Astrophysical Fluid Dynamics

Xander Byrne

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1 Fundamentals

1.1 Fluids, generally

To treat a system as a fluid (rather than a kinetic system), we require that the mean free path of the particles is much smaller than the system's scale: $L \gg \lambda$. (The mean free path can also be written $\lambda = 1/n\sigma$.) Equivalently, dividing through by the average *relative* speed $\bar{v}$, we require that the timescale over which the system changes is much greater than the collision timescale: $L/\bar{v} \gg \tau \equiv \lambda/\bar{v}$. As a result, particles share out their velocity very effectively with nearby particles, giving very similar *bulk* velocities $\mathbf{u}$ on small scales. This leads to the concept of fluid elements: parcels of fluid small enough that all fluid variables are spatially constant throughout the parcel.

We distinguish between two time derivatives:

- **Eulerian** $\frac{\partial q}{\partial t}$: The change over time in a quantity q at a particular point in space $\mathbf{x}$. From one moment to the next, one fluid element moves away from $\mathbf{x}$ and is replaced by a new fluid element, which may have a different value of q than the original fluid element had when it was at $\mathbf{x}$.

- **Lagrangian** $\frac{\mathrm{D}q}{\mathrm{D}t}$: The change over time in a quantity q for a particular (moving!) fluid element. Rather than defining q for all the fluid elements (which would be weird as they are moving), it is simpler to define q as a function of position and time $q = q(\mathbf{x}, t)$. Thus as a fluid element moves from $\mathbf{x}$ to $\mathbf{x} + \delta\mathbf{x}$ in a time δt, its value of q will change both due to the time-dependence of $q(\mathbf{x}, t)$, and by virtue of moving to a different point. The fluid element's value will change by

$$\begin{aligned}\Delta q &= q(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - q(\mathbf{x}, t) = q(\mathbf{x}, t) + \mathbf{u}\delta t \cdot \boldsymbol{\nabla} q + \delta t \frac{\partial q}{\partial t} + \mathcal{O}(\delta t^2) - q(\mathbf{x}, t) \\ &= \delta t \left(\frac{\partial q}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} q + \mathcal{O}(\delta t) \right) \qquad \Rightarrow \qquad \frac{\mathrm{D}q}{\mathrm{D}t} \equiv \lim_{\delta t \to 0} \frac{\Delta q}{\delta t} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} q\end{aligned}$$

We will make several simplifying assumptions to all subsequent analysis:

- **Ideal**: Fluids are adiabatic (so $p \propto \rho^\gamma$), and inviscid. Heat flows are important in AFD, but are complicated. Viscosity is important in shocks (see §5.3), but we will sidestep it.

- **Perfect**: p is related to ρ and T by $p = \frac{k}{\mu}\rho T$, for mean molecular weight $\mu \sim 10^{-24} g$.

- **Perfectly Conducting**: The conductivity $\sigma \to \infty$. (The fluid is often a plasma.)

- **Non-relativistic**: $|\mathbf{u}| \ll c$

1.2 Hydrodynamics

1.2.1 Mass Conservation

Consider an arbitrary fixed (Eulerian) volume V; the mass contained within is $M = \int_V \rho \, \mathrm{d}V \Rightarrow \frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V$. The mass flux (mass per area per time) at any given point is $\rho \mathbf{u}$, so

$$\frac{\partial M}{\partial t} = -\oint_{\partial V} \rho \mathbf{u} \cdot \mathrm{d}\mathbf{S} = -\int_V \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \, \mathrm{d}V \qquad \Rightarrow \qquad \int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \right) \mathrm{d}V = 0$$

Now because V is arbitrary, it can be infinitesimal; dividing by this volume:

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0} \qquad \Rightarrow \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \boldsymbol{\nabla} \cdot \mathbf{u} \tag{$\mathfrak{C}$}$$

– The Continuity Equation $\mathfrak{C}$. Note that incompressible fluids have $\frac{\mathrm{D}\rho}{\mathrm{D}t} \equiv 0 \Rightarrow \boldsymbol{\nabla} \cdot \mathbf{u} = 0$

1.2.2 Thermodynamics

An adiabatic fluid has $p = K\rho^\gamma$, where K and γ are global constants. $\gamma > 1$ is given by $c_p/c_V = \partial s/\partial T \,|_p / \, \partial s/\partial T \,|_V$, or alternatively by $1 + 1/n$ for polytropic index $n = f/2$, where f is the number of degrees of freedom (e.g. 3 for a monatomic gas; 5 for a diatomic).

The entropy per unit mass is generally $\boxed{s = c_V \ln\left(p\rho^{-\gamma}\right)}$. For an adiabatic fluid, the constant K is then given by e^{s/c_V}.

The sound speed c_s is given by $\boxed{c_s^2 \equiv \partial p/\partial \rho \,|_s}$. For an adiabatic fluid, $c_s^2 = \gamma p/\rho$.

We can use adiabaticity and $p = p(\rho, s)$ to get the so-called Energy Equation $\mathfrak{E}$, an equation for the evolution of p:

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \left.\frac{\partial p}{\partial \rho}\right|_s \frac{\mathrm{D}\rho}{\mathrm{D}t} + \left.\frac{\partial p}{\partial s}\right|_p \underbrace{\frac{\mathrm{D}s}{\mathrm{D}t}}_{0} = -\left.\frac{\partial p}{\partial \rho}\right|_s \rho \boldsymbol{\nabla} \cdot \mathbf{u} = -\gamma p \boldsymbol{\nabla} \cdot \mathbf{u} \quad \Rightarrow \quad \boxed{\frac{\partial p}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} p + \gamma p \boldsymbol{\nabla} \cdot \mathbf{u} = 0} \tag{$\mathfrak{E}$}$$

1.2.3 Momentum Equation

Consider a (Lagrangian) fluid element, with mass $\rho \, \mathrm{d}V$; the acceleration of the element is $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$. $\mathfrak{N}2\mathfrak{L}$ gives

$$\rho \, \mathrm{d}V \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathrm{d}\mathbf{F} \qquad \Rightarrow \qquad \int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \, \mathrm{d}V = \int_V \mathrm{d}\mathbf{F} \equiv \mathbf{F} \equiv \mathbf{F}_p + \mathbf{F}_\mathbf{g} + \mathbf{F}_\mathbf{B}$$

where we have split the force into pressure, gravitational, and magnetic parts.

$$\mathbf{F}_p = \oint_{\partial V} -p \, \mathrm{d}\mathbf{S} = -\int_V \boldsymbol{\nabla} p \, \mathrm{d}V \qquad \mathbf{F}_\mathbf{g} = \int_V -\rho \boldsymbol{\nabla} \Phi \, \mathrm{d}V \qquad \mathbf{F}_\mathbf{B} = \int_V \mathbf{f}_\mathbf{B} \, \mathrm{d}V$$

$$\Rightarrow \int_V \left(\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p + \rho \boldsymbol{\nabla} \Phi - \mathbf{f}_\mathbf{B} \right) \mathrm{d}V = 0 \qquad \Rightarrow \qquad \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Phi + \mathbf{f}_\mathbf{B}$$

where $\mathbf{f}_\mathbf{B}$ is the magnetic force per unit volume, deduced in the next section.

1.3 Magnetohydrodynamics

1.3.1 Maxwell's Equations in AFD

In cgs units (used throughout), three of Maxwell's Equations are:

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

We also have Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, where σ is the conductivity of the material the fluid is made of. We would measure this by subjecting a tank of the fluid in a lab to an $\mathbf{E}$ field and measuring $\mathbf{J}$; if the fluid is moving we would do that while running along with it: $\mathbf{J}' \equiv \sigma \mathbf{E}'$ is the definition of σ, where prime means in the rest frame of the fluid at velocity $\mathbf{u}$. However, we approximate our fluid to be made of a perfectly conducting material (e.g. plasma), so $\sigma \to \infty$. To ensure that none of the fields become infinite, we thus require $\mathbf{E}' = \mathbf{0}$. There is nothing wrong with $\mathbf{E}$ fields in general, but they cannot exist in the fluid's rest frame, as the infinitely large currents would move charges around until the $\mathbf{E}'$ field gets cancelled out. What do we get out of this? In a frame not comoving with the fluid, the Lorentz transform gives

$$\mathbf{E}' \equiv \gamma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{0} \qquad \Rightarrow \qquad \boxed{\mathbf{E} = -\frac{\mathbf{u} \times \mathbf{B}}{c}} \qquad \Rightarrow \qquad |\mathbf{E}| = \frac{|\mathbf{u}|}{c}|\mathbf{B}| \ll |\mathbf{B}|$$

Note that in cgs, $[\mathbf{E}] = [\mathbf{B}]$ dimensionally. From this, the 2nd Maxwell equation above gives:

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \left(-\frac{\mathbf{u} \times \mathbf{B}}{c}\right) \qquad \Rightarrow \qquad \boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})} \qquad (\mathfrak{F})$$

Consider a surface $\mathcal{S}$ which moves with the fluid (by a vector $\mathbf{u}\,\mathrm{d}t$ in a time $\mathrm{d}t$); consider the flux $\Psi = \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$. The (Lagrangian) derivative of this flux will be due to two factors: changes in the $\mathbf{B}$ field itself, and changes in the surface as it moves. The change in the former is simply $\int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t}\,\mathrm{d}t \cdot \mathrm{d}\mathbf{S}$; the second uses the fact that the differential change of the surface is $-\,\mathrm{d}\boldsymbol{\ell} \times \mathbf{u}\,\mathrm{d}t$, so the change in flux due to this is $-\oint_{\partial\mathcal{S}} \mathbf{B} \cdot (\mathrm{d}\boldsymbol{\ell} \times \mathbf{u}\,\mathrm{d}t)$. Thus

$$\begin{aligned}\frac{\mathrm{D}\Psi}{\mathrm{D}t} &= \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} - \oint_{\partial\mathcal{S}} \mathbf{B} \cdot (\mathrm{d}\boldsymbol{\ell} \times \mathbf{u}) = \int_{\mathcal{S}} \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} + \oint_{\partial\mathcal{S}} (\mathbf{B} \times \mathbf{u}) \cdot \mathrm{d}\boldsymbol{\ell} \\ &= \oint_{\partial\mathcal{S}} (\mathbf{u} \times \mathbf{B} + \mathbf{B} \times \mathbf{u}) \cdot \mathrm{d}\boldsymbol{\ell} = 0\end{aligned}$$

Hence $\mathfrak{F}$ is called the *flux freezing equation.* A more rigorous definition is given on Wikipedia.

Now consider the final Maxwell equations above; comparing the magnitudes of two terms:

$$\frac{\left|\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right|}{|\nabla \times \mathbf{B}|} \sim \frac{\frac{1}{c}|\mathbf{E}|/\tau}{|\mathbf{B}|/\ell} \sim \frac{\ell/\tau}{c}\frac{|\mathbf{E}|}{|\mathbf{B}|} \sim \frac{|\mathbf{u}|}{c}\frac{|\mathbf{u}|}{c} \sim \frac{|\mathbf{u}|^2}{c^2} \ll 1 \qquad \Rightarrow \qquad \boxed{\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}}$$

1.3.2 Lorentz Force

In cgs units, the Lorentz force is given by $\frac{1}{c}q_i\mathbf{u}_i \times \mathbf{B}$ for a single charge; notably $\perp \mathbf{B}$. Summing over all charges in a fluid element and dividing by the volume, we have that the magnetic force per volume $\mathbf{f}_{\mathbf{B}} = \frac{1}{c}\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$. Thus the full momentum equation is:

$$\boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p - \rho\nabla\Phi + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}} \qquad (\mathfrak{M})$$

$\mathbf{f_B}$ can be broken down into two terms, using $\boxed{(\nabla \times \mathbf{B}) \times \mathbf{B} \equiv (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla\left(\frac{1}{2}\mathbf{B}^2\right)}$.

We have a *magnetic tension* force/volume $\frac{1}{4\pi}(\mathbf{B} \cdot \nabla)\mathbf{B}$: wherever the $\mathbf{B}$ lines are curved, this force acts towards the centre of curvature, and because flux freezing "ties" the $\mathbf{B}$ field to the fluid, the tension force encourages the field lines to straighten. There is also a term $\nabla(\mathbf{B}^2/8\pi)$; being a force/volume and a gradient, this is like a pressure $p_\mathbf{B} = \mathbf{B}^2/8\pi$, described as the *magnetic pressure*; it is also interpretable as the gradient of the magnetic energy density $\mathbf{B}^2/8\pi$. The *beta* of a plasma $\beta \equiv p/p_\mathbf{B} = 8\pi p/\mathbf{B}^2$ quantifies the relative importances of gas and magnetic pressure: $\beta \gg 1$ means magnetic pressure is unimportant.

