AP

Def

1. Discrete time Markov chain
2. Continuous time random process
3. Jump time J_n
4. Jump chain Y
5. CTMC
6. Transition probability, time-homogeneous
7. $P(t+s) = P(t)P(s)$
8. Holding time S_x: Memoryless property

1. Poisson Process $[S_i \sim exp(\lambda)]$
2. Markov property for Poisson process / strong markov $(X_{s+t} - X_s)_{t\geq 0}$,
3. Three equivalent def: Holding time, infinitesimal, X_t
4. Superposition (Def 3)
5. Thinning (Def 2)
6. Conditional on the event $(X_t = n)$, we have $f(t_1, \ldots, t-n) = \frac{n!}{t^n} 1(0 \leq t_1 \leq \ldots \leq t_n \leq t)$

1. Birth Process, $[S_i \sim exp(q_i)]$, simple birth process
2. $T = inf_k T_k$, $T \sim Exp(\sum_k qk)$; The infimum is attained at a point T_K almost surely, and $P(K=n) = q_n / \sum_k qk$; T and K are independent
3. $\zeta := \sum_n S_n < \infty$ explosion
4. Explosiveness equivalence in $\sum \frac{1}{q_i}$
5. Equivalent three defs

1. Q-matrix
2. def of minimal CTMC with Q-matrix and initial distribution
3. Three constructions
4. Non-explosive if three conditions

(a) Y is ~
(b) Conditional on $Y_0, \ldots, Y_n$, $S_1, \ldots, S_{n+1}$ are indep ~ $Exp(q_{Y_{i-1}})$

1. Kolmogorov's forward & backward equations
2. Finite set $P(t) = e^{tQ}$
3. Let I be a finite state space and Q be a matrix. Then it is a Q-matrix iff P(t) = ^ tQ is a stochastic matrix for all t.
4. Let X be a right-continuous process with values in a finite set I, and let Q be a Q-matrix on I. Then the following are equivalent

1. Class structure
2. Recurrent and transient

$\mathbb{E}[\mathbb{1}[X_t=x]]$

3. x is recurrent for X if and only if $\int_0^\infty p_{xx}(t)dt = \infty$, and x is transient for X if and only if $\int_0^\infty p_{xx}(t)dt < \infty$
4. If $|I|$ is finite, then $\lambda Q = 0$ if and only if $\lambda P(s) = \lambda$ for all $s \geq 0$
5. Positive recurrent equivalent to non-explosive + invariant distribution

inv for Y

$\mu = q\pi$ inv for X

not necessarily distribution, just measure

Queue

1. $M/M/1: q(i, i+1) = \lambda, q(i, i-1) = \mu; M/M/\infty: q(i, i+1) = \lambda, q(i, i-1) = i\mu$
2. Let $\rho = \lambda/\mu$. Then the queue length X (for a $M/M/1$ process) is transient if and only if $\rho > 1$, recurrent if and only if $\rho \leq 1$ and positive recurrent if and only if $\rho < 1$. X is non-explosive. In the positive recurrent case, the invariant distribution is $\pi(n) = (1-\rho)\rho^n, n = 0, 1, \ldots$. And if $\rho < 1$, and $X_0 \sim \pi$, then the wait time (including service time) for a customer that arrives at time t is $Exp(\mu - \lambda)$.
3. $M/M/\infty$, The queue length X_t is positive recurrent for all $\mu > 0, \lambda > 0$ with invariant distribution $Poi(\rho)$ where $\rho = \lambda/\mu$
   DB, $\mu_i = \pi_i q_i$, show μ is normalisable
4. (Burke's Theorem). Consider an M/M/1 queue with $\mu > \lambda > 0$ or an $M/M/\infty$ queue with $\mu, \lambda > 0$. At equilibrium ($i.eX_0 \sim \pi$), D is a Poisson process of rate λ and X_t is independent of $(D_s : s \leq t)$.
5. (X, Y) is positive recurrent if and only if $\lambda < \mu_1$ and $\lambda < \mu_2$. In this case, the invariant distribution is given by $\pi(m, n) = (1-\rho_1)\rho_1^m(1-\rho_2)\rho_2^n$ where $\rho_1 = \lambda/\mu_1, \rho_2 = \lambda/\mu_2$
6. traffic equation: $\bar{\lambda}_i = \bar{\lambda}_i + \sum_{j=1, j\neq i}^{N} \lambda_j p_{ji}$
7. Jackson network positive recurrent
8. $M/G/1$ queue: transition matrix, recurrent

Renewal

1. Definition ξ (holding time) T_n (jump time), N_t: renewal process
2. If $E\xi = 1/\lambda < \infty$ then as $t \to \infty$, $N_t/t \to \lambda$ almost surely
3. Size-biased picking: S_i, $Y_i = S_i/S_n$ U, $\hat{Y}_i$
4. $f_{\hat{Y}}(y) \propto y f_{Y1}(y)$, $\quad P(\hat{Y} \in dy) = nyP(Y_1 \in dy)$
5. Let X be a non-negative random variable with distribution μ and $EX = m < \infty$, Then the size-biased distribution of μ is $\hat{\mu}(dy) = y\mu(dy)/m$. We write $\hat{X}$ for a random variable with distribution $\hat{\mu}$
6. $A(t) = t - T_{N_t}, E(t) = T_{N_{t+1}} - t, L(t) = T_{N_{t+1}} - T_{N_t} = A(t) + E(t)$
7. r.v. is arithmetic $if P(\xi \in kZ) = 1$ for some $k > 1, k \in Z$
8. $(L(t), E(t)) \to (\hat{\xi}, U\hat{\xi})$, $P(U\hat{\xi} \leq y) = \lambda \int_0^y P(\xi > z)dz$
9. Renewal Reward
10. (ξ_i, R_i); $R(t) = \sum_{i=1}^{N_t} R_i$
11. $R(t)/t \to E[R]/E[\xi]$
12. $\gamma(t) \to \lambda E(R\xi)$; $\gamma(t) = E[R_{N_{t+1}}]$
13. regenerative
14. Little's formula: (τ_n), N: arrival process, W_i: waiting time(including service); long-run queue $L = \int_0^t X_s ds/t$, waiting time $W = (W_1 + \ldots + W_n)/n$, arrival rate $\lambda = N_t/t$. Then $L = W\lambda$