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Probability & Measure 2011:

1.) (a) Define: ~~$\|f\|_p =$~~ $L^p = \{ f : E \to \mathbb{R} : f \text{ is measurable}, \|f\|_p < \infty \} / \sim$

$f \sim g$ iff $\|f-g\|_p = 0$

Since L^p is a vector space wrt $\|\cdot\|_p$ norm, we only require completeness to show it is a Banach space.

Claim: L^p is complete.

Let $(f_n)_{n\geq 1}$ be a cauchy sequence in L^p.

Pick subsequence $(n_k)_{k\geq 1}$ of $\mathbb{N}$ s.t.

$$\forall m \geq n_k, \quad \|f_m - f_{n_k}\|_p < 1/2^k$$

$$\therefore \sum_{k\geq 1} \|f_{n_k} - f_{n_{k+1}}\|_p \leq 1 < \infty$$

$$\Rightarrow \Big\| \sum_{k\geq 1} |f_{n_k} - f_{n_{k+1}}| \Big\|_p < \infty \Rightarrow \sum_{k\geq 1} |f_{n_k}(x) - f_{n_{k+1}}(x)| < \infty \text{ A.S}$$

~~If~~ ~~Define~~ Let $\Lambda = \{ x : \sum_{k\geq 1} |f_{n_k}(x) - f_{n_{k+1}}(x)| < \infty \}$.

If $x \in \Lambda$: $(k_1 \leq k_2)$ $|f_{n_{k_1}}(x) - f_{n_{k_2}}(x)| \leq \sum_{r=k_1}^{k_2-1} |f_{n_{r+1}}(x) - f_{n_r}(x)|$

$$\leq \sum_{r\geq k_1} |f_{n_{r+1}}(x) - f_{n_r}(x)| \to 0 \text{ as } k_1 \to \infty$$

$\therefore (f_{n_k}(x))_{k\geq 1}$ is cauchy in $\mathbb{R}$ $\Rightarrow$ $f_{n_k}(x) \to f(x)$ in $\mathbb{R}$.

If $x \notin \Lambda$: Define $f(x) = 0$.

Fatou: $\|f\|_p^p = \mu(|f|^p) = \mu(|f|^p 1_\Lambda) =$

$$\mu(\liminf_n |f_n|^p 1_\Lambda) \leq \liminf_n \mu(1_\Lambda |f_n|^p) \leq \liminf_n \|f_n\|_p^p$$

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But $\| f_{n_k} \|_p \le \sum_{r=1}^{k-1} \| f_{n_r} - f_{n_{r+1}} \|_p \le 1 \Rightarrow \| f \|_p < \infty.$

$\therefore f \in L^p$

$\| f_{n_k} - f \|_p = \| \liminf_{r} f_{n_k} - f_{n_r} \| \le \liminf_r \| f_{n_k} - f_{n_r} \|$

$\le 1/2^k \to 0$ as $k \to \infty.$

$\therefore f_{n_k} \to f$ in L^p

Since limit of Cauchy subsequence = limit of Cauchy sequence $\therefore f_n \to f$ in L^p

(b) Convergence in distribution is equivalent to pointwise convergence of characteristic function

Fix $u \in \mathbb{R}$: Let $X_n \xrightarrow{\mathbb{P}} X$;

$$|\mathbb{E}[e^{iuX_n} - e^{iuX}]| \le \mathbb{E}[|e^{iuX_n} - e^{iuX}| \mathbb{1}_{|X_n - X| \ge \delta}] + \mathbb{E}[|e^{iuX_n} - e^{iuX}| \mathbb{1}_{|X_n - X| < \delta}]$$

$$\le 2\,\mathbb{P}(|X_n - X| \ge \delta) + \mathbb{E}[|u| \cdot |X_n - X| \mathbb{1}_{|X_n - X| < \delta}]$$

$$\le \delta \cdot |u| + 2\,\mathbb{P}(|X_n - X| \ge \delta) \quad (\delta > 0 \text{ arbitrary})$$

$\therefore \lim_{n\to\infty} |\mathbb{E}[e^{iuX_n} - e^{iuX}]| = \delta |u| \Rightarrow LHS = 0.$

$\therefore X_n \xrightarrow{\mathbb{P}} X \Rightarrow \mathbb{E}[e^{iuX_n}] \to \mathbb{E}[e^{iuX}] \Leftrightarrow X_n \xrightarrow{d} X.$

(c) $\mathbb{E}[e^{isY_n}] = \prod_{j=1}^{n} \mathbb{E}[e^{isd_j X_n}] \quad -(1)$

$X_n \sim N(0,1)$: $\mathbb{E}[e^{isX_n}] = e^{\frac{1}{2}(is)^2} = e^{-s^2/2}$

$\therefore (1) = \prod_{j=1}^{n} e^{-s^2 d_j^2/2} = e^{-\frac{s^2}{2} \sum_{k=1}^{n} d_k^2} \to e^{-\frac{s^2}{2}\sigma^2}$ as $k \to \infty.$

$\therefore$ Characteristic function of $Y_n \to$ Charac' $\mathbb{E}[e^{is N(0,\sigma)}]$

$\Rightarrow Y_n \xrightarrow{d} N(0, \sigma)$

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Claim: $(Y_n)_{n\geq 1}$ is L^2 cauchy.

$$\mathbb{E}[|Y_n - Y_m|^2] = \mathbb{E}\left[\left|\sum_{k=m+1}^{n} X_k\right|^2\right] - (*)$$

Since $\mathbb{E}[\cancel{X_k} \sum_{k=m+1}^{n} X_k] = 0$: $(*) = Var\left(\sum_{k=m+1}^{n} X_k\right) = \sum_{k=m+1}^{n} Var(X_k)$ (independent)

$$\leq \sum_{k\geq m+1} Var(X_k) = \sum_{k\geq m+1} \alpha_k^2 \to 0 \text{ as } k \to \infty$$

$\therefore$ Cauchy ; L^2 complete $\Rightarrow \exists Y$ s.t $L^2 \to Y$ in L^2.

Since $Y_n \xrightarrow{L^2} Y$: $\mathbb{P}(|Y_n - Y| \geq \varepsilon) \leq \|Y_n - Y\|^2 / \varepsilon^2 \to 0$.

$\therefore$ $Y_n \to Y$ in $\mathbb{P}$ $\Rightarrow$ $Y_n \to Y$ in distribution

$\therefore$ $Y \sim N(0, \sigma^2)$ $\blacksquare$

2.) (a) Let X be a set:

$\mathcal{A} \subseteq \mathcal{P}(X)$ is a:

(1) π-system: $\emptyset \in \mathcal{A}$, $\forall A_i \in \mathcal{A}$, $A_1 \cap A_2 \in \mathcal{A}$

(2) d-system: $X \in \mathcal{A}$; $\forall A_i \in \mathcal{A}$: $A_1 \subseteq A_2 \Rightarrow A_2 \setminus A_1 \in \mathcal{A}$

$A_k \subseteq A_{k+1}$ for $k \in \mathbb{N} \Rightarrow$

$\bigcup_{k\geq 1} A_k \in \mathcal{A}$

Dynkin's π-lemma: Let $\mathcal{A}$ be a π-system, $\mathcal{D}$ be a d-system containing $\mathcal{A}$. $\mathcal{D}$ contains $\sigma(\mathcal{A})$.

By taking the intersection of all ~~$\mathcal{D}$, systems~~ d-systems containing $\mathcal{A}$, we can let $\mathcal{D}$ be the smallest d-system containing $\mathcal{A}$.

$$\mathcal{D}_1 = \{B \in \mathcal{D} : \forall A \in \mathcal{A},\ A \cap B \in \mathcal{D}\}$$

Since $\mathcal{A}$ is a π-system, $\pi \subseteq \mathcal{D}_1$ by definition

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$\forall A \in \mathcal{A}: \quad A \cap X = A \in \mathcal{A} \subseteq \mathcal{D} \Rightarrow X \in \mathcal{D}_1$

If $B_1 \subseteq B_2$, $B_i \in \mathcal{D}_1$: $\forall A \in \mathcal{A}$, $(B_2 \setminus B_1) \cap A = B_2 \cap A \setminus B_1 \cap A$

$B_i \setminus A \in \mathcal{D}$, $B_1 \setminus A \subseteq B_2 \setminus A \Rightarrow (B_2 \setminus B_1) \cap A \in \mathcal{D}$.

If $B_n \subseteq B_{n+1}$, $B_n \in \mathcal{D}_1$: $\forall A \in \mathcal{A}$, $\bigcup_{n \geq 1} B_n \cap A = \bigcup_{n \geq 1} B_n \cap A \in \mathcal{D}$

as $(B_n \cap A)_{n \geq 1}$ is increasing, $B_n \cap A \in \mathcal{D}$.

$\therefore \bigcup_{n \geq 1} B_n \in \mathcal{D}_1 \Rightarrow \mathcal{D}_1$ is a d system containing $\mathcal{A}$.

$\therefore \mathcal{D}_1 = \mathcal{D}$

Let $\mathcal{D}_2 = \{ B \in \mathcal{D} : \forall A \in \mathcal{D}, A \cap B \in \mathcal{D} \}$:

$\therefore \mathcal{A} \subseteq \mathcal{D}_2$ (from above).

By the same argument above, $\mathcal{D}_2$ is a d-system $\Rightarrow$

$\mathcal{D}_2 = \mathcal{D}$.

$\therefore \mathcal{D}$ is closed wrt finite $\cap \Rightarrow \mathcal{D}$ is a π-system $\Rightarrow$

$\mathcal{D}$ is a σ-algebra. $\therefore \sigma(\mathcal{A}) \subseteq \mathcal{D}$.

(b) $\mathcal{E}_1 \otimes \mathcal{E}_2 = \sigma(\{ A \times B : A \in \mathcal{E}_1, B \in \mathcal{E}_2 \})$

~~$\mu_1 \otimes \mu_2 (C) = \mu_1 (x \to \mu_2 (\{ y \to 1_C (x,y)))$~~

Fubini: Let $(E_i, \mathcal{E}_i, \mu_i)$ be measure spaces. If $f: E_1 \times E_2 \to \mathbb{R}$ is ≥ 0, measurable (wrt $\mathcal{E}_1 \otimes \mathcal{E}_2$),

$\mu_1 \otimes \mu_2 (f) = \mu_1 (x \mapsto \mu_2 (y \mapsto f(x,y))) = \mu_2 (y \mapsto \mu_1 (x \mapsto f(x,y)))$

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(c) Consider the following composition of maps:

$$E \times \mathbb{R} \xrightarrow{(f \otimes i)} \mathbb{R} \times \mathbb{R}$$

$$(x, y) \to (f(x), y) \to 1_{0 \le y \le f(x)}$$

Since f, identity is measurable: $f \otimes i$ is measurable

$A = \{ (z, y) : 0 \le y \le z \}$;

A is closed $\Rightarrow A \in \mathcal{B}(\mathbb{R})^{\otimes 2}$

$\therefore G = (f \otimes i)^{-1}(A)$ is measurable $\Rightarrow G \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$

$$\int_E f \, d\mu = \int_E \int_0^\infty 1_{f(z) \ge y} \, dy \cdot d\mu =$$

$$= \int_{E \times \mathbb{R}} 1_{0 \le y \le f(z)} \, d\mu \otimes d\lambda = \mu \otimes \lambda (G)$$

3.) (a) Let (Ω, F, μ) be a probability space; $(X_n)_{n \ge 1}$ be independent R.V. $\subset F$.

Let $F_n = \sigma(X_1, \dots X_n)$, $\mathcal{G}_n = \sigma(X_{n+k}; k \in \mathbb{N})$

Since $(X_n)_{n \ge 1}$ independent: F_n independent of $\mathcal{G}_n$.

Let $\tau = \bigcap_{n \ge 1} \mathcal{G}_n$:

Koglomorov 0-1: If $A \in \tau$, $\mu(A) = 0$ or 1

$F_\infty = \sigma(X_n : n \in \mathbb{N})$: $\therefore \bigcup_{n \ge 1} F_n$ is a π-system

$\sigma(X_n) \subseteq F_n \Rightarrow \bigcup_{n \ge 1} F_n$ generates F_∞.

Since F_n independent of τ, $\bigcup_{n \ge 1} F_n$ is independent of τ.

$\bigcup_{n \ge 1} F_n$ is a π-generator: F_∞ independent of τ.

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But $\mathcal{T} \subseteq \mathcal{F}_\infty$: $\mathcal{T}$ independent of $\mathcal{T}$.

$\therefore$ $\cancel{\mathbb{P}(A}$ $\mu(A \cap A) = \mu(A) = \mu(A)^2 \Rightarrow \mu(A) \in \{0,1\}$

(b) If $A_n \subseteq A_{n+1}$: Let $C_1 = A_1$, $C_{n+1} = A_{n+1} \setminus \bigcup_{k=1}^{n} A_k$ $(n \geq 1)$

$\therefore$ $A_n = \bigcup_{k=1}^{n} C_k$, $(C_k)_{k\geq 1}$ pairwise disjoint.

$\therefore$ $\mu(A_n) = \sum_{k=1}^{n} \mu(C_k) \to \sum_{k\geq 1} \mu(C_k) = \mu(\bigcup_{k\geq 1} C_k) = \mu(\bigcup_{k\geq 1} A_k)$

$\therefore$ $\mu(\bigcup_{k\geq 1} A_k) = \lim_{n\to\infty} \mu(A_n)$

If $B_{n+1} \subset B_n$: $B_n^c \subset B_{n+1}^c$

$\therefore$ $\underbrace{\mu(\bigcup_{k\geq 1} B_k^c)}_{} = \lim_{k\to\infty} \mu(B_k^c) = \lim_{k\to\infty} \mu(E) - \mu(B_k)$

$= \mu(E) - \mu(\bigcap_{k\geq 1} B_k)$

$\mu(E) < \infty$: $\mu(\bigcap_{k\geq 1} B_k) = \lim_{k\to\infty} \mu(B_k)$

(c) Let F_n be distribution of $S_n/\sqrt{n}$.

$(X_n)_{n\geq 1}$ iid, $\mathbb{E}[X_n] = 0$, $\mathrm{Var}(X_n) < \infty \Rightarrow$ $\frac{X_1 + \dots + X_n}{\sqrt{n}}$ converges in distribution

Central limit theorem : $F_n \to$ distribution of $N(0,\sigma)$ (F)

$\mathbb{P}(A_n) \geq \underbrace{1 - F_n(k)}_{\mathbb{P}(S_n/\sqrt{n} > k)}$

Since ~~F has~~ F is cont. : $F_n(k) \to F(k)$

& Since $F(k) > 0$: Pick $0 < c < F(k)$

$\therefore$ $\lim_{n\to\infty} \mathbb{P}(A_n) > c \Rightarrow$ For n suff. large, $\mathbb{P}(A_n) \geq c$.

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(iv) $\mathbb{P}(A_n \text{ occurs i.o.}) = \mathbb{P}\left(\bigcap_{n\geq 1} \bigcup_{m\geq n} A_m\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m\geq n} A_m\right)$

$$\geq \liminf_n \mathbb{P}(A_n) \geq c > 0.$$

Claim: ~~An~~ A_n occurs. i.o. is a tail event.

Fix $N \in \mathbb{N}$: Claim A_n occurs i.o is $\sigma(X_{N+k} : k \in \mathbb{N})$ measurable.

$$\limsup_n \underbrace{\frac{X_1 + \dots + X_N}{\sqrt{n}}}_{\to 0} + \frac{X_{N+1} + \dots + X_n}{\sqrt{n}} \geq k \quad \text{iff} \quad \limsup_n \frac{X_{N+1} + \dots + X_n}{\sqrt{n}} \geq k.$$

$\therefore$ Done.

$\therefore \mathbb{P}(A_n \text{ occurs i.o.}) = 0$ or 1 (Kolgomorov 0-1) $\Rightarrow \mathbb{P}(A_n \text{ occurs i.o.}) = 1.$

(v) $\therefore \mathbb{P}\left(S_n/\sqrt{n} \geq k \text{ i.o.} \right) = \mathbb{P}\left(\limsup_n S_n/\sqrt{n} \geq k \right) = 1$

$\therefore \mathbb{P}\left(\limsup_n S_n/\sqrt{n} = \infty\right) = \mathbb{P}\left(\bigcap_{k\geq 1} \left\{ \limsup_n S_n/\sqrt{n} \geq k \right\}\right)$

$$= \lim_{m\to\infty} \mathbb{P}\left(\limsup_n S_n/\sqrt{n} \geq k\right) = 1$$

$\therefore$ Done.

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4) (a) ~~F~~ Let f_n be ≥ 0, measurable functions:

(Fatou): $\liminf_n \mu(f_n) \geq \mu(\liminf_n f_n)$

(Proof): $\mu(\inf_{m \geq n} f_m) \leq \mu(f_m)$ for $m \geq n$

$\therefore \mu(\inf_{m\geq n} f_m) \leq \inf_{m \geq n} \mu(f_m)$

Since $\inf_{m: m\geq n} f_m$ is ≥ 0, increasing function: Monotone convergence theorem $\Rightarrow$

$\lim_{n\to\infty} \mu(\inf_{m\geq n} f_m) = \mu(\lim_{n\to\infty} \inf_{m \geq n} f_m) \Rightarrow \liminf_n \mu(f_n) \geq \mu(\liminf_n f_n)$

(Lebesgue Dominated Convergence Theorem): If f_n measurable, $f_n \to f$ A.S., $\exists$ integrable g st $g \geq |f_n|$:

$\mu(f) = \lim_{n\to\infty} \mu(f_n)$

$g \pm f_n \geq 0$: $\liminf_n \mu(g \pm f_n) \geq \mu(\liminf_n g \pm f_n)$

LHS $= \mu(g) + \liminf_n \pm \mu(f_n)$; RHS $= \mu(g \pm f) = \mu(g) \pm \mu(f)$

$\therefore$ $\mu(g) - \limsup_n \mu(f_n) \geq \mu(g) - \mu(f)$] $\limsup_n \mu(f_n) \leq \mu(f) \leq \liminf_n \mu(f_n)$

$\mu(g) + \liminf_n \mu(f_n) \geq \mu(g) + \mu(f)$

$\therefore \mu(f_n) \to \mu(f)$

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Since max, min are cont. functions:

$f_n(x) \to f(x) \Rightarrow \text{Max}\{f_n(x), 0\} \to \text{Max}\{f(x), 0\}$

(similiarly for min)

$\therefore f_n^{\pm} \to f^{\pm}$

$\mu(|f|) = \mu(f^+ + f^-) = \mu(\liminf_n f_n^+) + \mu(\liminf_n f_n^-)$

$\leq \liminf_n \mu(f_n^+) + \liminf_n \mu(f_n^-)$

$\leq \liminf_n(\mu(f_n^+ + f_n^-)) = \lim_n \mu(f_n^+ + f_n^-) = \mu(|f|)$

If ~~$\limsup_n \mu(f_n^+) > \liminf_n \mu(f_n^-)$: $\exists\, \varepsilon > 0$, n_k subsequence~~

~~s.t. $\mu(f_{n_k}^+) >$~~

~~But $\limsup_n \mu(f_n^+) + \limsup_n \mu(f_n^-)$~~

$\therefore \liminf_n \mu(f_n^{\pm}) = \mu(f^{\pm})$

If $\limsup_n \mu(f_n^+) > \mu(f^+) = a$:

$\exists$ subsequence n_k s.t. $\mu(f_n^+) > a + \varepsilon$ (for some $\varepsilon > 0$)

$\therefore \lim_{k\to\infty} \mu(f_{n_k}^+ + f_{n_k}^-) = \limsup_{k\to\infty} \mu(f_{n_k}^+ + f_{n_k}^-) \geq a + \varepsilon + \liminf_k \mu(f_{n_k}^-)$

$= a\ \varepsilon + \mu(|f|)$

$\therefore$ Contradiction $\Rightarrow \mu(f_n^+) \to \mu(f^+)$!

Similiarly: $\mu(f_n^-) \to \mu(f^-)$!

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* Easier way: $\mu($ ~~$k f_n^+ + 4 f^+ - 2(f_n^+ - f_n^-$~~ $\liminf_n \mu(f_n^+) \geq \mu(f^+)$

~~$\liminf_n \mu(f_n^+ - f^+) \geq \mu(f_n)$~~ $0.$

$\liminf_n$

$$\underbrace{\liminf_n \mu(|f_n| - f_n^{\pm})}_{\mu(|f|) - \limsup_n \mu(f_n^{\pm})} \geq \mu(f^{\mp}) \quad \Bigg\} \quad \limsup_n \mu(f_n^{\pm}) \leq \mu(f^{\pm})$$

(c) $\liminf_n \mu(|f_n| + |f| - |f - f_n|) \geq 2\mu(|f|)$

$$LHS = 2\mu(|f|) - \limsup_n \mu(|f - f_n|) \geq 2\mu(|f|)$$

$\therefore \quad 0 \geq \limsup_n \mu(|f - f_n|) \geq \liminf_n \mu(|f - f_n|) \geq 0$

$\therefore \quad \lim_n \mu(|f - f_n|) = 0.$

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Probability & Measure 2012

1.) (a) Jensen Inequality (Probabilistic):

If X is an integrable R.V. with values in I:

$\mathbb{E}[c(X)] \geq c(\mathbb{E}[X])$

Proof: Let $x \in I^{\circ}$. If $y < x < z$, $y, z \in I$:

Let $x = \lambda y + (1-\lambda) z$. $\Rightarrow$ $f(x) \leq \lambda f(y) + (1-\lambda) f(z)$

$\therefore \dfrac{f(x) - f(y)}{(1-\lambda)} \leq \dfrac{f(z) - f(x)}{\lambda}$, $(x-y) \cdot \lambda = (1-\lambda)(z-y)$

$\therefore \dfrac{f(x) - f(y)}{x - y} \leq \dfrac{f(z) - f(x)}{z - x}$; Pick $a = \sup\limits_{y<x} \left\{ \dfrac{f(x) - f(y)}{x - y} \right\}$.

$\therefore \forall y < x, \quad \forall z > x$: $f(y) \geq f(x) - (x-y) \cdot a$

$f(z) \geq f(x) + a(z-x)$

$\therefore \exists a, b$ s.t. $f \geq a \cdot + b$, $f(x) = ax + b$.

If X is A.S. constant, equality holds. Else, $\mathbb{E}[X] \in I^{\circ}$.

Pick a, b as described wrt $\mathbb{E}[X]$

$\therefore \mathbb{E}[c(X)] \geq \mathbb{E}[aX + b] = a\mathbb{E}[X] + b = c(\mathbb{E}[X])$

Equality: $\mathbb{E}[c(X)] = c(\mathbb{E}[X])$ iff $\exists A \subseteq I$, A is convex and $c|_A$ is linear.

Proof: If $\mathbb{P}(X \in A) = 1$, $c|_A$ linear, $\mathbb{E}$ is linear $\Rightarrow$ equality holds

~~Else~~: If equality: $\mathbb{E}[\underbrace{c(X) - aX - b}_{\geq 0}] = 0$

$\therefore \mathbb{P}(c(X) = aX + b) = 1$ ~~$\Rightarrow \exists A \subseteq I$ s.t. $c|_A(x) = ax+b$~~ (linear),

$\mathbb{P}(\cancel{X \in A}) = 1$ *. (Back)

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(b) $\mathbb{E}\left[\log\left(f(X)/f_0(X)\right)\right] \leq \log\left(\mathbb{E}\left[f(X)/f_0(X)\right]\right)$ ($x \mapsto -\log(x)$ is convex on $(0,\infty)$)

$= \log\left(\int_{\mathbb{R}} \frac{f(x)}{f_0(x)} f_0(x)\,dx\right) = \log(1) = 0.$

$\therefore \mathbb{E}[\log(f(X))] \leq \mathbb{E}[\log(f_0(X))]$

$\therefore f_0$ is a maximiser.

Unique: If f is an optimiser, $\exists$ convex A, $\log(f(x)/f_0(x))$ is linear on A, $\mathbb{P}(\log(f(x)/f_0(x)) \in A) = 1$

But if $|A| \geq 2$: log is strictly convex $\Rightarrow$ Not linear.

$\therefore$ $\log(f) - \log(f_0) = 0$ A.S.

$\therefore f = f_0$ A.S.

$\therefore f_0$ is unique maximiser.

* Equality case proof:

$\mathbb{P}(c(X) - aX - b = 0) = 1$

If $c(X_i) = aX_i + b$, $X_1 < X_0$:

$X_2 = \lambda X_0 + (1-\lambda)X_1 \Rightarrow c(X_2) \leq \lambda c(X_0) + (1-\lambda)c(X_1)$

$= aX_2 + b$

$\therefore$ Equality.

$\therefore c(X) = aX + b$ on a convex set A, $\mathbb{P}(X \in A) = 1$.

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2.) (a) $\hat{f}(u) = \int_{\mathbb{R}} e^{iux} e^{-ax^2} dx$

Since integrand is diff., with integrable derivative : $\hat{f}$ is ~~integrable~~ differatiable.

$\hat{f}(u)' = i \int_{\mathbb{R}} e^{iux} \left(\frac{-2ax}{-2a}\right) e^{-ax^2} dx = \frac{i}{-2a} \left[\left[e^{iux} e^{-ax^2} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} \cancel{iu}\, e^{-ax^2} e^{iux} dx \right]$

$= -u/2a \; \hat{f}(u)'$

$\therefore \hat{f}(u) = A \cdot e^{-u^2/4a}$; $\hat{f}(0) = \int_{\mathbb{R}} e^{-x^2/2(1/2a)} dx = \frac{1}{\sqrt{2a}} \int_{\mathbb{R}} e^{-z^2/2} dz$

$= \sqrt{\frac{2\pi}{2a}} \cdot \Rightarrow \hat{f}(u) = \sqrt{\pi/a}\, e^{-u^2/4a}$

(b) $G_t(z) = \frac{1}{\sqrt{2\pi t^2}} e^{-z^2/2t^2}$: $\hat{G}_t(u) = \frac{1}{\sqrt{2\pi t^2}} \sqrt{\pi/(1/2t^2)}\, e^{-u^2/4(1/2t^2)}$

$= e^{-t^2/2 \cdot u^2}$

$\therefore \hat{\hat{G}}_t(x) = \sqrt{2\pi/t^2}\, e^{-u^2/2t^2} = (2\pi)\, G_t(-x)$

(c) Fourier Inversion: Let $f, \hat{f}$ be integrable

$f(-x) \cdot (2\pi) = \hat{\hat{f}}(x)$ (A.E.)

(Proof): $G_t * f(x) \to f(x)$ in L^1

$LHS = \int_{\mathbb{R}} G_t(z) f(x-z) dz = \cancel{\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{G}_t(u) e^{-iuz} du\, f(x-z)\, dz}$

$= \int_{\mathbb{R}} G(z/t) f(x-z) dz/t = \int_{\mathbb{R}} G(z) f(x-tz) dz$

$$= \int_{\mathbb{R}} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuz}\, \hat{G}(u)\, du\, f(x-tz)\, dz$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{G}(u)\, e^{-iuz} f(x-tz)\, dz\, du \quad \Bigg] \quad \begin{aligned} x - tz &= \omega \\ \therefore z &= \frac{x-\omega}{t} \end{aligned}$$

$$= 1/2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{G}(u)\, e^{-i(u/t)(x-\omega)} f(\omega)\, \frac{d\omega}{t}\, du$$

$$= 1/2\pi t \int_{\mathbb{R}} \hat{G}(u)\, e^{-iux/t}\, \hat{f}(u/t)\, du \qquad u \to v \cdot t$$

$$= 1/2\pi \int_{\mathbb{R}} \hat{G}(v \cdot t)\, e^{-iv\cdot x}\, \hat{f}(v)\, dv \quad - (1)$$

$$\hat{G}(z) = e^{-z^2/2} \Rightarrow |\hat{G}(vt)| = |e^{-v^2t^2/2}| \leq 1, \to 0 \text{ as } t \to \infty$$

$|\text{Integral}| \leq |\hat{f}|$ (Integrable)

$\therefore$ Dominated conv. theorem: $\lim_{t\to\infty} (1) = 1/2\pi \int_{\mathbb{R}} e^{-ivx}\, \hat{f}(v)\, dv$

$$= 1/2\pi\, \hat{\hat{f}}(-x)$$

$\therefore f(x) = (2\pi)^{-1}\, \hat{\hat{f}}(-x)$.

(d) $f_{\alpha,\lambda}(x) = \lambda^{\alpha} e^{-\lambda x} x^{\alpha-1} 1_{[0,\infty)}$

~~We can integrate by parts to get:~~ $\cancel{\int_0^\infty f_{\alpha,\lambda}(x)}$

$\cancel{|\hat{f}_{\alpha,\lambda}(u)| = |\int_0^\infty \lambda^{\alpha} e^{-(\lambda - iu)x} x^{\alpha-1} 1_{[0,\infty)}\, dx|}$

Since fourier transforms are bounded, continuous and:

$\lim_{x\to 0^+} f_{\alpha,\lambda}(x) = \infty$: $f_{\alpha,\lambda}$ cannot be A.E. equal to a fourier transform.