2 Conservation Laws

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{F}_q = 0$$

The above expression describes a flux density $\mathbf{F}_q$ of the quantity q. We wish, for a given quantity q, to find the flux density $\mathbf{F}_q$. For example, $\mathfrak{C}$ gives the flux density of ρ as simply $\rho\mathbf{u}$.

A *material invariant* is a quantity which is constant for a given fluid element as it moves: $\frac{\mathrm{D}q}{\mathrm{D}t} = 0$; an example is s. Expanding the derivative, multiplying by ρ, and adding $q \times \mathfrak{C}$,

$$\rho\frac{\partial q}{\partial t} + \rho\mathbf{u} \cdot \nabla q + q\frac{\partial \rho}{\partial t} + q\nabla \cdot (\rho\mathbf{u}) = 0 \qquad \Rightarrow \qquad \frac{\partial(\rho q)}{\partial t} + \nabla \cdot (\rho q\mathbf{u}) = 0$$

so for every material invariant q, there is a quantity ρq with flux density $\rho q\mathbf{u}$.

2.1 Momentum Density Flux

The momentum density is $\rho\mathbf{u}$, a vector. Its flux density, $\mathbf{\Pi}$, is therefore a rank-2 tensor:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot \mathbf{\Pi} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial(\rho u_i)}{\partial t} + \partial_j\Pi_{ij} = 0$$

Firstly, note that due to $\mathfrak{C}$,

$$\frac{\partial(\rho u_i)}{\partial t} = \rho\frac{\partial u_i}{\partial t} - u_i\partial_j(\rho u_j) = \rho\frac{\mathrm{D}u_i}{\mathrm{D}t} - \partial_j(\rho u_i u_j) \qquad \Rightarrow \qquad \rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \partial_j(\rho u_i u_j - \Pi_{ij}) \equiv \partial_j T_{ij}$$

where $\mathbf{T} \equiv \rho\mathbf{u} \otimes \mathbf{u} - \mathbf{\Pi}$ is the *stress tensor*; it is not a flux tensor as $\mathbf{\Pi}$ is designed to be, but it is purely due to the external forces (p, Φ, $\mathbf{B}$) on the fluid, unlike $\mathbf{\Pi}$ which would also need a ram pressure term $\rho u_i u_j$. Now comparing the above equation for $\rho\frac{\mathrm{D}u_i}{\mathrm{D}t}$ with the momentum equation, we identify

$$\nabla \cdot \mathbf{T} = -\nabla p - \rho\nabla\Phi + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

Before proceeding to derive the components T_{ij} from the above, we establish a useful lemma for divergences of a tensor. For a general vector $\mathbf{A} = A_i$ and the tensor $\mathbf{A} \otimes \mathbf{A} = A_i A_j$:

$$\nabla \cdot \left(\mathbf{A} \otimes \mathbf{A} - \frac{1}{2}|\mathbf{A}|^2\boldsymbol{\delta}\right) = (\nabla \cdot \mathbf{A})\mathbf{A} + (\nabla \times \mathbf{A}) \times \mathbf{A}$$

where $\boldsymbol{\delta} = \delta_{ij}$. This can be derived from the expansion of $(\nabla \times \mathbf{A}) \times \mathbf{A}$ given earlier.

Now if Φ is externally imposed (e.g. we're near a big mass), it turns out that there is no general way to write $\rho\boldsymbol{\nabla}\Phi$ as the divergence of a tensor. However if the fluid is self-gravitating ($\boldsymbol{\nabla}^2\Phi = 4\pi G\rho$), there is; we therefore consider only self-gravitating fluids. Defining $\mathbf{g} \equiv -\boldsymbol{\nabla}\Phi$, we have $\boldsymbol{\nabla}\cdot\mathbf{g} = 4\pi G\rho$ and $\boldsymbol{\nabla}\times\mathbf{g} = \mathbf{0}$. Thus

$$\rho\boldsymbol{\nabla}\Phi = \frac{1}{4\pi G}(\boldsymbol{\nabla}\cdot\mathbf{g})\mathbf{g} = \frac{1}{4\pi G}\boldsymbol{\nabla}\cdot\left(\mathbf{g}\otimes\mathbf{g} - \frac{|\mathbf{g}|^2}{2}\boldsymbol{\delta}\right) \equiv \boldsymbol{\nabla}\cdot\mathbf{G}$$

Similarly, using $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we derive the *Maxwell stress tensor*

$$\frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\left(\mathbf{B}\otimes\mathbf{B} - \frac{|\mathbf{B}|^2}{2}\boldsymbol{\delta}\right) \equiv \boldsymbol{\nabla}\cdot\mathbf{M}$$

Finally, $\boldsymbol{\nabla}p = \boldsymbol{\nabla}\cdot(p\boldsymbol{\delta})$. Comparing with the expression for $\boldsymbol{\nabla}\cdot T$, we then have

$$\mathbf{T} = -p\boldsymbol{\delta} - \mathbf{G} + \mathbf{M}, \qquad G_{ij} = \frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}g_k g_k \delta_{ij}\right), \qquad M_{ij} = \frac{1}{4\pi}\left(B_i B_j - \frac{1}{2}B_k B_k \delta_{ij}\right)$$

$$\boldsymbol{\Pi} = \rho\mathbf{u}\otimes\mathbf{u} - \mathbf{T} = \rho\mathbf{u}\otimes\mathbf{u} + p\boldsymbol{\delta} + \mathbf{G} - \mathbf{M}$$

$$\Rightarrow \Pi_{ij} = \rho u_i u_j + p\delta_{ij} + \frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}g_k g_k \delta_{ij}\right) - \frac{1}{4\pi}\left(B_i B_j - \frac{1}{2}B_k B_k \delta_{ij}\right)$$

2.2 Energy Flux

We now find the flux of the energy density $\epsilon = \rho\left(\frac{1}{2}|\mathbf{u}|^2 + \Phi + e\right) + |\mathbf{B}|^2/8\pi$, where the thermal energy per unit mass $e = f\frac{1}{2}\frac{kT}{\mu} = \frac{1}{\gamma-1}\frac{p}{\rho}$ according to equipartition. We now allow Φ to be externally imposed. Now

$$\frac{\partial\epsilon}{\partial t} = \frac{\partial\rho}{\partial t}\left(\frac{1}{2}|\mathbf{u}|^2 + \Phi + e\right) + \rho\left(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\Phi}{\partial t} + \frac{\partial e}{\partial t}\right) + \frac{1}{4\pi}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

We now substitute:

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) \qquad \rho\frac{\partial\mathbf{u}}{\partial t} = -\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Phi + \frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}$$

$$\mathbf{u}\cdot[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}] = \frac{1}{2}(\mathbf{u}\cdot\boldsymbol{\nabla})\left(|\mathbf{u}|^2\right) \qquad \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = (\boldsymbol{\nabla}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{u}) = c(\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{E}$$

to give:

$$\frac{\partial\epsilon}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{u})\left(\frac{1}{2}|\mathbf{u}|^2 + \Phi + e\right) - (\rho\mathbf{u}\cdot\boldsymbol{\nabla})\left(\frac{1}{2}|\mathbf{u}|^2 + \Phi\right) - \mathbf{u}\cdot\boldsymbol{\nabla}p$$
$$+ \frac{c}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{E} + \rho\frac{\partial\Phi}{\partial t} + \rho\frac{\partial e}{\partial t} + \frac{1}{4\pi}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

Now consider $\partial e/\partial t$. Applying 2𝔏𝔗 per unit mass to a fluid element, we have

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t} - p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} \qquad \Rightarrow \qquad \frac{\partial e}{\partial t} = -\mathbf{u}\cdot\boldsymbol{\nabla}e + T\frac{\mathrm{D}s}{\mathrm{D}t} - \frac{p}{\rho}\boldsymbol{\nabla}\cdot\mathbf{u}$$

We also have $\partial\mathbf{B}/\partial t = -c\boldsymbol{\nabla}\times\mathbf{E}$, and $\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$. Substituting,

$$\begin{aligned}
\frac{\partial\epsilon}{\partial t} &= -\boldsymbol{\nabla}\cdot(\rho\mathbf{u})\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+e\right) - (\rho\mathbf{u}\cdot\boldsymbol{\nabla})\left(\frac{1}{2}|\mathbf{u}|^2+\Phi\right) - \mathbf{u}\cdot\boldsymbol{\nabla}p - p\boldsymbol{\nabla}\cdot\mathbf{u} - \rho\mathbf{u}\cdot\boldsymbol{\nabla}e \\
&\qquad - \frac{c}{4\pi}\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) + \rho T\frac{\mathrm{D}s}{\mathrm{D}t} + \rho\frac{\partial\Phi}{\partial t} \\
&= -\boldsymbol{\nabla}\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+e\right) + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right] - \boldsymbol{\nabla}\cdot(p\mathbf{u}) + \rho T\frac{\mathrm{D}s}{\mathrm{D}t} + \rho\frac{\partial\Phi}{\partial t} \\
&= -\boldsymbol{\nabla}\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+e+\frac{p}{\rho}\right) + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right] + \rho T\frac{\mathrm{D}s}{\mathrm{D}t} + \rho\frac{\partial\Phi}{\partial t}
\end{aligned}$$

Thus for an adiabatic fluid in a time-independent potential, the energy flux is given by

$$\mathbf{F}_\epsilon = \rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+e+\frac{p}{\rho}\right) + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \qquad \Rightarrow \qquad \frac{\partial\epsilon}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{F}_\epsilon = 0$$

The terms in the flux are easily recognisable as the fluxes of kinetic energy, potential energy, *enthalpy* $h = e + \frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$, and the cgs *Poynting vector* (the flux of electromagnetic energy).

For steady-state systems with $\mathbf{B} = \mathbf{0}$, $\mathfrak{C}$ gives $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$, and so the above equation gives

$$\boldsymbol{\nabla}\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+h\right)\right] = 0 \qquad \Rightarrow \qquad \mathbf{u}\cdot\boldsymbol{\nabla}\underbrace{\left(\frac{1}{2}|\mathbf{u}|^2+\Phi+h\right)}_{b} = 0$$

Thus the quantity b, the Bernoulli constant, is conserved along a streamline[1].

3 Spherical Accretion

Accretion involves stationary gas of density ρ_0 falling onto a gravitating centre at $r = 0$. We will neglect $\mathbf{B}$ fields, assume spherical symmetry, adiabaticity, and a Keplerian potential $\Phi = -GM/r$. We will seek steady-state solutions: $\partial/\partial t = 0$. $\mathfrak{C}$ gives:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho u) = 0 \qquad \Rightarrow \qquad r^2\rho u = \text{const.}$$

Therefore the *accretion rate* $\dot{M} \equiv -4\pi r^2\rho u$ is a constant (< 0 for an outflow). Differentiating,

$$2r\rho u + r^2\frac{\mathrm{d}\rho}{\mathrm{d}r}u + r^2\rho\frac{\mathrm{d}u}{\mathrm{d}r} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho\left(\frac{2}{r}+\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right) \qquad \Rightarrow \qquad \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho c_s^2\left(\frac{2}{r}+\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

$\mathfrak{M}$ gives:

$$\rho u\frac{\mathrm{d}u}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \rho c_s^2\left(\frac{2}{r}+\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right) \qquad \Rightarrow \qquad (u^2 - c_s^2)\frac{\mathrm{d}u}{\mathrm{d}r} = u\left(\frac{2c_s^2}{r} - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

We are most interested in *transonic flows*, which go from subsonic to supersonic. It will therefore be important to find the *sonic point* r_s, at which $u = c_{ss}$, where c_{ss} is the sound speed at r_s. From the above, this requires

$$\frac{2c_{ss}^2}{r_s} = \left.\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right|_{r_s} \qquad \Rightarrow \qquad c_{ss}^2 = \frac{GM}{2r_s}$$

[1]Note that fluid elements on different paths will in general have different b.

