Disks

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1 Orbital Dynamics

The Lagrangian $\mathcal{L}$ for orbits in an axisymmetric disk is

$$\mathcal{L} = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r,z)$$

From $\partial\mathcal{L}/\partial\phi = \partial\mathcal{L}/\partial t = 0$, calculus of variations gives the two conserved quantities:

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = h \equiv r^2\dot{\phi} \qquad\qquad \sum \dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \mathcal{L} = \varepsilon \equiv \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \Phi$$

the specific AM and energy. The Euler-Lagrange equations in the r and z directions give:

$$\ddot{r} = r\dot{\phi}^2 - \partial_r\Phi \qquad \ddot{z} = -\partial_z\Phi \qquad \text{Now define: } \Phi_{\text{eff}} \equiv h^2/2r^2 + \Phi$$

$$\Rightarrow \ddot{r} = -\partial_r\Phi_{\text{eff}} \qquad \ddot{z} = -\partial_z\Phi_{\text{eff}} \qquad \text{Also: } \varepsilon = \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \Phi_{\text{eff}}$$

Consider circular midplane orbits. These require

$$0 = \partial_{rc}\Phi_{\text{eff}} \Rightarrow \frac{h^2}{r^3} = \partial_{rc}\Phi; \qquad\qquad 0 = \partial_{zc}\Phi_{\text{eff}} \Rightarrow \partial_{zc}\Phi = 0$$

where ∂_{ic} is the i-derivative evaluated at the circular orbit at $(r,0)$. The second condition is automatically satisfied as we assume $\Phi(r,z) = \Phi(r,-z)$. The first condition shows that $\partial_{rc}\Phi > 0$ is required for circular orbits, and gives an expression for the AM of circular orbits of radius r in a given Φ, and more:

$$\boxed{h_c(r) = \sqrt{r^3\partial_{rc}\Phi}} \quad\Rightarrow\quad \boxed{\Omega_c(r) = \frac{h_c}{r^2} = \sqrt{\frac{\partial_{rc}\Phi}{r}}}; \quad \boxed{\varepsilon_c(r) = \frac{1}{2}r\partial_{rc}\Phi + \Phi = \frac{h_c^2}{2r^2} + \Phi}$$

An important property of disks is that adjacent orbits move at different speeds. We define the *shear rate* $S(r) \equiv -r\,\mathrm{d}\Omega_c/\mathrm{d}r$, which has dimensions of inverse time.

The equations of motion give the following oscillatory evolution of deviations δr and δz from circular orbits, at the frequencies Ω_r and Ω_z

$$\ddot{\delta r} = -\partial^2_{rrc}\Phi_{\text{eff}}\delta r \quad\Rightarrow\quad \Omega_r^2 \equiv \partial^2_{rrc}\Phi_{\text{eff}} \qquad\qquad \ddot{\delta z} = -\partial^2_{zzc}\Phi_{\text{eff}}\delta z \quad\Rightarrow\quad \Omega_z^2 = \partial^2_{zzc}\Phi_{\text{eff}}$$

Now $\partial_z\Phi_{\text{eff}} = \partial_z\Phi$, so we could just write $\Omega_z^2 = \partial^2_{zzc}\Phi$. Ω_r however can also be written

$$\Omega_r^2 = \frac{3h_c^2}{r^4} + \partial^2_{rrc}\Phi = \frac{3h_c^2}{r^4} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{h_c^2}{r^3}\right) = \frac{1}{r^3}\frac{\mathrm{d}h_c^2}{\mathrm{d}r}$$

$$= \frac{1}{r^3}\frac{\mathrm{d}(r^4\Omega_c^2)}{\mathrm{d}r} = 4\Omega_c^2 + 2r\Omega_c\frac{\mathrm{d}\Omega_c}{\mathrm{d}r} = 2\Omega_c(2\Omega_c - S)$$

with the interesting results at the ends of the above lines.

If $\Omega_r \neq \Omega_c$, orbits will precess. Periapses occur in time intervals $\Delta t = 2\pi/\Omega_r$, in which time ϕ has changed by $\Omega_c \Delta t = 2\pi(\Omega_c/\Omega_r - 1) \mod 2\pi$, so the *apsidal precession rate* is $\Delta\phi/\Delta t = \Omega_c - \Omega_r$. Similarly the *nodal precession rate* is $\Omega_c - \Omega_z$.

Keplerian orbits, with $\Phi(r,z) = -GM/\sqrt{r^2+z^2}$, have the following properties:

$$h_c = \sqrt{GMr} \qquad \Omega_c = \sqrt{\frac{GM}{r^3}} \qquad \varepsilon_c = -\frac{GM}{2r} \qquad S = \frac{3}{2}\Omega_c \qquad \Omega_r = \Omega_z = \sqrt{\frac{GM}{r^3}} = \Omega_c$$

And thus do not precess.

2 Accretion Disks

2.1 Fundamentals

Consider two particles (m_i, h_i, ε_i, $i = 1,2$) on circular orbits which exchange some mass $\mathrm{d}m$, angular momentum $\mathrm{d}h$ and orbital energy $\mathrm{d}\varepsilon$; mass and angular momentum must be conserved, though orbital energy will overall be dissipated into heat[1]. Before calculating the total energy change, we derive a cool lemma about ε_c and h_c:

$$\frac{\mathrm{d}\varepsilon_c}{\mathrm{d}h_c} = \frac{\mathrm{d}\varepsilon_c}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}h_c} = \left[\frac{h_c}{r^2}\frac{\mathrm{d}h_c}{\mathrm{d}r}\overbrace{-\frac{h_c^2}{r^3} + \partial_{rc}\Phi}^{0}\right]\frac{\mathrm{d}r}{\mathrm{d}h_c} = \frac{h_c}{r^2} = \Omega_c$$

$$\Rightarrow \mathrm{d}E = \mathrm{d}(m_i\varepsilon_{ci}) = \mathrm{d}m_i\,\varepsilon_{ci} + m_i\mathrm{d}\varepsilon_{ci} = \mathrm{d}m_i\,\varepsilon_{ci} + m_i\Omega_{ci}\mathrm{d}h_{ci} = \mathrm{d}m_i\,\varepsilon_{ci} + \Omega_{ci}\mathrm{d}H_{ci} - \mathrm{d}m_i\,\Omega_{ci}h_{ci}$$
$$= [(\varepsilon_c - \Omega_c h_c)_1 - (\varepsilon_c - \Omega_c h_c)_2]\,\mathrm{d}m_1 + [\Omega_{c1} - \Omega_{c2}]\,\mathrm{d}H_1$$

Now $\frac{\mathrm{d}}{\mathrm{d}r}(\varepsilon - \Omega_c h_c) = -h_c\,\mathrm{d}\Omega_c/\mathrm{d}r$, so in the usual case that $\Omega_c(r)$ monotonically decreases this Jacobic quantity monotonically increases. Suppose wlog $r_{c1} < r_{c2}$, so that the first squacket above is negative and the second is positive. $\mathrm{d}E < 0$ thus encourages $\mathrm{d}m_1 > 0$ and $\mathrm{d}H_1 < 0$: *mass is transferred inwards* and *AM is transferred outwards.*

2.1.1 Mass Conservation

$$0 = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(\rho u_\phi) + \frac{\partial}{\partial z}(\rho u_z)$$

Multiplying by r and integrating over all z and ϕ, we find

$$0 = \underbrace{\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty}\int_0^{2\pi} r\mathrm{d}\phi\,\mathrm{d}z\,\rho\right)}_{\mathcal{M}} + \underbrace{\frac{\partial}{\partial r}\left(\int_{-\infty}^{\infty}\int_0^{2\pi} r\mathrm{d}\phi\,\mathrm{d}z\,\rho u_r\right)}_{\mathcal{F}} + 2\pi r\underbrace{[\rho u_z]_{-\infty}^{\infty}}_{0} \Rightarrow \boxed{\frac{\partial\mathcal{M}}{\partial t} + \frac{\partial\mathcal{F}}{\partial r} = 0}$$

$\mathcal{M}(r,t)$ is the mass in a disk per unit radius. $\mathcal{F}(r,t)$ is the radial momentum per unit radius; accretion corresponds to $\mathcal{F} < 0$. With azimuthal symmetry, these are normally written:

$$\mathcal{M} = 2\pi r\overbrace{\int_{-\infty}^{\infty}\rho\,\mathrm{d}z}^{\Sigma(r,t)} = 2\pi r\Sigma \qquad\qquad \mathcal{F} = 2\pi r\overbrace{\int_{-\infty}^{\infty}\rho u_r\,\mathrm{d}z}^{\Sigma(r,t)\bar{u}_r(r,t)} = 2\pi r\Sigma\bar{u}_r = -\dot{M}$$

[1] Otherwise we could extract thermal energy from the particles down to speed them up, in violation of $2\mathfrak{LT}$.

