Topics in Astrophysics

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1 Timescales & Collisions

For any quantity Q, a rough timescale for its change is:

$$\tau = \frac{Q}{|\,\partial Q/\partial t\,|}$$

If $t \ll \tau$, for a different timescale of interest t (such as a system's age) then Q is effectively in equilibrium over t.

Some processes involve exponential decays $Q(t) = Q_0 e^{-t/\tau}$, such as radioactive decay. Others involve power laws as $Q(t) = Q_0(t/T)^{-\alpha}$ – such processes are said to have no "intrinsic scaling" or be "scale-free", as their timescale $\tau = t/\alpha$ changes with time and is of order the system's age.

1.1 Examples

- **Dynamical timescale**: can be derived by dimensional analysis to be of order

$$\boxed{[G] = L^3 M^{-1} T^{-2}} \qquad \Rightarrow \qquad \tau_{\text{dyn}} \sim \sqrt{\frac{R^3}{GM}}$$

- **Collisional timescale**: the collision rate can be shown equal to $n\sigma v$ where n is the number density of impactors, σ the collision cross-section, and v the relative speed. Hence the collision time is

$$\tau_{\text{coll}} = \frac{1}{n\sigma v}$$

  σ may be larger than the actual size of the object e.g. in gravitational focusing. An object coming from infinity at v_∞ and impact parameter b will collide with the body if

$$\pi b^2 \leq \pi R^2 \left(1 + \frac{2GM}{R v_\infty^2}\right)$$

  and so the collision cross-section is effectively larger than πR^2.

- **Thermal time**: in thermal equilibrium, heating and cooling processes are equal; if heating were switched off, it would take τ_{th} to exhaust its thermal energy. The thermal energy of a gas is $C_V T$; that of a photon gas is aT^4.

- **Sound crossing time**: pressure disturbances can be transmitted across a region of size R in time $\tau_{\text{sc}} = R/c_s$, where the speed of sound $c_s \sim \sqrt{p/\rho}$. For an ideal gas

$$p = \frac{\rho kT}{\mu m_H} \qquad \Rightarrow \qquad \tau_{\text{sc}} = R\sqrt{\frac{\mu m_H}{kT}}$$

- **Alfvén wave crossing time:** The Alfvén speed is $v_A = \sqrt{B^2/\rho\mu_0}$, hence the Alfvén wave crossing time is $R\sqrt{\rho\mu_0/B^2}$

- **Diffusion time:** Consider a random walk path, so that something travels a net vector $\mathbf{R} = \sum_{i=1}^{N} \boldsymbol{\lambda_i}$ where $|\boldsymbol{\lambda}_i| = \lambda \forall \boldsymbol{\lambda}_i$, $\langle \boldsymbol{\lambda}_i \rangle = \mathbf{0}$ and $\langle \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \rangle = 0$. Calculating the expectation value of $\mathbf{R}^2$ gives

$$\left\langle \mathbf{R}^2 \right\rangle = N\lambda^2$$

It therefore takes on average $N = R^2/\lambda^2$ steps to travel a distance R. If each step takes a time λ/v, then the diffusion timescale is

$$\tau_{\text{diff}} = \frac{R^2/\lambda^2}{\lambda/v} = \frac{R^2}{\lambda v}$$

For photons in the Sun, $\lambda = 1/(\kappa\rho)$, so the time taken to get out is $\sim R_\odot^2 \kappa\rho/c$.

- **Light crossing time** is the absolute minimum: $\tau_\gamma = R/c$

1.2 Applications

1.2.1 Light Echo

Consider a variable star surrounded by dust. If the star pulses, not only will we see the pulse directly from the star, but we will also see light from the pulse reflected off of the surrounding dust. This reflected light (the *echo*) will be seen a short time after the pulse itself, as the light must travel along a slightly longer path to reflect off the dust before it reaches us (see Figure 1). For dust at a radius R from the star and at an angle θ to the line of sight, this distance can be seen to be $R(1 - \cos\theta)$, so the time delay is $\tau = R(1 - \cos\theta)/c$.

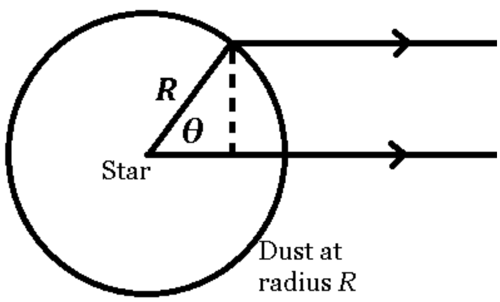

Figure 1 | Light Echo Geometry.

If we replace the variable star with a variable quasar, the dust will be orbiting in an accretion disc, with circular velocity $v = \sqrt{GM/R}$ ish. We can measure the line-of-sight velocity $v_\perp = \sqrt{GM/R}\sin\theta$, and plot this against $\tau = R(1 - \cos\theta)/c$. For all θ at a particular R, θ parametrises an ellipse with semi-axes R/c and $\sqrt{GM/R}$. The envelope of these ellipses can be found by deparametrising (eliminating θ: $\frac{Rv_\perp^2}{GM} + (1 - \frac{c\tau}{R})^2 = 1$), differentiating this equation wrt R ($\Rightarrow \frac{v_\perp^2}{GM} - \frac{2c\tau}{R^2}\left(1 - \frac{c\tau}{R}\right) = 0$) and eliminating R. This gives $\tau = \frac{GM}{c} v_\perp^{-2}$. Hence from the envelope we can deduce M, the mass of the quasar.

1.2.2 Two-Body Relaxation

This is a similar analysis to in the SDSG course, but it goes slightly differently.

When a star of mass m passes a large distance (or at high speed) past another at impact parameter b and speed v, it will acquire a slight perpendicular velocity of about the acceleration at b times the time spent around there:

$$\Delta v_\perp \sim \frac{Gm}{b^2}\frac{b}{v} = \frac{Gm}{bv}$$

Now it will take a number $N = (v/\Delta v_\perp)^2$ interactions for the star to be deflected by an amount comparable with its original velocity (like a random walk in v-space). The rate of interactions is $n\sigma v \sim nb^2 v$ where n is the number density, so the timescale for two-body relaxation is

$$\tau_{2br} \sim \left(\frac{bv^2}{Gm}\right)^2 \frac{1}{nb^2 v} = \frac{v^3}{G^2 m^2 n}$$

which nicely is independent of b.

2 Distributions

For a probability distribution function $p(q)$, the probability that $q \in [q_1, q_2]$ is

$$\int_{q_1}^{q_2} p(q)\,\mathrm{d}q = \int_{\ln q_1}^{\ln q_2} qp(q)\,\mathrm{d}(\ln q)$$

Where the latter form is useful on a logarithmic plot, used when $p(q) \propto q^{-\alpha}$. In such cases, the behaviour of $q^{1-\alpha}$ determines which end of the distribution dominates the integral; also the cumulative distribution is proportional to $q^{1-\alpha}$. If $\alpha > 1$, then the distribution drops off very quickly and lower-q contributions dominate; if $\alpha < 1$ the distribution either drops off slowly or increases, and hence higher-q contributions dominate.

The total "amount of q" contributed between q_1 and q_2 is

$$\int_{q_1}^{q_2} qp(q)\,\mathrm{d}q$$

2.1 Applications

2.1.1 Initial Mass Function

A common distribution is the initial mass function (IMF) $f(M)$, and above about $1M_\odot$ we observe $f(M) \propto M^{-\alpha}$ with $\alpha = 2.35$. The IMF is to be distinguished from the present day mass function (PDMF) because of the earlier death of high-mass stars. The total luminosity coming from stars between M_1 and M_2 is

$$\int_{M_1}^{M_2} L(M) f(M)\,\mathrm{d}M$$

We observe that $L(M) \propto M^{3.5}$, so the total luminosity $\propto [M^{3.5-2.35+1}] = [M^{2.15}]$.