Conservation of energy gives:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho u\left(\frac{1}{2}u^2+\Phi+h\right)\right)=0 \qquad \Rightarrow \qquad b=\frac{1}{2}u^2-\frac{GM}{r}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}=\text{const.}$$

where we have used $r^2\rho u$ =const., and identified the Bernoulli constant b; indeed we have a steady-state non-magnetic system so such a constant exists. In fact, because all radial streamlines are equivalent by spherical symmetry, b is a global constant. At the sonic point,

$$b=\frac{1}{2}c_{ss}^2-\frac{GM}{r_s}+\frac{1}{\gamma-1}c_{ss}^2=\frac{5-3\gamma}{2(\gamma-1)}c_{ss}^2=\frac{5-3\gamma}{4(\gamma-1)}\frac{GM}{r_s}$$

3.1 Bondi Accretion

Suppose as $r\to\infty$, $u\to 0$, $\rho\to\rho_0$, and $c_s\to c_{s0}=\sqrt{\gamma K\rho_0^{\gamma-1}}$, therefore $b=c_{s0}^2/(\gamma-1)$. We wish to derive the accretion rate $\dot{M}=-4\pi r^2\rho u$, and although it is a constant, clearly evaluating it at $r=\infty$ is not a way forward. The only other special points in the problem are $r=0$ (also not helpful) and $r=r_s$ where $u=-c_{ss}$; we can therefore write $\dot{M}=4\pi r_s^2\rho_s c_{ss}$. Now c_{ss} can be related to c_{s0} (and hence ρ_s to ρ_0) via b:

$$\frac{c_{s0}^2}{\gamma-1}=b=\frac{5-3\gamma}{2(\gamma-1)}c_{ss}^2 \qquad \Rightarrow \qquad c_{ss}^2=\frac{2}{5-3\gamma}c_{s0}^2 \qquad \Rightarrow \qquad \rho_s=\left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}}\rho_0$$

Finally, r_s is related to c_{ss} via $c_{ss}^2=GM/2r_s$, and c_{ss} is related to c_{s0} as above. Thus:

$$r_s^2=\left(\frac{GM}{2c_{ss}^2}\right)^2=\left(\frac{5-3\gamma}{2}\right)^2\left(\frac{GM}{2c_{s0}^2}\right)^2$$

$$\Rightarrow \dot{M}=4\pi\left[\left(\frac{5-3\gamma}{2}\right)^2\left(\frac{GM}{2c_{s0}^2}\right)^2\right]\left[\left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}}\rho_0\right]\left[\left(\frac{2}{5-3\gamma}\right)^{1/2}c_{s0}\right]$$

$$=\pi\rho_0 c_{s0}\underbrace{\left(\frac{GM}{c_{s0}^2}\right)^2}_{R_B}\underbrace{\left(\frac{5-3\gamma}{2}\right)^{-\frac{5-3\gamma}{2(\gamma-1)}}}_{f(\gamma)}=f(\gamma)\pi R_B^2\rho_0 c_{s0}$$

We have defined the *Bondi radius* R_B as it forms an effective cross-section πR_B^2 for the accretion. At this radius, the potential energy $GM/R_B=c_{s0}^2$ is of the same order as the thermal energy. For $r\lesssim R_B$ the gravity of the central object qualitatively dominates over gas pressure.

The function $f(\gamma)$ has interesting limits, found by allowing $f(\gamma)=\lim_{\epsilon\to 0}f(\gamma-\epsilon)$: we find $f(1)=\lim_{\epsilon\to 0}\left(1+\frac{3\epsilon}{2}\right)^{1/\epsilon}=e^{3/2}$ and $f\left(\frac{5}{3}\right)=\lim_{\epsilon\to 0}\left(\frac{\epsilon}{2}\right)^{-9\epsilon/4}=1$

3.1.1 Bondi-Littlewood Accretion

We now generalise to the case where the mass M is moving relative to the stationary gas at a speed V. We first modify the length scale R_B to R_{BL}, to account for the fact that gravity must overcome both thermal and kinetic energy in order for accretion to occur:

$$\frac{GM}{R_{BL}}\sim c_{s0}^2+V^2 \qquad \Rightarrow \qquad R_{BL}\sim\frac{GM}{c_{s0}^2+V^2} \qquad \Rightarrow \qquad \dot{M}\sim\pi\left(\frac{GM}{c_{s0}^2+V^2}\right)^2\rho_0\sqrt{c_{s0}^2+V^2}$$

For $V\ll c_{s0}$, this naturally tends to the previous result with $V=0$. For $V\gg c_{s0}$, the accretion rate is reduced by a factor $(V/c_{s0})^3$, as the central object ploughs through the medium without much chance for gas to fall in.

4 Axisymmetric MHD

We now look for steady-state solutions to the fluid equations which are also axisymmetric: $\partial/\partial t = \partial/\partial\phi = 0$; we will work in cylindrical coordinates (R, ϕ, z). It will be useful to consider vector fields as the sum of toroidal (like lines of latitude, parallel to $\hat{\mathbf{e}}_\phi$) and poloidal (like lines of longitude) components. For example,

$$\mathbf{B} = \mathbf{B}_\mathrm{p}(R, z) + B_\phi(R, z)\hat{\mathbf{e}}_\phi \qquad\qquad \mathbf{u} = \mathbf{u}_\mathrm{p}(R, z) + u_\phi(R, z)\hat{\mathbf{e}}_\phi$$

With no ϕ dependence, a magnetic field line forms a surface of revolution around the z axis – a *magnetic surface*. The intersection between this surface and any horizontal plane will be a horizontal circle. It will be convenient to define the quantity ψ as the magnetic flux through this horizontal circle:

$$\psi(R, z) \equiv \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_0^R 2\pi R' B_z(R', z)\,\mathrm{d}R'$$

We now emphasise the following equivalent statements:

- Interior to a given magnetic surface and travelling upwards, the flux is conserved. This is because there is never a component of $\mathbf{B}$ which is perpendicular to the magnetic surface (by construction), so no flux can escape.
- ψ is the same for all points on a given same magnetic surface; ψ can therefore be used to label the magnetic surfaces.
- $\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}\psi = 0$, as we will show now.

From its definition, we have $B_z = \frac{1}{2\pi R}\frac{\partial\psi}{\partial R}$. Now $\boldsymbol{\nabla} \cdot \mathbf{B}$ and $\partial/\partial\phi = 0$ impose $\boldsymbol{\nabla} \cdot \mathbf{B}_\mathrm{p} = 0$:

$$\begin{aligned}0 &= \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{2\pi R}\frac{\partial^2\psi}{\partial z\partial R} \\ &= \frac{1}{R}\frac{\partial}{\partial R}\left(RB_R + \frac{1}{2\pi}\frac{\partial\psi}{\partial z}\right) \quad\Rightarrow\quad RB_R + \frac{1}{2\pi}\frac{\partial\psi}{\partial z} = C(z) \quad\Rightarrow\quad B_R = -\frac{1}{2\pi R}\frac{\partial\psi}{\partial z} + \underbrace{\frac{C(z)}{R}}_{\substack{0\Rightarrow\text{regular}\\ \text{on axis}}}\end{aligned}$$

$$\Rightarrow B_R = -\frac{1}{2\pi R}\frac{\partial\psi}{\partial z} \qquad B_z = \frac{1}{2\pi R}\frac{\partial\psi}{\partial R}. \qquad \text{Now } \boldsymbol{\nabla}\psi \times \hat{\mathbf{e}}_\phi = \frac{\partial\psi}{\partial R}\hat{\mathbf{e}}_z - \frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_R$$

$$\Rightarrow \mathbf{B}_\mathrm{p} = \frac{1}{2\pi R}\boldsymbol{\nabla}\psi \times \hat{\mathbf{e}}_\phi$$

Thus ψ changes in a direction perpendicular to $\mathbf{B}_\mathrm{p}$ and $\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}\psi = 0$.

4.1 Surface Functions

By integrating the fluid equations in steady state, one finds that several fields are *surface functions*, of ψ alone.

Mass Loading k. Beginning with the flux-freezing equation $\mathfrak{F}$ in steady state:

$$\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) = \mathbf{0} \qquad\Rightarrow\qquad \underbrace{\mathbf{u}_\mathrm{p}\times\mathbf{B}_\mathrm{p}}_{\text{toroidal}} + \underbrace{u_\phi\hat{\mathbf{e}}_\phi\times\mathbf{B}_\mathrm{p} + \mathbf{u}_\mathrm{p}\times B_\phi\hat{\mathbf{e}}_\phi}_{\text{poloidal}} = \underbrace{\boldsymbol{\nabla}\chi}_{\text{pol.}} \qquad\Rightarrow\qquad \mathbf{u}_\mathrm{p}\times\mathbf{B}_\mathrm{p} = \mathbf{0}$$

where we have used $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\equiv\mathbf{0}$, introduced a scalar function[2] χ, and noted that $\boldsymbol{\nabla}\chi$ is poloidal with $\partial/\partial\phi=0$. The result is that $\mathbf{u}_\mathrm{p}\parallel\mathbf{B}_\mathrm{p}$, therefore the fluid is stuck on the same magnetic surface forever and cannot move off it. We write $\rho\mathbf{u}_\mathrm{p}=k(R,z)\mathbf{B}_\mathrm{p}$ where k is the *mass loading*, the ratio of the mass flux to the magnetic flux. Substituting into $\mathfrak{C}$:

$$0=\boldsymbol{\nabla}\cdot(\rho\mathbf{u}_\mathrm{p})=\boldsymbol{\nabla}\cdot(k\mathbf{B}_\mathrm{p})=0=\mathbf{B}_\mathrm{p}\cdot\boldsymbol{\nabla}k+k\boldsymbol{\nabla}\cdot\mathbf{B}_\mathrm{p}=\mathbf{B}_\mathrm{p}\cdot\boldsymbol{\nabla}k\qquad\Rightarrow\qquad\mathbf{B}_\mathrm{p}\cdot\boldsymbol{\nabla}k=0$$

Thus k is constant on a magnetic surface, and is hence a surface function $k=k(\psi)$.

Angular Velocity ω. We now have $\mathbf{u}\times\mathbf{B}=\hat{\mathbf{e}}_\phi\times(u_\phi\mathbf{B}_\mathrm{p}-B_\phi\mathbf{u}_\mathrm{p})=\boldsymbol{\nabla}\chi$. Taking the curl and using the definition of k,

$$\begin{aligned}\mathbf{0}=\boldsymbol{\nabla}\times\left[\hat{\mathbf{e}}_\phi\times\left(u_\phi-\frac{B_\phi k}{\rho}\right)\mathbf{B}_\mathrm{p}\right]&=\boldsymbol{\nabla}\times\left[\hat{\mathbf{e}}_\phi\times\overbrace{\left(\frac{u_\phi}{R}-\frac{B_\phi k}{\rho R}\right)}^{\equiv\omega(R,z)}\frac{\boldsymbol{\nabla}\psi\times\hat{\mathbf{e}}_\phi}{2\pi}\right]\\&\propto\boldsymbol{\nabla}\times\left[\hat{\mathbf{e}}_\phi\times(\omega\boldsymbol{\nabla}\psi\times\hat{\mathbf{e}}_\phi)\right]=\boldsymbol{\nabla}\times[\omega\boldsymbol{\nabla}\psi]=\boldsymbol{\nabla}\omega\times\boldsymbol{\nabla}\psi\\\Rightarrow\omega(R,z)=\omega(\psi)\equiv\frac{u_\phi}{R}-\frac{B_\phi k}{\rho R}\end{aligned}$$

Allowing us to write the fluid velocity in a suggestive form:

$$\Rightarrow\mathbf{u}\equiv\mathbf{u}_\mathrm{p}+u_\phi\hat{\mathbf{e}}_\phi=\frac{k(\psi)}{\rho}\mathbf{B}_\mathrm{p}+\frac{k(\psi)}{\rho}B_\phi\hat{\mathbf{e}}_\phi+R\omega(\psi)\hat{\mathbf{e}}_\phi=\frac{k(\psi)}{\rho}\mathbf{B}+R\omega(\psi)\hat{\mathbf{e}}_\phi$$

Now if we choose the particular magnetic surface with flux ψ and move to a frame rotating about the z-axis at $\omega(\psi)$, then in this frame the fluid velocity on this surface will be[3]

$$\mathbf{u}'=\frac{k(\psi)}{\rho}\mathbf{B}$$

which is $\parallel$ $\mathbf{B}$, so in this frame the fluid elements move along the $\mathbf{B}$ field lines. In the rest frame, if one imagines that the $\mathbf{B}$ field lines of a particular magnetic surface are wires being swung around the axis at ω, then the fluid elements would move like beads on those wires.

