No.: .... Date: ....

Applied Probability 2015

1) (a) Birth Death Chain: Let $S = \mathbb{Z}_{\geq 0}$ be state space,

$Q : S \times S \to \mathbb{R}$ defined by: ~~$Q(n,n+1) =$~~

$$Q(n,m) = 1_{m=n+1} \lambda_n + 1_{m=n-1} \mu_n + 1_{m=n} (-\lambda_n - \mu_n 1_{n \neq 0})$$

$(\lambda_n, \mu_n \geq 0)$

Let $(X_t)_{t \geq 0}$ be continuous markov chain with Q as a generator:

$(X_t)_{t \geq 0}$ is a birth death chain

(b) If $\underset{\sim}{\pi}$ is a solution to Detailed Balance Equations (DB):

$$\forall n \in S: \sum_{m \geq 0} \pi_m Q_{m,n} = \sum_{m \geq 0} \pi_n Q_{n,m} = \pi_n \sum_{m \geq 0} Q_{n,m} = 0.$$

$\therefore$ $\underset{\sim}{\pi}$ is invariant.

If $\underset{\sim}{\pi}$ is invariant: $(n \neq m)$ $Q_{n,m} \neq 0$ iff $|n-m| = 1$

$\therefore$ (DBE): $\forall n \in S$, $\pi_n \cdot Q_{n,n+1} = \pi_{n+1} Q_{n+1,n}$ $\quad -(*)$

$n=0$: $\sum_{m \geq 0} \pi_m Q_{m,0} = -\pi_0 \cdot \lambda_0 + \pi_1 \mu_1 \Rightarrow (*)$ True for $n=1$

If $(*)$ True for some $N-1$:

$$\sum_{m \geq 0} \pi_m Q_{m,N} = \pi_N (-\lambda_N - \mu_N) + \pi_{N-1} \lambda_{N-1} + \pi_{N+1} \mu_{N+1} = 0$$

$(*)$: $\pi_N \overset{\mu}{\cancel{\lambda}}_N = \pi_{N-1} \lambda_{N-1} \Rightarrow \pi_N \lambda_N = \pi_{N+1} \overset{\mu}{\cancel{\lambda}}_{N+1}$

$\therefore$ Induction complete $\Rightarrow$ $\underset{\sim}{\pi}$ satisfies DB.

(b) $S = \mathbb{Z}_{\geq 0}$: $(X_t)_{t \geq 0}$ is a Birth-Death Chain,

$\lambda_n = \lambda$, $\mu_n = \operatorname{Min}\{n, s\} \cdot \mu$

Chain is clearly irreducible. Let $(Y_n)_{n \geq 0}$ be jump chain.

No.: .... Date: ....

$\therefore$ $(X_t)_{t\geq 0}$ is recurrent iff $(Y_n)_{n\geq 0}$ is recurrent iff state s is recurrent (in $(Y_n)_{n\geq 0}$) iff $\mathbb{P}(\text{return to } s \mid Y_0 = s) = 1$

~~But~~ ~~$\mathbb{P}$(re~~

Claim: $\alpha = \mathbb{P}(\text{hit } s \mid Y_0 = s-1) = 1$; Consider jump chain with modification s.t. $S \to S-1$ with $\mathbb{P} = 1$.

$\therefore$ This does not affect α.

$\{1, \dots s\}$ is a finite, irreducible markov chain $\Rightarrow$ $\alpha = 1$

Claim: $\beta = \mathbb{P}(\text{hit } s \mid Y_0 = s+1) = 1$ iff $\lambda \leq S \cdot \mu$. Consider modification s.t. $S \to S+1$ with $\mathbb{P} = 1$ (will not change β)

$\therefore$ $\{k : k \geq s\}$ is ~~state~~ irreducible class ; $(Y_n)_{n\geq 0}$ on this class is a random walk ; $\mathbb{P}(+1) = \lambda / _{\lambda + S\mu}$, $\mathbb{P}(-1) = S\mu / _{\lambda + S\mu}$

$\therefore$ S is Recurrent iff $\lambda \leq S\mu$ $\Rightarrow$ ~~$\mathbb{P}$(hits~~ $\beta = 1$ iff $\lambda \leq S\mu$.

$\therefore$ $\mathbb{P}(\text{return to } s \mid Y_0 = s) = 1$ iff $\lambda \leq S\mu$.

If $\lambda \leq S\mu$: Chain is recurrent & Non-explosive.

$\therefore$ +ve invariant iff invariant distribution exists.

6 (a) : Suffice to solve ~~Invariant~~ D.B.

$$\therefore \quad \pi_n \cdot \lambda = \pi_{n+1} \mu \cdot (n+1) \qquad (n+1 \leq s)$$
$$= \pi_{n+1} \mu \cdot S \qquad (n+1 > s)$$

$$\therefore \quad \pi_k = \frac{(\lambda/\mu)^k}{k!} \pi_0 \qquad (0 \leq k \leq s)$$

$$\pi_{s+k} = \pi_s \cdot (\lambda/\mu s)^k$$

$\sum \pi_k < \infty$ iff $\sum (\lambda/\mu s)^k < \infty$ iff $\lambda < \mu \cdot s$

$\therefore$ $\lambda > S \cdot \mu$ (transient)

$\lambda = S\mu$ (null-recurrent)

$\lambda < S\mu$ (+ve - recurrent)

No.: Date:

If π is X- invariant : $\underset{\sim}{\pi} \cdot P = \pi$

$Q = \lambda \cdot P - \lambda I$ (I is idantity matrix)

$\therefore \underset{\sim}{\pi} \cdot Q = \lambda \left\{ \underset{\sim}{\pi} \cdot P - \underset{\sim}{\pi} \right\} = 0$

$\therefore \underset{\sim}{\pi}$ is Q or $(Y_t)_{t \geq 0}$ invariant.

(generating $(N_t)_{t \geq 0}$)

(c) $T_a = \sum_{k=1}^{\tau_a} \xi_k$, $(\xi_k)_{k \geq 1}$ is iid $Exp(1)$ R.V. λ

$(\overset{N}{X}_t)_{t \geq 0}$: $\overset{N}{X}_t = k$ for $t \in [\xi_1 + \dots + \xi_k, \xi_1 + \dots + \xi_{k+1})$

$(\xi_k)_{k \geq 1}$ independent of X

$\therefore \mathbb{E}[T_a] = \mathbb{E}[\mathbb{E}[T_a \mid \tau_a = n]] = \mathbb{E}[\mathbb{E}[\sum_{k=1}^{n} \xi_k \mid \tau_a = n]]$

$(\xi_n)_{n \geq 1}$ independent of τ: $\mathbb{E}[T_a] = \mathbb{E}[\mathbb{E}[n \xi]] = \mathbb{E}[\xi] \cdot \mathbb{E}[n \sim \tau_a]$

$\therefore \mathbb{E}[T_a] = \mathbb{E}[\tau_a] / \lambda$. #

No.: .... Date: ....

2.) (a) Let $(X_t)_{t\geq 0}$ be a ~~right~~ process with independent increments;

$$\left.\begin{aligned} \mathbb{P}(X_{t+h} - X_t = 0) &= 1 - \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 1) &= \lambda h + o(h) \end{aligned}\right\} \text{ Uniformly for all } t \in \mathbb{R}_t$$

$(X_t)_{t\geq 0}$ is a Poisson process of rate λ.

Let $X_t = N_t + M_t$: N_t, M_t have independent increments $\Rightarrow$ X_t has independent increments.

$$\mathbb{P}(X_{t+h} - X_t = 0) = \mathbb{P}(N_{t+h} - N_t = M_{t+h} - M_t = 0) = (1 - \lambda h + o(h)) \cdot (1 - \mu h + o(h))$$

$$= 1 - (\lambda + \mu) h + o(h)$$

$$\mathbb{P}(X_{t+h} - X_t = 1) = \mathbb{P}(\{N_{t+h} - N_t = 1,\ M_{t+h} - M_t = 0\}) +$$

$$\mathbb{P}(\{N_{t+h} - N_t = 0,\ M_{t+h} - M_t = 1\})$$

$$= (\lambda h + o(h))(1 - \mu h + o(h)) + (\mu h + o(h))(1 - \lambda h + o(h))$$

$$= (\lambda + \mu) h + o(h)$$

$\therefore$ $(X_t)_{t\geq 0}$ is a Poisson process of rate $\lambda + \mu$.

(b) At state x: $\mathbb{P}$ transition to y = $P(x,y)$

Let $r_x = P(x,x)$: We ~~expect~~ only move out of state x at a rate of λ, thinned by probability p.