2.1.2 Angular Momentum Conservation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\rho\nabla\Phi + \nabla\cdot\mathsf{T} \qquad \Rightarrow \qquad \frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u} - \mathsf{T}) = -\rho\nabla\Phi$$

where we have used mass conservation; T is the stress tensor – a symmetric 2nd-order tensor, of which the $r\phi$ component will be the most important. T encapsulates various momentum transport processes, including self-gravity, **B** fields, viscous stresses, and turbulence. The external gravitational field (due to the central object) remains as a source term. The azimuthal component of this equation multiplied by r^2 can be written[2]:

$$\frac{\partial}{\partial t}(\rho r^2 u_\phi) + \frac{\partial}{\partial r}\left[r^2(\rho u_r u_\phi - T_{r\phi})\right] + r^2\frac{\partial}{\partial z}[(\rho u_\phi u_z - T_{z\phi})] + r\frac{\partial}{\partial \phi}(\rho u_\phi^2 - T_{\phi\phi}) = 0$$

assuming $\partial\Phi/\partial\phi = 0$. Integrating over z and ϕ as above, using the definitions of $\mathcal{M}$, $\mathcal{F}$, and assuming $r u_\phi(r,\phi,z) = h(r)$ independent of z (and ϕ), we eventually find

$$\boxed{\frac{\partial \mathcal{M}}{\partial t}h + \frac{\partial}{\partial r}(\mathcal{F}h + \mathcal{G}) = 0} \qquad \mathcal{G}(r,t) \equiv \int_{-\infty}^{\infty}\int_0^{2\pi} -r^2 T_{r\phi}\,\mathrm{d}\phi\,\mathrm{d}z \equiv -2\pi\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

where the *mean effective kinematic viscosity* $\bar{\nu}(r,t)$ is defined by $\bar{\nu}\Sigma r\frac{\mathrm{d}\Omega}{\mathrm{d}r} = \int_{-\infty}^{\infty} T_{r\phi}\,\mathrm{d}z$. The flux of angular momentum $\mathcal{M}h$ is thus due to the advection of orbital momentum $\mathcal{F}h$ as well as a torque. Eliminating $\mathcal{M}$ from the equation for mass conservation gives

$$\mathcal{F}\frac{\mathrm{d}h}{\mathrm{d}r} + \frac{\partial\mathcal{G}}{\partial r} = 0 \qquad \Rightarrow \qquad \frac{\partial\mathcal{M}}{\partial t} = \frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\partial\mathcal{G}}{\partial r}\right]$$

Substituting the forms of $\mathcal{M}$ and $\mathcal{G}$ in terms of more ordinary variables, we find

$$\frac{\partial}{\partial t}(2\pi r\Sigma) = \frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\partial}{\partial r}\left(-2\pi\bar{\nu}\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)\right]$$

$$\Rightarrow \frac{\partial\Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\partial}{\partial r}\left(r^3\bar{\nu}\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)\right] \xrightarrow{\text{Keplerian}} \boxed{\frac{\partial\Sigma}{\partial t} = -\frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right]}$$

a quasi-diffusion equation. We stated earlier that $\bar{\nu} = \bar{\nu}(r,t)$; in fact usually $\bar{\nu} = \bar{\nu}(r,\Sigma)$. We see that if $\bar{\nu} = \bar{\nu}(r)$ then the diffusion equation is linear in Σ.

2.1.3 Energy Conservation

Consider the quantity $\frac{\partial}{\partial t}(\mathcal{M}\varepsilon) + \frac{\partial}{\partial r}(\mathcal{F}\varepsilon)$: this is a sort of Lagrangian rate of change of energy per unit radius. Using the above, and the fact that $\mathrm{d}\varepsilon = \Omega(r)\,\mathrm{d}h(r) \Rightarrow \varepsilon = \varepsilon(r)$ we find

$$\frac{\partial}{\partial t}(\mathcal{M}\varepsilon) + \frac{\partial}{\partial r}(\mathcal{F}\varepsilon) = \varepsilon\left(\frac{\partial\mathcal{M}}{\partial t} + \frac{\partial\mathcal{F}}{\partial r}\right) + \mathcal{F}\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} = -\frac{\partial\mathcal{G}}{\partial r}\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^{-1}\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} = -\frac{\partial\mathcal{G}}{\partial r}\Omega$$

$$\Rightarrow \frac{\partial}{\partial t}(\mathcal{M}\varepsilon) + \frac{\partial}{\partial r}(\mathcal{F}\varepsilon + \mathcal{G}\Omega) = \mathcal{G}\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

[2] Using the formula $[\nabla\cdot\mathsf{T}]_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{\phi r}) + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{\partial T_{\phi z}}{\partial z}$ for a symmetric tensor. This formula has surprise powers of two in the first term and needs some kinda tensor calculus to derive.

so $\mathcal{G}\Omega$ contributes to the energy flux, and the viscous torque acts to dissipate energy where the fluid shears. In the normal notation, the rate of dissipation per unit area is given by

$$\frac{\partial}{\partial t}(\Sigma\varepsilon) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma\bar{u}_r\varepsilon - r^3\bar{\nu}\Sigma\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) = -\bar{\nu}\Sigma\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2$$

where we note that the RHS is the viscosity times the shear rate S squared. The energy per unit area that is dissipated on the RHS will be dissipated at the two surfaces of the disk, defining an effective temperature: $2\sigma T_{\mathrm{eff}}^4 = \bar{\nu}\Sigma\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2$.

2.1.4 Boundary Conditions

A *free boundary* has $\mathcal{G} = 0$ there. For a black hole, orbits are unstable within $r = 3r_s$ (the *ISCO*); $\mathcal{G} \approx 0$ here. Non-magnetic stars rotate at an angular frequency $\Omega_* \ll \Omega_K(R_*)$, so the Ω of the disk will have to be dragged down to the star's Ω_* in a boundary layer. By Rolle's theorem there will have to be a point where $\mathrm{d}\Omega/\mathrm{d}r = 0 \Rightarrow \mathcal{G} = 0$ somewhere near $R_\star$.

Stars with significant $\mathbf{B}$ fields disrupt the disk somewhere within the magnetospheric radius, at which the accretion flow is diverted out of the plane and the accretion becomes polar. The linkage of the $\mathbf{B}$ field through the disc may create a magnetic torque here.

2.2 Steady Accretion

If $\partial/\partial t = 0$, then we have $\mathcal{F}(r) = -\dot{M}$ const. and $\mathcal{F}(r)h(r) + \mathcal{G}(r) =$ const. Hence if there is no torque at the inner boundary, $\mathcal{G}(r) = \dot{M}[h(r) - h(r_{\mathrm{in}})]$. Now for the Keplerian case, $\mathcal{G}(r) = -2\pi\bar{\nu}\Sigma r^3\left(-\frac{3}{2}\sqrt{GM/r^5}\right) = 3\pi\bar{\nu}\Sigma\sqrt{GMr} = 3\pi\bar{\nu}\Sigma h(r)$, so we have

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \frac{h(r_{\mathrm{in}})}{h(r)}\right) = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_{\mathrm{in}}}{r}}\right)$$

With the definition of the effective temperature, we find

$$\sigma T_{\mathrm{eff}}^4 = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \sqrt{\frac{r_{\mathrm{in}}}{r}}\right) \qquad \Rightarrow \qquad L \equiv \int_{r_{\mathrm{in}}}^{\infty} 2\pi r \cdot 2\sigma T_{\mathrm{eff}}^4 \,\mathrm{d}r = \frac{GM\dot{M}}{2r_{\mathrm{in}}}$$

The other half of the potential energy released by accretion is dissipated in the boundary layer.

2.3 Time-Dependent Accretion

2.3.1 Linear Diffusion Equation

Consider the linear case, where $\bar{\nu} = \bar{\nu}(r)$, and with no central torque $\mathcal{G}(r_{\mathrm{in}}) = 0$. This can be solved with a Green's function. Let $\Delta(r, r_0, t)$ be the solution $\mathcal{M} = 2\pi r\Sigma(r, t)$ of the diffusion equation with initial condition $\mathcal{M}(r, 0) = \delta(r - r_0)$. Then we have by definition

$$\mathcal{M}(r, t) = \int_0^{\infty} \Delta(r, r_0, t)\mathcal{M}(r_0, 0)\,\mathrm{d}r_0$$

where $\mathcal{M}(r_0, 0)$ comes from the initial condition $\mathcal{M}(r, 0)$.

In the case $\bar{\nu} = Ar$, the diffusion equation has a (relatively!) simple solution: letting

$$g = r^{1/2}\bar{\nu}\Sigma = Ar^{3/2}\Sigma = \frac{Ar^{1/2}}{2\pi}\mathcal{M}, \qquad y = \sqrt{\frac{4}{3A}}r^{1/2} \qquad \Rightarrow \qquad \frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}$$

The solution to this with $g(y,0) = \delta(y)$ is $\frac{1}{\sqrt{4\pi t}}\exp(-y^2/4t)$, so we might guess a solution $g(y,t) = \frac{1}{\sqrt{4\pi t}}\exp(-(y-y_0)^2/4t)$, where $y_0 = y(r_0)$. But this cannot be the full solution as $g \propto \mathcal{G}$, and this does not give $g(r_{\rm in}) = 0$. Using the method of images, however, we can put another source at $y = 2y_{\rm in} - y_0$, giving:

$$g(y, y_0, t) = \frac{1}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(y-y_0)^2}{4t}\right) - \exp\left(-\frac{(y+y_0-2y_{\rm in})^2}{4t}\right)\right]$$

This is the solution for $g(y,0) = \delta(y - y_0)$, but the actual boundary condition at $t = 0$ has an extra proportionality factor:

$$g(y,0) = \frac{Ar^{1/2}}{2\pi}\mathcal{M}(r,0) = \frac{Ar^{1/2}}{2\pi}\delta(r-r_0) = \frac{Ar_0^{1/2}}{2\pi}\frac{\delta(y-y_0)}{\sqrt{3Ar_0}} = \frac{1}{2\pi}\sqrt{\frac{A}{3}}\delta(y-y_0)$$

where we have used $\delta(f(x)) = \delta(x-x_0)/|f'(x_0)|$ to convert the δs. With this extra proportionality factor we then have

$$g(y, y_0, t) = \frac{1}{2\pi}\sqrt{\frac{A}{12\pi t}}\left[\exp\left(-\frac{(y-y_0)^2}{4t}\right) - \exp\left(-\frac{(y+y_0-2y_{\rm in})^2}{4t}\right)\right]$$

Thus $\Delta(r, r_0, t)$, the solution to all this when $\mathcal{M}(r,0) = \delta(r - r_0)$, is given by

$$\begin{aligned}\Delta(r, r_0, t) &= \frac{2\pi}{Ar^{1/2}}\frac{1}{2\pi}\sqrt{\frac{A}{12\pi t}}\left[\exp\left(-\frac{(y-y_0)^2}{4t}\right) - \exp\left(-\frac{(y+y_0-2y_{\rm in})^2}{4t}\right)\right] \\ &= \frac{1}{\sqrt{12\pi Art}}\left[\exp\left(-\frac{(\sqrt{r}-\sqrt{r_0})^2}{3At}\right) - \exp\left(-\frac{(\sqrt{r}+\sqrt{r_0}-2\sqrt{r_{\rm in}})^2}{3At}\right)\right]\end{aligned}$$

The full solution for $\mathcal{M}(r,t)$ is then the convolution of this Δ with $\mathcal{M}(r,0)$. Interestingly, in the limit of $3At \gg r$, we find

$$\Delta(r, r_0, t) \approx \frac{1}{\sqrt{12\pi Art}}\frac{4\sqrt{rr_0} + 4r_{\rm in} - 4\sqrt{rr_{\rm in}} - 4\sqrt{r_0 r_{\rm in}}}{3At} \equiv 2\pi r\Sigma = \frac{2\pi}{A}\bar{\nu}\Sigma$$

$$\Rightarrow \qquad \bar{\nu}\Sigma = \frac{1}{3\pi}\frac{\sqrt{rr_0} + r_{\rm in} - \sqrt{rr_{\rm in}} - \sqrt{r_0 r_{\rm in}}}{\sqrt{3\pi Art^3}} = \frac{\sqrt{r_0} - \sqrt{r_{\rm in}}}{\sqrt{3\pi At^3}}\frac{1}{3\pi}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right)$$

similarly to the steady case, with the prefactor being the now time-dependent accretion rate $\dot{M} \propto t^{-3/2}$.

