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Principle of Statistics 2014

1.) (a) Let $\mathcal{X} \subseteq \mathbb{R}$, $\left\{ f(\cdot, \theta) : \mathcal{X} \to \mathbb{R}_{\geq 0} : \theta \in \Theta \right\}$

be a model (regular)

$$I(\theta) = \mathbb{E}_\theta \left[\cancel{\tfrac{1}{f}\tfrac{\partial f}{\partial \theta}} \frac{\partial}{\partial \theta} (\log(f(X, \theta)))^2 \right]$$

If $\tilde{\Theta}$ is an unbiased estimator of θ:

$$\operatorname{Var}_\theta(\tilde{\Theta}) \geq 1/I(\theta)$$

(b) Let $f(\cdot, \theta)$ be density of $N(\theta, 1)$, $\theta \in \mathbb{R}$.

$$\log \cancel{\tfrac{\partial f}{\partial \theta}} f(x, \theta) = -\tfrac{1}{2}\log(2\pi) - (x-\theta)^2/2$$

$$\therefore \frac{\partial}{\partial \theta} \log(f(x, \theta)) = -(x - \theta)$$

$$\therefore I(\theta) = \mathbb{E}_\theta[(X - \theta)^2] = 1$$

Let $\tilde{\Theta}(X) = X$: $\mathbb{E}[X] = \theta \Rightarrow$ Unbiased

$\operatorname{Var}_\theta(\tilde{\Theta}) = \operatorname{Var}(X) = 1 \Rightarrow$ ~~fn~~ Equality attained.

(c) Let $f(\cdot, \theta)$ be density of $N(0, \theta)$, $\theta \in [0, \infty)$.

$\prod_{k=1}^{n} f(x, \theta) = \frac{1}{(2\pi\theta)^{n/2}} e^{-\frac{1}{2\theta}\sum_{k=1}^{n} X_k^2}$ is maximised by:

Max : $-\frac{n}{2}\log(2\pi\theta) - \frac{n}{2\theta} S_n$, $S_n = \frac{1}{n}\sum_{k=1}^{n} X_k^2$

$\therefore$ Pick : $\frac{n}{2} \cdot \frac{2\pi}{2\pi\theta} = \frac{n}{2\theta^2} S_n \Rightarrow \theta = S_n$

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Let $f(\cdot\,;(\mu,\theta))$ be density of $N(\mu,\theta)$, $\mu\in\mathbb{R}$, $\theta\geq 0$.

$\therefore\ \hat{\mu} = \frac{1}{n}\sum_{k=1}^{n} X_n$, $\quad \hat{\theta}^{*} = \frac{1}{n}\sum_{k=1}^{n}(X_k - \cancel{X_n}\ \hat{\mu})^2$

$$\mathbb{E}[\hat{\theta}^{*}] = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[(X_k-\mu)^2 + (\mu-\hat{\mu})^2 + 2(\mu-\hat{\mu})(X_k-\mu)\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[(X_k-\mu)^2 - (\mu-\hat{\mu})^2\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(\theta - \theta/n\right) = \frac{n-1}{n}\theta \neq \theta.$$

$\therefore$ Biased.

$$\mathrm{Var}(\hat{\theta}) \geq \frac{\frac{d}{d\theta}(\theta + B(\theta))^2}{I(\theta)}, \qquad B(\theta) = \mathbb{E}[\hat{\theta}] - \theta$$

(d) Let X be observation space, generated by density $f(\cdot,\theta)$, $\theta\in\Theta$

~~Given~~ Let A be an action space, $\delta: X\to A$ is a rule.
Given $L: A\times\Theta\to\mathbb{R}_{\geq 0}$, let $R(\delta,\theta) = \mathbb{E}_\theta[L(\delta(X),\theta)]$

δ is minimax if it minimises: $\sup_{\theta\in\Theta}\{R(\delta,\theta)\}$.

(e) $\hat{\theta}(X_1,\dots X_n) = \overline{X_n}$ is unbiased

$R(\overline{X_n},\theta) = \mathbb{E}_\theta[(\overline{X_n}-\theta)^2] = \frac{1}{n}$.

Suppose $\exists$ rule δ, $R(\delta,\theta) < 1/n$ $(\forall\theta)$:

~~Var~~ $R(\delta,\theta) = \mathbb{E}_\theta[(\delta-\theta)^2] = \mathbb{E}_\theta[(\delta-\mathbb{E}_\theta[\delta])^2 + (\mathbb{E}_\theta[\delta]-\theta)^2]$

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$= Var(\theta) + (B(\theta)^2)$

$Var(\theta) \geq I_n(\theta)^{-1} (1 + B'(\theta))^2 = \frac{1}{n} (1 + B'(\theta))^2$

$\therefore \frac{1}{n} > B(\theta)^2 + \frac{1}{n}(1 + B'(\theta))^2$

$\therefore B' < 0 \quad (\forall \theta \in \mathbb{R}_+)$

But $|B| < 1/\sqrt{n} \Rightarrow B' \to 0$ as $|\theta| \to \infty$ (Monotone decreasing)

$\therefore$ Contradiction: Pic θ_0 s.t $B'(\theta_0) < 0 \Rightarrow \lim_{\theta \to \infty} B'(\theta) \leq B'(\theta_0) < 0$

$\therefore \hat{X}_n$ is minimax

2.) (a) Let π be a prior, L be a loss function, $\delta : \mathcal{X} \to A$ be a rule.

$R(\delta, \theta) = \mathbb{E}_\theta [L(\delta(X), \theta)]$. If δ minimises

$\int_{\theta \in \Theta} \pi(\theta) R(\delta, \theta) \, d\theta$, δ is a bayes rule.

δ is inadmissible if $\exists \, \delta' : \mathcal{X} \to A$ s.t
$\forall \theta \in \Theta$ & $R(\delta', \theta) \leq R(\delta, \theta)$ and
$\exists \theta \in \Theta$ $R(\delta', \theta) < R(\delta, \theta)$

δ is admissible if it is not inadmissible

Suppose unique bayes rule δ is not admissible. (wrt prior π)

$\therefore \exists \delta'$ s.t. : $\int_\Theta \pi(\theta) R(\delta', \theta) \, d\theta \leq \int_\Theta \pi(\theta) R(\delta, \theta) \, d\theta$ as $R(\delta, \theta) \geq R(\delta', \theta)$

$\therefore$ But RHS minimal $\Rightarrow$ Equality $\Rightarrow \delta'$ is another bayes

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rule $\Rightarrow$ Contradiction

$\therefore$ Admissible.

(b) Let $\Theta_1 = \hat{\Theta}_{MLE}$:

$$f_{X_n}(x_1, \dots x_n ; \theta) = \prod_{k=1}^{n} \frac{e^{-\theta}}{x_k!} \theta^{x_k} = \frac{e^{-n\theta} \, \theta^{\sum_{k=1}^{n} x_k}}{\prod_{k=1}^{n} x_k!}$$

log, Diff. wrt θ: (equate to 0) $\Rightarrow$ $n = \left(\sum_{k=1}^{n} x_k \right) \cdot 1/\theta$

$\therefore$ $\Theta_1 = \bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$.

$\mathbb{E}_\theta [\Theta_1] = \theta$

If Θ_1 is a bayes rule for some prior, Θ_1 must be the posterior mean. ~~since~~

Θ_1 minimises : $\int_{[0,\infty)} \sum_{k \geq 0} \pi(\theta) \frac{e^{-\theta} \theta^k}{k!} (\delta(x) - \theta)^2 \, d\theta$

$= \sum_{k \geq 0} \int_0^\infty \pi(\theta) \frac{e^{-\theta} \theta^k}{k!} (* - \theta)^2 \, d\theta$

Diff. wrt $*$, equate to 0 : $\sum_{k \geq 0} \int_0^\infty \pi(\theta) \frac{e^{-\theta} \theta^k}{k!} (2(* - \theta)) \, d\theta = 0$

$\therefore \int_0^\infty$

$\therefore$ $\mathbb{E}_{\pi(\cdot | x_1, \dots x_n)} [\Theta_1] = \bar{X}_n$

$\therefore$ $\mathbb{E}_\theta [\Theta_1 \Theta] = \mathbb{E}[\theta \, \mathbb{E}[\Theta_1 | \theta]] = \mathbb{E}[\theta^2]$

$\mathbb{E}[\Theta_1 \theta] = \mathbb{E}[\Theta_1 \underbrace{\mathbb{E}[\theta | X_1, \dots X_n]}_{\Theta_1}] = \mathbb{E}[\Theta_1^2]$

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$\therefore \mathbb{E}[(\Theta - \Theta_1)^2] = 0 \Rightarrow \Theta_1$ is not random

But $\bar{X}_n$ is random $\Rightarrow$ Contradiction

(c) $\pi(\theta \mid X_1, \dots X_n) \propto \pi(\theta) \prod_{k=1}^{n} X_k \propto \theta^{\alpha-1} e^{-\lambda\theta} e^{-n\theta} \theta^{\sum_{k=1}^{n} X_k}$

$\therefore \sim \Gamma(\lambda + n, \sum_{k=1}^{n} X_k + \alpha)$ distribution

$$\mathbb{E}[\Gamma(\alpha, \lambda)] = \int_0^\infty \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)} \theta \, d\theta = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\cdot\lambda} = \alpha/\lambda$$

$\therefore$ Posterior estimator: $\left(\alpha + \sum_{k=1}^{n} X_k\right) / (\lambda + n)$ ~~is~~ defines a Bayes rule (Quadratic loss)

(d) $$\sqrt{n}\left(\hat{\Theta}_{Bayes} - \hat{\Theta}_{MLE}\right) = \sqrt{n}\left(\frac{\alpha + \sum_{k=1}^{n} X_k}{\lambda + n} - \frac{\sum_{k=1}^{n} X_k}{n}\right)$$

$$= \frac{\sqrt{n}}{n(\lambda+n)}\left(\alpha \cdot n + (n - n - \lambda)\sum_{k=1}^{n} X_k\right) = \underbrace{\frac{\alpha\sqrt{n}}{\lambda+n}}_{(3)} - \underbrace{\frac{\lambda\sqrt{n}}{\lambda+n}}_{(2)} \underbrace{\frac{1}{n}\sum_{k=1}^{n}(X_k)}_{(1)}$$

(3) $\to 0$ (A.S)

(2) $\to 0$ (A.S $\Rightarrow$ in $\mathbb{P}$)

(1) $\to \Theta_0$ (~~Strong~~ law of large #) (A.S., $\mathbb{P}$)

$(2)\cdot(1) \xrightarrow{Dist} 0$ (Slutsky) $\Rightarrow$ $(2)\cdot(1) \xrightarrow{\mathbb{P}} 0$

$\therefore$ Slutsky: $(3) - (2)(1) \to 0$ in Dist. $\Rightarrow \to 0$ in $\mathbb{P}$

(e) $$(*)\ \sqrt{n}(\hat{\Theta}_{Bayes} - \Theta_0) = \underbrace{\sqrt{n}(\hat{\Theta}_{Bayes} - \hat{\Theta}_{MLE})}_{\to 0 \text{ in } \mathbb{P}} + \underbrace{\sqrt{n}(\hat{\Theta}_{MLE} - \Theta_0)}_{\xrightarrow{(d)} N(0, \Theta_0)}$$ (Central limit theorem)

$\therefore$ Slutsky: $(*) \to N(0, \Theta_0)$ in distribution.