$\therefore$ If $\hat{f}_{\alpha,\lambda}$ integrable, $f_{\alpha,\lambda}(x) = (2\pi)^{-1}\, \hat{\hat{f}}_{\alpha,\lambda}(-x) \Rightarrow$ Contradiction

$\therefore \hat{f}_{\alpha,\lambda} \notin L^1$

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3.) (a) Let $(A_n)_{n\geq 1}$ be a sequence of events:

If $\sum_{k\geq 1} \mathbb{P}(A_n) < \infty$:

$$\mathbb{P}(A_n \text{ happens i.o.}) = \mathbb{P}\Big(\bigcap_{n\geq 1} \bigcup_{m\geq n} A_m\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\bigcup_{m\geq n} A_m\Big)$$

$$\leq \lim_{n\to\infty} \sum_{m\geq n} \mathbb{P}(A_m) = 0 \quad (\infty \text{ sum converges})$$

$\therefore \mathbb{P}(A_n \text{ i.o.}) = 0.$ (1^{st} borel cantelli)

Let $(A_n)_{n\geq 1}$ be sequence of independent events:

$$\sum_{n\geq 1} \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}(A_n \text{ i.o.}) = 1$$

$$\mathbb{P}(A_n \text{ occurs finitely}) = \mathbb{P}\Big(\bigcup_{n\geq 1} \bigcap_{m\geq n} A_m^c\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\bigcap_{m\geq n} A_m^c\Big)$$

$$= \lim_{n\to\infty} \prod_{m:m\geq n} (1-\mathbb{P}(A_m)) \leq \lim_{n\to\infty} \prod_{m:m\geq n} e^{-\mathbb{P}(A_m)} = \lim_{n\to\infty} e^{\underbrace{-\sum_{m\geq n} \mathbb{P}(A_m)}_{=0}}$$

$$= 0.$$

$\therefore \mathbb{P}(A_n \text{ occurs i.o.}) = 1$ (2^{nd} borel cantelli)

(b) $$\mathbb{E}[S_N^2] = \sum_{k=1}^{N} \mathbb{E}[\cancel{x}\, 1_{A_k}^2] + 2\sum_{1\leq k<l\leq N} \mathbb{E}[1_{A_k}\cdot 1_{A_l}]$$

$$\therefore \cancel{Var(S_N)}\ \mathbb{E}[S_n]^2 = \sum_{k=1}^{N} \mathbb{E}[1_{A_k}]^2 + 2\sum_{1\leq k<l\leq N} \mathbb{E}[1_{A_k}]\cdot\mathbb{E}[1_{A_l}]$$

$$\therefore Var(S_N) = \sum_{k=1}^{N} \left(\mathbb{E}[1_A^2] - \mathbb{E}[1_{A_k}]^2\right) \leq \sum_{k=1}^{N} \mathbb{E}[1_A] = \mathbb{E}[S_N]$$

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(d) Fix R: $\mathbb{P}(S_N < R)$ $T_N = N - S_N$.

$\therefore \operatorname{Var}(T_N) = \operatorname{Var}(S_N)$

$$\mathbb{P}(T_N \geq R) \leq \mathbb{E}[T_N^2]/R^2 = \operatorname{Var}(T_N) +$$

Let $\mu_N = \sum_{k=1}^{N} \mathbb{P}(A_k)$:

$$\mathbb{P}(|S_N - \mu_N| \geq \mu_N/2) \leq 4\operatorname{Var}(S_N)/\mu_N^2 \leq 4/\mu_N.$$

But $S_N < \mu_N/2 \Rightarrow |S_N - \mu_N| \geq \mu_N/2$

$$\therefore \mathbb{P}(S_N < \mu_N/2) \leq 4/\mu_N$$

Since $\lim_{N\to\infty} S_N = S_\infty \geq S_N$: $\mathbb{P}(S_\infty \geq \mu_N/2) \geq 1 - 4/\mu_N$

$$\therefore \mathbb{P}(S_\infty = \infty) = \lim_{N\to\infty} \mathbb{P}(S_\infty \geq \mu_N/2) \geq 1 \Rightarrow \text{Equality}$$

$$\therefore \lim_{N\to\infty} S_N = \infty \quad \text{A.S.}$$

$S_N = \infty$ iff A_n occurs i.o.

$$\therefore \mathbb{P}(A_n \text{ occurs i.o.}) = 1.$$

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4.) (a) - } 2011 paper.

(b) -

(c) Let $(E, \mathcal{E}, \mu) = ((0,1), \mathcal{B}((0,1)), \text{Lebesgue})$

$f_n = \cancel{1_{(0,1/n)} \cdot n^2}\ 1/x^2\ 1_{[1/n,1)}$

$\therefore f_n \to f$, $f(x) = 1/x$ pointwise.

$\| f_n \|_1 \leq n^2 < \infty$; But $\int_0^1 1/x^2\, dx = \infty \Rightarrow f \notin L^1$

(d) $f_n \to f$ in L^1

$\| f \|_1 \leq \| f_n - f \|_1 + \| f_n \|_1 \Rightarrow \| f \|_1 \leq 1$

$\| f \|_1 \geq \| f_n \|_1 - \| f_n - f \|_1 = 1 - \| f_n - f \|_1 \to 1$

$\therefore \| f \|_1 = 1$

Since $| f_n - f | \geq | f_n - f^+ |$, $f^+ = \text{Min}\{0, f\}$

$\| f_n - f \| \geq \| f_n - f^+ \| \Rightarrow f_n \to f^+$ in L^1

Since limits are unique in L^1: $f = f^+$ A. E.

$\therefore f \geq 0$ A. E.

$\therefore f \in \mathcal{D}$

(e) Using $(E, \mathcal{E}, \mu)$ in (c) :

$f_n = 1_{(0, 1/n)} \cdot n$; $\| f_n \|_1 = 1$, $f_n \to 0$ pointwise.

But $\| 0 \|_1 = 0 \Rightarrow f \notin \mathcal{D} \Rightarrow$ Not true.

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Probability & Measure 2013

1.) (a) Let A be a π system: Any $\mathcal{D}$ system containing A contains $\sigma(A)$.

(b) Let $\mathcal{D} = \{ B \in \mathcal{E} : \mu(B) = \nu(B) \}$: $\therefore A \subseteq \mathcal{D}$

Claim: $\mathcal{D}$ is a d-system.

$\mu(E) = \nu(E) = 1 \Rightarrow E \in \mathcal{D}$

~~If $B_1 \subseteq B_2$, $B \in \mathcal{D}$:~~

$\cancel{\mu(E \setminus B) = \mu(E) - \mu(B) = \nu(E) - \nu(B) = \nu(E \setminus B)}$

$\cancel{\therefore E \setminus B \in \mathcal{D}}$

If $B_1, B_2 \in \mathcal{D}$, $B_1 \subseteq B_2$: $\mu(B_2 \setminus B_1) = \mu(B_2) - \mu(B_1) = \nu(B_2) - \nu(B_1) = \nu(B_2 \setminus B_1)$

$*$ $\forall B_2 \subseteq E \Rightarrow \mu(B_2) = \nu(B_2) \leq 1 \Rightarrow \begin{cases} \nu(B_2) - \nu(B_1) = \nu(B_2 \setminus B_1) \\ \mu(B_2) - \mu(B_1) = \mu(B_2 \setminus B_1) \end{cases}$

If $B_n \in \mathcal{D}$, $B_n \subseteq B_{n+1}$: $\mu(\bigcup_{n \geq 1} B_n) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \nu(B_n)$

$= \nu(\bigcup_{n \geq 1} B_n)$

$\therefore \bigcup B_n \in \mathcal{D} \Rightarrow$ d-system.

Dynkin's lemma $\Rightarrow \sigma(A) = \mathcal{E} \subseteq \mathcal{D} \Rightarrow \mu \equiv \nu$.

(c) $(X_n)_{n \geq 1}$ be random variables:

$\cancel{\text{Independent: } \forall N \in \mathbb{N},\ |\Lambda| < \infty,\ \forall A_\alpha \in \mathcal{B}(\mathbb{R})\ (\alpha \in \Lambda):\ \mathbb{P}(\bigcap_{n=1} \bigcap_{\alpha \in \Lambda} \{X_\alpha \in A_\alpha\}) = \prod_{\alpha \in \Lambda} \mathbb{P}(\{X_\alpha \in A_\alpha\})}$

If $(\mathcal{F}_n)_{n \geq 1}$ are σ-algebras: independent iff $\forall A_n \in \mathcal{F}_n$,

$\mathbb{P}(\bigcap_{k=1}^{N} A_k) = \prod_{k=1}^{N} \mathbb{P}(A_k)$ $(N \in \mathbb{N})$

$(X_n)_{n \geq 1}$ independent iff $(\mathcal{F}_n)_{n \geq 1}$ independent, $\mathcal{F}_n = \sigma(X_n)$

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(d) $\mathbb{P}(Z_1 \le z) = \mathbb{E}[\mathbb{1}_{Z_1 \le z}]$; ~~WLO~~ By taking mod 1, we can WLOG $Y \in [0,1)$

$\therefore \mathbb{E}[\mathbb{1}_{Z_1 \le z}] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{X_1+y \le z} \mid Y=y]]$ - (1)

If $y \in [0, z]$: $X_1 \in [0, z-y] \cup [1-y, 1]$
$y \in (z, 1)$: $X_1 \in [1-y, 1-y+z]$ Interval length = z.

$\therefore (1) = \mathbb{E}[~~\mathbb{E}~~\,z] = z$.

$\therefore Z_1$ has distribution of $U[0,1]$ R.V.

(e) Claim: $\mathbb{P}(Z_1 \le z_1, \dots Z_n \le z_n) = \prod_{k=1}^{n} \mathbb{P}(Z_k \le z_k)$

LHS $= \mathbb{E}[\mathbb{E}[\mathbb{1}_{Z_1 \le z_1}, \dots \mathbb{1}_{Z_n \le z_n} \mid y = Y]]$

Since $(X_n)_{n \ge 1}$ independent: Eval $Z_k \le z_k \mid Y=y$ is fully determined by X_k (as described above.

$\therefore \mathbb{E}[\mathbb{1}_{Z_1 \le z_1}, \dots \mathbb{1}_{Z_n \le z_n} \mid Y=y] = \prod_{k=1}^{N} \mathbb{E}[\mathbb{1}_{Z_k \le z_k} \mid Y=y]$

$\therefore$ LHS $= \mathbb{E}\left[\prod_{k=1}^{n} z_k\right] = \prod_{k=1}^{n} \mathbb{E}[Z_k \le z_k]$

Since $\{X_n^{-1}(-\infty, a]_n : a \in \mathbb{R}\}_n \cup \{\emptyset\}$ is a π-generator of $\sigma(X_n)$,

and we have shown $\mathbb{P}(Z_1 \le z, \dots Z_n \le z_n) = \prod_{k=1}^{n} \mathbb{P}(Z_k \le z_k)$,

Measure on $\sigma(\overset{Z_1}{~~X_1~~})$: ~~$\mathbb{P}(Z_1 \in$~~ $\mu(A) = \mathbb{P}(Z_1 \in A;\ Z_2 \le z_2, \dots Z_n \le z_n)$
$= \mathbb{P}(Z_1 \in A) \cdot \prod_{k=2}^{n} \mathbb{P}(Z_k \le z_2) = \nu(A)$

$\mu = \nu$ on π-generator $\Rightarrow \mu = \nu$ on $\sigma(Z_1)$

Continue inductively: $(Z_n)_{n \ge 1}$ is independent.

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2.) (a) $\liminf_n f_n \geq 0$: $\forall n \in \mathbb{N}$, $m \geq n \Rightarrow f_m(x) \geq 0$

$\therefore \inf_{m \geq n} f_m(x) \geq 0 \Rightarrow \liminf_n f_n(x) \geq 0.$

$\liminf_n f_n^{-1}(-\infty, a] = \bigcap_{n \geq 1} \bigcup_{m \geq n} f_m^{-1}(-\infty, a]$

f_m measurable $\Rightarrow \bigcup_{m \geq n} f_m^{-1}(-\infty, a]$ measurable $\Rightarrow \liminf_n f_n^{-1}(-\infty, a]$ ~~is~~ $\in \mathcal{E}$.

Since $\sigma(\{\emptyset, (-\infty, a] : a \in \mathbb{R}\}) = \mathcal{B}(\mathbb{R})$, $\liminf_n f_n$ measurable.

(b) Let $f_n \geq 0$ be measurable, $f_n \leq f_{n+1}$:

$\lim_{n \to \infty} \mu(f_n) = \mu(\lim_{n \to \infty} f_n)$ (MCT)

(c) Fatou's Lemma: Let $f_n \geq 0$: be measurable, ~~integrable~~.

$\therefore \liminf_n \mu(f_n) \geq \mu(\liminf_n f_n)$

(Proof): $\forall m \geq n$, $\mu(f_m) \geq \mu(\inf_{m \geq n} f_m) \Rightarrow \inf_{m \geq n} \mu(f_m) \geq \mu(\inf_{m \geq n} f_m)$

$\inf_{m \geq n} f_m$ is increasing, ≥ 0, measurable : MCT $\Rightarrow \lim_{n \to \infty} \mu(\inf_{m \geq n} f_m) = \mu(\liminf_n f_n)$

$\therefore \liminf_n \mu(f_n) \geq \lim_{n \to \infty} \mu(\inf_{m \geq n} f_m) = \mu(\liminf_n f_n)$

(d) $f_n(x) \to f(x) \Rightarrow$ (Min is cont. function) $\text{Min}\{f_n(x), f(x)\} \to \text{Min}\{f(x)\} = f(x)$

(Fatou)

$\therefore \text{Min}\{f_n, f\} \leq f \Rightarrow \liminf_n \mu(f - f_n) \geq \mu(\liminf_n f - f_n) = 0.$

$\therefore \mu(f) - \limsup_n \mu(f_n) \geq 0 \Rightarrow \mu(f) \geq \limsup_n \mu(f_n)$

~~$\liminf_n \mu(f_n)$~~ $f_n \geq 0$: $\liminf_n \mu(f_n) \geq \mu(\liminf_n f_n) = \mu(f)$

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$\therefore \mu(f) = \liminf_n \mu(f_n) = \limsup_n \mu(f_n) = \lim_n \mu(f_n)$

$\therefore \mu(f_n) \to \mu(f)$

Consider:

(d) $\mu(f_n + f - |f_n - f|)$: $|f_n - f| \leq |f_n| + |f| = f_n + f.$

$\therefore f_n + f - |f_n - f| \geq 0.$

(Fatou): $\liminf_n \mu(f_n + f - |f_n - f|) \geq \mu(\liminf_n f_n + f - |f_n - f|)$

LHS $= 2\mu(f) - \limsup_n \mu(|f_n - f|)$, RHS $= 2\mu(f)$.

$\therefore 0 \leq \limsup_n \mu(|f_n - f|) \leq 0 \Rightarrow$ Equality.

$\therefore \mu(|f_n - f|) \to 0. \Rightarrow f_n \to f$ in L^1

* OR, $|f_n - f| = f - \text{Min}\{f_n, f\} + f_n - \text{Min}\{f_n, f\}$

$\therefore \mu(|f_n - f|) = \mu(f) + \mu(f_n) - 2\mu(\text{Min}\{f_n, f\})$

RHS $\to \mu(f) + \mu(f) - 2\mu(f) = 0.$

$\therefore \mu(|f_n - f|) \to 0.$

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3.) (a) $\varphi : \mathbb{R} \to \mathbb{C}, \quad \varphi(u) = \mathbb{E}[\underbrace{e^{iuX}}_{f(u,x)}]$

Theorem: ~~If $\forall x$, $u \mapsto e^{iux}$ is differentiable, and derivative $x \mapsto ix e^{iux}$~~ ~~$\forall u \in \mathbb{R}$, $x \mapsto e^{iux}$ is integrable,~~

Let $f : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ satisfy:

$\forall u \in \mathbb{R}, \quad f(u, \cdot)$ is integrable

$\forall x \in \mathcal{X}, \quad f(\cdot, x)$ is differentiable

$\forall u \in \mathbb{R}, \quad \frac{\partial f}{\partial u}(u, \cdot)$ is integrable

$\int f(u,x)\, \mu(dx)$ is diff. wrt u, derivative $= \int \frac{\partial f}{\partial u}(u,x)\, \mu(dx)$

$|e^{iuX}| = 1 \Rightarrow x \to e^{iux}$ is integrable

$|iuxe^{iux}| = |x|$ is integrable

$u \to e^{iux}$ is smooth

$\Rightarrow \varphi(u)$ is differentiable, $\varphi'(u) = \mathbb{E}[iXe^{iuX}]$

$\varphi'(0) = \mathbb{E}[iX] = 0$

(b) $\mathbb{E}[e^{iu(S_n/n)}] = \prod_{k=1}^{n} \mathbb{E}[e^{iu/n \cdot X_k}]$ (independence)

Let $\Phi(u) = \mathbb{E}[e^{iuX_1}]$ $\therefore \mathbb{E}[e^{iuS_n/n}] = \Phi(u/n)^n$

~~lim~~ Consider $n \log(\Phi(u/n))$. $\lim_{n\to\infty} \Phi(u/n) = \Phi(0) = 1$

$\therefore \lim_{n\to\infty} \log(\Phi(u/n)) = 0$

Since X is integrable, $\Phi(u)$ is differentiable on $\mathbb{R}$

$\therefore \lim_{z\to\infty} \frac{\Phi(u/z)}{1/z} = \lim_{z\to 0} \frac{\Phi(uz)}{z} = \Phi'(uz)\cdot u \Big|_{z=0}$

$= 0$

$\therefore$ By continuity of $x \to e^x$:

$$\lim_{n\to\infty} \cancel{\phi_n}\ \mathbb{E}\left[e^{iu S_n/n}\right] = e^0 = 1 = \mathbb{E}\left[e^{iu\cdot 0}\right]$$

Since convergence of characteristic function is equivalent to convergence of distributions: $S_n/n \to 0$ in distribution

Let $\varepsilon > 0$: $\quad \mathbb{P}\left(\left|S_n/n\right| \geq \varepsilon\right) = \mathbb{P}\left(S_n/n \geq \varepsilon\right) - \mathbb{P}\left(S_n/n \leq -\varepsilon\right)$

$$= 1 - \underbrace{\mathbb{P}\left(S_n/n < \varepsilon\right)}_{\cancel{\geq 1 - \mathbb{P}(S_n/n \leq \varepsilon/2) \to 0}} - \underbrace{\mathbb{P}\left(S_n/n \leq -\varepsilon\right)}_{\to 0.}$$

$$\leq 1 - \underbrace{\mathbb{P}\left(S_n/n \leq \varepsilon/2\right)}_{\to 1} - \underbrace{\mathbb{P}\left(S_n/n \leq -\varepsilon\right)}_{\to 0}$$

$$\to 0.$$

$\therefore$ $S_n/n \to 0$ in $\mathbb{P}$ $_\#$

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4.) (a) Birkhoff's Theorem: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T : \Omega \to \Omega$ be measure preserving

$f : \Omega \to \mathbb{R}$ be integrable.

$S_n(f) = \sum_{k=0}^{n-1} f \cdot T^{(k)}$: $S_n(f)/n \to \bar{f}$ A.S.

for some T-invariant $\bar{f}$.

(b) $\cancel{\mathbb{P}(2^{-n_1} + 2^{-n_2} + 2^{-n_r})}$

$Y_2 \cdot \frac{1}{2} + \frac{1}{2} X_1 = Y_1$

$\mathbb{P}(Y_2 \leq \frac{1}{8}, Y_1 \geq \frac{7}{8}) = 0$ as $Y_1 \leq \frac{1}{2} Y_2 + \frac{1}{2} = \frac{9}{16} < \frac{7}{8}$.

But $\mathbb{P}(Y_2 \leq \frac{1}{8}) > 0$, $\mathbb{P}(Y_1 \geq \frac{7}{8}) = \frac{1}{8} > 0$.

$\therefore$ Not independent

Claim: $Y_1 \sim U(0,1)$

We note that : $\{ [0, N/2^n) : 0 \leq N \leq 2^n, n \in \mathbb{N} \} = \Lambda$.

$\Lambda \cup \{ \phi \}$ is a π generator of $\mathcal{B}([0,1))$

$\cancel{\mathbb{P}(2^{n} Y_2 \in 2[N/2^n, N+1/2^n}$

$(X_1, X_2, \ldots)$ is the binary representation of $Y_1 \in [0,1]$

$\therefore \mathbb{P}(Y_1 \in [k/2^n, k+1/2^n)) = 1/2^n$ for $k+1 \leq 2^n$

$\therefore \mathbb{P}(Y_1 \in [0, N/2^n)) = N/2^n$

Since law of Y_1 agrees with law of $U[0,1)$ on a π-generator, $Y_1 \sim U[0,1)$

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(d) Let $T: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the left shift operator.

$$\therefore \mu(T^{-1}\{(X_1, \dots X_n) \in A\}) = \mu(\{(X_2, \dots X_{n+1}) \in A\}$$

$$= \mu((X_1, \dots X_n) \in A) \quad \text{as} \quad X_n \quad \text{iid}.$$

$\therefore$ T measure preserving

$f: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, $\quad f(\underline{X}) = \sum_{k=1}^{\infty} X_k/2^k$

$\therefore Y_n = f \circ T^{n-1}$

Claim: T is ergodic.

If $\cancel{\mu}(T^{-1}(A) = A$: $A \in \sigma(X_n : n \geq 1) \Rightarrow T^{-1}(A) \in \sigma(X_n : n \geq 2)$

$T^{-k}(A) = A \Rightarrow A \in \bigcap_{n \geq 1} \sigma(X_k : k \geq n)$ (Tail algebra)

Koglomorov 0-1 $\Rightarrow$ $\mathbb{P}(A) = 0$ or 1. $\therefore$ Done.

$\therefore S_n(f)/n \to \bar{f}$; T ergodic $\Rightarrow$ $\bar{f}$ is A.S. constant

Von-neumann: $S_n(f)/n \to \bar{f}$ in L_1

$S_n(f) \geq 0$: $\| S_n(f)/n \|_1 = \mathbb{E}[Y_1] = 1/2$

$$\therefore \|\bar{f}\|_1 = 1 \Rightarrow \bar{f} = 1/2 \quad (A.S.)$$

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Probability and Measure 2014

4.1 (a) Let $(X_n)_{n\geq 1}$ be iid R.V., $\mathbb{E}[X_1] = \mu < \infty$

~~$S_n/n \to \mu$ A.S.~~

$X_n \wedge R$

(ii) $\mathbb{E}[\,|X_n^{(R)}|\,] \leq R \Rightarrow$ Finite mean

$(X_n)_{n\geq 1}$ iid $\Rightarrow (X_n \wedge R)_{n\geq 1}$ iid.

$\therefore S_n^R/n \to \mathbb{E}[X_1 \wedge R]$ A.S.

$\mathbb{E}[X_1 \wedge R] \nearrow \mathbb{E}[X_1] = \infty$ (Monotone conv. theorem)

$\therefore \liminf_n S_n/n \geq \liminf_n S_n^R/n = \mathbb{E}[X_1 \wedge R]$ (For all $R>0$)

$\therefore \liminf_n S_n/n \geq \sup_{R>0} \mathbb{E}[X_1 \wedge R] = \infty$ (A.S.)

$\therefore S_n/n \to \infty$ A.S.

(iii) $\sum_{n\geq 0} \mathbb{P}(X_n \geq n) = \sum_{n\geq 0} \mathbb{P}(X_1 \geq n)$ ($(X_n)_{n\geq1}$ iid)

(Fubini, ≥ 0)

$= \sum_{n\geq 0}\sum_{k\geq 0} \mathbb{P}(X_1 \in [k,k+1))\, 1_{k\geq n} = \sum_{k\geq 0} \mathbb{P}(X_1 \in [k,k+1)) \cdot \sum_{n\geq 0} 1_{k\geq n}$

$= \sum_{k\geq 0} (k+1)\, \mathbb{P}(X_1 \in [k,k+1)) \geq \sum_{k\geq 0} \mathbb{E}[X_1 \cdot 1_{X_1 \in [k,k+1)}]$

$=$ (Fubini, ≥ 0) $\mathbb{E}\left[\sum_{k\geq 0} X_1 \cdot 1_{X_1 \in [k,k+1)}\right] = \mathbb{E}[X_1] = \infty$

Since X_n independent: $A_n = \{X_n \geq n\}$ are independent.

If $\mathbb{E}[X_n] = \infty$: $\forall R > 0$, $\mathbb{E}[X_n/R] = \infty$

$\therefore$ By repeating the argument: $\sum_{k\geq 0} \mathbb{P}(X_n/R \geq n) = \infty$

$\therefore$ Borel Cantelli II: $\{X_n/R \geq n\}$ happens i.o. A.S.

$\therefore \{X_n \geq R\cdot n\}$ occurs i.o. (A.S.)

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(iii) Let $B_n = \{ \omega : |S_n(\omega)|/n \geq R \}$, $R > 0$.

~~Suppose B_n occurs finitely: $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow$~~

If $|S_n(\omega)|/n < R$ for $n \geq N$:

$|X_n|/n \geq 2R$ i.o.

$\therefore \exists\ n' > N$ s.t. $|X_{n'}|/n' \geq 2R \Rightarrow \dfrac{|S_1 + \dots + S_{n'-1}|}{n'} \geq R$

$\therefore |S_1 + \dots + S_{n'-1}|/(n'-1) \geq R \Rightarrow$ Contradiction as $n' \geq N$

$\therefore |S_n(\omega)|/n \geq R$ finitely $\Rightarrow |X_n(\omega)|/n \geq 2R$ finitely

$\Rightarrow$ Measure 0 subset.

$\therefore B_n$ occurs i.o. A.S.

$\Rightarrow \limsup_n |S_n(\omega)|/n \geq R$

R arbitrary $\Rightarrow \limsup_n |S_n(\omega)|/n = \infty$.

(Glitch Paper)

3) (a) Θ is measure preserving if: Θ is measurable; $\forall A \in \mathcal{E}$,

$$\mu(A) = \mu(\Theta^{-1}(A))$$

Θ is ergodic if: Θ is measure preserving and $\forall A \in \mathcal{E}$,

$$\Theta^{-1}(A) = A \Rightarrow \mu(A) = 0 \text{ or } \mu(A^c) = 0$$

(b) Birkhoff Theorem: $f : E \to \mathbb{R}$ be integrable.

$$S_n(f) = \sum_{k=0}^{n-1} f \cdot T^{(k)}, \quad n \geq 1 \quad ; \quad S_0 = 0.$$

$$S_n(f)/n \to \bar{f}, \quad \& \ (A.S) \quad ; \quad \bar{f} \text{ is } T\text{- invariant.}$$

(ii) ~~A~~ $T^{-1}\{(x, y)\} = \{(x-a, y-b)\}$ (mod 1)

as $z \to z + a$ is bijective ($[0,1) \to [0,1)$)

$\therefore$ ~~A~~ $\forall A \in \mathcal{B}[0,1)$, $-a + A$ (mod 1) has the same measure as A as Lebesgue measure is translation invariant

$$\therefore \mu(T^{-1}(A_1 \times A_2)) = \mu(A_1 \times A_2)$$

But $G \subset \{A_1 \times A_2 : A_i \in \mathcal{B}[0,1)\}$) is a π - generator of ~~π~~ ~~σ-f~~ $\mathcal{B}[0,1)^2$.

μ, $\mu(T^{-1}(.))$ agrees on π-generators, $\mu([0,1)^2) = \mu(T^{-1}[0,1)^2) = 1 \Rightarrow$

$\mu = \mu \cdot T^{-1}$

Since $(x, y) \to ((x+a) 1_{x<a} + (x+a-1) 1_{x\geq a}, (y+a) 1_{y<a} + (y+a-1) 1_{y\geq a})$

is the sum of ~~measurable fn~~ product of cont., measurable functions

$\therefore T$ is measurable.

$\therefore T$ is measure preserving.

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Consider fourier transform: $f = 1_A$, A is T invariant

$$\hat{f}_{\mu}(n_1, n_2) = \int_0^1 \int_0^1 e^{2\pi i (n_1 x_1 + n_2 x_2)} f(x_1, x_2) \, \mu(d(x_1, x_2))$$

$$= \int_0^1 \int_0^1 e^{2\pi i (\underline{n} \cdot \underline{x})} f(\underline{x}) \, \mu(T^{-1}) \cdot (d(x_1, x_2)) \quad (T \text{ is } \mu\text{-preserving})$$

$$= \int_0^1 \int_0^1 e^{2\pi i (\underline{n} \cdot T\underline{x})} \cdot \underbrace{f \cdot T(x)}_{f(x)} \, \mu(d\underline{x}) = e^{2\pi i (n_1 \cdot a + n_2 \cdot a)} \hat{f}_{\mu}(n_1, n_2)$$

$\therefore e^{2\pi i (n_1 + n_2) \cdot a} = 1$; $\Rightarrow (n_1 + n_2) \cdot a \in \mathbb{Z}$

Scanned: If $n_1 + n_2 \neq 0$, $a \in G$ (reject) or $\hat{f}(n_1, n_2) = 0$.

$\therefore$

$$\therefore f(x_1, x_2) = \sum_{k \in \mathbb{Z}} \hat{f}(k, -k) \, e^{-2\pi i (k x_1 - k x_2)}$$

Not ergodic:

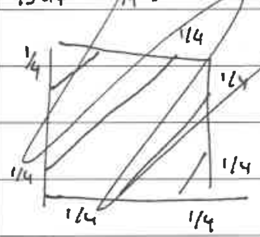
$$A = \{ (x, y) : x \in \mathbb{R}, |y - x| \leq 1/4 \} \pmod 1$$

$T^{-1}((x, y)) = (x - \alpha, y - \alpha)$; $|x - \alpha - (y - \alpha)| = |x - y| \leq 1/4$.

$\therefore T^{-1}(A) \subseteq A$

$T^{-1}(\underbrace{(x+\alpha, y+\alpha)}_{|x+\alpha - (y+\alpha)| \leq 1/4}) = (x, y) \Rightarrow T^{-1}(A) = A$

But A: ~~$\mu(A) = 2 \times \frac{1}{4}$~~



If we integrate A and move part around,

$$\mu(A) = \mu(\{ (x, y) : y \in [x - 1/4, x + 1/4], x \in [0, 1] \})$$

$= 1/2 \neq 0, 1 \Rightarrow$ Not ergodic.

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$$\bar{f}_{(x_1,x_2)} = \int_0^1 f(x_1+t,\ x_2+t) \pmod 1 \ dt$$

2.) (a) Let $(E, \mathcal{E}, \mu)$ be a measure space, $f_n, f : E \to [0,\infty]$ is measurable, ≥ 0

If $\forall x \in E$: $(f_n(x))_{n\geq 1}$ is increasing, $f_n(x) \to f(x)$;
f_n integrable : $\mu(f_n) \to \mu(f)$

Let $g \geq 0$ be a simple function: $g(x) = \sum_{r=1}^{N} a_r \cdot 1_{A_r}$ $(a_r > 0)$

$$V(B) = \mu(1_B \cdot g)$$

Claim: V is a measure

$V(\emptyset) = \mu(0 \cdot g) = 0$

$\forall B \in \mathcal{E}$, $\mu(1_B \cdot g) \geq \mu(0) = 0$.