2.1.2 Collisional Cascade

Assume the number of asteroids of a size a has number density distribution $n(a) \propto a^{-b}$. We assume that fragmentational collisions occur only between asteroids of similar orders of magnitude (a smaller asteroid will simply scrape a much larger one), that velocity is independent of size, and that the rate of mass loss is independent of size.

The number of asteroids in a logarithmic bin size $\mathrm{d}(\ln a)$ is $a^{1-b}\,\mathrm{d}\ln a$, and the mass contained within is $a^{4-b}\,\mathrm{d}(\ln a)$. When two asteroids of size $\sim a$ collide and fragment, they leave this bin. The rate of these collisions is

$$n\sigma v \propto a^{1-b}a^2a^0\,\mathrm{d}(\ln a) \propto a^{3-b}\,\mathrm{d}(\ln a)$$

In each of these collisions the two asteroids in question are removed from that bin, so the rate of mass loss from that bin is $\sim$

$$a^{4-b}a^{3-b}\,\mathrm{d}(\ln a) = a^{7-2b}\,\mathrm{d}(\ln a)$$

As we assumed that the rate of mass loss is independent of a, this fixes $a = 7/2$.

3 Tides

3.1 Tidal Acceleration

Consider two masses M_1 and M_2 orbiting their mutual CoM and at a distance a from each other. We will consider the tidal forces exerted on M_2 as a result of M_1. For a circular orbit, centrifugal balance for M_2 gives:

$$\frac{GM_1}{a^2} = \Omega^2\frac{M_1 a}{M_1+M_2} \qquad \Rightarrow \qquad \Omega^2 = \frac{G(M_1+M_2)}{a^3}$$

Consider a point located at coordinates $(x, y), x, y \ll a$ relative to the centre of M_2, which wlog we put along the x-axis. The acceleration at the point (x, y) is approximately

$$\begin{aligned}\ddot{x} &= -\frac{GM_1}{(a+x)^2+y^2}\frac{a+x}{\sqrt{(a+x)^2+y^2}} + \Omega^2\frac{M_1 a}{M_1+M_2} \approx -\frac{GM_1}{(a+x)^2} + \Omega^2\frac{M_1 a}{M_1+M_2} \\ &\approx \underbrace{-\frac{GM_1}{a^2} + \Omega^2\frac{M_1 a}{M_1+M_2}}_{0 \text{ by centrifugal balance}} + \frac{2GM_1}{a^3}x = \frac{2GM_1}{a^3}x\end{aligned}$$

which is repulsive: relative to the centre of M_2, the point (x, y) is not being attracted as much and so appears to be flung away from the centre. In the y-direction:

$$\ddot{y} = -\frac{GM_1}{(a+x)^2+y^2}\frac{y}{\sqrt{(a+x)^2+y^2}} \approx -\frac{GM_1}{a^3}y$$

which is attractive (due to the y-component of the gravitational force) and has half the magnitude of the x-acceleration. The tendency of a tide is therefore to stretch along the line of attraction and squeeze perpendicular to it. The accelerations above can be accounted by a "tidal potential":

$$\boxed{\Phi_T = \frac{GM_1}{a^3}\left(-x^2 + \frac{1}{2}y^2\right)}$$

A fluid will fill up to an equipotential surface, the sum of the tidal and gravitational potentials. For instance, the oceans on Earth bulge a distance Δ higher where the Moon is overhead[1], than where it is perpendicular. Say the Moon is over Accra (0° longitude, $(x, y) = (R_2, 0)$) where the ocean is at a height $R_2 + \Delta$, and in Dhaka (90°, $(x, y) = (0, R_2)$) the ocean is at simply R_2. For the equipotential to pass through both, we require:

$$-\frac{GM_2}{R_2+\Delta} - \frac{GM_1}{a^3}(R_2+\Delta)^2 = -\frac{GM_2}{R_2} + \frac{GM_1}{a^3}\frac{1}{2}{R_2}^2$$

which can be expanded to 1st order and solved for Δ; alternatively we can use the fact that the gravitational potential difference will be about $g\Delta = \frac{GM_2}{{R_2}^2}\Delta$, which can be set equal to minus the tidal potential difference. Either way, using the assumption that $R_2 \ll a$, we obtain

$$\boxed{\frac{\Delta}{R_2} = \frac{3}{2}\frac{M_1}{M_2}\left(\frac{R_2}{a}\right)^3}$$

For the Moon's tides on Earth this is about 60cm, though coastal resonances alter this. We see that $\Delta \propto M_1/a^3 \propto \rho_1 {R_1}^3/a^3 \propto \rho_1\theta^3$, where θ is the angular size as seen from M_2. The Moon and the Sun have about the same θ, so the ratio of the tidal bulges caused by the two is $\rho_M/\rho_\odot \approx 3$. Thus the Moon dominates the tides; the Sun provides second-order Spring and Neap tides.

3.2 Roche Potential

Consider a circular binary system with M_1 and $M_2 \ll M_1$, a apart. In the frame corotating with the orbital frequency $\omega = \sqrt{G(M_1+M_2)/a^3}$. In 2D, the centrifugal force is $\omega^2\mathbf{r}$, so we can include a fictitious potential $-\frac{1}{2}\omega^2 r^2$, giving an overall *Roche potential* (Figure 2).

$$\Phi_R(\mathbf{r}) = -\frac{GM_1}{|\mathbf{r}-\mathbf{r}_1|} - \frac{GM_2}{|\mathbf{r}-\mathbf{r}_2|} - \frac{1}{2}\omega^2 r^2$$

This potential has 5 stationary (*Lagrange*) points, two of which (L1 & L2) lie on the same equipotential and are approximately equidistant from M_2 if $M_2 \ll M_1$. L1,2,3 are unstable, whereas L4,5 are stable: despite being maxima in the Roche potential, this does not account for Coriolis; if a body at L4/5 is perturbed its velocity will in fact bring it back.

At L1, the attraction of M_1 is balanced by the centrifugal force and the attraction of M_2. Setting L1 at a distance x on the near side of M_2 and recalling that $|\mathbf{r}_2| = \frac{M_1}{M_1+M_2}a$, we have

$$\begin{aligned}
\frac{GM_1}{(a-x)^2} &= \frac{GM_2}{x^2} + \omega^2\left(\frac{M_1}{M_1+M_2}a - x\right) \\
\Rightarrow \frac{GM_2}{x^2} &\approx \frac{GM_1}{a^2} + \frac{2GM_1}{a^2}\frac{x}{a} - \frac{GM_1}{a^2} + \frac{G(M_1+M_2)}{a^2}\frac{x}{a} \approx \frac{3GM_1}{a^3}x \\
\Rightarrow & \boxed{x_{L1} = \left(\frac{M_2}{3M_1}\right)^{1/3} a}
\end{aligned}$$

[1] Or at the point antipodal to this sublunar point

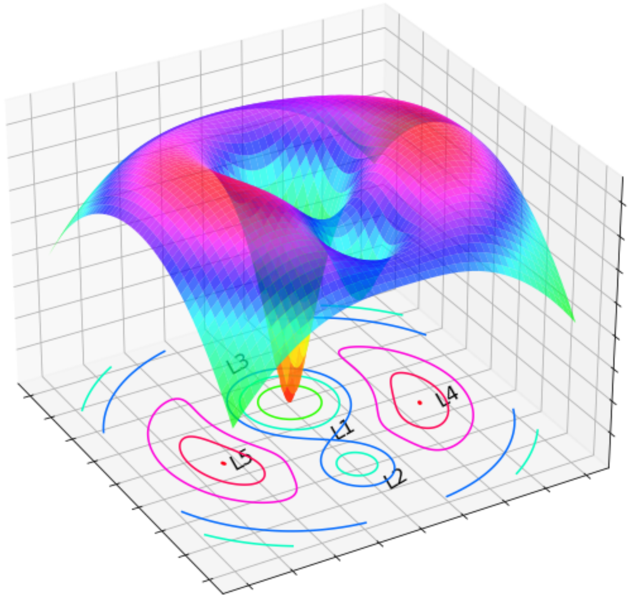

Figure 2 | Roche Potential and Lagrange Points. In this diagram, $M_2 = 0.4M_1$.