$\therefore$ Exit rate $= (1-p) \cdot \lambda$

$$\mathbb{P}(X \to y) = P(x,y) / 1-p \qquad (y \neq x)$$

Q matrix : Let S be state space;

$$\left.\begin{aligned} (x \neq y) \quad Q(x,y) &= P(x,y) \cdot \lambda \\ Q(x,x) &= -\lambda(1 - P(x,x)) \end{aligned}\right\} Q \text{ is generator of } (Y_t)_{t\geq 0}$$

No.: Date:

3.) (a) Infinitesimal Definition: ~~Let $(\varepsilon_n)_{n\geq 1}$ be $\overset{iid}{,}$ sequence of Ber(p) R.V., independent of X.~~

Since $(X_t)_{t\geq 0}$ has independent increments, $(Y_t)_{t\geq 0}$ has independent increments as well.

$$\mathbb{P}(Y_{t+h} - Y_t = 0) = \mathbb{P}(X_{t+h} - X_t = 0) + \mathbb{P}(X_{t+h} - X_t = 1, \text{reject}).$$

$$= (1-\lambda h + o(h)) + (1-p)(\lambda h) + o(h)$$

$$= 1 - (\lambda p)h + o(h)$$

$$\mathbb{P}(Y_{t+h} - Y_t \geq 1) = \mathbb{P}(Y_{t+h} - Y_t \geq 2, \text{at least 2 accepts})$$

$$\leq \mathbb{P}(X_{t+h} - X_t \geq 2) = o(h)$$

$$\therefore \mathbb{P}(Y_{t+h} - Y_t = 1) = (\lambda p)h + o(h)$$

$\therefore$ Poi($\lambda \cdot p$ rate) process

$$\mathbb{P}(X_t - Y_t = n, Y_t = m) = \mathbb{P}(X_t = n+m, \text{ m accepts out of } n+m)$$

$$= \frac{e^{\lambda t}\lambda^{n+m}}{(n+m)!} \frac{(n+m)!}{n!\,m!} p^n (1-p)^m = \frac{e^{\lambda p}(\lambda p)^m}{m!} \frac{e^{\lambda(1-p)}(\lambda(1-p))^n}{n!}$$

$$= \mathbb{P}(X_t - Y_t = n) \cdot \mathbb{P}(Y_t = m)$$

$\therefore$ $(X_t - Y_t, Y_t)$ independent (fixed t)

(b) By applying (a) repeatedly to n bins: ~~$(M_n)_{n\geq}$~~ # of balls in Bin k

$(Y^k_t)_{t\geq 0}$ is Poi(rate 1/n), $(Y^k : 1 \leq k \leq n)$ are independent.

~~$\mathbb{P}(M_n \geq \alpha) = \mathbb{P}(\text{Poi}(1/n) \geq \alpha)^n$~~

At time n: $Y^k_n \sim \text{Poi}(1)$

$$\therefore \mathbb{P}(M_n \geq \alpha) = \mathbb{P}(\text{Poi}(1) \geq \alpha)^n \leq e^{n\{\alpha - \alpha\log\alpha\}}$$

~~If~~ ~~∀~~ Let $\log(n) = x$: $n\left[\alpha - \alpha\log\alpha\right] = e^{x}\left[(1+\varepsilon)x/_{\log x}\ \left(1-\log\left(1+\varepsilon\ {}^{\log x}/_{\log x}\right)\right)\right]$

$$= e^{x}\,\frac{x(1+\varepsilon)}{\log x}\left[1-\log(1+\varepsilon)-\log(x)+\log(\log(x))\right] \quad -(*)$$

But $e^{x}\cdot x(1+\varepsilon) \to +\infty$ as $x \to \infty$;

$$\frac{(1-\log(1+\varepsilon)) - \log(x) + \log(\log(x))}{\log(x)} =$$

$$-1 + \underbrace{\frac{(1-\log(1+\varepsilon))}{\log(x)}}_{\to 0 \text{ as } x\to\infty} + \underbrace{\frac{\log(z)}{z}}_{\because x\to\infty \Rightarrow z\to\infty;}$$

$$= -1 + \lim_{z\to\infty} \frac{\log(z)}{z} = -1 + \lim_{z\to\infty} \frac{1}{z} = -1$$

$\therefore (*) \to -\infty$

$$\therefore 0 \leq \lim_{n\to\infty} \mathbb{P}\left(M_n \geq (1+\varepsilon)\,{}^{\log n}/_{\log(\log n)}\right).$$

$$\leq \lim_{x\to\infty} e^{e^{x}\frac{x(1+\varepsilon)}{\log x}\left[1-\log(1+\varepsilon)-\log(x)+\log(\log x)\right]}$$

$$= 0$$

$$\therefore \lim_{n\to\infty} \mathbb{P}\left(M_n \geq (1+\varepsilon)\,{}^{\log n}/_{\log(\log n)}\right) = 0.$$

No.: .... Date: ....

$\hat{\varepsilon}$ density : $f(x) = x \cdot \lambda^2 e^{-\lambda x}$

$$\mathbb{E}[\hat{\varepsilon}] = \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx = \lambda \left\{ [-x^2 e^{-\lambda x}]_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx \right.$$

$$= 2\,\mathbb{E}[\varepsilon] = 2/\lambda$$

$\therefore \lim_{t\to\infty} \mathbb{E}[A_2 - A_1 |_t] = 2/\lambda$.

$\{e_1, \dots e_N\}$

(c) Q - matrix: $Q(\underset{\sim}{x}, \underset{\sim}{x} + e_i - e_j) = \mu_j \cdot P_{ji}$ x_j ($x_j \geq 1$)

$$Q(\underset{\sim}{x}, \underset{\sim}{x}) = - \sum_{k=1}^{N} 1_{x_k \neq 0}\, \mu_k x_k$$

R - matrix ($K = 1$) : $Q(e_i, e_j) = \mu_i P_{ij}$

~~Balance $\lambda e_i\, Q(e_i, e_j) = \lambda e_j\, Q(e_j, e_i)$~~

~~Invariant Equation:~~ π_j

$\cancel{\pi_{e_i} \mu_i = \sum_{j\neq i} P_{ji} \pi_{e_j} \mu_j}$

Since there are finite states, invariant measure $\underset{\sim}{\lambda}$ exists

$P_{ij} > 0 \Rightarrow \forall\ e_i, e_j :\ Q(e_i, e_j) > 0 \Rightarrow$ Irreducible

Irreducible, finite state jump chain $\Rightarrow$ Recurrent $\Rightarrow$ $(X_t)_{t\geq 0}$

recurrent $\Rightarrow$ Invariant measure is unique (up to constant)

Finite states $\Rightarrow$ $\sum \underset{\sim}{\lambda} < \infty$ $\Rightarrow$ $\exists$ Unique invariant distribution

(d) $\pi(\underset{\sim}{n}) = \prod_{i=1}^{N} \lambda_i^{n_i} / n_i!$

$\cancel{\sum_{\underset{\sim}{n}} \pi(\underset{\sim}{n})\, Q(\underset{\sim}{n}, \underset{\sim}{m}) =}$

No.: .... Date: ....

$$\sum_{x \in \mathbb{Z}^d} \frac{1}{\max\{1, \|x\|_\infty^\alpha\}} < \infty \qquad -(1)$$

If $\alpha < 0$: $\max\{1, \|x\|_\infty^\alpha\} = 1 \Rightarrow (1) = \infty$

Let $\alpha < 0$: (1) iff $\sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|x\|_\infty^\alpha} = \sum_{n \geq 1} \{n^2 - (n-1)^2\} \cdot \frac{1}{n^\alpha}$ $(d=2)$

$$= \sum_{n \geq 1} \frac{2n-1}{n^\alpha} < \infty \quad \text{iff} \quad \alpha > 2$$

(1) iff $\sum_{n \geq 1} \frac{2}{n^\alpha} < \infty$ iff $\alpha > 1$

$\therefore$ the recurrent iff $\alpha > d$ #

2.) (a) $M(\lambda)/M(\mu)/\infty$ Queue: Let $S = \mathbb{Z}_{\geq 0}$ be state space, $(X_t)_{t \geq 0}$ is a $(M(\lambda)/M(\mu)/\infty)$ queue if it is generated by Q matrix:

$$Q(n, m) = \lambda \cdot 1_{m=n+1} + (n\mu) 1_{m=n-1} + 1_{n=n}(-\lambda - n\mu)$$

Invariant: $\pi_n = e^{-(\lambda/\mu)} \cdot (\lambda/\mu)^n / n!$

Burke's process: Let $(D_t)_{t \geq 0}$ be the departure process

$(D_t)_{t \geq 0}$ is a Poisson rate λ process.