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3.) (a) Wilk's Theorem: Let $\Lambda_n(\{\theta_0\}, \Theta) = 2\log \dfrac{\sup\{f(x_1,..x_n;\theta):\theta\in\Theta\}}{f(x_1,..x_n;\theta_0)}$

$\Lambda_n \to \chi^2$ in distribution (wrt $\mathbb{P}_{\theta_0}$)

$\cancel{\Lambda_n = 2\log(f(\underset{\sim}{x_k};\hat\theta_n)) - 2\log(f(\underset{\sim}{x};\theta_0))}$; $q(x,\theta) = \log(f(x,\theta))$

$\cancel{= -2\left[(\theta_0-\hat\theta_n)\cdot \underbrace{q'(x,\hat\theta_n)}_{=0} + \frac{1}{2}(\theta_0-\hat\theta_n)^2 q''(x,\tilde\theta_n)\right]},$

$\cancel{\tilde\theta_n \in [\theta_0, \hat\theta_n]}$ ~~segment. (Taylor's theorem)~~

$\cancel{\therefore \Lambda_n = (\sqrt{n}(\hat\theta_n-\theta_0))^2 \cdot \frac{1}{n}}$

Let $\log(f(x,\theta)) = q(x,\theta)$:

$$\Lambda_n = -2\sum_{k=1}^{n}\left[q(X_k,\theta_0) - q(X_k,\hat\theta_n)\right]$$

$$= -2\left[(\theta_0-\hat\theta_n)\sum_{k=1}^{n} q'(X_k,\hat\theta_n) + \frac{1}{2}(\theta_0-\hat\theta_n)^2\sum_{k=1}^{n} q''(X_k,\tilde\theta_n)\right],$$

(where $\tilde\theta_n \in [\theta_0, \hat\theta_n]$)

$$= \underbrace{(\sqrt{n}(\hat\theta_n-\theta_0))^2}_{\xrightarrow{(d)} \cancel{N(0,\, I(\theta_0)^{-1})}} \underbrace{\left(-\frac{1}{n}\sum_{k=1}^{n} -q''(X_k,\theta_0)\right)}_{\xrightarrow{A.S.} I(\theta_0)} + \underbrace{\cancel{\frac{1}{n}\sum_{k=1}^{n} -q''(X_k,\tilde\theta_n) + q''(X_k,\theta_0)}}_{(*)}$$

$$(*) = \frac{1}{n}\sum_{k=1}^{n}\left(-q''(X_k,\tilde\theta_n) - \mathbb{E}[-q''(X_k,\tilde\theta_n)]\right)$$

$$+ \mathbb{E}[-q''(X,\tilde\theta_n)] - \mathbb{E}[-q''(X,\theta_0)]$$

(regularity) line 1 $\to 0$ by uniform law of large #,
line 2 $\to 0$ as $\tilde\theta_n \to \theta_0$ A.S., $\theta \to \mathbb{E}[-q''(X,\theta)]$ is cont.

$\therefore$

~~Stats~~ ~~Since~~ ~~Statistic~~ ~~$\Lambda_n \to$~~

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Cont. map theorem: $(\sqrt{n}(\hat{\theta}_n - \theta_0))^2 \xrightarrow{(D)} Z^2$, $Z \sim N(0, I(\theta)^{-1})$

$\therefore (\Lambda_n \to N(0, I(\theta_0)^{-1})\, I(\theta_0)\, N(0, I(\theta_0)) \sim \chi^2$

(b) Max. : $(2\pi\theta)^{-n/2} e^{-\frac{1}{2\theta}\sum_{k=1}^{n} X_k^2} \equiv$ Max $-\frac{n}{2}\log(\theta) - \frac{1}{2\theta}\sum_{k=1}^{n} X_k^2$

$\therefore -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{k=1}^{n} X_k^2 = 0 \Rightarrow$ Pick $\hat{\theta}_n = \frac{1}{n}\sum_{k=1}^{n} X_k^2$

Consider hypothesis $\Theta = (0, \infty)$, $\Theta_1 = \{1\}$

$$\Lambda_n = 2\log\left(\frac{1}{\sqrt{2\pi\hat{\theta}}^{\,n}} e^{-\frac{1}{2\hat{\theta}}(X_1^2 + \dots + X_n^2)} \Big/ \frac{1}{\sqrt{2\pi}^{\,n}} e^{-\frac{1}{2}(X_1^2+\dots+X_n^2)}\right)$$

$$= \log\left(\frac{1}{\hat{\theta}^n} e^{-n} \Big/ e^{-n\hat{\theta}}\right) = n\hat{\theta} - n - n\log(\hat{\theta})$$

$$= -n\left(\log(\hat{\theta}) - (\hat{\theta} - 1)\right) \to \chi^2 \text{ in distribution}$$

4) (a) Define Pseudo. Inversion: $F: \mathbb{R} \to [0,1]$

$$F^{-1}(x) = \inf\{\omega \in \mathbb{R} : F(\omega) \geq x\}$$

$\therefore F^{-1}(x) \leq \omega$ iff $F(\omega) \geq x$

Let $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2} = F(\omega)$:

Define $X_k^* = F^{-1}(U_k^*)$:

Since $(U_k^*)_{k\geq 1}$ are iid, $(F^{-1}(U_k^*))_{k\geq 1}$ are iid.

$\mathbb{P}(X_k^* \leq \omega) = \mathbb{P}(F^{-1}(U_k^*) \leq \omega) = \mathbb{P}(U_k \leq F(\omega)) = F(\omega)$

$\therefore X_k^*$ has distribution F $(\sim N(0,1))$

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(h) $I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{(\pi + |X_k|)^{1/4}}$, $X_k = F^{-1}(U_k)$ as defined.

$\therefore$ We only need to simulate $U_1, \dots U_N$ to calculate I_N.

Accuracy: $\mathbb{P}\left(|I_N - I| > \frac{1}{N^{1/4}} \right) \le \frac{Var(I_n)}{1/\sqrt{N}} = \frac{1}{\sqrt{N}} Var\left((\pi+|Z|)^{-1/4} \right)$,

$Z \sim N(0,1)$ (Tchebyschev)

But $Var\left((\pi + |Z|)^{-1/4} \right) = \mathbb{E}\left[\frac{1}{\sqrt{\pi + |Z|}} \right] - \mathbb{E}\left[\frac{1}{(\pi+|Z|)^{1/4}} \right]^2$

$\le \mathbb{E}\left[\frac{1}{\sqrt{\pi + |Z|}} \right] \le \mathbb{E}\left[\frac{1}{\sqrt{\pi}} \right] < 1.$

$\therefore \mathbb{P}\left(|I_n - I| > 1/N^{1/4} \right) < 1$ $\cancel{\times}$

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Principle of Statistics 2015

1.) (a) $\hat{\Theta}^{JS}(X) = \left(1 - \frac{(P-2)}{\|X\|^2}\right) \cdot X$

$$R(\tilde{\Theta}, \theta) = \sum_{k=1}^{P} \mathbb{E}_\theta\left[|X_k - \theta_k|^2\right] = P$$

$$R(\hat{\Theta}^{JS}, \theta) = \mathbb{E}\left[\left\{\left\|X - \theta - \frac{(P-2)X}{\|X\|^2}\right\|\right\}^2\right] =$$

$$\mathbb{E}\left[\|X-\theta\|^2\right] + (P-2)^2\,\mathbb{E}\left[\frac{\|X\|^2}{\|X\|^4}\right]$$

$$-2(P-2)\,\mathbb{E}\left[\frac{X^T \cdot (X-\theta)}{\|X\|^2}\right]$$

For $j = 1, \dots P$: $\mathbb{E}\left[\frac{(X_j - \theta_j)\cdot X_j}{\sum_{k=1}^{P} X_j^2}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{X_j\,(X_j - \theta_j)}{\|X\|^2}\,\middle|\, X_1, \dots X_{j-1}, X_{j+1}\right]\right]$

$$\overset{(*)}{=} \mathbb{E}\left[\mathbb{E}\left[\frac{\|X\|^2 - 2X_j^2}{\|X\|^4}\,\middle|\, X_1, \dots \bar{X}_j, \dots\right]\right] = \mathbb{E}\left[\frac{1}{\|X\|^2} - \frac{2X_j^2}{\|X\|^4}\right]$$

$$\therefore R(\hat{\delta}^{JS}, \theta) = P + (P-2)^2\,\mathbb{E}\left[\frac{1}{\|X\|^2}\right] - 2(P-2)\sum_{j=1}^{P}\left(\mathbb{E}\left[\frac{1}{\|X\|^2}\right] - 2\,\mathbb{E}\left[\frac{X_j^2}{\|X\|^4}\right]\right)$$

$$= P + (P-2)^2\,\mathbb{E}\left[\frac{1}{\|X\|^2}\right] - 2(P-2)\left[(P)\,\mathbb{E}\left[\frac{1}{\|X\|^2}\right] - 2\,\mathbb{E}\left[\frac{1}{\|X\|^2}\right]\right]$$

$$= P - (P-2)^2\,\mathbb{E}_\theta\left[\frac{1}{\|X\|^2}\right]$$

But $\mathbb{E}\left[\frac{1}{\|X\|^2}\right] \geq \mathbb{E}\left[\frac{1}{\|X\|^2}\,\mathbb{1}_{\|X\| \in [1,2]}\right] \geq \frac{1}{\sqrt{2\pi}^P}\, e^{-\frac{1}{2}\cdot 2^2} \cdot \frac{1}{2^2} > 0$

$$\therefore R(\hat{\delta}^{JS}, \theta) = P - (P-2)^2\,\mathbb{E}_\theta\left[\frac{1}{\|X\|^2}\right] < P.$$

$\therefore$ Dominates $\tilde{\Theta}$.

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(a) (*) Stein's Lemma: If $X \sim N(\mu, 1)$ (1 dim), g is bounded, diff., g' & $\mathbb{E}[g'] < \infty$:

$$\mathbb{E}[g(X)(X-\mu)] = \mathbb{E}[g'(X)]$$

(b) Claim: $\mathbb{E}_\theta\left[\frac{1}{\|X\|^2}\right] \to 0$ as $\|\theta\| \to \infty$

WLOG: $Z \sim N(0, I_p)$ (Density G: $G(x) = \frac{1}{\sqrt{2\pi}^p} e^{-\frac{1}{2}(|z|^2)}$

We know: $\mathbb{E}\left[\frac{1}{\|Z\|^2}\right] < \infty$ for $p \geq 3$.

$$\therefore \int_{\bar{D}(0,1)} \frac{1}{(2\pi)^{p/2}} \frac{1}{|z|^2} e^{-\frac{1}{2}|z|^2} dz = M < \infty$$

$$\therefore \int_{\bar{D}(0,1)} \frac{1}{(2\pi)^{p/2}} \frac{1}{|z|^2} dz \geq M \geq \int_{\bar{D}(0,1)} \frac{1}{(2\pi)^{p/2}} \frac{1}{|z|^2} e^{-\frac{1}{2}} dz.$$

$\therefore$ $\|\theta\| \geq 2N$:

$$\mathbb{E}\left[\frac{1}{\|Z-\theta\|^2}\right] \leq \mathbb{E}\left[\frac{1}{N^2} 1_{Z \in \bar{D}(0,N)}\right] + \int_{\bar{D}(\theta,1)} \frac{1}{(2\pi)^{p/2}} \frac{1}{|z|^2} e^{-\frac{1}{2}\|\theta - 1\|^2} dz$$

$$+ \mathbb{E} \int_{\mathbb{R}^p \setminus \bar{D}(0,N)} 1 \cdot \frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2}\|z\|^2} dz$$

$$\leq \frac{1}{N^2} + M e^{-\frac{1}{2}(2N-1)^2} + \mathbb{P}(|Z| > N) \quad \text{as } N$$

$\to 0$ as $N \to \infty$.

$$\therefore \mathbb{E}_\theta\left[\frac{1}{\|X\|^2}\right] = \mathbb{E}\left[\frac{1}{\|Z-\theta\|^2}\right] \to 0 \text{ as } \theta \to \infty$$

$$\therefore \sup_{\theta \in \mathbb{R}^p} \left\{ R(\delta^{JS}, \theta) \right\} \geq p = \lim_{\theta \to \infty} \left\{ R(\delta^{JS}, \theta) \right\}$$

$$\leq \sup_{\theta \in \mathbb{R}^p} \left\{ R(\delta^{JS}, \theta) \right\}$$

$\therefore$ Equality.