If $B_n \in \mathcal{E}$, B_n disjoint: $V(\bigcup_{n\geq 1} B_n) = \mu(\sum_{r=1}^{N} a_r 1_{A_r} \cdot 1_{\bigcup_{n\geq 1} B_n})$

$=$ (Integral linear wrt simple functions) $\sum_{r=1}^{N} a_r \mu(1_{A_r} 1_{\bigcup B_n})$

$= \sum_{r=1}^{N} a_r \mu(\bigcup_{n\geq 1} B_n \cap A_r) = \sum_{r=1}^{N} a_r \sum_{n\geq 1} \mu(B_n \cap A_r) = \sum_{n\geq 1} \mu(\sum_{r=1}^{N} a_r 1_{B_n \cap A_r})$

$= \sum_{n\geq 1} \mu(g \cdot 1_{B_n}) = \sum_{n\geq 1} V(B_n)$

Countable additive $\Rightarrow$ V is a measure.

Let g be a simple function, $0 \leq g \leq f$. Fix $\varepsilon > 0$.

$$A_n = \{x \in E : \forall m \geq n,\ f_m(x) \geq (1-\varepsilon)\, g(x)\}$$

Since $(f_n)_{n\geq 1}$ is increasing in n, $A_n \subseteq A_{n+1}$

$\forall x \in E$, $f_n(x) \to f(x) \geq g(x)$

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∴ $f_n(x) \geq (1-\varepsilon)\, g(x)$ for n large enough

∴ $\bigcup_{n\geq 1} A_n = E$

∴ $V(E) = \lim_{n\to\infty} V(A_n)$ (V is measure wrt g)

$= \lim_{n\to\infty} \mu(\cancel{1_g}\, g \cdot 1_{A_n}) \leq \lim_{n\to\infty} \mu(\cancel{1_{A_n}}\, \cancel{f_n}\, 1_{A_n} \frac{f_n(x)}{(1-\varepsilon)})$

∴ $(1-\varepsilon)\cdot V(E) \leq \liminf_{n\to\infty} \mu(1_{A_n} \cdot f_n) \leq \liminf_{n\to\infty} \mu(f_n)$

~~But~~ ∴ ~~But~~ $(1-\varepsilon) \cdot \sup_{\substack{0 \leq g \leq f, \\ g \text{ simple}}} V_g(E) \leq \liminf_{n\to\infty} \mu(f_n)$

ε arbitrary $\Rightarrow \mu(f) \leq \liminf_{n\to\infty} \mu(f_n)$

$f_n \leq f \Rightarrow \overset{\sup}{\liminf_{n\to\infty}} \mu(f_n) \leq \mu(f)$

$\mu(f) = \lim_{n\to\infty} \mu(f_n)$

(b) $\mu(A) = \mu_1\left(x \mapsto \mu_2(y \to 1_A(x,y))\right)$ (product measure)

$\mathcal{E}_1 \otimes \mathcal{E}_2 = \sigma(\{A_1 \times A_2 : A_i \in \mathcal{E}_i\})$

μ is countable additive:

Let $(A_n)_{n\geq 1}$ be disjoint measurable

Fix $x \in E_1$: $\cdot \to 1_{\bigcup_{k=1}^{n} A_k}(x,\cdot)$ is ≥ 0, measurable increasing.

∴ $\lim_{n\to\infty}$ $x \mapsto \mu_2(y \to 1_{A \bigcup_{k=1}^{n} A_k}(x,y))$ is ≥ 0, measurable, increasing

$x \mapsto \mu_2(y \to 1_{\bigcup_{k\geq 1} A_k}^{(x,y)}) = x \mapsto \lim_{n\to\infty} \mu_2(y \to 1_{\bigcup_{k=1}^{n} A_k}^{(x,y)})$

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Let $g_x(y) = \mathbb{1}_{\bigcup_{k=1}^{\infty} A_k}(x,y)$:

$$g_x^{(n)}(y) = \mathbb{1}_{\bigcup_{k=1}^{n} A_k}(x,y) \nearrow g_x(y)$$

$$\therefore \mu_2(g_x^{(n)}) \nearrow \mu_2(g_x) \quad \text{(Monotone convergence)}$$

Bu $\therefore$ $x \mapsto \mu_2(g_x^{(n)})$ is increasing in n, ≥ 0.

$$\therefore \mu_1(x \mapsto \mu_2(g_x^{(n)})) \nearrow \mu_1(\lim_{n\to\infty} x \mapsto \mu_2(g_x^{(n)}))$$

$$= \mu_1(x \mapsto \mu_2(g_x))$$

$$= \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$\text{LHS} = \mu_1(x \mapsto \mu_2(y \to \mathbb{1}_{\bigcup_{k=1}^{n} A_k}(x,y)) = \sum_{k=1}^{n} \mu_1(x \mapsto \mu_2(y \to \mathbb{1}_{A_k}(x,y))$$

$$= \sum_{k=1}^{n} \mu(A_k)$$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k) = \sum_{k\geq 1} \mu(A_k) = \mu\left(\bigcup_{k\geq 1} A_k\right).$$

$\therefore$ Countable additive.

1.) (a) Let $\mathcal{I}$ be set of all open sets of $\mathbb{R}$:

$$\mathcal{B}(\mathbb{R}) = \bigcap \left\{ \mathcal{A} \in P(P(\mathbb{R})) : \begin{array}{l} \mathcal{A} \text{ is a } \sigma\text{-algebra}, \\ \mathcal{I} \subseteq \mathcal{A} \end{array} \right\}$$

Let $\Lambda = \{\phi\} \cup \left\{ \bigcup_{k=1}^{n} (a_k, b_k] : n \in \mathbb{N}, \ a_k, b_k \in \mathbb{R}, \ a_k < b_k < a_{k+1} \right\}$

Λ is a ring; $\mu\left(\bigcup_{k=1}^{n} (a_k, b_k]\right) = \sum_{k=1}^{n} b_k - a_k$

L - measure defined from extending μ to outer - measure via Cartheodory Extension Theorem.

(b) Let $B \in \mathcal{B}(\mathbb{R})$, $B \subseteq [0,1]$: Fix $\varepsilon > 0$

$\exists$ $A_k \in \Lambda$ (disjoint) s.t $B \subseteq \bigcup_{k \geq 1} A_k$, $\mu(B) + \varepsilon/4 > \sum_{k \geq 1} \mu(A_k)$

Since Λ is a ring, we can WLOG, $A_k = (a_k, b_k]$

Pick $N \in \mathbb{N}$ s.t $\sum_{k > N} \mu(A_k) < \varepsilon/4$

Let $C = \bigcup_{k=1}^{N} A_k \cap [0,1]$, $C' = \bigcup_{k=1}^{N} A_k$

$\mu(B \triangle C) \leq \mu(B \setminus C) + \mu(C \setminus B) \leq \mu\left(\bigcup_{k=1}^{\infty} A_k \setminus B\right) + \mu(B \cap C^c \cap C'^c) + \mu(B \cap C' \cap C^c)$ $\mu \in C'$

1^{st} term: $\mu\left(\bigcup_{k \geq 1} A_k\right) = \mu\left(\bigcup_{k \geq 1} A_k \setminus B\right) + \mu(B) \Rightarrow \mu\left(\bigcup_{k=1} A_k \setminus B\right) < \varepsilon/4$

$\mu(B \setminus (C \cup C')^c) = \mu(B \setminus C'^c) \leq \mu\left(\bigcup_{k=1}^{\infty} A_k \setminus C'\right) < \varepsilon/4$

$\mu(B \cap C' \cap C^c) \leq \mu(C' \setminus C) \leq \mu\left(\bigcup_{k=1}\right.$

$$\bigcup_{k=1}^{\infty} A_k \setminus [0,1] \subseteq \bigcup_{k=1}^{\infty} A_k \setminus B$$

$\therefore \mu(LHS) \le \varepsilon/4$

$$\mu\Big(\bigcup_{k=1}^{N} A_k \cap [0,1] \;\Delta\; B\Big) \le \underbrace{\mu\Big(\bigcup_{k=1}^{\infty} A_k \setminus B\Big)}_{\le\ \varepsilon/4} + \mu\Big(B \setminus \bigcup_{k=1}^{N} A_k \cap [0,1]\Big)$$

$$\mu\Big(B \setminus \bigcup_{k=1}^{N} A_k \cap [0,1]\Big) \le \mu\Big([0,1] \cap \Big(\bigcup_{k\ge 1} A_k \setminus \bigcup_{k=1}^{n} A_k\Big)\Big)$$

$$\le \mu\Big(\bigcup_{k>N} A_k\Big) < \varepsilon/4$$

$$\therefore \mu\Big(\bigcup_{k=1}^{N} A_k \cap [0,1] \;\Delta\; B\Big) \le \varepsilon/2 < \varepsilon.$$

$A_k \cap [0,1]$ are (disjoint) intervals in $[0,1]$.

(c) Suppose otherwise: Pick $I_1, \dots I_n$ s.t. $\mu(B \,\Delta \bigcup_{k=1}^{n} I_k) \le \varepsilon$ ~~$=\emptyset$~~, I_k disjoint

$\therefore$ ~~$\varepsilon \ge$~~ If $\mu(B \cap I_k) = \mu(I_k)/2$:

$$\mu\Big(B \cap \bigcup_{k=1}^{n} I_k\Big) = \frac{1}{2}\mu\Big(\bigcup_{k=1}^{n} I_k\Big)$$

$$\mu\Big(\bigcup_{k=1}^{n} I_k \setminus B\Big) = \mu\Big(\bigcup_{k=1}^{n} I_k\Big) - \mu\Big(B \cap \bigcup_{k=1}^{n} I_k\Big) = \frac{1}{2}\mu\Big(\bigcup_{k=1}^{n} I_k\Big)$$

$$\mu\Big(B \setminus \bigcup_{k=1}^{n} I_k\Big) = \mu(B) - \mu\Big(\bigcup_{k=1}^{n} I_k \cap B\Big) = -\frac{1}{2}\mu\Big(\bigcup_{k=1}^{n} I_k\Big) + \mu(B)$$

$\therefore \mu(B) \le \varepsilon$ (ε arbitrary)

$\therefore \mu(B) = 0.$

$\mu(B \cap [0,1]) = 0 \ne \frac{1}{2}\mu([0,1])$

$\therefore$ Contradiction

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1.) (a) Let X be a set: A

π system: $A \in P(P(X))$ satisfies $\emptyset \in A$ and $\forall A_i \in A$, $A_1 \cap A_2 \in A$

d-system: $D \in P(P(\overset{X}{\cancel{A}}))$ satisfies $X \in D$ and $\forall A_i \in A$:

$A_1 \subseteq A_2 \Rightarrow A_2 \setminus A_1 \in A$

$A_n \subseteq A_{n+1} \Rightarrow \bigcup_n A_n \in A$.

(*)

(b) Let $(E, \mathcal{E}, \mu)$ be a measure space, $f_n, f: E \to \mathbb{R}$ be integrable. If $\exists$ integrable g s.t. $g \geq |f|, |f_n|$ $(\forall x \in E)$ and $f_n \to f$ A.E.: $\lim_{n\to\infty} \mu(f_n) = \mu(f)$. (Dominated convergence theorem)

(a) (*) σ-algebra: $F \in P(P(\cancel{}X))$ satisfies $\emptyset, X \in F$;

$\forall A_n \in F$: $A_n^c \in F$

$\bigcup_n A_n \in F$

(c) No. Let $E = \mathbb{R}$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$.

$\mu(\{n\}) = 0$ $(n \in \mathbb{Z})$, $\mu(\mathbb{Z}) = 1$ (by definition of μ)

But σ-additivity of measure: $1 = \mu(\mathbb{Z}) = \mu(\bigcup_{n\in\mathbb{Z}} n) = \sum_{n\in\mathbb{Z}} \mu(n) = 0$

$\therefore$ Contradiction $\Rightarrow$ Not a valid measure.

(d) f is cont. on a compact set $\Rightarrow$ Bounded.

$\therefore$ Let $\|f\|_\infty = M$

$\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, $|f(g(x)^n)| \leq M \Rightarrow$ Dominated by M

$\int_0^1 M = M < \infty \Rightarrow$ Integrable

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If $g(x) = 1$: $\lim_{n\to\infty} f(g(x)^n) = f(\lim_{n\to\infty} g(x)) = f(1)$

Else : $\lim_{n\to\infty} g(x)^n = 0 \Rightarrow \lim_{n\to\infty} f(g(x)^n) = f(\lim_{n\to\infty} g(x)^n) = f(0)$

Dominated converge theorem: $\lim_{n\to\infty} \int_0^1 f(g(x)^n)\, dx = \int_0^1 \lim_{n\to\infty} f(g(x)^n)\, dx$

$$= \int_0^1 1_{g(x)=1} \cdot f(1) \cdot 1_{g(x)=0} f(0)\, dx$$

$$= f(1) \cdot \underbrace{\mu(\{x : g(x) = 1\})}_{\in [0,1]} + f(0) \cdot \underbrace{\mu(\{x : g(x) = 0\})}_{\in [0,1]}$$

$\therefore$ Limit exists, $\in [f(0), f(1)]$

2) (a) $f \in L^p$ if $\int_E |f|^p \, d\mu < \infty$; $\|f\|_p = \left(\int_E |f|^p \, d\mu\right)^{1/p}$

(b) Let $p \in [1, \infty]$: $q = \infty$ if $p = 1$, $q = 1$ if $p = \infty$;
Else $q = p/_{p-1}$

If $f \in L^p$, $g \in L^q$: $\int_E |f| \cdot |g| \cdot d\mu \le \|f\|_p \cdot \|g\|_q$ (Hölder)

(c) $\left(\int_0^1 |f(y)|\,|x-y|^{-\alpha} \, dy\right)^2 \le \underbrace{\int_0^1 |f(y)|^2 \, dy}_{<\infty} \cdot \int_0^1 |x-y|^{-2\alpha} \, dy$ — (1)

$\int_0^1 |x-y|^{-2\alpha} \, dy = \int_{-x}^{-x+1} |y|^{-2\alpha} \, dy = \cancel{\text{fen } \alpha} \cdot \Phi(x)$

$\Phi(x) \le \int_{-R}^{R} |y|^{-2\alpha} \, dy$ for $R > |x| + 1$

$= 2 \int_0^R |y|^{-2\alpha} \, dy = \frac{2}{1-2\alpha} \left[y^{1-2\alpha} \right]_0^R = \frac{2R^{1-2\alpha}}{1-2\alpha} < \infty$

$(\alpha \in (0, 1/2))$

$\therefore \Phi(x) < \infty \Rightarrow (1) < \infty \quad (\forall x \in \mathbb{R}) \Rightarrow \int_0^1 |f(y)| \cdot |x-y|^{-\alpha} \, dy < \infty$

(ii) $|g(x+h) - g(x)| = \left| \int_0^1 f(y) \left(|x+h-y|^{-\alpha} - |x-y|^{-\alpha} \right) dy \right|$

$\le \int_0^1 |f(y)| \cdot \left(|x+h-y|^{-\alpha} - |x-y|^{-\alpha} \right) dy$

$\le$ (Hölder) $\cancel{\int_0^1 |f|}$ $\|f\|_2 \cdot \sqrt{\int_0^1 (|x+h-y|^{-\alpha} - |x-y|^{-\alpha})^2 \, dy}$

$= \|f\|_2 \cdot I(h)^{1/2}$

(iii) Claim : $\lim_{h \to 0} I(h)^{1/2} = 0$

$I(0) = 0$. WLOG, $h \neq 0$.

$$I(h) = \int_0^1 \left(|x+h-y|^{-d} - |x-y|^{-d}\right)^2 \mathbb{1}_{y \in \bar{B}(x, 2h)}\, dy$$

$$+ \int_0^1 \mathbb{1}_{|y-x|>2h} \left(|x+h-y|^{-d} - |x-y|^{-d}\right)^2 dy$$

2^{nd} term: Integrand $\leq \operatorname{Max}\left(|h|^{-d} + |2h|^{-d}\right)^2 < \infty$

$\therefore$ Dominated

$$\lim_{h \to 0} |x+h-y|^{-d} = |x-y|^{-d} \quad \text{for } x \neq y$$

as $|x-y| > 2h > 0$, $z \mapsto |z|^{-d}$ is cont. away from 0.

$\therefore$ 2^{nd} term $\to 0$ as $h \to 0$

Let $\varepsilon > 0$ be arbitary:

$$\int_0^1 \left(|x+h-y|^{-d} - |x-y|^{-d}\right)^2 \mathbb{1}_{|x-y|>\varepsilon}\, dy = I_1(h)$$

For $|h| < \varepsilon/2$: $|x+h-y|^{-d}$, $|x-y|^{-d} < \left(2/\varepsilon\right)^d$

$\therefore$ Integrand $\leq \left(2 \cdot (2/\varepsilon)^d\right)^2 < \infty$ (Dominated)

Since $|x-y| > \varepsilon$: $\lim_{h \to 0} |x+h-y|^{-d} = |x+h|^{-d}$.

$\therefore \lim_{h \to 0} I_1(h) = 0$ (Dominated conv. theorm)

$$I_2(h) = \int_0^1 \mathbb{1}_{|x-y| \leq \varepsilon} \left(|x+h-y|^{-d} - |x-y|^{-d}\right)^2 dy$$

$$\leq 2 \int_0^1 \mathbb{1}_{|x-y| \leq \varepsilon} |x+h-y|^{-2d} + |x-y|^{-2d}$$

$$\leq 4 \int_{\mathbb{R}} \mathbb{1}_{|x-y| \leq 2\varepsilon} |x-y|^{-2d}\, dy \qquad (|h| < \varepsilon)$$

$$\leq 4 \int_{-2\varepsilon}^{2\varepsilon} |x|^{-2d}\, dx = 8 \int_0^{2\varepsilon} |x|^{-2d}\, dx = \frac{8}{1-2d} \left[x^{1-2d}\right]_0^{2\varepsilon}$$

$$= \frac{8}{1-2d} \left[(2\varepsilon)^{1-2d}\right]$$

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$\therefore \limsup_{h\to 0} I(h) \le \frac{8}{1-2\alpha} (2\varepsilon)^{1-2\alpha}$ (Arbitary $\varepsilon > 0$)

$\therefore$ LHS $\le \inf_{\substack{h\to 0 \\ \varepsilon > 0}}$ (RHS) $= 0$.

$\therefore$ LHS $= 0 \Rightarrow \lim_{h\to\infty} I(h) = 0$

$\therefore \lim_{h\to 0} |g(x+h) - g(x)| = 0 \Rightarrow g$ is cont.

3.) (a) $\forall A \in \mathcal{E}: \mu(T^{-1}(A)) = \mu(A)$ (T is measure preserving)
T is measurable map

A is invariant if $T^{-1}(A) = A$

$\mathcal{I} = \{ A \in \mathcal{E} : T^{-1}(A) = A \}$

$T^{-1}(\phi) = \phi$
$T^{-1}(E) = E$ $\Big\}$ $\phi, E \in \mathcal{I}$

If $A \in \mathcal{I}$: $T^{-1}(E \setminus A) = E \setminus T^{-1}(A) = E \setminus A \Rightarrow A^c \in \mathcal{I}$

If $A_n \in \mathcal{I}$: $T^{-1}\left(\bigcup_{n\ge 1} A_n\right) = \bigcup_{n\ge 1} T^{-1}(A_n) = \bigcup_{n\ge 1} A_n \Rightarrow \bigcup_{n\ge 1} A_n \in \mathcal{I}$

$\therefore \mathcal{I}$ is a σ-algebra

(b) If $x \in [0,1)$, $2x \in [0,1)$ or $[1,2)$

$\therefore T^{-1}\{x\} = \{ x/2, \frac{1}{2} + x/2 \}$

~~Claim: T is measure~~

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Let $\mathcal{A} = \{\phi\} \cup \{[k\cdot 2^{-n}, (k+1)\cdot 2^{-n}) : 0 \le k < 2^n,\ n \in \mathbb{N}\}$.

$\mathcal{A}$ is a π-generator of $\mathcal{B}[0,1) = \mathcal{E}$;

& $T^{-1}[k\cdot 2^{-n}, (k+1)2^{-n}) = [k\cdot 2^{-(n+1)}, (k+1)2^{-(n+1)}) \cup [\frac{1}{2} + k\cdot 2^{-n}, \frac{1}{2} + (k+1)2^{-n})$

$\therefore \in \mathcal{A} \subseteq \mathcal{E} \Rightarrow T$ is measurable

$\mu(T^{-1}[k\cdot 2^{-n}, (k+1)2^{-n})) = 2^{-n} = \mu([k\cdot 2^{-n}, (k+1)2^{-n}))$

$\therefore \mu = \mu(T^{-1}(\cdot))$ on π-generator set; $\mu(E) = \mu(T^{-1}(E)) = 1 < \infty$

$\therefore \mu = \mu(T^{-1}(\cdot))$ on $\mathcal{E}$. $\Rightarrow T$ is measure preserving

(c) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X_n : \Omega \to \mathbb{R}^d$ be iid R.V.

Consider: $\Phi : \Omega \to \mathbb{R}^{d\mathbb{N}}$, $(\Phi(\omega))_n = X_n(\omega)$

Let $\mathcal{E} \in P(P((\mathbb{R}^d)^{\mathbb{N}}))$, $\mathcal{E} = \sigma(X_n : n \geq 1)$.
(Φ measurable by construction)

$T : (\mathbb{R}^d)^{\mathbb{N}} \to (\mathbb{R}^d)^{\mathbb{N}}$, $T((x_n)_{n\geq 1}) = (x_{1+n})_{n \geq 1}$
be the shift map

Define $\mu : \mathcal{E} \to [0,1]$, $\mu = \mathbb{P}(\Phi^{-1}(\cdot))$ (Image measure);

~~$(\Omega, \mathcal{F}, \mathbb{P}, T)$~~ $(\mathbb{R}^{d\mathbb{N}}, \mathcal{E}, \mu, T)$ is the canonical model.

T is the bernoulli shift map

(d) Measure preserving: Since $(X_n)_{n \geq 1}$ are iid,

$\mathbb{P}((X_{n+1}, \dots X_{n+k}) \in A) = \mathbb{P}((X_{N+1}, \dots X_{N+k}) \in A)$

as law of $(X_{n+1}, \dots X_{n+k})$ is independent of n

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$\mathcal{E}$ is generated by π - system:

$$\mathcal{A} = \bigcup_{n \geq 1} \sigma(X_k : 1 \leq k \leq n).$$

But $\mu(T^{-1} (\{ (X_1, \dots X_n) \in A \})) = \mu(\{ (X_2, \dots X_n) \in A \}$

$= \mu(\{ (X_1, \dots X_n) \in A \})$

$\therefore \mu, \mu(T^{-1}(\cdot))$ agrees on $\mathcal{A} \Rightarrow$ Agrees on $\mathcal{E}$.

Also, $\forall T^{-1} : \mathcal{A} \to \mathcal{A} \Rightarrow T^{-1}$ is measurable.

Claim: T is ergodic.

Koglomorov 0-1 law: Let $(X_n)_{n \geq 1}$ be $\forall$ independent R.V.

$\forall$ $\mathcal{T} = \bigcap_{n \geq 1} \sigma(X_k : k \geq n)$

If $A \in \mathcal{T}$: $\mathbb{P}(A) = 0$ or 1

A If $A \in \mathcal{E}$ is T - invariant : $A \in \sigma(X_k : k \geq 1)$

But $T^{-r}(A) \in \sigma(X_k : k \geq r+1) \Rightarrow A \in \sigma(X_k : k \geq r+1)$

$\therefore A \in \bigcap_{n \geq 1} \sigma(X_k : k \geq n) \Rightarrow \mathbb{P}(A) = 0$ or 1

$\therefore$ Ergodic.

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4.) (a) Let $(E, \mathcal{E}, \mu)$ be a measure space, $f_n : E \to \mathbb{R}$ be measurable, $f_n \geq 0$

Fatou's Lemma: $\liminf \mu(f_n) \geq \mu(\liminf f_n)$

(b) $\forall g: \mathbb{R}^d \to \mathbb{R}$, g bounded, continuous:

$X_n \Rightarrow X$ (weak conv.) if $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$

Central limit theorem: Let $(X_n)_{n \geq 1}$ be sequence of iid R.V.,

$\mathbb{E}[X_1] = 0$ and $Var(X_1) = \mathbb{E}[X_1^2] = \sigma^2 < \infty$.

$$S_n = \sum_{k=1}^{n} X_k : \quad S_n/\sqrt{n} \to N(0, \sigma^2) \text{ weakly.}$$

(Proof): Theorem: $Y_n \Rightarrow Y$ iff $\varphi_{Y_n}(u) \to \varphi_Y(u)$ $(\forall u \in \mathbb{R}^d)$,

$Y_n, Y: \Omega \to \mathbb{R}^d$ are R.V., φ is characteristic function

Let $\phi(u) = \mathbb{E}[e^{iu \cdot X_1}]$:

$$\phi_{S_n/\sqrt{n}}(u) = \Phi(u) = \prod_{k=1}^{n} \phi_{X_k}(u/\sqrt{n}) = \phi(u/\sqrt{n})^n$$

Since X_1 is square integrable: 2^{nd} derivative of e^{iuX_1} is integrable.

$u \to |-X_1^2||e^{iuX_1}| \leq |X_1^2|$ (Dominating function) $\Rightarrow$

$\phi''(u)$ is cont.

$\therefore$ Taylor series: $\phi(u/\sqrt{n}) = \phi(0) + \frac{u}{\sqrt{n}} \underbrace{\phi'(0)}_{\mathbb{E}[X_1]=0} + \frac{u^2}{2n} \underbrace{\phi''(0)}_{-\mathbb{E}[X_1^2] = -\sigma^2} + o(u^2/n)$

$$\therefore \phi(u/\sqrt{n}) = 1 - u^2/n \, (\sigma^2/2) + o(u^2/n)$$

$$= e^{-\sigma^2/2 \cdot u^2/n + o(u^2/n)}$$

$$\therefore \phi_{S_n/\sqrt{n}}(u) = e^{-\sigma^2/2 \cdot u^2 + o(u^2)} \to e^{-\sigma^2/2 \cdot u^2} \quad (n \to \infty)$$

But $\mathbb{E}[e^{iuN(0,\sigma^2)}] = e^{-\sigma^2/2 \, u^2} \Rightarrow S_n/\sqrt{n} \to N(0, \sigma^2)$ weakly

$\therefore$ Done.

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(c) $\cancel{2\varphi(0) - \varphi(h_n) - \varphi(-h) / h_n^2 = \mathbb{E}\left[2 - e^{ih_n} - e^{-ih_n} / h_n^2\right]}$

$\cancel{= \mathbb{E}\left[2 - 2\cos(h_n) / h_n^2\right]}$

$$\frac{1}{h_n^2}\left(2\varphi(0) - \varphi(h_n) - \varphi(-h_n)\right) = \mathbb{E}\left[\frac{2 - e^{ih_nX} - e^{-ih_nX}}{h_n^2}\right]$$

$$= \mathbb{E}\left[\underbrace{\frac{2 - 2\cos(h_n X)}{h_n^2}}_{\geq 0.}\right]$$

Fatou's lemma: $\liminf_n \dfrac{2\varphi(0) - \varphi(h_n) - \varphi(-h_n)}{h_n^2} \geq \mathbb{E}\left[\liminf_n \dfrac{2 - 2\cos(h_n X)}{h_n^2}\right]$

But: $h_n \to 0$; $\displaystyle\lim_{h_n \to 0} \frac{2 - 2\cos(h_n X)}{h_n^2} =$ (L'Hôpital, ~~h_n~~ Cos. is smooth)

$$\lim_{h \to 0} \frac{2\sin(h_n X) \cdot X}{2h_n}$$

$$= \lim_{h \to 0} \frac{2\cos(h_n X) \cdot X^2}{2} = X^2$$

$$\therefore\ \infty > \liminf_n \frac{2\varphi(0) - \varphi(h_n) - \varphi(-h_n)}{h_n^2} \geq \mathbb{E}[X^2]$$

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1.) (a) Let $A = \{ x \in E : \liminf_n f_n(x) < \limsup_n f_n(x) \}$

$f_n \to f$ A.E if $\mu(A) = 0$

$f_n \to f$ in μ if : $\forall$ ~~Given~~ $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} \mu(\{ x \in E : |f_n(x) - f(x)| \geq \varepsilon \}) = 0$$

(b) Let $E = \mathbb{R}$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$, $f_n(x) = 1_{|x| \geq n}(x)$; μ be Lebesgue measure.

Claim: $f_n \to f$ A. E.

$\forall x \in \mathbb{R}$, pick $n \in \mathbb{N}$ s.t. $n > |x|$: $\forall m \geq n$, $f_m(x) = 0$.

$\therefore \lim_{n \to \infty} f_n(x) = 0$.

Claim: $f_n \not\to f$ (in μ)

$\mu(\{ x \in E : |f_n(x) - f(x)| \geq 1/2 \}) = \infty$ $(\forall n \in \mathbb{N})$

$\therefore$ Not convergence in μ.

(c) Let $E = [0, 1)$, $\mathcal{E} = \mathcal{B}([0,1))$, μ be Lebesgue measure.

$$f_n(x) = 1_{x \in [\sum_{k=1}^{n} 1/k, \sum_{k=1}^{n+1} 1/k+1)} \pmod 1$$

Since $\sum_{k=1}^{N} 1/k \to \infty$, $\forall x \in [0,1)$, $x \in [\sum_{k=1}^{n} 1/k, \sum_{k=1}^{n+1} 1/k+1)$

infinitely often, $x \notin [\sum_{k=1}^{n} 1/k, \sum_{k=1}^{n+1} 1/k)$ i.o.

