Graph Theory

1. $\Delta(G), \delta(G)$
2. Tree is a connected, acyclic graph = maximal acyclic graph = minimum connected graph
3. leaf
4. Every tree has a leaf
5. $e(T) = |T| - 1$
6. Spanning tree

1. Bipartite graph
2. A graph G is bipartite if and only if G contains no odd cycles
3. circuit: An odd circuit contains an odd cycle

Matching

1. Hall's theorem (augmenting path, alternating path)
2. cor: A k-regular bipartite graph contains a perfect matching.
3. deficiency; Let G be a bipartite graph. Then G contains a matching saturating $|A| - d$ vertices in A if and only if for all $A_0 \subseteq A$, we have $|N(A_0)| \geq |A_0| - d$ [hint: Find d extra vertices]
4. system of distinct representatives
5. left and right coset

Connectivity

1. cut vertex
2. separator
3. Peterson graph: $\kappa(G) = 3$.
4. Menger's Theorem, First Form: Let $G = (V, E)$ be a graph, with distinct and non-adjacent $a, b \in V$. If every ab separator in G has size at least k then we can find k independent ab paths
5. Let $G = (V, E)$ be a graph. Then G is k-connected if and only for all $u, v \in V$ with $u \neq v$, there exists k independent uv-paths
6. Edge cut
7. Cut edge
8. Edge connectivity: $\lambda(G)$
9. k-edge-connected
10. Edge form: Let $G = (V, E)$ be a graph, and u, v be distinct vertices of G. If every set of edges $F \subseteq E$ that separates u from v has size greater than or equal to k, then there exists k edge disjoint paths from u to v.
11. Line Graph

Planar graph

1. Planar graph
2. Euler's formula
3. $|E| \leq 3|V| - 6$
4. $K_5, K(3,3)$ is Non-Planar [hint: K_{33} each face >= four edges]
5. subdivision: if non-planar, then so is subdivision

Graph Colouring

1. chromatic number $\chi(G)$
2. $\chi(G) \leq \Delta(G) + 1$. Sharp.
3. If $\delta(G) < \Delta(G)$. Then $\chi(G) \leq \Delta(G)$ [order the distance]
4. Brook's thm: Let G be a connected graph that is not complete or an odd cycle. Then $\chi(G) \leq \Delta(G)$
5. Six Colour Theorem [Euler formula]
6. Planar graph: five colour theorem [hint: clockwise, $x_1 x_3, x_2 x_4$]
7. Heawood's Theorem: If G is a graph drawn on a surface of Euler characteristic E, with $E \leq 0$, then $\chi(G) \leq [7 + \sqrt{}(49 - 24E)]/2$.
8. $\chi'(G)$: edge coloring
9. Vizing's theorem: $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

Extremal Graph Theory

1. Eulerian circuit is a circuit in a graph G that crosses each edge exactly once.
2. Euler's theorem: A connected graph has an Eulerian circuit if and only if every vertex has even degree. [hint: induction]
3. a Hamiltonian cycle in G is a cycle that visits each vertex exactly once
4. Dirac's theorem: Let G be a graph with $n \geq 3$ such that $\delta(G) \geq n/2$. Then G contains a Hamiltonian cycle. [hint: longest path not a cycle, Seek $x_i \sim x_1, x_{i+1} \sim x_l$]
5. Let G be a connected graph. Let $k < n$ and assume $\delta(G) \geq k/2$. Then $G \geq P_k + 1$.
6. Let G be a graph. If $e(G) > n/2(k-1)$, then G contains a path of length k. [hint: induction on contrapositive, use the result above]

Turan

1. Mental's theorem: If $e(G) > n^2/4$, then $G \supset K_3$, and this is sharp
2. Turan's theorem: If $e(G) > (1 - 1/r)n^2/2$, then $K_{r+1} \subset G$, and this is sharp [hint: induction, split K_r and others, count edges in between]
3. complete r-partite graph
4. $Z(n,t)$ to be the maximum number of edges in a bipartite graph with n vertices in each part and no $K_{t,t}$ subgraph
5. We have $Z(n,t) \leq t^{1/t} n^{2-1/t} + tn$ for all n [hint: $\sum_{S \subseteq A, |S|=t} |\cap_{x \in S} N(x)| = \sum_y \frac{d(y)}{t}$]
6. For infinitely many n we have $Z(n,2) \geq cn^{3/2}$ for $c > 0$. [hint: $L = \{(x, ax+b) | x \in Z/pZ\}$]
7. extremal function $ex(n,H) = \max\{e(G) || G| = n, H \not\subset G\}$

8. Erdos-Stone: Let H be a graph with $\chi(G) = r$, and $r \geq 2$. Then $lim_{n \to \infty} ex(n, H) / \frac{n}{2} = 1 - \frac{1}{r-1}$

Ramsey

1. Colour each of the edges of K_6 red or blue . Then there must be a monochromatic triangle.
2. Define $R(t)$ the tth Ramsey number to be the smallest n for which every 2-colouring of K_n contains a monochromatic K_t. So $R(3) = 6$.
3. $R(t)$ is finite and $R(t) \leq 4^t$ [hint: $R(s,t) \leq R(s-1,t) + R(s,t-1)$, $R(s,t) \leq \binom{s+t-2}{s-1}$]
4. Infinite Ramsey Theorem: for every 2-colouring of complete countable graph $G = (N, N^2)$ there exists an infinite set $X \subseteq N$ so that X^2 is monochromatic [hint: take (x_i, c_i) each step]
5. $R(t) \geq 2^{t/2}$ [hint: probabilistic]

Probabilistic Method

1. Binomial Random Graph $G(n, p)$
2. $Z(n,t) \geq (1/4) n^{2-2/(t+1)}$ [modification method]
3. Girth, independent set, $\chi(G) \geq n / \alpha(G)$
4. For every $k, G \in N$, there exists a graph G with girth(G) ≥ g and $\chi(G) \geq k$
5. $G(n, p(n))$ contains $K_{t,t}$ condition
6. $G(n, p(n))$ connectedness condition

Algebraic Method

1. diameter: maximal distance
2. Moore graph is a k-regular graph on $k^2 + 1$ vertices that has diameter 2
3. Adjacency matrix: symmetric, trace = 0, eigenvalue sum = 0
4. $\frac{1}{n} \sum_x x \in V d(x) \leq \lambda_{max}(G) \leq \Delta(G)$ [hint: $\lambda_{max} = max_{x:|x|^2=1} x^T Ax$, take $w = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1)$; RHS maximal coordinate of Ax_{max}
5. $\lambda_{max}(G) = \Delta(G)$ if and only if G is Δ-regular; $\lambda_{min}(G) = \Delta(G)$ if and only if G is Δ-regular and bipartite
6. Moore graph of degree k: $\frac{1}{2}(k^2 \pm \frac{k^2 - 2k}{\sqrt{4k-3}})$ are integers.