$\forall T \geq 0$: X_T is independent of $(D_s)_{0 \leq s \leq T}$

(b) Since departure process is a Poisson process: We can model it as a renewal process, inter-renewal time of $\mathcal{E}_n$, $(\mathcal{E}_n)_{n \geq 1}$ is iid $\mathrm{Exp}(\lambda)$ R.V.

$A_2(t) - A_1(t)$ distribution $\rightarrow$ Size biased distribution of $\mathcal{E}$

No.: .... Date: ....

Applied Probability 2013

1.) (a) Consider the jump chain: $(Y_n)_{n\geq 0}$

$$P(n, n+1) = \frac{1}{1+\varepsilon_n}, \quad P(n+1, n) = \frac{\varepsilon_{n+1}}{1+\varepsilon_{n+1}}$$

Let $h_k = \mathbb{P}((Y_n)_{n\geq 0} \text{ returns to } 0 \mid Y_0 = k)$: $h_0 = 1$,

$h_k \in [0,1]$

Recurrence : $h_k = \frac{\varepsilon_k}{1+\varepsilon_k} + \frac{1}{1+\varepsilon_k} h_{k+1}$

$(1+\varepsilon_k)\cdot h_k = \varepsilon_k + h_{k+1} \Rightarrow (h_k - h_{k+1}) = \varepsilon_k (1-h_k) \geq 0$

$\therefore (h_k)_{k\geq 0}$ is decreasing sequence.

Since markov chain is irreducible : Chain is transient iff state 0 is transient iff state 0 is Jump-chain transient

$$\mathbb{P}(\exists n \geq 1 \text{ s.t } Y_n = 0 \mid Y_0 = 0) = 1 - \prod_{k=0}^{\infty} \frac{1}{1+\varepsilon_k} \quad -(*)$$

If $\sum_{k\geq 0} \varepsilon_k < \infty$: $\varepsilon_k \to 0 \Rightarrow \exists N \in \mathbb{N}$ st $k \geq N \Rightarrow \log(1+\varepsilon_k) \geq \varepsilon_k/2$

$$\therefore \prod_{k=0}^{\infty} \frac{1}{1+\varepsilon_k} \geq \prod_{k=0}^{N-1} \frac{1}{1+\varepsilon_k} \cdot e^{\sum_{k\geq N} \varepsilon_k/2} > 0.$$

$\therefore (*) < 1 \Rightarrow 0$ is transient $\Rightarrow (X_t)_{t\geq 0}$ is transient.

If $\sum_{k\geq 0} \varepsilon_k = \infty$: Let $h_k = \mathbb{P}((Y_n)_{n\geq 0} \text{ returns to } 0 \mid Y_0 = k)$, $h_0 = 1$

$h_k \in [0,1]$

$h_k - h_{k+1} = \varepsilon_k (1-h_k) \geq 0 \Rightarrow (h_k)_{k\geq 0}$ decreasing.

$\therefore h_k - h_{k+1} \geq \varepsilon_k \cdot (1-h_1) \quad (h_1 \geq h_k)$

$\therefore h_1 - h_{k+1} \geq (1-h_1) \sum_{r=1}^{k} \varepsilon_r \to \infty$ if $h_1 \neq 1$ (reject!)

$\therefore h_1 = 1 \Rightarrow$ Chain is recurrent at 0

$\Rightarrow$ Recurrent.

(b) Chain is transient: (A.S) $\exists$ N s.t. $\forall$ $Y_N = 0$, $Y_{N+k} = k$

$\therefore$ $\cancel{\zeta = \sum_{k=1}^{N} J_N + \sum_{k\ge1} \frac{Z_{N+k}}{(\varepsilon_{k-1}+1)\lambda_{k-1}},\ (Z_k)_{k\ge1} \sim \text{iid Exp}(1)}$

$\therefore$ ζ ~~to~~ Let $(J_n)_{n\ge0}$ be jump times

~~Birth chain~~

$$\zeta = J_N + \sum_{k\ge N+1} \frac{Z_k}{\lambda_{k-1}(1+\varepsilon_{k-1})}, \quad (Z_k)_{k\ge1} \text{ is iid Exp}(1) \text{ R.V.}$$

Consider a birth only chain, $q(n,n+1) = \lambda_n(1+\varepsilon_n)$

A.S. Explosive iff $\sum_{n\ge1} \frac{1}{\lambda_n(1+\varepsilon_n)} < \infty$

$\zeta < \infty$ iff $\sum_{k\ge1} Z_k/\lambda_{k-1}(1+\varepsilon_{k-1}) < \infty$ ~~iff~~ ; RHS is the total time of birth chain as described above

$\therefore$ Critera: $\sum_{k\ge0} 1/\lambda_k(1+\varepsilon_k) < \infty$ iff A.S. Explosive.

(c) $\cancel{\pi_n = \pi_{n-1}\cdot\lambda_{n-1}\ (n\ne0) \Rightarrow \pi_0 = 1,\ \pi_n = \prod_{k=0}^{n-1}\lambda_k}$

$\cancel{\pi_0 = \sum_{n\ge1}\pi_n(\lambda_n\varepsilon_n)}$

Invariant equation: $\pi_n\cdot\lambda_n(1+\varepsilon_n) = \pi_{n-1}\cdot\lambda_{n-1}$ $(n\ge1)$

$$\therefore \pi_n = \pi_{n-1}\frac{\lambda_{n-1}}{\lambda_n(1+\varepsilon_n)} = \pi_0\prod_{k=0}^{n-1}\frac{\lambda_k}{\lambda_{k+1}(1+\varepsilon_k)} = \frac{\lambda_0}{\lambda_n}\prod_{k=0}^{n-1}\frac{1}{1+\varepsilon_k}\pi_0$$

$$\overset{\lambda_0}{\cancel{(1+\varepsilon_0)}}\pi_0 = \sum_{k\ge1}\pi_k\cdot\lambda_k\varepsilon_k = \pi_0\cdot\lambda_0\underbrace{\sum_{k\ge1}\frac{\varepsilon_k}{(1+\varepsilon_0)\cdots(1+\varepsilon_{k-1})}}_{=1}$$

$\therefore$ π_n invariant.

Special case: $\pi_n = \pi_0\prod_{k=0}^{n-1}\frac{1}{1+d/(n+1)} = \pi_0\prod_{k=0}^{n-1}\frac{n+1}{n+1+d}$

If $d\le1$: $\prod_{k=0}^{n-1}\frac{n+1}{n+1+d} \ge \prod_{k=0}^{n-1}\frac{n+1}{n+2} = \frac{1}{n+1} \Rightarrow \sum\pi_n \ge \sum\frac{1}{n+1} = \infty$

$\therefore$ No invariant distribution $\Rightarrow$ Not +ve recurrent

No.: Date:

If $\alpha > 1$: $\sum \xi_n = \infty \Rightarrow$ Recurrent $\Rightarrow$ Non-explosive

General stirling approx.: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$(n+\alpha)! \sim \sqrt{2\pi(n+\alpha)}\left(\frac{n+1}{e}\right)^{n+\alpha}$$

$\therefore \sim \sum 1/n^{\alpha} < \infty$

$\therefore$ Invariant ~~Measure~~ Distribution Exists, (Suspicious)

2) (a) Poisson Process: Let $(\xi_n)_{n\geq 1}$ be iid $\mathrm{Exp}(\lambda)$ R.V.,

$$J_0 = 0, \quad J_n = \sum_{k=1}^{n} \xi_k$$

Define $X_t = k$ for $t \in [J_k, J_{k+1})$: $(X_t)_{t\geq 0}$ is a Poisson process.