$\therefore \liminf_n f_n(x) = 0 < 1 = \limsup_n f_n(x)$

$\therefore f_n \not\to f \Rightarrow$ Not everywhere conv.

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(d) Let ~~$x\in$~~ Λ be the set of points where $f_n \not\to f$.

$\therefore \mu(\Lambda) = 0$.

If $x \in \Lambda^c$: $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| \leq 1/k$.

$\therefore x \in E_{N,k}^c$.

$\therefore \Lambda^c \subseteq \bigcup_{N\geq 1} E_{N,k}^c \Rightarrow \bigcap_{N\geq 1} E_{N,k} \subseteq \Lambda$.

Since $\mu([0,1]) < \infty$: $\bigcap_{m\geq n}$ is ~~decreasin~~ decreasing sequence,

$\lim_{n\to\infty} \mu(E_{n,k}) = \mu\left(\bigcap_{n\geq 1} E_{n,k}\right) \leq \mu(\Lambda) = 0$.

(ii) ~~Let $A \supseteq \{x \in E$~~ :

Fix $\varepsilon > 0$: For each $k \in \mathbb{N}$, pick $R_k \in \mathbb{N}$ s.t.

$\mu(E_{R_k,k}) < \varepsilon / 2\cdot 2^k$

Let $A = \bigcup_{k\geq 1} E_{R_k,k}$: $\mu(A) \leq \sum_{k\geq 1} \mu(E_{R_k,k}) < \varepsilon/2 < \varepsilon$.

Claim: $f_n \to f$ uniformly.

Given $\varepsilon_1 > 0$, pick $k_1 > 1/\varepsilon_1$: $\forall m \geq R_{k_m}$, $\forall x \in [0,1]\setminus A$,

$x \in E_{R_k,k}^c \Rightarrow |f_m(x) - f(x)| \leq 1/k < \varepsilon_1$

$\therefore$ Uniform convergence.

(iii) False: Use example in part (b)

~~$\forall f$~~ $\forall n \in \mathbb{N}$, $\mu(\{x : |f_n - f(x)| \geq 1/2\}) = \infty$.

$\therefore$ If $\mu(A) < \varepsilon < \infty$, $\mu(\{x \in \mathbb{R}\setminus A : |f_n(x) - f(x)| \geq 1/2\}) = \infty$

for all $n \in \mathbb{N}$.

$\therefore$ No uniform convergence.

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2.) (a) Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be convex.

If $(E, \mathcal{E}, \mu)$ is a probability space: Jensen's inequality $\Rightarrow$
$\forall$ integrable $g: E \to \mathbb{R}$, $\mu(f(g)) \geq f(\mu(g))$

$\| f \|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$; Let $(E, \mathcal{E}, \mu)$ be a measure space,

$f \in L^p$ iff $\| f \|_p < \infty$

False: If $\| f - g \|_p = 0$, $f \neq g$ everywhere.
Let $E = [0,1]$, $\mathcal{E} = \mathcal{B}(E)$, $\mu =$ Lebesgue measure
$f = 1_{x=0}$, $g = 0$ $\Rightarrow$ $f \neq g$ (at 0)
$\int |f-g|^p \, dx = \int 1_{x=0} \, dx \leq \int 1_{[0, 1/n]} \, dx = 1/n$ $(\forall n \in \mathbb{N})$
$\therefore \| f - g \|_p^p = 0 \Rightarrow \| f - g \|_p = 0$

Claim: $f = g$ A.E.

Let $A_n = \{ x \in E : |f(x) - g(x)| \geq 1/n \}$: A_n is increasing,
$\bigcup_{n \geq 1} A_n = \{ x \in E : f \neq g \}$

$\therefore \mu(\bigcup_{n \geq 1} A_n) = \lim_{n \to \infty} \mu(A_n)$

If LHS > 0: $\exists N \in \mathbb{N}$ s.t. $\mu(A_N) > 0$.
$\therefore \| f - g \|_p^p \geq \int |f-g|^p 1_{A_N} \geq \frac{1}{N^p} \mu(A_N) > 0 \Rightarrow \| f - g \|_p \neq 0$.

$\therefore$ Contradiction $\Rightarrow$ $\mu(\bigcup_{n \geq 1} A_n) = 0$. $\Rightarrow$ Equal A.E.

Let $q \in (1, \infty)$ s.t. $1/p + 1/q = 1$: (Hölder) $f \in L^p$, $g \in L^q \Rightarrow \int |f| \cdot |g| \leq \| f \|_p \cdot \| g \|_q$
If $\| f \|_p = 0$, $f = 0$ A.E $\Rightarrow \int |f| \cdot |g| = 0 = \| f \|_p \cdot \| g \|_q$
$\therefore$ WLOG, $\| f \| > 0$

$\therefore \int \frac{|f| \cdot |g|}{\| f \|_p^p} = \int 1_{|f| > 0} \frac{|g|}{|f|^{p-1}} \underbrace{\frac{|f|^p}{\| f \|_p^p}}_{\text{IP measure}}$

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$x \mapsto x^q$ is convex for $q > 1$

$$\therefore \frac{\int |fg|}{\|f\|_p^p} \le \left(\int 1_{|f|>0} \frac{|g|^q}{|f|^{(p-1)\cdot q}} |f|^p / \|f\|^p \right)^{1/q} = \left(\int 1_{|f|>0} \frac{|g|^q}{|f|^p} \cdot \frac{|f|^p}{\|f\|^p} \right)^{1/q}$$

$$\le \|f\|^{-p/q} \left(\int |g|^q \right)^{1/q}$$

$$\therefore \int |fg| \le \|f\|_p^{p(1-1/q)} \|g\|_p^q = \|f\|_p \cdot \|g\|_q \quad \text{※}$$

(b) $\|f\|_\infty = \sup \{ \theta \ge 0 : \mu(\{x : |f(x)| \ge \theta\}) > 0 \}$

~~Let $\varepsilon > 0$ be arbitrary:~~

$f \in L^p$: $r = \cancel{q/p}\ p/q > 1$. Pick $s \in (1, \infty)$ s.t. $1/r + 1/s = 1$

$$\therefore \int |f|^q \le \left(\int |f|^{q(\cdot r)} \right)^{1/r} \cdot \left(\int (1)^s \right)^{1/s}$$

$$\therefore \|f\|_q^q \le \mu(E)^{1/s} \|f\|_{p}^{p/r} < \infty.$$

$$\therefore f \in L^q$$

$\|f\|_p \to \|f\|_\infty$

If $\|f\|_\infty = \infty$: $\forall N \in \mathbb{N}$, $\exists A \subseteq E$ s.t. $|f|_A(x)| \ge N$ for $x \in N$, $|A| > 0$.

$$\therefore \|f\|_p^p \ge \int |f|^p \cdot 1_A \ge N^p \cdot \mu(A)$$

$$\therefore \|f\|_p \ge N \cdot (\mu(A))^{1/p} \; ; \; \forall x \in (0, \infty), \; x^{1/p} \to 1 \text{ as } p \to \infty$$

$$\therefore \liminf_{p \to \infty} \|f\|_p \ge N \; ; \; N \text{ arbitrary} \Rightarrow \|f\|_p \to \infty = \|f\|_\infty$$

Suppose $\|f\|_\infty < \infty$: Let $\varepsilon > 0$ be arbitrary, $A_\varepsilon = \{ x \in E : |f(x)| \ge (1-\varepsilon)\|f\|_\infty \}$

$$\therefore \|f\|_p^p \ge \int |f|^p \cdot 1_{A_\varepsilon} \ge (1-\varepsilon)^p \|f\|_\infty^p \cdot \underbrace{\mu(A_\varepsilon)}_{>0}$$

$$\therefore \|f\|_p \ge (1-\varepsilon) \|f\|_\infty \, \mu(A_\varepsilon)^{1/p}$$

$$\therefore \liminf_{p} \|f\|_p \ge (1-\varepsilon) \|f\|_\infty \quad (1)$$

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Since ε was arbitrary, $\liminf \| f \|_p \geq \| f \|_\infty$

$$\| f \|_p^p = \int |f|. \geq \| f \|_\infty^p \cdot \mu(E)$$

$$\therefore \| f \|_p \geq \| f \|_\infty \cdot \mu(E)^{1/p}$$

$$\therefore \limsup_p \| f \|_p \geq \| f \|_\infty \limsup_p \mu(E)^{1/p} = \| f \|_\infty$$

$\therefore \| f \|_p \to \| f \|_\infty$

(c) Pick $\theta \in (0,1)$ st $r = \theta \cdot p + (1-\theta) \cdot q$

$$\therefore \int |f|^r = \int |f|^{\theta \cdot p} |f|^{(1-\theta) \cdot q} \quad ; \quad 1/\theta, \ 1/(1-\theta) \in (0,1)$$

(Hölder): $\leq \| f^{\theta \cdot p} \|_{1/\theta} \cdot \| f^{(1-\theta) \cdot q} \|_{1/1-\theta} = (\int |f|^p)^\theta (\int |f|^q)^{1-\theta}$

$< \infty$

$\therefore f \in L^r$

(d) Let $E = [1, \infty)$, $\mathcal{E} = \mathcal{B}(E)$, $\mu =$ Lebesgue measure.

Let $f(x) = \frac{1}{\sqrt{x}}$; $q = 3/2$

$$\therefore \int_1^\infty |f|^{3/2} = \int_1^\infty x^{-3/4} \geq \int_1^\infty \frac{1}{x} = \infty$$

$\therefore f \notin L^q$

Let $p = 4$: $p > q$

$$\int_1^\infty |f|^4 = \int_1^\infty \frac{1}{x^2} = 1 < \infty \Rightarrow f \in L^p$$

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4.) (a) $f * g(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy$ (if integral well-defined)

If $f \in L^1$: $\hat{f}(u) = \int_{\mathbb{R}^d} f(x)\, e^{iux}\, dx \quad (u \in \mathbb{R}^d)$

Claim $\widehat{f * g} = \hat{f} \cdot \hat{g}$

$$LHS = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(y)\, g(x-y)\, e^{iu(x-y+y)}\, dx \cdot dy$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)\, g(z)\, e^{iuz}\, e^{iuy}\, dz\, dy = \hat{f}(u) \cdot \hat{g}(u)$$

If $f, \hat{f} \in L^1$: $f(x) = (2\pi)^{-1}\, \hat{\hat{f}}(-x)$ (Fourier inversion)

(b) If $f \in L^2 \cap L^1$: $\|f\|_2^2 = (2\pi)^{-d} \|\hat{f}\|_2^2$

$\|\hat{f}\|_2 = (2\pi)^{1/2} \|f\|_2$

$$\int |f|^2 = \int_0^1 e^{-2x}\, dx = \left[e^{-2x}/_{-2} \right]_0^1 = \tfrac{1}{2}(1 - e^{-2})$$

$$\therefore \|\hat{f}\|_2 = \left((2\pi)^{1/2} \left(\frac{1-e^{-2}}{2} \right) \right)^{1/2} = \sqrt{\pi(1-e^{-2})}$$

(c) $\forall u \in \mathbb{R}^d$: $|\hat{f}_n(u) - \hat{f}(u)| = |(\hat{f}_n - \hat{f})(u)| \leq \int_{\mathbb{R}^d} |f_n - f| \cdot 1 = \|f_n - f\|_1$

$$\therefore \|\hat{f}_n - \hat{f}\|_\infty \leq \|f_n - f\|_1 \to 0 \Rightarrow \hat{f}_n \to \hat{f} \text{ uniformly}$$

(d) Let X_n be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$:

$X_n \Rightarrow X$ (weak conv.) iff $\forall$ bounded cont. $g: \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$$

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$X_n \Rightarrow X$ iff $\mathbb{E}[e^{iuX_n}] \to \mathbb{E}[e^{iuX}]$

Central limit theorem: Let $(X_n)_{n \geq 1}$ be iid R.V., $\mathbb{E}[X_n] = 0$ and $Var(X_n) = \sigma^2 < \infty$.

$S_n = \sum_{k=1}^{n} \cancel{S_k} X_k$: Claim $\cancel{\sqrt{n}}\ \frac{1}{\sqrt{n}}(S_n) \to N(0, \sigma^2)$ weakly. in dis.

(Proof): $\mathbb{E}[e^{iu S_n/\sqrt{n}}] = \prod_{j=1}^{n} \mathbb{E}[e^{i(u/\sqrt{n})X_j}]$ — (1)

Let $\Phi(u) = \mathbb{E}[e^{iuX_1}]$:

$(1) = \Phi(u/\sqrt{n})^n$

$\cancel{\lim_{n\to\infty} \Phi(u/\sqrt{n})}$

Claim: $\Phi(u)$ is cont.

If $u_n \to u$, e^{iuX_1} is bounded $\Rightarrow$ integrable

Dominated convergence theorem $\Rightarrow$ $\Phi(u_n) \to \Phi(u)$ $\Rightarrow$ Cont. on $\mathbb{R}^d$

Claim: Φ is differentiable.

$u \to e^{iuX_n}$ is differentiable; derivative $u \to X_n e^{iuX_n}$ is integrable ($\mathbb{E}[|X_n|] \leq \mathbb{E}[|X_n|^2]^{1/2} < \infty$)

$\therefore \Phi'(u) = \mathbb{E}[iX_1 e^{iuX_1}]$

Similarly, Φ is twice diff. ; $\Phi''(u) = -\mathbb{E}[X_1^2 e^{iuX_1}]$, and is also cont.

$\therefore \cancel{\Phi(u)} = \Phi(u/\sqrt{n}) = \cancel{\mathbb{E}[1 + \Phi(0) + \tfrac{u}{\sqrt{n}}\Phi'(0)]} + \frac{u^2}{2n}\Phi''(0)$, $\underbrace{\Phi'(0)}_{=0}$, $\underbrace{\Phi''(0)}_{-1}$

(Taylor) : $\Phi(u/\sqrt{n}) = 1 + u/\sqrt{n} \cdot (0) - u^2/2n + o(u^2/n)$

$\therefore \Phi(u/\sqrt{n})^n = \left(e^{-u^2/2n + o(u^2/n)}\right)^n = e^{-u^2/2 + o(u^2)} \to e^{-u^2/2}$

But $\mathbb{E}\left[e^{iu\cdot N(0,1)}\right] = e^{-u^2/2}$

$\therefore \quad S_n/\sqrt{n} \to N(0,1)$

3.) (a) $\mathcal{B} = \bigcap \left\{ A \in P(P(X)) : \begin{array}{l} A \text{ contains open sets of } X, \\ A \text{ is a } \sigma\text{-algebra} \end{array} \right\}$

$f: X \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is borel if $\forall A \in \mathcal{B}(\mathbb{R})$, $f^{-1}(A) \in \mathcal{B}(X)$
(X is a topological space)

(b) Define relation on $[\frac{1}{3}, \frac{2}{3}]$: $x \sim y$ iff $x - y \in \mathbb{Q}$] ($|x-y| \le 1/3$)

Let $\Lambda = \{$ representative element from each $\sim$ equivalence class $\}$

$\therefore \forall q \in \mathbb{Q}$, :

If $\Lambda \in \mathcal{L}(\mathbb{R})$:

Consider $q + \Lambda = \Lambda_q$ for $q \in [-1/3, 1/3] \cap \mathbb{Q}$.

$\therefore$ Λ_q disjoint for distinct q, $\Lambda_q \subseteq [0,1]$

$$\bigcup_{q \in [-1/3, 1/3] \cap \mathbb{Q}} \Lambda_q = [1/3, 2/3] \Rightarrow 1/3 = \sum_{q \in [-1/3, 1/3] \cap \mathbb{Q}} \mu(\Lambda_q) = \sum_{q \in [-1/3, 1/3] \cap \mathbb{Q}} \mu(\Lambda)$$

(Translation invariance) $\Rightarrow 1/3 = \infty \cdot \mu(\Lambda)$ (reject!)

$\therefore \Lambda \notin \mathcal{L}(\mathbb{R})$

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Let $x \in [0,1]$, $(a_n^x)_{n\geq 1}$ be expansion in base 3

(Use ~~Without~~ ∞ "2" tails)

$\therefore C = \{ x \in [0,1] : (a_n^x)_{n\geq 1}$ contains only 0 or 2 $\}$

$= \bigcap_{N\geq 1} \{ x \in [0,1] : (a_n^x)_{n=1,\dots N}$ contain 0 or 2 $\}$

$= \bigcap_{N\geq 1} \bigcup_{k=0}^{3^N} C_N.$

$C_1 = [0, 1/3] \cup [2/3, 1]$; $C_{n+1} = \frac{1}{3} \cdot C_n \cup \frac{2}{3} + \frac{1}{3} \cdot C_n$

Since C_n is closed, C is a countable $\cap$ of closed sets $\Rightarrow$

$C \in \mathcal{B}([0,1]) \subseteq \mathcal{L}[0,1]$

$\therefore$ Lebesgue measurable

(ii) $|C| \leq |C_{n+1}| = 2/3 |C_n| = (2/3)^n \cdot |C_1| = (2/3)^{n+1}$

$\therefore |C| \leq \inf_{n\geq 1} (2/3)^n = 0.$ $\Rightarrow$ Measure 0.

(iii) If $A \subseteq C$:

$\forall B \subseteq \mathbb{R}$: μ^* be outer measure. (μ^* is countable sub-additive)

$\mu^*(B) \leq \underbrace{\mu^*(B \cap A)}_{\leq \mu^*(A) \leq \mu(C) = 0} + \mu^*(B \cap A^c) = \mu^*(B \cap A^c) \leq \mu^*(B)$

$\therefore A \in \mathcal{L}$ (Lebesgue measurable)

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(iv) f is Borel:

a_k is the k^{th} digit of binary expansion (without ∞ "1" tails)

$f_k(x) = \sum_{r=1}^{k} \frac{2a_r}{3^r}$; f_k ~~is~~ ~~has~~ is constant on

intervals ~~$[k \cdot 2^{-n}, (k+1)2^{-n}), \{1\}$~~ $(0 \le k < 2^n, n \in \mathbb{N})$
$\{1\}$, $[r \cdot 2^{-k}, (r+1)2^{-k})$, $0 \le r < 2^k$.

(~~measur~~ Borel) $\Rightarrow$ f_k is simple $\Rightarrow$ f_k is borel

limit of borel function is borel $\Rightarrow$ f is borel

(iv) f is injective:

If $f(x_1) = f(x_2)$, binary expansion of x_1, x_2 agree

$\therefore x_1 = x_2$

Let N be non lebesgue measurable set constructed in (b)

If $f(N) \in \mathcal{B}[0,1]$: $f^{-1}(f(N)) = N \in \mathcal{B}([0,1])$

$\therefore$ Contradiction

But $f(N) \subseteq C \Rightarrow f(N) \in \mathcal{L}[0,1]$ $\times$

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1.) (a) $\mathcal{B}(\mathbb{R})$: Let τ be all open sets of $\mathbb{R}$, $\sigma(\tau)$ be the smallest σ-algebra containing τ.

$\mathcal{B}(\mathbb{R}) = \sigma(\tau)$ Yes (or a σ-algebra generated by the topology on $\mathbb{R}$)

$*$ $\mathcal{B}(\mathbb{R}) = \bigcap \Lambda$, $\Lambda = \left\{ A \in \mathbb{P}(\mathbb{P}(\mathbb{R})) : \begin{array}{l} A \text{ is a } \sigma\text{-algebra,} \\ \tau \subseteq A \end{array} \right\}$

f is borel if : $\forall A \in \mathcal{B}(\mathbb{R}),\ f^{-1}(A) \in \mathcal{E}$
~~Given $A \subseteq \mathbb{P}(\mathbb{R})$, A is a π-system, $\sigma(A) = \mathcal{B}(\mathbb{R})$: $\forall A \in A$, $f^{-1}(A) \in \mathcal{E}$.~~ Yes.

(b) Let $A = \{\phi\} \cup \{(-\infty, a] : a \in \mathbb{R}\}$
$f^{-1}(\phi) = \phi \in \mathcal{E}$

$x \in f^{-1}(-\infty, a]$ iff $\lim_{n\to\infty} f_n(x) \leq a$, iff $\forall N \in \mathbb{N}$, $\exists M \in \mathbb{N}$ s.t. $m \geq M \Rightarrow f_m(x) \leq a + 1/N$. Yes.

$$\therefore f^{-1}(-\infty, a] = \bigcap_{N\in\mathbb{N}} \bigcup_{M\in\mathbb{N}} \bigcap_{m\geq M} \underbrace{f_m^{-1}(-\infty, a + 1/N]}_{\in \mathcal{E}}$$ Yes.

Since $\mathcal{E}$ is closed wrt countable union, intersections, $f^{-1}(-\infty, a] \in \mathcal{E}$.

A is a π-system, A generates $\mathcal{B}(\mathbb{R}) \Rightarrow$ $\forall A \in \mathcal{B}(\mathbb{R})$, $f^{-1}(A) \in \mathcal{E}$ as well

$\therefore f$ measurable Yes.

(c) $$R_n^{-1}\{0\} = \bigcup_{\substack{k=0,\\ k \text{ even}?}}^{2^n - 1} [k \cdot 2^{-n}, (k+1)\cdot 2^{-n}) \in \mathcal{B}([0,1))$$ ← all possible sequences of the first $n-1$ digits have to be enumerated

Since $R_n = 0$ or 1, $R_n^{-1}\{1\} = R_n^{-1}\{0\}^c \in \mathcal{B}([0,1))$

$\therefore R_n$ measurable as $\forall A \in P(\mathbb{R})$, ~~$R_n^{-1}(A) \in \{\phi, R_n^{-1}\{$~~
$R_n^{-1}(A) \in \sigma(\{R_n^{-1}\{0\}\}) \subseteq \sigma(\mathcal{B}([0,1)))$
Yes.

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$\therefore$ R_n is borel. Yes.

(d) $f(x_1, x_2) \in \{0, \infty\} \cup \mathbb{N} \Rightarrow f \geq 0$ Yes.

If A $B \subseteq \mathbb{N}$, $|B| < \infty$:

$\{(x, y) \in [0,1)^2 : x, y \text{ binary form disagrees at } B \text{ only}\} = \Lambda(\cancel{x,y}, B)$

$\cancel{\bigcap_{k \in B} (R_k^{-1}\{0\}^2 \cup R_k^{-1}\{1\}^2)}$

Let $C_k = R_k^{-1}\{0\}^2 \cup R_k^{-1}\{1\}^2 \in \mathcal{B}([0,1)^2) = \mathcal{E}$ Yes.

$\therefore \Lambda(\cancel{x,y}, B) = \left(\bigcap_{k \in B} C_k\right) \cap \left(\bigcap_{k \in B^c} C_k^c\right) \in \mathcal{E}$. Yes.

$f^{-1}\{n\} = \bigcup_{B \subseteq \mathbb{N}, |B| = n} \Lambda(B) \in \mathcal{E}$ as there are countable subsets of size n in $\mathbb{N}$. Yes.

$f^{-1}\{\infty\} = [0,1)^2 \setminus \bigcup_{n \geq 0} f^{-1}(n) \in \mathcal{E}$ Yes.

$\therefore$ $\forall A \subseteq [0, \infty]$, $f^{-1}(A) \in \sigma(\{f^{-1}(\alpha) : \alpha = 0, \infty; \alpha \in \mathbb{N}\})$

$\subseteq \mathcal{E}$.

$\therefore$ f is measurable. Yes. Even shorter paths to a solution exist.

(e) $A_n = \{(x, y) \in [0,1)^2 : R_n(x) \neq R_n(y)\}$

$|A_n| = \frac{1}{2} \Rightarrow \sum_{n \geq 1} |A_n| = \infty$

Yes.

Note : $\{x : R_1(x) = a_1, \ldots R_n(x) = a_n\} = [0.\bar{a}_1 \ldots \bar{a}_n, 0.\bar{a}_1 \ldots \bar{a}_n + 2^{-n})$

$\therefore |\{R_1(x) = a_1, \ldots R_n(x) = a_n\}| = \frac{1}{2^n} = \prod_{k=1}^{n} |\{R_k(x) = a_k\}|$

Yes.

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$\therefore (R_k(\omega))_{k\geq 1}$ is independent $\Rightarrow$ $((R_k(x), R_k(y)))_{k\geq 1}$ is independent

$\therefore A_n$ is independent sequence of events

Borel Cantelli II: A_n happen i.o. A.S.

$\therefore \left| \{ (x, y) : f(x,y) = \infty \} \right| = 1$ Yes.

2.) (a) $\hat{f}(u) = \int_{\mathbb{R}^d} f(x)\, e^{iux}\, dx$

(b) Fourier inversion is valid if:

$f(x) = (2\pi)^{-d} \hat{f}(u)$ A.E. (on condition $\hat{f}$ is well defined)

(c) $\hat{g}_t(u) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} \prod_{k=1}^{d} \left(e^{-x_k^2/2dt}\, e^{iu_k \cdot x_k} \right) dx_1 \dots dx_d$

$= \prod_{k=1}^{d} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t}}\, e^{-x_k^2/2dt}\, e^{iu_k x_k}\, dx_k$

Let $f(u_k) = \int_{\mathbb{R}^d} (2\pi t)^{-1/2}\, e^{-x_k^2/2t}\, e^{iu_k x_k}\, dx_k$

$f(u)$: integrand is diff., derivative is integrable $\Rightarrow f$ is differentiable

$f'(u) = \int_{\mathbb{R}} (2\pi t)^{-1/2}\, e^{-x_k^2/2t}\, (i x_k)\, e^{iux}\, dx$

$=$ (integrate by parts) $\int_{\mathbb{R}} \frac{i}{\sqrt{2\pi t}} (\dots)(+t)(iu)\, e^{iux}$

$= -ut\, f(u)$

$\therefore f(u) = c \cdot e^{-u^2 t/2}$; $f(0) = 1 \Rightarrow f(u) = e^{-u^2 t/2}$

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$\therefore \hat{g}_t(u) = e^{-t/2 \|u\|^2} = e^{-\|u\|^2/2(1/t)}$

$\therefore \hat{\hat{g}}_t(u) = (2\pi/t)^{+d/2} e^{-\frac{1}{2t}\|u\|^2} = (2\pi)^d (2\pi t)^{-d/2} e^{-\frac{1}{2}\|-u\|^2}$

$= (2\pi)^d g_t(u)$

$\therefore$ Valid.

$\widehat{f * g_t}(u) = \hat{f}(u) \cdot \hat{g}_t(u)$

$f * g_t(u) = \int_{\mathbb{R}^d} g_t(z) f(u-z) \, dz$

$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (2\pi)^{-d} \hat{\hat{g}}_t(\theta) e^{-i\theta \cdot z} d\theta \; f(u-z) \, dz$

$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (2\pi)^{-d} \hat{g}_t(\theta) f(w) e^{-i\theta u} e^{i\theta w} dw \cdot d\theta$

$= (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{g}_t(\theta) e^{-i\theta u} \hat{f}(\theta) \, d\theta$

$= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{g_t * f}(\theta) e^{-i\theta u} d\theta = (2\pi)^{-d} \hat{\hat{g_t * f}}(-u)$

$\therefore$ Done

(d) If $f_1(x) = (2\pi)^{-d} \hat{\hat{f}}(-x)$:

$|\hat{f}(u)| \le \int_{\mathbb{R}^d} |f(x)| \cdot 1 \, dx = \|f\|_1$

$\therefore \|\hat{f}\|_\infty \le \|f\|_1$

$\therefore$ If inverse exists ($\hat{f} \in L_1$) $\Rightarrow \|\hat{\hat{f}}\|_\infty < \infty \Rightarrow f$ is bounded.

Let $u_n \to u$: $\int_{\mathbb{R}^d} \underbrace{f(x) e^{iu_n x}}_{|\cdot| = |f| \in L^1} dx \longrightarrow \hat{f}(u)$ (Dominated conv. theorem)

$\therefore \hat{f}$ is cont.

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$f(x) = (2\pi)^{-d} \hat{f}(-x)$ A.E. $\Rightarrow$ f has a cont., bounded version

(c) No: If $g = f$ A.E., $f = 0$ on $(-\infty, 0)$ $\Rightarrow$ $\exists\, x_n < 0$, $x_n \to 0$ s.t. $g(x_n) = 0$.

But $f = 1$ on $(0,1)$ $\Rightarrow$ $\exists\, y_n > 0$, $y_n \to 0$ s.t. $g(y_n) = 1$.

$\therefore$ g not cont. at 0. $\Rightarrow$ ~~f~~ f ~~cannot be~~ is not fourier invertible.

3) (a) Given $\varepsilon > 0$, $\exists\, M \geq 0$ s.t.

$$\sup\left\{ \mathbb{E}\left[|X| \cdot 1_{|X| \geq M}\right] : X \in \mathcal{X} \right\} < \varepsilon.$$

Yes.

(b) Let $1 \leq p \leq \infty$: Pick q s.t. $1/p + 1/q = 1$ ($p = \infty \to q = 1$, $p = 1 \to q = \infty$)

Hölder: { If $f \in L^p$, $g \in L^q$, $\mu(|f| \cdot |g|) \leq \|f\|_p \cdot \|g\|_q$.

Yes.

~~If~~ WLOG, $p \leq q$:

If $p = 1$: $\mu(|f| \cdot |g|) \leq \mu(|f| \cdot \|g\|_\infty) = \|g\|_\infty \cdot \|f\|_1$ Yes.

If $p > 1$: $1/p + 1/q = 1$

$$\int |f| \cdot |g| = \int |f|^p \cdot |g| \cdot 1_{|f|>0} \frac{1}{|f|^{p-1}}$$

~~Would the measure~~ This measure could be defined for 1_A, $A \in \mathcal{E}$

If $\|f\|_p = 0$: $f = 0$ A.E. $\Rightarrow$ Inequality true Yes.

$f \neq 0$ A.E.: $$\int \frac{|f| \cdot |g|}{\|f\|_p^p} = \int \frac{1_{|f|>0}\, |g|}{|f|^{p-1}} \underbrace{\frac{|f|^p}{\|f\|_p^p}\, d\mu}_{\mathbb{P}\text{- measure}} \leq$$

Yes.