If R_2 exceeds this (Roche, or Hill, or tidal) radius, whether by expanding (increasing R_2) or by spiralling inwards (decreasing a), then M_1 will pull off bits of M_2 on the near side, and on the far side bits of M_2 will be flung off by the centrifugal force.

It turns out that if $M_1 \sim M_2$, as is often the case for X-ray binaries, L1 is slightly nearer to M_2 than L2 is. As such, matter will be drawn out of M_2 from L1 before it gets flung out the back at L2.

3.3 Tidal Energy

The energy dissipated in the tide on M_2 due to M_1 is roughly:

$$E_{\text{tide}} \approx m_{\text{bulge}} g h \approx \left(\underbrace{4\pi {R_2}^2 \Delta}_{\approx V_{\text{bulge}}} \cdot \underbrace{\frac{M_2}{\frac{4}{3}\pi {R_2}^3}}_{\approx \rho_{\text{bulge}}} \right) \left(\frac{G M_2}{{R_2}^2} \right) \Delta \propto \frac{G {M_2}^2}{{R_2}^3} \Delta^2 \propto \frac{G {M_1}^2 {R_2}^5}{a^6}$$

From this we see that the ratio of the energy dissipated in M_2 to M_1 is

$$\frac{E_{\text{tide, 2}}}{E_{\text{tide, 1}}} = \left(\frac{M_1}{M_2} \right)^2 \left(\frac{R_2}{R_1} \right)^5$$

Hence for a pair of MS stars (which have $M \propto R$), more energy is dissipated in the larger body.

3.3.1 Tidal Capture

If M_2 is initially unbound and flies by M_1, enough energy may be dissipated through tidal interactions on M_2 to *tidally capture* it. To go from an unbound orbit of energy $\frac{1}{2} M_2 v_\infty^2$ to a bound orbit of energy < 0, we require $E_{\text{tide}} > \frac{1}{2} M_2 v_\infty^2$, so:

$$\text{Tidal Capture:} \qquad \frac{M_2 v_\infty^2}{2} < \frac{G {M_1}^2 {R_2}^5}{a_p^6} \qquad \Rightarrow \qquad v_\infty a_p^3 \lesssim \sqrt{\frac{G {M_1}^2 {R_2}^5}{M_2}}$$

In a cluster, the dispersion is $v_\infty \sim 10$km/s. Thus for two Sun-like stars we require $a_p \lesssim 3.5R_\odot$. Hence tidal capture binaries can only form either very tightly, or if R_1 is very compact, such as for a white dwarf (this is how X-ray binaries begin).

For M_1 a Sun-like star and M_2 a rogue Jupiter-like ($10^{-3}M_\odot$, $0.1R_\odot$) planet, and $v_\infty \approx 30$km/s, this gives about $a_p \lesssim R_\odot$. It is thus difficult to tidally capture rogue planets.

3.3.2 Tidal Destruction

Further than being captured, enough energy may be dissipated to destroy the object: if E_{tide} at pericentre is around the binding energy of M_2 (of order $G{M_2}^2/R_2$), then the body will be tidally destroyed. This condition is

$$\text{Tidal Destruction:} \quad \frac{G{M_1}^2{R_2}^5}{a_p^6} \gtrsim \frac{G{M_2}^2}{R_2} \qquad \Rightarrow \qquad R_2 \gtrsim \left(\frac{M_2}{M_1}\right)^{1/3} a_p$$

which is compatible with the calculation in the previous subsection, where the RHS was identified as the Roche limit (to within a factor of $3^{1/3}$) What happens to M_2 depends on E_{tide}: if it is above $G{M_2}^2/R_2$ then M_2 will be unbinded and destroyed; if

$$\text{Tidal Capture (intact):} \quad \frac{G{M_2}^2}{R_2} \gtrsim E_{\text{tide}} \gtrsim \frac{1}{2}M_2 v_\infty^2$$

then M_2 will be tidally captured intact; if $E_{\text{tide}} < \frac{1}{2}M_2v_\infty^2$ then M_2 will remain unbound to M_1 and will drift away (though eventually reaching a speed $< v_\infty$).

If the double inequality above sandwiches shut, then it will not be possible to tidally capture intact – tidal effects will either destroy M_2 or fail to even capture it. This condition is

$$\text{Intact Tidal Capture Impossible:} \quad \frac{1}{2}M_2 v_\infty^2 \lesssim \frac{G{M_2}^2}{R_2} \qquad \Rightarrow \qquad v_\infty \lesssim \sqrt{\frac{2GM_2}{R_2}}$$

So if the body's original velocity is far above its own escape velocity, it will either remain free or be destroyed.

If a star M_2 passes by a star M_1 which hosts a planetary disc, a rule of thumb is that M_2 will transfer enough energy to the disc to unbind all of the disc material that is at a radius greater than M_2's pericentre. This may reduce M_2's energy below 0, leading to it becoming bound. It turns out that gravitational focusing is strong in these interactions, and so the collision cross-section for a disc of radius R_d is

$$\sigma \approx \pi R_d^2\left[\frac{2GM_1}{R_d v^2}\right] = \frac{2\pi G M_1 R_d}{v^2}$$

Hence the rate at which such collisions (potentially binary-forming events) occur for a given star is thus

$$\Gamma = n\sigma v = \frac{2\pi n G M_1 R_d}{v}$$

Multiplying by the number of stars (and technically dividing by 2 to avoid double-counting but we're way beyond that now) gives the total rate of collisions in the whole cluster.

When a star passes within $a_p \lesssim \left(\frac{M_1}{M_2}\right)^{1/3} R_2$ of a black hole of mass M_1 it gets tidally destroyed (=disrupted). If this a_p is less than the Schwarzschild radius $2GM_1/c^2$ then the star will be swallowed first. This condition is

$$\text{No Tidal Disruption:} \quad \left(\frac{M_1}{M_2}\right)^{1/3} R_2 < \frac{2GM_1}{c^2} \qquad \Rightarrow \qquad M_1 > \sqrt{\frac{c^6 {R_2}^3}{8G^3 M_2}}$$

which for a Sun-like star is of order $10^8 M_\odot$. If M_1 is less than this, the star may get tidally disrupted if it passes within the a_p above. At a radius r, if the star's velocity vector lies within a *loss cone*, it will pass within a_p and be disrupted. By angular momentum conservation, this loss cone is defined by an angle θ_L where

$$rv \sin\theta_L = \ell_{\text{crit}}$$

where ℓ_{crit} is a critical pericentre angular momentum which apparently is independent of r and v. Anyway this means a thinner loss cone (smaller $\sin\theta_L$) for larger r.