Q-matrix Definition: $X_0 = 0$; $Q(n, n+1) = \lambda$, $Q(n,n) = -\lambda$,
$Q(n,m) = 0 \quad (m \neq n, n+1)$

Infinitesimal: $(X_t)_{t\geq 0}$ has independent incraments and

$$\left.\begin{aligned} \mathbb{P}(X_{t+h} - X_t = 0) &= 1 - \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 1) &= \lambda h + o(h) \end{aligned}\right] \text{ uniformly (in } t) \text{ as } h \to 0$$

Poisson: $(X_t)_{t\geq 0}$ has stationary, independent incraments.
$\forall t \geq 0$, $X_t \sim \mathrm{Poi}(\lambda t)$

(b) $P_k(t) = \mathbb{P}(X_t = k)$

$$P_k(t+h) = \sum_{i=0}^{k} P_i(t)\, \mathbb{P}(X_{t+h} - X_t = k-i)$$

$$= P_k(t)\,(1-\lambda h) + \cancel{o(h)} + P_{k-1}(t)\,(\lambda h) + o(h) \quad (k \geq 1)$$

$$= P_k(t)\,(1-\lambda h) + o(h) \quad (k=0)$$

No.: Date:

$$\frac{P_k(t+h) - P_k(t)}{h} = -\lambda(t)\, P_k(t) + 1_{k\neq 0}\, \lambda(t)\, P_{k-1}(t) + o(h)$$

$\therefore$ $P_k(t)$ is cont; Right derivative exists

$P_{k-1}(t-h)$

$t \to t-h$: Left derivative exists, $= \lim_{h\to 0} -\lambda(t-h)\, P_k(t-h) + 1_{k\neq 0}\, \lambda(t-h)\, P_{k-1}(t-h)$

$= -\lambda(t)\left\{P_k(t) - P_{k-1}(t)\right\}$

$\therefore$ $P_k'(t) = \lambda(t)\left\{-P_k(t) + P_{k-1}(t)\, 1_{k\neq 0}\right\}$

$P_0'(t) = -\lambda(t)\, P_0(t) \Rightarrow P_0(t) = A\, e^{-\Lambda(t)}$, $P_0(0) = 1 \Rightarrow A = 1$

Let $G(t, z) = \sum_{k\geq 0} P_k(t)\, z^k$.

$$\frac{\partial G}{\partial t} = \sum_{k\geq 0}\left\{-\lambda(t)\, P_k(t) + 1_{k\neq 0}\, P_{k-1}(t)\, \lambda(t)\right\} z^k$$

$$= -\lambda(t)\, G(t, z) + \lambda(t)\, z\, G(t, z)$$

$\therefore$ $G_t = \lambda(t)\, G\{z-1\} \Rightarrow G(t, z) = A\, e^{(z-1)\int_0^t \lambda(s)\, ds}$

$G(0, z) = 1 \Rightarrow A = 1$

$\therefore$ MGF of $X(t)$ = MGF of $Poi(\Lambda(t))$ R.V.

$\therefore$ $X(t) \sim Poi(\Lambda(t))$

No.: Date:

(c) Fix t : Let $(Y_n)_{n\geq 1}$ be subsequence of X, $(Y_n)_{n\geq 1}$ are elements $> t$. ; $(Z_n)_{n\geq 1}$ be elements $\leq t$

$\therefore$ $(R_s)_{0\leq s\leq t}$ is a function of $(Z_n)_{n\geq 1}$, $(R_s)_{s>t}$ is a function of $(Y_n)_{n\geq 1}$

$\therefore$ $(R_s)_{0\leq s\leq t}$, $(R_s)_{s>t}$ are independent.

$\therefore$ $R_{t_1} - R_{t_2}$ independent of $(R_s)_{0\leq s\leq t_2}$ $(t_1 > t_2 \geq 0)$

$\therefore$ Independent increments.

$$\mathbb{P}(X_{t+h} - X_t = 0) = \mathbb{P}(Y_1 \geq t+h) = \int_{t+h}^{\infty} f(x)\,dx \Big/ \int_t^{\infty} f(x)\,dx$$

$$= \frac{1 - F(t+h)}{1 - F(t)}$$

$$= 1 + \frac{F(t) - F(t+h)}{1 - F(t)} = 1 - \underbrace{h\, f(t) \big/ 1 - F(t)}_{\lambda(t)} + o(h) \quad \text{(Mean - value theorem)}$$

$$\mathbb{P}(X_{t+h} - X_t \geq 2) \leq \mathbb{P}(Y_1, Y_2 \in [t, t+h]) = \left[\int_{t+h}^{t+h} f(x)\,dx \Big/ \int_t^{\infty} f(x)\,dx\right]^2$$

$$= \left(h f(t) \big/ 1 - F(t) + o(h)\right)^2 = o(h)$$

$$\therefore \mathbb{P}(X_{t+h} - X_t = 1) = 1 - \mathbb{P}(X_{t+h} - X_t = 0) + o(h)$$

$$= h \cdot \lambda(t) + o(h)$$

$\therefore$ (b): $R_t \sim \mathrm{Poi}(\Lambda(t))$

$$\Lambda(t) = \int_0^t \frac{f(s)}{1 - F(s)}\,ds = -\Big[\log(1 - F(s))\Big]_0^t = -\log(1 - F(t))$$

$\therefore$ $\mathrm{Poi}(-\log(1 - F(t)))$

No.: Date:

Proof of hint:

$$\forall\ X_n : \left\langle \begin{array}{lll} p & (>t) & f(s)/1-F(t) \\ 1-p & (<t) & f(s)/F(t) \end{array} \right. \Bigg|$$

We redefine the process $(X_n)_{n\geq 1}$:

Let $(\varepsilon_n)_{n\geq 1}$ be iid $Ber(0,p)$ R.V., $p = F(t)$

$(a_n)_{n\geq 1}$ be iid, density $f(s)/F(t)$

$(b_n)_{n\geq 1}$ be iid, density $f(s)/1-F(t)$

$$\left.\begin{array}{llll} X_n = a_n & \text{if} & \varepsilon_n = 1 \\ \quad = b_n & \text{if} & \varepsilon_n = 0 \end{array}\right] \text{Same distribution as original } (X_n)$$

Ch\ $\mathbb{P}(Y_1 \in A_1, \dots Y_n \in A_n) = \cancel{\mathbb{P}(a_1 \in A_1, \dots a_n \in A_n)}$.

$$\sum_{n_1,\dots n} \mathbb{P}(Y_1 = X_{n_1}, \dots Y_n = X_{n_n}) \cdot \underbrace{\mathbb{P}(a_{n_1} \in A_1, \dots a_{n_n} \in A_n)}_{= \mathbb{P}(a_1 \in A_1, \dots a_n \in A_n)}$$

$= \mathbb{P}(a_1 \in A_1, \dots a_n \in A_n)$

(Similarly for Z_n)

$$\mathbb{P}((Y_1, \dots Y_n) \in A, (Z_1, \dots Z_m) \in B) = \sum_{\in G} \mathbb{P}((Y_1, \dots Y_n) = (X_{i_1}, \dots X_{i_n}), (Z_1, \dots Z_m) = (X_{j_1}, \dots X_{j_m}))$$

$$\cdot \mathbb{P}((a_{i_1}, \dots a_{i_n}) \in A, \mathbb{P}(b_{j_1}, \dots b_{j_m}) \in B)$$

$= \mathbb{P}((a_1, \dots a_n) \in A) \cdot \mathbb{P}((b_1, \dots b_m) \in B)$ δ.

No.: .... Date: ....

4) (a) M/M/1: ~~Arrivals follow a $Poi(\lambda)$~~ process, ~~and each~~

Let $S = \mathbb{Z}_{\geq 0}$: Q-matrix : $Q(n,n+1) = \lambda$, $n \geq 0$

$Q(n,n-1) = \mu$ (, $n \geq 1$)

$Q(n,n) = -(\lambda+\mu)$

$(X_t)_{t\geq 0}$ be the process generated by Q: $(X_t)_{t\geq 0}$ is a $(M,M,1)$ Queue.

Consider Jump chain : $\left(\frac{\lambda}{\lambda+\mu}, \frac{\mu}{\lambda+\mu}\right)$ walk on S

$\therefore$ Recurrent iff $\mu/\lambda+\mu \geq 1/2$ iff $\mu \geq \lambda$

$\therefore (X_t)_{t\geq 0}$ is recurrent iff $\lambda \leq \mu$. (Recurrent $\Rightarrow$ Non-Explosive)

Invariant Distribution: $\pi_n \cdot \lambda = \pi_{n+1} \cdot \mu$ (Detailed Balance)

$\therefore \pi_n = (1-\lambda/\mu)(\lambda/\mu)^n$ is an invariant measure;

(Distribution) iff $(\lambda/\mu) < 1$

$\therefore$ +ve recurrent iff $\lambda < \mu$.

Burke's Theorem: If $(X_t)_{t\geq 0}$ is a M/M/1(λ,μ) Queue, +ve recurrent and in equilibrium: Let $(D_t)_{t\geq 0}$ be departure process

$D_t \sim$ Poisson ~~(λ)~~ process of rate λ; X_t is independent of $(D_s)_{0\leq s\leq t}$ $(\forall\, t\geq 0)$

(Proof): Let ~~$(\hat{X}_t)$~~ $(\hat{X}_s)_{s\in[0,t]}$ be time reversal

Since we solved for detailed balance, $(X_s)_{0\leq s\leq t}$ is time reversible $\Rightarrow$ $(\hat{X}_s)_{s\in[0,t]}$, $(X_s)_{0\leq s\leq t}$ have the same distribution

But arrivals of $\hat{X}$ corresponds to departures of X.