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by Jensen's inequality

$$\left(\int \left(\frac{1_{|f|>0}\,|g|}{|f|^{p-1}}\right)^q \frac{|f|^p}{\|f\|_p^p}\,d\mu\right)^{1/q}$$

$(q\cdot(p-1)=p$

$\frac{p+q}{q\cdot p}=1$

$$=\left(\int |g|^q/\|f\|_p^p\,1_{|f|>0}\,d\mu\right)^{1/q} \le \|g\|_q\,\|f\|^{-p/q}$$ Yes.

$$\therefore \int \mu(|f|\cdot|g|) \le \|g\|_q\cdot\|f\|_p^{p(1-1/q)} = \|g\|_q\cdot\|f\|_p.$$ Yes.

(c) $\mathcal{X}$ is L^p bounded if: $\sup\{\|X\|_p : X\in\mathcal{X}\}=M<\infty$

$p>1 \Rightarrow q\neq\infty$ Yes.

$$\mathbb{E}[|X|\cdot 1_{|X|\ge R}] \le \|X\|_p\cdot\mathbb{E}[1^q_{|X|\ge R}]^{1/q}$$

$$\le M\cdot\mathbb{P}(|X|\ge R)^{1/q}$$ Yes, by Roger-Hölders ineq.

(Markov): ~~$M^{p/q}\,\mathbb{P}(|X|\ge M) \le \|X\|^p_q = M\cdot\mathbb{P}(|X|\ge M)^{1/q}$~~

$$\mathbb{P}(|X|\ge R) \le \cancel{\|X\|_p^p}\ \|X\|_p^p/R^p$$ Yes, for $R>0$.

$$\therefore \mathbb{E}[|X|\cdot 1_{|X|\ge R}] \le M\cdot M^{p/q}/R^{p/q} = M^p/R^{p/q}$$ Yes.

Given $\varepsilon>0$, pick R large s.t. $R>(M^p/\varepsilon/2)^{q/p}$:

$$\sup\{\mathbb{E}[|X|\,1_{|X|\ge R}] : X\in\mathcal{X}\} \le \varepsilon/2<\varepsilon$$

$\therefore$ Uniformly Integrable Yes.

(d) Let $\Omega=(0,1)$, $\mathcal{F}=\mathcal{B}((0,1))$, $\mathbb{P}=$ Lebesgue measure.

$$X_n(x)=1_{(0,1/n)}\cdot n$$

$\|X_n\|_1=1 \Rightarrow \mathcal{X}$ is L^1 bounded Yes.

$$\mathbb{E}[|X_n|\cdot 1_{|X_n|\ge M}]=1 \quad \text{for } n\ge M$$

$$\therefore \sup\{\mathbb{E}[|X_n|\cdot 1_{|X_n|\ge M}] : n\in\mathbb{N}\}\ge 1 \quad \text{for all } M.$$

$\therefore$ Not uniformly integrable Yes.

4) (a) $\mu_\Theta(\Phi) = \mu(\Theta^{-1}(\Phi)) = \mu(\Phi) = 0$

Since $\Theta^{-1}(\Phi) \in \mathcal{E}$: $\mu_\Theta(\Phi) \in [0, \mu(E)]$ Yes.

If $(A_n)_{n\geq 1}$ is disjoint, $A_n \in \mathcal{E}$:

$$\mu_\Theta\Big(\bigcup_{n\geq 1} A_n\Big) = \mu\Big(\Theta^{-1}\Big(\bigcup_{n\geq 1} A_n\Big)\Big) = \mu\Big(\bigcup_{n\geq 1} \Theta^{-1}(A_n)\Big)$$

$(A_n)_{n\geq 1}$ disjoint $\Rightarrow$ $(\Theta^{-1}(A_n))_{n\geq 1}$ is disjoint Yes.

$$\mu_\Theta\Big(\bigcup_{n\geq 1} A_n\Big) = \sum_{n\geq 1} \mu(\Theta^{-1}(A_n)) = \sum_{n\geq 1} \mu_\Theta(A_n)$$

Yes.

$\therefore$ Valid measure

by countable additivity of μ

(b) If Θ is measure preserving: $\mathcal{A} \subseteq \mathcal{E} \Rightarrow \forall A \in \mathcal{A}$, $\mu(\Theta^{-1}(A)) = \mu(A)$

Yes.

If $\mu(\Theta^{-1}(A)) = \mu(A)$ for $A \in \mathcal{A}$:

Let $\mathcal{D} = \{ B \in \mathcal{E} : \mu(B) = \mu(\Theta^{-1}(B)) \}$: $\mathcal{A} \subseteq \mathcal{D}$. Yes.

If $B_1 \subseteq B_2$, $B_i \in \mathcal{D}$: $\cancel{\mu(\Theta^{-1}(B_2 \cap B_1^c)) = \mu(\Theta^{-1}(B_2) \cap \Theta^{-1}(B_1))}$

$$\mu(\Theta^{-1}(B_2 \setminus B_1)) + \underbrace{\mu(\Theta^{-1}(B_1))}_{\mu(B_1)} = \underbrace{\mu(\Theta^{-1}(B_2))}_{\mu(B_2)}$$

$$\therefore \mu(\Theta^{-1}(B_2 \setminus B_1)) = \mu(B_2) - \mu(B_1) = \mu(B_2 \setminus B_1)$$

$\therefore B_2 \setminus B_1 \in \mathcal{D}$ Yes.

If $B_n \in \mathcal{D}$, $B_n \subseteq B_{n+1}$: Countable additivity $\Rightarrow$

$$\mu\Big(\Theta^{-1}\Big(\bigcup_{n\geq 1} B_n\Big)\Big) = \lim_{n\to\infty} \mu(\Theta^{-1}(B_n)) = \lim_{n\to\infty} \mu(B_n) = \mu\Big(\bigcup_{n\geq 1} B_n\Big)$$

$\mu(\Theta^{-1}(E)) = \mu(E)$ Yes.

$\therefore \mathcal{D}$ is a d-system. Dynkin's π-lemma $\Rightarrow$ $\sigma(\mathcal{A}) = \mathcal{E} \subseteq \mathcal{D}$

$\therefore \mu(\Theta^{-1}\cdot) = \mu$ on $\mathcal{E}$. Yes.

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(c) Let $(E, \mathcal{E}, \mu)$ be finite measure space, $T: E \to E$ be measure preserving ~~ergodic~~

If $f: E \to \mathbb{R}$ is ~~me~~ integrable:

$$S_n(f) = \sum_{k=0}^{n-1} f \cdot T^{K} \quad (n \geq 1), \quad S_0(f) = 0.$$

$$S_n^*(f) = \sup_{0 \leq k \leq n} S_k(f)$$

Maximal inequality: $\int 1_{S_n^*(f) > 0} f(x)\, dx \geq 0$ for $n \geq 1$

$$\int 1_{\{\sup_{0 \leq k \leq n} S_k(f) > 0\}} f(x)\, dx \geq 0$$

Yes.

Birkhoff: $S_n(f)/n \to \bar{f}$ A.S., $\bar{f}$ is T-invariant and $\mu(|\bar{f}|) \leq \mu(|f|)$

(d) (i) ~~$\Theta^{-1}([k2^{-n}, (k+1)2^{-n}))$~~ =

Let $R_n(x)$ be the n^{th} digit in binary expansion

X: $\Theta^{-1}(\{R_1(x) = a_1, \dots R_{2N}(x) = a_{2N}\})$ (~~at~~ even

$= \cancel{\Theta^{-1}} \{R_2(x) = a_1, \dots R_{2N}(x) = a_{2N-1}, R_1(x) = a_2, \dots R_{2N-1}(x) = a_{2N}\}$ X:

$\therefore$ ~~Equal~~ measure.

~~Since $[k_x 2^{-n}, (k+1)2^{-n}) \in \sigma(R_m : m \geq 1)$ for $k \in [0, \dots 2^n - 1]$, $n \geq 1$~~

~~$\Lambda = \{\phi\} \cup \{[k 2^{-n}, (k+1)2^{-n}) : 0 \leq k < 2^n, n \geq 1\}$ is a π-system~~

~~$\Theta\, \mu_\Theta = \mu$ on Λ and $\sigma(\Lambda) = \mathcal{B}([0,1))$~~

~~$[0, a) = \bigcup_{n \geq 1} \bigcup_{0 \leq k < a \cdot 2^n - 1}$ ~~for a~~ $[k 2^{-n}, (k+1)2^{-n})$ $\Rightarrow \{\phi\} \cup \{[0, a) : a \in [0,1]\} \in \sigma(\Lambda)$~~

~~Since $\sigma(\{\phi\} \cup \{[0, a) : a \in [0,1]\})$ ~~ge~~ $= \mathcal{B}([0,1))$, $\mu_\Theta = \mu$ on $\mathcal{B}([0,1))$~~

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$\therefore$ ~~θ is measure preserving.~~

Both sets have measure $(1/2)^{2N}$. Yes.

$\therefore \mu_\theta, \mu$ agree on $\sigma\left(\bigcup_{n\geq 1} \sigma(R_1, \dots R_{2n})\right)$ (π generator of $\sigma(R_n : n\geq 1)$)

$\therefore \mu_\theta = \mu$ on $\{\sigma(R_n : n \geq 1)\}$ Yes.

But $\Lambda = \{\phi\} \cup \{[0,a) : a \in (0,1]\}$ ~~a~~ $\alpha\in\mathbb{R}$ is generated by $\{\phi\} \cup \{[k\cdot 2^{-n}, (k+1)2^{-n}) : n \in \mathbb{N}, 0 \leq k < 2^n\}$ and latter is in $\sigma(R_n : n \geq 1)$

$\therefore \mu_\theta = \mu$ on $\mathcal{B}([0,1))$ Yes.

(iii) Not ergodic: If $R_{n1}(x) = R_2(x) = 1$

$\theta^{-1}\{x: R_1(x) = R_2(x) = 1\} = \{x: R_1(x) = R_2(x) = 1\} = A$

But $|\{R_1 = R_2 = 1\}| = 1/4 \notin \{0,1\}$. Yes.

Thus we have found a set $A \in \mathcal{E}$ such that $A = \theta^{-1}(A)$, which also violates ergodicity condition.

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Probability & Measure 2018

1.) (a) $\mathbb{E}[(X-a)^2] = \mathbb{E}[(X-\mu)^2 + (\mu-a)^2 + 2(X-\mu)(\mu-a)]$,

$\mu = \mathbb{E}[X]$

Note: $|\mu| \leq \mathbb{E}[|X|] \leq \mathbb{E}[|X|^2]$ (Hölder).

$\therefore \mathbb{E}[(X-a)^2] = \underbrace{\mathbb{E}[(X-\mu)^2]}_{Var(X)} + (\mu-a)^2$ is minimised at $a=\mu$,

$\inf_{a \in \mathbb{R}} \{ \mathbb{E}[(X-a)^2] \} = Var(X)$.

(b) $\hat{f}(u) = \int_{-1}^{1} e^{iux} \, dx = \left[\frac{e^{iux}}{iu} \right]_{-1}^{1} = \frac{1}{iu} (e^{iu} - e^{-iu})$

$= \frac{i \cdot 2 \sin u}{iu} = 2 \sin u / u$

(c) We have: $\| f \|_2 = (2\pi)^{-d} \| \hat{f} \|_2$ (Parseval: $f \in L^1$)

$\therefore \int_{\mathbb{R}} |\hat{f}(u)|^2 du = 4 \int_{\mathbb{R}} \sin^2 u / u^2 \, du = (2\pi)^{2 \times 1} \cdot \int_{\mathbb{R}} 1_{[-1,1]}^2 \, dx$

$= (2\pi) \cdot 2$

$\therefore \int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = 2 \int_{0}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \pi$

$\therefore \int_{0}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \pi / 2$.

(d) $|\mathbb{E}[e^{iuX}]| \leq \mathbb{E}[|e^{iuX}|] = 1$

Equality holds $\overset{(*)}{\text{iff}}$ $e^{iuX} = \text{constant } z$ A.S.

iff $uX \in \{ \theta + 2\pi k : k \in \mathbb{Z} \}$ A.S. for some $\theta \in \mathbb{R}$

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Proof of (*):

~~$|\mathbb{E}[Z]| \le \mathbb{E}[|Z|]$~~ (Jensen's)

Equality: $|\cdot|$ must be linear on some convex A, $\mathbb{P}(Z \in A) = 1$

$|a+b| \le |a| + |b|$, equality iff $a = c \cdot b$, $c \in \mathbb{R}$ ~~c cons~~

$\therefore |Z|$ ~~$Z \in \mathbb{R}_{\geq 0}$~~

$|\lambda a| = \lambda |a|$ iff $\lambda \geq 0$.

$\therefore Z \in \mathbb{R}_{\geq 0} \cdot e^{i\theta}$ for some θ.

(ii) uX $\in \theta_1 + 2\pi k_1$ ~~$\Rightarrow vX \in v\theta/u + (2\pi k_1) v/u$~~

$vX \in \theta_2 + 2\pi k_2$

By removing a measure 0 set if necessary:

~~$\forall \omega \in \Omega$: $\frac{uX - \theta_1}{2\pi} \in k_1 \mathbb{Z}$, $\frac{vX - \theta_2}{2\pi} \in \mathbb{Z}$~~

Suppose $X(\omega_1) \neq X(\omega_2)$: $u(X(\omega_1) - X(\omega_2)) \in 2\pi k_1$

$$\therefore \frac{u(X(\omega_1) - X(\omega_2))}{v(X(\omega_1) - X(\omega_2))} \in \mathbb{Q} \Rightarrow u/v \in \mathbb{Q}$$

$\therefore$ Contradiction

$\therefore X$ is constant A.S.

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2.) (a) Hölder: $p=q=2$

$$\mathbb{E}[|Z \cdot X_n|] \le \sqrt{\mathbb{E}[|Z|^2] \cdot \mathbb{E}[|X_n|^2]} \cancel{< \infty}.$$

$$\le \sqrt{\mathbb{E}[|Z|^2]} < \infty.$$ Yes.

Also by Cauchy-Schwarz.

(b) Let $\langle X, Y \rangle = \mathbb{E}[XY]$, $\|X\|_2^2 = \mathbb{E}[X^2]$

If $\|X\|^2 = \alpha$, $\|y\|^2 = 1$, $|\langle x, y\rangle| \le 1$: $\|x+y\|^2 \le \alpha + 1 + 2|\langle x,y\rangle| = \alpha + 3$

Construct $\cancel{Y_n}$ X_{n_k} as follow:

$\cancel{Y_1 = X_1}$ $X_{n_1} = X_1$

Given $X_{n_1}, \dots X_{n_r}$ satisfying $\|X_{n_1} + \dots + X_{n_r}\|_2^2 \le 3r$:

Consider sequence: $\langle X_{n_1} + \dots + X_{n_r}, X_m \rangle_{m \ge n_r + 1}$ — (1)

Since $\|X_{n_1} + \dots + X_{n_r}\|_2^2 < \infty$, condition implies (1) $\to 0$ as $m \to \infty$

$\therefore$ Pick n_{r+1} s.t. $|(1)| \le 1$ Yes.

$$\therefore \|X_{n_1} + \dots + X_{n_{r+1}}\|_2^2 \le 3r + \|X_{n_{r+1}}\|^2 + 2 \le 3(r+1)$$ Yes.

$$\left\|\frac{1}{N} Y_N\right\|_2^2 = \frac{1}{N^2}\left\|\sum_{k=1}^{N} X_{n_k}\right\|_2^2 \le 3N/N^2 \to 0.$$

$\therefore$ Done. Yes.

(c) $X_n \to X$ in $\mathbb{P}$ probability: $\exists$ subsequence X_{n_k} s.t. $X_{n_k} \to X$ A.S. Yes.

$$\mathbb{E}[\liminf_k |X_{n_k}|^2] \le \liminf_k \mathbb{E}[|X_{n_k}|^2] \le 1$$ (Fatou)

$\therefore \mathbb{E}[|X|^2] \le 1 \Rightarrow X \in \mathcal{L}^2$ Yes.

(d) $\forall$ $\|X_n - X\|_1 = \|(X_n - X)\mathbb{1}_{|X_n - X| \ge M}\|_1 + \|(X_n - X)\mathbb{1}_{|X_n - X| < M}\|_1$

$$\le (\text{Hölder}) \underbrace{\|X_n - X\|_2}_{\le \|X_n\|_2 + \|X\|_2 \le 2} \cdot \|\mathbb{1}_{|X_n - X| \ge M}\|_2 + \|(X_n - X)\mathbb{1}_{|X_n - X| < M}\|_1$$ Yes.

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$$\leq 2\, \mathbb{P}(|X_n - X| \geq M)^{1/2} + \mathbb{E}[\,|X_n - X|\, 1_{|X_n - X| \leq M}] \quad -(2)$$

Yes.

Dominated conv. theorem: $|X_n - X| \cdot 1_{|X_n - X| \leq M} \leq M$, M integrable due to finite measure

$\therefore \lim_{n\to\infty} \mathbb{E}[\,|X_n - X|\, 1_{|X_n - X| \leq M}] = 0.$ More arguments?

$X_n \xrightarrow{\mathbb{P}} X \Rightarrow \mathbb{P}(|X_n - X| \geq M)^{1/2} \to 0$ Yes.

$\therefore \lim_{n\to\infty} (2) = 0$ not the clearest notation

$\therefore X_n \xrightarrow{L^1} X.$ Yes.

Convergence in L^2: Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}((0,1))$, $\mathbb{P}$ = Lebesgue measure

$\therefore X_n = 1_{(0,1/n)} \cdot \sqrt{n}$

$\|X_n\|_2 = \left(\int_0^{1/n} n \, dx\right)^{1/2} = 1$; Yes. $\quad X_n \to 0$ everywhere $\Rightarrow X_n \to 0$ in $\mathbb{P}$ Yes.

But $\|X_n\|_2 = 1 \not\to 0$. $\therefore X_n \not\xrightarrow{L^2} 0$. Yes.

(d) $\mathbb{E}[Y_N^2] = \mathbb{E}[(Y_N - \mathbb{E}[Y_N])^2] + \cancel{\mathbb{E}[}\mathbb{E}[Y_N]^2$ not entirely clear

~~Var~~ $\cancel{\mathbb{E}[(Y_N}$ $\mathrm{Var}(Y_N) \leq 1/N \to 0$. Yes, by independence of X_i

$\therefore$ $\cancel{\mathbb{E}[Y_N^2] \to}$

$$\mathbb{E}[(Y_n - Y)^2] = \mathbb{E}[(Y_n - \mathbb{E}[Y_n])^2] + \mathbb{E}[(Y - \mathbb{E}[Y])^2] + (\mathbb{E}[Y] - \mathbb{E}[Y_n])^2$$

Yes.

1st term $= 1/n \to 0$ as $n \to \infty$

2nd term $= \mathrm{Var}(Y) \geq 0$

3rd term ≥ 0

$Y_n \xrightarrow{L^2} Y$ iff Y is constant, & $\mathbb{E}[Y_n] \to Y$. Yes.

$\therefore$ Condition: $\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}[X_k]$ is convergent as $N \to \infty$

Yes.

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3.) (a) Let T be the set of all open sets of $\mathbb{R}$:

$\sigma(T)$ is the smallest σ-algebra containing T

E is borel iff $E \in \sigma(T)$ Yes.

(b) E is Lebesgue measurable iff ~~$\forall A \subseteq \mathbb{R}: m(A) = m(A \cap E) + m(A \cap E^c)$~~ $= \text{Inf}\{m(X):$

~~(b)~~

$\forall A \subseteq \mathbb{R}, \quad m(A) = m(E \cap A) + m(E^c \cap A)$ Yes.

(c) $\forall q \in \mathbb{Q}: \{q\}$ is closed $\Rightarrow \in \mathcal{B}$ Yes.

$\therefore \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \in \mathcal{B}$ ($\mathbb{Q}$ is countable, $\mathcal{B}$ closed wrt countable unions)

$$\underbrace{|\mathbb{Q}|}_{m(\mathbb{Q})} = \sum_{q \in \mathbb{Q}} \underbrace{|\{q\}|}_{m(\{q\})} = 0$$ Yes.

Since $\mathbb{Q} \in \mathcal{B}$: $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{B}$ (closed wrt complements) Yes.

$|\mathbb{Q}| + |\mathbb{R} \setminus \mathbb{Q}| = |\mathbb{R}| = \infty \Rightarrow |\mathbb{R} \setminus \mathbb{Q}| = \infty$ Yes.

$\Rightarrow [1/n, n] \in \mathcal{B}$

Each $[1/n, n]$ is closed $\Rightarrow \bigcap_{n \geq 2} [1/n, n] \in \mathcal{B}$ ($\mathcal{B}$ closed wrt countable $\cap$)

$|\bigcap_{n} [1/n, n]| = |[1/2, 2]|$ as $[1/n, n] \subseteq [1/{n+1}, n+1]$

$\therefore |\bigcap_{n} [1/n, n]| = 3/2$ Yes.

(d) $f: \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable if:

$\forall A \in \mathcal{B}(\mathbb{R}), \quad f^{-1}(A) \in \mathcal{B}(\mathbb{R})$ Yes.

(e) For $k \in \mathbb{N}$: $G_n = \{ x \in \mathbb{R} : \exists \delta > 0$ s.t. $\forall y_i \in D(x, \delta)$, $|y_1 - y_2| < 1/n$ ~~...~~ $\}$

$|f(y_1) - f(y_2)| < \frac{1}{n}$

δ could be written as $\frac{1}{m}$ for the sake of countability

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Claim: G_n is an open set.

Let $x \in G_n$: $\exists\ \delta_x > 0$ s.t. $y_i \in D(x, \delta_x) \Rightarrow |f(y_1) - f(y_2)| < 1/n$

$\therefore$ Let $z \in D(x, \delta_x/4)$: $\forall\ w_i \in D(z, \delta_x/4)$: $w_i \in D(x, \delta_x)$

$\therefore |f(w_1) - f(w_2)| < 1/n \Rightarrow D(x, \delta_x/4) \in G_n$

$\therefore G_n$ open $\Rightarrow \in \mathcal{B}(\mathbb{R})$ Yes.

Claim: $\{x : f \text{ cont. at } x\} = \bigcap_{n \geq 1} G_n$ Yes.

f cont. at x iff $\forall\ n \in \mathbb{N}$, $\exists\ \delta_{x,n} > 0$ s.t. $\forall z \in D(x, \delta_{x,n})$, $|f(z) - f(x)| < 1/n$ ~~for~~ ~~$\forall n \in \mathbb{N}$~~, Yes.

$|f(z_1) - f(z_2)| \leq |f(z_1) - f(x)| + |f(z_2) - f(x)| < 2/n$ Yes.

$\therefore$ Equivalent to : $\forall\ n \in \mathbb{N}$, $\exists\ \delta_{x,n} > 0$ s.t. $\forall z_i \in D(x, \delta_{x,n})$, $|f(z_1) - f(z_2)| < 1/n$.

$\therefore$ Done Yes.

$G_n \in \mathcal{B}(\mathbb{R}) \Rightarrow \bigcap_{n\geq 1} G_n \in \mathcal{B}(\mathbb{R}) \Rightarrow$ Borel set. Yes.

(f) Suppose $0 < \beta$, $\beta \notin E - E$: $\forall\ e \in E$, $\beta + e \notin E \Rightarrow (\beta + E) \cap E = \phi$

~~$\beta \in E - E$ iff $-\beta \in E - E$~~

~~$\therefore (\beta + E \cup -\beta + E) \cap E = \phi$~~

~~Consider: $[0,1] \cap \{(\beta + E) \cup (-\beta + E)\}$~~

~~If $\beta \in [0, 1/2)$: $[0,1] \cap (\beta + E) \supseteq ([0, 1-\beta] \cap E) + \beta$~~

~~$[0,1] \cap (-\beta + E) \supseteq ([\beta, 1] \cap E) - \beta$~~

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Pick (a_k, b_k) s.t $\sum_{k=1}^{\infty} |(a_k, b_k)| \leq |E| + \varepsilon_1$,

$(1-\varepsilon)|E| + (1-\varepsilon)\cdot\varepsilon_1 \overset{?}{\cancel{<}} |E|$ ($>$?)

(WLOG): $|E| \in (0, \infty)$. E l.x, consider $E \cap (n, n+1]$, $n \in \mathbb{Z}$.

One of the intervals has measure in $(0, \infty)$

$\therefore \left(\varepsilon_1 < \dfrac{\varepsilon |E|}{1-\varepsilon} \right)$

$\therefore$ Contradiction Yes.

$\therefore \exists\ a, b$ s.t $|E \cap (a,b)| > (1-\varepsilon)|a,b|$ Yes.

By translating and scaling: $E_1 = E \cap (a,b)$ $\therefore$ Pic, $(\varepsilon = 1/4)$
Yes.

$1/b\,(E_1 - a) = 1/b\,(-a + E) \cap (0,1)$, $|1/b\,(-a+E) \cap (0,1)| > 3/4$.

Can use f) part here.

$3/4$ rescale the found interval to $(0,1)$? (and also translate)

$\therefore (-1,1) \subseteq {-a+E}/b - {-a+E}/b = {E-E}/b$

$\therefore (-b, b) \subseteq E - E \Rightarrow \exists$ open interval around 0.

$\dfrac{1}{b-a}$ for arriving at $(0,1)$ interval?

$[0,1] \cap (\beta + E) = \beta + ([0, 1-\beta] \cap E) \subseteq [\beta, 1]$ - (1)

$[0,1] \cap (-\beta + E) = -\beta + ([\beta, 1] \cap E) \subseteq [0, 1-\beta]$ - (2)

(1), (2) : ~~are~~ ~~disjoint~~ $\pm\beta + E$ disjoint from $E \Rightarrow$

$\mu((1)), \mu((2)) \le 1 - \mu(E) = 1/2 - \alpha$

$\therefore \mu(E \cap [0, 1-\beta]), \mu(E \cap [\beta, 1]) \le 1/2 - \alpha$

$\mu(\beta + E) = \mu(E)$ (Extend Lebesgue measure to $\mathbb{R}$) Yes.

$\therefore \mu((\beta + E) \cup E) = 1 + 2\alpha \Rightarrow \mu((\beta + E \cup E) \cap [0,1]) + \mu((\beta + E \cup E) \cap (1, \infty))$

$= 1 + 2\alpha$ Yes.

We assume that $\beta \notin E - E$ and thus $\beta + E$ and E are disjoint

1st term $\le 1 \Rightarrow$ 2nd term $\ge 2\alpha \Rightarrow \mu(\beta + E) \ge 2\alpha \Rightarrow \beta \ge 2\alpha$

Yes.

But $\beta \in E - E$ iff $-\beta \in E - E$.

$\therefore (-2\alpha, 2\alpha) \subseteq E - E$ Yes. Initially can assume that $\beta \in (-2\alpha, 2\alpha)$.

(9) Since E is a borel set, $\mathcal{B}(\mathbb{R})$ generated by the set of all finite unions of open intervals.

$\therefore \forall E \in \mathcal{B}(\mathbb{R})$, $\exists$ ~~disjoint~~ (a_k, b_k) s.t.

$E \subseteq \bigcup_{k \ge 1} (a_k, b_k)$, $\mu(E) + \varepsilon \ge \sum_{k=1}^{\infty} \mu(a_k, b_k)$ Yes.

$\therefore \cancel{\exists N \in \mathbb{N} \text{ s.t. } \mu(E) + \varepsilon \ge \sum_{k=1}^{N} \mu(a_k, b_k) \ge \mu(E) - \varepsilon}$

Suppose: $\exists \varepsilon > 0$ s.t. $\forall a, b \in \mathbb{R}$, $|E \cap (a,b)| \le (1-\varepsilon)|(a,b)|$

$\therefore \sum_{k=1}^{\infty} |(a_k, b_k)| \ge (1-\varepsilon)^{-1} \sum_{k=1}^{\infty} |(a_k, b_k) \cap E| \ge (1-\varepsilon)^{-1} |E \cap \bigcup_{k=1}^{\infty} (a_k, b_k)|$

$= (1-\varepsilon)^{-1} |E|$

Yes.

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4.) (a) (i) $\forall A \in \mathcal{A}$, $\mu^*(A) = \mu(T^{-1}(A))$ Yes. ✓

(ii) $\forall A \in \mathcal{A}$: $T^{-1}(A) = A \Rightarrow \mu(A) \in \{0,1\}$

$\mu(A) = 0$ or $\mu(A^c) = 0$ as we know nothing about the measure

(b) Let $f: X \to \mathbb{R}$ be integrable, $S_n(f) = \sum_{k=0}^{n-1} f \circ T^{(k)}$

$\frac{1}{n} S_n(f) \to \tilde{f}$ A.S., $\tilde{f}$ is T- invariant.

We are working on a 6-finite measure space and with T being a measure preserving transformation. And $\int |\tilde{f}| \, d\mu \le \int |f| \, d\mu$.

(c) $X = [0,1)$, $\mathcal{A} = \mathcal{B}([0,1))$, $\mu =$ Lebesgue measure

$T(x) = 2x \pmod 1$

T is measure preserving:

$T^{-1}\left([k \cdot 2^{-n}, (k+1) 2^{-n}) \right) = [k \cdot 2^{-n-1}, (k+1) 2^{-n-1}) \cup \left(\tfrac{1}{2} + [k \cdot 2^{-n-1}, (k+1) 2^{-n-1}) \right)$

$\therefore \mu(T^{-1}[k \cdot 2^{-n}, (k+1) 2^{-n})) = \mu([k \cdot 2^{-n}, (k+1) 2^{-n}))$

Yes.

Since $\mathcal{B}([0,1))$ is generated by π system:

$\Lambda = \{ [k \cdot 2^{-n}, (k+1) 2^{-n}) : n \in \mathbb{N}, 0 \le k \le 2^n - 1 \} \cup \{\phi\}$,

measures $\mu(\cdot)$, $\mu(T^{-1}(\cdot))$ agree on $\Lambda \Rightarrow \mu \equiv \mu(T^{-1}(\cdot))$

Yes.