3.4 Transfer of Angular Momentum

Consider two general systems transferring E and J. The total energy change will be $\Delta E = \left(\frac{\partial E_2}{\partial J_2} - \frac{\partial E_1}{\partial J_1}\right)\Delta J$, where ΔJ and ΔE are transferred from system $1 \to 2$. It can be shown that the quantities $\partial E/\partial J$ are equal to the relevant angular frequency Ω – e.g. for a rotating body, $\partial E/\partial J$ is the angular frequency of rotation; for a system of two orbiting bodies $\partial E/\partial J$ is the angular velocity of the orbit. Thus $\Delta E = (\Omega_2 - \Omega_1)\Delta J$.

Now tides dissipate energy away from rotations (via friction into heat), so $\Delta E < 0$ and $\Omega_2 - \Omega_1$ has the opposite sign to ΔJ. Hence for $\Delta J > 0$ (transfer of J to system 2) we require $\Omega_2 - \Omega_1 < 0$, i.e. system 1 rotates faster. Hence *angular momentum is passed from faster to slower systems* in terms of their angular velocity.

For example, the Earth rotates (about 27 times) faster than the orbit of the Moon, so angular momentum is transferred from the Earth's rotation to the Moon's orbit. This causes the Earth's day to slow down and the Moon's orbit to creep out[2]. The physical mechanism behind this is friction between the Earth's fast-rotating floor and the oceans, which drags the high tide further east than the sublunar point; the Moon exerts a torque on the oblique bulge to drag it back, slowing the Earth's orbit.

A stable situation is where $\Omega_1 = \Omega_2$. This leads to tidal locking – for the Moon, the angular frequencies of its orbit and its rotation are equal, so that the same side always faces the Earth. Eventually the Earth's rotation will also slow to become equal to the same angular frequency as the Moon's rotation and orbit, so that the same side of the *Earth* faces the Moon; Pluto and Charon have already reached this stage of being "totally tidally despun".

4 Cluster Dynamics

From the virial theorem, $2T + \Omega = 0$. As such, $E = T + \Omega = -T$, and $\partial E/\partial T = -1$. As the kinetic energy T is a way of defining the "temperature" of a group of stars, we see that the heat capacity of a virialised system is negative. This is an unstable situation, as if a region of stars loses energy, it will become "hotter" as it sinks to the centre and gains kinetic energy. This will increase the region's temperature (despite losing energy overall as its potential becomes more

[2]The angular momentum is transferred to a rather than Ω. In fact, the Moon's angular velocity *decreases*! This causes the gap in angular velocities to get larger and it becomes a runaway process

negative), increasing the temperature gradient and hence the rate of loss of energy... The result is that the cores of clusters become exponentially hotter and denser over time, and eventually implode.

This situation can be avoided by the formation of binaries, which tends to fling stars back outwards to replenish the outer halo of a cluster. The energy of a cluster is about $-G(100M)^2/r_c$, and that of a binary is about $-GM^2/r_b$, and $r_c \sim 10^4 r_b$, so the energy stored in a binary is sufficient to reinflate the cluster.

5 The Solar System

5.1 The Planets

Only some of the inner planets have retained an atmosphere. Mercury instead has an *exobase*, which has lost the title of atmosphere because the mean free path of the molecules is larger than the pressure scale height and so the molecules are essentially collisionless. The mean free path is given by $\ell = (\sigma n)^{-1} = kT/\sigma p$; the pressure scale height is found from:

$$g = \frac{1}{\rho}\boldsymbol{\nabla} p = \frac{kT}{\mu m_H}\boldsymbol{\nabla} \ln \rho \qquad \Rightarrow \qquad \rho = \rho_0 \exp\left(\frac{g\mu m_H}{kT} z\right) \qquad \Rightarrow \qquad H = \frac{kT}{\mu m_H g}$$

So we see that an atmosphere requires $p > \mu m_H g/\sigma$.

The outer planets are split between gas giants of H and He, and ice giants of water, ammonia, and methane. They are all far less dense than the inner planets, indicating a partition of chemistry in the protoplanetary disk.

Unlike H and He, metals aren't blown away as the star forms, so the metals provide a less biased view of what the protoplanetary environment was like. The metal content of the Sun is of order 10^{28}kg, whereas that of the planets is more like 10^{27}kg, so the efficiency of planet formation is only about 10%.

The (rotational) angular momentum of the Sun is $L_\odot \propto k^2 M_\odot R_\odot \Omega_\odot = (0.3)^2(2 \times 10^{30})(7 \times 10^8)^2(10^{-6}) \sim 10^{41}\text{kgm}^2\text{s}^{-1}$. The (orbital) angular momentum of Jupiter alone is $L_J = M_J a_J^2 \Omega_J = M_J a_J^2 \sqrt{GM_\odot/a_J^3} \sim 10^{43}\text{kgm}^2\text{s}^{-1}$, 100 times greater than that of the Sun. Yet the Sun's *mass* is 1000 times greater.

5.1.1 Minimum Mass Solar Nebula

The minimum mass solar nebula (MMSN) is the surface density $\Sigma(r)$ of the protoplanetary disk (PPD). This is achieved by "augmenting the [metal-enriched] planets with H and He to restore a solar composition" and "spreading the augmented planetary masses through zones surrounding their orbits" (Weidenschilling, 1977). The result is that the MMSN is

$$\Sigma_{\text{MMSN}} \sim 10^{21} r^{-3/2}$$

which fits with estimates from extrasolar PPDs.

5.2 Smaller Bodies

5.2.1 Size Distribution

More directly observed than the size distribution $N(r) \propto r^{-q}$ is the magnitude distribution $N(H) \propto 10^{\alpha H}$. The magnitude is $H = C - 2.5 \log F$, and the flux $F \propto r^2$, so $H = C' - 5 \log r$.

Thus

$$r^{-q}\,\mathrm{d}r \propto 10^{\alpha(-5\log r)}\,\mathrm{d}H = r^{-5\alpha}\overbrace{\frac{\mathrm{d}H}{\mathrm{d}r}}^{-5/r}\mathrm{d}r \propto r^{-5\alpha-1}\,\mathrm{d}r \qquad \Rightarrow \qquad q = 5\alpha+1$$

(the $+1$ was missing from the notes but it's in papers n stuff). Observations show $q = -7/2$ in the asteroid belt, as would be expected for a collisional cascade.

The Kuiper Belt has no large objects in it, and the whole thing has only 1% the mass of the Earth. This may be because the collision timescale is proportional to $(\Sigma\Omega)^{-1} \propto a^3$, so it would take ages to grow anything big.

5.2.2 Differentiation

Some asteroids (*chondrites*) contain pristine grains (*chondrules*), whose composition is representative of the PPD.

Other asteroids are differentiated into a metallic (mostly Fe) core and a silicate mantle. These give rise to different types of meteorite depending on whether it broke off the asteroid's core (iron meteorites) or the mantle (achondrites). This differentiation requires the sinking of the metal to the core, which requires the asteroid to melt somehow. There are two ways this could have happened:

GPE Release. The gravitational potential of a uniform sphere is $U = -\frac{3}{5}\frac{GM^2}{R}$. This is converted to thermal energy on formation, and the maximum temperature this could have raised the asteroid to depends on U and the specific heat of the asteroid C_pM. This would have been sufficient to melt the rocky planets, but not 4 Vesta.

Radionuclides. The temperature change due to radioactive decay is

$$\frac{\mathrm{d}\Delta T}{\mathrm{d}t} = \frac{Q(t)}{C_p} = \frac{Q_0X_0}{C_p}e^{-\lambda t} \qquad \Rightarrow \qquad \Delta T = \int_{t_0}^{\infty}\frac{\mathrm{d}\Delta T}{\mathrm{d}t}\,\mathrm{d}t = \frac{Q_0X_0}{\lambda C_p}e^{-\lambda t_0} \sim 10^3\mathrm{K}$$

where $Q_0 = 0.355\mathrm{Wkg}^{-1}$ is the initial rate of heat production per kg of ^{26}Al, $X_0 = 7\times10^{-7}$ is its initial abundance, $\lambda = \ln 2/(7\times10^5\mathrm{yr})$ is its decay constant, and t_0 is the time it takes to form the object. This would have been sufficient to melt 4 Vesta.