For the case $k=0$, corresponding to purely axisymmetric fluid flow ($\mathbf{u}=u_\phi\hat{\mathbf{e}}_\phi$, $\mathbf{u}_\mathrm{p}=\mathbf{0}$), we see that $u_\phi=R\omega(\psi)$. In other words, the fluid that is on the surface labelled ψ is "locked" into uniform corotation at a rate $\omega(\psi)$. This is known as *Ferraro's isorotation law.*

Angular Momentum $\ell(\psi)$. $\mathfrak{M}$ has lots of gradient terms in it. If we take $\hat{\mathbf{e}}_\phi\cdot\mathfrak{M}$, these will disappear as $\hat{\mathbf{e}}_\phi\cdot\boldsymbol{\nabla}=\partial/\partial\phi=0$. We are left with:

$$\rho\hat{\mathbf{e}}_\phi\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=\frac{1}{4\pi}\hat{\mathbf{e}}_\phi\cdot(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}$$

Both sides are of the same form; we simplify the LHS using[4] $\partial\hat{\mathbf{e}}_R/\partial\phi=\hat{\mathbf{e}}_\phi$ and $\partial\hat{\mathbf{e}}_\phi/\partial\phi=-\hat{\mathbf{e}}_R$:

$$\hat{\mathbf{e}}_\phi\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=\hat{\mathbf{e}}_\phi\cdot\left(u_R\frac{\partial}{\partial R}+u_z\frac{\partial}{\partial z}+\frac{u_\phi}{R}\frac{\partial}{\partial\phi}\right)(u_R\hat{\mathbf{e}}_R+u_z\hat{\mathbf{e}}_z+u_\phi\hat{\mathbf{e}}_\phi)$$

[2] $\mathbf{u}\times\mathbf{B}=-c\mathbf{E}$, so this scalar is in fact c times the electric potential

[3] $\mathbf{B}$ will also have a relativistic correction, but this will be $\mathcal{O}(\omega^2R^2/c^2)$, so the $\mathbf{B}$ field lines will look almost the same as in the non-rotating frame.

[4] But isn't $\partial/\partial\phi=0$? For all physical quantities, yes. Here we are differentiating basis vectors, and it is simply in their nature to vary with ϕ; this does not ruin the axisymmetry of the actual physical setup.

$$= \left(u_R\frac{\partial}{\partial R} + u_z\frac{\partial}{\partial z}\right)u_\phi + \frac{u_\phi}{R}\hat{\mathbf{e}}_\phi \cdot (u_R\hat{\mathbf{e}}_\phi - u_\phi\hat{\mathbf{e}}_R)$$

$$= \mathbf{u}_\mathrm{p} \cdot \boldsymbol{\nabla} u_\phi + \frac{u_R u_\phi}{R} = \frac{1}{R}\mathbf{u}_\mathrm{p} \cdot \boldsymbol{\nabla}(Ru_\phi)$$

The same steps can be taken with the RHS. Setting the two equal, we have:

$$\frac{\rho}{R}\mathbf{u}_\mathrm{p} \cdot \boldsymbol{\nabla}(Ru_\phi) = \frac{1}{4\pi R}\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}(RB_\phi) \qquad \Rightarrow \qquad k\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}(Ru_\phi) = \mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}\left(\frac{RB_\phi}{4\pi}\right)$$

$$\Rightarrow \mathbf{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\Big(\underbrace{kRu_\phi - \frac{RB_\phi}{4\pi}}_{\equiv k(\psi)\ell(R,z)}\Big) = 0 \qquad \Rightarrow \qquad \mathbf{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\ell = 0 \qquad \Rightarrow \qquad \ell(R,z) = \ell(\psi) \equiv Ru_\phi - \frac{RB_\phi}{4\pi k}$$

where we have twice exploited the fact that $k = k(\psi)$ and hence $\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla} k = 0$. From the above we see that the angular momentum integral has an extra term: B_ϕ causes a magnetic torque.

Bernoulli Integral $\varepsilon(\psi)$. Beginning with $\mathfrak{E}$,

$$0 = \frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + \Phi + h\right) + \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}\right] = \boldsymbol{\nabla} \cdot \left[\rho\mathbf{u}_\mathrm{p}\left(\frac{1}{2}\mathbf{u}^2 + \Phi + h\right) + \frac{c}{4\pi}[\mathbf{E} \times \mathbf{B}]_\mathrm{p}\right]$$

To calculate the poloidal component of the Poynting vector, note first that

$$\mathbf{E} = -\frac{\mathbf{u} \times \mathbf{B}}{c} = -\frac{1}{c}\left(\frac{k}{\rho}\mathbf{B} + R\omega\hat{\mathbf{e}}_\phi\right) \times \mathbf{B} = -\frac{R\omega}{c}\hat{\mathbf{e}}_\phi \times \mathbf{B}$$

is poloidal. $\mathbf{E} \times \mathbf{B}_\mathrm{p}$ will therefore be toroidal. The poloidal component of $\mathbf{E} \times \mathbf{B}$ is therefore

$$[\mathbf{E} \times \mathbf{B}]_\mathrm{p} = \underbrace{[\mathbf{E} \times \mathbf{B}_\mathrm{p}]_\mathrm{p}}_{\mathbf{0}} + \mathbf{E} \times B_\phi\hat{\mathbf{e}}_\phi = -\frac{R\omega}{c}B_\phi(\hat{\mathbf{e}}_\phi \times \mathbf{B}) \times \hat{\mathbf{e}}_\phi = -\frac{R\omega}{c}B_\phi\mathbf{B}_\mathrm{p}$$

Substituting this and $\rho\mathbf{u}_\mathrm{p} = k\mathbf{B}_\mathrm{p}$, and noting that $\boldsymbol{\nabla} \cdot \mathbf{B}_\mathrm{p} = 0$, we have

$$0 = \boldsymbol{\nabla} \cdot \left[\left(k\left(\frac{1}{2}\mathbf{u}^2 + \Phi + h\right) - \frac{R\omega}{4\pi}B_\phi\right)\mathbf{B}_\mathrm{p}\right] = \mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla}\Bigg[\overbrace{k\left(\frac{1}{2}\mathbf{u}^2 + \Phi + h\right) - \frac{R\omega B_\phi}{4\pi}}^{\equiv k(\psi)\varepsilon(R,z)}\Bigg]$$

$$\Rightarrow \varepsilon(R,z) = \varepsilon(\psi) \equiv \frac{1}{2}\mathbf{u}^2 + \Phi + h - \frac{R\omega B_\phi}{4\pi k}$$

We see again that B_ϕ imparts a correction to the intuitive form of the function.

Material Invariants. Finally, we will show that all material invariants are surface functions, using s as an example. Starting from the definition of a material invariant and using $\partial/\partial t = \partial/\partial\phi = 0$,

$$0 = \frac{\mathrm{D}s}{\mathrm{D}t} = \underbrace{\frac{\partial s}{\partial t}}_{0} + \mathbf{u} \cdot \boldsymbol{\nabla} s = \mathbf{u}_\mathrm{p} \cdot \boldsymbol{\nabla} s + u_\phi \underbrace{\frac{\partial s}{\partial \phi}}_{0} = \frac{k}{\rho}\mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla} s \qquad \Rightarrow \qquad \mathbf{B}_\mathrm{p} \cdot \boldsymbol{\nabla} s = 0 \qquad \Rightarrow \qquad s = s(\psi)$$

The same applies to all material invariants, so they are all surface functions too.

4.1.1 Magnetic Wind Launching

Consider the pair of functions:

$$\omega(\psi) \equiv \frac{u_\phi}{R} - \frac{B_\phi k}{\rho R} \qquad\qquad \ell(\psi) \equiv R u_\phi - \frac{R B_\phi}{4\pi k}$$

Eliminating B_ϕ between them:

$$B_\phi = \frac{\rho}{k}(u_\phi - R\omega) = 4\pi k\left(u_\phi - \frac{\ell}{R}\right) \qquad \Rightarrow \qquad u_\phi - R\omega = \frac{4\pi k^2}{\rho}\left(u_\phi - \frac{\ell}{R}\right)$$

The prefactor is the square of the *poloidal Alfvén number*[5] A, in terms of $\mathbf{v}^2_{\mathbf{A}p} \equiv \mathbf{B}_\mathrm{p}/\sqrt{4\pi\rho}$:

$$A^2 \equiv \frac{\mathbf{u}_\mathrm{p}^2}{\mathbf{v}^2_{\mathbf{A}p}} = \mathbf{u}_\mathrm{p}^2\frac{4\pi\rho}{\mathbf{B}_\mathrm{p}^2} = \frac{4\pi k^2}{\rho} \quad \Rightarrow \quad u_\phi - R\omega = A^2\left(u_\phi - \frac{\ell}{R}\right) \quad \Rightarrow \quad u_\phi = \frac{R^2\omega - A^2\ell}{R(1-A^2)}$$

We see that for $A \gg 1$ (fast poloidal flow/weak $\mathbf{B}_\mathrm{p}$), $u_\phi \approx \ell/R$ and the flow preserves angular momentum; if a fluid element moves to a region of the surface further away from the axis it will slow down. This is intuitive because in the limit $\mathbf{B}_\mathrm{p} \to 0$ the situation is simply hydrodynamic and there is no $\mathbf{B}_\mathrm{p}$ field to mess with the conservation of angular momentum. For $A \ll 1$ (slow poloidal flow/strong $\mathbf{B}_\mathrm{p}$, or the $k = 0$ of Ferraro's isorotation law above), $u_\phi \approx R\omega$; if a fluid element moves to a region of the surface further away from the axis it will speed up its u_ϕ to keep the same ω. The surfaces will then isorotate.

In the transonic flow of §3, fluid goes from subsonic to supersonic, passing along a radial path through a sonic point r_s (the locus of sonic points on all radial paths forms a *sonic sphere*) where $u = c_s$. In *trans-Alfvénic flow*, fluid goes from *sub-Alfvénic* ($\mathbf{u}_\mathrm{p} < \mathbf{v}_{\mathbf{A}p}$) to *super-Alfvénic* ($\mathbf{u}_\mathrm{p} > \mathbf{v}_{\mathbf{A}p}$), passing along a magnetic surface through an Alfvén point at axial distance $R = R_A(\psi)$ (the locus of Alfvén points on a magnetic surface forms a horizontal circle by axisymmetry; the locus of these circles on all magnetic surfaces forms an *Alfvén surface* cutting through the magnetic surfaces) on which $\mathbf{u}_\mathrm{p} = \mathbf{v}_{\mathbf{A}p}$. At an Alfvén point, $A = 1 \Rightarrow R_A(\psi) = \sqrt{\ell(\psi)/\omega(\psi)}$.

Another surface function is $\varepsilon'(\psi) \equiv \varepsilon(\psi) - \omega(\psi)\ell(\psi)$, the *Jacobi Integral*. This evaluates to:

$$\varepsilon' = \frac{1}{2}\mathbf{u}^2 + \Phi + h - R\omega u_\phi = \frac{1}{2}\mathbf{u}_\mathrm{p}^2 + \frac{1}{2}(u_\phi - R\omega)^2 + \Phi_\mathrm{cg} + h$$

where $\Phi_\mathrm{cg} = \Phi - \frac{1}{2}(R\omega)^2$. Near the disk, the fluid is sub-Alfvénic (so $u_\phi \approx R\omega$), and cold (so $h \approx 0$). Thus $\varepsilon'(\psi) = \frac{1}{2}\mathbf{u}_\mathrm{p}^2 + \Phi_\mathrm{cg}$ near to the disk. To launch the fluid off the disk, it needs to increase its $\mathbf{u}_\mathrm{p}$, to eventually reach $\mathbf{v}_{\mathbf{A}p}$ at the Alfvén surface. Thus it will have to lose some Φ_cg. Taylor expanding about the point $P = (R_0, 0)$, assuming force balance at P and symmetry in the $z = 0$ plane:

$$\Phi_\mathrm{cg}(R,z) \approx \Phi_\mathrm{cg}(P) + \frac{1}{2}\left.\frac{\partial^2\Phi_\mathrm{cg}}{\partial R^2}\right|_P (R-R_0)^2 + \frac{1}{2}\left.\frac{\partial^2\Phi_\mathrm{cg}}{\partial z^2}\right|_P z^2 = \Phi_\mathrm{cg}(P) + \frac{1}{2}\Omega_0^2\left(z^2 - 3(R-R_0)^2\right)$$

where we have substituted the Keplerian $\Phi(R,z) = -GM(R^2+z^2)^{-1/2}$, and used $\Omega_0^2 = GM/R_0^3$. We see that P is a saddle point of the centrifugal potential, with separatrices at gradients of $\pm\sqrt{3}$ through P, an angle of $\pi/3$ from the disk. Therefore Φ_cg decreases (allowing $\mathbf{u}_\mathrm{p}^2$ to increase while maintaining ε') if the magnetic surface that the fluid is being launched on makes an angle of less than $\pi/3$ with the disk; this is therefore the condition for a magnetic wind to be launched.