~~x ↦ 2x~~

Consider $(0,1) \to (0,2) \to (0,1)$

$x \xrightarrow{\alpha} 2x \xrightarrow{\beta} 2x \pmod 1$

α is cont. $\Rightarrow$ measurable

$\beta = (id - 1_{x \ge 1})$ is sum of measurable functions

$\Big\}$ $T = \beta \circ \alpha$ is measurable.

Yes.

$T^{-1}\{x\} = \{ x/2, \ 1/2 + x/2 \}$, $\therefore |T^{-1}\{x\}| = 2 > 1$

Yes.

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(d) Claim : T is surjective.

If $x \notin \mathrm{Im}(T)$: $T^{-1}\{x\} = \phi \Rightarrow \mu(T^{-1}\{x\}) = \mu(\{x\}) = 0$.

$\therefore$ Contradiction. Yes.

X finite : T surjective iff T bijective Yes.

(e) $\mu(\mathbb{N}) = 1 = \sum_{n \geq 1} \mu(\{n\}) \Rightarrow \exists\ N$ s.t. $\mu(N) > 0$.

Yes.

Let $x_0 = \mu$ N : $X_r = T^{(r)}(N)$

$\mu(\{x_{r+1}\}) = \mu^{*}(T^{-1}(x_{r+1})) \geq \mu(\{x_r\}) \Rightarrow \mu(\{x_r\}) \geq \mu(\{x_0\})$

Yes.

If $(x_n)_{n\geq 0}$ distinct, $\mu(\mathbb{N}) \geq \sum_{n\geq 0} \mu(x_0) = \infty$ (reject)

$\therefore \exists M \in \mathbb{N}$, $R \in \mathbb{N}$ s.t. $x_M = x_R$, $R < M$, $\{x_R, \dots x_{M-1}\}$ distinct.

if necessary

WLOG (Redefine x_0) : $R = 0$ Yes.

$\mu(\{x_0\}) = \mu(T^{-1}\{x_0\}) \geq \mu(\{x_{M-1}\}) \dots \geq \mu(\{x_0\})$.

$\therefore$ Equality holds Yes.

$\therefore\ 0 \leq n \leq M-1$: $\mu(\{T^{-1}\{x_n\} \setminus \{x_{n-1\ (\mathrm{mod}\ M)}\}) = 0$.

$\forall m \geq 0$: $T^{-m}\{x_n\}$ contains $x_{n-m\ (\mathrm{mod}\ M)}$

$\therefore \mu(T^{-m}\{x_n\} \setminus \{x_n\}) = 0$

$\Lambda = \bigcup_{m\geq 0} T^{-m}\{x_0, \dots x_{M-1}\}$ is T-invariant.

$\mu(\Lambda) \geq \mu(x_0) > 0 \Rightarrow \mu(\Lambda) = 1$

But $\mu(\Lambda) = \mu(\{x_0, \dots x_{M-1}\}) + \mu(\bigcup_{m\geq 0} (T^{-m}\{x_0, \dots x_{M-1}\} \setminus \{x_0, \dots x_{M-1}\}))$

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Let $\Lambda = \{ x_0, \dots x_{M-1} \}$:

$\forall r \geq 0, \forall x_k \in \Lambda : \mu(T^{-r}\{x_k\}) = \mu(\{x_k\})$

$x_{k-r \pmod M} \in T^{-r}\{x_k\} \Rightarrow \mu(T^{-r}\{x_k\} \setminus \Lambda) = 0.$

by periodicity?

$\therefore \mu(\bigcup_{r\geq 0} T^{-r}\{x_k\} \setminus \Lambda) = 0 \Rightarrow \mu(\bigcup_{r \geq 0} T^{-r}(\Lambda) \setminus \Lambda) = 0.$

Yes.

But $\bigcup_{r \geq 0} T^{-r}(\Lambda)$ is T-invariant:

$\Lambda \subseteq T^{-r}(\Lambda)$: $T^{-1}(\bigcup_{r\geq 0} T^{-r}(\Lambda)) = \bigcup_{r \geq 1} T^{-r}(\Lambda)$

for a particular r value?

$= \bigcup_{r \geq 0} T^{-r}(\Lambda)$

Yes.

$\therefore \mu(\bigcup_{r\geq 0} T^{-r}(\Lambda)) = 0, 1$; But LHS $\geq \mu(\{x_0\}) > 0 \Rightarrow = 1$

Yes.

$\therefore \mu(\Lambda) + \underbrace{\mu(\bigcup_{r\geq 0} T^{-r}(\Lambda) \setminus \Lambda)}_{0} = 1 \Rightarrow \mu(\Lambda) = 1$

$|\Lambda| < \infty \Rightarrow$ Done. Yes.

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No.: Dace Ruklisa   Date: Queens'

Probability & Measure 2019

1.) (a) $d=1$: $\mathbb{E}[e^{iux}] = \int_{\mathbb{R}} e^{iux} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = f(u)/\sqrt{2\pi}$ Yes.

Since integrand is differentiable, derivative integrable: ← integrable derivative could be justified

$$f'(u) = i\int_{\mathbb{R}} x e^{iux} e^{-x^2/2}\,dx = i\left[[-e^{iux}e^{-x^2/2}]_{\mathbb{R}} + iu\int_{\mathbb{R}} e^{iux}e^{-x^2/2}\,dx \right]$$ Yes.

$$= -u\, f(u)$$ Yes.

$$\therefore f(u) = e^{-u^2/2}\cdot A\ ;\ f(0) = \sqrt{2\pi}$$ Yes.

$$\therefore \mathbb{E}[e^{iux}] = e^{-u^2/2}.$$ $\sigma^2 = 1$; $\sigma\cdot N(0,1) \sim N(0,\sigma^2)$

$$\therefore \mathbb{E}[e^{iu N(0,\sigma^2)}] = e^{-\sigma^2 u^2/2}$$ Yes.

↑ random variable could be used

General d: Since individual components have 0 covariance, they are independent as X is Gaussian

$$\therefore \mathbb{E}[e^{i\underline{u}\cdot\underline{x}}] = \prod_{j=1}^{d} \mathbb{E}[e^{iu_j\cdot x_j}] = \prod_{j=1}^{d} e^{-u_j^2/2\,\cdot\,\sigma^2}$$

$$= e^{-(\|u\|^2/2)\cdot\sigma^2}$$ Yes.

(b) $\Phi_X(u) = \mathbb{E}[e^{i\underline{X}^T\cdot\underline{u}}]$:

Claim: $\|u_1\|_2 = \|u_2\| \Rightarrow \Phi_X(u_1) = \Phi_X(u_2)$

Pick orthogonal A s.t. $A\cdot u_1 = u_2$.

$$\therefore \Phi_X(u_2) = \mathbb{E}[e^{iX^T A\cdot u_1}] = \mathbb{E}[e^{i(A^T X)^T\cdot u_1}]$$ Yes.

Since law of X, $A^T X$ is the same, $\Phi_X(u_2) = \Phi_{A^T X}(u_1)$
$= \Phi_X(u_1)$ //

$\therefore \Phi_X(u)$ is a function of $\|u\|^2 = u_1^2 + \dots + u_d^2$ Yes.

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$\Phi_X(u) = f(\|u\|^2)$

Claim: f is cont.

Also note that f takes real values as $\phi(u) = \phi(-u) = \overline{\phi(u)}$ Yes.

Let $r_n \geq 0$, $r_n \to r$ $(r_n^2 \to r^2)$

Since Φ_X is continuous: $\Phi_X(r_n(1,0,\dots)) \to \Phi_X(r(1,0,\dots))$

$\therefore f(r_n^2) \to f(r^2) \Rightarrow f$ is cont. ~~($f(r) = \Phi_X(\sqrt{r}, 0, \dots) \Rightarrow f$ defined on $[0,\infty)$)~~ Yes.

(ii) $f(r_1 + r_2) = \Phi_X((\sqrt{r_1}, \sqrt{r_2}, 0 \dots))$

$= \mathbb{E}[e^{i\sqrt{r_1}X_1} \cdot e^{i\sqrt{r_2}X_2}]$

$= \mathbb{E}[e^{i\sqrt{r_1}X_1}] \cdot \mathbb{E}[e^{i\sqrt{r_2}X_2}]$ (independence of X_1, X_2)

$= \Phi_X((\sqrt{r_1}, 0 \dots)) \cdot \Phi_X((0, \sqrt{r_2}, 0 \dots))$

$= f(r_1) \cdot f(r_2)$ Yes.

(iii) (i) Claim: $f : [0,\infty) \to \mathbb{R}$

$\|u\|^2 \in [0,\infty) \Rightarrow \mathrm{Dom}(f) \subseteq [0,\infty)$

$\|(\sqrt{r}, 0, \dots)\| = r \Rightarrow \mathrm{Dom}(f) = [0,\infty)$

We have $\mathbb{E}[e^{ir \cdot X_k}] = \mathbb{E}[e^{-irX_k}] \Rightarrow$ Im. part $= 0$.

$\therefore \mathrm{Im}(f) \subseteq \mathbb{R}$ Yes.

(iii) $|\mathbb{E}[e^{iu \cdot X}]| \leq \mathbb{E}[|e^{iu \cdot X}|] = 1$ Yes.

$\therefore$

~~$\forall r \geq 0$: $f(r/2)^2 = f(r) \Rightarrow f(r) \geq 0$~~

~~If $f(r) = 0$: $f(r+s) = f(r) \cdot f(s) = 0$ $(\forall s \geq 0)$~~

$\forall r \geq 0$: $f(r) = f(r/2)^2 \geq 0$ by ii) part

If $\exists r \geq 0$, $f(r) = 0$: $f(r) = 0 \Rightarrow f(r/2)^2 = 0 \Rightarrow f(r/2) = 0$.

$\therefore f(r/2^n) = 0$, $r/2^n \to 0$

for $n \in \mathbb{N}$

f cont. $\Rightarrow f(r/2^n) \to f(0) = 1 \neq 0$.

$\therefore f(r) \neq 0 \Rightarrow f(r) \in (0,1]$ Yes.

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From analysis: $g : [0, \infty) \to \mathbb{R}$ satisfy $g(a+b) = g(a) + g(b)$,

g cont. $\Rightarrow \exists\, c$ s.t. $g(x) = cx$

$-\log(f) \in \cancel{(-\infty, 0]}\ [0, \infty)$

$-\log(f(r_1 + r_2)) = -\log(f(r_1)) - \log(f(r_2)) \Rightarrow -\log(f(r)) = cr.$

$\therefore f(r) = e^{-cr}$, $c \in \mathbb{R}$: $f(r) \in (0,1] \Rightarrow c \geq 0$ Yes.

(iv) $\phi_X(u) = e^{-|u|^2/2 \cdot (2c)}$

Since characteristic function determines law:

$X \sim N(0, I_d \cdot 2c)$. Yes.

(* ?): If $\operatorname{Var}(X_i) = \sigma^2$, $2c = \sigma^2 \Rightarrow X \sim N(0, \sigma^2)$

When $c=0$ the law of X is a Dirac mass at 0.

2) (a) ~~or~~ $\cancel{A_1 \otimes A_2 = \sigma(\{E_1 \times E_2 : E_i \in A_i}$

$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\{A_1 \times A_2 : A_i \in \mathcal{A}_i\})$ is the product σ-algebra

$\mu_1 \otimes \mu_2(A) = \mu_1(x \mapsto \mu_2(y \mapsto 1_A(x,y)))$

If $A_i \in \mathcal{A}_i$: $\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(x \mapsto \mu_2(y \mapsto 1_{A_1}(x) \cdot 1_{A_2}(y)))$

$= \mu_1(x \mapsto 1_{A_1}(x))\ \mu_2(y \mapsto 1_{A_2}(y))$

$\therefore$ Well defined on $\Lambda = \{A_1 \times A_2 : A_i \in \mathcal{A}_i\}$

i.e. for the collection of all rectangles

Since Λ is a π-system, any product measure is finite (μ_i are probability measures), product measure is uniquely defined.

Yes.

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(b) U has law μ:

μ: Let $X : \Omega \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be R.V.

$\mu_X = \mathbb{P}(X^{-1}(\cdot))$

μ_X is defined on $\mathcal{B}(\mathbb{R})$ Yes, and this is the law of X.

U, V independent if: $\forall A_1, A_2 \in \mathcal{B}(\mathbb{R})$,

Yes. $\mathbb{P}(U \in A_1, V \in A_2) = \mathbb{P}(U \in A_1)\mathbb{P}(V \in A_2)$

or $\mu_X(\langle X_1, \dots, X_n \rangle) = \mu_{X_1}(X_1) \times \dots \times \mu_{X_n}(X_n)$ for laws

(c) Consider the measure:

$$\cancel{V(A) = \int_0^{1/2}\int_0^{1/2} \mu_1 \otimes \mu_2 (A \cdot 1_{xy})}$$

$$\cancel{= 2\mu_1 \otimes \mu_2 \left((x,y) \to 1_A(x,y) \cdot 1_{xy \geq 0} \right)}$$

$$\cancel{V(A_1 \times X) = 2\mu_1(A_1) \cdot \mu_2([0, 1/2]) = \mu(A_1)}$$

$$\cancel{V(A_2 \times X \times A_1) = 2\mu_1(A_1) \cdot \mu(}$$

~~Let A ∈~~

$$V(A) = 2\int_{[0,1/2]^2 \cup [-1/2,0]^2} 1_A \, dx\, dy \quad (\text{Lebesgue measure } dx\, dy)$$

Yes.

$$\forall B \in \mathcal{A}: \quad V(B \times X) = \int_{[-1/2,1/2]} 1_B(x) \underbrace{\int_{[-1/2,1/2]} 1_{xy \geq 0} \cdot 2 \, dy}_{= 1_{x \neq 0}} \, dx$$

by Fubini's theorem

$$= \int 1_B(x)\, 1_{x \neq 0} \, dx = \mu(B)$$

Similarly $V(X \times B) = \mu(B)$ Yes.

$$V(\cancel{[-3/4, -3/4] \times [3/4}\ [-1/2, -1/4] \times [1/4, 1/2]) = 0$$

$\therefore$ Not lebesgue measure Yes.

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(d) $n(I\times J) \le n(I\times X) + n(X\times J) + \cancel{|X|+|J|} \Rightarrow n(I\times J) \le |I|,\ |J|$

$|I|+|J| = n(I\times X) + n(X\times J) = n(I\times \overset{X}{\cancel{J}}) + n(I^c\times J) + n(I\times J)$

$\le n(I\times J) + n(I\times X) + n(I^c\times X) = 1 + n(I\times J)$ Yes.

$\therefore n(I\times J) \in [\,|I|+|J|-1,\ \operatorname{Min}\{|I|, |J|\}\,]$

Yes.

(e) Let $U_1, \dots U_d$ be iid $[0, 1/2]$ R.V.,

$\varepsilon_1, \dots \varepsilon_{d-1}$ be iid R.V., $\mathbb{P}(\varepsilon_1 = -1) = \mathbb{P}(\varepsilon_1 = 1) = 1/2$.

$\varepsilon_d = 1/\varepsilon_1 \dots \varepsilon_{d-1}$ ~~$(\mathbb{P}(\varepsilon_d = 1) = \mathbb{P}(\varepsilon_d = -1) = 1/2)$~~

Yes. Ver Calculate EV:

Let $X_k = U_k \cdot \varepsilon_k$:

$X_k \sim U[-1/2, 1/2]$: $\mathbb{P}(X_k \le r)$, $r<0$ $= \mathbb{P}(\varepsilon_k = -1,\ \overset{U_k}{\cancel{X_k}} \ge -r)$

$= \frac{1}{2}\cdot 2\ (\frac{1}{2}+r) = r - (-1/2)$

$r \ge 0$: $\mathbb{P}(\varepsilon_k = 1,\ U_k \le r) + \mathbb{P}(\varepsilon_k = -1)$

$= \frac{1}{2} + \frac{1}{2}\cdot 2r = r - (-1/2)$

Yes.

$*$ ε_d same dist as ε_1: $\mathbb{P}(\varepsilon_d = 1) = \mathbb{E}[\,\mathbb{E}[\mathbb{1}_{\varepsilon_1 \dots \varepsilon_{d-1} = 1} \mid \varepsilon_2, \dots \varepsilon_{d-1}]]$

$= 1/2$.

Yes.

Also: $(\varepsilon_1, \dots \varepsilon_{d-1})$ independent.

$(\varepsilon_1, \dots \varepsilon_{k-1}, \varepsilon_{k+1}, \dots \varepsilon_d)$ is also independent Yes.

$\mathbb{P}(\varepsilon_1 = a_1, \dots \varepsilon_{k-1} = a_{k-1}, \varepsilon_{k+1} = a_{k+1}, \dots \varepsilon_d = a_d)$

$= (\frac{1}{2})^{d-2} \cdot \mathbb{P}(\varepsilon_k = 1/a_1 \dots a_{k-1} a_{k+1} \dots a_{d-1} a_d) = (\frac{1}{2})^{d-1}$

$= \prod_{r : r \ne k} \mathbb{P}(\varepsilon_r = a_r)$ Yes.

Projection to Π_i: $(X_1 \times \dots \times X_{k-1} \times X_{k+1} \times \dots \times X_d) \sim (U[-1/2, 1/2])^{d-2}$ (independent)

$\therefore$ Lebesgue measure

Yes.

But $\mathbb{P}(X_k \in [-1/2, 0] : 1 \le k \le d) = 0 \Rightarrow$ Not lebesgue.

odd and even d values would require different examples

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3.) (a) X, Y have the same law iff $\forall$ bounded, cont. $f: \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ Yes.

Let $\chi_n \in C_c^\infty$, $\mathrm{Supp}(\chi_n) \subseteq \bar{D}(0, n+1)$, $\chi_n = 1$ on $\bar{D}(0,n)$, $\chi_n \in [0,1]$ Yes.

$\therefore \chi_n \cdot f = f_n$ is compactly supported on $\bar{D}(0, n+1)$, $\chi_n \cdot f$ f_n is cont.

$$\therefore \mathbb{E}[\chi_n \cdot f(X)] = \mathbb{E}[\chi_n \cdot f(Y)]$$ Yes.

Let $M = \|f\|_\infty$: $|\chi_n \cdot f(X)| \leq M$ (Dominating) Convergence

$$\therefore \lim_{n\to\infty} \mathbb{E}[\chi_n \cdot f(X)] = \mathbb{E}[\lim_{n\to\infty} \chi_n f(X)] = \mathbb{E}[f(X)]$$

Similarly: $\mathbb{E}[\chi_n \cdot f(Y)] \to \mathbb{E}[f(Y)]$ as $n \to \infty$

$$\therefore \mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$$ Yes.

(b) Fix $\varepsilon > 0$:

$$RHS = \int \hat{g}(t) \int f(Z - \varepsilon t) \, \mu_Z(dz) \cdot dt$$

$$= \iiint g(u) e^{iut} f(Z - \varepsilon t) \, du \, \mu_Z(dz) \, dt$$ Yes.

Sub $t = \frac{1}{\varepsilon}(Z - x)$: $dt = -\frac{1}{\varepsilon} dx$

$$-\frac{1}{\varepsilon} \iiint g(u) e^{i(\frac{u}{\varepsilon})(Z-x)} f(x) \, dx \, du \, \mu_Z(dZ) \quad \text{(Fubini)}$$

variable substitution? $= (u \to s \cdot \varepsilon) = \iiint g(\varepsilon \cdot s) e^{isz} e^{-isx} f(x) \, dx \, ds \, \mu_Z(dz)$

$$= \text{(Fubini)} \iint g(\varepsilon \cdot s) e^{-isx} \varphi_Z(s) \, ds \, dx.$$ $f(x)$

Finite integrals could be justified for Fubini's theorem.

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Take limit $t \to 0$ on both side:

~~LHS: $|\hat{g}(t)| \cdot |\mathbb{E}[f(Z-\varepsilon t)]| \leq \|f\|_\infty \cdot |\hat{g}(t)|$ integrable ($\forall \varepsilon > 0$)~~

~~RHS: $g(\varepsilon s) \|f\| \cdot |\varphi_Z(s)| \leq \|f\|_\infty$~~

~~$\therefore$ (Dominated convergence theorem (DCT)):~~

~~$\lim_{\varepsilon \to 0} LHS = \mathbb{E}[f(Z$~~

$|\hat{g}(t)| \cdot |f(Z-\varepsilon t)| \leq \|f\|_\infty |\hat{g}(t)|$ is integrable Yes.

$\therefore$ Dominated convergence theorem (DCT):

$$\lim_{\varepsilon \to 0} LHS = \int \mathbb{E}[\lim_{\varepsilon \to 0} \hat{g}(t) f(Z-\varepsilon t)]\, dt = \int \hat{g}(t)\, \mathbb{E}[f(Z)]\, dt$$

$$= \|\hat{g}\|_1 \cdot \mathbb{E}[f(Z)]$$ Yes.

~~$RHS = \frac{1}{\varepsilon} \iint g(s) e^{-i(s/\varepsilon)x} \varphi_Z(s/\varepsilon)\, ds\, dz$~~

~~$\int \{g(\varepsilon s)\} * \hat{\varphi}_Z(s)\}$~~

$$\therefore \mathbb{E}[f(Z)] = \frac{1}{\|\hat{g}\|_1} \lim_{\varepsilon \to 0} \iint g(\varepsilon s) f(x) e^{-isx} \varphi_Z(s)\, ds\, dz$$

$\therefore \mathbb{E}[f(Z)]$ is determined by $\varphi_Z \Rightarrow$ Law of Z determined by φ_Z.

* Pick $g \neq 0$, s.t. $g = \hat{g}(x) \Rightarrow \hat{g} \neq 0$. Yes.

By part a) if $\varphi_X = \varphi_Y$, then X and Y have the same law.

$\therefore$ ~~$\mathbb{E}$~~

(c) $\varphi_Z(s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(isZ)^k}{k!}\right]$; Since power series of e^{ix} is uniformly convergent on compact set, Z is bounded: ($\mathbb{E}$ integrals over compact set)

Yes.

Swap $\mathbb{E}, \sum$: $\varphi_Z(s) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!} \mathbb{E}[Z^k]$

Could be justified by dominated convergence Yes.

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$\therefore \varphi_Z$ determined by $\{ \mathbb{E}[Z^k], k \geq 0 \}$

$\therefore$ If X, Y have equal moments : X, Y have the same law. Yes.

(d) Since $X \geq 0$: ~~X^i~~ ~~$X \ln$~~ ~~e~~ $Y = e^{-X}$ is ≥ 0, bounded.

$\therefore$ ~~$\forall$~~ Law of Y is determined by $\mathbb{E}[Y^k]$, $k \geq 0$.

$\{ \mathbb{E}[e^{-kX}] : k \in \mathbb{N} \cup \{0\} \} = \{ \varphi_X(k) : k \in \mathbb{Z}_{\geq 0} \}$?

$\therefore \varphi_{X_1} = \varphi_{X_2} \Rightarrow e^{-X_1}, e^{-X_2}$ have the same law

$\Rightarrow X_1, X_2$ have the same law ($z \to e^{-z}$ on $[0, \infty)$ is cont., continuously invertible)

$E(e^{-kX}) = E(e^{-kY})$ for $\forall k \in \mathbb{Z}_{\geq 0}$ Yes. $\Rightarrow \varphi_X = \varphi_Y$ and the laws are the same for X and Y

(e) $\mathbb{E}[e^{-sX_r}] = \int_0^\infty \frac{e^{-x} x^{k_r}}{k_r!} e^{-sx} \, dx$

$= \int_0^\infty (1+s)^{-k_r} \frac{e^{-(1+s)x} ((1+s)x)^{k_r}}{k_r!} \, dx$

$= (1+s)^{-(k_r+1)}$ Yes.

$\therefore \mathbb{E}[e^{-s(X_1 + \dots + X_r)}] = (1+s)^{-\sum_{r=1}^{n} (k_r+1)} =$ Laplace tf of $f(\cdot, (n-1) + k_1, \dots + k_r)$

by X_i independence

The new random variables take values in $[0,1]$, which is beneficial.

Yes.

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4.) (a) $X_n \xrightarrow{L^2} X$: ~~$|\mathbb{E}[X_n^2 - X^2]|$~~

~~$|\mathbb{E}[X_n^2 - X^2]| \leq \mathbb{E}[$~~

$\| X_n \| = \sqrt{\mathbb{E}[X_n^2]}$, $\langle X, Y \rangle = \mathbb{E}[X \cdot Y]$.

If $\| X_n - X \| \to 0$:

$\pm(\| X_n \| - \| X \|) \leq \| X_n - X \| \Rightarrow | \| X_n \| - \| X \| | \leq \| X_n - X \| \to 0.$

$\therefore \| X_n \|^2 \to \| X \|^2$

If $\mathbb{E}[X_n^2] \to \mathbb{E}[X^2]$:

$2(X_n^2 + X^2) \geq (X_n - X)^2$

$\therefore \liminf_n \mathbb{E}\left[2(X_n^2 + X^2) - (X_n - X)^2\right] \geq \mathbb{E}\left[\cancel{\liminf_n 2(X_n^2 + X^2) - (X_n - X)^2}\right]$

(Fatou)

$\therefore 4\,\mathbb{E}[X^2] - \limsup_n \mathbb{E}[(X_n - X)^2] \geq \mathbb{E}[2(X^2 + X^2)]$

$\therefore 0 \leq \limsup_n \| X_n - X \|_2^2 \leq 0 \Rightarrow \lim_n \| X_n - X \|_2^2 = 0$

$\therefore X_n \to X$ in L^2

(b) Claim: ~~$F_\mu(x) \geq \omega$ iff $x \geq X_\mu(\omega)$~~

~~F_μ is right cont. ~~ $\Rightarrow \{x \in \mathbb{R} : F_\mu(x) \geq \omega\} = [X_\mu(\omega), \infty)$~~

~~$\therefore$ Done.~~

~~$X_\mu^{-1}(-\infty, a]$: $X_\mu(z) \leq a$ iff $z \leq F_\mu(a)$~~

~~$\therefore X_\mu^{-1}(-\infty, a] = (-\infty, F_\mu(a)]$~~

~~$\therefore \mathbb{P}(X_\mu^{-1}(-\infty, a]) = \mu(-\infty, F_\mu(a)\}] = F$~~

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(b) Law on X_μ: $X_\mu(\omega) \le a$ iff $\omega \le F_\mu(a)$

~~$X_\mu(\omega) \le a$ iff $F_\mu(a) \ge \omega$~~

$\therefore$ ~~$\mathbb{P}(\{X_\mu(\omega) \le a\}) =$~~

$\mathbb{P}(X_\mu \le a) = \mathbb{P}(\{\omega : X_\mu(\omega) \le a\}) = \mathbb{P}(\{\omega : \omega \le F_{\mu^{-1}}(a)\})$

$= \cancel{F_{\mu^{-1}}(a)} = F_{\mu^{-1}}(a) - 0 = \mu((-\infty, a])$

$\therefore$ μ agrees with law of X_μ on $\Lambda = \{\phi\} \cup \{(-\infty, a] : a \in \mathbb{R}\}$

Since Λ is a π generator of $\sigma(\mathbb{R})$; ~~$\mu, \mathbb{P}$~~

$\mu(\mathbb{R}) = \mathbb{P}(X_\mu \in \mathbb{R}) = 1$, $\mu = \mathbb{P}$ on $\mathcal{B}(\mathbb{R})$

(ii) ~~If $X_n \to X_\nu$ in L^2:~~

~~Claim: $X_n \to X$ A.S.~~

Claim: $X_{\mu_n} \to X_\nu$.

~~We note: X_{μ_n}, X_ν are increasing functions of X_ν has countable discontinuities~~

If $\int x^2 \, d\mu_n \to \int x^2 \, d\nu$, ; $\mu_n \to \nu$ weakly iff $\mu_n \to \nu$ A.S.

(a) (ii) $\Rightarrow$ (a) (i). $\therefore$ ~~$\mu_n \to$~~ $X_{\mu_n} \to X_\nu$ in L^2.

If $X_{\mu_n} \to X$ in L^2:

(a)(i) $\Rightarrow$ (ii) : $\|X_{\mu_n}\|_2^2 \to \|X_\nu\|_2^2$.

Also: $\|X_{\mu_n} - X\|_1 \le \|X_{\mu_n} - X\|_2 \Rightarrow X_{\mu_n} \to X_\nu$ in L^1.

$X_\nu \in L^1 \Rightarrow X_{\mu_n} \to X_\nu$ in probability $\Rightarrow X_{\mu_n} \to X_\nu$ in distribution

$\Rightarrow X_{\mu_n} \to X_\nu$ ~~as~~ weakly.

Since

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This is true for any R.V. with laws (μ_n, ν)

Skorokhod representation: If we construct R.V. by

$$\tilde{X}_n(\omega) = \inf\{x : \omega \le F_{\mu_n}(x)\} \quad \text{(similarly for } \nu)$$

$$\tilde{X}_n \to \tilde{X}_\nu \quad \text{A.S}$$

But $X_{\mu_n} \overset{d}{=} \tilde{X}_n$, $X_\nu \overset{d}{=} \tilde{X}_\nu$

$$\therefore \; X_{\mu_n} \to X_\nu \quad \text{A.S}$$

* Proof: $\tilde{X}$ is left cont., increasing $\Rightarrow$ Set of points of discontinuity is countable $\Rightarrow$ Measure 0.

If ω $\tilde{X}$ cont. at ω: Fix $\varepsilon > 0$.

$\exists\, \delta > 0$ s.t. $|\omega_1 - \omega| < \delta \Rightarrow |\tilde{X}(\omega_1) - \tilde{X}(\omega)| < \varepsilon$

Pick $x^{\pm} \in \{$points of cont. of $F_\nu\}$ (dense in $\mathbb{R}$)

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1.) (a) $L^2(X, F, \mu) = \{ f : X \to \mathbb{R} : f \text{ is measurable wrt } F, \ \mu(|f|^2) < \infty \}$

Yes.

$\langle f, g \rangle = \mu(f \cdot g)$: $\langle f, g \rangle = \langle g, f \rangle$; $\langle \cdot, \cdot \rangle$ is linear in both arguments and $\langle f, f \rangle = 0$ iff $f = 0$ A.S.