The mass remaining in the disk at time t is given by

$$\int_{r_{\rm in}}^{\infty} \Delta(r, r_0, t)\,\mathrm{d}t = \frac{1}{\sqrt{12\pi At}}\left[\int_{-\xi_{\rm in}}^{\infty}\sqrt{12At}e^{-\xi^2}\mathrm{d}\xi - \int_{\xi_{\rm in}}^{\infty}\sqrt{12At}e^{-\xi^2}\mathrm{d}\xi\right] = \frac{1}{\sqrt{\pi}}\int_{-\xi_{\rm in}}^{\xi_{\rm in}} e^{-\xi^2}\mathrm{d}\xi$$

which is just $\mathrm{erf}(\xi_{\rm in})$; we have substituted ξ^2 to be the exponent of each integral and defined $\xi_{\rm in} = \left(\sqrt{r_0} - \sqrt{r_{\rm in}}\right)/\sqrt{3At}$. In the small-argument limit $\mathrm{erf}(x) \to x$, so for large times the mass remaining in the disk decays as t^{-1}. It can be apparently shown that $\int_0^t \dot{M}(r_{\rm in}, t)\,\mathrm{d}t = \mathrm{erfc}(\xi_{\rm in})$, verifying that mass has been conserved. Further, the angular momentum remaining in the disk, $\int_{r_{\rm in}}^{\infty}\sqrt{GMr}\Delta\,\mathrm{d}r = h_0 - h_{\rm in}\mathrm{erfc}(\xi_{\rm in})$: the initial angular momentum minus that of the matter when it falls to $r_{\rm in}$.

2.3.2 Non-Linear Diffusion Equation

Generally, $\bar{\nu} = Ar^a\Sigma^b$. If we let $r_{\rm in} \to 0$, the problem becomes scale-free. If there is no central torque the angular momentum is conserved:

$$\sqrt{GM}\int_0^\infty r^{1/2}2\pi r\Sigma\,\mathrm{d}r \equiv \sqrt{GM}C \quad \text{for some } C: \quad [C] = ML^{1/2}$$

We now use dimensional analysis to construct a characteristic length scale $R(t)$ using the constants A, C, and time t.

$$[\bar{\nu}] = L^2T^{-1} \qquad \Rightarrow \qquad [A] = L^2T^{-1}\cdot L^{-a}\cdot M^{-b}L^{2b} = M^{-b}L^{2-a+2b}T^{-1}$$

Thus the quantity $[AC^bt] = L^{2-a+5b/2}$. The size of the disk thus scales as $t^{2/(4-2a+5b)}$. The mass of the disk scales as $C/L^{1/2} \propto t^{-1/(4-2a+5b)}$. There are general but very complicated solutions, which tend to the above scalings for large times (when the initial conditions have faded into the past and the problem loses its scale).

2.4 The z-Direction

2.4.1 Hydrostatic Equilbrium

In the z-direction, the force balance in the steady state is $\rho\,\partial\Phi/\partial z = -\,\partial p/\partial z$ Now the potential will be $\Phi(r,z) \approx \Phi(r,0) + \frac{1}{2}\partial^2_{zz}\Phi(r,0)z^2 \Rightarrow \partial_z\Phi = \partial^2_{zz}\Phi(r,0)z = \Omega_z^2 z$, so

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z$$

This can be solved if $p(\rho)$ is known. For an isothermal gas, $p = c_s^2\rho$, so $p \propto \rho \propto \exp(-z^2/2H^2)$, where $H = c_s/\Omega_z$. To consider more general cases, we define:

$$\Sigma \equiv \int_{-\infty}^{\infty}\rho\,\mathrm{d}z \qquad P \equiv \int_{-\infty}^{\infty}p\,\mathrm{d}z \qquad \Sigma H^2 = \int_{-\infty}^{\infty}\rho z^2\,\mathrm{d}z$$

defining the *vertically integrated pressure* P and the scaleheight H as the standard deviation of $\rho(z)$. These quantities have some dimensional redundancy:

$$P = \underbrace{[zp]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} z\frac{\partial p}{\partial z} = \Omega_z^2\int\rho z^2\,\mathrm{d}z = \Sigma\Omega_z^2H^2$$

This can be dedimensionalised by writing

$$\tilde{z} = z/H \qquad \rho(z) = \hat{\rho}\tilde{\rho}(\tilde{z}), \qquad \hat{\rho} = \frac{\Sigma}{H}, \qquad p(z) = \hat{p}\tilde{p}(\tilde{z}), \qquad \hat{p} = \frac{P}{H}$$

$$\Rightarrow \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{z}} = -\tilde{\rho}\tilde{z}, \qquad \int\tilde{\rho}\,\mathrm{d}\tilde{z} = \int\tilde{p}\,\mathrm{d}\tilde{z} = \int\tilde{\rho}\tilde{z}^2\,\mathrm{d}\tilde{z} = 1$$

The isothermal model ($\rho \propto p$) corresponds to a normalised Gaussian $\tilde{\rho} = \tilde{p} = \frac{1}{\sqrt{2\pi}}\exp(-\tilde{z}^2/2)$. A constant density model, with $\rho = $ const. up to a certain $|z|$, has the solution $\tilde{\rho} = 1/2\sqrt{3}$, $\tilde{p} = (3-\tilde{z}^2)/4\sqrt{3}$ for $|\tilde{z}| < \tilde{z}_0 = \sqrt{3}$, and $\tilde{\rho} = \tilde{p} = 0$ outside.

2.4.2 Radiative Transfer

Radiation diffuses upwards in the disk due to temperature gradients:

$$F_z = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}$$

Neglecting small ($\mathcal{O}(H/r)^2$) contributions from e.g. $\partial F_r/\partial r$, energy conservation gives

$$\frac{\partial F_z}{\partial z} = \rho\bar{\nu}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2$$

Together with hydrostatic equilibrium, this gives three equations in ρ, p, F_z, T, and κ. This set is closed by assuming an equation of state $p = p(\rho, T)$ (e.g. $p = \frac{k\rho T}{\mu m_p} + \frac{4\sigma}{3c}T^4$) and a function for $\kappa = \kappa(\rho, T)$ (e.g. Kramers' $\kappa = C\rho T^{-7/2}$). Further work on this is non-examinable in 22/23.

2.5 Scalings

A thin disk has aspect ratio $H/r \ll 1$. Hydrostatic equilibrium: $p/H \sim \rho\Omega^2 H \Rightarrow c_s \sim H\Omega$.

The dimensions of $[\bar{\nu}] = L^2T^{-1}$, so $[\rho\bar{\nu}] = ML^{-1}T^{-1}$. The dimensions of $[p] = ML^{-1}T^{-2}$, so $[\rho\bar{\nu}] = [p/\Omega]$. The alpha viscosity prescription prescribes a constant proportionality:

$$\bar{\nu} = \alpha\frac{p}{\rho\Omega} \sim \alpha\frac{c_s^2}{\Omega} \sim \alpha c_s H$$

True molecular viscosity $\bar{\nu} \sim v_\nu \ell_\nu$ is negligible for disks, but the effective viscosity due to turbulence can be thought of as being due to molecular motions over the length of a typical eddy. Now we require $v_\nu < c_s$ for subsonic turbulence, and we require $\ell_\nu < H$ for the eddy to fit into the disk, so we expect $\bar{\nu} \lesssim c_s H$ and hence $\alpha \lesssim 1$ for turbulent viscosity alone (other effective viscosities, due to self-gravity of $\mathbf{B}$ fields, could have $\alpha > 1$).

The orbital Mach number $\sim r\Omega/c_s \sim (H/r)^{-1} \gg 1$: highly supersonic. Typical accretion velocities $|\bar{u}_r| \sim \bar{\nu}/r \sim \alpha c_s(H/r) \ll c_s$: highly subsonic. Thus $|\bar{u}_r| \ll c_s \ll r\Omega$.

We assumed earlier that $u_\phi = r\Omega(r)$; this is justified by showing that the relative contributions of the pressure gradient to the centripetal force is

$$\frac{\partial p/\partial r}{\rho r\Omega^2} \sim \frac{c_s^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2 \ll 1 \qquad \Rightarrow \qquad u_\phi = r\Omega\left[1 + \mathcal{O}(\frac{H}{r})^2\right]$$

There are three important timescales in astrophysical disks:

- **Dynamical timescale:** $t_d \sim \Omega^{-1} \sim H/c_s$. This is the timescale of orbital motion, and the timescale over which vertical hydrostatic equilibrium is established.
- **Thermal timescale:** $t_T \sim$ (internal energy / area)/(dissipation rate / area). We can write this as $\sim P/\bar{\nu}\Sigma\Omega^2 \sim c_s^2/\bar{\nu}\Omega^2 \sim c_s/\alpha H\Omega^2 \sim 1/\alpha\Omega \sim \alpha^{-1}t_d > t_d$. This is the timescale over which vertical thermal balance is established.
- **Viscous timescale:** $t_\nu \sim r^2/\bar{\nu} \sim \alpha^{-1}(H/r)^{-2}t_d \gg t_T$. This is the timescale over which matter moves radially in the disk, and hence Σ evolves.

Thin disks with $\alpha < 1$ thus have $t_d < t_T \ll t_\nu$; all three timescales also increase with r.