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2.) (a) Max. $\prod_{k=1}^{n} \binom{n}{x} \Theta^{x} (1-\Theta)^{n-x}$

$\therefore$ Max. $x\log(\theta) + (n-x)\log(1-\theta)$

(Diff wrt θ): $x/\theta - (n-x)/1-\theta = 0$

$\therefore x(1-\theta) = (n-x).\theta$

$\therefore n\theta = x \Rightarrow \hat{\theta} = x/n$

$I(\theta) = -\mathbb{E}[\ (x\log(\theta) + (n-x)\log(1-\theta))''\ ]$

$= -\mathbb{E}\left[-x/\theta^2 - \frac{(n-x)}{(1-\theta)^2} \right] = \frac{1}{\theta^2}(\theta \cdot n) + \left(\frac{n}{(1-\theta)^2} - \frac{n\theta}{(1-\theta)^2}\right)$

$= \frac{n}{\theta} + \frac{n}{1-\theta} = n\left(\frac{1}{\theta(1-\theta)}\right)$

(b) $U[0,1]$ prior:

Posterior: $\pi(\theta|x) \propto 1 \cdot \theta^{x}(1-\theta)^{n-x}$

$\therefore$ Beta $(x+1,\ n-x+1)$

Mean: $\mathbb{E}[\beta(x+1, n-x+1)] = \frac{x+1}{n+2}$ (Not $\hat{\theta}_{MLE}$)

Mode: $x/(x+1)+(n-x+1)-2 = \cancel{x}/\cancel{n+x}\ x/n$ ($\hat{\theta}_{MLE}$)

(c) $I(\theta)^{1/2}$ prior: $\pi(\theta|x) \propto \frac{1}{\sqrt{\theta(1-\theta)}} \theta^{x}(1-\theta)^{n-x} = \theta^{x-1/2}(1-\theta)^{n-x-1/2}$

$\therefore \pi \sim \beta(x+1/2,\ n-x+1/2)$

Mean: $x+1/2 / n+1$

Mode: $\frac{\mathrm{Max}(0, x-1/2)}{\mathrm{Max}(x-1/2, 0) + \mathrm{Max}(n-x-1/2)}$ $n-x+x=n \geq 1 \Rightarrow n-1/2$ or $n-x-1/2 > 0$

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Both are not $\hat{\theta}_{MLE}$

(c) Prior: $\beta(\sqrt{n}/2, \sqrt{n}/2)$

$$\pi(\theta \mid X) \propto \cancel{X}\theta^{\sqrt{n}/2 - 1} (1-\theta)^{\sqrt{n}/2 - 1} \theta^{x} \theta^{n-x}$$

$$\therefore \pi \sim \beta\left(\sqrt{n}/2 + x, \sqrt{n}/2 + (n-x)\right)$$

Mean: $\dfrac{\sqrt{n}/2 + x}{n + \sqrt{n}}$

Mode: $\dfrac{\Lambda(\sqrt{n}/2 + x - 1, 0)}{\Lambda(\sqrt{n}/2 + x - 1, 0) + \Lambda(\sqrt{n}/2 + n - x, 0)}$

$(\sqrt{n}/2 + x + \sqrt{n}/2 + (n-x) > n \geq 1 \Rightarrow \sqrt{n}/2 + x$ or $\sqrt{n}/2 + n - x > 0$.

(Not $\hat{\theta}$)

(d) Claim: $\hat{\theta}_1 = \dfrac{\sqrt{n}/2 + X}{n + \sqrt{n}}$ is unique minimax ($\Rightarrow$ All other estimators not minimax)

$\hat{\theta}_1$ is unique posterior (quadratic loss) of prior $\beta(\sqrt{n}/2, \sqrt{n}/2) \Rightarrow$ Unique bayes rule

Constant Risk: $R(\hat{\theta}_1, \theta) = \mathbb{E}\left[\left(\dfrac{\sqrt{n}/2 + X}{n + \sqrt{n}} - \theta\right)^2\right]$

$$= \left(\frac{1}{n + \sqrt{n}}\right)^2 \mathbb{E}\left[\left(\frac{\sqrt{n}}{2} + X - n\theta - \sqrt{n}\theta\right)^2\right] = \frac{1}{(n+\sqrt{n})^2} \mathbb{E}\left[(X - n\theta)^2 + \left(\frac{\sqrt{n}}{2} - \sqrt{n}\theta\right)^2\right]$$

$$= \frac{1}{(1+\sqrt{n})^2}\left(\theta(1-\theta) + \left(\frac{1}{2} - \theta\right)^2\right) = \frac{1}{(1+\sqrt{n})^2}\left(\theta - \theta^2 + \theta^2 - \theta + \frac{1}{4}\right.$$

$$= \frac{1}{4}(1+\sqrt{n})^{-2}$$

$\therefore$ Unique minimax.

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3.) (a) Let $m = \operatorname{Min}\left\{|S(\theta_0+1)|, |S(\theta_0-1)|\right\}$, $m>0$, $\delta>0$ be arbitary

~~If $|\hat{\theta}_n$ ...~~

If $|\hat{\theta}_n - \theta_0| > 1$: ~~...~~

$S_n(\theta_0 \pm 1)$ have the same sign $\Rightarrow$

$$\operatorname{Sup}\left\{|S_n(\theta) - S(\theta)| : \theta \in \{\pm 1 + \theta_0\}\right\} \geq m.$$

($S(\theta_0 \pm 1)$ have opp. signs)

If $\delta \leq |\hat{\theta}_n - \theta_0| \leq 1$: $|S_n(\hat{\theta}_n) - S(\hat{\theta}_n)| = |S(\hat{\theta}_n)| \geq$

$$\operatorname{Inf}\left\{|S(\theta)| : \delta \leq |\theta - \theta_0| \leq 1\right\} = m_2 > 0.$$

Uniform LLN: $\displaystyle\operatorname{Sup}_{\delta \leq |\theta - \theta_0| \leq 1}\left\{|S_n(\theta) - S(\theta)|\right\} \to 0$ A.S., in $\mathbb{P}$

$\therefore$ $\mathbb{P}(|\hat{\theta}_n - \theta_0| > \delta) \leq \mathbb{P}(|\hat{\theta}_n - \theta_0| > 1) + \mathbb{P}(|\hat{\theta}_n - \theta_0| \in [\delta, 1])$

$\leq \mathbb{P}\left(|S_n(\theta_0+1) - S(\theta_0+1)| \geq m \text{ or } |S_n(\theta_0-1) - S(\theta_0+1)| \geq m\right) +$

$\mathbb{P}\left(\displaystyle\operatorname{Sup}_{|\theta-\theta_0| \leq 1}\left\{|S_n(\theta) - S(\theta)|\right\} \geq m_2\right)$

$\to 0$ (1^{st} line $\leq 1 - \mathbb{P}(|S_n(\theta_0+1) - S_n(\theta_0+1)| \geq m)$
$- \mathbb{P}(|S_n(\theta_0-1) - S(\theta_0-1)| \geq m)$)

$\therefore$ $\theta_n \to \theta_0$ in $\mathbb{P}$

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(b) $\cancel{\Theta_0 \text{ is root of } \underbrace{\mathbb{E}[\log(f(X,\theta))]'}_{S(\theta)}, \; f \text{ is density of } N(\theta_0,1)}$

$\cancel{S(\theta) = \mathbb{E}_{\theta_0}\left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}(X-\theta)^2 \right] = -\frac{1}{2}\log(2\pi) - \frac{1}{2}(\theta_0-\theta)^2 - \frac{1}{2}}$

$\cancel{\text{is cont., } S(\theta) = -(\theta_0-\theta) = \theta - \theta_0 \text{ satisfies criteria.}}$

$\cancel{\text{Score} = \text{constant} \; S_n = \sum_{k=1}^{n} -\frac{(X_k-\theta)^2}{2}} \; S_n(\theta) =$

(b) f = density of $N(\theta, 1)$

$T = \mathbb{E}[\log(f(X,\theta))]' = \mathbb{E}[-\frac{1}{2}(X-\theta)^2] = -\frac{1}{2}[(\theta_0-\theta)^2+1]$

θ_0 root of $T'(\theta) = \theta_0 - \theta$, $T'(\theta)$ satisfies criteria of S.

$\frac{d}{d\theta}\log(f(X_1)\dots f(X_n)) = \frac{d}{d\theta}\left(-\frac{1}{2}[(X_1-\theta)^2 + \dots + (X_n-\theta)^2]\right)$

$= \sum_{k=1}^{n} X_k - n\theta$ satisfy criteria of S_n

$\hat{\theta}_n$ = root of S_n

$\therefore \hat{\theta}_n \to \theta_0$ in $\mathbb{P}$ $\Rightarrow$ Consistent

(c) $f = \prod_{k=1}^{n} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(X_{1k}-\mu_k)^2 + (X_{2,k}-\mu_k)^2\right]}$

$\therefore \hat{\mu}_k = \frac{1}{2}(X_{1,k} + X_{2,k})$

Max. $-n\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{k=1}^{n} \underbrace{(X_{1,k}-\hat{\mu}_k)^2 + (X_{2,k}-\hat{\mu}_k)^2}_{*}$

Diff wrt σ^2: $+\frac{n}{\sigma^2} = \frac{1}{2\sigma^4}\sum_{k=1}^{n}(*)$

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$\therefore \hat{\sigma}^2 = \frac{1}{n} \cancel{\sum_{k=1}^{2}} \sum_{k=1}^{n} S_k^2$

(d) $\hat{\sigma}^2 = \frac{1}{2n} \sum_{k=1}^{n} (X_{1,k}^2 + X_{2,k}^2 + 2\hat{\mu}_k^2 - 2(X_{1,k} + X_{2,k}) \hat{\mu}_k)$

$= \frac{1}{2n} \left[\sum_{k=1}^{n} X_{1,k}^2 + X_{2,k}^2 \cancel{+ 2 \sum_{k=1}^{n} \hat{\mu}_k} - 2\sum_{k=1}^{n} \hat{\mu}_k^2 \right]$

$\cancel{\{X_{1,k}\}_{k=1}}$ $\cancel{\mathbb{E} [X_{i,k}^2] = \sigma^2 + \mu_k^2}$

$\cancel{\hat{\mu}_k \sim N (\mu_k, \sigma^2/2) \Rightarrow \mathbb{E} [\hat{\mu}_k^2] = \sigma^2/2 + \mu_k^2}$

$\cancel{LLN: \hat{\sigma}^2 \to \frac{1}{2} (2\sigma^2 - 2}$

$= \frac{1}{2n} \left(\sum_{k=1}^{n} (X_{1,k}^2 + X_{2,k}^2 - \frac{1}{2} (X_{1,k} + X_{2,k})^2 \right)$

$= \frac{1}{4n} \left[\sum_{k=1}^{n} X_{1,k}^2 + X_{2,k}^2 - 2 X_{1,k} X_{2,k} \right] = \frac{1}{4n} \sum_{k=1}^{n} (X_{1,k} - X_{2,k})^2$

$X_{1,k} - X_{2,k} \sim N (0, 2\sigma^2)$

~~CLT~~ LLN: $\frac{1}{n} \sum_{k=1}^{n} Z_n \to \mathbb{E} [Z]$, $Z_n \sim N (0, 2\sigma^2)^2$

$\therefore \hat{\sigma}_n^2 \to \frac{1}{4} [2\sigma^2 + 0^2] = \sigma^2/2 \neq \sigma^2$

(e) Each sample of $\underset{\sim}{X}_k$ are not identically distributed

$\therefore$ Consistency Theorem may not apply.

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4.) (a) $(X_k^b)_{k=1,\dots n}$ is sampled from space $\{X_1, \dots X_n\}$ with measure $\mathbb{P}_n$, $\mathbb{P}_n(X_k) = 1/n$ $(k=1,\dots n)$

C_n construction: Let $\overline{X_n^b}$ be mean of X_k^b R.V.

Pick R_n s.t. $\mathbb{P}_n(|X_k^b - \overline{X_n^b}| \leq R_n/\sqrt{n}) = 1-\alpha$.

$\mathbb{P}(\mu \in |\mu - \overline{X_n^b}| \leq R_n/\sqrt{n}) \to 1-\alpha$ A.S.

~~Let $\overline{X_n^b}$ be $= \mathbb{E}_{\mathbb{P}_n}[X_k^b]$: Pick R_n s.t. $\mathbb{P}(|X_k^b - \overline{X_n}| \leq R_n/\sqrt{n}) = 1-\alpha$~~

$\overline{X_{n,b}} = \frac{1}{n}\sum_{k=1}^{n} X_k^b$: Pick R_n s.t. $\mathbb{P}_n(|* - \overline{X_n}| \leq R_n/\sqrt{n}) = 1-\alpha$

$\mathbb{P}(|\mu - \overline{X_n}| \cdot \sqrt{n} \leq R_n) \to 1-\alpha$

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4.) (a) Given $X_1, \dots X_n$: Construct samples $(X_{k,b})_{k=1,\dots n}$ ${}_{b}^{\infty}$

$X_{k,b}$ are iid, $\sim P_n$; $P_n(X_{\{X_k\}}) = 1/n$ for $k = 1, \dots n$.