6 Exoplanets

6.1 Demographics

Most planets seem to have $M \gg M_E$, $r < r_E$, as shown in Figure 3. Three populations emerge:

- **Hot Jupiters:** 10^2-$10^3 M_E$ and 10^{-2}-10^{-1}au. About 1% of stars have these easy planets. They are expected to have migrated during their lifetime.
- **Super Earths:** 1-10M_E. May be more like Neptune than Earth. More likely to have formed in situ and not moved much.
- **Ice Giants:** 10^2-$10^4 M_E$ and 1-10au.

There are notable dearths of planets:

- **Inner Edge:** $\lesssim 10^{-2}$au
- **Outer Edge:** $\gtrsim$ 10au (this is likely a measurement effect)

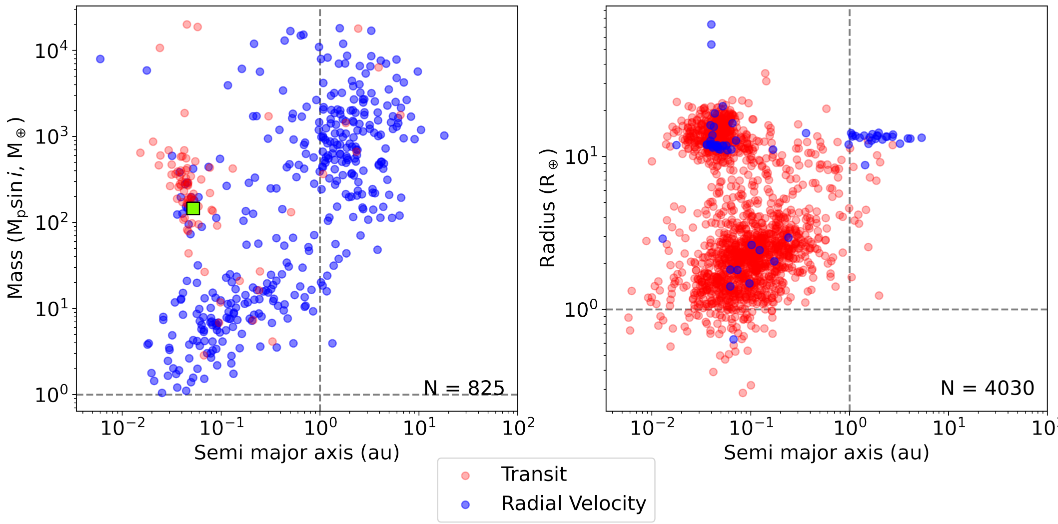

Figure 3 | Exoplanet Demographics.

- **Period Valley:** 0.1-1au
- **Planetary Desert:** 10-100M_E.
- **Fulton Gap:** 1.5-2R_E (not that visible in the above data)

Further, the planet-star ratio peaks at around 10^{-5}, between Earth and Neptune: Super Earths are the most common type of planet.

To figure out if these data match the MMSN, we spread the mass of a planet around an annulus of radius $r \pm \Delta r$ with $\Delta r = r/2$, which has surface area $\approx 2\pi r \Delta r = \pi r^2$. The surface density associated with that planet is then $\Sigma_i = M_i/\pi r_i^2$. Observationally we find $\Sigma \propto r^{-3/2}$ as with the MMSN, though with a proportionality constant about 5$\times$ larger.

With the relations $\Sigma \propto r^{-3/2}$, $\Sigma \sim Mr^{-2}$ and Kepler's $r^3 \propto T^2$, we have $M \propto r^{1/2} \propto T^{1/3}$ – a mass-period relation which works ok.

6.2 Limitations of Detection Methods

The main exoplanet detection methods are **radial velocity** and **transit**. In each case, the signal strength (the amplitude of the oscillations of spectral lines, or the amplitude of a flux reduction) depends to first order on the parameters of the system (e.g. M, r) and to second order on the details (e.g. eccentricity), which will only influence the detailed shape of the signal; they will only affect the signal strength by a factor $g \sim 1$, modifying the signal strength A to gA. If there is some noise σ associated with each measurement and one takes N measurements, the overall noise on the measurements will be $\sigma/\sqrt{N}$. The signal-to-noise ratio is then

$$\mathrm{SN} = \frac{gA}{\sigma/\sqrt{N}} = gN^{1/2}\frac{A}{\sigma}$$

6.2.1 Radial Velocity

For a circular orbit of M_P around M_* at a distance a_P, the star's velocity is

$$v_* \approx \frac{M_P}{M_*}\sqrt{\frac{GM_*}{a}}$$

If the orbit is inclined by an angle i, we will only observe a maximum line-of-sight velocity of amplitude $v_* \sin i \propto M_P a^{-1/2} \sin i$.

A spectral line at frequency f_0 will be shifted to $f' = f_0/(1+v_*/c)$, so in terms of the shift $\Delta f = f_0 - f'$, we have

$$\frac{\Delta f}{f_0} = 1 - \frac{1}{1+v_*/c} \qquad \Rightarrow \qquad \frac{v_*}{c} = \left(1 - \frac{\Delta f}{f_0}\right)^{-1} - 1$$

The noise, originating from stellar convection and the instruments, doesn't depend on the properties of the planet. To detect the planet, the SN must be above some critical value. Rearranging, we obtain

$$\mathrm{SN}_{\mathrm{crit}} \frac{\sigma}{\sqrt{N}} = gA \propto M_P a^{-1/2} \sin i$$

The minimum detectable mass thus varies as $M_{P,\mathrm{min}} \sin i \propto a^{1/2}$. As expected, planets further out must be more massive to be detectable. This explains a diagonal lower limit in Figure 3(a), which is about $M_P \sin i \propto a^{1/2}$.

The spectral resolution R of RV spectrometers is the maximum $f_0/\Delta f$ detectable, and is about 10^5. Expanding the expression for the radial velocity above, we find $\frac{v_*}{c} \approx \frac{1}{R}$, so the minimum detectable v_* is about $c/r \sim 10^8/10^5 = 1\mathrm{kms}^{-1}$. This looks bad as Jupiter causes $v_* \sim 10\mathrm{kms}^{-1}$ and Earth $\sim 1\mathrm{kms}^{-1}$; luckily other information (like the line widths) enables greater accuracy.

6.2.2 Transits

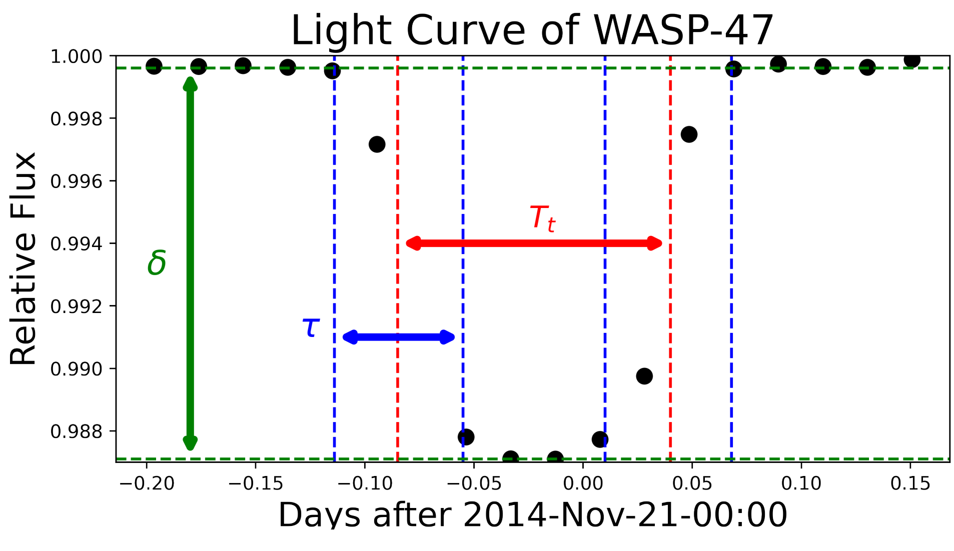

Figure 4 | Kepler Data from a Transit of WASP-47, by what I think is WASP-47b.