2.6 Instabilities

2.6.1 Viscous Instability

Suppose we find a solution to the general time-dependent diffusion equation $\Sigma_0(r,t)$. Is it a stable solution? Linearising the equation (substituting $\Sigma = \Sigma_0 + \Sigma'$ where $\Sigma' \ll \Sigma_0$) gives:

$$\frac{\partial \Sigma'}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\beta r^{1/2}\bar{\nu}\Sigma'\right)\right], \qquad \beta \equiv \frac{1}{\bar{\nu}}\left.\frac{\partial(\bar{\nu}\Sigma)}{\partial \Sigma}\right|_{\Sigma_0} = \left.\frac{\partial \ln(\bar{\nu}\Sigma)}{\partial \ln \Sigma}\right|_{\Sigma_0}$$

The factor β thus alters the effective diffusivity of the surface density. If $\beta < 0$, the disk becomes anti-diffusive, with all the mass piling into a δ function. This viscous instability will cause the disk to break apart into rings. Viscous instability is typically much less important than thermal instability.

2.6.2 Thermal Instability

Suppose $\alpha \ll 1$, so that $t_d \ll t_T \ll t_\nu$. We will now work on thermal timescales, thus assuming the disk to be in vertical hydrostatic equilibrium and with Σ constant in time.

The heating and cooling rates $\mathcal{H}$ and $\mathcal{C}$ are likely to be functions of $\bar{\nu}$ and Σ, or equivalently $\bar{\nu}\Sigma$ and Σ; in fact $\mathcal{H} = \mathcal{H}(\bar{\nu}\Sigma)$ alone. The curve on which $\mathcal{H}(\bar{\nu}\Sigma) = \mathcal{C}(\bar{\nu}\Sigma, \Sigma)$ defines a curve in the $\bar{\nu}\Sigma$-Σ plane, a small step along which has $\mathrm{d}\mathcal{H} = \mathrm{d}\mathcal{C}$:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}(\bar{\nu}\Sigma)}\mathrm{d}(\bar{\nu}\Sigma) = \frac{\partial \mathcal{C}}{\partial(\bar{\nu}\Sigma)}\mathrm{d}(\bar{\nu}\Sigma) + \frac{\partial \mathcal{C}}{\partial \Sigma}\mathrm{d}\Sigma \qquad \Rightarrow \qquad \frac{\partial}{\partial(\bar{\nu}\Sigma)}(\mathcal{H} - \mathcal{C}) = \frac{\partial \Sigma}{\partial(\bar{\nu}\Sigma)}\frac{\partial \mathcal{C}}{\partial \Sigma} = \frac{1}{\beta\bar{\nu}}\frac{\partial \mathcal{C}}{\partial \Sigma}$$

Suppose some heat is added to the system, on thermal timescales t_T. Σ is unable to readjust on these timescales, so there must be a change in $\bar{\nu}\Sigma$ – this change is in practice an increase, so $\bar{\nu}\Sigma$ increases. If $\mathcal{H} - \mathcal{C}$ increases as a result of $\bar{\nu}\Sigma$ increasing (i.e. if $\partial(\mathcal{H} - \mathcal{C})/\partial(\bar{\nu}\Sigma) > 0$), then the system will heat up more and become unstable. The thermal instability therefore occurs if

$$\frac{1}{\beta\bar{\nu}}\frac{\partial \mathcal{C}}{\partial \Sigma} > 0 \qquad \Rightarrow \qquad \beta < 0$$

where we have assumed[3] $\left.\partial\mathcal{C}/\partial\Sigma\right|_{\bar{\nu}\Sigma} > 0$. This is the same condition as for viscous instability, but VI occurs on viscous timescales, so if $\beta < 0$ then the thermal instability dominates.

Some non-examinable work gives $\bar{\nu} \propto r\Sigma^{2/3}$ in a disk with Thomson opacity, so that $\beta = 5/3$ and the disk is thermally and viscously stable. Cooler disks may instead have S-shaped $\bar{\nu}\Sigma$–Σ cooling balance curves, with an intermediate instability region. Apparently this leads to limit cyclic behaviour. Suppose we start on the lower ("cool") branch of the S and Σ is slowly rising. When we reach the corner, β becomes negative and the system becomes thermally unstable, heating up a lot and quickly increasing $\bar{\nu}\Sigma$, jumping up to the top branch (on which $\beta > 0$ again) over a thermal timescale. Apparently Σ then decreases until the other corner is turned and $\beta < 0$ again, at which point we jump down quickly to the cool branch again. Not fully sure of the mechanisms here, but this cycle might be causing periodic outbursts as seen in cataclysmic variable stars and X-ray binaries.

[3] This assumption comes from $\Sigma \sim \bar{\nu}^{-1} \propto (\alpha T)^{-1}$, so $\left.\partial\Sigma/\partial T\right|_{\bar{\nu}\Sigma} < 0$, and the fact that typically $\partial\mathcal{C}/\partial T > 0$

3 Shearing Sheet

3.1 Local Model

We now zoom in to a small patch of the disk at a radius r_0, and change frames to one instantaneously orbiting at angular frequency Ω. About this point, we prescribe the radial coordinate $x = r - r_0$ and azimuthal coordinate $y = r_0(\phi - \Omega t)$ as locally Cartesian coordinates. Substituting these coordinates into the Lagrangian, it will become to second order

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r,z) \\ &= \frac{1}{2}r_0^2 - \Phi(r_0,0) + r_0\Omega\dot{y} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + 2\Omega x\dot{y} + \frac{1}{2}(\Omega^2 - \partial^2_{rr0}\Phi)x^2 - \frac{1}{2}(\partial^2_{zz0}\Phi)z^2\end{aligned}$$

where a derivative with subscript 0 is evaluated at $(r_0, 0)$ and we have cancelled $r_0\Omega^2$ with $\partial_{r0}\Phi$. In the Euler-Lagrange equations, the constant terms will be differentiated and not contribute; the only linear term is $r_0\Omega\dot{y}$, but this will only enter in the term $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{y}}$, which is 0. Thus the only part of this Lagrangian which will contribute to the motion will be

$$\mathcal{L}_2 = \underbrace{\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + 2\Omega x\dot{y} + \frac{1}{2}(\Omega^2 - \partial^2_{rr0}\Phi)x^2}_{\mathcal{L}_h} + \underbrace{\frac{1}{2}\dot{z}^2 - \frac{1}{2}(\partial^2_{zz0}\Phi)z^2}_{\mathcal{L}_z}$$

Where we highlight that this separates into horizontal and vertical Lagrangians $\mathcal{L}_h$ and $\mathcal{L}_z$, so the vertical motion is decoupled from the x, y. The Euler-Lagrange equations then give:

$$\Rightarrow \ddot{x} = 2\Omega\dot{y} + \left(\Omega^2 - \partial^2_{rr0}\Phi\right)x; \qquad \ddot{y} + 2\Omega\dot{x} = 0; \qquad \ddot{z} = -(\partial^2_{zz0}\Phi)z$$

The Φ derivatives are a bit annoying, so we evaluate them here. Radial force balance gives $\partial_{r0}\Phi = r\Omega^2 \Rightarrow \partial^2_{rr0}\Phi = \Omega^2 + 2\Omega r\frac{\mathrm{d}\Omega}{\mathrm{d}r} = \Omega^2 - 2\Omega S$. We can also write $\partial^2_{zz0}\Phi = \Omega_z^2$. Thus

$$\boxed{\ddot{x} - 2\Omega\dot{y} = 2\Omega S x} \qquad \boxed{\ddot{y} + 2\Omega\dot{x} = 0} \qquad \boxed{\ddot{z} = -\Omega_z^2 z}$$

These equations can also be thought of as emerging from a combination of Coriolis force and tidal potential $\Phi_t(x,z) = -\Omega S x^2 + \frac{1}{2}\Omega_z^2 z^2$. Recalling that $S \equiv -r\,\mathrm{d}\Omega/\mathrm{d}r > 0$, the tidal potential thus has a saddle point at the (x, y, z) origin.

Circular midplane orbits correspond to trajectories with constant x and $z = 0$. The first equation shows that for such trajectories $\dot{y} = -Sx$: adjacent orbits move at different speeds, and the system shears at a rate $-S$.

The general solution makes use of the fact that $p_y \equiv \frac{\partial\mathcal{L}_2}{\partial\dot{y}} = \dot{y} + 2\Omega x$ is a constant in time, as is clear from the second equation. Substituting p_y into the x-Euler-Lagrange equation gives

$$\ddot{x} - 2\Omega(p_y - 2\Omega x) = 2\Omega S x \qquad \Rightarrow \qquad \ddot{x} + \underbrace{2\Omega(2\Omega - S)}_{\Omega_r^2\text{, see §1}}x = 2\Omega p_y$$

$$\Rightarrow x(t) = \frac{2\Omega p_y}{\Omega_r^2} + \Re\left[Ae^{-i\Omega_r t}\right] \qquad \Rightarrow \qquad y(t) = y_0 - S\frac{2\Omega p_y}{\Omega_r^2}t + \Re\left[\frac{2\Omega A}{i\Omega_r}e^{-i\Omega_r t}\right]$$

where we note that y only enters the equations of motion as its derivatives so it always has an unspecified integration constant, reflecting the azimuthal symmetry of the disk. Note also that the coefficients of the exponentials are $\pi/2$ out of phase. These solutions therefore correspond to oscillations about orbital motion with frequency Ω_r; there may also be vertical oscillations, which are clearly at frequency Ω_z.

Other than p_y, two other conserved quantities are the horizontal and vertical Hamiltonians:

$$\varepsilon_h \equiv \sum_i \dot{q}_i \frac{\partial \mathcal{L}_h}{\partial \dot{q}_i} - \mathcal{L}_h = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \Omega S x^2 \qquad \varepsilon_z \equiv \sum_i \dot{q}_i \frac{\partial \mathcal{L}_z}{\partial \dot{q}_i} - \mathcal{L}_z = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\Omega_z^2 z^2$$

p_y and $\varepsilon_h + \varepsilon_z$ can be related to the global constants h and $\varepsilon - \Omega h$ respectively by expanding to second order in (x, y, z).

Aside from azimuthal symmetry, the disk also has vertical symmetry ($z \mapsto -z$), translational symmetry in x if accompanied by Galilean boost in y ($x \mapsto x+c; y \mapsto y-cSt$), rotational symmetry about the z-axis ($x \mapsto -x; y \mapsto -y$: the local model doesn't know where the centre is) and scale invariance ($\mathbf{x} \mapsto c\mathbf{x}$; we are looking over scales much smaller than r_0 to linearise, so c cancels out by design).

