Spatial Poisson Process

1. Definition. A random countable subset $\Pi \subseteq R^d$ is called a Poisson process with constant intensity $\lambda > 0$ if for all sets $A \in B(R^d)$: (a) $N(A) := \#(A \cap \Pi) \sim Poi(\lambda|A|)$; (b) For any $A_1, \ldots, A_k \in B(R^d)$ disjoint, $N(A_1), \ldots, N(A_k)$ are independent. If $|A| = \infty$then we interpret (a) as $N(A) = \infty$ with probability 1
2. $N_t = N([0, t])$ for Poisson
3. Definition. Let $\lambda : R^d \to R$ be a non-negative and measurable function such that $\Lambda(A) := \int_A \lambda(x)dx < \infty$ for all bounded $A \in B(R^d)$. Then Π is a non-homogeneous Poisson process with intensity function λ if for all$A \in B(R^n)$ (a) $N(A) = \#(A \cap \Pi) \sim Poi(\Lambda(A))$; (b) For any $A_1, \ldots, A_k$ disjoint Borel sets, $N(A_1), \ldots, N(A_k)$ are independent. Λ is called the mean measure of the Poisson process.
4. Superposition theorem
5. Mapping theorem: $\Lambda(f^{-1}(y)) = 0$, $\mu(B) := \Lambda(f^{-1}(B)) < \infty$ for bounded B.
6. Conditional on $\#(\Pi \cap A) = n$, the n points in $\Pi \cap A$ have the same distribution as n points chosen independently from A according to the probability distribution $\nu(B) = \Lambda(B)/\Lambda(A) = \int_B \lambda(x)/\Lambda(A)dx, B \subseteq A$
7. Colouring Theorem: colour x with probability $\gamma(x)$, then intensity $\gamma(x)\lambda(x)$
8. Renyi's theorem: if $P(\Pi \cap A = \emptyset) = e^{-\Lambda(A)}$ for all bounded Borel sets A then Π is a Poisson process with mean measure Λ

Spatial Poisson

AP.

X pos rec $\not\Rightarrow$ Y pos rec:

$q_n = (1+|n|)^2$.

$\leftarrow \quad \rightarrow \quad \frac{1}{2}$ on $\mathbb{Z}$.

$\pi(n) = \frac{1}{(1+|n|)^2}$. invariant distri.

Y pos recu $\not\Rightarrow$ X pos rec.

$\vdash \xrightarrow{\frac{1}{3}}$ $\xleftarrow[\frac{2}{3}]{}$ on $\mathbb{N}_{\geq 0}$.

$q_n = \cancel{9^n} \cdot \left(\frac{1}{9}\right)^n$.

X not pos rec:

$\mathbb{E} \geq \cancel{\left(\frac{1}{9}\right)^n \cdot \frac{4}{9}}$

$9 \cdot \frac{4}{9} + 9^2 \cdot \left(\frac{4}{9}\right)^2 + \cdots \to \infty$.

DB $\Leftrightarrow$ reversible $\Rightarrow$ invarient distri.

But invariant ~~distri~~ measure $(\lambda Q=0)$ $\not\Rightarrow$ DB holds.

Only in B&D chain, π invariant measure $\Leftrightarrow$ DB.

c irreducible positive recurrent $\Leftrightarrow$ invariant distri. (for Y)

non-explo + invariant distri $\Leftrightarrow$ positive rec (for X)

~~AP 2015.~~ Superposition

AP 2015

P2, S II, 24K.

(i) Poisson process with rate λ:

$(X_t)_{t\geq 0}$ be a process with indep increment.

s.t. $\begin{cases} \mathbb{P}(X_{t+h}-X_t=0)=1-\lambda h+o(h) \\ \mathbb{P}(X_{t+h}-X_t=1)=\lambda h+o(h) \end{cases}$

uniformly for all $t\in\mathbb{R}_+$ for $h\downarrow 0$.

Let $X_t=N_t+M_t$.

$$\mathbb{P}(X_{t+h}-X_t=0)=\mathbb{P}(N_{t+h}-N_t=0)\,\mathbb{P}(M_{t+h}-M_t=0)$$
$$=(1-\lambda h+o(h))(1-\mu h+o(h))$$
$$=1-(\lambda+\mu)h+o(h).$$

$$\mathbb{P}(\sim=1)=\mathbb{P}(=0)(=1)+\mathbb{P}(=1)(=0)$$
$$=(\lambda+\mu)h+o(h)$$

So $\sim$

Thinning: Use infinitesimal def.

For Y_t: $\mathbb{P}(Y_{t+h}-Y_t=0)$

$$= \mathbb{P}(X_{t+h}-X_t=0)+\mathbb{P}(X_{t+h}-X_t \geq 1, \text{not chosen})$$

$$= 1-\lambda h+(1-p)\lambda h+o(h)$$

$$=1-p\lambda h+o(h)$$

$$\mathbb{P}(Y_{t+h}-Y_t=1)=\mathbb{P}(X_{t+h}-X_t=1, \text{chosen})$$
$$+\mathbb{P}(X_{t+h}-X_t\geq 2, \text{only chosen } 1)$$
$$= p\lambda h+o(h)$$

$$\mathbb{P}(Y_{t+h}-Y_t\geq 2)=o(h)$$

So ✓. Similarly for X_t-Y_t.