$\therefore$ Valid Inner product. Yes. Completeness of this space has to be claimed as well.

(b) Let $Z_1 \sim \text{Ber}(0,1)$: $Z_2 = 1 - Z_1$ has the same law

$\mathbb{P}(Z_i = 0) = \mathbb{P}(Z_i = 1) = 1/2$.

But $\mathbb{P}(Z_1 = 0, Z_2 = 0) = 0 \neq \mathbb{P}(Z_1 = 0) \cdot \mathbb{P}(Z_2 = 0)$

$\therefore$ Not independent Yes.

This section asks for two real variables.

(c) $\cancel{\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} f(Z_k)\right] = \mathbb{E}\left[f(Z_1)\right] = \int_0^1 f(\lambda)\, d\lambda}$

$$\therefore D(f) = \cancel{\mathbb{E}[}\ \text{Var}\left(\frac{1}{n}\sum_{k=1}^{n}\left(f(Z_k) - \underbrace{\int_0^1 f(\lambda)\, d\lambda}_{\mathbb{E}[f(Z_k)]}\right)\right)$$

$$= \sum_{k=1}^{n} \text{Var}\left(\frac{f(Z_k) - \int_0^1 f(\lambda)\, d\lambda}{n}\right)$$ (independence of $Z_k \Rightarrow (f(Z_k) - \int_0^1 f(\lambda))/n$ independent)

$$= \frac{1}{n}\text{Var}\left(f(Z_1) \cancel{- \int_0^1 f(\lambda) d\lambda}\right)$$ Yes.

Not needed.

$$= \frac{1}{n}\left\{\mathbb{E}[f(Z_1)^2] - \mathbb{E}[f(Z_1)]^2\right\} = \frac{1}{n}\left\{\int_0^1 f(\lambda)^2\, d\lambda - \left[\int_0^1 f(\lambda)\, d\lambda\right]^2\right\}$$

(d) $$D(f_{p,q}) = \cancel{V}\ \mathbb{E}\left[\left(\frac{1}{n}\sum_{k=1}^{n} f_{p,q}(Z_k)\right)^2\right]$$ as the mean is zero here

$$\cancel{= \frac{k^2}{2n^2}\mathbb{E}\left[\left(\sum_{k=1}^{n}\left(1_{I_p}(Z_k) - 1_{I_q}(Z_k)\right)\right)^2\right]}$$

$$= \frac{k^2}{2n^2}\left\{\mathbb{E}\left[\left(\sum_{i=1}^{n} 1_{I_p}(Z_i)\right)^2 + \left(\sum_{j=1}^{n} 1_{I_q}(Z_j)\right)^2 - 2\left(\sum_{i=1, j=1}^{n} 1_{I_p}(Z_i)\, 1_{I_q}(Z_j)\right)\right]\right\}$$

Yes.

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$$= \frac{k^2}{2n^2} \left\{ \frac{2n}{k} + \cancel{2\sum_{i<j} 1_{I_p}(Z_i)\, 1_{I_q}(Z_j)} - 2\,\mathbb{E}\left[\sum_{i \neq j} 1_{I_p}(Z_i)\, 1_{I_q}(Z_j) \right.\right.$$

$$\mathbb{E}\left[1_{I_p}(Z_i) \right] = 1/k$$

$$\left.\left. + \sum_{i \neq j} 1_{I_p}(Z_i)\, 1_{I_p}(Z_j) + \sum_{i \neq j} 1_{I_q}(Z_i)\, 1_{I_q}(Z_j) \right] \right\}$$

(Not Sure).

Now it is possible to take a sum over all pairs $p < q$.
Then rearrange sums.

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2.) (a) $\mathcal{E} \in P(P(X))$ is a σ algebra if:

$$\left.\begin{array}{l} \phi,\ X \in \mathcal{E} \\ A \in \mathcal{E} \text{ iff } A^c \in \mathcal{E} \\ \text{If } A_n \in \mathcal{E}: \bigcup_n A_n \in \mathcal{E} \end{array}\right\}$$

Let $\mathcal{E}$ be a ~~a~~ σ-algebra: $\mu: \mathcal{E} \to [0, \infty]$ is a measure

if: $\mu(\phi) = 0$

$(\forall n)\ A_n \in \mathcal{E},\ (A_n)_{n \geq 1}$ disjoint $\Rightarrow \mu(\bigcup_{n \geq 1} A_n) = \sum_{n \geq 1} \mu(A_n)$

μ is a probability measure if: μ is a measure, $\mu(E) = 1$

(b) Let $\mathcal{A} \in P(P(X))$ be a a ring: $\mu: \mathcal{A} \to [0, \infty]$ be a ~~sa~~ countable-additive set function, $\mu(\phi) = 0$.

~~A~~ Let $\mu^*: P(X) \to [0, \infty]$, $\mu^*(A) = \text{Inf}\left\{\sum_{k \geq 1} \mu(A_k) : X \subseteq \bigcup_{k \geq 1} A_k\right\}$

$(\text{Inf } \phi = \infty)$.

Let $\mathcal{M} = \left\{ B \in P(X) : \forall Z \in P(X),\ \mu^*(Z) = \mu^*(Z \cap B) + \mu^*(Z \cap B^c) \right\}$.

Caratheodory: $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}$ is a σ-algebra, μ^* is an extension of μ to a measure on $\mathcal{M}$.

(c) Claim: $\mathcal{C}$ is a σ-algebra.

ϕ, ~~\$~~ $X \in \mathcal{B}$: $\mu(\phi \Delta \phi) = \mu($~~\$~~$X \Delta X) = 0 \Rightarrow \phi, X \in \mathcal{B}$

If $A \in \mathcal{C}$: Given $\varepsilon > 0$, pick B s.t. $\mu(A \Delta B) < \varepsilon$

But $\mu(A^c \Delta B^c) = \mu(A^c \cup B^c \setminus (A^c \cap B^c))$

$A^c \cup B^c \setminus (A^c \cap B^c) = [A^c \cup B^c] \cap [A \cup B]$

$= [A \cup B] \setminus (A \cap B)^c$

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$\therefore\ \mu(A^c \triangle B^c) = \mu(A \triangle B) < \varepsilon$

$\therefore\ A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}.$

Let $A_n \in \mathcal{G}$: ~~Given $\varepsilon > 0$, pick $B_n \in \mathcal{B}$ s.t. $\mu(A_n \triangle B_n) < \varepsilon/2^n$.~~

By ~~picking $\tilde{A}_n = A_n \setminus (A_1 \cup \dots A_{n-1})$, we can WLOG $(A_n)_{n\geq 1}$~~ disjoint.

Let $A = \bigcup_{n\geq 1} A_n \in \mathcal{F}$: $\mu(A) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^{n} A_k) \leq 1$

Pick N s.t. $\cancel{\mu(A \setminus \bigcup_{k=1}^{N} A_k)}$ $\mu(A) - \mu(\bigcup_{k=1}^{N} A_k) < \varepsilon/3$.

For $1 \leq k \leq N$: Pick $B_k \in \mathcal{B}$ s.t. $\mu(A_k \triangle B_k) < \varepsilon/3N$.

$\therefore\ \bigcup_{k=1}^{N} B_k = B \in \mathcal{B}.$

$\mu(A \triangle B) = \mu(A \setminus B) + \mu(B \setminus A) \leq \cancel{\mu(\bigcup_{k>N} A \setminus \bigcup_{k>N} A_k) + \mu(\bigcup_{k=1}^{N}}$

$\underbrace{\mu(A \setminus \bigcup_{k=1}^{N} A_k)}_{< \varepsilon/3} + \mu(\bigcup_{k=1}^{N} A_k \setminus B) + \mu(B \setminus \bigcup_{k=1}^{N} A_k)$

$< \varepsilon/3 + \sum_{k=1}^{N} \{ \mu(A_k \setminus B) + \mu(B_k \setminus \bigcup_{k=1}^{N} A_k) \}$

$< \varepsilon/3 + \sum_{k=1}^{N} \{ \mu(A_k \setminus B_k) + \mu(B_k \setminus A_k) \} < 2\varepsilon/3 < \varepsilon.$

$\therefore\ A \in \mathcal{G}.$

$\therefore\ \mathcal{G}$ is a σ-algebra $\Rightarrow \sigma(\mathcal{B}) \subseteq \mathcal{G}.$

Let $X = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, μ = Lebesgue measure.

$\mathcal{B} = \{$ finite union of ~~...~~ $(a,b]$, $(-\infty, c]$, (d,∞) sets $\}$

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$\mathcal{B}$ is a boolean algebra;

Let $A = \bigcup_{n \in \mathbb{Z}} (n, n+\frac{1}{2})$: Given $\bigcup_{k=1}^{N} (a_k, b_k] = B$:

$\mu(A \triangle B) = \infty$ (infinite $(n, n+\frac{1}{2})$ intervals outside B)

$\mu(A \triangle (B \cup (-\infty, c]) = \infty$
$\mu(A \triangle (B \cup [d, \infty)) = \infty$
$\mu(A \triangle (B \cup [d,\infty) \cup (-\infty, c))) = \infty$

Pick a $M \geq |a_1|, \dots |a_N|, |b_1|, \dots |b_N|$ Outside $[-M, M]$, we have infinite $(n, n+\frac{1}{2})$ intervals, symmetric difference with $(-\infty, c]$ or $[d, \infty)$ has ∞ measure.

(ii) Let $A \in \mathcal{L}_{[0,1]}$: From caratheodory extension theorem, ~~$\exists \varepsilon/2$ s.t.~~

$m^*(A) \leq 1 \Rightarrow \exists B_n \in \mathcal{B}$ s.t. $A \subseteq \bigcup_n B_n$ and

$$\sum_n \overset{m}{\mu}(B_n) \leq m(A) + \varepsilon/2.$$

Pick N s.t. $\sum_{n>N} m(B_n) < \varepsilon/2$: $\bigcup_{k=1}^{N} B_k \in \mathcal{B}$.

$$m^*\left(\bigcup_{k=1}^{N} B_k \triangle A\right) \leq \underbrace{\overset{m}{\mu}\left(\bigcup_{k\geq 1} B_k \setminus A\right)}_{(*) + \overset{m}{\mu}(A) = \overset{m}{\mu}(\bigcup_{k\geq1} B_k)} + \underbrace{\overset{m}{\mu}\left(A \setminus \bigcup_{k=1}^{N} B_k\right)}_{\leq \mu(\bigcup_{k\geq1} B_k \setminus \bigcup_{k=1}^{N} B_k) < \varepsilon/2}$$

$$\leq \overset{m}{\mu}\left(\bigcup_{k\geq1} B_k\right) - \overset{m}{\mu}(A) + \varepsilon/2 \leq \varepsilon/2 + \left(\sum_{k\geq1} \overset{m}{\mu}(B_k)\right) - m(A)$$
$$< \varepsilon.$$

($\sigma(\mathcal{B}) \subseteq \mathcal{L}_{[0,1]}$, m is countable additive on $\mathcal{L}_{[0,1]}$)

$\therefore A \in \mathcal{C} \Rightarrow \mathcal{L}_{[0,1]} \subseteq \mathcal{C} \subseteq \mathcal{L}_{[0,1]} \Rightarrow$ Equality

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3.) (a) System is Ergodic if: $\forall A \in \mathcal{A}$, $T^{-1}(A) = A \Rightarrow m(A) = 0$ or $m(A^c) = 0$.

(b) Koglomorov 0-1: Let $(X_n)_{n\geq 1}$ be independent r.v.;

$\mathcal{T} = \bigcap_{n\geq 1} \sigma(X_k : k \geq n)$: If $A \in \mathcal{T}$, $\mathbb{P}(A) = 0$ or 1

(c) Consider the canonical model: $(\mathbb{R}^{\mathbb{N}}, \bigotimes_{k=1}^{\infty} \mathcal{B}(\mathbb{R}), \bigotimes_{k=1}^{\infty} \mu_{X_k})$,

μ_{X_k} is the law of X_k.

Let $T(\underline{x})_k = x_{k+1}$: ~~T is measure preserving,~~ T is ergodic:

Let $T^{-1}(A) = A$: $A \in \sigma(X_n : n \geq 1)$

But $T^{-r}(A) \in \sigma(X_n : n \geq r+1) \Rightarrow A \in \sigma(X_n : n \geq r+1)$

(True $\forall r \geq 0$): $A \in \bigcap_{n\geq 1} \sigma(X_k : k \geq n) = \mathcal{T} \Rightarrow \mathbb{P}(A) = 0$ or 1

$\therefore$ T is ergodic

(c) (i) $T(x) \in A$ iff $2x \in A$ or $2 - 2x \in A$ iff $x \in \frac{1}{2} \cdot A$ or $x \in 1 - \frac{1}{2} \cdot A$

$A \subseteq [0,1] \Rightarrow \frac{1}{2} A \subseteq [0, \frac{1}{2}]$

$\therefore T^{-1}(A) = \frac{1}{2} \cdot A \cup (1 - \frac{1}{2} A)$ (Disjoint)

$m(T^{-1}(A)) = m(\frac{1}{2} \cdot A) + m(1 - \frac{1}{2} \cdot A) = \frac{1}{2} m(A) + \frac{1}{2} m(A)$ as Lebesgue measure is translation invariant and $m(\lambda(\cdot)) = \lambda\, m(\cdot)$ ($\forall \lambda \in \mathbb{R}$)

$\therefore m(T^{-1}(A)) = m(A) \Rightarrow$ Invariant measure

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(ii) $T^{-0}(0,\tfrac{1}{2}) = (0,\tfrac{1}{2})$ (Disjoint union of 2^0 open intervals of length $\frac{1}{2^{0+1}}$)

A = If $T^{-n}(0,\tfrac{1}{2})$ = Disjoint union of 2^n open intervals of length $\frac{1}{2^{n+1}}$:

$T^{-1}(T^{-n}(0,\tfrac{1}{2})) = \underbrace{\tfrac{1}{2}A}_{\subset [0,\frac{1}{2})} + \underbrace{(1-\tfrac{1}{2}A)}_{\subseteq (\frac{1}{2},1]}$; $\tfrac{1}{2}A$, $(1-\tfrac{1}{2}A)$ disjoint.

$\tfrac{1}{2}A$, $(1-\tfrac{1}{2}A)$ are both 2^n disjoint union of length $\frac{1}{2^{n+1+1}}$

$\therefore T^{-(n+1)}(0,\tfrac{1}{2})$ is a disjoint union of 2^{n+1} open intervals of length $2^{-(n+2)}$

$\therefore$ ~~$\mathbb{P}(X_n=$~~ Induction complete: $T^{-n}(0,\tfrac{1}{2})$ is a disjoint union of 2^n open intervals of length $2^{-(n+1)}$

$\therefore \mathbb{P}(X_n = 1) = 2^n \cdot 2^{-(n+1)} = \frac{1}{2} \Rightarrow X_k$ identically distributed

Claim: ~~(independ~~ $\forall n \geq 0$, $(0, \tfrac{1}{2^{n+1}}) \subseteq T^{-n}(0,\tfrac{1}{2})$,
$[\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}) \subseteq (T^{-n}(0,\tfrac{1}{2}))^c$

$(n=0)$: ~~T~~ $(0,\tfrac{1}{2})$ satisfies claim

If true for some n: $T^{-1}(T^{-n}(0,\tfrac{1}{2})) = \tfrac{1}{2}T^{-n}(0,\tfrac{1}{2}) \cup (1-\tfrac{1}{2}T^{-n}(0,\tfrac{1}{2}))$

$\therefore \tfrac{1}{2}\cdot(0,\tfrac{1}{2^{n+1}}) = (0,\tfrac{1}{2^{n+2}}) \subseteq T^{-(n+1)}(0,\tfrac{1}{2})$

If $x \in [\tfrac{1}{2^{n+2}}, \tfrac{1}{2^{n+1}}) \in T^{-(n+1)}(0,\tfrac{1}{2}] \Rightarrow 2x \in T^{-(n)}(0,\tfrac{1}{2})$

But $2x \in [\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}) \Rightarrow$ Contradiction

$\therefore$ Induction complete

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$\mathbb{P}(X_0, \dots X_n = 1) = m\left(\bigcap_{k=0}^{n} T^{-k}(0, 1/2)\right)$

Claim: $\bigcap_{k=0}^{n} T^{-k}(0, 1/2) = (0, 1/2^{n+1})$ $(n \geq 0)$

(True for $n=0$)

If valid for some n: $\bigcap_{k=0}^{n+1} T^{-k}(0, 1/2) = (0, 1/2^{n+1}) \cap T^{-(n+1)}(0, 1/2)$

$= (0, 1/2^{n+1})$ (from previous part)

$\therefore$ Induction complete.

$\therefore$ If $A_n = \Phi$ on $\{X_n = 1\}$: $\mathbb{P}\left(\bigcap_{k=0}^{N} A_k\right) = \prod_{k=0}^{N} \mathbb{P}(A_k)$.

$= 2^{-(N+1)} \mathbb{1}_{A_0, \dots A_N \neq \Phi}$

$\therefore \{\{\Phi, A_n\} : 0 \leq n \leq N\}$ independent $\Rightarrow \{\sigma(\{\Phi, A_n\}) : 0 \leq n \leq N\}$ independent

$\therefore (X_1, \dots X_N)$ independent

(True $\forall N$): $(X_n)_{n \geq 1}$ iid.

Claim: $\sigma(X_n : n \geq 0) = \mathcal{A}$

$\sigma(X_0) = \{\Phi, \Phi\ (0, 1/2), \{0\} \cup \{1\}\ [1/2, 1]$

$\sigma(X_0, X_1)$: $(0, 1/4), (3/4, 1), \{0\} \cup [1/4, 1/2], \{1\} \cup [1/2, 3/4]$

$[1/4, 1/2), \{0\} \cup \{1\} \cup [1/2, 3/4], \{0, 1/2\}, [1/4, 1/2)$,

$(0, 1/4), [1/4, 1/2), (3/4, 1), \{0, 1\} \cup [1/2, 3/4)$

$\sigma(X_0, X_1, X_2)$: $(0, 1/8), [1/8, 1/4), [1/4, 3/8], (3/8, 1/2), (1/2, 5/8), [5/8, 3/4) \cup \{0, 1\}$,

$(3/4, 7/8], (7/8, 1)$

* I think diadic sets are in $\sigma(X_n : n \geq 1) \Rightarrow$ Generate $\mathcal{B}([0,1])$

* Unable to prove it.

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(iii) { System is ergodic: Let $\overset{A}{B}$ be an invariant set in $\mathcal{B}([0,1])$
$= \sigma(X_n : n \geq 0)$

$T^{-1}(\overset{A}{B}) \in \sigma(X_n : n \geq 1)$, ... $T^{-N}(\overset{A}{B}) \in \sigma(X_n : n \geq N)$

$\therefore A \in \bigcap_{N \geq 0} \sigma(X_n : n \geq N) = \mathcal{I}$

Kolmogorov 0-1 $\Rightarrow \mathbb{P}(A) = 0$ or 1 //

This theorem is stated with $E(X_n) = \mu$: this case can be reduced to $E(X_n) = 0$.

4.) (a) Let $(X_n)_{n \geq 1}$ be iid, $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^4] < M$:
$S_n = X_1 + \dots + X_n$: $S_n/n \to 0$ A.S. (Strong law) Yes.

$$\mathbb{E}[S_n^4] = \sum_{k=1}^{n} \mathbb{E}[X_k^4] + 6 \sum_{\substack{i \neq j \\ 2}}^{n} \mathbb{E}[X_i^2 X_j^2] + 0$$ (odd terms have 0 expectancy)

$$\leq n \cdot M + 6 \binom{n}{2} \left(\sqrt{\mathbb{E}[X_i^4] \cdot \mathbb{E}[X_j^4]} \right)$$ by Cauchy-Schwarz

$$= M \left\{ n + 3n(n-1) \right\} \leq 3n^2 \cdot M$$ Yes.

$\therefore \mathbb{E}[S_n^4/n^4] \leq 3M/n^2 \Rightarrow \mathbb{E}\left[\sum_{n \geq 1} S_n^4/n^4\right] < \infty$. by monotone convergence

$\therefore \sum_{n \geq 1} (S_n^4/n^4) < \infty$ A.S. $\Rightarrow \sum_n (S_n/n)^4 \to 0$ A.S. $\Rightarrow S_n/n \to 0$ A.S.

$\therefore$ Done. Yes.

(b) (i) Claim: $(S_n)_{n \geq 1}$ is cauchy in L^2 ($n > m \geq N$) $n < m$ here

$$\| S_n - S_m \|_2^2 = \| \sum_{k=n+1}^{m} a_k X_k \|_2^2 = \mathrm{Var}\left(\sum_{k=n+1}^{m} a_k X_k\right) \quad (\mathbb{E}[X_k] = 0)$$

$$= \sum_{k=n+1}^{m} a_k^2 \mathrm{Var}(X_k) = \sum_{k=n+1}^{m} a_k^2 \leq \sum_{k \geq N} a_k^2 \to 0 \text{ as } N \to \infty$$
by X_i independence

$\therefore L^2$ cauchy Yes.

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$L^2(\Omega, F, \mathbb{P})$ complete: $\cancel{X}$ $S_n \to S_\infty$ in L^2. Yes.

$\| S_n - S \|_1 = \mu(|S_n - S| \cdot 1) \leq \| S_n - S \|_2 \cdot 1$ (Cauchy Schwartz)

$\therefore S_n \to S$ in L^1 Yes.

~~Characteristic Function:~~ $\cancel{\mathbb{E}[e^{izS_n}] = \prod_{k=1}^{n} \mathbb{E}[e^{iz a_k X_k}]}$

$S_n \in L^1$, $S \in L^1$: $\cancel{A}$ $S_n \to S$ in $L^1 \Rightarrow \cancel{X_n} \to \cancel{X}$ in $\mathbb{P}$

Yes. $\Rightarrow X_n \to X$ in law $\cancel{x}$ These claims could be briefly justified.

(ii) $\| S_n \|_4^4 = \mathbb{E}[S^4]$: $\cancel{X}$ $S_n \to S$ in $\mathbb{P} \Rightarrow \exists$ subsequence $S_{n_k} \to S$ A.S. Yes.

$\therefore \mathbb{E}[\liminf_k S_k^4] \leq \liminf_k \mathbb{E}[S_{n_k}^4]$ - (*) by Fatou's lemma

But $\mathbb{E}[S_n^4] = \mathbb{E}[\sum_{k=1}^{n} a_k^4 + 6 \sum_{i<j}^{n} a_i^2 a_j^2]$

$\mathbb{E}[S_n^2] = \sum_{k=1}^{n} a_k^2 + \cancel{2\sum_{i<j}^{n} a_i a_j} + 0.$ Yes.

$\therefore \mathbb{E}[S_n^2]^2 = \sum_{k=1}^{n} a_k^4 + 2\sum_{i<j}^{n} a_i^2 a_j^2 \Rightarrow 3\,\mathbb{E}[S_n^2]^2 \geq \mathbb{E}[S_n^4]$ Yes.

$\therefore (*) \leq \liminf_k \{ 3\,\mathbb{E}[S_{n_k}^2]^2 \}$ Yes.

Since $S_{n_k} \to S$ in L^2: $\| S_{n_k} \|_2 \to \| S \|_2 \Rightarrow \mathbb{E}[S_{n_k}^2] \to \mathbb{E}[S^2]$ also from $S_{n_k} \to S$ a.s.

$\therefore (*) \leq 3 \lim_{k \to \infty} \mathbb{E}[S_{n_k}^2]^2 = 3\,\mathbb{E}[S^2]^2$.

$\therefore \| S \|_4 \leq 3^{1/4} \| S \|_2$ $\cancel{x}$ Yes.

(iii) Convergence in law $\equiv$ pointwise Convergence of characteristic function. Yes.

$\mathbb{E}[e^{iz(\sum_{k=1}^{n} a_k Y_k)}] = \prod_{k=1}^{n} \mathbb{E}[e^{i a_k z Y_k}] = \prod_{k=1}^{n} e^{-a_k^2 z^2/2}$ by Y_k independence

$= e^{-z^2/2 (\sum_{k=1}^{n} a_k^2)} \to e^{-z^2/2 \sum_{k=1}^{\infty} a_k^2}$ Converges to $\cancel{A}$ char. func. of $N(0, \sum_{k=1}^{\infty} a_k^2)$ (Limit)

Yes.

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Probability & Measure 2021

1.) (a) Let $(E, \mathcal{E}, \mu)$ be a measure space;

$f_n : E \to [0,\infty]$ be measurable, ~~A.E.~~

Fatou's Lemma: $\liminf_n \mu(f_n) \geq \mu(\liminf_n f_n)$

(Proof): $\forall N \in \mathbb{N}$, $\inf_{m \geq n} f_m \leq f_k$, $k \geq n$

$\therefore \mu(f_k) \geq \mu(\inf_{m\geq n} f_m) \quad (\forall k \geq n)$

$\therefore \inf_{m \geq n} \mu(f_m) \geq \mu(\inf_{m \geq n} f_m)$

Sequence: $(\inf_{m\geq n} f_m)_n$ is increasing, non-negative sequence of measurable functions

Monotone convergence: $\lim_{n\to\infty} \mu(\inf_{m\geq n} f_m) = \mu(\lim_{n\to\infty} \inf_{m \geq n} f_m) = \mu(\liminf_n f_n)$

$\therefore \mu(\liminf_n f_n) \leq \lim_{n\to\infty} \inf_{m\geq n} \mu(f_m) = \liminf_n \mu(f_n)$ *

* Monotone convergence: If $f_m : E \to [0,\infty]$ is measurable,

$f_m \leq f_{m+1} \quad (\forall m \geq 1)$ A.E ; $f_m \to f$ (A.E.)

$\mu(f_m) \nearrow \mu(f)$

(b) Let $E = \mathbb{N}$, $\mathcal{E} = P(\mathbb{N})$, μ be counting measure

$f_n = 1_{x \geq n}$: $\mu(f_n) = \infty \Rightarrow \liminf_n \mu(f_n) = \infty$

But $\forall k \in \mathbb{N}$, $f_n(k) = 0$ for $n > k \Rightarrow \liminf_n f_n \equiv 0$.

$\therefore \mu(\liminf_n f_n) = 0$

$\therefore$ Strict inequality.

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(c) Fatou's Lemma:

$$\liminf_n \mathbb{E}[X_n] \geq \mathbb{E}[\liminf_n X_n]$$

$X_n \to X$ A.S. $\Rightarrow \liminf_n X_n = X \Rightarrow$ RHS $= \mathbb{E}[X]$

$\therefore \mathbb{E}[X] \leq \liminf_n \mathbb{E}[X_n]$; $\inf_{m \geq n} \mathbb{E}[X_n] \leq \sup_{m \leq n} \mathbb{E}[X_n] \leq \sup_n \mathbb{E}[X_n]$

$\therefore \mathbb{E}[X] \leq \lim_{n\to\infty} \inf_{m\geq n} \mathbb{E}[X_n] \leq \sup_n \mathbb{E}[X_n]$

2.) (a) Define f_n :

$$f_n = \sum_{k=0}^{n 2^n - 1} (k \cdot 2^{-n}) \mathbb{1}_{f(x) \in [k\cdot 2^{-n}, (k+1)\cdot 2^{-n})} + n \mathbb{1}_{f(x) \geq n}$$

f_n is simple: $\{x : f(x) \in [k \cdot 2^{-n}, (k+1) 2^{-n})\} \in \mathcal{E}$

is as f is borel measurable and

$[k \cdot 2^{-n}, (k+1) 2^{-n}) \in \mathcal{B}([0,\infty))$

$\{x : f(x) \geq n\} = f^{-1}[n, \infty) \in \mathcal{B} \in [0,\infty)$

Claim: $f_n \to f$ pointwise.

If $f(x) < \infty$: Pick $N > f(x)$. If $x \in [k\cdot 2^{-n}, (k+1)\cdot 2^{-n})$

for $n \geq N$, $0 \leq k < 2^n \cdot n$, $|f_n(x) - f(x)| \leq 2^{-n} \to 0$

$\therefore f_n(x) \to f(x)$

If $f(x) = \infty$: $f_n(x) = n \nearrow \infty$.

$\therefore$ Pointwise converge everywhere.

~~By construction, f_n~~ $\leq f$.

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If f is integrable: $\{x : |f(x)| = \infty\}$ has 0-measure.

Let $A = \{x \in E : f(x) \neq \infty\}$: $\mu(A^c) = 0$.

Let $f_n = \sum_{k=0}^{n2^n - 1} (k.2^n) \, 1_{f^{-1}[k \cdot 2^{-n}, (k+1)2^{-n})}$

From (a): $f_n \to f$ for $f(x) \neq \infty \Rightarrow f_n \to f$ ~~A.E.~~ everywhere on A

Claim: $f_n \to f$ in L_1

We note: $f_n \leq f$ everywhere.

$\therefore \mu(|f_n - f|) = \mu(1_A \cdot |f_n - f|)$ $\quad (\mu(A^c) = 0)$

Integrand $1_A |f_n - f| \leq 1_A (|f_n| + |f|) \leq 1_A \cdot 2|f|$ $\quad (\forall n \geq 1)$

$\therefore$ Dominated by $2|f|$, f is integrable

Dominated convergence theorem: $\lim_{n\to\infty} \mu(1_A |f_n - f|) = \mu(\lim_{n\to\infty} 1_A |f_n - f|)$

$\therefore \lim_{n\to\infty} \mu(|f_n - f|) = \mu(0) = 0$

$\therefore f_n \to f$ in L^1.

(c) ~~$\alpha = \sup\{|f(x)| : x \in E\} + 1$~~:

$f_n = \sum_{k=0}^{\alpha \cdot 2^n - 1} (k \cdot 2^{-n}) \, 1_{f(x) \in [k \cdot 2^{-n}, (k+1)2^{-n})}$, $\quad \alpha = 1 + \lceil \sup \|f\|_\infty \rceil$

$\forall x \in E$: $f(x) \leq \alpha - 1 < \alpha \Rightarrow f \in \bigcup_{k=0}^{\alpha 2^n - 1} [k \cdot 2^{-n}, (k+1)2^{-n})$

$\therefore |f_n(x) - f(x)| \leq 2^{-n}$ $(\forall x \in E) \Rightarrow \|f_n - f\|_\infty \leq 2^{-n} \to 0$

$\therefore f_n \to f$ uniformly.