3.2 Satellites

With a second massive body on a circular orbit ($x = y = 0$), the motion of test particles is dominated by the gravity of two bodies. The equations of motion in the plane become

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial \Psi}{\partial x} \qquad \ddot{y} + 2\Omega\dot{x} = -\frac{\partial \Psi}{\partial y}$$

where $\Psi = -GM_s/\sqrt{x^2 + y^2}$ is the satellite potential.

3.2.1 Epicyclic Excitation

Recall the general solution in the absence of Ψ, which we showed in the previous section to be

$$x(t) = x_0 + \Re\left[Ae^{-i\Omega_r t}\right], \qquad y(t) = y_0 - Sx_0 t + \Re\left[\frac{2\Omega}{i\Omega_r} Ae^{-i\Omega_r t}\right]; \qquad x_0 = \frac{2\Omega p_y}{\Omega_r^2}$$

Note then also that

$$\dot{x} = \Re\left[-i\Omega_r Ae^{-i\Omega_r t}\right] = \Omega_r \Im\left[Ae^{-i\Omega_r t}\right], \qquad \dot{y} = -Sx_0 - 2\Omega\Re\left[Ae^{-i\Omega_r t}\right]$$

The epicyclic amplitude A is then given by

$$Ae^{-i\Omega_r t} = \Re\left[Ae^{-i\Omega_r t}\right] + i\Im\left[Ae^{-i\Omega_r t}\right] = -\frac{\dot{y} + Sx}{2\Omega - S} + \frac{i}{\Omega_r}\dot{x} \quad \Rightarrow \quad A = \left[-\frac{\dot{y} + Sx}{2\Omega - S} + \frac{i}{\Omega_r}\dot{x}\right]e^{i\Omega_r t}$$

$$\begin{aligned}
\Rightarrow \dot{A} &= \left[-\frac{\ddot{y} + S\dot{x}}{2\Omega - S} + \frac{i}{\Omega_r}\ddot{x} + i\Omega_r\left(-\frac{\dot{y} + Sx}{2\Omega - S} + \frac{i}{\Omega_r}\dot{x}\right)\right]e^{i\Omega_r t} \\
&= \left[-\frac{\ddot{y} + 2\Omega\dot{x}}{2\Omega - S} + i\frac{\ddot{x} - 2\Omega\dot{y} - 2\Omega Sx}{\Omega_r}\right]e^{i\Omega_r t} = \left[\frac{1}{2\Omega - S}\frac{\partial \Psi}{\partial y} - \frac{i}{\Omega_r}\frac{\partial \Psi}{\partial x}\right]e^{i\Omega_r t}
\end{aligned}$$

where we have substituted in from the equations of motion. Consider a test particle initially travelling on an unperturbed circular orbit at impact parameter x_0, such that without Ψ the particle would just move at constant x_0 and $y = -Sx_0t$ (corresponding to $A = 0$). The satellite will however induce a small epicyclic perturbation, of amplitude ΔA, which we now deduce. Substituting in Ψ, we find

$$\dot{A} = \frac{GM_s}{(x^2+y^2)^{3/2}}\left[\frac{y}{2\Omega - S} - \frac{ix}{\Omega_r}\right]e^{i\Omega_r t} \approx -i\frac{GM_s}{\Omega_r x_0^2}\left(1+S^2t^2\right)^{-3/2}\left(1 - i\frac{2\Omega}{\Omega_r}St\right)e^{i\Omega_r t}$$

where we have assumed that x and y will not deviate too much from their unperturbed values and simplified using the definition of Ω_r. The total ΔA will be the integral of $\dot{A}$ from $t = -\infty$ to ∞, and from inspection the real part of $\dot{A}$ will integrate out, leaving

$$\begin{aligned}\Delta A &= -i\frac{GM_s}{\Omega_r x_0^2}\int_{-\infty}^{\infty}\left(1+S^2t^2\right)^{-3/2}\left(\cos(\Omega_r t) + \frac{2\Omega S}{\Omega_r}t\sin(\Omega_r t)\right)\mathrm{d}t \\ &= -i\frac{GM_s}{\Omega_r x_0^2 S}\underbrace{\left[f(\Omega_r/S) - \frac{2\Omega}{\Omega_r}f'(\Omega_r/S)\right]}_{C=3.36\text{ for Keplerian}}; \qquad f(k) \equiv \int_{-\infty}^{\infty}(1+x^2)^{-3/2}\cos(kx)\,\mathrm{d}x\end{aligned}$$

The energy is related to $|A|^2$. In the unperturbed case, recall that $A = \left[-\frac{\dot{y}+Sx}{2\Omega-S} + \frac{i}{\Omega_r}\dot{x}\right]e^{i\Omega_r t}$. A lot of algebra gives:

$$\begin{aligned}|A|^2 &= \frac{1}{(2\Omega-S)^2}\left(\dot{y}^2 + 2Sx\dot{y} + S^2x^2\right) + \frac{1}{\Omega_r^2}\dot{x}^2 = \frac{1}{\Omega_r^2}\left[\left(\dot{x}^2 + \dot{y}^2 - 2\Omega Sx^2\right) + \frac{2\Omega S}{\Omega_r^2}\left(\dot{y}+2\Omega x\right)^2\right] \\ &= \frac{1}{\Omega_r^2}\left[2\varepsilon_h + \frac{2\Omega S}{\Omega_r^2}p_y^2\right] \qquad \Rightarrow \qquad \varepsilon_h = \frac{1}{2}\Omega_r^2|A|^2 - \frac{\Omega S}{\Omega_r^2}p_y^2\end{aligned}$$

The quantity ε_h is preserved for unperturbed orbits as A is constant.

For perturbed orbits, the conserved quantity is instead $\varepsilon_h + \Psi$. As $t \to \pm\infty$, $\Psi \to 0$, so to preserve $\varepsilon_h + \Psi$ the perturbation also cannot change ε_h. The value of p_y must thus change according to

$$\Delta\left(p_y^2\right) = \frac{{\Omega_r}^4}{2\Omega S}\Delta\left(|A|^2\right) = 2p_y\Delta p_y$$

to keep $\varepsilon_h = 0$. Now a particle on an unperturbed circular orbit has $A = 0$ and

$$p_y = \dot{y} + 2\Omega x = (2\Omega - S)x_0 = \frac{\Omega_r^2}{2\Omega}x_0$$

$$\Rightarrow \Delta p_y = \frac{1}{2}\frac{2\Omega}{\Omega_r^2 x_0}\frac{{\Omega_r}^4}{2\Omega S}\Delta\left(|A|^2\right) = \frac{\Omega_r^2}{2Sx_0}\left(\frac{CGM_s}{\Omega_r x_0^2 S}\right)^2 = C^2\frac{(GM_s)^2}{2S^3x_0^5}$$

This could be derived similarly using the *impulse approximation*. The change in x-velocity is of order $(-GM_s/x_0^2) \times (1/S)$: the acceleration $\times$ the time (roughly). Conservation of energy requires $\Delta(v_x^2) + \Delta\left(v_y^2\right) = 0$ across the whole encounter, so using the fact that initially $v_y = -Sx_0$, we have

$$0 = \left(\frac{GM_s}{x_0^2 S}\right)^2 - 2Sx_0\Delta v_y \qquad \Rightarrow \qquad \Delta v_y = -\frac{(GM_s)^2}{2S^3x_0^5}$$

all that is missing is the dimensionless $C^2 \approx 11.3$.

3.2.2 Interactions with the Disk

At a radius x, the azimuthal force per unit x is proportional to

$$\begin{aligned}&\frac{\text{velocity change}}{\text{particle}} \times \frac{\text{number of particles} \times \text{particle mass}}{\Delta x} \times \text{encounter rate}\\ &\sim \frac{(CGM_s)^2}{2S^3x^5} \times \frac{\Sigma}{\Delta x} \times |Sx\Delta x| \quad \propto \quad +x^{-4}\mathrm{sgn}(x)\end{aligned}$$

And hence particles on the outside *gain* y-momentum, encouraging them to move further out. Conversely, particles on the inside *lose* momentum, encouraging them to spiral in. The satellite's angular momentum change by an equal and opposite amount; the inner and outer disk torques usually balance out to cause the satellite to net lose angular momentum, and migrate inwards.

Dust moves through the gas present in the disk, experiencing a drag force. This modifies the equations of motion to

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x + \gamma(u_x - \dot{x}), \qquad \ddot{y} + 2\Omega\dot{x} = \gamma(u_y - \dot{y}), \qquad \ddot{z} = -\Omega_z^2 z + \gamma(u_z - \dot{z})$$

where $\mathbf{u}$ is the gas velocity and γ is a drag coefficient. The vertical oscillations are clearly damped, as are the planar oscillations in the case that $\mathbf{u} = -Sx\hat{\mathbf{e}}_y$.

If we allow deviations for the gas velocity $\mathbf{u} = [-Sx + v_y(x)]\hat{\mathbf{e}}_y$, and the dust velocity $\dot{\mathbf{x}} = -Sx\hat{\mathbf{e}}_y + \mathbf{w}$, the equations of motion become

$$\dot{w}_x = 2\Omega w_y - \gamma w_x \qquad \dot{w}_y + (2\Omega - S)w_x = \gamma(v_y - w_y) \qquad \dot{w}_z = -\Omega_z^2 z - \gamma w_z$$

A steady-state solution to this is $w_z = z = 0$ in the z-direction, and $w_y = \frac{\gamma}{2\Omega}w_x$ and $w_x = \frac{\gamma}{2\Omega - S}(v_y - w_y)$ in the plane. This solves to give

$$w_x = \frac{2\Omega\gamma}{\Omega_r^2 + \gamma^2}v_y \qquad w_y = \frac{\gamma^2}{\Omega_r^2 + \gamma^2}v_y$$

If the Stokes number $\sim \Omega_r/\gamma$ is much less than 1, then $w_x \approx 0$ and $w_y \approx v_y$. Maximum w_x (radial motion, note that likely $v_y < 0$ due to gas pressure) is achieved for $\Omega_r \approx \gamma$, under which circumstances the dust should fall into the centre in a matter of centuries, unless some non-monotonicities in the pressure $P(x)$ are able to affect v_y.