Claim: $Y_t \perp X_t-Y_t$ for fixed t:

~~$\mathbb{P}(Y_t=m, X_t=n-m)$~~

$$\mathbb{P}(Y_t=m, X_t-Y_t=n)=\binom{n+m}{m}p^m(1-p)^n\frac{e^{-\lambda t}(\lambda t)^{m+n}}{(n+m)!}$$
$$=\cdots$$
$$=\mathbb{P}(Y_t=n)\,\mathbb{P}(X_t-Y_t=m)$$

~~By Markov,~~ N.T.S. $\mathbb{P}(Y_{t_1}=n_1,\dots,Y_{t_k}=n_k, X_{t_1}-Y_{t_1}=m_1,\dots,X_{t_k}-Y_{t_k}=m_k)$

$$=\mathbb{P}(Y_{t_1}=n_1,\dots,Y_{t_k}=n_k)\,\mathbb{P}(\cdots)$$

enough to show the claim, by Markov property.

Joint density proof:

- Conditioned on $(X_t = n)$, $J_1, \dots, J_n$ distributed as order stats of n i.i.d $U[0,t]$ r.v.

i.e. $f(t_1, \dots, t_n) = \frac{n!}{t^n} \mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)$

Proof: Joint density for $(S_1, \dots, S_{n+1})$ is

$$\lambda^{n+1} e^{-\lambda(S_1 + \dots + S_{n+1})} \mathbb{1}(S_i \geq 0 \ \forall i)$$

$$= \lambda^{n+1} e^{-\lambda J_{n+1}} \mathbb{1}(0 \le J_1 \le J_2 \le \dots \le J_{n+1})$$

Then $\mathbb{P}((J_1, \dots, J_n) \in A, X_t = n)$ for $A \subseteq \mathbb{R}$

$$= \mathbb{P}((J_1, \dots, J_n) \in A, J_n \le t < J_{n+1})$$

$$= \int_A \int_t^\infty \lambda^{n+1} e^{-\lambda t_{n+1}} \mathbb{1}[0 \le t_1 \le \dots \le t_n \le t)\, dt_{n+1}\, dt_1 \dots dt_n$$

$$= \int_A \lambda^n e^{-\lambda t} \mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)\, dt_1 \dots dt_n.$$

So $\mathbb{P}((J_1, \dots, J_n) \in A \mid X_t = n)$

$$= \frac{\mathbb{P}((J_1, \dots, J_n) \in A, X_t = n)}{\mathbb{P}(X_t = n)}$$

$$= \frac{\sim}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} = \int_A \frac{n!}{t^n} \mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)\, dt_1 \dots dt_n$$

$\mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots X_0 = x_0)$

$=$

④

3 definitions of Poisson:

① $S_i \sim Exp(\lambda)$, $J_n = \sum_{i=1}^{n} S_i$, $X_t = \max\{n : J_n \leq n < J_{n+1}\}$

② $\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$

$\mathbb{P}(\cdots = 1) = \lambda h + o(h)$.

③ indep and stationary increment s.t. ~~$(X_{s+t} - X_s)_{t \geq 0} \sim Poi(\lambda t)$ for all~~

$X_t \sim Poi(\lambda t)$ for all $t \geq 0$

① $\Rightarrow$ ② : ✓

② $\Rightarrow$ ③ :

~~$\lim_{h \to 0} \frac{X_{t+h} - X_t}{h} =$~~

$P_j(t) = \mathbb{P}(X_t = j)$.

$$\begin{aligned}
P_j(t+h) &= \mathbb{P}(X_{t+h} = j) \\
&= \sum_{i=0}^{j} \mathbb{P}(X_t = j-i)\,\mathbb{P}(X_{t+h} - X_t = i) \\
&= P_j(t)(1 - \lambda h + o(h)) + P_{j-1}(t)(\lambda h + o(h)) + o(h)
\end{aligned}$$

$$\Rightarrow \frac{P_j(t+h) - P_j(t)}{h} = -\lambda P_j(t) + \lambda P_{j-1}(t) + o(1)$$

$$P_j'(t) = -\lambda P_j(t) + \lambda P_{j-1}(t)$$

③ ⇒ ①: ~~For $t_1 \le \cdots \le t_k$, $n_1 \le \cdots$~~

$$\mathbb{P}(X_{t_1}=n_1, \ldots, X_{t_k}=n_k)$$

$$= \mathbb{P}(X_{t_1}=n_1)\,\mathbb{P}(X_{t_2}-X_{t_1}=n_2-n_1)\cdots$$

$$\mathbb{P}(X_{t_k}-X_{t_{k-1}}=n_k-n_{k-1})$$

$\sim Poi(\lambda t_1)$ $\qquad \sim Poi(\lambda(t_2-t_1))$

So ③ determines finite-dim distr of process X ✓

So ③ determines X. ① ⇒ ③. So ③ ⇔ ①.

