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Planar graph

1. Planar graph
2. Euler's formula "connected plane graph".
3. $|E| \leq 3|V| - 6$ ~~"conne~~
4. $K_5, K(3,3)$ is Non-Planar [hint: K_{33} each face >= four edges]
5. subdivision: if non-planar, then so is subdivision

Graph Colouring

1. chromatic number $\chi(G)$
2. $\chi(G) \leq \Delta(G) + 1$. Sharp.
3. If $\delta(G) < \Delta(G)$. Then $\chi(G) \leq \Delta(G)$ [order the distance]
4. G connected Brook's thm: Let G be a connected graph that is not complete or an odd cycle. Then $\chi(G) \leq \Delta(G)$
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5. Six Colour Theorem [Euler formula]
6. Planar graph: five colour theorem [hint: clockwise, x_1x_3, x_2x_4] $\geq 2-2g$.
7. Heawood's Theorem: If G is a graph drawn on a surface of Euler characteristic E, with $E \leq 0$, then $\chi(G) \leq [7 + \sqrt{(49 - 24E)}]/2$.
8. $\chi'(G)$: edge coloring $V-E+F \geq 2-2g$.
9. 20 Vizing's theorem: $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

Extremal Graph Theory

1. Eulerian circuit is a circuit in a graph G that crosses each edge exactly once.
2. Euler's theorem: A connected graph has an Eulerian circuit if and only if every vertex has even degree. [hint: induction] — has a cycle: G is not a tree.
3. 17 a Hamiltonian cycle in G is a cycle that visits each vertex exactly once
4. Dirac's theorem: Let G be a graph with $n \geq 3$ such that $\delta(G) \geq n/2$. Then G contains a Hamiltonian cycle. [hint: longest path not a cycle, Seek $x_i \sim x_1, x_{i+1} \sim x_l$]
5. Let G be a connected graph. Let $k < n$ and assume $\delta(G) \geq k/2$. Then $G \geq P_k$ ~~+1~~
6. Let G be a graph. If $e(G) > \cancel{n/2(k-1)}$ $\frac{n(k-1)}{2}$, then G contains a path of length k. [hint: induction on contrapositive, use the result above]

Turan

1. 22 Mental's theorem: If $e(G) > n^2/4$, then $G \supset K_3$, and this is sharp
2. 20 Turan's theorem: If $e(G) > (1 - 1/r)n^2/2$, then $K_{r+1} \subset G$, and this is sharp [hint: induction, split K_r and others, count edges in between]
3. complete r-partite graph
4. $Z(n,t)$ to be the maximum number of edges in a bipartite graph with n vertices in each part and no $K_{t,t}$ subgraph
5. 18 We have $Z(n,t) \leq t^{1/t}n^{2-1/t} + tn$ for all n [hint: $\sum_{S \subseteq A, |S|=t} |\cap_{x \in S} N(x)| = \sum_y \frac{d(y)}{t}$]
6. For infinitely many n we have $Z(n,2) \geq cn^{3/2}$ for $c > 0$. [hint: $L = \{(x, ax+b) | x \in Z/pZ\}$]
7. extremal function $ex(n,H) = \max\{e(G) || G| = n, H \not\subset G\}$

Graph Theory

1. $\Delta(G), \delta(G)$
2. Tree is a connected, acyclic graph = maximal acyclic graph = minimum connected graph
3. leaf
4. Every tree has a leaf
5. $e(T) = |T| - 1$
6. Spanning tree

1. Bipartite graph
2. A graph G is bipartite if and only if G contains no odd cycles
3. circuit: An odd circuit contains an odd cycle

Matching

21, 19, 17.

1. Hall's theorem (augmenting path, alternating path)
2. cor: A k-regular bipartite graph contains a perfect matching.
3. deficiency; Let G be a bipartite graph. Then G contains a matching saturating $|A| - d$ vertices in A if and only if for all $A_0 \subseteq A$, we have $|N(A_0)| \geq |A_0| - d$ [hint: Find d extra vertices]
4. system of distinct representatives
5. left and right coset

Connectivity

1. cut vertex
2. separator
3. Peterson graph: $\kappa(G) = 3$.
4. 20, 18, 16. Menger's Theorem, First Form: Let $G = (V, E)$ be a graph, with distinct and non-adjacent $a, b \in V$. If every ab separator in G has size at least k then we can find k independent ab paths
5. Let $G = (V, E)$ be a graph. Then G is k-connected if and only for all $u, v \in V$ with $u \neq v$, there exists k independent uv-paths
6. Edge cut
7. Cut edge
8. Edge connectivity: $\lambda(G)$
9. k-edge-connected
10. Edge form: Let $G = (V, E)$ be a graph, and u, v be distinct vertices of G. If every set of edges $F \subseteq E$ that separates u from v has size greater than or equal to k, then there exists k edge disjoint paths from u to v.
11. Line Graph