A transit light curve has several important features, shown in Figure 4 above:

- **Transit Depth $\delta = \Delta F/F_0$.** This gives the size of the planet relative to its host star, as $\delta = R_P^2/R_*^2$.

- **Transit Time T_t.** This is measured FWHM. Assuming the orbit is edge-on,

$$T_t = \frac{2R_*}{v_P} = \frac{2R_*}{2\pi a/T} = \frac{R_* T}{\pi a}$$

- **Duty Cycle $f_P = T_t/T = R_*/\pi a$.** The fraction of the planet's orbital period spent transiting the planet.
- **Duration of Ingress/Egress τ.** The time the planet takes to enter full transit. Again assuming an edge-on orbit:
$$\tau = \frac{2R_P}{v_P} = \frac{R_P T}{\pi a}$$
from which we also find that $\tau/T_t = R_P/R_* = \delta^{1/2}$.

The Earth has $\delta \sim 10^{-4}$, $T_t = 13$hrs and $\tau \sim 10$mins.

If the orbit is not edge-on ($i \neq 90°$), things are more complex. The transit may not even be visible from Earth: if the radii of planet, star, and orbit are known, the probability of transit is apparently $p(\text{transit}) \approx R_*/a$. Also, the quantity b is defined for a transit so that the projected distance between the centre of the star and the planet at conjunction is $bR_* = a\cos i$. For larger b, the planet will only be blocking out the redder limbs at the top of the star, giving different light curves at different wavelengths.

Again, the noise σ is independent of the planet's properties, however N, the number of measurements made of the transit, is proportional to the duty cycle f_P. The signal strength is δ. The limiting signal-to-noise ratio is then

$$\text{SN}_{\text{crit}} \propto f_P^{1/2}\delta \propto a^{-1/2}R_P^2 \qquad \Rightarrow \qquad R_P \propto a^{1/4}$$

explaining a diagonal lower limit in Figure 3(b).

These considerations of biases in the data have not explained many features of the exoplanet demographics, and so must relate to the planet formation process.

6.3 Metallicity

I really have no idea why this section is here. There's a bit about galactic chemical evolution but it's not worth the effort.

The metallicity Z of something is the proportion of its *mass* consisting of >He. For instance the Sun is 99% H and He so has a metallicity of $Z_\odot \approx 0.01$.

The quantity [Fe/H] is defined in the following way:

$$[\text{Fe/H}] = \log_{10}\left(\frac{N_{\text{Fe}}/N_{\text{H}}\big|_*}{N_{\text{Fe}}/N_{\text{H}}\big|_\odot}\right)$$

So a quantity that is *more* iron-rich than the Sun will have a *positive* [Fe/H]. Note that [Fe/H] uses the *number of atoms*, whereas Z uses mass.

The composition of planets should reflect that of the star, and indeed that of the molecular cloud from whence it formed. Metal-rich stars appear to be more likely to host giant planets, suggesting that their growth is aided by metals somehow.

In the "core accretion model" of giant planet formation, we start with a $Z = 1$ metal core of $M_c = 10M_E$, and then accrete a stellar-metallicity (Z_*) envelope to make up the rest of the planet's mass. Hence the metallicity of the overall planet is

$$Z_P = \frac{\overbrace{M_c + (M_P - M_c)Z_*}^{\text{metal mass in planet}}}{M_P} = Z_* + \frac{M_c(1 - Z_*)}{M_P} \qquad \Rightarrow \qquad \frac{Z_P}{Z_*} = 1 + \frac{M_c}{M_P}\frac{1 - Z_*}{Z_*}$$

Observationally though, $Z_P/Z_* \propto M_P^{-1/2}$. Perhaps giant planets preferentially accrete metals over H/He, such as sweeping up planetesimals.

7 Protoplanetary Disks

The precursor to a protoplanetary disk is a turbulent molecular cloud. On collapse, there will be some non-zero angular momentum so the cloud collapses into a disk. Radially, the disk is supported by centrifugal force; vertically it is supported by pressure gradients.

7.1 Temperature

7.1.1 Radial

No one seems to be consistent with the definitions of intensity, radiance, flux, flux density etc. and the method given in the slides is incomprehensible as well as not quite compatible with the notes. Anywho, the latter gives the disk temperature $T_d(r)$ as

$$T_d = \frac{T_*}{2\pi}\left(\frac{R_*}{r}\right)^{3/4} \propto r^{-3/4}$$

The net torque τ_{net} on an annulus Δr is $\frac{\mathrm{d}\tau}{\mathrm{d}r}\Delta r$, and so the rate of work done is $\tau_{\text{net}}\Omega$:

$$\tau_{\text{net}}\Omega = \frac{\mathrm{d}\tau}{\mathrm{d}r}\Omega\Delta r = \left[\frac{\mathrm{d}(\tau\Omega)}{\mathrm{d}r} - \tau\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right]\Delta r$$

It is (apparently) the second term which is important, as it is the differential angular velocity across the annulus that creates friction. The dissipation rate per unit area $D(r)$ is simply this work rate divided by the area from which the heat is radiated: $2 \times 2\pi r\Delta r$ (the disk radiates from both sides). Thus

$$D(r) = \frac{\tau}{4\pi r}\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

In the steady state, this is equal to σT_d^4, so

$$T_d = \left[\frac{\tau}{4\pi\sigma r}\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right]^{1/4}$$

Continuing this requires expressions for τ and $\Omega(r)$. From fluid dynamics, the viscous force is $2\pi r\nu\Sigma \cdot r\ \mathrm{d}\Omega/\mathrm{d}r$, so the viscous torque is $2\pi r^3\nu\Sigma\ \mathrm{d}\Omega/\mathrm{d}r$. Let $\Omega(r)$ be the Keplerian $\sqrt{GM_*/r^3}$ and $\mathrm{d}\Omega/\mathrm{d}r = -\frac{3}{2}\Omega/r$. Then[3]

$$T_d = \left[\frac{2\pi r^3\nu\Sigma}{4\pi\sigma r}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2\right]^{1/4} = \left[\frac{\nu\Sigma}{2\sigma}r^2\frac{9}{4}\frac{GM_*}{r^5}\right]^{1/4} = \left[\frac{9}{8}\frac{GM_*\nu\Sigma}{\sigma}\right]^{1/4} r^{-3/4} \propto r^{-3/4}$$

as with the radiation calculation.