[5]Note that this is not a surface function

5 Waves

5.1 Linear Waves

Consider a static and uniform[6] background state with $\rho = \rho_0$ constant, similarly for $\mathbf{u}$, p, $\mathbf{B}$, with $\mathbf{u}_0 = \mathbf{0}$ and neglecting gravity, in such a way that $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{F}$ and $\mathfrak{M}$ are all satisfied. Now consider small perturbations on this state, so that $\rho(\mathbf{x}, t) = \rho_0 + \delta\rho(\mathbf{x}, t)$, and similarly for $\mathbf{u}$, p, $\mathbf{B}$. To first order, $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{F}$ and $\mathfrak{M}$ become respectively

$$\frac{\partial \delta\rho}{\partial t} = -\rho_0 \boldsymbol{\nabla} \cdot \delta\mathbf{u} \qquad \frac{\partial \delta p}{\partial t} = -\gamma p_0 \boldsymbol{\nabla} \cdot \delta\mathbf{u} \qquad \frac{\partial \delta\mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta\mathbf{u} \times \mathbf{B}_0)$$

$$\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}\delta p + \frac{1}{4\pi}(\mathbf{B}_0 \cdot \boldsymbol{\nabla})\delta\mathbf{B} - \frac{1}{4\pi}\boldsymbol{\nabla}(\mathbf{B}_0 \cdot \delta\mathbf{B})$$

recalling that $\mathbf{B}_0$ is a spatial constant. We now introduce the variable $\boldsymbol{\xi}$, defined such that $\boxed{\delta\mathbf{u} \equiv \partial\boldsymbol{\xi}/\partial t}$. Now because ρ_0 etc. are time-independent, we can simply integrate the first three of the above over time:

$$\delta\rho = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \qquad \delta p = -\gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \qquad \delta\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - \mathbf{B}_0(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})$$

Substituting these into the perturbed momentum equation,

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \gamma p_0 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + \frac{1}{4\pi}\Big[(\mathbf{B}_0 \cdot \boldsymbol{\nabla})((\mathbf{B}_0 \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - \mathbf{B}_0(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})) - \boldsymbol{\nabla}\big(\mathbf{B}_0 \cdot (\mathbf{B}_0 \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - \mathbf{B}_0^2(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})\big)\Big]$$

Consider a plane wave perturbation:

$$\Rightarrow \boldsymbol{\xi} = \hat{\boldsymbol{\xi}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \qquad \boldsymbol{\nabla} \to i\mathbf{k} \qquad \frac{\partial}{\partial t} \to -i\omega$$

$$\Rightarrow -\rho_0 \omega^2 \boldsymbol{\xi} = -\gamma p_0 \mathbf{k}(\mathbf{k}\cdot\boldsymbol{\xi}) - \frac{1}{4\pi}\left[(\mathbf{B}_0 \cdot \mathbf{k})((\mathbf{B}_0 \cdot \mathbf{k})\boldsymbol{\xi} - \mathbf{B}_0(\mathbf{k} \cdot \boldsymbol{\xi})) - \mathbf{k}\big(\mathbf{B}_0 \cdot (\mathbf{B}_0 \cdot \mathbf{k})\boldsymbol{\xi} - \mathbf{B}_0^2(\mathbf{k} \cdot \boldsymbol{\xi})\big)\right]$$

$$\Rightarrow \left(\rho_0\omega^2 - \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{4\pi}\right)\boldsymbol{\xi} = \left[\left(\gamma p_0 + \frac{\mathbf{B}_0^2}{4\pi}\right)(\mathbf{k} \cdot \boldsymbol{\xi}) - \frac{(\mathbf{B}_0 \cdot \boldsymbol{\xi})(\mathbf{B}_0 \cdot \mathbf{k})}{4\pi}\right]\mathbf{k} - \frac{(\mathbf{B}_0 \cdot \mathbf{k})(\mathbf{k} \cdot \boldsymbol{\xi})}{4\pi}\mathbf{B}_0 \quad (*)$$

To obtain dispersion relations, we need to eliminate $\boldsymbol{\xi}$. To do this, we construct two equations in $\mathbf{k} \cdot \boldsymbol{\xi}$ and $\mathbf{B}_0 \cdot \boldsymbol{\xi}$, and eliminate. Taking the scalar product of $(*)$ with $\mathbf{k}$ and $\mathbf{B}_0$ respectively,

$$\left(\rho_0\omega^2 - \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{4\pi}\right)(\mathbf{k} \cdot \boldsymbol{\xi}) = \left[\left(\gamma p_0 + \frac{\mathbf{B}_0^2}{4\pi}\right)(\mathbf{k} \cdot \boldsymbol{\xi}) - \frac{(\mathbf{B}_0 \cdot \boldsymbol{\xi})(\mathbf{B}_0 \cdot \mathbf{k})}{4\pi}\right]\mathbf{k}^2 - \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2(\mathbf{k} \cdot \boldsymbol{\xi})}{4\pi}$$

$$\Rightarrow \left[\rho_0\omega^2 - \left(\gamma p_0 + \frac{\mathbf{B}_0^2}{4\pi}\right)\mathbf{k}^2\right](\mathbf{k} \cdot \boldsymbol{\xi}) + \frac{(\mathbf{B}_0 \cdot \mathbf{k})\mathbf{k}^2}{4\pi}(\mathbf{B}_0 \cdot \boldsymbol{\xi}) = 0$$

$$\left(\rho_0\omega^2 - \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{4\pi}\right)(\mathbf{B}_0 \cdot \boldsymbol{\xi}) = \left[\left(\gamma p_0 + \frac{\mathbf{B}_0^2}{4\pi}\right)(\mathbf{k} \cdot \boldsymbol{\xi}) - \frac{(\mathbf{B}_0 \cdot \boldsymbol{\xi})(\mathbf{B}_0 \cdot \mathbf{k})}{4\pi}\right](\mathbf{B}_0 \cdot \mathbf{k}) - \frac{(\mathbf{B}_0 \cdot \mathbf{k})(\mathbf{k} \cdot \boldsymbol{\xi})}{4\pi}\mathbf{B}_0^2$$

[6] Uniform but not necessarily isotropic due to $\mathbf{B}$'s direction.

$$\Rightarrow -\gamma p_0(\mathbf{B}_0 \cdot \mathbf{k})(\mathbf{k} \cdot \boldsymbol{\xi}) + \rho_0 \omega^2 (\mathbf{B}_0 \cdot \boldsymbol{\xi}) = 0$$

$$\Rightarrow \begin{pmatrix} \left[\rho_0\omega^2 - \left(\gamma p_0 + \frac{\mathbf{B}_0^2}{4\pi}\right)\mathbf{k}^2\right] & \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{k}^2}{4\pi} \\ -\gamma p_0(\mathbf{B}_0 \cdot \mathbf{k}) & \rho_0\omega^2 \end{pmatrix} \begin{pmatrix} \mathbf{k}\cdot\boldsymbol{\xi} \\ \mathbf{B}_0\cdot\boldsymbol{\xi} \end{pmatrix} = \mathbf{0}$$

where we have expressed the two as a matrix equation. There are now two possibilities:

- $\mathbf{k}\cdot\boldsymbol{\xi} = \mathbf{B}_0\cdot\boldsymbol{\xi} = 0$. These are transverse waves oscillating perpendicular to $\mathbf{B}$. Substituting into $(*)$ gives $\omega^2 = \frac{(\mathbf{B}_0\cdot\mathbf{k})^2}{4\pi\rho} = (\mathbf{v_A}\cdot\mathbf{k})^2$, where we have defined the *Alfvén velocity*:

$$\mathbf{v_A} \equiv \frac{\mathbf{B}_0}{\sqrt{4\pi\rho_0}}$$

  This dispersion relation is independent of p; indeed $\delta p = -\gamma p_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = -i\gamma p_0 \mathbf{k}\cdot\boldsymbol{\xi} = 0$, so the pressure is not perturbed. Furthermore, $\delta\mathbf{B} = (\mathbf{B}_0\cdot\mathbf{k})\boldsymbol{\xi} \Rightarrow \delta\mathbf{B}\cdot\mathbf{B}_0 = (\mathbf{B}_0\cdot\mathbf{k})(\boldsymbol{\xi}\cdot\mathbf{B}_0) = 0 \Rightarrow \delta(\mathbf{B}^2) = 0$, so the magnetic pressure is also not perturbed.

  The restoring force for this wave must therefore be the magnetic tension: the fluid oscillates perpendicular to the $\mathbf{B}_0$ lines, causing the $\mathbf{B}$ lines to bend into a wave shape; magnetic tension then acts to snap the $\mathbf{B}$ lines straight, in the direction of the restoring force required. The phase velocity and group velocity are respectively defined by:

$$\mathbf{v}_p = \frac{\omega}{|\mathbf{k}|}\hat{\mathbf{k}} \qquad\qquad \mathbf{v}_g = \boldsymbol{\nabla}_\mathbf{k}\omega$$

  so we have $\mathbf{v}_p = \pm|\mathbf{v_A}|\cos(\theta)\hat{\mathbf{k}}$ and $\mathbf{v}_g = \pm\mathbf{v_A}$, where θ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$. As indicated by the dispersion relation, this means that when $\mathbf{B}_0 \perp \mathbf{k}$, the wave does not propagate. Also, whereas $\mathbf{v}_p$ can be at an angle to the $\mathbf{B}_0$ field, $\mathbf{v}_g \parallel \mathbf{B}_0$, so the wave's energy transfer is along the $\mathbf{B}_0$ field lines.

- The matrix determinant is 0. This gives the following quadratic in ω^2:

$$0 = \omega^4 - \left(c_s^2 + \mathbf{v}_\mathbf{A}^2\right)\mathbf{k}^2\omega^2 + c_s^2(\mathbf{v_A}\cdot\mathbf{k})^2\mathbf{k}^2$$
$$\Rightarrow \omega^2 = \frac{1}{2}\left[c_s^2 + \mathbf{v}_\mathbf{A}^2 \pm \sqrt{\left(c_s^2 + \mathbf{v}_\mathbf{A}^2\right)^2 - 4c_s^2\mathbf{v}_\mathbf{A}^2\cos^2\theta}\right]\mathbf{k}^2$$

  For the non-magnetic case $\mathbf{B}_0 = \mathbf{v_A} = \mathbf{0}$, we have $\omega^2 = c_s^2\mathbf{k}^2$. Going back a bit to $(*)$, we find in fact $\rho_0\omega^2\boldsymbol{\xi} = \gamma p_0\mathbf{k}$, so these waves are longitudinal. These are vanilla sound waves, and can propagate isotropically.

  Generally, the two solutions do not coincide and represent the fast and slow *magnetosonic waves*: both p and $\mathbf{B}$ contribute to the restoring force.

  The angular dependence of the phase velocities of the Alfvén and fast and slow magnetosonic waves is plotted in a Friedrichs-Spiderman diagram (see Figure 1), named after the two individuals who inspired them. We see that fast magnetosonic waves propagate almost isotropically, but that slow magnetosonic and Alfvén waves are strongly channelled by the $\mathbf{B}$ field.