~~$\lambda^n e$~~ Joint ✓.

Birth Process:

Explosive ⇔ $\sum \frac{1}{q_i} = \infty$.

If $\sum \frac{1}{q_i}$, then $\mathbb{E}[\zeta] < \infty$,

$\zeta < \infty$ a.s.

If $\sum \frac{1}{q_i} = \infty$,

$$\cancel{\mathbb{E}[e^{-\sum \xi_i}]} \quad \mathbb{E}[e^{-\xi}]$$

$$= \mathbb{E}[e^{-\sum_{i}^{\infty} \xi_i}] = \lim_{n\to\infty} \prod_{i=1}^{n} \mathbb{E}(e^{-\xi_i}]$$

$$\le \lim \prod \frac{1}{1+q_i}$$

$$= 0.$$

$$\int e^{-x} \cdot \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda+1}.$$

Q-matrix ✓.

2. (a) Jump chain $Y_n = X_{J_n}$ is discret tie MC with transit matr P, initia λ.

(b) Holding time $S_n = \exp(q_{Y_{n-1}})$

Conditional.

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x reccurrent for $X \iff \int_0^\infty P_{xx}(t)dt = \infty$.

proof: $\int_0^\infty P_{xx}(t)\,dt$

$$\cancel{= \int_0^\infty \mathbb{P}(X_t = x \mid X_0 = x)\,dt}$$

$$= \int_0^\infty \mathbb{E}_x[\mathbb{1}[X_t = x]]\,dt$$

$$= \mathbb{E}_x \int_0^\infty \mathbb{1}[X_t = x)\,dt$$

$$= \mathbb{E}_x \sum_{n=0}^{\infty} \mathbb{1}(Y_n = x) S_{n+1}$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_x[\mathbb{P}_x(Y_n = x)\ \mathbb{E}_x[S_{n+1} \mid Y_n = x] = \sum_{n=0}^{\infty} P_{xx}(n) \frac{1}{q_x}.$$

Since recurrent $\iff$ $\cancel{\sum P_{xx}}$ $\sum P_{xx}(n) = \infty$. ✓

4. If $|I|$ finite.

$$\lambda Q = 0.$$

$$\Rightarrow \lambda e^{tQ} = \lambda \checkmark.$$

$\lambda P(s) = \lambda.$ $\qquad \sum_{j\in I} \lambda_j p_{ji}'(s) = 0$

$$= \sum_{j\in I}\sum_{k\in I} \lambda_j p_{jk}(s) q_{ki}$$

$$= \sum_{k\in I} \lambda_k q_{ki}$$

So $(\lambda Q)_i = 0. \checkmark$

$\pi Q = 0$ iff $\mu P = \mu$, $\mu_x = q_x \pi_x$.

Proof. $(\Rightarrow)$ $\sum_{j\in I} \pi_j q_{ji} = 0.$

$$P_{ji} = \begin{cases} \frac{q_{ji}}{q_{jj}}, & j \neq i \\ 0, & j = i. \end{cases}$$

$$(\mu P)_i = \sum_{j\in I} \mu_j P_{ji} = \sum_{j\in I} q_j \pi_j P_{ji}$$

$$= \sum_{j\in I} q_j \pi_j \frac{q_{ji}}{q_{jj}} 1[j\neq i]$$

$$= \sum_{j\in I} \pi_j q_{ji} 1[j \neq i]$$

$$= q_i \pi_i = \mu_i.$$

(⇐) If $\mu P = \mu$,

$$
\begin{aligned}
(\pi Q)_i &= \pi_j Q_{ji} \\
&= \sum_{j \in I} \frac{\mu_j}{q_j} Q_{ji} \\
&= \sum \mu_j \cdot \frac{Q_{ji}}{q_j} \\
&= \cancel{\sum_{j \neq i} \mu_j \frac{Q_{ji}}{q_i} - \mu_i} \\
&= \sum_{j \neq i} \mu_j \frac{Q_{ji}}{q_j} + \cancel{\mu_i Q_{ji}} \, \mu_i \frac{Q_{ii}}{q_i} \\
&= \mu_i - \mu_i = 0.
\end{aligned}
$$

Burke's Thm: $M/M/1$ queue with $\mu > \lambda > 0$

or $M/M/\infty$ queue with $\mu, \lambda > 0$.

At equilibrium D is a poisson of rate λ,

X_t is independent of $(D_s : s \le t)$.

Proof: X is B-D process, π invariant dist,

so X reversible, $(\hat{X}_t)_{0 \le t \le T} \overset{d}{=} (X_t)_{0 \le t \le T}$,

where $\hat{X}_t = X_T - X_t$.

So arrival for $\hat{X}_t$ ~~is Poi~~ ~ Poisson process with intensity λ,

$\hat{X}_t = D_T - D_{T-t}$.