$\therefore$ Departure process of $(X_s)_{0\leq s\leq t}$ has the same distribution as arrival process of $\hat{X}$ $\Rightarrow$ Poisson λ process

No.: Date:

$\therefore \hat{X}_0$ is independent of arrival process of $(\hat{X}_s)_{0 \le s \le t}$

$\therefore X_t$ independent of departure ~~fro~~ process $(D_s)_{0 \le s \le t}$

(b) Jackson's Theorem: Let $\tilde{\lambda}_i = \sum_{j \ne i} \tilde{\lambda}_j P_{ji} + \lambda_i$ be a system of linear equations. $\exists$ unique $_{\ge 0}$ solutions to $\tilde{\lambda}_i$

If $\tilde{\lambda}_i / \mu_i < 1$: $(1 \le i \le n)$

Queue system ~~its in equilibrium with~~ ~~invariant~~ is +ve recurrent with invariant distribution $\pi_{\underset{\sim}{n}} = \prod_{k=1}^{n} (1 - \tilde{\lambda}_i / \mu_i)(\tilde{\lambda}_i / \mu_i)^{n_i}$

In equilibrium: Departure process (from queue to outside world) are independent poisson processes of rate $\tilde{\lambda}_i \cdot P_{i,0}$.

$(P_{i0} = \sum_{j \ne i} -P_{ij} + 1)$

Process $(L_1, \dots L_n)$ is not independent:

$\mathbb{P}($ ~~no~~ $L_1, \dots L_{n-1} = 0$ on $[0,1]$, no external arrivals to ~~Q~~ L_n on $[0,1]$, 1 arrival to L_n on $[0,1]) = 0$

But $\mathbb{P}(L_1, \dots L_{n-1} = 0$ on $[0,1]) > 0$, $\mathbb{P}($no external arrival, 1 internal arrival to L_n on $[0,1]) > 0$

$\therefore$ Not independent

POP bazic

No.: .... Date: ....

Let $(\hat{L}_1, \dots \hat{L}_n)_{0 \le s \le t}$ be reversed process at equilibrium

$$\hat{Q}_{\underline{n}, \underline{n}+e_i-e_j} = \pi_{\underline{n}+e_i-e_j} / \pi_n \quad Q_{n+e_j-e_i} \qquad (n_j \ge 1)$$

$$= \cancel{(\ldots)} (\tilde{\lambda}_i/\mu_i) / (\tilde{\lambda}_j/\mu_j) \cancel{\ldots} P_{ij} \mu_i$$

$$= P_{ij} (\tilde{\lambda}_i / \tilde{\lambda}_j \mu_i)$$

$$\hat{Q}_{\underline{n}, \underline{n}+e_i} = (\tilde{\lambda}_i/\mu_i) P_{i0} \mu_i = P_{i0} \tilde{\lambda}_i$$

$$\hat{Q}_{\underline{n}, n-e_j} = 1/(\tilde{\lambda}_j/\mu_j) \quad \lambda_j = \lambda_j \mu_j / \tilde{\lambda}_j$$

$\therefore (\hat{L}_1, \dots \hat{L}_n)$ is also a Jackson Network

Departure process of $(L_1, \dots L_n)_{0 \le s \le t}$ $\equiv$ Arrival process of $(\hat{L}_1, \dots \hat{L}_n)$ $\Rightarrow$ Independent Poisson process

L_i departure process =

$\hat{L}_i$ arrival $\equiv$ Poisson $P_{i0} \tilde{\lambda}_i$

$L_t = \hat{L}_0$, $\hat{L}_0$ independent of arrivals in $(\hat{L}_{(s)})_{0 \le s \le t}$.

$\therefore$ L_t independent of departures in $0 \le s \le t$.

POP bazic

No.: .... Date: ....

No.: .... Date: ....

Applied Probability 2014

1.) (a) Let $(\mathcal{E}_n)_{n\geq 1}$ be iid Exp (1) R.V. ; S be state space,

$\forall s \in S, \quad q(s) = -Q_{ss}$

Define $P : S \times S \to [0,1]$, $P(a,b) = Q(a,b)/q(a)$;

Let $(Y_n)_{n\geq 0}$ be discrete markov chain with transition matrix P, $Y_0 = X_0$.

Define : $J_0 = 0$, $J_n = \sum_{k=1}^{n} \mathcal{E}_k / q(Y_{k-1})$:

Let $X_t = Y_k$ for $t \in [J_k, J_{k+1})$.

$(X_t)_{t\geq 0}$ is a cont. markov chain with generator Q

(b) Let $S = \cancel{\mathbb{Z}_{\geq 0}}\ \mathbb{N}$:

~~With population n: from n+1 we have n → n-1 at a rate Sn~~

If population = n : It can either go to n+1 or n-1

(n+1): There are n processes leading to +1 ⇒ ~~n → n+1~~

$\text{Min } Z = \text{Min}\{\xi_1, \ldots \xi_n\}$, $\xi_i \sim \text{Exp}(\lambda)$

$\mathbb{P}(Z \geq t) = \mathbb{P}(\xi \geq t)^n = e^{-\lambda n t} \Rightarrow Z \sim \text{Exp}(n\cdot\lambda)$

$\therefore$ $n \to n+1$ at rate $n\cdot\lambda$

(n-1): In time h, $\mathbb{P}(n+1 \to n) = S_n \cdot h$

$\therefore$ $n+1 \to n$ at a rate of S_n.

Q - matrix : $Q(n, n+1) = \lambda n$

$Q(n, n-1) = S_n$

$Q(n,n) = -(\lambda n + S_n)$

$Q(n,m) = 0$

POP bazic

No.: .... Date: ....

Invariant Distribution:

(Birth - death cycle): Π is invariant iff detailed balance holds

$$\pi_n \cdot (\lambda \cdot n) = \pi_{n+1}(\delta_{n+1}) \Rightarrow \pi_{n+1} = \pi_n (\lambda n)/\delta_{n+1}$$

$$= \lambda^n \cdot n! / \delta_{n+1} \dots \delta_2$$

$\therefore$ Invariant distribution exists iff $\sum_{n \geq 1} \frac{\lambda^n n!}{\cancel{\delta_{n+1} \dots} \prod_{k=2}^{n+1} \delta_k} < \infty$.

If $\delta_n = \mu(n-1)$: $\pi_{n+1} = \pi_n \cdot (\lambda n)/\mu(n) = (\lambda/\mu)\, \pi_n$

$\therefore \pi_{n+1} = \pi_1 (\lambda/\mu)^n$

$$\sum_{n \geq 1} \pi_n = \cancel{\pi_1 \left(\frac{1}{1 - \lambda/\mu}\right)} \; \pi_1 \sum_{n \geq 0} (\lambda/\mu)^n$$

Invariant distribution exists iff $\lambda < \mu$.

If $\lambda < \mu$: $\pi_n = (1 - \lambda/\mu)(\lambda/\mu)^{n-1}$

3.) (a) Let $(\mathcal{E}_n)_{n\geq 1}$ be iid $Exp(\lambda)$ R.V.:

$J_0 = 0$, $J_n = \sum_{k=1}^{n} \mathcal{E}_k$; $(X_t)_{t\geq 0}$ defined by: $X_t = k$ for $t \in [J_k, J_{k+1})$

$N_t \sim Poi(\lambda t)$

$T_1 = \mathcal{E}_1$, $T_{k+1} - T_k \sim \mathcal{E}_{k+1}$

$\therefore (T_1, T_2 - T_1, \dots T_n - T_{n-1}) = (\mathcal{E}_1, \dots \mathcal{E}_n)$

$f_{T_1, T_2 - T_1, \dots T_n - T_{n-1}}(y_1, \dots y_n) = \lambda^n e^{-\lambda(y_1 + \dots + y_n)}$ (product of independent R.V.)

$\wedge \ (T_1, \dots T_n)^T = \begin{pmatrix} 1 & & 0 \\ 1 & 1 & \\ \vdots & & \ddots \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 - T_1 \\ \vdots \\ T_n - T_{n-1} \end{pmatrix}$

$\therefore f_{T_1, \dots T_n}(y_1, y_1 + y_2, \dots y_1 + \dots + y_n) = \left| \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \right| \cdot f_{T_1, T_2 - T_1, \dots T_n - T_{n-1}}(y_1, \dots y_n)$

$= \lambda^n e^{-\lambda(y_1 + \dots + y_n)}$ | where $y_i \geq 0$.