3.3 Hydrodynamics

We now explore how fluids behave in local coordinates. The equation of motion in the rotating frame is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla\Phi_t - \frac{1}{\rho}\nabla p + \bar{\nu}\nabla^2\mathbf{u}$$

where we have assumed uniform $\bar{\nu}$. For an incompressible fluid ($\nabla\rho = \mathbf{0} \Rightarrow \nabla \cdot \mathbf{u} = 0$), the basic shear flow is a velocity field $\mathbf{u}_0 = -Sx\hat{\mathbf{e}}_y$. In this state, $\nabla\Phi_t$ balances $2\boldsymbol{\Omega} \times \mathbf{u}$ in the plane and $\frac{1}{\rho}\nabla p(z)$ in the z-direction. Consider general perturbations to this shear flow: $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}(\mathbf{x}, t)$ and $p = p_0 + \rho\psi(\mathbf{x}, t)$. Substituting these into the equation of motion gives

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v} \cdot \nabla + 2\boldsymbol{\Omega}\times\right)\mathbf{v} - v_x S\hat{\mathbf{e}}_y = -\nabla\psi + \bar{\nu}\nabla^2\mathbf{v} \qquad \nabla \cdot \mathbf{v} = 0$$

$$
\begin{aligned}
\Rightarrow \left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y &= -\frac{\partial \psi}{\partial x} + \bar{\nu}\boldsymbol{\nabla}^2 v_x \\
\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x &= -\frac{\partial \psi}{\partial y} + \bar{\nu}\boldsymbol{\nabla}^2 v_y \\
\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_z &= -\frac{\partial \psi}{\partial z} + \bar{\nu}\boldsymbol{\nabla}^2 v_z
\end{aligned}
$$

Notice that the Lagrangian derivatives on the LHSs show advection due to both $\mathbf{u}_0$ and $\mathbf{v}$.

3.3.1 Shearing Waves & Hydrodynamic Instability

Consider plane wave solutions to these equations with a generally time-dependent real wavevector $\mathbf{k}(t)$. Substituting $\mathbf{v}(\mathbf{x},t) = \Re\left[\tilde{\mathbf{v}}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right]$, into the advection term (in vector form),

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} &= \Re\left[\left(\dot{\tilde{\mathbf{v}}} + i\left(\dot{\mathbf{k}}\cdot\mathbf{x}\right)\tilde{\mathbf{v}} - iSxk_y\tilde{\mathbf{v}}\right)e^{i\mathbf{k}\cdot\mathbf{x}}\right] + \Re\left[\tilde{\mathbf{v}}e^{i\mathbf{k}\cdot\mathbf{x}}\right]\cdot\Re\left[i\mathbf{k}\tilde{\mathbf{v}}e^{i\mathbf{k}\cdot\mathbf{x}}\right] \\
&= \Re\left[\left(\dot{\tilde{\mathbf{v}}} + i\left(\dot{\mathbf{k}}\cdot\mathbf{x} - Sk_yx\right)\tilde{\mathbf{v}}\right)e^{i\mathbf{k}\cdot\mathbf{x}}\right] + \Re\Big[\underbrace{\mathbf{k}\cdot\tilde{\mathbf{v}}}_{0\because\boldsymbol{\nabla}\cdot\mathbf{v}=0}e^{i\mathbf{k}\cdot\mathbf{x}}\Big]\Re\left[i\tilde{\mathbf{v}}e^{i\mathbf{k}\cdot\mathbf{x}}\right]
\end{aligned}
$$

where to go to the second line we have used $\mathbf{k}\in\mathbb{R}^3$ to move it through the $\Re$ brackets and kill the 2nd term, though this $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ term will not generally disappear for incompressible fluids or superpositions of multiple plane waves. We now look for simple solutions in which $\mathbf{k}$ is such that the innermost bracket in the first term goes to 0 (for all $\mathbf{x}$). This requires

$$
\dot{k}_x = Sk_y, \qquad \dot{k}_y = \dot{k}_z = 0 \qquad \Rightarrow \qquad k_x(t) = k_{x0} + Sk_yt, \qquad k_y, k_z = \text{consts.}
$$

so k_x is the only time-dependent part of $\mathbf{k}(t)$. What this looks like in practice is a tilting of the wavefronts – a wave with $k_x = 0$ initially will have its (horizontal) wavefronts sheared, causing azimuthal oscillations to gradually converge to radial oscillations.

Anyway the advective term now becomes just $\Re\left[\dot{\tilde{\mathbf{v}}}e^{i\mathbf{k}\cdot\mathbf{x}}\right]$. The equations of motion are then:

$$
\begin{aligned}
\dot{\tilde{v}}_x - 2\Omega\tilde{v}_y &= -ik_x\tilde{\psi} - \bar{\nu}k^2\tilde{v}_x \\
\dot{\tilde{v}}_y + (2\Omega - S)\tilde{v}_x &= -ik_y\tilde{\psi} - \bar{\nu}k^2\tilde{v}_y \qquad\qquad k_x\tilde{v}_x + k_y\tilde{v}_y + k_z\tilde{v}_z = 0 \\
\dot{\tilde{v}}_z &= -ik_z\tilde{\psi} - \bar{\nu}k^2\tilde{v}_z
\end{aligned}
$$

The $\bar{\nu}$ terms can be taken over to the LHS and simplified with an integrating factor:

$$
\frac{\mathrm{d}\tilde{v}_i}{\mathrm{d}t} + \bar{\nu}k^2\tilde{v}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{v}_i\exp\left(\int\bar{\nu}k^2\,\mathrm{d}t\right)\right]\exp\left(-\int\bar{\nu}k^2\,\mathrm{d}t\right)
$$

where $k(t)\sim Sk_yt$ for large t. Writing $\tilde{\mathbf{v}}\equiv\hat{\mathbf{v}}\exp\left(-\int\bar{\nu}k^2\,\mathrm{d}t\right)$, and similarly for ψ, we now have

$$
\dot{\hat{v}}_x - 2\Omega\hat{v}_y = -ik_x\hat{\psi} \qquad\qquad \dot{\hat{v}}_y + (2\Omega - S)\hat{v}_x = -ik_y\hat{\psi} \qquad\qquad \dot{\hat{v}}_z = -ik_z\hat{\psi}
$$

Together with the time derivative of $\mathbf{k}\cdot\mathbf{x} = 0$, these equations give after a page of algebra $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2v_x) + \Omega_r^2k_z^2v_x = 0$. For $k_y = 0$ (axisymmetric disturbances), $\hat{\mathbf{v}}$ will have exponential instability for $\Omega_r^2 < 0$ (which may supercede the viscous damping exponential decay), or stable oscillations if $\Omega_r^2 > 0$. Apparently for non-axisymmetric disturbances the growths/decays are algebraic. Keplerian disks, with $\Omega_r^2 > 0$, are expected to be stable to this hydrodynamic instability.

3.3.2 Vortices

Consider the incompressible shearing sheet in 2 dimensions:

$$\Rightarrow \left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \bar{\nu}\boldsymbol{\nabla}^2 v_x$$
$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \bar{\nu}\boldsymbol{\nabla}^2 v_y$$

together with $\partial v_x/\partial x + \partial v_y/\partial y = 0$. We define the streamfunction $\chi(x, y, t)$ by $v_x = \partial\chi/\partial y$ and $v_y = -\,\partial\chi/\partial x$, automatically satisfying incompressibility. The perturbation to the vorticity $\boldsymbol{\nabla}\times\mathbf{u}$ is then

$$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{e}}_z = -\boldsymbol{\nabla}^2\chi\hat{\mathbf{e}}_z \equiv \zeta\hat{\mathbf{e}}_z$$

Taking the curl of the 2D equation of motion ($\frac{\partial}{\partial x}$[$y$-component] $-$ $\frac{\partial}{\partial y}$[$x$-component]) above removes the $\boldsymbol{\nabla}\psi$ terms and many others[4], leaving a diffusion equation in ζ:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\zeta = \bar{\nu}\boldsymbol{\nabla}^2\zeta$$

Together with Poisson's equation $\boldsymbol{\nabla}^2\chi = -\zeta$ and the definition of χ, we can solve for $\mathbf{v}$ and ζ.

Zonal flows are axisymmetric ($\partial/\partial y = 0$) and so have $v_x = \partial\chi/\partial y = 0$ and hence the above becomes $\frac{\partial\zeta}{\partial t} = \bar{\nu}\frac{\partial^2\zeta}{\partial x^2}$: a diffusion equation with diffusivity $\bar{\nu}$.

Ansatzing $\zeta(\mathbf{x}, t) = \Re\left[\tilde{\zeta}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right]$ into the above, assuming as above that $\dot{\mathbf{k}}\cdot\mathbf{x} = Sk_y x$ to get that one term to cancel, and using the fact that if $\zeta \propto e^{i\mathbf{k}\cdot\mathbf{x}}$ then so will $\mathbf{v}$ and hence $\mathbf{k}\cdot\mathbf{v} = 0$ as before, we eventually find simply that $\dot{\tilde{\zeta}} = -\bar{\nu}k^2\tilde{\zeta}$. Hence

$$\tilde{\zeta}(t) \propto \exp\left(-\int \bar{\nu}k^2\,\mathrm{d}t\right)$$

Now $\tilde{\chi} = \tilde{\zeta}/k^2$, and $|\tilde{\mathbf{v}}| \propto k\tilde{\chi}$, so $|\tilde{\mathbf{v}}|^2 \propto \exp\left(-\int \bar{\nu}k^2\,\mathrm{d}t\right)/k^2$. Recalling that $k^2(t)$ is a quadratic, $|\tilde{\mathbf{v}}|^2$ may increase for some time (depending on the (initial) wavevector) before being damped away by the viscous exponential.

If a question comes up in the exam about rotating vortex patches, I'm not answering it.

3.4 Density Waves

We now relax the assumptions of incompressibility and no self-gravity, but neglect viscosity. Still working in 2D, mass conservation becomes:

$$\frac{\partial\Sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\Sigma\mathbf{u}) = 0$$

and the equation of motion becomes

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}\phi_t - \boldsymbol{\nabla}\phi_d - \frac{1}{\Sigma}\boldsymbol{\nabla}P$$

where $P = \int \rho\,\mathrm{d}x$, $\phi_t = -\Omega S x^2$ is the midplane tidal potential, and $\phi_d = \Phi_d(x, y, 0, t)$ is the midplane of the 3D disk self-gravity potential, which satisfies $\boldsymbol{\nabla}^2\Phi_d = 4\pi G\Sigma\delta(z)$.

