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Applied Probability 2019

1.) (a) We note that $(T_n)_{n\geq 0}$ is the ~~hold~~ jump times,
$(V_n)_{n\geq 0}$ is the holding time of the continuous markov chain

$\therefore$ At state $i$, holding time $\sim Exp(q(i))$ $\therefore$ Moves away at rate $i$

Chain moves to state $j$ $(j\neq i)$ at ~~a~~ with probability $P_{i,j}$

By applying thinning: $i$ moves to $j$ at rate $q(i)\, P_{i,j}$

$\therefore$ Process has Q-matrix: $Q_{ij} = q(i)\, P_{ij}\, 1_{j\neq i} - q(i)\, 1_{j=i}$

(4/4) $\therefore$ Markov Chain (continuous)

(0/1) why is $X_t$ defined for all $t\geq 0$?

(b) $X$ is irreducible if $\forall\, x, y \in S$, $\exists$ sequence $x = x_0, x_1, \ldots = x_n = y$

(2/2) s.t. $Q(x_i, x_{i+1}) > 0$ for $0 \leq i < n$

(2/2) $x \in S$ is recurrent: $\mathbb{P}(\{t : X(t) = x\}$ is unbounded$) = 1$

(2/2) +ve recurrent: $\mathbb{E}_x[\inf\{~~t\geq 0~~\, t\geq T_1 : X_t = x\}] < \infty$,

$T_1 = \inf\{t\geq 0 : X_t \neq X_0\}$

(c) ~~If~~ $X$ is irreducible iff $\forall\, x, y \in S$, $\exists\, x_0, x_1, \ldots x_n$ s.t. $x = x_0$, $y = x_n$
and $Q(x_i, x_{i+1}) > 0$ $(i\geq 0)$

iff $\forall\, x, y \in S$, $\exists\, x = x_0, \ldots x_n = y$ s.t.
$P(x_i, x_{i+1}) > 0$ $(i \geq 0)$ as $P(x_i, x_{i+1}) = Q(x_i, x_{i+1})/q(x_i)$

and $x_i \neq x_{i+1}$ else $Q(x_i, x_{i+1}) < 0$.

(3/3) iff $Y$ is irreducible.

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If ~~X~~ $Y$ is recurrent: $\forall x \in S$, $(Y_n)_{n\geq 0}$ visits $x$ i.o. (A.S.) where $Y_0 = x$.

$\therefore$ If $X_0 = Y_0 = 0$, $X_{J_n} = x$ for infinite ~~#~~ $n$
$(J_n)_{n\geq 0}$ are jump times.

$Y$ recurrent $\Rightarrow$ $X$ non explosive $\Rightarrow$ $J_n \to \infty$ A.S.
$\therefore$ $\{t: X_t \geq 0\}$ is unbounded (A.S.)

If $Y$ is transient: $\exists x \in S$, $\exists N \in \mathbb{N}$ s.t. $\mathbb{P}_x(Y_n \neq x \text{ for } n \geq N) > 0$

$\wedge$ $\{Y_n \neq x \text{ for } n \geq N\} \Rightarrow \{t: X_t = x\} \subseteq [0, J_{\#N}]$

$J_N$ bounded A.S. $\Rightarrow$ $\mathbb{P}_x(\{\cancel{X}\ t: X_t = x\} \text{ bounded}) >$

$\mathbb{P}_x(Y_n \neq x \text{ for } n \geq N) > 0$

(3/3) $\therefore$ $X$ transient.

(d) If $\pi \cdot Q = 0$

~~Lea~~ $\sum_{i: i\neq j} \pi_i Q_{i,j} = \pi_j\, q(j)$

$$LHS = \sum_{i: i\neq j} \pi_i \cdot q_i \, Q_{ij}/q(i) = \sum_{i: i\neq j} \pi_i q_i P_{ij} = \sum_i \pi_i q_i P_{ij}$$

$(P_{ii} = 0)$

$\therefore$ $(\pi_i q_i)_{i\in S}$ is an invariant measure ~~as~~ of $Y$.

$Y$ +ve recurrent $\Rightarrow$ Invariant measure unique (up to constant)

$$\Lambda = \sum_{i\in S} \pi_i q_i$$

(2/3)

$\left(\pi_i q_i / \Lambda\right)_{i\in S}$ is the invariant distribution

(18) you were asked to give invariant of $X$ using invariant of $y$...

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2.) (a) $S = \mathbb{Z}_{\geq 0}$ is state space :

$$P(i,j) = \lambda_i / \lambda_i + \lambda_i p_i \quad \text{if } j = i+1$$
$$= \lambda_i p_i / \lambda_i + \lambda_i p_i \quad \text{if } j = 0, \; i \neq 0$$
$$= 0 \quad (\text{otherwise})$$

(2/2)

WLOG:
( $p_0 = 0$ )

(b) $\mathbb{P}(N \to 0) > 0$ for $N \geq 1$. ( " $N \to 0$ " represents chain moving from N to 0 )

But $\mathbb{P}(0 \to N) = 0$. ∴ If we watch the reversed process, it cannot have the same distribution as the original one as $N \to 0$ is not possible in the reversed version.

Not needed.

(b) Reversed Q-matrix : $\hat{Q}(i,j) = \pi_j / \pi_i \; Q(j,i)$

(2/2)

$$\therefore \hat{Q}(n,0) = \pi_j/\pi_i \; Q(0,n) = 0 \neq Q(n,0) > 0$$
for $n \geq 1$

(c) Consider jump chain: $P(i,j) = 1/1+p_i$ if $j = i+1$
$= p_i / 1+p_i$ if $j = 0$

Since chain is clearly irreducible: $0 \to 1, .. \to n \to 0 \Rightarrow 0$ communicates with all states. ⇒ Irreducible

Y recurrent iff 0 is Y- recurrent ( Y is jump chain)
iff $\mathbb{P}_0$ ( Y returns to 0) = 1
iff $\mathbb{P}_0$ ( Y never returns to 0) = 0

(4/4)

iff $\prod_{k \geq 1} 1/1+p_k = 0$ iff $\prod_k (1+p_k) = \infty$.

∴ X transient iff Y transient iff $\prod_k (1+p_k) < \infty$

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(d) $\prod_k (1+P_k) < \infty \Rightarrow$ Chain is transient. Let $(J_n)_{n\geq 0}$ be jump times

Consider jump chain: (A.S.) $\exists\, N$ s.t. $\cancel{X_N}\ Y_{N+k} = k$.

$\therefore\ \zeta = \sum_{k=1}^{N-1} J_N + \sum_{k\geq 0} \mathcal{E}_{N+k}/\lambda_k(1+P_k)$ , $(\mathcal{E}_n)_{n\geq 1}$ is iid Exp(1), independent of jump chain

Since $J_N < \infty$ A.S. : $\zeta = \infty$ iff $\sum_{k\geq 0} \mathcal{E}_{k+N}/\lambda_k(1+P_k) = \infty$

Consider a birth chain, $Q(n,n+1) = \lambda_n(1+P_n)$

Chain is explosive A.S. if $\sum_n 1/\lambda_n(1+P_n) < \infty$

Chain is non-explosive if $\sum_n 1/\lambda_n(1+P_n) = \infty$

But chain explosive iff $\zeta = \sum_{k\geq 0} \mathcal{E}'_k/\lambda_k(1+P_k) < \infty$

$(\mathcal{E}'_k)_{k\geq 1} \sim$ iid Exp(1))

$\therefore \sum_{k\geq 0} \mathcal{E}_{k+N}/\lambda_k(1+P_k) < \infty$ iff $\sum_{k\geq 0} 1/\lambda_k(1+P_k) < \infty$

Since this is independent of N: Chain is A.S. explosive if $\sum_{k\geq 0} 1/\lambda_k(1+P_k) < \infty$, non-explosive otherwise