$\cancel{\mathbb{P}(T_1 \in A_1, \dots T_n \in A_n \wedge N_t = n) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} 1_{0 \leq x_1 \leq x_2 \leq \dots x_n} \lambda^n e^{-\lambda(x_1 + \dots + x_n)} 1_{x_n \leq t < x_{n+1}} \cdot \lambda e^{-\lambda x_{n+1}} \, dx_{n+1} \dots dx_1}$

$\cancel{= \int_{A_1} \dots \int_{A_n} 1_{0 \leq x_1 \dots \leq x_n \leq t} \lambda^n e^{-\lambda x_n} \cdot \int_t^\infty 1_{t < x_{n+1}} \lambda}$

$\mathbb{P}(T_1 \in A_1, \dots T_n \in A_n, N_t = n) = \mathbb{P}(T_1 \in A_1, \dots T_n \in A_n, T_n < t \leq T_{n+1})$

$= \int_{A_1} \dots \int_{A_n} \int_t^\infty 1_{0 \leq x_1 \dots \leq x_n \leq t} \lambda^{n+1} e^{-\lambda x_{n+1}} \, dx_{n+1} \dots dx_1$

$= \int_{A_1} \dots \int_{A_n} 1_{0 \leq x_1 \dots \leq x_n \leq t} \lambda^n e^{-\lambda t} \, dx_n \dots dx_1$

No.: .... Date: ....

$$\mathbb{P}(T_1 \in A_1, \dots T_n \in A_n \mid \{N_t = n\}) =$$

$$\int_{A_1 \dots} \int_{A_n} 1_{0 \le x_1 \dots \le x_n \le t} \lambda^n e^{-\lambda t} \, dt \Big/ e^{-\lambda t} (\lambda t)^n / n!$$

$$= \int_{A_1 \dots} \int_{A_n} \underbrace{1_{0 \le x_1 \dots \le x_n \le t} \, n!/t^n}_{\text{Density of order statistics of } n \text{ iid } U[0,t] \text{ R.V.}} \, dt$$

$$\therefore \text{Joint Density}: f_{T_1, \dots T_n}(x_1, \dots x_n) \mid \{N_t = n\} = 1_{0 \le x_1 \dots \le x_n} \, n!/t^n$$

(b) $$\mathbb{E}\left[e^{\theta \sum_{j=1}^{N(t)} g(T_j, X_j)}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\theta \sum_{j=1}^{n} g(T_j, X_j)} \mid N(t) = n\right]\right]$$

$(T_1, \dots T_j) \mid \{N_t = n\} \sim$ order statistic on $U_1, \dots U_n$, (U_i are iid $U[0,t]$ R.V.)

$$\therefore \mathbb{E}\left[e^{\theta \sum_{j=1}^{n} g(T_j, X_j)} \mid N_t = n\right] = \mathbb{E}\left[e^{\theta \sum_{j=1}^{n} g(U_{\sigma(j)}, X_j)}\right],$$

σ is the ordering of $(U_1, \dots U_n)$

$$= \mathbb{E}\left[e^{\theta \sum_{j=1}^{n} g(U_j, X_{\sigma^{-1}(j)})}\right] = \mathbb{E}\left[e^{\theta \sum_{j=1}^{n} g(U_j, X_j)}\right]$$

as $(X_1, \dots X_n)$ are iid.

$$= \mathbb{E}\left[e^{\theta g(U, X)}\right]^n$$

$$\mathbb{E}\left[e^{\theta g(U,X)}\right] = \frac{1}{t} \int_0^t \mathbb{E}\left[e^{\theta g(y, Y)}\right] dy = \Lambda.$$

$$\therefore \mathbb{E}\left[e^{\theta \sum_{j=1}^{N(t)} g(T_j, X_j)}\right] = \mathbb{E}\left[\Lambda^{N(t)}\right] = e^{-\lambda t(1-\Lambda)} = e^{\lambda(\Lambda - 1)}$$

$$= e^{\lambda \left\{ \int_0^t \mathbb{E}[g(y, X)] \, dy - 1 \right\}}$$

Let S be # of students at time T

$$S = \sum_{j=1}^{N(T)} 1_{T-T_j \le X_j}$$

$$\therefore \mathbb{E}[e^{\theta \cdot S}] = \cancel{\mathbb{E}} \, e^{\lambda \int_0^t \mathbb{E}[e^{\theta 1_{T-y \le X}}] - 1 \, dy}$$

$$\mathbb{E}[e^{\theta 1_{T-y \le X}}] = 1 + (e^{\theta} - 1) \, \mathbb{P}(X \ge T-y)$$

$$\therefore \mathbb{E}[e^{\theta \cdot S}] = \cancel{\mathbb{E}} \, e^{\lambda \int_0^t \mathbb{P}(X \ge T-y) \, dy \, (e^{\theta}-1)}$$

If $Z \sim Poi(\mu)$, $\mathbb{E}[e^{\theta Z}] = \sum_{k \ge 0} \frac{e^{-\mu} \mu^k}{k!} e^{\theta k} = e^{-\mu} e^{\mu e^{\theta}} = e^{\mu(e^{\theta}-1)}$

$\therefore S \sim Poi\left(\lambda \int_0^t \mathbb{P}(X \ge T-y) \, dy\right) \Rightarrow$ Poisson R.V.

(Pick T to represent time interval of 9 am $\to$ 5 pm)

4.) (a) $M(\lambda)/M(\mu)/1$ Queue:

Consider Jump chain: Random Walk on $\mathbb{Z}_{\ge 0}$, $\mathbb{P}(+1) = \lambda/\lambda+\mu$, $\mathbb{P}(-1) = \mu/\lambda+\mu$.

$\therefore$ Transient iff $\lambda/\lambda+\mu > 1/2$ iff $\lambda > \mu$.

Since jump chain recurrent iff $(X_t)_{t \ge 0}$ is recurrent:

$\lambda > \mu$: Transient

$\lambda < \mu$: Recurrent.

If $\lambda < \mu$: Detailed Balance Equation $\pi_n \cdot \lambda = \mu \pi_{n+1}$

$\therefore \pi_n = (\lambda/\mu)^n \cdot \pi_0$ (Invariant Measure)

$\therefore$ Invariant distribution iff $\lambda/\mu < 1$; Recurrent $\Rightarrow$ Non-explosive;

$\therefore \lambda = \mu$: null-recurrent

$\lambda < \mu$: +ve recurrent; Invariant Dist. $\pi_n = (1 - \lambda/\mu)(\lambda/\mu)^n$

No.: .... Date: ....

* $M(\lambda)/M(\mu)/1$ Definition: State Space $= \mathbb{Z}_{\geq 0}$,

Q matrix: $Q(n, n+1) = \lambda$ $(\cancel{A} \; n \geq 0)$

$Q(n, n-1) = \mu$ $(n \geq 1)$

$Q(n, n) = \lambda + 1_{n>0}\mu$ $(n \geq 0)$

$Q(n, m) = 0$ (otherwise)

$M/M/\infty$ Queue: State Space $\mathbb{Z}_{\geq 0}$, Q matrix:

$Q(n, n+1) = \lambda$ $(n \geq 0)$

$Q(n, n-1) = n \cdot \mu$ $(n \geq 1)$

$Q(n, n) = \lambda + n\mu$ $(n \geq 0)$

$Q(n, m) = 0$ (otherwise)

$\pi_n = (\lambda/\mu)^n / n!$: $\dfrac{\pi_n \, Q_{n,n+1}}{\pi_{n+1} \, Q_{n+1,n}} = \dfrac{n+1}{(\lambda/\mu)} \cdot \dfrac{\lambda}{(n+1)\mu} = 1$

$\therefore$ $\underline{\pi}$ is in detailed balance, $\Rightarrow$ Invariant

$\sum_{n \geq 0} \pi_n = e^{(\lambda/\mu)} < \infty \Rightarrow$ ~~Invariant~~ $\left(\pi_n / e^{\lambda/\mu}\right)_n$ is an invariant distribution

We know: If $\underline{\pi}$ is $(X_t)_{t \geq 0}$ invariant, $(\pi_n \cdot q_n)_{n \geq 0}$ is jump chain invariant

Let $\tilde{\pi}_n = (\lambda/\mu)^n \cdot (\lambda + n\mu) / n!$; $\tilde{\pi}$ is jump chain invariant

$$\sum_{n \geq 0} \tilde{\pi}_n = \lambda \sum_{n \geq 0} \frac{(\lambda/\mu)^n}{n!} + \lambda \sum_{n \geq 1} \frac{(\lambda/\mu)^{n-1}}{(n-1)!}$$

$$= 2\lambda \, e^{(\lambda/\mu)} < \infty.$$

Since Invariant distribution exist for jump chain, jump chain is +ve recurrent $\Rightarrow$ $\mathbb{N}$ Recurrent $\Rightarrow$ $(X_t)_{t \geq 0}$ is recurrent $\Rightarrow$ Non-explosive

Invariant distribution of $(X_t)_{t \geq 0}$ exists, Non-explosive $\Rightarrow$ +ve recurrent.