7.1.2 Vertical

Suppose all the energy of the disk is liberated at $z = 0$ and radiatively diffuses up to a height at which $\tau = 1$ and $T = T_d$. Suppose $h/r \ll 1$ so that vertical gradients matter most. The power flux is $F(z(\tau = 1)) = \sigma T_d^4$ at the surface, but it must be this at all heights in the disk, otherwise some layer would accumulate energy and heat up until the whole disk equilibrates again. The radiative diffusion equation is

[3]The lecturer really messed this derivation up

$$F(z) = -\frac{16\sigma T^3}{3\kappa\rho(z)}\frac{\mathrm{d}T}{\mathrm{d}z} = \sigma T_d^4 \qquad \Rightarrow \qquad -\frac{16}{3}\int_{T(z)}^{T_d} T^3\,\mathrm{d}T = T_d^4\,\overbrace{\kappa\int_{z(\tau)}^{z(\tau=1)}\rho(z)\,\mathrm{d}z}^{\tau-1\approx\tau}$$

$$\Rightarrow T(\tau) = T_d\left(1+\frac{3\tau}{4}\right)^{1/4} \approx T_d\left(\frac{3\tau}{4}\right)^{1/4}$$

This temperature variation means that different species which condense at different temperatures will do so at different heights.

7.2 Shape

The disk doesn't just collapse down to infinitesimal thickness because eventually the disk's pressure can vertically support the disk. The vertical component of the star's gravity is $g_z = \frac{GM_*}{r^3}z$, and hydrostatic equilibrium requires $\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g_z$. Thus, now assuming temperature is independent of z for some reason,

$$\frac{kT_d}{\mu m_H}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{GM_*}{r^3}z \qquad \Rightarrow \qquad \rho = \rho_0 \exp\left(-\frac{z^2}{2h^2}\right) \quad \text{where} \quad h^2 = \frac{kT_d}{\mu m_H}\frac{r^3}{GM_*}$$

is the scale height and ρ_0 is the midplane density. The scale height therefore varies as $h \propto T_d^{1/2} r^{3/2} \propto r^{-3/8} r^{3/2} \propto r^{9/8}$. The aspect ratio $h/r \propto r^{1/8}$ is thus nearly constant, "flaring" only weakly. At Earth's position we apparently have $T_d \sim 100$K, so $h/r \sim 0.1$, so the disk would have been ~ 0.1au thick here.

8 Solid Dynamics & Planet Formation

There is evidence for transport of material within disks:

- Comets feature *refractory* (melted) material despite being 100s of au away
- The period valley probably reflects planet migration
- The Earth is rich in volatiles despite its proximity to the Sun
- Assuming the MMSN, a Jupiter-mass planet would need to be at 10au to sweep up enough mass from an annulus the size of its Roche lobe, yet HJs are found at < 0.1au

The two main components of the disk are gas and dust grains.

8.1 Gas

The radial force experienced by gas is not only gravitational, but also from pressure gradients in the disk. This modifies the equation of motion to:

$$\frac{v_{\phi g}^2}{r^2} = \frac{GM_*}{r} + \frac{1}{\rho_g}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{v_K^2}{r} + \frac{1}{\rho_g}\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \Rightarrow \qquad v_{\phi g}^2 = v_K^2(1-\eta) \quad \text{where} \quad \eta \equiv -\frac{1}{v_K^2}\frac{r}{\rho_g}\frac{\mathrm{d}p}{\mathrm{d}r}$$

As $\mathrm{d}p/\mathrm{d}r < 0 \Rightarrow \eta > 1$, the gas velocity will be *sub-Keplerian*. In fact,

$$\eta \sim \frac{1}{(r\Omega)^2}\frac{r}{\rho_0}\frac{p_0}{r} = \frac{1}{r^2}\overbrace{\frac{r^3}{GM_*}\frac{kT_d}{\mu m_H}}^{h^2} = \frac{h^2}{r^2}$$

Thus at 1au, where $h/r \approx 0.1$, the gas velocity is reduced by a factor $\sqrt{0.99}$ with respect to v_K, giving a headwind of about 100ms^{-1} for dust grains, which want to travel at v_K.

8.2 Dust

If a dust grain of radius a has a velocity v relative to the gas, it will experience a drag force. In a time $\mathrm{d}t$ it will sweep through a mass $\mathrm{d}m = \rho(\pi a^2)(v\,\mathrm{d}t)$ of gas particles. For energy conservation we expect that the work done on the grain, $F\,\mathrm{d}x = Fv\,\mathrm{d}t$, will be given to the gas molecules and thus proportional to $\frac{1}{2}\,\mathrm{d}m\,v^2$. So

$$Fv\,\mathrm{d}t = C_d\frac{1}{2}\,\mathrm{d}m\,v^2 = C_d\frac{1}{2}\big(\rho\pi a^2 v\,\mathrm{d}t\big)v^2 \qquad \Rightarrow \qquad F = -\frac{\pi a^2}{2}C_d\rho v^2$$

The *drag coefficient* C_d takes different forms depending on the busyness of the gas. The Epstein regime has $a < \lambda$ (a dilute gas), which gives $C_d = \frac{8}{3}\frac{c_s}{v} \propto v^{-1}$. The Stokes regime has $a > \lambda$ (a busy gas) and C_d as some function of the Reynolds number $\mathrm{Re} = av/\nu \propto v$.

8.2.1 Vertical Motion

Frictional Timescale. This is defined by grain momentum over (Epstein) drag force:

$$t_{\text{fric}} = \frac{mv}{\frac{\pi a^2}{2}C_d\rho_g v^2} = \frac{2\big(\frac{4}{3}\pi a^3\rho_d\big)}{\pi a^2\big(\frac{8}{3}\frac{c_s}{v}\big)\rho_g v} = \frac{a}{c_s}\frac{\rho_d}{\rho_g}$$

Using $\rho_g \sim 10^{-7}, \rho_d \sim 10^3, c_s \sim 10^3, a \sim 10^{-6}$ (all SI), we find $t_{\text{fric}} \sim 10\text{s}$, which is very short compared to the lifetime of the disc ($10^6\text{yr} \sim 10^{13}\text{s}$). The consequence is that for even cm-sized a, the dust is strongly "coupled" to the gas – that is to say moving along with it.

Settling Timescale. Consider a grain at the scale height and falling towards the midplane at terminal velocity, that is, when the gravitational force is equal to the (Epstein) drag force:

$$\frac{GM_* mz}{r^3} = \frac{\pi a^2}{2}\frac{8c_s}{3v_T}\rho_g v_T^2 \qquad \Rightarrow \qquad v_T = \frac{GM_*}{r^3}\left(\frac{4\pi}{3}\rho_d a^3\right)\frac{3}{4\pi a^2 c_s\rho_g}z = \Omega^2\frac{a}{c_s}\frac{\rho_d}{\rho_g}z$$

This gives a timescale at 1au of $t_{\text{set}} = z/v_T \sim 10^{13}\text{s} \sim 10^6\text{yr}$, comparable to the lifetime of the disk and way longer than t_{fric}.

This analysis does not account for turbulence. If the disk is very turbulent then settling grains will be suspended away from the midplane. Turbulence can be modelled like diffusion, whose timescale is $t_{\text{turb}} \sim z^2/D$. By setting $t_{\text{turb}} = t_{\text{set}}$, one can find a critical diffusion constant $D_{\text{crit}} \propto a$, above which turbulence wins. Intuitively, larger grains thus need turbulence to be stronger to prevent them from settling.

Growth Timescale. As the grain falls at v_T, it encounters smaller grains which stick to it, and the grain grows. In a time $\mathrm{d}t$, the grain encounters a mass $\mathrm{d}m = \pi a^2 v_T(f\rho_g)\,\mathrm{d}t$ of smaller grains, where $f = \rho_d/\rho_g \sim 0.01$. Thus the rate of mass growth is

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \pi a^2\left(\Omega^2\frac{a}{c_s}\frac{\rho_d}{\rho_g}z\right)f\rho_g = \frac{3}{4}\frac{\Omega^2}{c_s}fzm$$

Now the grain tends to grow much quicker than it settles (apparently), so z can be taken as constant, so $\frac{\mathrm{d}m}{\mathrm{d}t} \propto m$ and the grain grows exponentially. Within 100yr, the grain grows from micron to millimetre.