Let $\bar{X}_b = \frac{1}{n}\sum_{k=1}^{n} X_{k,b}$: Pick R_n s.t.

$$\mathbb{P}_n\left(\sqrt{n}\,|\bar{X}_b - \bar{X}_n| \le R_n\right) = 1-\alpha$$

$\mathbb{P}(|\mu - \bar{X}_n|\cdot\sqrt{n} \le R_n) \to 1-\alpha$ A.S. $\Big\}$ $C_n = [\bar{X}_n - R_n/\sqrt{n}, \bar{X}_n + R_n/\sqrt{n}]$

(b) $X_k^b = \sum_{r=1}^{n} \mathbb{1}_{U_{k} \in (\frac{r-1}{n}, \frac{r}{n}]} \cdot X_r$

$\mathbb{P}(X_k^b = X_r) = 1/n$ $(r = 1, \dots n)$; U_k independent $\Rightarrow$ X_k^b independent

(c) Fix $N \in \mathbb{N}$: Pick $-\infty = x_0 < x_1, \dots \dots \le x_N = \infty$, s.t. $F(x_k) = k/N$.

(x_k exists as F is cont.)

$\therefore$ For $r = 0, \dots N$: $F_n(x_r) \xrightarrow{(n)} F(x_r)$

Pick $M \in \mathbb{N}$ s.t. $n \ge M \Rightarrow$ For $r = 0, \dots N$, $|F(x_r) - F_n(x_r)| < 1/N$.

$\forall x \in \mathbb{R}$, pick r s.t. $x_r \le x < x_{r+1}$

$n \ge M$:

$\therefore |F_n(x) - F(x)| \le |F_n(x_r) - F(x_r)| + |F_n(x) - F_n(x_r)| + |F(x) - F(x_r)|$

$\le 1/N + |F_n(x_{r+1}) - F_n(x_r)| + |F(x_{r+1}) - F(x_r)|$

$\le 1/N + |F_n(x_{r+1}) - F(x_{r+1})| + 2|F(x_{r+1}) - F(x_r)| + |F_n(x_r) - F(x_r)|$

$\le 5/N$ (indep. of x)

Since N arbitrary: $\|F_n - F\|_\infty \to 0$. as $n \to \infty$.

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(d) $\cancel{\sqrt{n}\,(\overline{X}_{n,b} - \overline{X}_n)}$

$$\sup_{t \in \mathbb{R}^n} \left\{ \left| \mathbb{P}_n \left(|\overline{X}_{n,b} - \overline{X}_n| \cdot \sqrt{n} \le t \,\middle|\, X_1, \dots X_n \right) - \Phi(t) \right| \right\} \to 0 \quad A.S.$$

Φ is cdf of $N(0, Var(X_k))$

(e) Claim : $R_n \to \beta$, $\Phi(\beta) - \Phi(-\beta) = 1-\alpha$.

$$\Lambda(t) = \Phi(t) - \Phi(-t)$$

$$\left| \mathbb{P}_n \left(\sqrt{n}\, |\overline{X}_{n,b} - \overline{X}_n| \le R_n \right) - \Lambda(R_n) \right| \to 0 \quad A.S.$$

$\therefore \Lambda(R_n) \to 1-\alpha$ A.S. ; Λ cont., increasing $\Rightarrow$ Invertible

$\therefore R_n \to \Lambda^{-1}(1-\alpha) = \beta$, $\Lambda(\beta) = 1-\alpha$.

Claim: $\mathbb{P}(\mu \in C_n) \to 1-\alpha$.

$$\mathbb{P}\left(|\mu - \overline{X}_n| \cdot \sqrt{n} \le R_n \right) = \mathbb{P}\left(\underbrace{\frac{\Lambda^{-1}(1-\alpha)}{R_n} \cdot \sqrt{n}\, |\mu - \overline{X}_n|}_{(1)} \le \Lambda^{-1}(1-\alpha) \right)$$

$\frac{\Lambda^{-1}(1-\alpha)}{R_n} \to 1$ A.S. $\Rightarrow \to 1$ in $\mathbb{P}$;

CLT: $\sqrt{n}\,(\mu - \overline{X}_n) \to N(0, \cancel{\sigma^2}\ Var(X))$ (in dist.)

$\therefore (1) \to N(0, Var(X))$ in dist.

$\therefore \mathbb{P}(\mu \in C_n) \to \Lambda(\Lambda^{-1}(1-\alpha)) = 1-\alpha$ A.S.

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Principles of Statistics 2016

1.) (a) $f(X_1, \dots X_n ; \theta) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}\sum_{k=1}^{n}(X_k - \theta)^2}$

$\therefore$ Max. $-\frac{1}{2}\sum_{k=1}^{n}(X_k - \theta)^2 \equiv$ Min. $\sum_{k=1}^{n}(\theta^2 - 2\theta X_k + X_k^2)$

$\therefore$ Pick $\theta = \frac{1}{n}\sum_{k=1}^{n} X_k = \hat{\theta}_n$

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{\sqrt{n}}{n}\sum_{k=1}^{n}\underbrace{(X_k - \theta)}_{\sim N(0,1),\ \text{independent}} \sim N(0,1)$$

$\therefore$ Converges to $N(0,1)$

(b) Consider: $\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n) = \sqrt{n}\, 1_{|\hat{\theta}_n| \le n^{-1/4}} (-\hat{\theta}_n)$

$\therefore \sqrt{n}\,|\tilde{\theta}_n - \hat{\theta}_n| \le \frac{\sqrt{n}}{n^{1/4}} 1_{|\hat{\theta}_n| \le n^{-1/4}}$

$\mathbb{P}_\theta(\sqrt{n}\,|\tilde{\theta}_n - \hat{\theta}_n| \ge \varepsilon) \le \mathbb{P}_\theta(n^{1/4} 1_{|\hat{\theta}_n| \le n^{-1/4}} \ge \varepsilon)$

$= \mathbb{P}_\theta(|\hat{\theta}_n| \le n^{-1/4}) = \mathbb{P}_\theta(\frac{1}{\sqrt{n}}|X_1 + \dots + X_n| \le \sqrt{n})$

Case 1: $\theta = 0$

Consider

$\mathbb{P}(\underbrace{\frac{\sqrt{n}}{n}|X_1 + \dots + X_n| \cdot 1_{|\hat{\theta}_n| > n^{-1/4}}}_{(*)} \ge \varepsilon)$

$(*) = 0$ if $|\hat{\theta}_n| \le n^{-1/4}$

$= \sqrt{n}\,|\hat{\theta}_n| > n^{1/4} \ge \varepsilon$ (for n suff large) if $|\hat{\theta}_n| > n^{-1/4}$

$\therefore \mathbb{P}(\sqrt{n}\,|\hat{\theta}_n| \ge \varepsilon) = \mathbb{P}(|\hat{\theta}_n| > n^{-1/4}) \le \frac{Var(\hat{\theta}_n)}{n^{-1/2}} = \frac{1}{\sqrt{n}} \to 0.$

$\therefore \sqrt{n}(\hat{\theta}_n - \theta) \to 0$ in $\mathbb{P}$ if $\theta = 0$

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Case 2: $\theta \neq 0$.

$$\sqrt{n}\,(\hat{\theta}_n - \tilde{\theta}_n) = \sqrt{n}\,\hat{\theta}_n \cdot \mathbb{1}_{|\hat{\theta}_n| \leq n^{-1/4}}$$

Similarly: $\mathbb{P}(\sqrt{n}\,|\hat{\theta}_n - \tilde{\theta}_n| > \varepsilon) = \mathbb{P}(|\hat{\theta}_n| \leq n^{-1/4})$

$$\leq \frac{1}{\sqrt{2\pi/n}} e^{-\frac{1}{2/n}(|\theta| - \frac{1}{n^{1/4}})^2} \quad - (*)$$

For n suff. large, (~~for $\theta \neq 0$~~)

($\hat{\theta}_n \sim N(\theta, 1/n)$)

~~$\log(*) = -\frac{1}{2}\log(n/2\pi) - \frac{(|\theta| - \frac{1}{n^{1/4}})^2 n}{2} \to -\infty$ as $n \to \infty$~~

$\therefore$ ~~$(*) \to 0$~~ (*): For n suff. large s.t. $|\theta| > 1/n^{1/4}$

~~$(*) \leq \sqrt{\frac{n}{2\pi}} e^{-n/2(|\theta| - 1/n^{1/4})^2}$~~

~~$\therefore$ (Slutsky) $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0,1)$~~

$$(*) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{n|\theta|^2}{2}} \cdot e^{\frac{|\theta|\, n^{3/4}}{2}}\, e^{-\sqrt{n}/2}$$

$$= \underbrace{\sqrt{\frac{n}{2\pi}}\, e^{-\sqrt{n}/2}}_{\to 0} \cdot \underbrace{e^{-|\theta|/2\,(n|\theta| - n^{3/4})}}_{\to 0}$$

Slutsky: $\left.\begin{array}{l} \sqrt{n}\,(\hat{\theta}_n - \tilde{\theta}_n) \to 0 \text{ in } \mathbb{P} \\ \sqrt{n}\,(\hat{\theta}_n - \theta_0) \to N(0,1) \text{ in dist.} \end{array}\right] \sqrt{n}\,(\tilde{\theta}_n - \theta) \to N(0,1)$ in dist.

(c) $\mathbb{E}_\theta\left[n|\hat{\theta}_n - \theta|^2\right] = 1 \Rightarrow \lim_{n\to\infty} \sup_{\theta \in \Theta} \left[n\,\mathbb{E}_\theta |\hat{\theta}_n - \theta|^2\right] = 1.$

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(Full Solve):

$$\mathbb{E}_\theta\left[(\tilde{\theta}-\theta)^2\right] = \mathbb{E}_\theta\left[(\hat{\theta}-\theta)^2\right] + \mathbb{E}_\theta\left[\theta^2 \cdot 1_{|\hat{\theta}| \le n^{-1/4}}\right]$$
$$- \mathbb{E}_\theta\left[(\hat{\theta}-\theta)^2 1_{|\hat{\theta}| \le n^{-1/4}}\right]$$

$$= \frac{1}{n} - \mathbb{E}_\theta\left[(\hat{\theta}^2 - 2\theta\hat{\theta}) 1_{|\hat{\theta}| \le n^{-1/4}}\right]$$

$$n\,\mathbb{E}_\theta\left[(\tilde{\theta}-\theta)^2\right] = 1 + \theta^2 \cdot n\, \mathbb{P}_\theta(|\hat{\theta}| \le n^{-1/4}) - \int_{\theta-n^{1/4}}^{\theta+n^{1/4}} (z-\theta)^2 \frac{e^{-\frac{1}{2}(z-\theta)^2}}{\sqrt{2\pi}}\,dz$$

Diff. wrt θ: $2\theta \cdot n\; \mathbb{P}_\theta(|\hat{\theta}| \le n^{-1/4}) + n\theta^2$

$$\hat{\theta} \sim N(\theta, 1/n) \Rightarrow \sqrt{n}\cdot\hat{\theta} \sim N(\sqrt{n}\theta, 1)$$

$$n\,\mathbb{E}_\theta\left[(\tilde{\theta}-\theta)^2\right] = 1 + \mathbb{E}_\theta\left[n\theta^2 1_{|\hat{\theta}| \le n^{-1/4}}\right] - \mathbb{E}_\theta\left[(\sqrt{n}(\hat{\theta}-\theta))^2 1_{\hat{\theta} \le n^{-1/4}}\right]$$

$$= 1 + n\theta^2 \underbrace{\mathbb{P}\left(|N(\sqrt{n}\theta, 1)| \le n^{1/4}\right)}_{\int_{\sqrt{n}\theta - n^{1/4}}^{\sqrt{n}\theta + n^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz} - \underbrace{\int_{n^{-1/4}+\sqrt{n}\theta}^{n^{-1/4}+\sqrt{n}\theta} z^2 \frac{e^{-z^2/2}}{\sqrt{2\pi}}}_{\text{indep}}$$

$$\sqrt{n}\,n\theta^2 \left[f(n^{1/4}+\sqrt{n}\theta) - f(-n^{1/4}+\sqrt{n}\theta)\right] + 2n\theta\,(*) - \int$$

$$(-z^2 + n\theta^2) e^{-\frac{1}{2}(z-\sqrt{n}\theta)^2}$$

$$(2n\theta + (z-\sqrt{n}\theta)(z^2 - n\theta^2)) e^{-\frac{1}{2}(z-\sqrt{n}\theta)^2}$$

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Let $\theta_n = n^{-1/4}$:

$$\mathbb{E}_{\theta_n}\left[n(\tilde{\theta}_n - \theta)^2 \right] = 1 + n\theta^2 \, \mathbb{P}_\theta(|\hat{\theta}_n| \leq n^{-1/4})$$

$$- \mathbb{E}\left[n(\hat{\theta}_n - \theta)^2 \, 1_{|\hat{\theta}_n| \leq n^{-1/4}} \right]$$

1st term: $\mathbb{P}_\theta(|\hat{\theta}_n| \leq n^{-1/4}) = \mathbb{P}(\sqrt{n}\,|\hat{\theta}_n| \leq n^{1/4})$

$$= \int_{\cancel{\sqrt{n}\theta_0} - n^{-1/4} + n^{1/4}}^{2n^{1/4}} f = F(2n^{1/4}) - \tfrac{1}{2} \to \tfrac{1}{2}$$

2nd term: $\displaystyle\int_0^{2n^{1/4}} z^2 f(z) \to \frac{1}{2}$

$\therefore \lim_{n\to\infty} \mathbb{E}_{\theta_n}\left[n(\tilde{\theta}_n - \theta_n)^2 \right] = 1 + \frac{1}{2}\sqrt{n} - \frac{1}{2} \to \infty$

$\therefore \lim_n \sup_\theta n\,\mathbb{E}_\theta (T_n - \theta)^2 = \infty$ for $T_n = \tilde{\theta}_n$

We will prefer $\hat{\theta}_n$ as its minimax risk is lower.