(8/8)

(c) $n \geq 1$ : $\pi_n \cdot \lambda_n(1+P_n) = \pi_{n-1} \cdot \lambda_{n-1}$

$\therefore\ \pi_n = \lambda_{n-1}/\lambda_n(1+P_n) \cdot \pi_{n-1} = \frac{\lambda_{n-1} \cdots \lambda_0}{\lambda_n \cdots \lambda_1} \cdot \frac{\pi_0}{(1+P_n)\cdots(1+P_1)}$

$= \frac{\lambda_0 \pi_0}{\lambda_n \prod_{k=1}^{n}(1+P_k)}$

Check:

$\therefore\ \cancel{\pi_0 = 1},\ \pi_0\lambda_0 = \sum_{n\geq 1} \pi_n \lambda_n P_n$

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$$\text{RHS} = \lambda_0 \pi_0 \sum_{n\geq 1} P_n \Big/ \prod_{k=1}^{n} (1+P_k)$$

$$1 - \sum_{n=1}^{N} P_n \Big/ \prod_{k=1}^{n} (1+P_k) = 1 \Big/ \prod_{k=1}^{n} (1+P_k)$$

$$\therefore \text{RHS} = \text{LHS} \quad \text{iff} \quad \prod_{k=1}^{\infty} (1+P_k) = \infty$$

$$\therefore \text{Invariant measure exists iff} \prod_{k=1}^{\infty} (1+P_k) = \infty$$

~~In Invariant distribution exists ⇒ Invariant measure exists~~

~~Invariant measure ex~~

Invariant distribution exists iff $\prod_{k=1}^{\infty} (1+P_k) = \infty$ and

$$\sum_{n\geq 1} 1 \Big/ \lambda_n \prod_{k=1}^{n} (1+P_k) < \infty.$$

4(4)

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3.) (a) Reversible in Equilibrium: ~~Fix T≥0~~. $(X_t)_{t\geq 0}$ is reversible in equilibrium if $\forall\, T \geq 0$, $(X_t)_{t\in[0,T]}$, $(X_{T-t})_{t\in[0,T]}$ have the same distribution

(2/2)

Detailed Balance: Let $Q$ be generator

$\forall\, x, y \in S$ = state space: $\pi_x Q(x,y) = \pi_y Q(x,y)$ (~~x≠y~~)

(2/2)

If $\underline{\pi}$ is in detailed balance: $\sum_x \pi_x Q_{xy} = \sum_x \pi_y Q_{yx} = \pi_y \sum_x Q_{yx} = 0$

$\therefore\ \underline{\pi}$ is invariant

(3/3)

(b) Generator: $Q$ ; $S = \{0, \dots N+S\}$.

$Q(n, n+1) = \lambda$

$Q(n~~+1~~, n-1) = \text{Min}\{n, s\}\cdot\mu$

$Q(n,n) = -(\lambda \mathbb{1}_{n\neq N+S} + \text{Min}\{n,s\}\mu)$

$Q = 0$ (otherwise)

There are finite states; ~~or~~ Since $Q(n, n\pm 1) > 0$ for ~~all~~ $n, n\pm 1 \in S$ : $X$ ~~(Q)~~ is irreducible.

Finite states $\Rightarrow$ ~~X~~ ~~is~~ Jump chain recurrent $\Rightarrow$ $X$ recurrent

(1/2)

$\therefore$ Any invariant measure is unique up to constant ~~(as non-explosive)~~ non-explosive?

Solve Detailed Balance:

$0 \leq k \leq s-1$: $\pi_k \cdot \lambda = \pi_{k+1}(k+1)\mu \Rightarrow \pi_{k+1} = \pi_0 \cdot (\lambda/\mu)^{k+1} / (k+1)!$

$k \geq s$: $\pi_k \lambda = \pi_{k+1}(s\mu) \Rightarrow \pi_k = \pi_s (\lambda/s\mu)^{k-s}$

$\therefore$ ~~$\pi$~~ $\underline{\pi}$: $\pi_k = \pi_0 (\lambda/\mu)^k / k!$ $(0 \leq k \leq s)$

$= \pi_0 (\lambda/\mu)^s / s! \cdot (\lambda/s\mu)^{k-s}$ $(k \geq s)$

After normalising: Distribution is unique invariant distribution

$$\Lambda = \sum_{k=0}^{S} (\lambda/\mu)^{k}/k! + \sum_{k=S+1}^{S+N} (\lambda/\mu)^{S}/S! \, (\lambda/S\mu)^{k-S} < \infty$$

(6/6) $\tilde{\pi}_k = \pi_k / \Lambda$ is invariant distribution

(1/1) Since we have solved detailed balance, $X$ is reversible in equilibrium (*)

(*): $X$ is reversible in equilibrium iff Detailed Balance has a solution

(19/2)

4.) (a) $\Pi$ is a Poisson process if:

$\forall$ Bounded, Measurable $A \subseteq \mathbb{R}^n$ : $|\Pi \cap A| \sim Poi(\int_A \lambda)$

If $A_m \in \mathcal{B}(\mathbb{R}^n)$, $(A_m)_{m\geq 1}$ disjoint : $(\Pi \cap A_m)_{m \geq 1}$ is independent sequence of R.V.

(5/5)

(b) If $\forall x \in \mathbb{R}^s$, $\int_{f^{-1}\{x\}} \lambda \, d\cancel{t} = 0$ and

$\forall$ Bounded, measurable $B \subseteq \mathbb{R}^s$ : $\int_{f^{-1}(B)} \lambda < \infty$

(6/6) Mean measure: $\tilde{\Lambda}(B) = \int_{f^{-1}(B)} \lambda = \Lambda \cdot f^{-1}(B)$

$\therefore \tilde{\Lambda} = \Lambda \cdot f^{-1}$ (set function)

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(c)

$f^{-1}\{r\} = \{ \underset{\sim}{x} \in \mathbb{R}^2 : |\underset{\sim}{x}|^2 = r^2 \}$ has 0 - Lebesgue measure

$\forall R \geq 0$ : $\mu ( f^{-1} [0, R^2] ) = \pi R^2 < \infty$ ($\mu$ is Lebesgue measure)

$\therefore \forall$ Bounded $B$, $\mu ( f^{-1} (B)) < \infty$

$\therefore f(\Pi)$ is a Poisson process

$\tilde{\Lambda}(B) = \Lambda \cdot f^{-1}(B)$

If $B = [0, R^2]$ : $\Lambda \cdot f^{-1}([0, R^2]) = \mu ( \{ \underset{\sim}{x} : |\underset{\sim}{x}|^2 \leq R^2 \})$

$= \pi \cdot R^2$

$\therefore \Lambda \cdot f^{-1} = \pi \cdot \mu$ on $A = \{ [0, R^2] : R \geq 0 \} \cup \{ \phi \}$

$A$ is a $\pi$ - system, $\mu$ $\sigma$-finite measure $\Rightarrow \Lambda \cdot f^{-1} = \pi \cdot \mu$ on $\sigma(A)$

$= \mathcal{B}(\mathbb{R}^2)$

$\therefore \tilde{\Lambda}(B) = \mu(B) \Rightarrow$ Homogenous $\pi$ - intensity process

Since $f(\Pi)$ is homogenous poisson process :

~~$\mathbb{P}(R_k^2 \leq x)$~~ ~~(R~~ Let $S_k = R_k^2$ :

~~$S_0, S_1, S_2, \ldots, S_{n+1} - S_n$~~

$S_0 = 0$ : $(S_{n+1} - S_n)_{n \geq 0} \sim \operatorname{Exp}(\pi)$ R. V.