8.2.2 Azimuthal Motion

Dust grains will experience an azimuthal drag due to their speed ($\approx v_K$) being different to that of the gas, which benefits from gas pressure. The rate of change of (specific) angular momentum is then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(rv_K) = \frac{\mathrm{d}r}{\mathrm{d}t}\left[v_K + r\overbrace{\frac{\mathrm{d}v_K}{\mathrm{d}r}}^{-v_K/2r}\right] = v_{rd}v_K\left[1 - \frac{1}{2}\right] = \frac{1}{2}v_{rd}v_K$$

This will be equal to the (specific) torque, equal to $r\times$ the (specific) force, which is of order $(v_{\phi d} - v_{\phi g})/t_{\text{fric}}$. Thus

$$v_{\phi d} - v_{\phi g} = -\frac{v_{rd}v_K t_{\text{fric}}}{2r}$$

8.2.3 Radial Motion

The dust is not expected to accelerate in the radial direction: $\mathrm{d}v_{rd}/\mathrm{d}t = 0$. The radial forces on the dust (which therefore sum to 0) are centrifugal, gravitational and frictional:

$$0 = \frac{v_{\phi d}^2}{r} - \overbrace{\frac{v_K^2}{r}}^{\text{grav.}} - \frac{v_{rd} - v_{rg}}{t_{\text{fric}}} = \frac{v_{\phi d}^2}{r} - \frac{v_{\phi g}^2}{r(1-\eta)} - \frac{v_{rd} - v_{rg}}{t_{\text{fric}}} \approx \frac{v_{\phi d}^2 - v_{\phi g}^2}{r} - \eta\frac{v_{\phi g}^2}{r} - \frac{v_{rd} - v_{rg}}{t_{\text{fric}}}$$

$$\approx \frac{\overbrace{(v_{\phi d} + v_{\phi g})}^{\approx 2v_K}\overbrace{(v_{\phi d} - v_{\phi g})}^{-v_{rd}v_K t_{\text{fric}}/2r}}{r} - \eta\frac{v_{\phi g}^2}{r} - \frac{\overbrace{v_{rd} - v_{rg}}^{v_{rd} \gg v_{rg}}}{t_{\text{fric}}} \approx -\frac{v_{rd}v_K^2 t_{\text{fric}}}{r^2} - \eta\frac{v_K^2}{r} - \frac{v_{rd}}{t_{\text{fric}}}$$

$$\Rightarrow \frac{v_{rd}}{v_K} \approx -\frac{\eta v_K/r}{\frac{1}{t_{\text{fric}}} + \frac{v_K^2 t_{\text{fric}}}{r^2}} = -\frac{\eta}{\left(\frac{v_K t_{\text{fric}}}{r}\right)^{-1} + \left(\frac{v_K t_{\text{fric}}}{r}\right)} \equiv -\frac{\eta}{\tau_s^{-1} + \tau_s}$$

so dust drifts inwards at a rate depending on η and τ. η simply depends on r, whereas τ depends also on t_{fric} and hence on a. The maximum v_{rd} is clearly for $\tau_s = 1$, which corresponds to a size of a few metres at 1au, for which $v_K \sim 100\text{ms}^{-1}$, which would need only 100yrs to reach the Sun. This is called the *drift barrier*, as to form things larger than this would require rapid growth around this size, or falling quickly into the Sun. However, this analysis neglects things like radial gas motion and the influence of large planets on the gas.

8.3 Planetesimal Growth

8.3.1 Collisions

Depending on the specific energy $Q = mv^2/2M$ of a collision between masses $m < M$, two planetesimals may merge, break off and reform as a rubble pile, or be totally destroyed/“dispersed”. The latter requires $Q \geq Q_D$, where Q_D is a measure of how hard it is to break apart. Q_D depends on a in a characteristic way:

- **Strength-dominated: $a \lesssim 1$km.** Q_D is dependent only on the material strength. As a increases the number of fractures in the planetesimal increases, making it easier to break apart: Q_D decreases.

- **Gravitationally-controlled:** $a \gtrsim$ **1km.** In this regime the ability of gravity to hold it all together becomes important in preventing dispersion – in other words overcoming the (specific) gravitational binding energy $Q_B(\propto (GM^2/a)/M \propto M/a \propto a^2)$ becomes the main barrier, rather than the material strength. Thus Q_D increases with a.

Thus planetesimals of size $\sim$ 1km are the easiest to break apart. This presents another barrier to planet formation: the *fragmentation barrier*; once a growing planet reaches a size $\sim$ 1km, it becomes susceptible to fragmentation. This may be avoided if one is past the ice line of a system, as ice is sticky and discourages fragmentation.

8.3.2 Accretion Rate

As mentioned previously, gravitational focusing modifies the effective collision cross-section by a gravitational focusing factor (GFF) $= (1 + v_{\rm esc}^2/\sigma^2)$, where σ is the velocity dispersion among the planetesimals.

The accretion rate $\dot{M}$ is the product of the planetesimal density, the planetesimal velocity, and this effective cross-section:

$$\dot{M} = \rho_p \sigma \pi a^2 \left(1 + \frac{v_{\rm esc}^2}{\sigma^2}\right)$$

In this swarm of planetesimals, they will be bobbing up and down around the disc, on a vertical scale similar to their scale height h_p and a timescale similar to Ω_K, thus $\sigma \sim h_p \Omega_K$. Also, the planetesimal surface density $\Sigma_p \sim \rho_p/2h_p$. Thus

$$\dot{M} \sim \Omega_K \Sigma_p a^2 \left(1 + \frac{v_{\rm esc}^2}{\sigma^2}\right)$$

where we see that $\dot{M}$ depends on σ only through the GFF. Using $\Sigma_p \propto r^{-3/2}$ and $\Omega_K \propto r^{-3/2}$, we find $\dot{M} \propto r^{-3}$, a strong dependence!

Supposing the GFF is independent of mass for some reason, we then have $\dot{M} \propto a^2 \propto M^{2/3}$, with the solution $M(t) \propto t^3$ and $a \propto t$.

Supposing instead that GFF $\gg 1$ and σ is constant (which seems more reasonable to me!), we have GFF $\propto v_{\rm esc}^2 \propto Ma^{-1}$. Then $\dot{M} \propto a^2 M a^{-1} \propto M^{4/3}$, which has the solution $M(t) = \left(M_0^{-1/3} - kt\right)^{-3}$. As this blows up in finite t another regime must take over at some point.

8.3.3 Isolation Mass

The isolation mass is the mass a planet has when it exhausts its orbital surroundings (an annulus of radius $r \pm \Delta r \approx r \pm R_{\rm Roche}$) of planetesimals. The planetesimal mass in this annulus is approximately

$$M_{\rm iso} \approx 2\pi r \cdot 2\Delta r \cdot \Sigma_p = 4\pi r \left(\frac{M_{\rm iso}}{M_*}\right)^{1/3} r\Sigma_p \qquad \Rightarrow \qquad M_{\rm iso} \approx (4\pi\Sigma_p)^{3/2} M_*^{-1/2} r^3$$

At $r = 1$au, and using $\Sigma_p \sim 10^2$ for some reason, we find $M_{\rm iso} \sim 10^{-1.5} M_E$, about the size of Mars' core. At $r = 5$au, $M_{\rm iso} \sim 5M_E$, which is about the size of Jupiter's core.

To create planets decently larger than a few M_E, we require a new construction process, e.g. planet-planet collisions.