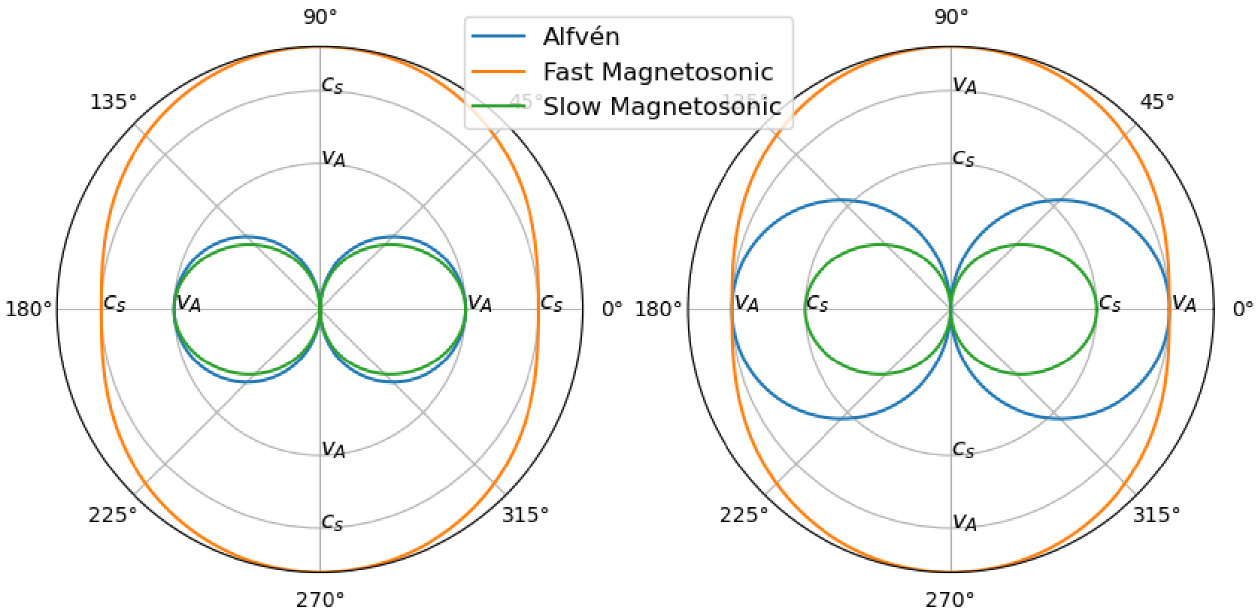

Figure 1 | **Friedrichs-Spiderman Diagrams.** The speeds of the Alfvén, fast and slow magnetosonic waves are plotted as a function of angle. In the left diagram, $c_s > \mathbf{v_A}$; in the right $c_s < \mathbf{v_A}$. In each diagram, the outer boundary is at $\sqrt{c_s^2 + \mathbf{v_A^2}}$.

5.2 Nonlinear Waves

5.2.1 Simple Waves

We restrict our analysis of nonlinear waves to a 1D adiabatic fluid with no $\mathbf{B}$ field or gravity, and an initial velocity field[7] $u_0(x)$. We further specialise to the case of a *simple wave*, where there is a 1-to-1 correspondence between velocity and density: $\rho = \rho(u)$; $u = u(\rho)$; the initial density distribution $\rho_0(x) = \rho(u_0(x))$. Further, for an adiabatic gas, $p = p(\rho)$ and $c_s^2 = \gamma p/\rho$ is also a function of ρ, so if *any* out of the set $\{\rho, u, p, c_s\}$ is known (at some particular point), then the rest are known automatically. $\mathfrak{C}$ and $\mathfrak{M}$ give

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} = -\frac{\mathrm{d}(\rho u)}{\mathrm{d}\rho}\frac{\partial \rho}{\partial x} = -\left(u + \rho\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)\frac{\partial \rho}{\partial x} \qquad \frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\left(u + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}u}\right)\frac{\partial u}{\partial x} = -\left(u + \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}u}\right)\frac{\partial u}{\partial x}$$

Now consider the 2D x-t plane. At any time, we could pick an x value, and the fluid will have some value of, for example, ρ (and corresponding values of p, u, c_s...). If we join together all the points in x-t space with, say, $\rho = \rho_1$, what will the gradient of that curve be in x-t space[8]? It would be $\partial x/\partial t\,\big|_\rho$ for the value $\rho = \rho_1$. According to the cyclic chain rule and the above,

$$\left.\frac{\partial x}{\partial t}\right|_\rho = -\frac{\partial\rho/\partial t\,\big|_x}{\partial\rho/\partial x\,\big|_t} = u + \rho\frac{\mathrm{d}u}{\mathrm{d}\rho} \qquad \text{Similarly,} \qquad \left.\frac{\partial x}{\partial t}\right|_u = -\frac{\partial u/\partial t\,\big|_x}{\partial u/\partial x\,\big|_t} = u + \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}u}$$

where we have identified the unspoken notation $\partial\rho/\partial t \equiv \partial\rho/\partial t\,\big|_x$. Now because u is a function of ρ, if ρ is equal to the constant ρ_1 all along that curve then u will also be a constant, $u(\rho_1)$,

[7] Well, it's a scalar so I guess technically a "speed field"

[8] this is the speed at which a compression or rarefaction would move along the axis

along that curve, so $\partial x/\partial t\,\big|_{\rho} = \partial x/\partial t\,\big|_{u} = \partial x/\partial t\,\big|_{p}$... Equating these two expressions, we have

$$\rho\frac{\mathrm{d}u}{\mathrm{d}\rho} = \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}u} \qquad \Rightarrow \qquad \frac{\mathrm{d}u}{\mathrm{d}\rho} = \pm\frac{c_s}{\rho} \qquad \Rightarrow \qquad u(\rho) = \pm\int\frac{c_s(\rho)}{\rho}\mathrm{d}\rho = \pm\frac{2}{\gamma-1}c_s(\rho) + u_*$$

$$\Rightarrow u(\rho) = \pm\frac{2}{\gamma-1}(\gamma K)^{1/2}\rho^{\frac{\gamma-1}{2}} + u_* \qquad\qquad c_s(u) = \pm\frac{\gamma-1}{2}(u-u_*)$$

where we have used $c_s \propto \rho^{\frac{\gamma-1}{2}}$ to evaluate the integral, and defined the arbitrary constant u_*, which relates the initial velocity and density distributions $u_0(x)$ and $\rho_0(x)$. From the above formula for $u(\rho)$, we see that although $u_0(x)$ and $\rho_0(x)$ cannot be specified completely independently[9], a completely arbitrary constant can be legally added to $u(x)$. This probably says something deep about the symmetry of the fluid equations under Galilean transformations. Anyway the above formulae finally relate u to ρ and c_s.

Now consider the function[10] $x = x(u,t)$. Recall

$$\left.\frac{\partial x}{\partial t}\right|_u = u + \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}u} = u \pm c_s(u) = \frac{\gamma+1}{2}u - \frac{\gamma-1}{2}u_*$$

$$\Rightarrow \qquad x(u,t) = \left(\frac{\gamma+1}{2}u - \frac{\gamma-1}{2}u_*\right)t + f(u)$$

which is an implicit relation for $u(x,t)$, which unfortunately is inaccessible in explicit form; $f(u)$ is set by the initial condition: at $t=0$, $u(x,0) \equiv u_0(x)$, and $x(u,0) = f(u)$; thus $f(u)$ and $u_0(x)$ are inverses:

$$\Rightarrow x(u,t) = \left(\frac{\gamma+1}{2}u - \frac{\gamma-1}{2}u_*\right)t + u_0^{-1}(u)$$

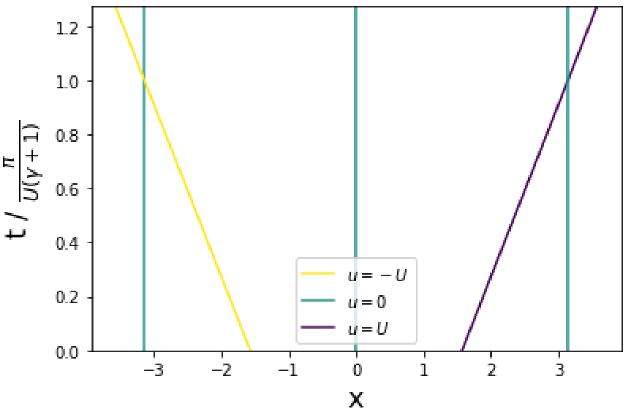

Figure 2 | **Intersection of some velocity loci.** Some of the loci of points with $u = -U, 0, U$ are shown. Intersection of these loci occurs at $t = \frac{\pi}{U(\gamma+1)}$, at which points the velocity becomes multivalued. Only a few loci are shown here; this is not the earliest time where a locus intersection occurs (see §5.2.2). In this plot $u_* = 0$ for simplicity; changing u_* does not affect the intersection time, but rather slants all of the lines equally, as would occur after a Galilean boost.

Whereas $u_0(x)$ and $u(x,t)$ must clearly be single-valued (if u were multivalued at some x value, where would the fluid go from there?!), the function $x(u,t)$ may be multivalued. Suppose $u_0(x) = U\sin x$: the values of x at which $u=0$ (*stagnation points*) are at a certain time t given by $x(u=0,t) = -\frac{\gamma-1}{2}u_*t + n\pi$, for $n \in \mathbb{Z}$. These will be straight lines in x-t space, with a gradient $-\frac{\gamma-1}{2}u_*$. In other words, the stagnation points travel at constant speed. Similarly, seeking the loci where $u = U$, we have $x(u=U,0) = (\frac{\gamma+1}{2}U - \frac{\gamma-1}{2}u_*)t + \pi/2 + 2\pi m$ for $n \in \mathbb{Z}$. These are also straight lines in x-t space, but with importantly with a different gradient and starting point (x-intercept, at $t=0$). As such, there will come a time when the lines cross. For the branches with $n=1$ and $m=0$, the two lines intersect at a time $t = \frac{\pi}{\gamma+1}\frac{1}{U}$ (see Figure 2).

[9] $u_0(x) \propto \rho_0(x)^{\frac{\gamma-1}{2}}$+const., so we can't have e.g. $u_0 \propto \ln(\rho_0)$

[10] This function is the answer to: given a time t, at what value(s) of x is the fluid moving at speed u?

This is a problem. Suppose more generally that the loci of $u = u_1$ and $u = u_2$ cross at location x after a time t. At this point, $u(x,t)$ will become multivalued, as the point lies on both the $u = u_1$ and $u = u_2$ loci. Ultimately this occurs because the x-values at which u has a particular value propagate along the axis at a speed

$$v \equiv \left.\frac{\partial x}{\partial t}\right|_u = u \pm c_s(u) = \frac{\gamma+1}{2}u - \frac{\gamma-1}{2}u_*$$

which is u-dependent. More precisely, it is a monotonically-increasing function of u, so any peaks in the velocity distribution travel along the axis faster, eventually "catching up" with the troughs. A sinusoidal initial velocity distribution will therefore evolve into a sawtooth shape; the blades then bulldoze through the slower regions of the fluid at the maximum v. This is the most simple illustration of a *shock*.

5.2.2 Inviscid Burger's Equation

To derive the earliest time at which shocks occur, we first derive the *inviscid Burger's equation*. Considering the above quantity v, we derive:

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \pm \frac{\mathrm{d}c_s}{\mathrm{d}u}\frac{\partial u}{\partial t} = \left(1 \pm \frac{\mathrm{d}c_s}{\mathrm{d}u}\right)\frac{\partial u}{\partial t} \qquad \frac{\partial v}{\partial x} = \left(1 \pm \frac{\mathrm{d}c_s}{\mathrm{d}u}\right)\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v/\partial t}{\partial v/\partial x} = \frac{\partial u/\partial t}{\partial u/\partial x} = -\left(u + \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}u}\right) = -(u \pm c_s) = -v \qquad \Rightarrow \qquad \boxed{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0} \qquad (\mathfrak{B})$$

This equation has the implicit solution $v(x,t) = V(x - vt)$, for an arbitrary function V:

$$\frac{\partial v}{\partial t} = V'(x-vt)\left[-\frac{\partial v}{\partial t}t - v\right] = -\frac{vV'}{1+V't} \qquad \frac{\partial v}{\partial x} = V'(x-vt)\left[1 - \frac{\partial v}{\partial x}t\right] = \frac{V'}{1+V't}$$

The function V can be fixed by the initial conditions: if $u_0(x)$ is known, then $V(x) = u_0(x) \pm c_s(u_0(x)) = \frac{\gamma+1}{2}u_0(x) - \frac{\gamma-1}{2}u_*$. The implicit solution is then $v = \frac{\gamma+1}{2}u_0(x-vt) - \frac{\gamma-1}{2}u_*$

For a shock to form, we require $\partial u/\partial x \to \infty \Rightarrow \partial v/\partial x \to \infty$, or

$$0 = \frac{1}{\partial v/\partial x} = \frac{1+V't}{V} \qquad \Rightarrow \qquad t_s = -\frac{1}{V'}$$

The earliest time at which a shock occurs (minimum t_s) therefore depends on the minimum (i.e. most negative) value of V', which makes sense as if a part of the fluid is propagating much faster than the part in front ($V' \ll 0$), they will collide quicker than if two adjacent parts of the fluid are travelling at similar speeds ($V' \approx 0$). If a part of the fluid is propagating *slower* than the part in front ($V' > 0$) then it will fall behind the part in front and never catch up.