Since time reversal of a Poisson is Poisson, and f.d.d is determined as $T > 0$ is arbitrary, we have $(D_t)_{0 \le t \le T}$ is Poisson process of rate λ in $[0, T)$.

Independence: $X_0 \perp (A_s : 0 \le s \le T)$

$\Rightarrow \hat{X}_0 \perp (\hat{A}_s)$, i.e. X_T independent of $(D_t)_{0 \le t \le T}$.

Renewal (Reward) Process:

ξ_i iid, $T_n = \sum_{i=1}^{n} \xi_i$, $N_t = \max\{n : T_n \le t < T_{n+1}\}$,

$\frac{N_t}{t} \to \lambda$ a.s.:

$\frac{T_{N_t}}{N_t} \le t$ $T_{N_t} \le t < T_{N_t+1}$.

$$\frac{T_{N_t}}{N_t} \le \frac{t}{N_t} \le \frac{T_{N_t+1}}{N_t}.$$

~~So T_{N_t}~~ $\frac{T_{N_t}}{N_t} \to \frac{1}{\lambda}$ a.s by SLLN,

$\to \frac{1}{\lambda}$ by SLLN,

$\Rightarrow \sim$

Reward: (ξ_i, R_i) be i.i.d pairs, $R(t) = \sum_{i=1}^{N_t} R_i$.

$$\frac{R(t)}{t} \to \lambda \mathbb{E}R \quad \text{a.s}$$

$$\frac{\mathbb{E}R(t)}{t} \to \lambda \mathbb{E}R \quad \text{a.s.}$$

$$r(t) \to \lambda \mathbb{E}[R\xi].$$

Little's Formula.

N: arrival
W_i: waiting time

L: long run queue.

X: queue, start from 0, regenerative with regeneration times (T_n)

N: arrival of X; W_i: waiting time of ith customer.

$$L = W\lambda,$$

$$L = \text{queue} = \frac{1}{t}\int_0^t X_s \, ds$$

$$W = \frac{W_1 + \dots + W_n}{n}$$

$$\lambda = \frac{N_t}{t}.$$

Note: regenerative: $\exists\, T_n$ s.t.

$$(X_{t+T_n})_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0}.$$

and indep of.

Spatial:

Def: random countable subset $\Pi \subseteq \mathbb{R}^d$,

s.t. $\forall A \subseteq \mathbb{R}^d$ measurable,

a) $\#|\Pi \cap A| \sim Poi(\lambda|A|)$.

b) Disjoint A_i, $N(A_i)$ indep.

If $|A| = \infty$, $N(A) = \infty$ with prob 1.

$\Lambda(A) = \int_A \lambda(x) dx$. intensity function λ.

a) $\#|\Pi \cap A| \sim Poi(\Lambda(A))$.

b) ~

Λ: mean measure.

Superposition:

claim: $\mathbb{P}(\Pi_1 \cap \Pi_2 \cap A \neq \phi) = 0$.

Proof: Let $Q_{k,n} = \prod_{i=1}^{3} (k_i 2^{-n}, (k_i+1) 2^{-n}]$

$$\mathbb{P}(\Pi_1 \cap \Pi_2 \cap A \neq \phi) \leq \sum_{k \in \mathbb{Z}^3} \mathbb{P}(\Pi_1 \cap \Pi_2 \cap A \cap Q_{k,n} \neq \phi)$$

$$\leq \sum_{k \in \mathbb{Z}^3} \mathbb{P}(|\Pi_1 \cap A \cap Q_{k,n}| \geq 1) \mathbb{P}(|\Pi_2 \cap A \cap Q_{k,n}| \geq 1)$$

$$= \sum_{k \in \mathbb{Z}^3} (1 - e^{-\int_{A \cap Q_{k,n}} z_1 dx})(1 - e^{-\int_{A \cap Q_{k,n}} z_2 dx})$$

~~$\leq z_1 z_2 |A||Q_{k,n}| \to 0$ as $n \to \infty$.~~

$$\leq \underbrace{(\max_{k \in \mathbb{Z}^d} \Lambda_1(Q_{k,n} \cap A))}_{M_n(A)} \sum_{k \in \mathbb{Z}^d} \Lambda_2(Q_{k,n} \cap A)$$

$M_n(A) \leq C|Q_{k,n} \cap A| \leq C|Q_{k,n}| = C 2^{-nd} \to 0$.

Let Π be non-homo Poisson process with λ.

Mapping: $f: \mathbb{R}^d \to \mathbb{R}^s$

$$\Lambda(f^{-1}\{y\}) = 0 \quad \forall y \in \mathbb{R}^s.$$

$$\mu(B) : \Lambda(f^{-1}(B)) < \infty \quad \forall B \in \mathcal{B}(\mathbb{R}^s)$$

Then $f(\Pi)$ is non-homo on $\mathbb{R}^s$ with mean measure μ.

Conditional property:

Given $\#(\Pi \cap A) = n$,

n points in $\Pi \cap A$ have same distribution

$$v(B) = \frac{\Lambda(B)}{\Lambda(A)} = \frac{\int_B \lambda(x) dx}{\Lambda(A)}.$$

Coloring:

color x ~~with~~ Red with $r(x)$

...