$\tilde{\theta}_n$ may be super-efficient at 0 but ~~is~~ has high risk around $\pm n^{-1/4}$

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2.) (a) Let $\{ f(\cdot, \theta) : \theta \in \Theta \}$ be a regular model on $\mathcal{X} \subseteq \mathbb{R}^{n}$,

$\tilde{\Theta}$ be an unbiased estimator.

$\mathrm{Var}_\theta(\tilde{\Theta}) \geq I(\Theta)^{-1}$ (Crame Rao)

(b) Proof: Let $q(x, \theta) = \log(f(x, \theta))$

$$\mathbb{E}_\theta\left[\frac{d}{d\theta} q(X, \theta) \right] = \int_{\mathcal{X}} f(x, \theta) \frac{\frac{d}{d\theta} f(x, \theta)}{f(x, \theta)} \, d\theta = \frac{d}{d\theta}(1) = 0.$$

$$\therefore \mathrm{Var}_\theta\left(\frac{d}{d\theta} q(x, \theta)^2 \right) = \mathbb{E}_\theta\left[\left(\frac{d}{d\theta} q(x, \theta) \right)^2 \right] = I(\theta)$$

Cauchy Schwarz : $\mathrm{Var}(\tilde{\Theta}) \cdot I(\theta) \geq \mathrm{Cov}\left(\tilde{\Theta} \cdot \frac{d}{d\theta} q\right)^2$

$$\mathrm{Cov}(\tilde{\Theta} \;\; q') = \int_{\mathcal{X}} f(x, \theta) \, \tilde{\Theta}(x) \frac{\frac{d}{d\theta} \log(f(x, \theta))}{f(x, \theta)} \, d\theta$$

$$= \frac{d}{d\theta} \int_{\mathcal{X}} f(x, \theta) \, \tilde{\Theta}(x) \, dx = \frac{d}{d\theta}(\theta) = 1.$$

$$\therefore \mathrm{Var}_\theta(\tilde{\Theta}) \geq I(\theta)^{-1}$$

(b) $\mathbb{E}[\bar{X}_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[X_i] = \theta$

$$\mathbb{E}\left[\sum_{k=1}^{n} (X_k - \bar{X}_n)^2 \right] = \mathbb{E}\left[\sum_{k=1}^{n} \cancel{(X_k^2 - \ldots)} (X_k^2 - 2X_k \bar{X}_n + \bar{X}_n^2) \right]$$

$$= \mathbb{E}\left[\left(\sum_{k=1}^{n} X_k^2 \right) - n \bar{X}_n^2 \right] = n(\mathrm{Var}(X) + \mathbb{E}[X]^2) - n(\mathrm{Var}(\bar{X}_n) + \mathbb{E}[\bar{X}_n]^2)$$

$$= n(\theta + \theta^2) - n(\theta/n + \theta^2) = (n-1)\theta$$

$\therefore \mathbb{E}[S^2] = \theta \Rightarrow T_a$ is unbiased

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$$Var(\bar{X}_n) = \frac{1}{n}\theta$$

$I(\theta)$: (1 sample) : $\cancel{\mathbb{E}\left[\frac{d}{d\theta}\log\left(\frac{e^{-\theta}\theta^{x_1}}{x_1!}\right)\right.}$

$$\log\left(\frac{e^{-\theta}\theta^{x}}{x!}\right) = -\theta + x\log(\theta) - \log(x!)$$

(2^{nd} derivative) : $-x/\theta^2 \Rightarrow I(\theta) = -\mathbb{E}[x/\theta^2] = 1/\theta$

$\therefore Var(\bar{X}_n) = I_n(\theta)^{-1} \leq Var(T_n)$ (Cramer Rao)

(c) $$\mathbb{E}[(\tilde{\theta}_n - \theta)^2] = \mathbb{E}[(\tilde{\theta}_n - \mu)^2 + (\mu - \theta)^2]$$

$$= B(\theta)^2 + Var(\tilde{\theta}_n) = \underbrace{B(\theta)^2}_{\geq 0} + \frac{\left(\frac{d}{d\theta}(\theta + B(\theta))\right)^2}{n/\theta}$$

$$\geq \mathbb{E}_\theta[(\bar{X}_n - \theta)^2]\left(\underbrace{(1+B')^2}_{B' \geq 0 \Rightarrow (1+B')^2 \geq 1}\right) \geq \mathbb{E}_\theta[(\bar{X}_n - \theta)^2]$$ //.

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3.) (a) $\Pi(\theta | X_1, \dots X_n) \sim \frac{1}{\sqrt{2\pi V^2}} e^{-\frac{1}{2V^2}\theta^2} \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(X_k - \theta)^2} \cdot C$ (C constant)

$$\propto e^{-\frac{1}{2V^2}\theta^2} \cdot e^{-\frac{1}{2}\left(\sum_{k=1}^{n} \theta^2 - 2\theta X_k\right)}$$

$$\propto e^{-\frac{1}{2V^2}\left[\theta^2 + nV^2\theta^2 - 2V^2 n \bar{X}_n \cdot \theta\right]}, \quad \bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$\propto e^{-\frac{1}{2V^2}\cdot(1+nV^2)\left(\theta - \frac{V^2}{1+nV^2}\bar{X}_n\right)^2}$$

$$\therefore \Pi(\cdot | X_1, \dots X_n) \sim N\left(\frac{V^2}{1+nV^2}\bar{X}_n, \; V^2/_{1+nV^2}\right)$$

(b) Bernstein Von Mises Theorem: Prior is cont. at (all) θ, $\pi(\theta) > 0$.

Let ϕ_n be pdf of $N(\hat{\theta}_n, \frac{1}{n} I(\theta_0)^{-1})$, θ_0 is true value.

$$\int_\Theta |\Pi(\theta | X_1, \dots X_n) - \phi_n(\theta)| \, d\theta \to 0 \quad \text{A.S.}$$

Construction of C_n: Define $\Phi : [0,\infty) \to [0,1)$,

$$\Phi(x) = \int_{-x}^{x} f_{N(0,\, I(\theta_0)^{-1})}(\theta)\, d\theta$$

is increasing, cont. ($\Rightarrow$ Inverse exists, Inverse cont.)

Pick R_n s.t. $\mathbb{P}_{\Pi(\cdot | X_1,\dots X_n)}\left(|\theta - \hat{\theta}_n| \cdot \sqrt{n} \le R_n\right) = 1-\alpha$, $\alpha = 0.05$

Claim: $R_n \to \Phi^{-1}(1-\alpha)$ A.S.

$$|\Phi(R_n) - (1-\alpha)| = \left| \int_{\hat{\theta}_n - R_n/\sqrt{n}}^{\hat{\theta}_n + R_n/\sqrt{n}} \Pi(\theta | X_1, \dots X_n)\, d\theta - (1-\alpha) \right| +$$

$$\left| \int_{\hat{\theta}_n + [-R_n/\sqrt{n},\, R_n/\sqrt{n}]} \Pi(\theta | X_1, \dots X_n) - \phi_n(\theta)\, d\theta \right|$$

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$$\le \int_{\mathbb{R}} \left| \Pi(\theta \mid X_1, \dots X_n) - \Phi_n(\theta) \right| d\theta \to 0 \quad A.S.$$

$\therefore \quad \Phi(R_n) \to 1-\alpha \quad A.S \Rightarrow$ (Cont. map theorem) $\quad R_n \to \Phi^{-1}(1-\alpha) \quad A.S.$

Claim: $\mathbb{P}\left(\theta_0 \quad |\theta_0 - \hat{\theta}_n| \le R_n/\sqrt{n}\right) = \mathbb{P}\left(\frac{\Phi^{-1}(1-\alpha)}{R_n} \sqrt{n}\,|\theta_0 - \hat{\theta}_n| \le \Phi^{-1}(1-\alpha)\right)$

$\frac{\Phi^{-1}(1-\alpha)}{R_n} \to 1$ in $\mathbb{P}$; $\sqrt{n}\,[\theta_0 - \hat{\theta}_n] \to N(0, I(\theta)^{-1})$ in dist. (CLT)

Slutsky : $\frac{\Phi^{-1}(1-\alpha)}{R_n} \cdot \sqrt{n}\,(\theta_0 - \hat{\theta}_n) \xrightarrow{(d)} N(0, I(\theta)^{-1})$

$\therefore \mathbb{P}(\theta_0 \in C_n) \to 1-\alpha \quad A.S.$

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4.) (a) Let Θ be a parameter space, π be a prior on Θ.

$\delta: X \to \overset{A}{\cancel{\Theta}}$ be a rule, $L: \Theta \times A \to [0, \infty)$ is a loss function

$R(\delta, \theta) = \mathbb{E}_\theta [L(\theta, \delta(X))]$

If δ minimises $\overset{R(\delta)_\pi =}{\wedge} \int_{\theta \in \Theta} \pi(\theta) \, R(\delta, \theta) \, d\theta$, δ is a π - bayes rule.

π_1 is least favourable if $\forall$ prior π_2,

$R(\delta_{\pi_1})_{\pi_1} \geq R(\delta_{\pi_2})_{\pi_2}$

(b) ~~Suppose $\exists$ prior π s.t. $R(\delta_{\pi'})_{\pi'} \geq R(\delta_{\pi})_{\pi}$:~~

~~$R(\delta_\pi)_\pi = \sup_{\theta \in \Theta} \{ R(\delta_\pi, \theta) \} \leq \sup_{\theta \in \Theta} \{ R(\delta_{\pi'}, \theta) \}$~~

Let π_1 be any prior:

$R(\delta_{\pi_1})_{\pi_1} \leq R(\delta_\pi)_{\pi_1} \leq \sup_{\theta \in \Theta} \{ R(\delta_\pi, \theta) \} = R(\delta_\pi)_\pi$

$\therefore$ Least favourable

(c) Let prior be $\beta(a, b)$, density $\propto x^{a-1} \cdot (1-x)^{b-1}$ on $[0, 1]$

Bayes rule: $\pi(\theta \mid X) \propto \theta^{\alpha-1} (1-\theta)^{\beta-1} \times \theta^x (1-\theta)^{n-x}$

$\sim \beta(\alpha + x, \beta + (n-x))$

$\delta(x) = \dfrac{\alpha + x}{\alpha + \beta + n}$ is the bayes rule.