For the fluid we have been considering, t_s is the minimum of $\frac{\gamma+1}{2}u_0'(x)$. For the $u_0 = U\sin x$ Ansatz from earlier, we find $t_s = \frac{2}{U(\gamma+1)}$, a bit earlier than the $t = \frac{\pi}{U(\gamma+1)}$ at which two of the loci we (arbitrarily) chose intersected.

5.2.3 Riemann Invariants

We now generalise somewhat to a no longer simple, but still adiabatic and non-magnetic, gas, so $u \neq u(\rho) \neq u(c_s)$. We now derive the *Riemann Invariants* $J_\pm$, apparently useful quantities. $\mathfrak{M}$ and $(\pm c_s/\rho) \times \mathfrak{C}$ give

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \qquad \pm\frac{c_s}{\rho}\left(\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right) = 0$$

$$\text{Summing,}\quad \Rightarrow \frac{\partial u}{\partial t} \pm \frac{c_s}{\rho}\frac{\partial \rho}{\partial t} + u\frac{\partial u}{\partial x} \pm c_s\frac{\partial u}{\partial x} + u\left(\pm\frac{c_s}{\rho}\frac{\partial \rho}{\partial x}\right) \pm u\frac{c_s}{\rho}\frac{\partial \rho}{\partial x} + c_s\frac{c_s}{\rho}\frac{\partial \rho}{\partial x}$$

$$\frac{\partial}{\partial t}\left(u \pm \int \frac{c_s}{\rho}\mathrm{d}\rho\right) + (u \pm c_s)\frac{\partial}{\partial x}\left(u \pm \int \frac{c_s}{\rho}\mathrm{d}\rho\right) = 0$$

$$\text{Define } J_\pm \equiv u \pm \int \frac{c_s}{\rho}\mathrm{d}\rho \qquad \Rightarrow \qquad \frac{\partial J_\pm}{\partial t} + (u \pm c_s)\frac{\partial J_\pm}{\partial x} = 0$$

Note that $u = (J_+ + J_-)/2$. Now consider the quantity $\partial x/\partial t\,\big|_{J_\pm}$: from the above and the triple product rule, this is equal to $-(\partial J_\pm/\partial t)/(\partial J_\pm/\partial x) = u \pm c_s$. Curves of constant $J_\pm$ in x-t space thus have gradients $u \pm c_s$. Unlike for a simple gas, where constant u meant constant $\partial x/\partial t\,\big|_u$, constant $J_\pm$ does not mean constant $\partial x/\partial t\,\big|_{J_\pm}$, so these lines will be curved. These lines crisscross the x-t plane, and are apparently useful for numerical integration.

5.3 Shock Waves

Shock waves are mathematical discontinuities in fluid quantities, caused by non-ideal fluid behaviour due to e.g. sharp gradients in ρ or $\mathbf{u}$. Viscosity contributes an extra term to the momentum equation: $\nu\big[\boldsymbol{\nabla}^2\mathbf{u} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\big]$, where the kinematic viscosity ν is usually very small for astrophysical fluids. Thus this term only becomes important for areas where $\mathbf{u}$ changes very quickly, and can be neglected elsewhere. Viscosity is thus important within shocks, where it mediates an irreversible dissipation of differential kinetic energy into heat, acting as an entropy source. We will not look at the details what goes on *within* the shock, simply what happens on either side; this enables us to use familiar formulae here.

The kinematic viscosity is of order $\nu \sim \lambda c_s$, which makes sense as viscosity is a microscopic process. To estimate the width W of the shock, we balance the viscosity term of the momentum equation with the advective term, and find that W is very small, of order the mean free path:

$$\nu\boldsymbol{\nabla}^2\mathbf{u} \sim (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \qquad \Rightarrow \qquad \lambda c_s\frac{c_s}{W^2} \sim \frac{c_s^2}{W} \qquad \Rightarrow \qquad W \sim \lambda$$

5.3.1 Shock Jump Conditions

Consider an infinite planar shock, wlog propagating in the x-direction at velocity $\mathbf{V} \parallel \hat{\mathbf{x}}$ in some rest frame, so that $\partial_y = \partial_z = 0$. We will work in the frame of the shock, so that the already-shocked fluid is travelling at velocity $\mathbf{u}_1$ and the as-yet-unshocked fluid is travelling at $\mathbf{u}_2$; to transform back to the rest frame one should add $\mathbf{V}$ to the velocities; if the unshocked fluid is stationary in the rest frame then we will have $\mathbf{u}_2 = -\mathbf{V}$.

We now wish to relate ρ, p, $\mathbf{u}$ and $\mathbf{B}$ in the shocked fluid to those in the unshocked fluid; we neglect gravity. In the frame of the shock, everything is in steady state $\Rightarrow \partial/\partial t = 0$. Recalling that some variables can be assigned a flux $\mathbf{F}$ such that $\partial q/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F}_q = 0$, we therefore have $\partial F_x/\partial x = 0$. Integrating over the shock, we therefore have $F_{x2} - F_{x1} \equiv [F_x] = 0$.

$$\mathfrak{C}: \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \quad \Rightarrow [\rho u_x] = 0$$

$$\mathfrak{M}: \quad \frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = \mathbf{0} \quad \Rightarrow [\Pi_{xx}] = 0 \Rightarrow \qquad \left[\rho u_x^2 + p + \frac{\mathbf{B}^2 - 2B_x^2}{8\pi}\right] = 0$$

$$\Rightarrow [\Pi_{xy}] = 0 \Rightarrow \qquad \left[\rho u_x u_y - \frac{B_x B_y}{4\pi}\right] = 0 \quad (\text{sim. } z)$$

$$\mathfrak{E}: \qquad \frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{F}_\epsilon = 0 \quad \Rightarrow [F_{\epsilon,x}] = 0 \Rightarrow \qquad \left[\rho u_x \left(\frac{1}{2}|\mathbf{u}|^2 + h\right) + \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})_x\right] = 0$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow [B_x] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + c\nabla \times \mathbf{E} = \mathbf{0} \quad \Rightarrow [E_y] = 0 \quad (\text{sim. } z)$$

where we have twice noted the symmetry of the setup under $y \leftrightarrow z$. This gives us $1+(1+2)+1+1+2 = 8$ relations between the $1+3+1+3$ variables ρ, $\mathbf{u}$, p, $\mathbf{B}$ on either side (recalling that $\mathbf{E} = \mathbf{u}\times\mathbf{B}/c$). Thus if we know all 8 of the values of these variables on one side, we can deduce all of the 8 values on the other side.

Note that $[s] = c_V[\ln(p\rho^{-\gamma})] \neq 0$. Although the fluid on either side is ideal, viscosity generates entropy within the shock.

5.3.2 Non-Magnetic Normal Shocks

Setting $\mathbf{B} = \mathbf{0}$, $u_x = u$, $u_y = u_z = 0$ and dividing some jump conditions by the first, we find

$$[\rho u] = 0 \qquad \left[u + \frac{p}{\rho u}\right] = 0 \qquad \left[\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right] = 0$$

We define the *Mach number* of a shock as $\mathcal{M} \equiv u_{x1}/c_{s1}$. The sound speed is $c_s^2 = \gamma p/\rho$, so

$$\left[u\left(1+\frac{1}{\gamma\mathcal{M}^2}\right)\right] = 0 \qquad \left[u^2\left(\frac{1}{2}+\frac{1}{(\gamma-1)\mathcal{M}^2}\right)\right] = 0$$

$$u_1\left(1+\frac{1}{\gamma\mathcal{M}^2}\right) = u_2\left(1+\frac{1}{\gamma\mathcal{M}_2^2}\right) \qquad u_1^2\left(\frac{1}{2}+\frac{1}{(\gamma-1)\mathcal{M}^2}\right) = u_2^2\left(\frac{1}{2}+\frac{1}{(\gamma-1)\mathcal{M}_2^2}\right)$$

$$(u_1-u_2)\left(1+\frac{1}{\gamma\mathcal{M}^2}\right) = \frac{u_2}{\gamma}\left(\frac{1}{\mathcal{M}_2^2}-\frac{1}{\mathcal{M}^2}\right) \qquad (u_1^2-u_2^2)\left(\frac{1}{2}+\frac{1}{(\gamma-1)\mathcal{M}^2}\right) = \frac{u_2^2}{\gamma-1}\left(\frac{1}{\mathcal{M}_2^2}-\frac{1}{\mathcal{M}^2}\right)$$

Dividing the 2nd equation by the 1st:

$$(u_1+u_2)\frac{\frac{1}{2}+\frac{1}{(\gamma-1)\mathcal{M}^2}}{1+\frac{1}{\gamma\mathcal{M}^2}} = \frac{\gamma}{\gamma-1}u_2 \qquad \Rightarrow \qquad \frac{u_1}{u_2} = \frac{(\gamma+1)\mathcal{M}^2}{(\gamma-1)\mathcal{M}^2+2} = \frac{\rho_2}{\rho_1}$$

where in the final equality we have used the first jump condition. Note that for a *strong* shock ($\mathcal{M} \gg 1$), the ratio does not increase arbitrarily large, but up to a maximum of $(\gamma+1)/(\gamma-1)$, equal to 4 for a monatomic gas with $\gamma = 5/3$.

5.4 Supernovae

5.4.1 Initial Phase

A supernova is a sudden explosion of $M_{\text{ej}} \sim 1M_\odot$ in a spherically symmetric manner, with an energy $E_0 \sim 10^{51}\text{erg}$; the ejecta thus explodes at $v_{\text{ej}} = \sqrt{2E_0/M_{\text{ej}}} \sim 10^4\text{kms}^{-1} \gg 1\text{kms}^{-1} \sim c_{s,ISM}$, causing an outwardly propagating shock bubble in the ISM. Initially, the ejection speed is roughly constant, as the ISM into which the ejecta is flung has a much lower density. However, the shock accumulates material from the ISM, with a density $\rho_0 \sim 1m_P\text{cm}^{-3}$; when the mass of this accumulated material is of order M_{ej}, its inertia becomes significant enough for the shock to notice that the ISM is actually there, rather than just barrelling through it. This transition occurs when the shock bubble has reached a radius R_T, where

$$\frac{4}{3}\pi\rho_0 R_T^3 \sim M_{\mathrm{ej}} \qquad \Rightarrow \qquad R_T \sim 2\mathrm{pc}$$

and after a time $t_T \sim R_T/v_{\mathrm{ej}} \sim 200\mathrm{yr}$. By then, the explosion's energy has been almost entirely dissipated to the thermal energy of the ISM behind the shock, and it subsequently slows down in a very characteristic way.

5.4.2 Adiabatic Sedov-Taylor Phase

The only important variables post-transition are E_0 and ρ_0. The time evolution of the bubble's radius $R(t)$ can be derived by dimensional analysis:

$$\{R\} = \mathbb{L}, \quad \{t\} = \mathbb{T}, \quad \{E_0\} = \mathbb{M}\mathbb{L}^2\mathbb{T}^{-2}, \quad \{\rho_0\} = \mathbb{M}\mathbb{L}^{-3} \quad \Rightarrow \quad R(t) = \alpha\left(\frac{E_0}{\rho_0}t^2\right)^{1/5} \propto t^{2/5}$$

for some constant α, which we will deduce later.