Then Π has intensity $r(x)\lambda(x)$

Proof: Given point red. Independence of points, color indep.

$$\bar{r} = \int_A \frac{r(x)\lambda(x)}{\Lambda(A)} dx$$

Thus, binomial with parameters $1-\bar{r}, n$

$$\mathbb{P}(N_r = n_r, N_b = n_b) = \mathbb{P}(N(A) = n) \frac{n!}{n_r! n_b!} \bar{r}^{n_r} (1-\bar{r})^{n_b}$$

$$= e^{-\Lambda(A)\bar{r}} \frac{(\ )^{n_r}}{n_r!} \cdots$$

So $\sim Poi(\bar{r}\Lambda(A))$, $Poi((1-\bar{r})\Lambda(A))$

AP 2016. ~~Q~~ [Nice Q.]

P1, S II, 26J.

(a) CTMC with infinitesimal generator Q.

jump train Y:

$Y \sim \text{Markov}(P)$

with $P_{ij} = -\frac{Q_{ij}}{Q_{ii}}$ $(i \neq j)$.

~~Holding time S_i~~ Let $(\xi_n)_{n \geq 1}$ be iid

~~$S_i \sim Exp(-Q_{ii})$~~. Exp(1).

Jump times $J_n = \sum_{i=1}^{n} \frac{\xi_i}{q(Y_{i-1})}$.

$X_t = \sum_{n=1}^{\infty} Y_n \mathbb{1}(J_n \leq t < J_{n+1})$.

(b) If x transient for Y,

Let $N = \sup\{n : Y_n = x\} < \infty$.

Then $\mathbb{P}(X_t \neq x : t \geq J_n) = 1$.

So transient for X.

(c) If x pos recurrent, ~~$\mathbb{E}_x[M_x]$~~ $m_x = \mathbb{E}_x T_x < \infty$

~~pos recurrent, $\mathbb{E}_x[T_x] < \infty$.~~

$S = \mathbb{Z}$.

$Q(n, n\pm 1) = \frac{(1+|n|)^2}{2}$,

$Q(n,n) = (1+|n|)^2$.

Discrete : SRW on $\mathbb{Z}$ $\Rightarrow$ Null recurrent.

For $(X_t)_{t\geq 0}$: Consider $\pi_n = \frac{1}{(1+|n|)^2}$,

$\pi_n Q(n, n\pm 1) = \pi_{n\pm 1} Q(n\pm 1, n) = \frac{1}{2}$

$\Rightarrow \pi$ satisfy detailed balance

$\Rightarrow$ invariant measure.

$\sum \frac{1}{(1+|n|)^2} < \infty \Rightarrow$ ~~non-explosive~~ invariant distribution

~~Clearly $q_n < \infty$~~,

Clearly $\sup q_n < \infty$, So non-explosive + invariant measure

$\Rightarrow$ pos recurrent.

(d) ez.

(e) Note that $\pi_x = \frac{1}{q(x)}$ solves DB.

So +ve recurrent iff $\sum_{x\in\mathbb{Z}^d} \frac{1}{\min\{1, \|x\|^\alpha\}} < \infty$.

$n^d - (n-1)^d = O(n^{d-1})$.

$\sum n^{d-1} \cdot \frac{1}{n^\alpha} < \infty \Rightarrow \alpha > d$.

pos recurrent when

AP 2016. P2, 5II 25J.

V good Qn.

(a) bookwork

(b) size-biased picking

(c) Finite states $\Rightarrow$ invariant measure exists.

$P_{ij} > 0 \Rightarrow$ irreducible.

$\Rightarrow$ Recurrent

jump chain

$\Rightarrow (X_t)$ recurrent

$\Rightarrow$ Invariant measure unique up to const.

Finite states $\Rightarrow \sum \lambda_i < \infty$

$\Rightarrow \exists$ unique invariant distribution.

AP2016

P3, SII. 24J.

(a) Thinning: Let $(X_t)_{t\geq 0}$ be a Poisson process with rate λ ~~$(Z_n)_{n\in\mathbb{N}, \geq 0}$ be~~

(Z_i) be iid r.v. $\sim$ Ber(P).

~~Then consider~~

Let Y be a Poisson process with values in $\{0, \dots, \}$ which jumps at t iff X_t jumps at t & $Z_{X_t}=1$

Then Y is Poisson process with λP, $X-Y$ indep Poi process rate $\lambda(1-P)$

Superposition:

Let X & Y be two indep Poisson process with λ & μ. Then $Z = X+Y$.

3 Def: ① $(X_t)_{t\geq 0} =$ $\max\{n : J_n \leq t\}$.

$S_1, S_2, \dots$ iid $\sim$ Exp(λ)

$J_n = \sum_{i=1}^{n} S_i$. Jump chain $Y_n = n$.

② $\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$

$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$ as $h \downarrow 0$.