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$$R(\delta, \theta) = \mathbb{E}_\theta\left[\left(\frac{\alpha+x}{\alpha+\beta+n} - \theta\right)^2\right] = (\alpha+\beta+n)^{-2}\,\mathbb{E}_\theta\left[\left(\alpha+x-(\alpha+\beta+n)\theta\right)^2\right]$$

$$= (\alpha+\beta+n)^{-2}\,\mathbb{E}_\theta\left[\left(x-n\theta + (\alpha-(\alpha+\beta)\theta)\right)^2\right]$$

$$= (\alpha+\beta+n)^{-2}\;\gamma\left[\underbrace{n\theta(1-\theta)}_{n\theta-n\theta^2} + \underbrace{(\alpha-(\alpha+\beta)\theta)^2}_{(\alpha+\beta)^2\theta^2 - 2\alpha(\alpha+\beta)\theta + \alpha^2}\right]$$

$$\left.\begin{matrix} n = 2\alpha(\alpha+\beta) \\ n = (\alpha+\beta)^2 \end{matrix}\right] \quad \alpha+\beta = \sqrt{n}, \quad \alpha = \beta = \sqrt{n}/2. \quad \text{gives constant risk.}$$

$$\therefore\; R(\delta_\pi)_\pi = \sup_{\theta\in\Theta}\left[R(\delta_\pi, \theta)\right]$$

$\therefore$ Least favourable.

Principle of Statistics 2017

1.) (a) Posterior : $\pi(\theta \mid \cancel{X_1,\dots,X_n}) \propto \pi(\theta)\, f(\cancel{X_1,\dots,X_n} \mid \theta)$

$$\propto \theta^{a-1}(1-\theta)^{b-1}\,\theta^{x}(1-\theta)^{n-x}$$

$$= \theta^{a+x-1}(1-\theta)^{n-x+b-1}$$

$\therefore \pi(\cdot \mid X) \sim \beta(a+x,\ b+n-x)$ (Beta distribution)

(b) $R(\hat{p}, p) = \mathbb{E}_p[L(\hat{p}, p)]$

$R(\hat{p},)_\pi = \int_0^1 \pi(p)\, R(\hat{p}, p)\, dp$ (Baye's risk.)

(ii) Posterior : $\pi \sim \beta(1+x,\ 1+n-x)$

$\therefore$ Minimise: $\mathbb{E}_\pi\left[\frac{(a-p)^2}{p(1-p)}\right]$, $p \sim \pi$

$$(*) = \int_0^1 C\, p^{x}(1-p)^{n-x}\frac{(a-p)^2}{p(1-p)}\,dp \quad (C \text{ is constant})$$

$$= \int_0^1 C\, p^{x-1}(1-p)^{n-x-1}(a^2-2ap+p^2)\,dp$$

Min. $a^2\Big(\underbrace{\int_0^1 p^{x-1}(1-p)^{n-x-1}\,dp}_{=\frac{(x-1)!\,(n-x-1)!}{(n-1)!}}\Big) - 2a\underbrace{\int_0^1 p^{x}(1-p)^{n-x-1}\,dp}_{=\frac{x!\,(n-x-1)!}{n!}}$

$\therefore$ Min. $a^2\left(\frac{n}{x}\right) - 2a \Rightarrow$ Pick $a = x/n$.

$\therefore \delta(x) = x/n$ is a posterior optimiser $\Rightarrow$ Minimise bayes risk.

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Risk: $R(\delta, p) = \mathbb{E}_p\left[\frac{(X/n - p)^2}{p\,(1-p)}\right] = \frac{1}{n^2 p(1-p)} \underbrace{\mathbb{E}_p\left[(X-np)^2\right]}_{Var(X) = np(1-p)}$

$= \frac{1}{n}$ is constant.

(iii) Since δ has constant risk:

$$R(\delta)_\pi = \sup_{p \in [0,1]} \left\{ R(\delta, p) \right\}$$

~~If δ_1 is minimax~~ : $\forall$ rule δ_1 :

~~$\forall p \in [0,1]$~~,

$$\sup_{p \in [0,1]} \left\{ R(\delta_1, p) \right\} \geq R(\delta_1)_\pi \geq R(\delta)_\pi = \sup_{p \in [0,1]} \left\{ R(\delta, p) \right\}$$

$\therefore$ δ is minimax. $\quad (\delta(X) = X/n.)$

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2.) (a) $$\mathbb{E}[X_1] = \int_\theta^{\theta+1} (1-\alpha)(x-\theta)^{-\alpha}\, x \, dx$$

$$= \left[(1-\alpha)\frac{(x-\theta)^{1-\alpha}}{1-\alpha} x \right]_\theta^{\theta+1} - \int_\theta^{\theta+1} (x-\theta)^{1-\alpha}\, dx$$

$$= \theta + 1 - \left[\frac{(x-\theta)^{2-\alpha}}{2-\alpha} \right]_\theta^{\theta+1} = \theta + 1 - \frac{1}{2-\alpha}$$

$$\mathbb{E}[X_1^2] = \int_\theta^{\theta+1} (1-\alpha)(x-\theta)^{-\alpha} x^2\, dx = \left[(x-\theta)^{1-\alpha} x^2 \right]_\theta^{\theta+1} - 2\int_\theta^{\theta+1} x (x-\theta)^{1-\alpha} dx$$

$$= (\theta+1)^2 - 2\cancel{/1-\alpha}\; \frac{2}{2-\alpha} \int_\theta^{\theta+1} \frac{x (x-\theta)^{1-\alpha}}{\cancel{1+(1-\alpha)}} \cdot (1+(1-\alpha))\, dx$$

$$= (\theta+1)^2 - \frac{2}{2-\alpha}\left[\theta + 1 - \frac{1}{3-\alpha} \right]$$

$$\therefore \operatorname{Var}(X_1) = (\theta+1)^2 - \frac{2}{2-\alpha}(\theta+1) + \frac{2}{(2-\alpha)(3-\alpha)} - (\theta+1)^2 + \frac{2(\theta+1)}{2-\alpha} - \frac{1}{(2-\alpha)^2}$$

$$= \frac{1}{(2-\alpha)^2(3-\alpha)}\left[2(2-\alpha) - (3-\alpha) \right] = \frac{1-\alpha}{(2-\alpha)^2(3-\alpha)}$$

$$\mathbb{E}[\tilde{\theta}_n] = \theta + 1 - \frac{1}{2-\alpha} + c(\alpha), \quad \text{pick } c(\alpha) = \frac{1}{2-\alpha} - 1 :$$

$\tilde{\theta}_n$ is unbiased.

(b) Consistent: Law of large #: $\bar{X}_n \to \mathbb{E}[X_1] = \theta + 1 - \frac{1}{2-\alpha}$ A.S.

$$\therefore \tilde{\theta}_n \to \theta + 1 - \frac{1}{2-\alpha} + c(\alpha) = \theta \quad \text{A.S.}$$

$$\tilde{\theta}_n = \frac{1}{n}\sum_{k=1}^{n} (X_k + c(\alpha)), \quad Y_k = X_k + c(\alpha)$$

$$\mathbb{E}[Y_k] = \theta, \quad \operatorname{Var}(Y_k) = \operatorname{Var}(X_k) \Rightarrow \mathbb{E}[Y_k^2] < \infty.$$

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$(\mathbb{E}[Y_k],\ \forall k\ \mathbb{E}[Y_k^2] < \infty)$

Central limit theorem: $\sqrt{n}\left[\frac{1}{n}\sum_{k=1}^{n} Y_k - \theta\right] \to N(0, Var(Y_1))$ in distribution ~~distributio~~

$\therefore \to N\left(0, \frac{1-\alpha}{(2-\alpha)^2(3-\alpha)}\right)$ (in distribution)

(c) MLE : $\text{Max} \sum_{k=1}^{n} \log\left(\cancel{(1-\alpha)}\,(X_k-\theta)^{-\alpha}\right)$, Pick θ s.t. $\begin{cases}\theta \le X_{(1)}, \\ \theta + 1 \ge X_{(n)}\end{cases}$

$$\text{Max} \ -\alpha \sum_{k=1}^{n} \log(X_k - \theta) \equiv \text{Min} \sum_{k=1}^{n} \log(X_k - \theta)$$

$\therefore$ Pick $\theta = X_{(1)}$.

$\therefore \hat{\theta}_n = X_{(1)}$

$$\mathbb{P}(\hat{\theta}_n - \theta > t) : \mathbb{P}(\text{Min}(X_1, .. X_n) > t+\theta) = \mathbb{P}(X_1 > t+\theta)^n$$

$$= \int_{\theta+t}^{\theta+1} (1-\alpha)(x-\theta)^{-\alpha}\,dx^{\,n} = \left(\left[(x-\theta)^{1-\alpha}\right]_{\theta+t}^{\theta+1}\right)^n$$

$$= (1-t^{1-\alpha})^n.$$

$$\mathbb{E}[\hat{\theta}_n] = \theta + \int_0^1 \mathbb{P}(\hat{\theta}_n - \theta > x)\,dx = \theta + \int_0^1 (1-x^{1-\alpha})^n\,dx$$

$$\ge \theta + \int_0^{1/2} (1-x^{1-\alpha})^n\,dx \ge \theta + \frac{1}{2}\left(1-\frac{1}{2}^{(1-\alpha)}\right)^n > \theta$$

$\therefore$ Biased

(d) $\mathbb{P}(\hat{\theta}_n - \theta > t\,n^{-\beta}) = (1-(t\,n^{-\beta})^{1-\alpha})^n = (1 - t^{1-\alpha} \cdot n^{-\beta(1-\alpha)})^n$

Pick $\beta = 1/1-\alpha$: $\lim_{n\to\infty} \mathbb{P}(\hat{\theta}_n - \theta > t/n^{\beta}) = \lim_{n\to\infty}\left(1 - (t^{1-\alpha})/n\right)^n$

$$= e^{-t^{1-\alpha}}$$

~~$N^{\alpha} \to 1/2$~~ $(\hat{\theta}_n - \theta)$ converges to 0 at rate $n^{1/1-\alpha}$, $\frac{1}{1-\alpha} \ge \frac{1}{2}$

$1/1-\alpha > 1 > 1/2 \Rightarrow$ F Much faster compared to $\tilde{\theta}_n$.

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3.) (a) $S_n(\theta) = \frac{d}{d\theta} \log(f(X_1, \dots X_n ; \theta))$

$$\mathbb{E}[S_n(\theta_0)] = \int_{\mathcal{X}} f(\underline{x}, \theta_0) \frac{f'(\underline{x}, \theta_0)}{f(\underline{x}, \theta_0)} dx = \frac{d}{d\theta} \int_{\mathcal{X}} f'(\underline{x}, \theta_0) dx$$

$$= \frac{d}{d\theta}(1) = 0.$$

$(\underline{X} = (X_1, \dots X_n))$

(b) $$I(\theta) = \mathbb{E}_\theta\left[\left(\frac{d}{d\theta} \log(f(X,\theta))\right)^2\right]$$

$$\mathbb{E}_\theta[\log(f(X,\theta))''] = \mathbb{E}_\theta\left[\left(\frac{f'(X,\theta)}{f(X,\theta)}\right)'\right]$$

$$= \mathbb{E}_\theta\left[\frac{f''(X,\theta)}{f(X,\theta)} - \left(\frac{f'(X,\theta)}{f(X,\theta)}\right)^2\right] = \int_X f''(X,\theta)\, dx - I(\theta)$$

$$= -I(\theta) + \frac{d^2}{d\theta^2}\left(\int_X f(X,\theta)\, d\theta\right) = -I(\theta)$$

$$\therefore I(\theta) = -\mathbb{E}_\theta\left[\frac{d^2}{d\theta^2}(\log(f(X,\theta)))\right]$$

(c) $$\hat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{arg\,max}} \left\{ \prod_{k=1}^{n} f(X_k, \theta) \right\}$$

$\hat{\theta}_n \to \theta_0$ in probability. $\forall \varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta_0| \geq \varepsilon) = 0.$

$\sqrt{n}(\hat{\theta}_n - \theta_0) \to N(0, I(\theta_0)^{-1})$ in distribution

$$\forall t \in \mathbb{R} : \mathbb{P}(\sqrt{n}(\hat{\theta}_n - \theta_0) \leq t) \to \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi / I(\theta_0)}} e^{-\frac{(x)^2}{2 I(\theta_0)^{-1}}} dx$$

as $n \to \infty$.

