{q}{\epsilon_0}} \quad \boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \quad \boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{j}} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad \text{(Maxwell)}$$

Looking at characteristic scales, we see from Maxwell 3 above that $E/B \sim u$. This enables one to simplify the momentum equation and Maxwell 4 for the case of non-relativistic plasmas ($u \ll c$), so that the terms outside the boxes become negligible.

Taking the curl of Maxwell 4, we obtain:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0\sigma[\boldsymbol{\nabla} \times \mathbf{E} + \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})] \qquad \Rightarrow \qquad \boxed{\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})} + \frac{1}{\mu_0\sigma}\boldsymbol{\nabla}^2\mathbf{B}$$

where the term outside is neglected as σ is assumed large. This is analogous to the Helmholtz equation for vorticity – hence in an analogous way to the Kelvin circulation theorem the magnetic flux through a surface carried with the field is constant. This is called *flux freezing*.

8.2 Plasma Waves

The magnetic force/volume is $\mathbf{f}_{\text{mag}} = \mathbf{j} \times \mathbf{B} = \mu_0^{-1}\left[-\boldsymbol{\nabla}\left(\frac{1}{2}B^2\right) + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}\right]$. The first term is simply analogous to a magnetic pressure $B^2/2\mu_0$; the second is *magnetic tension*, which apparently acts to straighten bent field lines[5].

Introducing linear perturbations $\Delta\Box = \Box_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ to the new momentum equation, continuity, the flux freezing equation, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, assuming a barotropic equation of state and neglecting gravity

$$\omega\rho_0\mathbf{u}_1 = c^2\rho_1\mathbf{k} + \mu_0^{-1}\mathbf{B}_0 \times (\mathbf{k} \times \mathbf{B}_1) \qquad \omega\rho_1 = \rho_0\mathbf{k}\cdot\mathbf{u}_1 \qquad \omega\mathbf{B}_1 = -\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0) \qquad \mathbf{k}\cdot\mathbf{B}_1 = 0$$

which lead to the following complicated expression for $\mathbf{u}_1$:

$$\omega^2\mathbf{u}_1 = c^2(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k} + (\mu_0\rho_0)^{-1} \times \left[\left(B_0^2(\mathbf{k} \cdot \mathbf{u}_1) - (\mathbf{B}_0 \cdot \mathbf{k})(\mathbf{B}_0 \cdot \mathbf{u}_1)\right)\mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k})(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{B}_0 + (\mathbf{B}_0 \cdot \mathbf{k})^2\mathbf{u}_1\right]$$

[5]The operator $\mathbf{B} \cdot \boldsymbol{\nabla}$ is the derivative along a field line

8.2.1 $\mathbf{k} \parallel \mathbf{B}_0$: Alfvén Waves

Taking the dot product with $\mathbf{k}$, we obtain[6] simply $\omega^2 = c^2k^2$: the $\mathbf{B}$ fields play no part. This is because these waves have $\mathbf{u}_1 \parallel \mathbf{k} \parallel \mathbf{B}_0$.

Instead taking the cross product with $\mathbf{k}$ (removing terms proportional to $\mathbf{k}$ and $\mathbf{B}_0$) and cancelling $\mathbf{k} \times \mathbf{u}_1$ gives $\omega^2 = v_A^2k^2$, where $v_A = B_0/\sqrt{\mu_0\rho_0}$ is the *Alfvén speed.* Noting that $\mathbf{k} \cdot \mathbf{B}_1 = 0 \Rightarrow \mathbf{B}_1 \perp \mathbf{B}_0$, we see that this describes non-dispersive magnetic *Alfvén waves* in the $\mathbf{B}$ field lines, where the restoring force is magnetic tension.

8.2.2 $\mathbf{k} \perp \mathbf{B}_0$: Fast Magnetosonic Waves

In this case, $\mathbf{B}_0 \cdot \mathbf{k} = 0$ so the above expression simplifies to to $\omega^2\mathbf{u}_1 = [c^2 + B_0^2/\mu_0\rho_0](\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k}$, so $\mathbf{u}_1 \parallel \mathbf{k} \perp \mathbf{B}_0$. Dotting with $\mathbf{k}$ gives the following dispersion relation:

$$\omega^2 = \left(c^2 + \frac{B_0^2}{\mu_0\rho_0}\right)k^2 = \left(c^2 + v_A^2\right)k^2$$

Here we have gas pressure (in c^2) and magnetic pressure (in $v_A^2/\mu_0\rho_0$) acting together for a compressive, longitudinal wave, faster than both sound and Alfvén waves.

[6] This is done most easily by writing $\mathbf{B}_0 = B_0\hat{\mathbf{e}}$ and $\mathbf{k} = k\hat{\mathbf{e}}$