$\pi_n = e^{-(\lambda/\mu)} (\lambda/\mu)^n / n!$

No.: .... Date: ....

(ii) Let X_t be the length of taxi in queue, at time t:

Arrival rate = ~~3~~ 2;

Service rate = Arrival rate of customers = 3 (Single Server)

$\therefore$ $M(2)/M(3)/1$ Queue, (+ve recurrent).

$\pi_n = \frac{1}{3}\left(\frac{2}{3}\right)^n$ is invariant distribution

~~For~~ In the long run: $\mathbb{P}(X_t = 0) \to \pi_0 = \frac{1}{3}$

$\therefore$ $\mathbb{P}$(customer leaves) $\to \frac{1}{3}$.

$\therefore$ $\mathbb{P}$(arriving customer gets a taxi) $= \frac{2}{3}$

Average # of taxis: $\displaystyle\sum_{n\geq 0} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n \cdot n = \left(\frac{2}{3}\right) \sum_{n\geq 1} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1} n$

$\displaystyle = \cancel{\left(\frac{2}{3}\right)}\ \cancel{2} \ \cancel{\text{ke}} -1 + \sum_{n\geq 1} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1} n$

$= -1 + \frac{1}{(1/2)} = 2.$

$\mathbb{E}$[Time between ~~two~~ 0 taxi] $= \dfrac{1}{\pi_0 \cdot q_0} = \dfrac{1}{\frac{1}{3}\cdot 2} = \frac{3}{2}$

$\mathbb{E}$[Time to 1^{st} taxi arrival from 0 queue length] $= \frac{1}{2}$

$\therefore$ $\mathbb{E}$[Time from taxi arrival $\to$ break] $= 1$

No.: .... Date: ....

* 2015 Q4 Part 1

4.) (a) Consider jump chain :

~~(c)(a)~~ Condition on $\{T_A = \infty\}$: ~~then~~ ~~the new~~

$\cancel{\tilde{P}(x,y) = \mathbb{P}(Y_1 = y}$

$\cancel{\mathbb{P}(Y_{n+1} = y \mid Y_n = x, T_A) = \mathbb{P}(}$ P

$\mathbb{P}(Y_0 = y_0, \dots Y_n = y_n ; T_A = \infty)$

$= P_{y_0, y_1}, \dots P_{y_{n-1} y_n} \cdot \mathbb{P}(T_A = \infty \mid Y_0 = y_n)$

$\therefore \mathbb{P}(Y_0 = y_0, \dots Y_n = y_n \mid T_A = \infty) = \prod_{k=1}^{n} P_{y_{k-1} y_k} \cdot h_{y_k} / h_{y_{k-1}}$

$\therefore \mathbb{E}_x[\tau_y] = h_x \cdot \mathbb{E}_x[\tau_y \mid \tau_A < \infty] + (1-h_x)\, \mathbb{E}_x[\tau_y \mid \tau_A = \infty]$

$\mathbb{E}_x[\tau_y \mid \tau_A = \infty] = \sum_{k \geq 0} \cancel{\mathbb{E}}[\ \mathbb{P}(Y_k = y \mid \tau_A = \infty)$

$= (1-h_y) / (1-h_x)\ \mathbb{E}[\tau_y]$

$\therefore \mathbb{E}_x[\tau_y] = h_x\ \mathbb{E}[\tau_y \mid \tau_A < \infty] + (1-h_y)\ \mathbb{E}[\tau_y]$ //

Applied Probability 2016

1.) (a) Let S be state space, $q(x) = -Q_{xx}$ $(x \in S)$, $P(x,y) = \frac{1_{x \neq y} Q(x,y)}{q(x)}$

$(Y_n)_{n \geq 0}$ be jump chain: $Y_0 = X_0$, & P is transition probability.

Let $(\mathcal{E}_n)_{n \geq 1}$ be iid Exp(1) R.V.:

$$J_0 = 0, \quad J_n = \sum_{k=1}^{n} \mathcal{E}_k / q(Y_{k-1}) \quad : \quad X_t = Y_k \text{ for } t \in [J_k, J_{k+1})$$

(b) If state x is $(Y_n)_{n \geq 0}$ transient: $\mathbb{P}($~~[illegible]~~$)$

x is non-absorbing;

Let $T = \sup\{n : Y_n = x\}$; $T < \infty$ A.S. (x is Transient)

$\therefore$ If $t > J_{T+1}$, $X_t \neq x$

$\therefore$ $\{t : X_t = x\} \subseteq [0, J_{T+1}]$ ⇒ ~~Bounded A.S.~~

$\therefore$ ~~x is (X_t)_{t≥0} transient~~

Since $J_{T+1} < \infty$ A.S. ~~A.S.~~ if $T < \infty$ A.S.: $\{t : X_t = x\}$

is bounded A.S.

$\therefore$ x is $(X_t)_{t \geq 0}$ transient.

(c) Let $S = \mathbb{Z}$, $Q(n, n \pm 1) = \frac{(1+|n|)^2}{2}$, $Q(n,n) = (1+|n|)^2$.

Discrete chain: $(Y_n)_{n \geq 0}$ is a symmetric random walk on $\mathbb{Z}$ ⇒ Null-recurrent.

$\therefore$ $(X_t)_{t \geq 0}$ is recurrent; Consider $\pi_n = \frac{1}{(1+|n|)^2}$:

$\pi_n Q(n, n \pm 1) = \pi_{n \pm 1} Q(n \pm 1, n) = \frac{1}{2}$ ⇒ π is in detailed balance

$\therefore$ Invariant measure.

$\sum_{n \in \mathbb{Z}} \frac{1}{(1+|n|)^2} < \infty$ ⇒ Invariant distribution ⇒ & +ve recurrent

($\cancel{\pi}$ recurrent ⇒ not explosive)

$\therefore$ (Counterexample)

No.: .... Date: ....

(d) Consider jump chain: Random Walk on $\mathbb{Z}$, $\mathbb{P}(+1) = 2/3$, $\mathbb{P}(-1) = 1/3$

∴ Jump chain is transient ⇒ $(X_t)_{t \geq 0}$ is transient.

Solve: $\pi_n (2 \cdot 3^{|n|}) = \pi_{n+1} \cdot 3^{|n+1|}$

$(n \geq 0)$ ∴ $\pi_{n+1} = \frac{2 \cdot 3^n}{3^{n+1}} \pi_n = (2/3) \pi_n$

∴ $\pi_n = (2/3)^n \pi_0$

$(n < 0)$: $\pi_n = \pi_{n+1} \, 3^{-(n+1)} \big/ 2 \cdot 3^{-n} = \pi_{n+1}/6.$

∴ $\sum_{n \in \mathbb{Z}} \pi_n < \infty$ ⇒ Invariant Distribution exists

(e) Let $h_x = \mathbb{E}_x[T_0]$:

$h_n = 1/3^{|n|+1} + 2/3 \, h_{n+1} + 1/3 \, h_{n-1}$

∴ $-1/3^{|n|} = 2 h_{n+1} - 3 h_n + 1$ $(2m-1)(m-1)$

∴ $h_n = A + B(\frac{1}{2})^n + C$

If $\mathbb{P}(T_0 = \infty) > 0$: ~~Since chain is recurrent,~~ If $\mathbb{E}_0[T_0] < \infty$, 0 is +ve recurrent;

Since chain is irreducible: All states are +ve-recurrent ⇒ Recurrent

∴ Contradiction

(e) Jump Chain: Symmetric Random Walk on $\mathbb{Z}^d$

∴ Recurrent iff $d \leq 2$.