[4]There's a term that I'm not sure how to cancel in vector form; it might be easiest just to evaluate the derivatives of the components rather than doing some cool vector shit

3.4.1 Conservation of Vortensity

Using $(\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u} = \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \frac{1}{2}\boldsymbol{\nabla}\left(|\mathbf{u}|^2\right)$, and assuming a barotropic equation of state $P = P(\Sigma)$, we can rewrite this equation of motion as

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\nabla}\times\mathbf{u} + 2\boldsymbol{\Omega})\times\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{u}|^2 - \phi_t - \phi_d - \int \frac{\mathrm{d}P}{\Sigma}\right)$$

Curling kills the $\boldsymbol{\nabla}$: $$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{u}) = -\boldsymbol{\nabla}\times\left[(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})\times\mathbf{u}\right]$$

Now in this analysis, $\mathbf{u}$ is in the x-y plane, but $\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}$ is in the z-direction, so many of the terms will vanish when this is expanded. What remains is

$$\begin{aligned}
\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{u}) &= -(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})(\boldsymbol{\nabla}\cdot\mathbf{u}) - (\mathbf{u}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}) \\
\Rightarrow \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}) &= (\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})\frac{\partial\Sigma/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}\Sigma}{\Sigma} \\
\Rightarrow 0 &= \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}) - \frac{(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})}{\Sigma}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\Sigma \\
&= \Sigma\left[\frac{1}{\Sigma}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}) - \frac{(\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})}{\Sigma^2}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\Sigma\right] \\
\Rightarrow \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)&\left(\frac{\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega}}{\Sigma}\right) = 0
\end{aligned}$$

So the quantity in the brackets, $f = (\boldsymbol{\nabla}\times\mathbf{u}+2\boldsymbol{\Omega})/\Sigma$, known as the *vortensity*, is conserved.

3.4.2 Density Wave Dispersion Relation

In 2D Fourier space, where $\tilde{\square}(k_x,k_y,z,t) = \iint \square(x,y,z,t)e^{-ik_x x}e^{-ik_y y}\,\mathrm{d}x\,\mathrm{d}y$ and $k^2 = k_x^2 + k_y^2$, Poisson's equation for the disk potential gives

$$\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{\Phi}_d = 4\pi G\tilde{\Sigma}\delta(z) \qquad \Rightarrow \qquad \tilde{\Phi}_d = -\frac{2\pi G\tilde{\Sigma}}{k}e^{-k|z|} \qquad \Rightarrow \qquad \tilde{\phi}_d = -\frac{2\pi G\tilde{\Sigma}}{k}$$

Consider now a uniform base state of a disk, with Σ, P = consts. and $\mathbf{u} = -Sx\hat{\mathbf{e}}_y$. In terms of perturbations $\mathbf{v}$, Σ', P' and Φ'_d, the linearised forms of conservation of mass, Poisson's equation, and the equation of motion become

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\Sigma' + \Sigma\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \qquad \boldsymbol{\nabla}^2\Phi'_d = 4\pi G\Sigma'\delta(z),$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\mathbf{v} - Sv_x\hat{\mathbf{e}}_y + 2\boldsymbol{\Omega}\times\mathbf{v} = -\boldsymbol{\nabla}\phi'_d - \frac{c_s^2}{\Sigma}\boldsymbol{\nabla}\Sigma'$$

where $P' = v_s^2\Sigma'$, so $v_s^2 = \mathrm{d}P/\mathrm{d}\Sigma$, the adiabatic sound speed of the base state, is a constant. These can be solved with shearing waves as before: substituting $\Sigma' = \mathfrak{R}\left[\tilde{\Sigma}'(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right]$ and similarly for $\mathbf{v}$ and ϕ'_d, and letting $\dot{\mathbf{k}}\cdot\mathbf{x} = Sk_y x$ as before, these equations become

$$\dot{\tilde{\Sigma}}' + i\Sigma\mathbf{k}\cdot\tilde{\mathbf{v}} = 0, \qquad \tilde{\phi}'_d = -\frac{2\pi G\tilde{\Sigma}'}{k}, \qquad \dot{\tilde{\mathbf{v}}} - S\tilde{v}_x\hat{\mathbf{e}}_y + 2\boldsymbol{\Omega}\times\tilde{\mathbf{v}} = -i\mathbf{k}\left(\tilde{\phi}'_d + \frac{v_s^2}{\Sigma}\tilde{\Sigma}'\right)$$

Specialise now in axisymmetric (radial) waves, where $k_y = 0$ and hence $k = |k_x| = \text{const}$. The only time-dependent quantities in the equations become the amplitudes $\tilde{\Sigma}'$ and so on, so we let these be proportional to $e^{-i\omega t}$, so that overall we are essentially looking at wavey perturbations $\propto e^{i(k_x x - \omega t)}$. Substituting and simplifying, we get a dispersion relation for ω:

$$\omega^2 = v_s^2 k^2 - 2\pi G \Sigma k + \Omega_r^2$$

Stable oscillations have $\omega^2 > 0$: we see that for short wavelengths the waves are approximately sound waves ($\omega^2 \approx v_s^2 k^2$), and for long wavelengths the waves are approximately inertial, following the global gravity ($\omega^2 \approx \Omega_r^2$). These oscillations in Σ' (*density waves*) are hence inertial-acoustic waves.

3.4.3 Gravitational Instability

In the intermediate wavelength regime, self-gravity may destabilise the disk, with the negative part proportional to Σ. If $\exists k_x : \omega^2 < 0$, then that k_x mode will grow exponentially. From the quadratic, we find that this gravitational instability (GI) can occur if the *Toomre parameter*:

$$Q \equiv \frac{\Omega_r v_s}{\pi G \Sigma}$$

falls below 1. This parameter quantifies the irrelevance of self-gravity: if much below 1 then it is very important and causes instabilities.

When an instability occurs, the surface density grows exponentially, creating rings. Differentiating the dispersion relation, the fastest-growing mode (most negative ω^2) occurs for $k_{\max} = \pi G \Sigma / v_s^2 = \Omega_r / Q v_s \sim 1/QH$, where H is the scale height. The length scale of gravitational instabilities will therefore be $2\pi / k_{\max} \sim 2\pi QH \sim 10H$.

Note that $Q \propto v_s \propto T^{1/2}$. If Q is initially stably high but then falls below a critical value which is about 2, it turns out that before an instability can occur, non-axisymmetric spiral density waves emerge. These waves apparently cause shocks and energy dissipation, raising the temperature, and raising Q. As such Q can stabilise at this value of about 2: the disk "thermostatically regulates".

4 Magnetic Fields

Now neglect both compressibility *and* viscosity. Under these conditions, the Maxwell equations and Ohm's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

together lead to the *induction equation*:

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{B} = \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{u} + \eta \boldsymbol{\nabla}^2 \mathbf{B}$$

where $\eta = 1/\mu_0 \sigma$. Per unit volume, the Lorentz force can be written as the divergence of the Maxwell stress tensor, which has components $M_{ij} = B_i B_j / \mu_0 - B_k B_k \delta_{ij} / 2\mu_0$. Hence

$$\boldsymbol{\nabla} \cdot \mathsf{M} = \frac{1}{\mu_0} \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B} - \boldsymbol{\nabla} \left(\frac{|\mathbf{B}|^2}{2\mu_0} \right)$$

is the Lorentz force/volume. The first term is *magnetic tension*; the second is due to magnetic pressure $p_{\mathbf{B}} \equiv |\mathbf{B}|^2/2\mu_0$. Adding these terms into the equation of motion in the local model,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla \Phi_t - \frac{1}{\rho}\nabla(p + p_{\mathbf{B}}) + \frac{1}{\mu_0 \rho}\mathbf{B} \cdot \nabla \mathbf{B}$$

We will only consider solutions to this and the induction equation which are horizontally invariant: $\partial/\partial x = \partial/\partial y = 0$, and hence $\partial u_z/\partial z = \partial B_z/\partial z = 0$ as $\mathbf{u}$ and $\mathbf{B}$ are divergenceless. We will write $\mathbf{u} = -Sx\hat{\mathbf{e}}_y + \mathbf{v}$ (so $\partial v_z/\partial z = 0$ also), and use the material derivative $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$. The components of the equation and motion and induction equation are then[5]:

$$\begin{aligned} \frac{\mathrm{D}v_x}{\mathrm{D}t} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_x}{\partial z} & \frac{\mathrm{D}B_x}{\mathrm{D}t} &= B_z \frac{\partial v_x}{\partial z} + \eta \frac{\partial^2 B_x}{\partial z^2} \\ \frac{\mathrm{D}v_y}{\mathrm{D}t} + (2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_y}{\partial z} & \frac{\mathrm{D}B_y}{\mathrm{D}t} &= B_z \frac{\partial v_y}{\partial z} + \eta \frac{\partial^2 B_y}{\partial z^2} - SB_x \\ \frac{\partial v_z}{\partial t} &= -\Omega_z^2 z - \frac{1}{\rho}\frac{\partial}{\partial z}(p + p_{\mathbf{B}}) & \frac{\partial B_z}{\partial t} &= 0 \end{aligned}$$

The final one of these gives that B_z is not only a constant in x and y, and not only z (as $\partial B/\partial z = 0$) but t as well: B_z is a complete constant. This reflects the uniform magnetic flux through the disk.

Consider now a disk with upper and lower surfaces $z = z_{\pm}(t)$, beyond which $p = 0$. Integrating the z-component of the equation of motion and using $\Sigma = \rho(z_+ - z_-)$, we can find

$$\frac{\partial^2}{\partial t^2}\left(\frac{z_+ + z_-}{2}\right) = -\Omega_z^2\left(\frac{z_+ + z_-}{2}\right) - \frac{|\mathbf{B}_+|^2 - |\mathbf{B}_-|^2}{2\mu_0 \Sigma}$$

– that is, the centre of mass oscillates about an equilibrium position which is likely $z = 0$.