$\therefore S_n$ is the sum of $n$ iid $\operatorname{Exp}(\pi)$ R. V.

$$\mathbb{P}(R_k \leq x) = \mathbb{P}(R_k^2 \leq x^2) = \mathbb{P}(\operatorname{Poi}(x^2 \pi) \geq k)$$

$$= \sum_{n \geq k} e^{-x^2 \pi} (x^2 \pi)^n / n!$$

$$\therefore f_{R_k}(x) = \sum_{n \geq k} \frac{1}{n!} e^{-x^2 \pi} \left[ -2x\pi (x^2 \pi)^n + \underset{(2x\pi)}{n} (x^2 \pi)^{n-1} \right]$$

$$= \sum_{n \geq k} 1/n! \, (2x\pi)(x^2 \pi)^{n-1} (n - 2x\pi)$$

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$$= (2x\pi) \cdot \left\{ \sum_{n \geq k} \frac{1}{(n-1)!} (x^2\pi)^{n-1} - \sum_{n \geq k} \frac{1}{n!} (x^2\pi)^n \right\} e^{-x^2\pi}$$

(9/9)

$$= \frac{(2x\pi)}{(k-1)!} (x^2\pi)^{k-1} e^{-x^2\pi}$$

(20/20)

2018 Paper 3

(a) If $Q_n \neq 0$: $(n+1)^{th}$ service starts immediately.

$Q_{n+1} = (Q_n - 1)$ ($n^{th}$ customer serviced)

$+ A_n$ (arrivals during $n^{th}$ service time)

Else: Wait until customer arrives. $Q_n \to Q_n + 1$

(Same as above)

$\therefore Q_{n+1} = Q_n + 1 - 1 + A_n = Q_n + A_n$

(4/4)

$$\therefore Q_{n+1} = Q_n + A_n - \mathbb{1}_{Q_n \neq 0}$$

$A_n \sim Poi(\lambda \cdot S)$

$$\mathbb{E}[A_n] = \mathbb{E}[Poi(\lambda \cdot S)] = \mathbb{E}[\mathbb{E}[Poi(\lambda S) \mid S]] = \lambda \cdot \mathbb{E}[S] = \rho$$

$$\mathbb{E}[A_n^2] = \mathbb{E}[\mathbb{E}[Poi(\lambda \cdot S)^2 \mid S]] = \mathbb{E}[(\lambda \cdot S + \lambda^2 S^2)]$$
$$= \rho + \lambda^2 \mathbb{E}[S^2]$$

(4/4)

(b) If $Q_n$ in equilibrium:

$$\mathbb{E}[Q_{n+1}] = \mathbb{E}[A_n] + \mathbb{E}[Q_n] - \mathbb{E}[\mathbb{1}_{Q_n \neq 0}] = \mathbb{E}[Q_n]$$

$$\therefore \mathbb{E}[\mathbb{1}_{Q_n \neq 0}] = \rho$$

$$\mathbb{E}[Q_n^2] = \mathbb{E}[Q_{n+1}^2] = \mathbb{E}[A_n^2] + \mathbb{E}[Q_n^2] + \mathbb{E}[\mathbb{1}_{Q_n \neq 0}] +$$
$$2\,\mathbb{E}[A_n \cdot Q_n] - 2\,\mathbb{E}[\mathbb{1}_{Q_n \neq 0} \cdot A_n] - 2\,\mathbb{E}[Q_n \cdot \mathbb{1}_{Q_n \neq 0}]$$

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$\therefore \ \rho + \lambda^2 \mathbb{E}[S^2] + \rho + 2\,\mathbb{E}[A_n]\,\mathbb{E}[Q_n] - 2\,\mathbb{E}[A_n]\,\mathbb{E}[\mathbb{1}_{Q_n \neq 0}] - 2\,\mathbb{E}[Q_n] = 0$

($A_n$, $Q_n$ independent)

$\therefore \ 2\rho + \lambda^2 \mathbb{E}[S^2] + 2(\rho - 1)\,\mathbb{E}[Q_n] - 2\rho^2 = 0$

(12/12) $\therefore \ \mathbb{E}[Q_n] = \rho + \lambda^2 \mathbb{E}[S^2] / 2(1-\rho)$

Good

* Highly sus part: Mean waiting time $\neq$ : $\mathbb{E}[Q_n]/\lambda$ = Average ~~queue~~

$\mathbb{E}[Q_n]$ = Average # of people in queue (Not really)

$\lambda$ = arrival intensity.

Equilibrium exists $\Rightarrow$ Regenerative

Little: Average wait time $= \mathbb{E}[Q_n]/\lambda$

$\therefore$ Average wait (without service) time $= -\rho + \mathbb{E}[Q_n]/\lambda$

$= \frac{1}{2} \lambda \,\mathbb{E}[S^2] / (1-\rho)$

Seems to work, but $Q_n$ is length immediately after someone leaves $\Rightarrow$ "True" average queue should be longer.

* Proper method: Consider waiting time (without service)

$\mathbb{E}[\text{wait time}] = \mathbb{E}[\text{Queue length}] \cdot \mathbb{E}[S] + \mathbb{E}[R]$,

$R$ is remaining time of service.

Little: $\mathbb{E}[\text{Queue length}] = \mathbb{E}[\text{Wait time}] \cdot \lambda$

$\therefore \ \mathbb{E}[\text{wait time}] = \mathbb{E}[R] / (1 - \lambda \mathbb{E}[S]) = \mathbb{E}[R]/(1-\rho)$

Consider $\int_0^{T_n} R(t)\, dt$, $T_n$ is time after $n^{th}$ service

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$\therefore$ { By adding triangle area:

$$\int_0^{T_n} R(t)\,dt = \sum_{k=1}^{n} S_k^2/2$$

In the long run: # of service completed at time t
→ # of arrivals up to t.

$$\therefore \frac{1}{T_n}\int_0^{T_n} R(t)\,dt = \frac{n}{T_n}\;\frac{1}{n}\int_0^{T_n} R(t)\,dt$$

$n/T_n \to \lambda$ ,

$$\therefore \mathbb{E}\left[\frac{1}{T_n}\int_0^{T_n} R(t)\,dt\right] \to \lambda/2\;\mathbb{E}[S^2]$$

$$\therefore \mathbb{E}[R] \to \lambda/2\;\mathbb{E}[S^2]$$

$\therefore$ $\mathbb{E}[\text{Wait time}] = \lambda/2\;\mathbb{E}[S^2]$ ※

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Applied Probability 2020

1.) (a) N is a birth process iff it is a cont. markov chain with:

State space = $\mathbb{Z}_{\geq 0}$

Q-matrix generator: $Q(n,n+1) = \lambda_n$

$Q(n,n) = -\lambda_n$

$Q(n,m) = 0$ (otherwise)

Let $(J_n)_{n\geq 0}$ be jump times, $S_n = J_n - J_{n-1}$ $(n \geq 1)$

$(S_n)_{n\geq 1}$ are independent, $S_n \sim \mathrm{Exp}(\lambda_{n-1})$

(5/5) $\zeta = \sum_{n\geq 1} S_n$ : N is non-explosive iff $\mathbb{P}(\zeta = \infty) = 1$

(b) If $\sum_{n=0}^{\infty} 1/\lambda_n = \infty$ : Let $A_n = \{S_{n+1} \geq 1/\lambda_n\}$ ; $\mathbb{P}(A_n) = e^{-\lambda_n \cdot (1/\lambda_n)} = e^{-1}$

$\therefore \sum_{n\geq 0} \mathbb{P}(A_n) = \infty \Rightarrow$ (1st Borel Cantelli) $A_n$ occurs i.o.