If $d \leq 2$: $\pi_x = 1/q(x)$ solves detailed balance

∴ +ve recurrent iff $\sum_{x \in \mathbb{Z}^d} \frac{1}{\min_{ax}\{1, \|x\|^\alpha\}} < \infty.$ $-(*)$

Since $\|\cdot\|_2$ is equivalent to $\|\cdot\|_\infty$, $(*)$ is equivalent to

No.: .... Date: ....

$$\sum_{i=1}^{N} 1_{n_i>0} \cancel{\pi_{n-e_i}} \sum_{j\neq i} \pi_{n+e_j-e_i}\, \mu_j\, (n_j+1)\, P_{ji} =$$

$$\cancel{\sum_{i=1}^{N} 1_{n_i>0}\, \pi_n \left\{ \sum_{j\neq i} \frac{(\lambda_j/\mu_j)_{n_j+1}}{(\lambda_i/\mu_i)}\, \mu_j P_{ji} \right\}}$$

$$= \cancel{\sum_{i=1}^{N} 1_{n_i>0}\, \pi_n\, (\mu_i/\lambda_i) \sum_{j\neq i} \lambda_j P_{ji}}$$

$$\sum_{i=1}^{N} 1_{n_i>0}\, \pi_n \sum_{j\neq 0} \frac{(\lambda_j/n_j+1)}{(\lambda_i/n_i)}\, \mu_j P_{ji}\, (n_j+1)$$

$$= \sum_{i=1}^{N} 1_{n_i>0}\, \pi_n\, \frac{n_i}{\lambda_i} \underbrace{\sum_{j\neq 0} \lambda_j \mu_j P_{ji}}_{=\mu_i\lambda_i} = \sum_{i=1}^{N} 1_{n_i>0}\, \pi_n\, \frac{n_i \mu_i}{\lambda_i}\lambda_i$$

$= \left(\sum_{i=1}^{N} 1_{n_i\neq 0}\, \mu_i n_i\right) \pi_n \Rightarrow$ Solves Invariant Equation.

(d) Not independent: Let $A^{(k)}$ be length of queue k.

$\mathbb{P}(A^{(k)}(1) = k) > 0$

But $\prod_{k=1}^{N} \mathbb{P}(A^{(k)}(1) = k) > 0$, $\mathbb{P}(A^{(k)}(1) = k \ (1 \le k \le N)) = 0$

$\therefore$ Not independent

3.) (a) Let $(N_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$ be independent Poisson processes of rate λ, μ respectively:

Superposition: $(N_t + M_t)_{t\geq 0}$ is a Poisson process of rate $\lambda+\mu$.

Thinning: Let $(\alpha_n)_{n\geq 1}$ be independent of $(N_t)_{t\geq 0}$, iid $Be(p)$ R.V.

~~Let Y_t be points in N_t~~

Let Y be process in ~~$\mathbb{N}$~~ $\mathbb{Z}_{\geq 0}$, with jump at time t iff N ~~X~~ jumps at t, $Z_{X_t} = 1$.

~~Z~~ Y is a Poisson rate $p\lambda$ process, $Z = X - Y$ is a Poisson rate $(1-p)\cdot\lambda$ process; Z, Y independent.

Proof of Superposition: $(N_t)_t$, $(M_t)_t$ have independent increments $\Rightarrow$ $(N_t + M_t)_{t \geq 0}$ has independent increments

$$\mathbb{P}(N_{t+h} + M_{t+h} = N_t + M_t) = \mathbb{P}(N_{t+h} = N_t, M_{t+h} = M_t)$$
$$= (1 - \lambda h + o(h))(1 - \mu h + o(h))$$
$$= 1 - (\lambda + \mu)h + o(h) \quad \text{(Uniformly in } h)$$

$$\mathbb{P}(N_{t+h} + M_{t+h} - N_t - M_t = 1) = \mathbb{P}(N_{t+h} - N_t = 1, M_{t+h} - M_t = 0) + \mathbb{P}(N_{t+h} - N_t = 0, M_{t+h} - M_t = 1) + o(h)$$
$$= (\lambda h + o(h))(1 - \mu h + o(h)) + (\mu h + o(h))\cdot(1 - \lambda h)$$
$$= (\lambda + \mu)h + o(h)$$

(Uniformly in all t)

$\therefore$ (Infinitesimal Definition): $(N_t + M_t)_{t \geq 0}$ is a Poisson process of rate $\lambda + \mu$.

(b) Let $(X_t)_t$, $(Y_t)_t$ be independent Poisson process of rate λ.

Define: $N_t = X_t \quad (t \geq 0)$

$N_t = Y_{-t} \quad (t < 0)$

$N_t - N_s$: If $t, s \geq 0$: ~~have the same sign~~, $N_t - N_s = X_t - X_s \sim \mathrm{Poi}(\lambda(t-s))$

≤ 0: $N_t - N_s = Y_{-s} - Y_{-t} \sim \mathrm{Poi}(\lambda(t-s))$

$s < 0 < t$: $N_t - N_s = X_t - Y_s$

No.: .... Date: ....

Let $N_t = X_t \quad (t \geq 0)$

$= -Y_{-t} \quad (t \leq 0)$

If $0 \leq s \leq t$: $N_t - N_s = X_t - X_s \sim X_{t-s}$ (indp, stat. increment)

$\sim Poi(\lambda(t-s))$

$0 \geq t \geq s$: $N_t - N_s = -\cancel{X}Y_{-t} + \cancel{X}Y_{-s} \sim Y_{-(s-t)} \sim Poi(\lambda(t-s))$

$t \geq 0 \geq s$: $N_t - N_s = N_t - N_0 + N_0 - N_s = X_t + \cancel{X}Y_{-s}$

$\sim Poi(\lambda(t-s))$ (Superposition)

($N_0 = 0$ by definition)

~~Any increment is independent, stationary as $(X_t)_{t\geq0}$, $(Y_t)_{t\geq0}$ have independent, stationary increments~~

(Independent Increments): $t_0 < t_1 \dots < t_k = 0 < t_{k+1} \dots < t_N$

$(N_{t_1} - N_{0,t_0}, \dots N_{t_N} - N_{t_{N-1}}) = (Y_{-t_0} - Y_{-t_1}, \dots Y_{-t_{k-1}} - Y_0 ; X_{t_{k+1}} - X_0,$

$\dots X_{t_N} - X_{t_{N-1}}) \Rightarrow$ Independent

Since $N_t - N_s \sim Poi(\lambda(t-s))$ is only dependent on $t-s$; Increments are stationary.

$\therefore$ Process Exists

(c) Let $\tilde{N}$ be shifted process:

$\tilde{N}_t - \tilde{N}_s = N_{t-c} - N_{s-c} \sim Poi(\lambda(t-s))$

& Increments remain stationary, independent as

~~$(\tilde{N}_{t_1} - t_0$~~ $(\tilde{N}_{t_1} - \tilde{N}_{t_0}, \dots \tilde{N}_{t_n} - \tilde{N}_{t_{n-1}}) = (N_{t_1-c} - N_{t_0-c}, \dots N_{t_n-c} - N_{t_{n-1}-c})$

$\Rightarrow$ Independent, stationary increments

$\therefore \tilde{N}$ is the same process

No.: Date:

(d) Thinning on each Point: ($\mathbb{P} = 1/2$)

$N \to X,\ N-X$; $X, N-X$ are independent $Poi(\lambda/2)$ processes

Shift X by $+1$, $N-X$ by -1: Remains Independent $Poi(\lambda/2)$ processes

Combine: $Poi(\lambda)$ process. $\Rightarrow$ Remains a $Poi(\lambda)$ process

4) (a) Let $(\xi_n)_{n\geq 1}$ be iid, ≥ 0 R.V., $\mathbb{E}[\xi_1] > 0$.

$N_t = \operatorname{Max}\{k \geq 0 : \xi_1 + \dots + \xi_k \leq t\}$ is a renewal process

$$\sum_{k=1}^{N_t} \xi_k \leq t \leq \sum_{k=1}^{N_t+1} \xi_k$$

Claim: $N_t \to \infty$ A.S.

$\exists\ \varepsilon > 0$ s.t. $\mathbb{P}(\xi_1 > \varepsilon) > 0 \Rightarrow \sum_{n\geq 1} \mathbb{P}(\xi_n > \varepsilon) = \infty$

2^{nd} Borel Cantelli $\Rightarrow$ $\xi_n > \varepsilon$ i.o. $\overset{A.S.}{\Rightarrow}$ $N_t \to \infty$ A.S.

$\therefore$ (Strong law) $\frac{1}{N(t)} \sum_{k=1}^{N_t} \xi_k \to \lambda^{-1}$ A.S.

$\frac{N(t)+1}{N(t)} \to 1$ (A.S.) $\Rightarrow$ $\frac{1}{N(t)} \sum_{k=1}^{N_t+1} \xi_k \to \lambda^{-1}$ A.S.

$\therefore$ $t/N(t) \to 1/\lambda$ A.S. $\Rightarrow$ $N(t)/t \to \lambda$ A.S. $\blacksquare$