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(d) Claim: $\tilde{\theta}_n \to \theta_0$ in $\mathbb{P}$

~~$\mathbb{P}(|\tilde{\theta}_n - \hat{\theta}_n| \geq \varepsilon) \leq \mathbb{P}(|\tilde{\theta}_n - \hat{\theta}_n|$~~

$\mathbb{P}(|\tilde{\theta}_n - \hat{\theta}_n| \geq \varepsilon) = 0$ for $1/n < \varepsilon$ (n suff. large)

$\therefore \tilde{\theta}_n - \hat{\theta}_n \to 0$ in $\mathbb{P}$

Slutsky: $\tilde{\theta}_n = \underbrace{(\tilde{\theta}_n - \hat{\theta}_n)}_{\to 0 \text{ in } \mathbb{P}} + \underbrace{\hat{\theta}_n}_{\to \theta_0 \text{ in } \mathbb{P}} \longrightarrow \theta_0$ in dist $\Rightarrow \to \theta_0$ in $\mathbb{P}$

Cont map theorem: $\psi(\tilde{\theta}_n) \to \psi(\theta_0)$ (ψ cont.) in $\mathbb{P}$

$|\sqrt{n}(\psi(\tilde{\theta}_n) - \psi(\hat{\theta}_n))| = \sqrt{n}\,|\psi'(\theta_n')| \cdot |\tilde{\theta}_n - \hat{\theta}_n|$

$\leq \frac{1}{\sqrt{n}} |\psi'(\theta_n')|$ — $(*)$

$\theta_n' \in [\tilde{\theta}_n, \hat{\theta}_n] \Rightarrow |\theta_n' - \hat{\theta}_n| \leq 1/n$

$\therefore \theta_n' - \hat{\theta}_n \to 0$ in $\mathbb{P}$ $\Rightarrow$ $\theta_n' - \theta_0 = \underbrace{(\theta_n' - \hat{\theta}_n)}_{\to 0 \text{ in d}} + \underbrace{(\hat{\theta}_n - \theta_0)}_{\to 0 \text{ in d}} \to 0$ (in d)

$\therefore \theta_n' - \theta_0 \to 0$ in dist.

(cont. map theorem)

ψ' cont. $\Rightarrow$ $\psi'(\theta_n') \to 0$ in dist. $\Rightarrow$ $(*) \to 0$ in dist.

$\therefore |\sqrt{n}(\psi(\tilde{\theta}_n) - \psi(\hat{\theta}_n))| \to 0$ in dist. $\Rightarrow \to 0$ in $\mathbb{P}$

Plug in: $\sqrt{n}(\psi(\hat{\theta}_n) - \psi(\theta_0)) \to N(0, I(\theta_0)^{-1}) \cdot \psi'(\theta_0)$ (in dist.)

$\therefore$ Slutsky: $\sqrt{n}(\psi(\tilde{\theta}_n) - \psi(\theta_0)) \to N(0, \psi'(\theta_0)^2 I(\theta_0)^{-1})$

in distribution

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4.) (a) Max. $\prod_{k=1}^{n} e^{-\frac{1}{2}(X_k - \theta)\cdot \Sigma^{-1}(X_k - \theta)}$ equivalent to:

Min. $\sum_{k=1}^{n} (X_k - \theta)^T \cdot \Sigma (X_k - \theta)$

Diff wrt θ: $\sum_{k=1}^{n} \cancel{(\ldots)_j^{(-1)}} (\Sigma \cdot (X_k - \theta))_j + ((X_k - \theta)^T \cdot \Sigma)_j \cancel{(\ldots)_j^{(-1)}}$

$= -2 \sum_{k=1}^{n} (\Sigma \cdot (X_k - \theta))_j = -2 \sum (\cdot (X_1 + \ldots + X_n - n\theta)) = 0$

$\therefore \hat{\theta} = \frac{1}{n} \sum_{k=1}^{n} X_k$ is the MLE

$\hat{\theta}$ sum of ~~iid~~ independent gaussian $\Rightarrow$ $\hat{\theta}$ is a gaussian

$\mathbb{E}[\hat{\theta}] = \theta$, $\mathrm{Cov}(\hat{\theta}) = \frac{1}{n^2} \sum_{k=1}^{n} \mathrm{Cov}(X_k, X_k)$

$= \frac{1}{n} \Sigma$

$\therefore \hat{\theta}_n \sim N_d(\theta, \frac{1}{n} \Sigma)$

(b) Pick β s.t. $\int_{\bar{D}(0,\beta)} f_{N_d(0,\Sigma)}(x) \, dx = 1 - \alpha$

$\therefore \mathbb{P}(\sqrt{n}(\hat{\theta}_n - \theta_0) \in \bar{D}(0,\beta)) = 1 - \alpha$

$\therefore C_n = \{ \theta : |\theta - \hat{\theta}_n| \cdot \ldots \leq \beta/\sqrt{n} \}$, $\mathbb{P}(\theta_0 \in C_n) = 1 - \alpha$

(c) ~~Max.~~ Min. $\sum_{k=1}^{n} \|(X_k - \theta)\|^2$ ~~s.t. $\|\theta\|^2 = 1$~~

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$$\sum_{k=1}^{n} \| X_k - \theta \|^2 = \sum_{k=1}^{n} \| X_k - \bar{X} \|^2 + \| \bar{X} - \theta \|^2 + 2 \langle X_k - \bar{X}, \bar{X} - \theta \rangle$$

$$= \sum_{k=1}^{n} \| X_k - \bar{X} \|^2 + n \| \bar{X} - \theta \|^2 + 2 \underbrace{\langle \sum_{k=1}^{n} (X_k - \bar{X}), \bar{X} - \theta \rangle}_{=0.}$$

Minimised by minimising $\| \bar{X} - \theta \|^2$:

(i) $\| \theta \|^2 = 1$: $\| \bar{X} - \theta \|^2 = \| \bar{X} \|^2 - 2 \langle \bar{X}, \theta \rangle + \| \theta \|^2$.

$\therefore$ Max. $\langle \bar{X}, \theta \rangle$: Pick $\theta = \bar{X} / \| \bar{X} \|$ ($\bar{X} \neq 0$)

$= (1, 0, \dots)$ ($\bar{X} = 0$)

Note: $\mathbb{P}(\bar{X} = 0) = 0$.

(ii) $\underline{v} \cdot \theta = 0$ (Project $\bar{X}$ onto plane $\underline{v} \cdot * = 0$)

$\pi(\bar{X}) = \bar{X} - \langle \bar{X}, V \rangle \cdot V$ ~~is a unit vector~~ of

$\hat{\theta} = \bar{X} - \langle \bar{X}, V \rangle \cdot V$ is minimal: $\forall \omega \in \Theta$,

$\| \bar{X} - \omega \|^2 = \| \bar{X} - \hat{\theta} \|^2 + \| \hat{\theta} - \omega \|^2$ as $V \perp (\hat{\theta} - \omega)$

$\therefore$ Done.

(d) If $\bar{X} = 0$: $\inf_{\theta \neq 0} \{ \| \theta \|^2 \} = 0 \Rightarrow \Rightarrow$ We can use MLE.

$$\Lambda = 2 \log \left(e^{-\frac{1}{2} \sum_{k=1}^{n} (X_k - \bar{X})^T (X_k - \bar{X})} \Big/ e^{-\frac{1}{2} \sum_{k=1}^{n} X_k^T \cdot X_k} \right)$$

$$= \sum_{k=1}^{n} \| X_k \|^2 - \| X_k - \bar{X} \|^2 = \sum_{k=1}^{n} \left(2 \langle X_k, \bar{X} \rangle - \| \bar{X} \|^2 \right)$$

$= 2n \| \bar{X} \|^2 - \| \bar{X} \|^2 = n \| \sqrt{n} \bar{X} \|^2$; $\sqrt{n} \bar{X} \sim N_d(0, I)$

$\therefore \| \sqrt{n} X_d \|^2 \sim \chi_d^2$

Wilk's Theorem : $\Lambda_n \to \chi_d^2$ ($n \to \infty$)

$\therefore$ Consistent.

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Principle of Statistics 2018

1.) (a) $f(*, \theta)$: $(*) = AA \quad (1-\theta)^2$

$= BB \quad (\theta)^2$

$= AB \quad 2(1-\theta)\cdot\theta$

(b) $n_* = \sum_{k=1}^{n} 1_{\text{Fly } k \text{ is } *}$: $\mathbb{E}[n_*] = n \cdot f(*, \theta)$

$\therefore \mathbb{E}[\hat{\theta}_{\omega}] = \underset{\sim}{\omega} \cdot \begin{pmatrix} (1-\theta)^2 \\ \theta^2 \\ 2\theta(1-\theta) \end{pmatrix} = \omega_{AA}(\theta^2 - 2\theta + 1) + \omega_{BB}\theta^2 + 2\theta(1-\theta)\omega_{AB}$

$= \theta^2[\omega_{AA} + \omega_{BB} - 2\omega_{AB}] + \theta[-2\omega_{AA} + 2\omega_{AB}] + [\omega_{AA}]$

Solve : $\underbrace{\begin{pmatrix} 1 & 1 & -2 \\ -2 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}}_{\text{Full rank}} \cdot \underset{\sim}{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

~~(Check)~~:

$\therefore$ Unique solution $\underset{\sim}{\omega} = \begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix}$

$\therefore \hat{\theta}_{\omega} = \frac{1}{n}\left[\omega\, n_{BB} + \frac{1}{2} n_{AB}\right]$

Strong law: $\frac{1}{n}\sum_{k=1}^{\hat{}} 1_{\text{Fly } n \text{ is } BB} \to \underbrace{\mathbb{P}(BB)}_{\theta^2}$ (A.S)

$\frac{1}{n}\sum_{k=1}^{\hat{}} 1_{\text{Fly } n \text{ is } AB} \to \underbrace{\mathbb{P}(AB)}_{2\theta(1-\theta)}$ (A.S)

$\therefore \hat{\theta} \to \theta^2 + \frac{1}{2}\cancel{2\theta}(2\theta)(1-\theta) = \theta$ in dist. $\Rightarrow \; \to \theta$ in $\mathbb{P}$

(Slutsky)

$\therefore \hat{\theta}$ is consistent.

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(c) $$f(n_{AA}, n_{BB}, n_{AB}) = \binom{n}{n_{AA},\, n_{BB},\, n_{AB}} \cdot (1-\theta)^{2 n_{AA}} \, \theta^{2 n_{\cancel{AA}BB}} \, 2^{n_{AB}} \, \theta^{n_{AB}} \, (1-\theta)^{n_{BA}}$$

$$= \underbrace{K}_{\text{constant in } \theta} \cdot \theta^{n_{AB} + 2n_{BB}} \, (1-\theta)^{n_{AB} + 2 n_{AA}}$$

Max. $(n_{AB} + 2n_{BB}) \log(\theta) + (n_{AB} + 2n_{AA}) \log(1-\theta)$

Diff wrt θ: $\dfrac{n_{AB} + 2n_{BB}}{\theta} = \dfrac{n_{AB} + 2n_{AA}}{1-\theta}$

$\therefore n_{AB} + 2n_{BB} = 2(n) \cdot \theta$

$\therefore \hat{\theta} = \frac{1}{2n}(n_{AB} + 2n_{BB}) = \hat{\theta}_{ML}$

(d) $$\hat{\theta} = \frac{1}{n} \sum_{k=1}^{\hat{n}} \underbrace{\tfrac{1}{2}\left(1_{\text{Fly } k \text{ is } AB} + 2 \, 1_{\text{Fly } k \text{ is } BB}\right)}_{Y_k}$$

$(Y_k)_{k \geq 1}$ is iid, $\mathbb{E}[Y_k] \leq \frac{1}{2}(1+2) < \infty$, $\mathbb{E}[Y_k^2] \leq 4 \cdot 3^2 < \infty$ } Apply Central limit theorem.

$$\sqrt{n}\,(\hat{\theta}_n - \cancel{\theta}\, \mathbb{E}[Y_1]) \to N(0, Var(Y_1))$$

$$\mathbb{E}[Y_k] = \theta(1-\theta) + 2\theta^2 = \theta$$

$$\mathbb{E}[Y_k^2] = \frac{1}{4}\mathbb{E}[1_{AB} + 4\,1_{BB} + 4\,\underbrace{1_{AB}1_{BB}}_{=0}]$$

$$= \frac{1}{4}(2\theta(1-\theta) + 4\theta^2) = \frac{\theta(1+\theta)}{2}$$

$\therefore$ Limit: $N(0, \frac{1}{2}\theta(1+\theta))$

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3.) (a) $I(\theta_0) = \mathbb{E}_{\theta_0}\left[\nabla \log(f(X,\theta_0)) \cdot \nabla \log(f(X,\theta_0))\right]$ where

$$f(X,\theta_0) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{1}{2}\|X-\theta\|^2}$$

$$\therefore \nabla \log(f(X,\theta_0)) = \nabla\left(-\frac{1}{2}\|X-\theta\|^2\right) = X-\theta$$

$$\therefore I(\theta_0) = \mathbb{E}_{\theta_0}\left[(X-\theta_0)\cdot(X-\theta_0)^T\right] = \mathrm{Cov}(X) = I$$