Now suppose instead that $v_z = 0$. The equations of motion and induction become:

$$\begin{aligned} \frac{\partial v_x}{\partial t} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_x}{\partial z} & \frac{\partial B_x}{\partial t} &= B_z \frac{\partial v_x}{\partial z} + \eta \frac{\partial^2 B_x}{\partial z^2} \\ \frac{\partial v_y}{\partial t} + (2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_y}{\partial z} & \frac{\partial B_y}{\partial t} &= B_z \frac{\partial v_y}{\partial z} + \eta \frac{\partial^2 B_y}{\partial z^2} - SB_x \end{aligned}$$

which are completely linear equations as B_z is a constant.

4.0.1 Steady-State Solutions

Setting all the time derivatives to 0 and solving for B_x, we find

$$\frac{\mathrm{d}^2 B_x}{\mathrm{d}z^2} + K^2 B_x = 0, \qquad K^2 = \frac{2\Omega S v_{\mathrm{A}z}^2}{\Omega_r^2 \eta^2 + v_{\mathrm{A}z}^4}, \qquad \mathbf{v}_{\mathrm{A}} \equiv \frac{1}{\sqrt{\mu_0 \rho}}\mathbf{B}$$

where we have defined the Alfvén velocity $\mathbf{v}_{\mathrm{A}}$. The solutions to the ODE for B_x are of course sinusoids. A common set of boundary conditions is the symmetric $B_x = \pm B_x^+$ and $B_y = \pm B_y^+$ at $z = \pm z^+$. With this, the solution to the time-independent equations can be found. The

[5] A subtlety emerges when evaluating $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$, because $\mathbf{u}$ does actually vary in x thanks to the $-Sx\hat{\mathbf{e}}_y$ term, giving a couple of terms with $-S$ coefficients scattered about in the y-equations. $\nabla \mathbf{u} \neq \nabla \mathbf{v}$, so be careful.

strategy is to get $B_x(z)$, then v_y, then to eliminate v_x from equations 2 and 3 above to derive an equation in B_y'' and B_x'' and solve for B_y, then finally to get v_x.

$$B_x = \frac{B_x^+}{\sin(Kz^+)}\sin(Kz) \qquad \Rightarrow \qquad v_y = -\frac{v_{\mathrm{A}z}^2}{2\Omega B_z}\frac{KB_x^+}{\sin(Kz^+)}\cos(Kz)$$

$$\Rightarrow B_y = \frac{(2\Omega - S)\eta}{v_{\mathrm{A}z}^2}\frac{B_x^+}{\sin(Kz^+)}\left[\sin(Kz) - \frac{\sin(Kz^+)}{z^+}z\right] + \frac{B_y^+}{z^+}z$$

$$\Rightarrow v_x = -\frac{\eta B_x^+}{B_z\sin(Kz^+)}\left[K\cos(Kz) - \frac{\sin(Kz^+)}{z^+}\right] + \frac{v_{\mathrm{A}z}^2}{(2\Omega - S)B_z}\frac{B_y^+}{z^+}$$

The poloidal $\mathbf{B}$ field (the radial B_x and the constant vertical B_z) bends inwards/outwards (depending on the sign of the constant B_x^+). The orbital speed u_y is no longer Keplerian ($v_y = 0$), but instead has a z-directed shear. The mean accretion velocity $\langle v_x \rangle$ is simply due to the final term ($\langle [\dots] \rangle$ averages out here), and is proportional to $v_{\mathrm{A}z}^2 B_y^+/B_z \propto B_z B_y^+$, which is a magnetic torque on the surface of the disk; this can potentially drive inflows or outflows within the disk.

4.0.2 Magnetocentrifugal Acceleration

Consider a disk with a steady $\mathbf{B}$ field[6]. Outside the disk, we have $\rho \to 0$, so the (original!) equations of motion and induction give $\partial B_x/\partial z = 0$, $\partial B_y/\partial z = 0$, and $\partial v_x/\partial z = 0$, $\partial v_y/\partial z = SB_x/B_z$. Thus

$$B_x = B_x^+, \qquad B_y = B_y^+, \qquad v_x = \text{const.}, \qquad v_y = \frac{SB_x^+}{B_z}z + \text{const.}$$

The force/mass in the direction of the magnetic field is then

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\cdot\mathbf{B} = 2\Omega v_y\frac{B_x}{|\mathbf{B}|} - (2\Omega - S)v_x\frac{B_y}{|\mathbf{B}|} - \Omega_z^2 z\frac{B_z}{|\mathbf{B}|} \propto \left[2\Omega S\left(\frac{B_x^+}{B_z}\right)^2 - \Omega_z^2\right]z + \text{const.}$$

The force along $\mathbf{B}$ due to B_y is constant, the force due to B_x is $\propto z$, and the force due to B_z is $\propto -z$. The $\mathbf{B}$ field points out of the disk and particles are being funnelled along it – if their acceleration *increases* as they travel up along the field lines, then their motion will be exponential and a jet/wind will be launched. This occurs if

$$2\Omega S\left(\frac{B_x^+}{B_z}\right)^2 > \Omega_z^2 \qquad \Rightarrow \qquad \left(\frac{B_x^+}{B_z}\right)^2 > \frac{\Omega_z^2}{2\Omega S} \xrightarrow{\text{Keplerian}} \frac{1}{3}$$

So if $\mathbf{B}$ field hits the disk at an angle greater than $\tan^{-1}(1/\sqrt{3}) = 30°$ to the vertical, it can launch an outflow.

[6]Not too sure of the assumptions made in this section: We were just working with $v_z = 0$ but the result here is an outflow, so that might not be it. Without this assumption, going back to the original equations now has some convective terms that I don't know what to do with... This phenomenon was treated better in the AFD course.

4.1 Magnetorotational Instability

Returning to the equations with $v_z = 0$, we now linearise them with small perturbations, again with the form $\mathbf{v} = \Re\left[\tilde{\mathbf{v}} e^{\lambda t + ikz}\right]$, and similarly for $\tilde{\mathbf{B}}$, but now with constant amplitudes $\tilde{\square}$. Recalling that B_z is a constant, we find

$$\lambda \tilde{v}_x - 2\Omega \tilde{v}_y = \frac{ikB_z}{\mu_0 \rho} \tilde{B}_x \qquad (\lambda + \eta k^2)\tilde{B}_x = ikB_z \tilde{v}_x$$

$$\lambda \tilde{v}_y + (2\Omega - S)\tilde{v}_x = \frac{ikB_z}{\mu_0 \rho} \tilde{B}_x \qquad (\lambda + \eta k^2)\tilde{B}_y = ikB_z \tilde{v}_y - S\tilde{B}_x$$

Eliminating all the amplitudes[7], we obtain the *magnetorotational dispersion relation*:

$$\left[\lambda(\lambda + \eta k^2) + \omega_{\mathrm{A}}^2\right]^2 + \left(\lambda + \eta k^2\right)^2 \Omega_r^2 - 2\Omega S \omega_{\mathrm{A}}^2 = 0, \qquad \omega_{\mathrm{A}} \equiv \mathbf{k} \cdot \mathbf{v}_{\mathrm{A}} = \frac{kB_z}{\sqrt{\mu_0 \rho}}$$

$$\Rightarrow \lambda^4 + \left[2\eta k^2\right]\lambda^3 + \left[2\omega_A^2 + \left(\eta k^2\right)^2 + \Omega_r^2\right]\lambda^2 + \left[2\left(\omega_A^2 + \Omega_r^2\right)\eta k^2\right]\lambda + \left[\omega_A^4 + \left(\eta k^2\right)^2 \Omega_r^2 - 2\Omega S \omega_A^2\right] = 0$$

According to the Routh-Hurwitz stability criterion, all the roots of a quartic $x^4 + ax^3 + bx^2 + cx + d = 0$ have a negative real part (and hence in our case would not lead to exponential instability) provided that all the coefficients are positive, *and* $abc - a^2 d - c^2 > 0$. Assuming that we are at least stable in an orbital sense (so that $\Omega_r^2 > 0$), all the coefficients are definitely positive except perhaps d, which is

$$d = k^4\left(v_{\mathrm{A}z}^4 + \eta^2 \Omega_r^2\right) - k^2\left(2\Omega_r S v_{\mathrm{A}z}^2\right) = \left(v_{\mathrm{A}z}^4 + \eta^2 \Omega_r^2\right) k^2 \left(k^2 - \frac{2\Omega_r S v_{\mathrm{A}z}^2}{v_{\mathrm{A}z}^4 + \eta^2 \Omega_r^2}\right)$$

The cutoff wavenumber in the brackets is the same as the wavenumber in the steady-state magnetic disk case, K^2. Thus for stability we require $k^2 > K^2$; perturbations on longer wavelengths will provoke magnetorotational instability (MRI). The other condition gives:

$$abc - a^2 d + c^2 = \left(2\eta k^2 \omega_A\right)^2 \left[\left(\eta k^2\right)^2 + 4\Omega^2\right] > 0$$

so $k^2 < K^2$ is the only way to achieve MRI.

Typically we require the magnetic perturbations to go to 0 at $z = \pm z^+$, which restricts the possible values of k to $k = n\pi/2z^+$. The minimum $|k|$ is then $\pi/2z^+$; if this is less than K then the disk can be unstable. Equivalently, if the disk is *thicker* than $z^+ > \pi/2K$, then large-enough-wavelength modes can fit in the disk and the instability can occur.

4.1.1 Ideal MHD

Ideal MHD has $\sigma \to \infty \Rightarrow \eta \to 0$. The dispersion relation becomes:

$$\lambda^4 + \left(2\omega_A^4 + \Omega_r^2\right)\lambda^2 + \left(\omega_A^4 - 2\Omega S \omega_A^2\right) = 0$$

Out of all the wavelength modes available (the ks, or the ω_As), how high can the growth rate be? The maximum growth rate will have $\partial\lambda^2/\partial\omega_A^2 = 0$, which gives

$$\omega_A^2 = \Omega S - \lambda^2 \qquad \Rightarrow \qquad \lambda^2 = \left(\frac{S}{2}\right)^2$$

where we have plugged the expression for the fastest-growing ω_A back into the dispersion relation. Weirdly, this result is independent of the magnetic field.

[7] It's actually not too difficult to do this with a matrix. It's a 4×4 determinant but it's not a very dense one. The matrix itself is also kinda pretty.