$\mathbb{P}(X_{t+h} - X_t \geq 2) = o(h)$

③ X has stationary & indep increment,

for all $t \geq 0$, $X_t \sim Poi(\lambda t)$.

Proof for Thinning:

Use def ②

Note: $Y \perp X-Y$ as,

$$\mathbb{P}(Y_t = n, X_t - Y_t = m) = \mathbb{P}(X_t = m+n, Y_t = n)$$

$$= \mathbb{P}(X_t = m+n)\,\mathbb{P}(Y_t = n \mid X_t = m+n)$$

$$= e^{-\lambda t}\frac{(\lambda t)^{m+n}}{(m+n)!}\binom{m+n}{n}p^n(1-p)^m$$

$$= e^{-\lambda t p}\frac{(\lambda t p)^n}{n!}\, e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!}$$

$$= \mathbb{P}(Y_t = n)\,\mathbb{P}(X_t - Y_t = m).$$

Superposition: Use def ③.

AP 2016.

P4, S IV. $\xi_1, \xi_2, \dots$, i.i.d.

$$\mathbb{E}(\xi_1) = \frac{1}{\lambda} < \infty.$$

$$T_n = \sum_{i=1}^{n} \xi_i.$$

$$N_t = \max\{n : T_n \le t\}.$$

~~Note:~~ $\cancel{T_n \le t \le T_{n+1}}$

Note: $\dfrac{T_{N_t}}{N_t} \le \dfrac{t}{N_t} \le \dfrac{T_{N_t+1}}{N_t}.$

By SLLN, $\dfrac{T_n}{n} \to \mathbb{E}\xi = \dfrac{1}{\lambda}$,

$N_t \to \infty$ as $t \to \infty$.

So $\dfrac{T_{N_t}}{N_t} \to \dfrac{1}{\lambda}$ a.s.

$\dfrac{T_{N_t+1}}{N_t+1} \cdot \dfrac{N_t+1}{N_t} \to \dfrac{1}{\lambda}$ a.s.

So $\dfrac{t}{N_t} \to \dfrac{1}{\lambda}$ a.s.

AP 2015

P1, SII, 241c

(a) Birth&Death chain:

X is a CTMC with

$Q(i,i+1), Q(i,i-1) > 0 \quad \forall i \geq 1.$

$Q(0,1) > 0.$

$Q(i,i) = -[Q(i,i+1) + Q(i,i-1)].$

$Q(i,j) = 0$ otherwise.

$q_i = -Q(i,i)$

~~Then~~ ~~Each~~ It has transition matrix P s.t.

$P_{i,i+1} = \frac{Q(i,i+1)}{q_i} \quad \forall i \geq 0$

$P_{i,i-1} = \frac{Q(i,i-1)}{q_i} \quad \forall i \geq 1.$

Y_n be discrete M.C. with transition matrix P, $Y_0 = X_0$

~~Holding time $S_i \sim Exp($~~

Let ξ_i be i.i.d. Exp(1) r.v.

~~$S_i = \frac{\xi_i}{q_i}$~~ $S_i = \frac{\xi_i}{q_{Y_{i-1}}}$, $i = 1, 2, \ldots$

$J_n = \sum_{i=1}^{n} S_i.$

$X_t = \sum_{n=1}^{\infty} \mathbb{1}[J_n \leq t < J_{n+1}] \cdot Y_n.$

or: $X_t = Y_n$ for $J_n \leq t < J_{n+1}$

~~(b) $M(\lambda)/M(\mu)/s$ queue~~

Claim: Measure π invariant for B&D chain iff it solves DB.

proof: π invariant

$\Rightarrow \pi Q = 0$.

$(\pi Q)_i = \sum_{j \in I} \pi_j Q_{ji}$

$= \pi_{(i+1)} Q_{(i+1),i} + \pi_{(i-1)} Q_{(i-1),i} + \pi_i Q_{ii} = 0 \quad \forall i = 1, 2, \ldots$

$Q_{ii} = -(Q_{i(i+1)} + Q_{i(i-1)})$

$\pi_{(i+1)} Q_{(i+1)i} - \pi_i Q_{i(i+1)}$

$= Q_i Q_{i(i-1)} - \pi_{(i-1)} Q_{(i-1)i}$

$= \ldots = \pi_1 Q_{10} - \pi_0 Q_{01} = 0$.

So solves DB.

~~DB: π~~

DB $\Rightarrow$ invariant always.