$\zeta = \sum_{n\geq 1} S_n \geq \sum_{n\geq 1} S_n \cdot 1_{S_n \geq 1/\lambda_{n-1}}$

$$\mathbb{E}\left[e^{-\sum_{n=1}^{\infty} \cancel{A_n} S_n}\right] = \prod_{n\geq 1} \mathbb{E}\left[e^{-S_n}\right] = \prod_{n\geq 1}\left(\frac{\lambda_{n-1}}{1+\lambda_{n-1}}\right)$$

$$= \left(\prod_{n\geq 1}\left(1 + 1/\lambda_{n-1}\right)\right)^{-1} \leq \left(\sum_{n\geq 1} 1/\lambda_{n-1}\right)^{-1} = 0.$$

$\therefore \sum_{n=1}^{\infty} S_n = \infty$ (A.S.) $\Rightarrow$ Non-explosive

If $\sum_{n\geq 0} 1/\lambda_n < \infty$ : $\mathbb{E}[\zeta] = \sum_{n\geq 0} 1/\lambda_n < \infty \Rightarrow \zeta < \infty$ A.S.

(5/5) $\therefore$ A.S. Explosive.

(c) $f(t) = \mathbb{E}[N(t)] = \sum_{k\geq 0} P_{0,k}(t) \cdot k$

$f'(t) = \sum_{k\geq 0} k\, P'_{0,k}(t) = \sum_{k\geq 0} k \sum_{j\geq 0} Q_{jk} P_{0,j}(t)$ (Kolmogorov forward)

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$$= \sum_{j\geq 0} P_{0,j}(t) \underbrace{\sum_{k\geq 0} Q_{j,k}\, k}$$

$$-Q_{j,j}\, j + (j+1)\, Q_{j,j+1} = \overset{(j+1)}{\cancel{(\alpha j + \beta)}} \quad \alpha j + \beta$$

$$= \beta \sum_{j\geq 0} P_{0,j}(t) + \alpha \sum_{j\geq 0} j\, P_{0,j}(t) = \beta + \alpha f(t)$$

$f' - \alpha f = \beta$ : $\cancel{\beta}$. $f' - \alpha f$ : $f(t) = C_1 e^{\alpha t}$ | $C_1$ constant

$\therefore f(t) = C_1 e^{\alpha t} - \beta/\alpha$ is a solution

(10/10)

(20 a)

$f(0) = 0$ : $C_1 = \beta/\alpha$ ; . $\cancel{C_1}$ : $f(t) = (\beta/\alpha)(e^{\alpha t} - 1)$ $_{\#}$

2.) (a) Let $J_0 = 0$, $J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}$ $(n \geq 0)$

$Y_n = X_{J_n}$ , $(Y_n)_{n\geq 0}$ is the jump chain

(4/4)

$X$ is irreducible if $\forall\, x, y \in S$ : $\exists$ sequence $x = x_0, \ldots x_N = y$

s.t $Q(x_i, x_{i+1}) > 0$ for $0 \leq i < N$.

$X$ is ~~irreducible~~ recurrent if: $\forall$ state $x$, $\mathbb{P}_x(\{t : X_t = x\}$ unbounded$) = 1$

If $Y$ is recurrent: $(X_t)_{t\geq 0}$ is non-explosive. ~~For state~~ $x$.

~~Since $\exists$ subsequence $n_k$ s.t $Y_{n_k} = x$~~

~~$\therefore \{t:$~~

Fix state $x$ : (A.S.) $\exists\, n_k$ s.t $\cancel{\forall}$ $Y_{n_k} = x$.

$\therefore \{J_{n_k} : k \geq 1\} \subseteq \{t : X_t = x\}$

LHS is A.S. unbounded $\Rightarrow$ $\{t : X_t = x\}$ is A.S. unbounded

$\therefore \mathbb{P}(\{\forall\, t : X_t = x\}$ unbounded$) = 1$

$\therefore X$ is recurrent.

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If $Y$ is transient: $\exists$ state $x$ s.t. $\mathbb{P}\left(\bigcup_{n\geq 1}\bigcap_{m\geq n}\{Y_m \neq x\}\right) > 0$

$$\left\{\{t: X_t = x\} \text{ bounded}\right\} \supseteq \bigcup_{n\geq 1}\{X_t \neq x \text{ for } t \geq J_n\} \cap \{J_1, .. J_n \neq \infty\}$$

$$= \bigcup_{n\geq 1}\left\{\bigcap_{m\geq n}\{Y_m \neq x\}\right\} \cap \left\{J_n \neq \infty : n \in \mathbb{N}\right\}$$

$$\therefore \mathbb{P}(\{t: X_t = x\} \text{ bounded}) \geq \mathbb{P}\left(\bigcup_{n\geq 1}\bigcap_{m\geq n}\{Y_m \neq x\}\right) - \sum_{n\geq 0}\mathbb{P}(J_n = \infty) = \mathbb{P}\left(\bigcup_{n\geq 1}\bigcap_{m\geq n}\{Y_m \neq x\}\right)$$

$$\neq 0.$$

(818) $\therefore X$ is also transient.

(h) Jump chain: Random walk on $\mathbb{Z}$, $\mathbb{P}(+1) = 2/3$, $\mathbb{P}(-1) = 1/3$

$\therefore$ Jump chain is transient $(\to -\infty) \Rightarrow X$ is also transient

$(\overset{n}{k} \geq 0)$: (Detailed Balance): $\pi_n \cdot 2 \cdot 3^n = \pi_{n+1}\, 3^{n+1} \Rightarrow \pi_{n+1} = \pi_n \left(2/3\right)$

$(n \leq 0)$: (Detailed Balance) $\pi_n \cdot \cancel{2}\ 3^{-n} = \pi_{n-1} \cdot 2\, 3^{-n+1}$

$$\therefore \pi_{n-1} = \pi_n / 6$$

$$\therefore \pi_n = (2/3)^n \pi_0 \quad (n \geq 0)$$
$$= (1/6)^{|n|} \pi_0 \quad (n \leq 0)$$

Solve Detailed Balance $\Rightarrow$ Invariant

$$\sum_n \pi_n = -1 + \frac{1}{1-2/3} + \frac{1}{1-1/6} = 2 + 6/5 < \infty$$

$\therefore$ Invariant distribution exists

Theorem: If $X$ is non-explosive existence of invariant distribution $\Rightarrow$

(818) $X$ is +ve recurrent $\Rightarrow X$ is recurrent

(20/20) $X$ transient $\Rightarrow$ (Not recurrent) $X$ is explosive.

No.: Date:

3.) (a) Renewal Process. Let $(\varepsilon_n)_{n\geq 1}$ be iid, $\geq 0$ R.V.

$S_k = \sum_{r=1}^{k} \varepsilon_r$

$N(t) = \text{Max}\{k \geq 0 : S_k \leq t\}$

Let $\{(\varepsilon_n, R_n)_{n\geq 1}$ be iid : $(\varepsilon_n)_{n\geq 1}$ be defines a renewal process.

Renewal reward theorem: $\frac{1}{t}\sum_{n=1}^{N(t)} R_n \to \mathbb{E}[R_1]/\mathbb{E}[\varepsilon_1]$ (A.S.)

$$\mathbb{E}\left[\frac{1}{t}\sum_{n=1}^{N(t)} R_n\right] \to \mathbb{E}[R]/\mathbb{E}[\varepsilon]$$

(b) Lawful - Good Method.

Let $\varepsilon_n = m$, $(\varepsilon_n)_{n\geq 1}$ forms a renewal process.

By the memoryless property of Exponential distributions, if the machine is working after a check, the time until breakdown is still an exponential $(\lambda)$ R.V.