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3.) (a) If $X_1, \dots X_N$ are independent: Let $A_k \in \mathcal{B}(\mathbb{R})$

$$\{ f_k(X_k) \in A_k : 1 \leq k \leq N \} = \{ X_k \in f_k^{-1}(A_k) : 1 \leq k \leq N \}$$

$$\therefore \mathbb{P}(\{ f_k(X_k) \in A_k : 1 \leq k \leq N \}) = \mathbb{P}(\{ X_k \in f_k^{-1}(A_k) : 1 \leq k \leq N \})$$

$$= \prod_{k=1}^{N} \mathbb{P}(X_k \in f_k^{-1}(A_k)) = \prod_{k=1}^{N} \mathbb{P}(f(X_k) \in A_k)$$

$\therefore (f_k(X_k))_{k=1, \dots N}$ are independent.

Let $\mu_1, \dots \mu_N$ be the laws of $X_1, \dots X_N$:

μ be the law of $(X_1, \dots X_n)$;

Claim: $\mu = \bigotimes_{i=1}^{N} \mu_i$

Let $\Lambda = \{ \prod_{k=1}^{N} A_k : A_k \in \mathcal{B}(\mathbb{R}) \}$: Λ is a π system,

generates $\mathcal{B}(\mathbb{R}^N)$ (by definition)

$$\mu(\prod_{k=1}^{N} A_k) = \mathbb{P}(X_1 \in A_1, \dots X_N \in A_N) = \prod_{k=1}^{N} \mathbb{P}(X_k \in A_k)$$

$$= \prod_{k=1}^{N} \mu_k(A_k)$$

$\therefore \mu$ agrees with $\bigotimes_{i=1}^{N} \mu_i$ on $\Lambda \Rightarrow \mu = \bigotimes_{i=1}^{N} \mu_i$ on $\mathcal{B}(\mathbb{R}^d)$

Let f_k be a bounded measurable:

$$\mathbb{E}[\prod_{k=1}^{N} f_k(X_k)] = \cancel{\int_{\mathbb{R}^N} \prod_{k=1}^{N} f_k(x_k) \, \mu(dx_1 \dots} \, dx_N)$$

$$= \mu((x_1, \dots x_N) \mapsto \prod_{k=1}^{N} f_k(x_k)) = \mu_1 \otimes \dots \mu_n (f_1(x_1) \dots f_n(x_n))$$

$$= \prod_{k=1}^{N} \mu_k(f(x_k)) = \prod_{k=1}^{N} \mathbb{E}[f_k(X_k)]$$

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If $\mathbb{E}\left[\prod_{k=1}^{N} f_k(X_k)\right] = \prod_{k=1}^{N} \mathbb{E}[f_k(X_k)]$:

$\forall\ A_k \in \mathcal{B}(\mathbb{R})$:

$\mathbb{P}(\cancel{A_k}\ X_k \in A_k : 1 \le k \le N) = \mathbb{E}\left[\prod_{k=1}^{N} \mathbb{1}_{X_k \in A_k}\right]$ ($\mathbb{1}_{X_k \in A_k}$ is measurable as $A_k \in \mathcal{B}(\mathbb{R})$)

$\therefore = \prod_{k=1}^{N} \mathbb{E}[\mathbb{1}_{X_k \in A_k}] = \prod_{k=1}^{N} \mathbb{P}(X_k \in A_k)$

$\therefore (X_1, \dots X_N)$ is independent.

(c) $L^2(\mathbb{P})$ is complete.

$\therefore$ $\sum_{k=1}^{N} X_k \to \sum_{k=1}^{\infty} X_k$ iff $\left(\sum_{k=1}^{N} X_k\right)_{N \ge 1}$ is L^2 cauchy

If $m \ge n \ge N$: $\cancel{\left\| \sum_{k=1}^{m} X_k - \sum_{k=1}^{n} X_k \right\|_2 \le \sum_{k=n+1}^{m} \| X_k \|_2}$

$\cancel{\le \sum_{k \ge N} \| X_k \|_2}$

$\cancel{\| X_k \|_2 = \sqrt{\ }}$

$$\left\| \sum_{k=1}^{m} X_k - \sum_{k=1}^{n} X_k \right\|_2^2 = \mathbb{E}\left[\left(\sum_{k=n+1}^{m} X_k\right)^2\right] = \mathbb{E}\left[\sum_{k=n+1}^{m} X_k^2 + 2 \underbrace{\sum_{n+1 \le i < j \le m} X_i X_j}_{\mathbb{E}[X_i] = \mathbb{E}[X_j] = 0}\right]$$

$$\le \sum_{k > n} \mathbb{E}[X_k^2] = \sum_{k > n} \sigma_k^2 \le \sum_{k \ge N} \sigma_k^2 \to 0 \text{ as } N \to \infty.$$

$\therefore$ Sequence is cauchy.

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* Hint method:

If $X_n \to X$ in L^2: $\|X_n - X\|_1 \le \|X_n - X\|_2 \Rightarrow X_n \to X$ in L^1

$X_n, X \in L^2 \Rightarrow X_n, X \in L^1$

$\therefore X_n \xrightarrow{\mathbb{P}} X \Rightarrow X_n \xrightarrow{d} X \Rightarrow \mathbb{E}[e^{iuX_n}] \to \mathbb{E}[e^{iuX}]$

$$\mathbb{E}[e^{iu(X_1+\dots+X_n)}] = \prod_{k=1}^{n} \mathbb{E}[e^{iuX_k}] = \prod_{k=1}^{n} e^{-u^2\sigma_k^2/2}$$

$$= e^{-u^2/2 \left(\sum_{k=1}^{n} \sigma_k^2\right)} \quad -(1)$$

If $\sum_{k=1}^{n} \sigma_k^2 = \infty$: $(1) \to 0 = \mathbb{E}[e^{iuX}]$ $(\forall u \in \mathbb{R})$

$\Rightarrow X = -\infty$ A.S. (reject, $X \in L^2$)

$\therefore \sum_{k\ge1} \sigma_k^2 < \infty$

If $\sum_{k=1}^{n} \sigma_k^2 < \infty$: $X_n \to N(0, \sum_{k\ge1} \sigma_k^2)$ Ans in dist.

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4.) (a) $\mathbb{P}(\limsup_n A_n) = \mathbb{P}\left(\bigcap_{n\geq 1} \bigcup_{m \geq n} A_m\right)$ - (1)

$\bigcup_{m\geq n} A_m$ is decreasing sequence ; $\mathbb{P}(\Omega) = 1 \Rightarrow (1) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m\geq n} A_m\right)$

$\therefore (1) \leq \lim_{n\to\infty} \sum_{m\geq n} \mathbb{P}(A_m) = 0$ as $\sum_{n\geq 1} \mathbb{P}(A_n) < \infty$.

$\therefore \mathbb{P}(\limsup_n A_n) = 0$

(b) If $X_n \to X$ A.S. : Fix $\varepsilon > 0$. ~~Let $A_n = \{ \omega : |X_n - X| \geq \varepsilon \}$~~

~~$B_n = \{ \omega \in \Omega : \limsup\ \exists m \geq n$ s.t. $|X_m(\omega) - X(\omega)| \geq \varepsilon \}$~~

~~$= \bigcup_{m\geq n} A_m$; $A_m = \{ \omega \in \Omega : |X_m(\omega) - X(\omega)| \geq \varepsilon \}$~~

Since $X_n \to X$ A.S. : $|X_n - X| < \varepsilon$ eventually.

$\therefore \mathbb{P}\left(\bigcup_{n\geq 1} \bigcap_{m\geq n} A_m\right) = 1$, $A_m = \{ \omega : |X_m(\omega) - X(\omega)| < \varepsilon \}$

$\mathbb{P}(|X_n(\omega) - X(\omega)| \geq \varepsilon) = \mathbb{P}(A_n^c)$ - (1)

We know : $\mathbb{P}\left(\bigcap_{n\geq 1} \bigcup_{m\geq n} A_m^c\right) = 0 \Rightarrow 0 = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m\geq n} A_m^c\right)$

$(1) \leq \mathbb{P}\left(\bigcup_{m\geq n} A_m^c\right) \to 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$.

$\therefore X_n \to X$ in $\mathbb{P}$

(c) If $X_n \to X$ in $\mathbb{P}$:

Claim: $\exists$ subsequence $(Y_n)_{n\geq 1}$, $Y_n = X_{k_n}$ s.t. $Y_n \to X$ A.S.

Pick ~~$k_{n+1} >$~~ ~~k_1~~ k_n s.t. $\mathbb{P}(|X_m - X| \geq 1/n) < 1/2^n$ for $m \geq k_n$

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(WLOG: $K_{n+1} > K_n$)

K_n exists as $\lim_{m\to\infty} \mathbb{P}(|X_m - X_n| < 1/n) = 0.$

Claim: $\{X_{K_n}\} \xrightarrow{(n)} X$ A. S.

If $\nexists\, X_{K_n} \not\to X$, $|X_{K_n}(\omega) - X(\omega)| \geq 1/N$ I.O. for some N.

$$\sum_{r\geq 1} \mathbb{P}(|X_{K_r}(\omega) - X(\omega)| \geq 1/N) \leq \sum_{r=1}^{N} \cancel{\mathbb{P}(|X_{K_r}(\omega) - X(\omega)| \geq 1/N)}$$

$$+ \sum_{r>N} \mathbb{P}(|X_{K_r} - X| \geq 1/r)$$

$$\leq N + \sum_{r>N} 1/2^r < \infty.$$

$\therefore$ A1 $\mathbb{P}(|X_{K_n}(\omega) - X(\omega)| \geq 1/N) = 0.$

$$\therefore \mathbb{P}(X_{K_n} \not\to X) \leq \sum_{N\geq 1} \mathbb{P}(|X_{K_n}(\omega) - X(\omega)| \geq 1/N \text{ I.O.}) = 0.$$

$\therefore X_{K_n} \to X$ A.S.

Since $X_n \to X$ in $\mathbb{P} \Rightarrow X_{n_k} \to X$ in $\mathbb{P}$, $\exists\, X_{n_{k_r}} \to X$ A.S.

and subsequence property valid

Suppose $X_n \not\to X$ in $\mathbb{P}$: $\exists\, n_k$, $\varepsilon > 0$ s.t. $\mathbb{P}(|X_{n_k} - X| \geq \varepsilon) \geq \delta > 0.$ for some δ.

But X_{n_k} contains A.S. conv. subsequence $X_{n_{k_r}} \to X$ A.S.

$\Rightarrow X_{n_{k_r}} \to X$ in $\mathbb{P}$

But $\mathbb{P}(|X_{n_{k_r}} - X| \geq \varepsilon) \geq \delta \Rightarrow$ Contradiction.

$\therefore$ Subsequence property $\Rightarrow X_n \to X$ in $\mathbb{P}$

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Probability & Measure 2022

1.) (a) Let $(\Omega, F, \mathbb{P})$ be a probability space:

~~$F_n \subseteq F$ be σ-alge~~

$(X_n)_{n\geq 1}$ be a sequence of independent R.V. on Ω;

$F_n = \sigma(X_k : k \geq n)$

$\therefore$ ~~$F_1 \subseteq$~~ $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$

Let $T = \bigcap_{k\geq 1} F_k$: $\forall A \in T$, $\mathbb{P}(A) = 0$ or 1 (Koglomorov 0-1)

Let $\mathcal{L}_n = \sigma(X_k : X < n)$: Since $(X_n)_{n\geq 1}$ independent, $\mathcal{L}_n$, ~~&~~ F_n are independent ; $T \subseteq F_n \Rightarrow T, \mathcal{L}_n$ independent.

$\bigcup_{n\geq 1} \mathcal{L}_n$ is a π system, generates $\sigma(X_k : k \geq n)$

$\forall A \in \bigcup_{n\geq 1} \mathcal{L}_n$, $\forall B \in T$: $A \in \mathcal{L}_N$ for some $N \Rightarrow A, B$ independent

$\therefore \bigcup_{n\geq 1} \mathcal{L}_n$, T independent $\Rightarrow \sigma(X_k : k \geq 1)$, T independent

$T \subseteq \sigma(X_k : k \geq 1) \Rightarrow T$ independent of itself.

$\therefore \mathbb{P}(A)^2 = \mathbb{P}(A) \Rightarrow \mathbb{P}(A) \in \{0, 1\}$

(b) We note that C is a π-system. If μ_i are probability measures on $(E, \sigma(C))$, $\mu_1 = \mu_2$ on ~~σ~~ $C \Rightarrow \mu_1 = \mu_2$ on $\sigma(C)$

But $\mu_i\left(\prod_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} m(A_n) \Rightarrow \mu_1 = \mu_2$ on $C \Rightarrow \mu_1 = \mu_2$ $\therefore$ Unique.

(Existence): Let $(X_n)_{n\geq 1}$ be an iid sequence of R.V. on $\mathbb{R}$ with law given by m.

Let μ be the law of $(X_n)_{n\geq 1}$ on $\mathbb{R}^{\mathbb{N}}$:

$$\mathbb{P}\left(\bigcap_{k=1}^{N} X_k \in A_k\right) = \prod_{k=1}^{N} \mathbb{P}(X_k \in A_k) \quad \text{(independence)}$$

$$= \prod_{k=1}^{N} \mathbb{P}(X_k \in A_k) = \prod_{k=1}^{N} m(A_k)$$

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If $A \in \mathcal{C}$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow A_n = \mathbb{R}$

$$\therefore \mathbb{P}(\underset{\sim}{X} \in A) = \mathbb{P}((X_1, \dots X_N) \in A_1 \times \dots \times A_N) = \prod_{k=1}^{N} \mu(A_k) \times \underbrace{\prod_{k>N} \mu(\mathbb{R})}_{=1}$$

$$\therefore \mu(A) = \prod_{k \geq 1} \mu(A_k) \qquad (A \in \mathcal{C})$$

$\therefore$ Measure μ exists.

(b) Θ is measurable: Let $A \in \cancel{\sigma}\mathcal{C}$: $T^{-1}(A) = \{ \mathbb{R} \times A \in \mathcal{C}$

$\therefore \forall A' \in \sigma(\mathcal{C})$, $T^{-1}(A') \in \sigma(\mathcal{C})$

~~Since $\forall A' \in \mathcal{C}$, $\mathbb{R} \times A' \in \mathcal{C}$. $A \in \sigma(\mathcal{C}) \Rightarrow \mathbb{R} \times A \in \sigma(\mathcal{C})$.~~

~~$\therefore T$ measurable.~~

$\forall A \in \mathcal{C}$: $A = A_1 \times \dots \times A_N \times \mathbb{R} \dots$

$$\therefore \mu(T^{-1}(A)) = \mu(\mathbb{R} \times A_1 \times \dots \times A_N \times \mathbb{R} \times \dots) = 1 \times \prod_{k=1}^{N} m(A_k) \times 1$$

$$= \mu(A)$$

μ, $\mu(T^{-1}(\cdot))$ agree on $\cancel{\sigma}\,\mathcal{C} \Rightarrow \mu = \mu(T^{-1}(\cdot))$ on $\sigma(\mathcal{C})$

$\therefore$ Measure preserving.

If $T^{-1}(A) = A$ $(A \in \sigma(\mathcal{C}))$: Using random variables $(X_n)_{n \geq 1}$ notation:

$$A \in \sigma(X_n : n \geq 1) \Rightarrow T^{-1}(A) \in \sigma(X_n : n \geq 2)$$

$$\therefore A = T^{-r}(A) \Rightarrow A \in \sigma(X_n : n \geq r+1) \Rightarrow A \in \bigcap_{k \geq 1} \sigma(X_n : n \geq k) = \tau$$

Koglomorov: $\mathbb{P}(A) = 0$ or 1.

$\therefore T$ is ergodic.

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2) Let $(E, \mathcal{E}, \mu)$ be a measure space:

If $f_n : E \to [0, \infty]$ ~~is~~ is measurable, $(f_n(x))_{n\geq 1}$ is monotone increasing, $f_n(x) \nearrow f(x)$: $\mu(f_n) \nearrow \mu(f)$

Let $g = \sum_{k=1}^{N} a_k 1_{B_k}$, $B_k \in \mathcal{E}$, $a_k > 0$.

Claim: $V(A) = \mu(g \cdot 1_A)$ is a measure

$V(\emptyset) = \mu(g \cdot 0) = 0$; $V(A) = \mu(g 1_A) \geq \mu(0) = 0$

If A_n disjoint: ~~V(A~~ $A = \bigcup_{n\geq 1} A_n$

$$V(A) = \mu\Big(\sum_{k=1}^{N} a_k 1_{B_k} 1_A\Big) = \sum_{k=1}^{N} a_k \mu(1_{B_k \cap A})$$ (μ_* is linear on simple functions)

$$= \sum_{k=1}^{N} a_k \sum_{n\geq 1} \mu(B_k \cap A_n) = \sum_{n\geq 1} \sum_{k=1}^{N} a_k \mu(1_{B_k \cap A_n})$$

$$= \sum_{n\geq 1} \mu(g \cdot 1_{A_n}) = \sum_{n\geq 1} V(A_n)$$

∴ Valid measure.

Let g be a simple function, $0 \leq g \leq f$:

Let $\varepsilon > 0$ be arbitrary: $A_n = \{x \in E : f_n(x) \geq (1-\varepsilon) g(x)\}$

Since $f_n \nearrow f$, $f_n \geq (1-\varepsilon) g$ eventually. $\Rightarrow \bigcup_n A_n = E$.

f_n increasing $\Rightarrow A_n \subseteq A_{n+1}$

$$\therefore V(E) = \lim_{n\to\infty} V(A_n) = \lim_{n\to\infty} \mu(g \cdot 1_{g \leq f_n/(1-\varepsilon)})$$

$$\therefore (1-\varepsilon) \cdot \underbrace{\mu(g)}_{V(E)} = \mu((1-\varepsilon) \cdot g) = \lim_{n\to\infty} \mu((1-\varepsilon) \cdot g \, 1_{(1-\varepsilon) g \leq f_n})$$

$$\leq \lim_{n\to\infty} \mu(f_n)$$

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$$\therefore \lim_{n\to\infty} \mu(f_n) \geq \sup_{\substack{0\leq g\leq f \\ g \text{ simple}}} \underbrace{\inf_{\varepsilon>0} (1-\varepsilon)\,\mu(g)}_{\mu(g)} = \mu(f)$$

$f \geq f_n \Rightarrow \mu(f) \geq \mu(f_n) \Rightarrow \mu(f) \geq \lim_{n\to\infty} \mu(f_n)$

$\therefore$ Equality.

(b) $g_n = f_n - f_1 \Rightarrow g_n \geq 0$, measurable, increasing to $f - f_1$

$\therefore \mu(g_n) \nearrow \mu(f - f_1)$ (Monotone conv. theorem)

$\therefore \mu(f_n) - \mu(f_1) \to \mu(f) - \mu(f_1)$

$\mu(f_1) < \infty \Rightarrow \mu(f_n) \to \mu(f)$ (Add $\mu(f_1)$ on both sides)

(c) Let $E = \mathbb{N}$, & $\mathcal{E} = \mathcal{P}(\mathbb{N})$, $\mu =$ "counting measure"

$f_n(x) = -1_{x>n} \Rightarrow \forall x$, $f_n(x)$ is increasing. ($\mathcal{E} = \mathcal{P}(\mathbb{N}) \Rightarrow$ All functions measurable)

$\mu(f_n) = -\infty$. ; $\forall x \in \mathbb{N}$, $n > x \Rightarrow f_n(x) = 0 \Rightarrow f_n \to 0$ Everywhere

$\mu(f) = 0 \neq \lim_n \mu(f_n)$

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Let g be cont., bounded:

$\mathbb{E}[g(c\cdot X_n)] \to \mathbb{E}[g(c\cdot X)]$ as $z \to cz \to g(cz)$ remains cont. bounded

Claim: $\mathbb{E}[g(c\cdot X_n) - g(X_n Y_n)] \to 0$.

$\cancel{\mathbb{E}[|g(X_n Y_n) - g(X_n c)|]}$

$\cancel{(X_n, Y_n) - (X_n, c) = (0, Y_n - c) \to 0 \text{ in } \mathbb{P}}$

$\mathbb{E}[|g(X_n Y_n) - g(X_n c)|] \leq \mathbb{E}[|g(X_n Y_n) - g(X_n c)| \, \mathbb{1}_{|X_n| \leq R} \cdot \mathbb{1}_{|Y_n - c| < \delta/R}]$

$+ \sup_{n} \|g\|_\infty^2 (\mathbb{E}[\mathbb{1}_{|X_n| \geq R}] + \mathbb{1}_{|Y_n - c| \geq \delta/R})$

$\therefore \lim_{n\to\infty} \text{LHS} \leq 2\|g\|_\infty (\mathbb{E}\,\mathbb{R}(\lim_{n\to\infty} \mathbb{P}(|X_n| > R^{-1/2}) + \mathbb{P}(|X_n| \leq -R)) + \lim_{n\to\infty} \mathbb{P}(|Y_n - c| \geq \delta/R)$

$+ \mathbb{E}[\varepsilon \cdot \mathbb{1}_{|X_n| \leq R} \mathbb{1}_{|Y_n - c| < \delta/R}]$

$\leq 2\|g\|_\infty (\underbrace{\mathbb{P}(|X| \geq R-1) + \cancel{\mathbb{P}(|X| \geq R+1)}}_{\cancel{\mathbb{P}(|X| \geq R-1)}} + 0) + \varepsilon$

(True for all $R > 0$:)

3.) (a) Let g be cont. bounded:

$\therefore \mathbb{E}[g(c\cdot X_n)] \to \mathbb{E}[g(c\cdot X)]$ as $z \to cz \to g(cz)$ is cont., bounded

Fix $R > 0$, $\varepsilon > 0$: $g|_{\bar{D}(0,2R)}$ is cont. on compact set $\Rightarrow$ Uniformly cont.

$\therefore \exists\, \delta > 0$ s.t. $|z_1 - z_2| < \delta$, $z_i \in \bar{D}(0, 2R) \Rightarrow |g(z_1) - g(z_2)| < \varepsilon$

$\mathbb{E}[|g(X_n Y_n) - g(X_n c)|] \leq \mathbb{E}[\mathbb{1}_{|X_n| \leq R} \cdot \mathbb{1}_{|Y_n - c| < \delta/R} |g(X_n Y_n) - g(X_n c)|]$

$+ 2\|g\|_\infty \mathbb{E}[\mathbb{1}_{|X_n| > R} + \mathbb{1}_{|Y_n - c| \geq \delta/R}]$

$\leq \varepsilon + 2\|g\|_\infty \mathbb{E}[\mathbb{1}_{X_n \geq R} + \mathbb{1}_{X_n \leq -R+1} + \mathbb{1}_{|Y_n - c| \geq \delta/R}]$

$\therefore \lim_{n\to\infty} \text{LHS} \leq \varepsilon + 2\|g\|_\infty \mathbb{P}(X \geq R-1) + \mathbb{P}(X < -(R-1)) + 0 \leq \varepsilon + 2\|g\|_\infty \mathbb{P}(|X| \geq R-1)$

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(True for all R): $\mathbb{P}(|X| \geq R-1) \to 0$ as $R \to \infty$

$\therefore \lim_{n\to\infty} LHS \leq \varepsilon$ (ε arbitrary) $\Rightarrow \lim_{n\to\infty} LHS = 0$

$\therefore \mathbb{E}[\,g(X_nY_n) - g(X_nc)\,] \to 0.$

$\therefore |\mathbb{E}[g(X_nY_n) - g(Xc)]| \leq |\mathbb{E}[g(X_nY_n) - g(X_nc)]| + |\mathbb{E}[g(X_nc) - g(Xc)]|$

$\to 0.$

$\therefore X_nY_n \to Xc$ in distribution

(b) Central limit theorem: $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \xrightarrow{d} N(0,1)$

$\frac{1}{n} \sum_{k=1}^{n} Z_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[Z_1^2] = 1$ (Weak law of large #)

If $X_n \to c$ ($c > 1$) in $\mathbb{P}$

$\mathbb{P}\left(\left|\frac{1}{X_n} - \frac{1}{c}\right| \geq \varepsilon\right) = \mathbb{P}\left(\frac{|X_n - c|}{|X_n|\cdot|c|} \geq \varepsilon\right) \leq \mathbb{P}\left(|X_n - c| \geq \varepsilon\cdot|c|\,|c|/2,\ |X_n| \geq c/2\right)$

$+ \mathbb{P}(|X_n - c| \geq \varepsilon/2) \to 0$ as $n \to \infty.$

$\therefore \frac{1}{\frac{1}{n}\sum_{i=1}^{n} Z_i} \to \frac{1}{1} = 1$ (in $\mathbb{P}$.)

(c) $\therefore \frac{\sqrt{n}\sum_{i=1}^{n} Z_i}{\sum_{i=1}^{n} Z_i^2} \to 1\cdot N(0,1) = N(0,1)$ in distribution

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4.) (a) $\int_{\mathbb{R}} |f * g_t(x)|\, dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |g_t(y)| \cdot |f(x-y)|\, dy\, dx$

$= (\text{Fubini}) \int_{\mathbb{R}}\int_{\mathbb{R}} |g_t(y)| \cdot |f(x-y)|\, dx\, dy = \int_{\mathbb{R}} |g_t(y)|\, \|f\|_1\, dy$

$= \|g_t\|_1 \cdot \|f\|_1 < \infty.$

$\therefore\ f * g_t \in L_1$

(b) Claim: $g_t(x) = (2\pi)^{-1}\, \hat{g}_t(-x)$

$\int_{\mathbb{R}} e^{-\alpha x^2} e^{iux}\, dx = \phi(u)$

$\phi'(u) = \int_{\mathbb{R}} ix\, e^{-\alpha x^2} e^{iux}\, dx$ ($u \to e^{-\alpha x^2} e^{iux}$ diff., derivative integrable)

$= \frac{i}{2\alpha}\left(0 + \int_{\mathbb{R}} iu\, e^{iux} e^{-\alpha x^2}\, dx\right) = -\frac{u}{2\alpha}\phi(u)$

$\therefore\ \phi(u) = e^{-u^2/4\alpha} \cdot \underbrace{\int_{\mathbb{R}} e^{-\alpha x^2}\, dx}_{\sqrt{2\pi/2\alpha}} = \sqrt{\alpha/\pi}$

$\int_{\mathbb{R}} e^{-x^2/2t} e^{iux}\, dx = \phi(u)$:

$\phi'(u) = \int_{\mathbb{R}} ix \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} e^{iux}\, dx = \left[\frac{i e^{iuy}}{-1}(-t)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\right]_{\mathbb{R}} + it \int_{\mathbb{R}} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} (iu)\, dx$

$\therefore\ \phi'(u) = -ut\, \phi(u) \Rightarrow \phi(u) = e^{-u^2 t/2}$

$\therefore\ \hat{g}_t(u) = e^{-u^2 t/2} \Rightarrow \hat{\hat{g}}_t(x) = \sqrt{2\pi/t}\, e^{-x^2/2t} = (2\pi)\, g_t(x)$

$f * g_t(x) = \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} e^{iuy}\, \hat{g}_t(u) \frac{du}{(2\pi)}\, dy$

$= \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{iuy}\, \hat{g}_t(u)\, f(x-y)\, dy\, du$

$= \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{iux} e^{-iuw}\, \hat{g}_t(u)\, f(w)\, dw\, du$

$= \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ e^{iux}\, \hat{g}_t(u)\, \hat{f}(-u)\, du \right. =$

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$$(2\pi)^{-1}\int_{\mathbb{R}} \underbrace{\hat{g}_t(-u)}_{=\hat{g}(u)}\ \hat{f}(-u)\ e^{iux}\,du = (2\pi)^{-1}\int_{\mathbb{R}} \hat{g}_t(u)\hat{f}(u)\, e^{-iux}\,du$$

$$= \widehat{\hat{f}\cdot\hat{g}_t}(-x)$$

But $\widehat{f*g} = \hat{f}\cdot\hat{g} \Rightarrow \widehat{\widehat{f*g_t}}(x) = (2\pi)\ f*g_t(-x)$

Claim: $f*g_t(x) \to f(x)$

$$\cancel{LHS = \int_{\mathbb{R}} f(x-y)\frac{1}{\sqrt{2\pi t}}e^{-y^2/2t}\,dy \quad (y/\sqrt{t} \to y)}$$

$$\cancel{= \int_{\mathbb{R}} f(x-\sqrt{t}\,y)\, g_1(y)\,dy = \int_{\mathbb{R}} f}$$

Since $f_1 \in L^1$: $f(x) = (2\pi)^{-1}\hat{\hat{f}}(-u)$ (A.E.)

$$\therefore |f*g_t(x) - f(x)| = (2\pi)^{-1}\left|\widehat{\hat{f}\hat{g}}(-x) - \hat{\hat{f}}(-x)\right| \le (2\pi)^{-1}\|\hat{f}\hat{g}_t - \hat{f}\|_1 \quad (A)\cdot$$

$$= (2\pi)^{-1}\int |\hat{f}(x)|\cdot|\hat{g}_t(x)-1|\,dx = (2\pi)^{-1}\int |\hat{f}|\cdot\underbrace{\left|e^{-x^2 t/2}-1\right|}_{\le 2}\,dx$$

$\therefore$ Integrand dominated by $|\hat{f}| \in L^1 \Rightarrow$ Dominated convergence theorem valid

$$\therefore \lim_{t\to 0}\|f*g_t - f\|_\infty \le \frac{1}{(2\pi)}\int |\hat{f}|\lim_{t\to 0}\left|e^{-tx^2/2}-1\right|\,dx = 0.$$

$\therefore$ $f*g_t \to f$ A.E. uniformly, on set $\{x : f(x) = \frac{1}{(2\pi)}\hat{\hat{f}}(-x)\}$.

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