Note: DB $\Leftrightarrow$ reversible

$\Rightarrow$ Invariant ~~distr~~ measure

But invariant measure $\not\Rightarrow$ DB / reversible

since chain may not be reversible

~~(a) Note: first part:~~

2015

P 2

(a) Note: 1st part:

$$Q(n, n+1) = \lambda_n$$

$$Q(n, n-1) = \mu_n.$$

$$Q(n,n) = -(\lambda_n + \mu_n).$$

(b) Try solving DB.

$$Q(n, n+1) = \lambda \quad \forall n \geq 0$$

$$Q(n, n-1) = \begin{cases} \mu n, & n \leq s \\ \mu s, & n > s. \end{cases}$$

$$Q(n,n) = \begin{cases} -(\lambda + \mu n), & n \leq s \\ -(\lambda + \mu s), & n > s. \end{cases}$$

Then want $\cancel{\pi(n, n+1)}$

$$\pi(n)\, Q(n, n+1) = \pi(n+1)\, Q(n+1, n) \quad \forall n.$$

$$\begin{cases} \pi(n) \cdot \lambda = \pi(n+1) \cdot \mu(n+1) & \text{for } n < s. \\ \pi(n)\, \lambda = \pi(n+1)\, \mu s & \text{for } n \geq s. \end{cases}$$

$$\pi(1) = \frac{\lambda}{\mu} \pi(0)$$

$$\pi(2) = \frac{\lambda}{2\mu} \pi(1) = \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{2!} \pi(0)$$

$$\cdots$$

$\cancel{\pi(s) = }$

$$\pi(n) = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \pi(0), & n \leq s \\ \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{s!} \cdot \left(\frac{1}{s}\right)^{n-\mu} \pi(0). \end{cases}$$

Since ~~Q~~ $\sup\{Q_{ii}\} < \infty$ it's non-explosive.

Thus invariant distri $\Rightarrow$ pos recurrent.

~~i.e.~~ ~~then~~ $\cancel{\sum \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!}}$

$\sum \pi(n) < \infty$

$\Leftrightarrow \sum \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{s!} \left(\frac{1}{s}\right)^{n-\mu} \pi(0) < \infty$

$\Leftrightarrow \sum \left(\frac{\lambda}{\mu s}\right)^n < \infty$.

$\Leftrightarrow \lambda < \mu s$.

So $\lambda < \mu s$, pos recurrent.

$\cancel{\lambda > \mu s}$ $\lambda = \mu s$, SRM with $p = \frac{1}{2}$,

Jump chain rec $\Rightarrow$ null recurrent.

$\lambda > \mu s$, jump chain transient

$\Rightarrow X$ transient.

(ii') $Y_t = X_{N_t}$.

$$\cancel{\mathbb{P}(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}}$$

Y jumps at same the as N_t jumps & $X_{N_t+1} \neq X_{N_t}$

which at state x is a thinning of

$\cancel{(1-p)\cdot 1}$ $1 \cdot (1 - K_{x,x})$.

So $\quad Q(x,y) = K(x,y)\cdot\lambda$

$\qquad Q(x,x) = -\lambda(1 - K_{x,x})$.

If $\pi K = \pi$, then

$$\begin{aligned}(\pi Q)_x &= \sum_{y\in S} \pi_y Q_{yx} \\ &= \Big(\sum_{\substack{y\in S\\ y\neq x}} \pi_y K(y,x)\cdot\lambda\Big) - \lambda(1-K(x,x))\pi_x \\ &= (\pi K)_x - \pi_x \\ &= \pi_x - \pi_x = 0.\end{aligned}$$

So π is invariant for Y.

AP Day 4.

1. ✓

2. (a) $\forall t \geq 0$, ~~X_t is~~ X_t is r.v. such that $X_t : \Omega \to I$.

 b) Right-continuous

 Defined by finite-dimensional distributions.

3. J_n: $J_0 = 0$, ~~ξ_1~~

$$J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}.$$

$$S_n = \begin{cases} J_n - J_{n-1}, & J_{n-1} < \infty \\ \infty, & \text{o.w.} \end{cases}$$

4. Jump chain: $Y_n = X_{J_n}$.

5. CTMC: Random process with Markov property

$$\mathbb{P}(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}, \dots, X_{t_0} = i_0)$$
$$= \mathbb{P}(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}).$$

6. Transitory probability $P_{ij}(s,t) = \mathbb{P}(X_t = j \mid X_s = i)$.

 Time-homogeneous if $P_{ij}(s,t) = P_{ij}(0, t-s)$.

Characterized by: (λ_i) Initial distribution

 Family of transition matrices $(P(t))_{t \geq 0}$
 $= (P_{ij}(t))_{t \geq 0}$.

(~~$P(t)$~~ called transition subgroup of MC.

7. $P(t+s) = P(t)\,P(s)$ for time-homogeneous:

$$P_{xz}(t+s) = \mathbb{P}(X_{t+s} = z \mid X_0 = x)$$

$$= \sum_{y \in \mathbb{Z}} \mathbb{P}(X_{t+s} = z \mid X_0 = x, X_t = y)\,\mathbb{P}(X_t = y \mid X_0 = x)$$

$$= \sum_{y \in \mathbb{Z}} \mathbb{P}(X_s = z \mid X_0 = y)\,\mathbb{P}(X_t = y \mid X_0 = x)$$

$$= \sum_{y \in \mathbb{Z}} P_{yz}(s) P_{xy}(t) = P_{x\cdot}(t)\, P_{\cdot z}(s)$$

AP Rum.

Defs. 2022 ✓.

2021.