$R_n \overset{\sim}{=} \text{Min}\{\text{Exp}(\lambda), \& m\}$ is the amount of time machine is working $\&$ between $m-1^{th}$, $m^{th}$ check

(10/10)

$\therefore$ Long run proportion of machine working:

(Renewal Reward theorem:) $\sum_{n=1}^{N(t)} R_n \leq R(t) \leq \sum_{n=1}^{N(t)+1} R_n(t)$

$\therefore R(t)/t \to \mathbb{E}[R]/m$ (A.S.)

$$\mathbb{E}[R] = e^{-\lambda M}\cdot m + \int_0^{m} \lambda e^{-\lambda x} x \, dx$$

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$$\therefore\ e^{-\lambda m} + \int_0^m \frac{1}{m} \lambda e^{-\lambda x} x \, dx \quad \text{proportion}$$

$*$ Evaluate: $\cancel{e^{-\lambda m}}$

$$\int_0^m x \lambda e^{-\lambda x} dx = [x e^{-\lambda x}]_m^0 + \int_0^m e^{-\lambda x}$$

$$= -m e^{-\lambda m} + \frac{1}{\lambda} [-e^{-\lambda x}]_0^m$$

$$= -m e^{-\lambda m} + 1/\lambda \{1 - e^{-\lambda m}\}$$

$$\therefore\ e^{-\lambda m} + \frac{1}{\lambda m}(1 - e^{-\lambda m}) - e^{-\lambda m}$$

$$= 1 - e^{-\lambda m} / \lambda m$$

(10/10)

(20 d)

Chaotic Evil Method:

Let $\varepsilon_n$ be time between $n-1^{th}$, $n^{th}$ repairs,

& $(d_n)_{n\geq 1}$ be working times of machine

$(\varepsilon_n, d_n)_{n\geq 1}$ forms a renewal reward theorem

$$\varepsilon_n = m \lceil d_n / m \rceil$$

$$\mathbb{E}[d_n] = 1/\lambda$$

$$\mathbb{E}[\varepsilon_n] = m \sum_{k=1}^{\infty} k \int_{(k-1)m}^{k(m)} \lambda e^{-\lambda x} dx$$

$$= m \sum_{k=1}^{\infty} k [e^{-\lambda x}]_{km}^{(k-1)m}$$

$$= m \sum_{k=1}^{\infty} k e^{-\lambda (k-1) m} - m \sum_{k=1}^{\infty} k e^{-\lambda k m}$$

$$= m \sum_{k=0}^{\infty} (k+1) e^{-\lambda m k} - m \sum_{k=0}^{\infty} k e^{-\lambda k m}$$

$$= m \sum_{k=0}^{\infty} (e^{-\lambda m})^k = \frac{m}{1 - e^{-\lambda m}} = \frac{m}{1 - e^{-\lambda m}}$$

$$\therefore\ \mathbb{P}\text{ working} = \frac{1/\lambda}{m / (1 - e^{-\lambda m})} = 1 - e^{-\lambda m} / \lambda m \ \#$$

No.: Date:

4.) (a) Customers arrive at times of a Poisson $\lambda$ process, and there is 1 server; Service times are iid, Exp ($\mu$)

(3/3)

State space $= \mathbb{Z}_{\geq 0}$, $Q(n,m) = \lambda \cdot 1_{m=n+1} + \mu 1_{m=n-1} + \cancel{(-\lambda-\mu) 1_{m=}} + (-\lambda - \mu 1_{n>0}) 1_{m=n}$

(b) Let $(d_n)_{n\geq1}$ be Exp ($\mu$) R.V.; $k \sim$ Geo ($\delta$) be independent

$$S = \sum_{r=1}^{k} d_r : \mathbb{E}[e^{\theta S}] = \mathbb{E}[\mathbb{E}[e^{\theta \mathrm{Exp}(\mu)}]^k : k \sim \mathrm{Geo}(\delta)]$$

$$= \mathbb{E}\left[\left(\frac{\lambda}{\lambda-\theta}\right)^k\right] = \sum_{k\geq1} \delta (1-\delta)^{k-1} (\lambda/_{\lambda-\theta})^k$$

$$= \frac{\lambda\delta}{\lambda-\theta} \frac{1}{1-(1-\delta)\lambda/\lambda-\theta} = \frac{\lambda\delta}{\lambda-\theta-(1-\delta)\lambda} = \frac{\lambda\delta}{\lambda\delta-\theta}$$

(MGF of Exp ($\lambda\delta$) R.V.)

(7/7)

Since the # of service / pax $\sim$ Geo($\delta$), the total service time is a Exp ($\lambda \cdot \delta$) R.V.

If we consider only the length of queue, it we can let a customer remain at the front and ~~rec~~ be serviced until he/she leaves the system.

$\therefore$ M($\lambda$) / M($\delta \cdot \mu$) / 1 queue.

If $\lambda < \delta\mu$: Invariant distribution exists e part (i)

$\pi_n = (\lambda/\delta\mu)^n$

~~Since Q matrix of M/M/1 queue is bounded, chain~~ is non-explosive. $\therefore$

Q matrix bounded $\Rightarrow$ Non-explosive; $\therefore$ Chain is +ve recurrent

Since M/M/1 is a birth death chain, invariant $\Rightarrow$ Detailed Balance

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$\therefore$ Equilibrium is possible (+ve recurrence), process reversible at equilibrium.

(~~Using~~ Customer remains at front until full completion):
Since departures corresponds to arrivals of reversed process, reversed process has the same distribution as the original process:

7/7

Departure process $\sim$ Poisson $\lambda$ - rate process
(Relative position of customers do not affect departure process)

In equilibrium: $\mathbb{P}(\text{queue empty}) = \cancel{1/\pi_0}\ \pi_0 = (1 - (\lambda/\delta\mu))$

$$\cancel{\mathbb{E}[\#\text{ of arrivals in } [0,1]] = \underbrace{\mathbb{E}[Poi(\lambda)]}_{\text{from outside}} + \mathbb{E}\left[\int_0^1 \mu\, 1_{\text{Queue not empty}}\, dt\right]}$$

$$\cancel{= \lambda + \mu \cdot (\lambda/\delta\mu) = \lambda(1 + 1/\delta)}$$

~~If arrival process poissonian:~~ $\cancel{Poi(\lambda(1+1/\delta))}$ ~~rate process.~~

$\therefore$ ~~Invariant queue length:~~ $\cancel{\tilde{\pi}_n = c\left(\frac{\lambda(1+1/\delta)}{\mu}\right)^n = c\left(\frac{\lambda(1+\delta)}{\delta\mu}\right)^n}$

$\therefore$ $\cancel{\tilde{\pi} \neq \pi}$ $\Rightarrow$ ~~Contradiction.~~

$\mathbb{E}[\text{Time to first arrival}]$:

If queue empty $(\mathbb{P}\ 1 - \lambda/\delta\mu)$: $\mathbb{E}[Exp(\lambda)] = 1/\lambda$
Else : $\mathbb{E}[\text{Min}\{Exp(\lambda), Exp(\mu)\}]$

$$= 1/(\lambda+\mu)$$

$$\therefore \mathbb{E} = (1 - \lambda/\delta\mu)\, 1/\lambda + \lambda/\delta\mu \cdot 1/(\lambda+\mu)$$

$$= 1/\lambda - \lambda/\delta\mu\,(1/\lambda - 1/(\lambda+\mu)) = 1/\lambda - \frac{\lambda\mu}{\delta\mu\lambda(\lambda+\mu)}$$

$$= 1/\lambda - 1/\delta(\lambda+\mu)$$

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(3/3)

$\therefore$ If Poissonian: $Poi\left( \cancel{1/\lambda} \; 1 / \underbrace{1/\lambda - 1/\delta(\lambda+\mu)}_{\delta \lambda_1} \right)$ process

~~Invariant: $(\lambda_1/\mu)^n \neq$~~

Invariant: $Geo(1 - \lambda_1/\mu) \neq Geo(1 - \lambda/\delta\mu)$

$\therefore$ Not Poissonian

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Applied Probability 2021

1.) (a) Claim: Let $A \subseteq \mathbb{R}^3$ be bounded, measurable. $|\Pi_1 \cap A| + |\Pi_2 \cap A| = |A \cap (\Pi_1 \cup \Pi_2)|$ (A.S.)