The two variables distinguishing one supernova from another (E_0 and ρ_0) cannot be used to construct a length scale. The problem is therefore "self-similar", and means that all variables $f(r,t)$ can be angesetzt as a function of the dimensionless similarity variable $\xi \equiv r/R(t)$, multiplied by a time-dependent dimensional constant: $f(r,t) = f_0(t)\tilde{f}(\xi)$. For example:

$$\rho(r,t) = \rho_0\tilde{\rho}(\xi) \qquad u(r,t) = \dot{R}(t)\tilde{u}(\xi) \qquad p(r,t) = \rho_0\dot{R}(t)^2\tilde{p}(\xi)$$

Note that we will only be interested in the region inside the bubble: $r < R$, $\xi < 1$; outside the bubble the ISM has not been affected by the blast wave and the variables simply take the background ISM values.

Before substituting these Ansätze[11] into the fluid equations, we first note some lemmata[11]:

$$\frac{\partial\tilde{f}}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} = -\frac{r}{R^2}\dot{R}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} = -\frac{\dot{R}}{R}\xi\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} \qquad \frac{\partial\tilde{f}}{\partial r} = \frac{\partial\xi}{\partial r}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} = \frac{1}{R}\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi}$$

$$R(t) \propto t^{2/5} \qquad \Rightarrow \qquad \frac{\dot{R}}{R} = \frac{2}{5t}, \quad \text{and} \quad \frac{\ddot{R}}{R} = -\frac{6}{25t^2} = -\frac{3}{2}\frac{\dot{R}^2}{R^2}$$

Substituting into $\mathfrak{C}$ and $\mathfrak{M}$, and noting that R depends on t but not ξ to help cancelling,

$$\begin{aligned}
0 &= \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho u\right) & 0 &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} \\
0 &= -\frac{\dot{R}}{R}\xi\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\xi} + \frac{1}{R^2\xi^2}\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(R^2\xi^2\tilde{\rho}\dot{R}\tilde{u}\right) & 0 &= \ddot{R}\tilde{u} - \xi\dot{R}^2\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \frac{\dot{R}^2}{R}\tilde{u}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \frac{\dot{R}^2}{R}\frac{1}{\tilde{\rho}}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\xi} \\
0 &= -\xi\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\xi} + \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\tilde{\rho}\tilde{u}\right) & 0 &= -\xi\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \tilde{u}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \frac{1}{\tilde{\rho}}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\xi} + \frac{R\ddot{R}}{\dot{R}^2} \\
0 &= (\tilde{u} - \xi)\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\xi} + \frac{\tilde{\rho}}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\tilde{u}\right) & 0 &= (\tilde{u} - \xi)\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \frac{1}{\tilde{\rho}}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\xi} + \frac{R\ddot{R}}{\dot{R}^2}
\end{aligned}$$

where we have neglected $\mathbf{B}$ fields and gravity.

[11] Languages!

Now for energy. Note $\epsilon = \rho\left(\frac{1}{2}u^2 + e\right) = \frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}p$, and $F_{\epsilon,r} = u\left(\frac{1}{2}\rho u^2 + \frac{\gamma}{\gamma-1}p\right)$. Thus

$$\frac{\partial \epsilon}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_{\epsilon,r}\right) = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial t}\Big(\underbrace{r^2\epsilon}_{\equiv q}\Big) + \frac{\partial}{\partial r}\Big(\underbrace{r^2 F_{\epsilon,r}}_{\equiv F}\Big) = 0 \qquad \Rightarrow \qquad \frac{\partial q}{\partial t} + \frac{\partial F}{\partial r} = 0 \quad (\mathfrak{e})$$

where we have defined $q \equiv r^2\epsilon$ and $F \equiv r^2 F_{\epsilon,r}$. They have the self-similar forms:

$$q = \rho_0 R^2 \dot{R}^2 \xi^2 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma-1}\tilde{p}\right) \qquad\qquad F = \rho_0 R^2 \dot{R}^3 \xi^2 \tilde{u}\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma}{\gamma-1}\tilde{p}\right)$$

Now defining the dimensionless variables $\tilde{q}(\xi)$ and $\tilde{F}(\xi)$ by:

$$q \equiv \rho_0 R^2 \dot{R}^2 \tilde{q} \qquad\qquad F \equiv \rho_0 R^2 \dot{R}^3 \tilde{F}$$

$$\Rightarrow \tilde{q} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma-1}\tilde{p}\right) \qquad\qquad \tilde{F} = \xi^2\tilde{u}\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma}{\gamma-1}\tilde{p}\right)$$

by comparison. Substituting into $\mathfrak{e}$, and using the easily derived $\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{R}^2\right) = -R\dot{R}^3$,

$$-\rho_0 R\dot{R}^3\tilde{q} + \rho_0 R^2\dot{R}^2\left(-\frac{\dot{R}}{R}\xi\frac{\mathrm{d}\tilde{q}}{\mathrm{d}\xi}\right) + \rho_0 R^2 \dot{R}^3\frac{1}{R}\frac{\mathrm{d}\tilde{F}}{\mathrm{d}\xi} = 0 \qquad \Rightarrow 0 = -\tilde{q} - \xi\frac{\mathrm{d}\tilde{q}}{\mathrm{d}\xi} + \frac{\mathrm{d}\tilde{F}}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tilde{F} - \xi\tilde{q}\right)$$

Thus $\tilde{F} = \xi\tilde{q}$; the integration constant is 0 (consider $\xi = 0$). Unwrapping $\tilde{F}$ and $\tilde{q}$,

$$\xi^2\tilde{u}\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma}{\gamma-1}\tilde{p}\right) = \xi^3\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma-1}\tilde{p}\right) \qquad \Rightarrow \qquad \tilde{p} = \frac{\gamma-1}{2}\frac{\tilde{u}-\xi}{\xi-\gamma\tilde{u}}\tilde{\rho}\tilde{u}^2$$

Substituting this into $\mathfrak{M}$ gives an equation in $\mathrm{d}\tilde{\rho}/\mathrm{d}\xi$, $\tilde{u}$ and $\mathrm{d}\tilde{u}/\mathrm{d}\xi$. Eliminating the $\tilde{\rho}$ terms with $\mathfrak{C}$ gives an ODE in $\tilde{u}$. Defining the variable $v \equiv \tilde{u}/\xi$, this ODE is apparently[12]

$$\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} = \frac{v(\gamma v - 1)[5 - (3\gamma - 1)v]}{\gamma(\gamma+1)v^2 - 2(\gamma+1)v + 2}$$

Perhaps surprisingly, this has an analytic, though implicit, solution, most easily derived by inverting and using partial fractions.

Before giving the solution, we must first discuss the boundary conditions. The boundary conditions are deduced by considering the conditions just on the inside of the shock bubble, at $r = R$, $\xi = 1$. The shock produced by a supernova is certainly a strong shock, with $\mathcal{M} \gg 1$. Thus we have $\rho(r = R) = \frac{\gamma+1}{\gamma-1}\rho_0$. As for the velocity, the velocity ratio is the inverse of the density ratio, but this is for velocities in the shock frame, whereas u here is defined relative to the $r = 0$ rest frame. In the shock frame, the ISM is moving towards $r = 0$ at a speed $\dot{R}$, so the shocked gas is moving towards $r = 0$ at $\frac{\gamma-1}{\gamma+1}\dot{R}$. Thus in the rest frame, the material is moving at $u(r = R) = \dot{R} - \frac{\gamma-1}{\gamma+1}\dot{R} = \frac{2}{\gamma+1}\dot{R}$. We therefore have

$$\tilde{\rho}(1) = \frac{\gamma+1}{\gamma-1} \qquad\qquad \tilde{u}(1) = \frac{2}{\gamma+1} \qquad\qquad \tilde{p}(1) = \frac{2}{\gamma+1} \qquad\qquad v(1) = \frac{2}{\gamma+1}$$

[12]Even *attempting* this derivation takes bloody ages

We can now deduce the full solution for $\xi(v)$, and hence the forms of $\tilde{\rho}(v)$ and $\tilde{p}(v)$. To simplify, we take a monatomic gas with $\gamma = 5/3$, leading to some gnarly powers:

$$\xi = \left(\frac{4v}{3}\right)^{-2/5} \left(\frac{20v}{3} - 4\right)^{2/13} \left(\frac{5}{2} - 2v\right)^{-82/195} \qquad \tilde{p} = \frac{3}{4096}(1-v)^{-5} \left(\frac{4v}{3}\right)^{6/5} \left(\frac{5}{2} - 2v\right)^{82/15}$$

$$\tilde{\rho} = \frac{1}{1024}(1-v)^{-6} \left(\frac{20v}{3} - 4\right)^{9/13} \left(\frac{5}{2} - 2v\right)^{-82/13} \qquad \tilde{u} = \xi v$$

v is now simply a parametrisation, without much physical meaning. What is the range of v? We see that $\xi(v = \frac{3}{5}) = 0$, and that $\xi(v = \frac{3}{4}) = 1$; we therefore conclude that v varies from 0.6 to 0.75. Plotting $\tilde{\rho}$, $\tilde{u}$ and $\tilde{p}$ against ξ over this range of v gives Figure 3. We see that $\tilde{\rho}$ is close to 0 except near $\xi = 1$, so most of the mass is swept into a thin shell near $r = R(t)$ at any given time. Further, $\tilde{u}$ is roughly proportional to ξ, giving a Hubble-esque flow. Finally, $\tilde{p}$ is close to its central value of ≈ 0.23 for most of the radial distance.

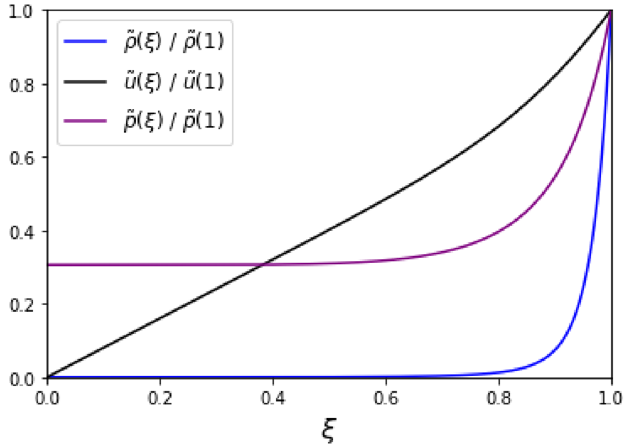

Figure 3 | Dimensionless Fluid Variables during a Supernova. Plotted are the implicit solutions $\tilde{\rho}(\xi)$, $\tilde{u}(\xi)$ and $\tilde{p}(\xi)$. We see that $\tilde{\rho}$ is very small except near $\xi = 1$, $\tilde{u} \sim \xi$, and $\tilde{p}$ is almost constant near to the centre of the bubble.

5.4.3 Energy Conservation

We have almost finished the analysis, but we need to scale things to find ξ's time dependence. We have established that $R = \alpha(E_0 t^2/\rho_0)^{1/5}$, but we have not yet established the constant α. We are neglecting energy losses during this phase, so

$$\begin{aligned} E_0 &= \int_0^{R(t)} 4\pi r^2 \epsilon \, \mathrm{d}r = 4\pi R^3 \int_0^1 \xi^2 \rho_0 \dot{R}^2 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma - 1}\tilde{p}\right) \mathrm{d}\xi \\ &= 4\pi\rho_0 \alpha^5 \frac{E_0}{\rho_0} \frac{4}{25} \int_0^1 \xi^2 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma - 1}\tilde{p}\right) \mathrm{d}\xi \\ \Rightarrow \alpha &= \left[\frac{16\pi}{25} \int_0^1 \xi^2 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{1}{\gamma - 1}\tilde{p}\right) \mathrm{d}\xi\right]^{-1/5} \end{aligned}$$

For the above solution with $\gamma = 5/3$, this integrates (numerically!) to about 1.188.

Thus with the supernova parameters E_0 and ρ_0 specified, we have established the evolution of $R(t)$. We can then use this and the formulae for the dimensionless (tilde) variables (in terms of the variable $\xi = r/R(t)$, which is now also known) to find $\rho(r, t)$, $u(r, t)$ and $p(r, t)$. Phew.

This *adiabatic Sedov-Taylor phase* lasts for $\sim$ 2000yr, before cooling processes such as Bremsstrahlung become important and the adiabaticity is lost. As it cools down, the expanding shell of material loses thermal pressure and compresses, but the inside of the shell is still hot, and "snowploughs" this dense shell outwards into the ISM; this is the *snowplough phase* of the supernova. Eventually, the shell slows below $c_{s,ISM}$ and essentially merges with the ISM.