P2.(a) Jump chain Y:

$Y_0 = X_0$.

$S_1 = \inf\{t : X_t \neq X_0\}$.

$J_n = \sum_{i=1}^{n} S_i$.

$S_{n+1} = \inf\{t : X_t \neq X_{J_n}, t \geq J_n\}$

$Y_i = X_{J_i}$.

It's a discrete-time MC:

$\mathbb{P}(Y_n = y_n \mid Y_{n-1} = y_{n-1}, \dots, Y_0 = y_0)$

$= \mathbb{P}(X_{J_n} = y_n \mid X_{J_{n-1}} = y_{n-1}, \dots, X_{J_0} = y_0)$

$= \mathbb{P}(X_{J_n} = y_n \mid X_{J_{n-1}} = y_{n-1})$

$\cancel{= \mathbb{P}(X_{J_n - J_{n-1}} = y_n \mid X_0 = y_{n-1})}$

$= \mathbb{P}(Y_n = y_n \mid Y_{n-1} = y_{n-1})$.

(b) X recurrent means $\forall x \in I$,

$$\mathbb{P}(\{t : X_t = x\} \text{ unbounded}) = 1.$$

Claim: X is recurrent iff Y is recurrent.

proof: ($\Leftarrow$) If Y recurrent,

$$\mathbb{P}(Y_n = x \text{ for infinitly } n) = 1.$$

Then ~~since~~ Let N_i be the index

s.t. $Y_{N_i} = x \quad \forall i$.

Then $\mathbb{E}\left[\sum_{i=1}^{k} S_{N_i}\right] = \frac{k}{q_{xx}} \to \infty$ as $k \to \infty$.

By SLLN, $\sum_{i=1}^{k} S_{N_i} \to \infty$ as $k \to \infty$,

so X recurrent.

($\Rightarrow$) If Y transient,

$$\sup(i : Y_i = x) < \infty$$

Say $\sup(i : Y_i = x) \leq N$.

Then $\sim$.

$$\mathbb{E} \int_0^\infty P_{00}(t)\,dt$$

$$= \mathbb{E}\left(\int^\infty \mathbb{E}\left[\mathbb{1}(X_t = 0 \mid X_0 = 0)\right] \cancel{\mathbb{1}\,\mathbb{P}(X_t=0 \mid X_0=0)} \right) dt$$

$$= \mathbb{E}_0 \int_0^\infty \mathbb{1}(X_t = 0 \mid \cancel{X_0 = 0})\,dt$$

$$= \mathbb{E}_0 \left[\sum_{n=0}^{\infty} \mathbb{1}(Y_n = 0) S_{n+1} \right]$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_0(Y_n = 0)\,\mathbb{E}_0\left[S_{n+1} \mid Y_n = 0\right]$$

$$= \sum_{n=0}^{\infty} P_{00}(n) \frac{1}{q_0}$$

2020 P2.

(i) X is CTMC on $\mathbb{Z}$ with $G=(g_{i,j})$.

Jump chain Y:

~~$J_0=0$~~

~~$J_{n+1}=\inf\{t: X_t: t\geq J_n; X_t\neq X_{J_n}\} = \sum_{i=1}^{n+1} S_i$, whe~~

~~$Y_n = X_{J_n}$.~~

~~$J_0=0$.~~

~~$S_i \sim \exp($~~

$J_0=0$

$J_{n+1}=\inf\{t: t\geq J_n: X_t\neq Y_{J_n}\}$

$Y_n=X_{J_n}$.

P3. Renewal-Process:

(ξ_i, R_i) i.i.d. pairs,

$T_n=\sum_{i=1}^{n}\xi_i$, $N_t=\max\{n: T_n\leq t<T_{n+1}\}$.

$R(t)=\sum_{i=1}^{N_t} R_i$.

Thm: $\frac{R(t)}{t}\to\frac{\mathbb{E}(R)}{\mathbb{E}(\xi)}$ a.s, $\frac{\mathbb{E}(R(t))}{t}\to\frac{\mathbb{E}(R)}{\mathbb{E}(\xi)}$,

$r(t)=\mathbb{E}(R_{N_t+1})$, $r(t)\to\frac{\mathbb{E}(R\xi)}{\mathbb{E}(\xi)}$.

2019.

P4,

(a) ① $\#(\Pi \cap A) \sim \mathrm{Poi}(\Lambda(A))$

② for all $A_1, \dots, A_n \in \mathbb{R}^d$ disjoint,
$\#(\Pi \cap A_i)$ are independent.

(b) $\Lambda(f^{-1}(y)) = 0$ for any $y \in \mathbb{R}^d$,

and $\Lambda(f^{-1}(A)) < \infty$ for all bounded measurable $B \subseteq \mathbb{R}^s$

If hold: mean measure of $f(\Pi)$
is $\mu(A) = \Lambda(f^{-1}(A))$.

$$\cancel{|f(\Pi) \cap B| = |\Pi \cap f^{-1}(B)| = \int_{f^{-1}(B)} \lambda(x)\,d(x)}.$$