(Activity of $\Pi_1, \Pi_2$: $\lambda, \mu$)

Pf Let $B_{N,k} = \prod_{i=1}^{3} [k_i \cdot 2^{-N}, (k_i+1) 2^{-N})$, $N \in \mathbb{N}$, $k \in \mathbb{Z}^3$

$\mathbb{P}(|\Pi_1 \cap \Pi_2 \cap A| \neq 0) \leq \sum_{k \in \mathbb{Z}^3} \mathbb{P}(|\Pi_1 \cap \Pi_2 \cap A \cap B_{N,k}| \neq 0)$

$\leq \sum_{k \in \mathbb{Z}^3} \mathbb{P}(|\Pi_i \cap A \cap B_{N,k}| > 0 \ : i = 1,2) \leq \sum_{k \in \mathbb{Z}^3} (1 - e^{\int_{A \cap B_{N,k}} \lambda \, dx}) \cdot (1 - e^{\int_{A \cap B_{N,k}} \mu \, dx})$

$\leq \sum_{k \in \mathbb{Z}^3} \left( \int_{A \cap B_{N,k}} \lambda \, dx \right) \cdot \left( \int_{A \cap B_{N,k}} \mu \, dx \right)$

$\leq \lambda \mu \sum_{k \in \mathbb{Z}^3} \sup_{k \in \mathbb{Z}^3} \left[ Vol(A \cap B_{N,k}) \right] Vol(A \cap B_{N,k})$

$\leq \lambda \mu \sup_{k \in \mathbb{Z}^3} \left[ Vol(A B_{N,k}) \right] \cdot \sum_{k \in \mathbb{Z}^3} Vol(A \cap B_{N,k})$

$= \lambda \mu \, Vol(A) \cdot 2^{-3N}$

(True $\forall N \in \mathbb{N}$): $\mathbb{P}(|\Pi_1 \cap \Pi_2 \cap A| \neq 0) \leq \inf_{N \geq 1} \left\{ \lambda \mu |A| 2^{-3N} \right\} = 0$

$\therefore \mathbb{P} = 0.$

$\therefore |\Pi_1 \cap A| + |\Pi_2 \cap A| = |A \cap (\Pi_1 \cup \Pi_2)|$

For fixed bounded measurable $A$: $|(\Pi_1 \cup \Pi_2) \cap A|$ is the sum of independent Poi$\left(\int_A z_i \, dx\right)$ R.V.

$\therefore \sim$ Poi$\left(\int_A (z_1 + z_2) \, dx\right)$

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If $(A_n)_{n\geq1}$ ~~independent~~ disjoint: $(|A_n \cap \Pi_i|)_{n\geq 1}$ are independent

$\Pi_1, \Pi_2$ independent: $\{|A_n \cap \Pi_i| : n\geq1, i=1,2\}$ are all independent

$\therefore (|A_n \cap (\Pi_1 \cup \Pi_2)|)_{n\geq1}$ ~~is~~ are independent

(7/7) $\therefore$ Ideal gas of ~~to~~ activity $z_1 + z_2$

(b) Consider characteristic functions:

$N(V_i) \sim \mathrm{Poi}(z \cdot |V_i|)$ ; Let $\sqrt{|V_i|} = d_i$ ;

$$\mathbb{E}\left[e^{i\theta\left(N(V_i) - \mathbb{E}[N(V_i)]\right)/\sqrt{|V_i|}}\right] = \mathbb{E}\left[e^{i\theta\left(\mathrm{Poi}(z d_i^2) - z d_i^2\right)/d_i}\right]$$

$$= e^{-i\theta z d_i}\,\mathbb{E}\left[e^{i\theta\,\mathrm{Poi}(z d_i^2)/d_i}\right]$$

$$= e^{-i\theta z d_i}\, e^{\left(e^{i(\theta/d_i)} - 1\right) z d_i^2}$$

$$\lim_{i\to\infty}\left\{\left(e^{i(\theta/d_i)} - 1\right) z d_i^2 - i\theta z d_i\right\}$$

$$= \lim_{i\to\infty}\left\{-\theta^2 z/2 + O(1/d_i)\right\} = -z\theta^2/2$$

$\therefore$ Converges pointwise to characteristic function of $N(0, z)$

(7/7) $\therefore \to N(0,z)$ in distribution

(c) Let $K = \mathrm{Supp}(g)$, $X = \sum_{x\in\Pi} g(x)$

$$\underset{\phi(\theta)=}{\mathbb{E}\left[e^{\theta X}\right]} = \mathbb{E}\left[\mathbb{E}\left[e^{\theta\sum_{x\in\Pi\cap K} g(x)} \,\Big|\, |\Pi\cap K| = n\right]\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\theta g(U)}\right]^n\right] ; \quad n \sim \mathrm{Poi}\left(\int_K z\right)$$

$$= e^{\left(\int_K z\,dx\right)\left(-1 + \mathbb{E}\left[e^{\theta g(U)}\right]\right)}, \quad U \sim \mathrm{Uniform}(K)$$

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$$= e^{z|K| \cdot \int_K \frac{1}{|K|} (e^{\theta g(x)} - 1)\, dx} = e^{z \int_K (e^{\theta g(x)} - 1)\, dx}$$

$$\phi'(\theta)\Big|_{\theta=0} : z \int_K g(x)\, dx = \mathbb{E}[X]$$

$$\phi''(\theta)\Big|_{\theta=0} : \int_K z\, g^2(x)\, dx + \left(\int_K z\, g(x)\, dx\right)^2 = \mathbb{E}[X^2]$$

$$\therefore \ \mathrm{Var}(X) = z \int_K g^2(x)\, dx = z \int_{\mathbb{R}^3} g^2(x)\, dx$$

$$\mathbb{E}[X] = z \int_{\mathbb{R}^3} g(x)\, dx$$

(6/6)

(20/20)

2.) (a) Let $J_0 = 0$, $J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}$ $(n \geq 1)$:

$Y_n = X_{J_n}$ ; $(Y_n)_{n\geq 0}$ is the jump chain.

Consider process $\tilde{X}$ generated as follows:

$P(x,y) = 1_{x \neq y} \cdot Q(x,y) / q(x)$, $q(x) = -Q(x,x)$

$(\tilde{Y}_n)_{n\geq 0}$ is a discrete markov chain, $\tilde{Y}_0 = X_0$ and transition matrix $= P$

Let $(\xi_n)_{n\geq 1}$ be iid $\mathrm{Exp}(1)$ R.V. ; $\tilde{J}_0 = 0$, $\tilde{J}_n = \sum_{k=1}^{n} \xi_k / q(\tilde{Y}_{k-1})$

$\tilde{X}_t = \tilde{Y}_k$ for $t \in [\tilde{J}_k, \tilde{J}_{k+1})$

$X$, $\tilde{X}$ have the same distribution; $\tilde{Y}$ is the jump chain of $\tilde{X}$ $\Rightarrow$ $\tilde{Y}$, $Y$ have the same distribution

$\tilde{Y}$ is a markov chain $\Rightarrow$ $Y$ is a markov chain

(4/4)

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(b) $X$ is recurrent if $\forall$ state $s$: $\mathbb{P}(\{t \geq 0 : X_t = s\}$ is unbounded$) = 1$

~~Since X is irreducible~~: Recurrence is ~~equivalent to 1 state~~ being recurr

If $(Y_n)_{n\geq 0}$ is recurrent: $Y_n$ visits state $s$ infinitely often

$\therefore \exists$ subsequence $n_k$ s.t. $J_{n_k} \in \{t \geq 0 : X_t = s\}$

$X$ non explosive: $J_n \to \infty \Rightarrow J_{n_k} \to \infty$ (A.S.)

$\therefore \{t \geq 0 : X_t = s\}$ is unbounded (A.S.)

$\therefore$ Recurrent.

If $(Y_n)_{n\geq 0}$ is transient: $Y_n$ visits state $s$ finitely often

If $Y_n \neq s$ for $n > N$: $\{t \geq 0 : X_t = s\} \subseteq [0, J_N]$,

$J_N < \infty$ (A.S.)

Since $\exists N$ s.t. $Y_n \neq s$ for $n > N$ A.S., $\{t \geq 0 : X_t = s\}$ is bounded

has $\mathbb{P} = 0$. $\therefore \mathbb{P}((X_{t})_{t\geq 0}$ is unbounded$) = 0$. $\therefore$ Not recurrent.

$$\int_0^\infty P_{00}(t)\,dt = \int_0^\infty \mathbb{E}_0[1_{X_t = 0}]\,dt = \mathbb{E}_0\left[\int_0^\infty 1_{X_t=0}\,dt\right]$$

$$= \mathbb{E}_{x_0}\left[\sum_{n\geq 1} J_n \cdot 1_{Y_n = 0}\right] = \frac{1}{q(0)} \cdot \mathbb{E}_0[\#\text{ of visits to } 0 \ (Y_n)_{n\geq 0}]$$

$= \infty \Rightarrow$ (0) 0 is $(Y_n)_{n\geq 0}$ recurrent

$\Rightarrow X$ - recurrent.

(6/6)

(c) By considering jump chain probabilities: Equivalent to consider

$q_{x,y} = e^{-|x-y|^2}$

$q_{x,y} = 1/(1+|x-y|)^2$

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$$\int_0^\infty P_{0,0}(t)\,dt = \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi} e^{-t\lambda(k)}\,dk\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} {}^{1}/_{\lambda(k)}\,dk$$

$$\lambda(k) = \sum_{z\in\mathbb{Z}} (1-e^{ikz})\,e^{-z^2} = R + iI, \qquad R, I \in \mathbb{R}$$

$$0 \le R(k) = \sum_{z\in\mathbb{Z}} (1-\cos kz)\,e^{-z^2} \qquad (\text{all terms} \ge 0)$$

$$\le \sum_{z\in\mathbb{Z}} \cancel{R(k)}\; k^2 z^2/_2\, e^{-z^2} = A\cdot k^2, \qquad A \text{ is constant in } k$$

$$|I(k)| \le \sum_{z\in\mathbb{Z}} |\sin(kz)|\cdot e^{-z^2} \le \sum_{z\in\mathbb{Z}} |k|\,|z|\,e^{-z^2} = B\cdot|k|$$

$$I(k) = -I(-k)$$

$$\therefore \int_0^\infty P_{0,0}(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{R+iI}\,dk = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R-iI}{R^2+I^2}\,dk$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{R(k)}{R^2(k)+I^2(k)}\,dk \ge \frac{1}{\pi}\int_0^{\pi}\frac{R(k)}{A^2k^4+B^2k^2}\,dk$$

~~* Sus port:~~ $\cancel{I_R = \frac{\sum_z (1-\cos kz)\,e^{-z^2}}{\sum_z \sin kz\, e^{-z^2}}}$

$$= \frac{1}{\pi}\sum_{z\in\mathbb{Z}} e^{-z^2}\int_0^{\pi}\frac{1-\cos kz}{A^2k^4+B^2k^2}\,dk$$

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$$I(k) = \sum_{z \in \mathbb{Z}} \sin(kz)\, e^{-z^2}$$

$$= 0 + \sum_{n \in \mathbb{N}} \left[ \sin(kn) + \sin(-kn) \right] e^{-\hat{z}^2} = 0$$

$$\therefore \int_0^\infty P_{00}(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{R(k)}\, dk \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{Ak^2}\, dk = \infty$$

(10/10) $\therefore$ 1st chain is recurrent.

(20/20)

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3.) (a) State space $= \mathbb{Z}_{\geq 0}$ :

Q matrix : $Q(n, n+1) = \lambda r^n$

$Q(n, n-1) = \min\{n, 2\} \cdot \mu \quad (n \geq 1)$

$Q(n, n) = \lambda \cdot r^n + \min\{n, 2\} \cdot \mu$

Since this is a birth-death cycle: $\pi$ is invariant iff $\pi$ satisfies Detailed Balance

$\pi_0 \cdot \lambda = \pi_1 \cdot \mu$

$\pi_n \lambda r^n = \pi_{n+1} (2\mu) \quad (n \geq 2)$

$\therefore \pi_{n+1} = (\lambda/2\mu)^n r^{1+\dots+n} \pi_1 = (\lambda/2\mu)^n r^{(n+1).n/2} \pi_1$

If $r = 1$: $\pi$ is an invariant distribution iff

$$\sum_{n\geq 1} (\lambda/2\mu)^n r^{(n+1)n/2} = \sum_{n \geq 1} (\lambda/2\mu)^n < \infty$$

iff $\lambda < 2\mu$.

If $r < 1$: Given $\lambda, \mu$ : Pick $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow$

$$r^{(n+1)/2} \cdot (\lambda/2\mu) < 1/2.$$

$$\therefore \sum_{n \geq 1} (\lambda/2\mu)^n r^{(n+1)\cdot n/2} = \sum_{n<N} (\lambda/2\mu)^n r^{(n+1)n/2} + \sum_{n\geq N} (\lambda/2\mu \, r^{(n+1)/2})^n$$

$$\leq \sum_{n<N} (\lambda/2\mu)^n r^{(n+1)n/2} + \sum_{n \geq N} (\tfrac{1}{2})^n < \infty.$$

$\therefore$ Invariant distribution exists $\forall \lambda, \mu$.

<7/7>

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$\therefore\ r < 1$ or $(r = 1,\ \lambda/2\mu < 1)$

(b) State space : $S = \{0, \dots N\}$ (Finite)

Q-matrix : $Q(n, n+1) = \lambda \quad (0 \le n < N)$ — $\lambda = 1$

$Q(n, n-1) = \mu \quad (0 < n \le N)$

$Q(n, n) = \lambda \cdot 1_{n<N} + \mu\, 1_{n>0}$

Since chain is irreducible : ~~Keep~~ Jump chain is recurrent $\Rightarrow$ $(X_t)_{t\ge0}$ Chain is recurrent.

$\therefore$ ~~U~~ Any invariant distribution is unique

~~$Q$ finite dim - matrix $\Rightarrow \exists$ solution to $\underset{\sim}{\lambda} \cdot Q = 0$ as $r(Q) < \dim(Q)$~~

by Detailed Balance : $\pi_n \cdot \lambda = \mu\, \pi_{n+1}$

$\therefore\ \pi_n = \left(\lambda/\mu\right)^n \pi_0 \quad (0 \le n \le N)$

$\pi_n = (\lambda/\mu)^n \Big/ \sum_{k=0}^{N} (\lambda/\mu)^k$ is an invariant distribution

(6/6)

(c) Since we have solved detailed balance, process is reversible at equilibrium. Let $\tilde{X}$ be reversed process.

Since departure of $(X_t)_{t\ge0}$ corresponds to arrivals in $(\tilde{X}_t)_{t\ge0}$, and $\tilde{X}$ have the same distribution as $X$ :

~~De~~ Arrival process at equilibrium : Customers arrive at rate $\lambda$; Each customers will be turned away with probability $p$ (if ~~queue~~ store is full), independently (chain in equilibrium)

why?

(7/7)

$\therefore$ ~~P~~ Arrival process is a thinned Poisson process $\Rightarrow$ Poissonian

$\therefore$ Departure process is also Poissonian

(20/20)

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* Commat: Last part seems a little suspicious. I am really appealing to the "Pasta" property.

4.) (a) $f_t(i, \lambda) = \mathbb{E}_i [ f(X(t), \lambda + L(t)) ]$

$= e^{-q_i t} f(i, \lambda + t e_i) + \int_0^t \sum_{j \neq i} Q_{ij} e^{-q_i s}$ ~~$f(i, \lambda + s e_i + L(t-s)) ds$~~

$\mathbb{E}_j [ f(X(t-s), \lambda + s e_i + L(t-s)) ] ds$

$e^{q_i t} f_t(i, \lambda) = f(i, \lambda + t e_i) + \sum_{j \neq i} \int_0^t Q_{ij} e^{q_i s} \mathbb{E}_j [ f(X(s)), \lambda + (t-s) e_i + L(s) ] ds$

$\therefore e^{q_i t} \{ q_i f_t(i, \lambda) + \frac{\partial f_t}{\partial t}(i, \lambda) \} = \frac{\partial f}{\partial \lambda_i}(i, \lambda + t e_i) +$ ~~$\sum_{j \neq i} \int_0$~~

$+ \sum_{j \neq i} \{ Q_{ij} e^{q_i t} f_t(j, \lambda) + \int_0^t Q_{ij} e^{q_i s} \frac{\partial}{\partial t} f_s(j, \lambda + (t-s) e_i) ds$

$n - e^{q_i t} \cdot \sum_j Q_{ij} f_t(j, \lambda) = \sum_{j \neq i} \int_0^t Q_{ij} e^{q_i s} \frac{\partial f}{\partial \lambda_i} f_s(j, \lambda + (t-s) e_i) ds$

$e^{q_i t} \frac{\partial f_t}{\partial t}(i, \lambda) +$

$\frac{\partial f_t}{\partial t} = Q \cdot f_t(\cdot, \lambda)$

POP bazic

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4.) (a) $\mathbb{E}_i\left[ f(X(t), \lambda + L(t)) \right] = e^{-q_i t} f(i, \lambda + t e_i) + \int_0^t \sum_{j \neq i} Q_{ij} e^{-q_i s} f_{t-s}(j, \lambda + s e_i)\, ds$

$$\therefore e^{q_i t} \cdot f_t(i, \lambda) = f(i, \lambda + t e_i) + \sum_{j \neq i} \int_0^t Q_{ij} e^{+q_i s} f_s(j, \lambda + (t-s) e_i)\, ds$$

Diff. wrt $t$:

$$e^{q_i t} \cdot \left\{ q_i f_t(i,\lambda) + \frac{\partial}{\partial t} f_t(i,\lambda) \right\} = \frac{\partial f}{\partial \lambda_i}(i, \lambda + t e_i) + \sum_{j \neq i} Q_{ij} e^{q_i t} f_t(j, \lambda) + \sum_{j \neq i} \int_0^t Q_{ij} e^{q_i s} \frac{\partial}{\partial \lambda_i} f_s(j, \lambda + (t-s) e_i)\, ds$$

$$\therefore \frac{\partial f_t}{\partial t}(i,\lambda) = \sum_{j \in I} Q_{ij} f_t(j,\lambda) + \underbrace{\frac{\partial f}{\partial \lambda_i}(i, \lambda + t e_i) + \int_0^t \sum_{j \neq i} Q_{ij} e^{q_i s} \frac{\partial f_s}{\partial \lambda_i}(j, \dots)\, ds}$$

$$= \mathbb{E}_i\left[ \frac{\partial f}{\partial \lambda_i}(X(t), \lambda) \right] = \frac{\partial}{\partial \lambda_i} f_t(i,\lambda)$$

(7/7)

$$\therefore \frac{\partial f_t}{\partial t} = M \cdot f_t(i,\lambda) \;*$$

(b) $\cancel{\int_{\mathbb{R}^n} e^{-\frac{1}{2} y^T Q y} \sum_{j \in I} \sum_{k \in I} y_i y_j Q_{jk} f(k, y^2/2)}$

Let $f$ be supported in $K$:

Consider the vector $\underset{\sim}{A} \in \mathbb{R}^n$, $A_j = y_i f(j, y^2/2)$

$$\nabla \cdot \underset{\sim}{A} = \sum_{j=1}^{n} y_i \frac{\partial f}{\partial \lambda_j}(j, y^2/2) \cdot y_j + f(i, y^2/2)$$

$$\nabla \cdot \left( e^{\frac{1}{2} y^T Q y} \cdot \underset{\sim}{A} \right) = e^{\frac{1}{2} y^T Q y} \nabla \cdot A + e^{\frac{1}{2} y^T Q y} (\underset{\sim}{Q} \cdot y)^T \cdot A$$

Apply Divergence theorem to the region $K$: $A_j = 0$ as $f = 0$ on $\partial K$.

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$$0 = \int_K \nabla\cdot\left(e^{\frac{1}{2}y^T Q y}\underset{\sim}{A}\right)dy = \int_{\mathbb{R}^n}\nabla\cdot\left(e^{\frac{1}{2}y^T Q y}\cdot A\right)dy$$ as $A=0$ outside $K$.

$$= \int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\cdot\left\{y_i\sum_{j=1}^n \frac{\partial f(j, y^2/2)}{\partial \lambda_j}y_j + f(i, y^2/2) + \sum_{j=1}^n\sum_{k=1}^n y_j Q_{jk}\, y_i\, f(k, y^2/2)\right\}dy$$

$$= \int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\, f(i, y^2/2) + y_i\sum_{j=1}^n\left(\sum_{k=1}^n y_j Q_{jk} f(k, y^2/2)\right) - \frac{\partial f(j, y^2/2)}{\partial\lambda_j}\Big) dy$$

$$\therefore \int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy} f(i, y^2/2)\,dy = -\int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\, y_i\sum_{j=1}^n y_j\, Mf(j, y^2/2)\,dy$$

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(c) $$\int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\left\{\mathbb{E}_i\left[g(\lambda + L(t))\,\mathbb{1}_{X(t)=j}\right] + y_i\sum_{j=1}^n y_j\frac{\partial}{\partial t}f(j, \tfrac{1}{2}y^2)\right\}dy = 0$$

as for $|\lambda| \geq t+T$, $\|\lambda + L(t)\|_1 \geq T \Rightarrow g = 0$ (C b) valid?

~~$\frac{\partial}{\partial t} f(i, y^2/2) \ \mathbb{E}_i[\ ]$~~

~~$\lim_{t\to\infty} \mathbb{E}_i[g(\lambda + L(t))\mathbb{1}_{X(t)=j}] = 0$ for $t \geq$~~

For fixed $y$: $\mathbb{E}_i\left[g(y^2/2 + L(t))\,\mathbb{1}_{X(t)=j}\right] = 0$ for $t \geq y^2/2 + T$

$(\|y^2/2 + L(t)\|_1 \geq T)$

$$\therefore \int_0^\infty \frac{\partial}{\partial t} f_t(j, \tfrac{1}{2}y^2)\,dt = -\mathbb{E}_i\left[g(y^2/2)\right]$$

$$\therefore \int_{\mathbb{R}^n} e^{+\frac{1}{2}y^TQy}\underset{y_iy_j}{\wedge}\mathbb{E}_i\left[g(y^2/2)\right]dy = \int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\, g(y^2/2)\cdot y_iy_j\,dy$$

$$= \int_0^\infty\int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\,\mathbb{E}_i\left[g(y^2/2 + L(t))\,\mathbb{1}_{X(t)=j}\right]dy\,dt =$$

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$$\int_{\mathbb{R}^n} e^{\frac{1}{2}y^TQy}\int_0^\infty \mathbb{E}_i\left[g(y^2/2 + L(t))\,\mathbb{1}_{X(t)=j}\right]dt\,dy$$

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* ~~How~~ Need some help on:

2022 Q2 part (a):