(b) $$i_n(\hat{\theta}_n) = \frac{1}{n}\sum_{k=1}^{n}\Big[\big(\nabla_\theta \log(f(X_k,\hat{\theta}_n)) \cdot \nabla_\theta \log(f(X_k,\hat{\theta}_n))\big) - \mathbb{E}_{\theta_0}\left[\nabla_\theta \log(f(X_k,\hat{\theta}_n))\nabla_\theta \log(f(X_k,\hat{\theta}_n))^T\right]\Big] \quad (*)$$

$$+ \mathbb{E}_{\theta_0}\left[\nabla_\theta \log(f(X,\hat{\theta}_n))\nabla_\theta \log(f(X,\hat{\theta}_n))^T\right] - \mathbb{E}_{\theta_0}\left[\nabla_\theta \log(f(X,\theta_0))\nabla_\theta \log(f(X,\theta_0))^T\right] + \quad (**)$$

$$+ I(\theta_0)$$

line 1: if $\hat{\theta}_n \in \bar{D}(\theta_0,1)$, $|*| \leq \frac{1}{n}\sum_{k=1}^{n} \sup_{\theta\in\bar{D}(\theta_0,1)}$

$$\underbrace{\left\{\nabla_\theta \log(f(X_k,\theta))\nabla_\theta \log(f(X_k,\theta))^T - \mathbb{E}_{\theta_0}\left[\nabla_\theta \log(f(X_k,\theta))\cdot\nabla_\theta \log(f(X_k,\theta))^T\right]\right\}}_{=\Lambda}$$

$$\mathbb{P}(\cancel{|\Lambda| \geq \varepsilon}\ (*) \geq \varepsilon) \leq \underbrace{\mathbb{P}(|\hat{\theta}_n-\theta_0| \geq \delta)}_{\to 0 \text{ as } \hat{\theta}_n \to \theta_0 \text{ in } \mathbb{P}} + \underbrace{\mathbb{P}(|\Lambda| \geq \varepsilon)}_{\text{Uniform L.LN. on compact } \bar{D}(\theta_0,1)}$$

Let $q(x,\theta) = \nabla_\theta \log(f(x,\theta)) \cdot \nabla_\theta \log(f(x,\theta))^T$.

Uniform LLN: $$\sup_{\theta\in\bar{D}(\theta_0,1)}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} q(X_k,\theta) - \mathbb{E}[q(X,\theta)]\right|\right\} \to 0 \quad \text{A.S.}$$

$$\therefore \mathbb{P}(|*| \geq \varepsilon) \leq \mathbb{P}(|\hat{\theta}_n-\theta_0| \geq 1) + \mathbb{P}\left(\sup\left|\frac{1}{n}\sum_{k=1}^{n} q(X_k,\hat{\theta}_n) - \mathbb{E}[q(X,\hat{\theta}_n)]\right| \geq \varepsilon\right)$$

$$\leq \underbrace{\mathbb{P}(|\hat{\theta}_n-\theta_0| \geq 1)}_{\to 0 \text{ as } \hat{\theta}_n \to \theta_0 \text{ in } \mathbb{P}} + \mathbb{P}\left(\sup_{\theta\in\bar{D}(\theta_0,1)}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} q(X_k,\theta) - \mathbb{E}[q(X_k,\theta)]\right|\right\} \geq \varepsilon\right)$$

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$\to 0$ as $n \to \infty$. $\therefore$ $(*) \to 0$ in $\mathbb{P}$

$(**)$: $\theta \to \mathbb{E}_{\theta_0}[q(X,\theta)]$ is cont. (regularity conditions)

$\therefore \mathbb{E}_{\theta_0}[q(X,\hat{\theta}_n) - q(X,\theta_0)] \to 0$ $\overset{(\mathbb{P})}{\text{Ass.}}$ as $\hat{\theta}_n \to \theta_0$ $\overset{(\mathbb{P})}{\text{Ass.}}$

(Cont. map theorem)

Slutsky: $\hat{i}_n \to I(\theta_0)$ in $\mathbb{P}$ ($I(\theta_0)$ is deterministic)

(c) $W_n(\theta) = n(\hat{\theta}_n - \theta_0)^T \cdot \hat{i}_n(\hat{\theta}_n) \cdot (\hat{\theta}_n - \theta)$

$W_n(\theta_0)$: $\sqrt{n}(\hat{\theta}_n - \theta_0) \to N(0, I)$ in dist.
$\hat{i}_n(\hat{\theta}_n) \to I(\theta_0)$ in $\mathbb{P}$

Let $D = \sqrt{I(\theta_0)}$ (D exists as $I(\theta_0)$ +ve semi-definite)

$\therefore D^{-1} \cdot \hat{i}_n(\hat{\theta}_n) \to D$ in dist. $\Rightarrow \to D$ in $\mathbb{P}$ (Slutsky)

$$W_n(\theta) = \| \underbrace{D^{-1} \hat{i}_n(\hat{\theta}_n) \cdot \sqrt{n}(\hat{\theta}_n - \theta)}_{A_n} \|^2$$

Slutsky: $A_n \to D^{-1} \hat{i}_n(\hat{\theta}_n) \cdot N(0, I(\theta_0)^{-1})$ in dist.
$= I^{-1} \cdot I \; N(0, I(\theta_0)^{-1}) = N(0, I)$

$\therefore$ Cont. map theorem: $W_n(\theta) \to N(0,I)^2 = \chi_p^2$ in distribution

(d) Given $\alpha \in (0,1)$, pick ξ s.t. $\mathbb{P}(\chi_p^2 \leq \xi) = 1-\alpha$.

$\therefore \mathbb{P}(\sqrt{n}(\hat{\theta}_n - \theta_0) \cdot \hat{i}_n(\hat{\theta}_n) \cdot (\hat{\theta}_n - \theta_0) \leq \xi) \to 1-\alpha$

$\therefore C_n = \{ \theta \in \mathbb{R}^p : (\hat{\theta}_n - \theta_0)^T \cdot \hat{i}_n(\hat{\theta}_n) \cdot (\hat{\theta}_n - \theta_0) \leq \xi / \sqrt{n} \}$ is a $1-\alpha$ confidence interval.

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2.) (a) $R(\hat{\theta}, \theta) = \mathbb{E}_\theta\left[\cancel{\lambda(}\ \|\hat{\theta} - \theta\|^2\right]$

MLE: Max $\frac{1}{\sqrt{2\pi}^{\,p}} e^{-\frac{1}{2}\|X-\theta\|^2} \Rightarrow \cancel{\text{Min. } \|X - \bar{X}\|^2 + \|\bar{X} - \theta\|^2 + 2\langle X - X}$

$\therefore$ Min. $\|X - \theta\|^2 \Rightarrow$ Pick $\theta = X$

$\hat{\theta} = X$; $R(\hat{\theta}, \theta) = \mathbb{E}_\theta\left[\|X - \theta\|^2\right] = \sum_{k=1}^{p} \mathbb{E}_\theta\left[|X_k - \theta_k|^2\right]$

$= \sum_{k=1}^{p} Var_\theta(X_k) = p.$

(b) $\delta(X) = \left(1 - \frac{(p-2)}{\|X\|^2}\right) X$:

$\mathbb{E}\left[\|\delta(X) - \theta\|^2\right] = \mathbb{E}\left[\left\|X - \theta - \frac{(p-2)}{\|X\|^2} X\right\|^2\right]$

$= \mathbb{E}\left[\|X - \theta\|^2 + \frac{(p-2)^2\|X\|^2}{\|X\|^4} - \frac{2(p-2)}{\|X\|^2}(X \cdot (X - \theta))\right]$

$= p + (p-2)^2\, \mathbb{E}\left[1/\|X\|^2\right] - 2(p-2) \underbrace{\mathbb{E}\left[\frac{X \cdot (X-\theta)}{\|X\|^2}\right]}_{(*)}$

$\mathbb{E}\left[\frac{X_j(X_j - \theta)}{\|X\|^2}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{X_j(X_j-\theta)}{\|X\|^2} \,\middle|\, X_{(-j)}\right]\right]$

$=$ (Stein's lemma) $\mathbb{E}\left[\mathbb{E}\left[\frac{\|X\|^2 - 2X_j^2}{\|X\|^4} \,\middle|\, X_{(-j)}\right]\right] = \mathbb{E}\left[\frac{1}{\|X\|^2} - \frac{2X_j^2}{\|X\|^4}\right]$

$\therefore (*) = \sum_{j=1}^{p} \mathbb{E}\left[\frac{1}{\|X\|^2} - \frac{2X_j^2}{\|X\|^4}\right] = (p-2)\, \mathbb{E}\left[\frac{1}{\|X\|^2}\right]$

$\therefore R(\delta, \cancel{x}\theta) = p - (p-2)\, \mathbb{E}_\theta\left[1/\|X\|^2\right] < p \quad (\forall \theta \in \mathbb{R}^p)$

$\therefore$ δ dominates $\hat{\theta} \Rightarrow \hat{\theta}$ inadmissible.

* δ_1 is not admissible if $\exists\, \delta_2$ s.t. $\forall \theta$, $R(\delta_1, \theta) \geq R(\delta_2, \theta)$, $\exists\, \theta$ s.t. $R(\delta_1, \theta) < R(\delta_2, \theta)$.

δ_1 is admissible if it is not not-admissible.

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(c) Posterior : $\pi(\theta \mid X) \propto e^{-\frac{1}{2c^2}\{\|\theta\|^2} \cdot e^{-\frac{1}{2}\|X-\theta\|^2}$

$$= e^{-\frac{1}{2c^2}\left[\|\theta\|^2 + c^2\|x\|^2 + c^2\|\theta\|^2 - 2c^2\langle x, \theta\rangle\right]}$$

$$\propto e^{-\frac{1}{2c^2}(1+c^2)\left[\|\theta\|^2 - \frac{2c^2}{1+c^2}\langle x, \theta\rangle\right]}$$

$$\propto e^{-\frac{1}{2\,c^2/1+c^2}\|\theta - \frac{c^2}{1+c^2}x\|^2}$$

$$\therefore \sim N\left(\frac{c^2 x}{1+c^2}, \; c^2/1+c^2 \cdot I\right)$$

Posterior optimiser (quadratic loss) : Mean $= c^2X/1+c^2$

$$R\left(\frac{c^2x}{1+c^2}, \theta\right) = \mathbb{E}\left[\frac{1}{(1+c^2)^2}|c^2x - c^2\theta - \theta|^2\right]$$

$$= \frac{1}{(1+c^2)^2}\mathbb{E}\left[c^4\|X-\theta\|^2 + \|\theta\|^2\right]$$

$$= \frac{\|\theta\|^2 + c^4 p}{(1+c^2)^2}$$

$$\therefore R\left(\frac{c^2x}{1+c^2}\right)_\pi = \frac{1}{(1+c^2)^2}\left[c^4 \cdot p + \mathbb{E}\left[\|\theta\|^2\right]\right] = \frac{pc^2}{1+c^2} \text{ ※}$$

(d) ~~Suppose $\sup_{\theta\in\mathbb{R}^p}\{R(\delta_1, \theta)\} < \sup_{\theta\in\mathbb{R}^p}\{R(\delta\hat{\theta}, \theta)\} = P$~~

~~$\mathbb{E}_\theta[\|\delta_1 - \theta\|^2] = \cancel{\mathbb{E}_\theta[Var(\delta_1) + \mathbb{E}[\delta_1 \theta \mathbb{E}[\delta_1]-\theta\|^2]}$~~

~~$= \sum_{k=1}^{p}\mathbb{E}_\theta[|(\delta_1)_k - \theta_k|^2] = \sum_{k=1}^{p} Var(\delta_{1,k}) + (\mathbb{E}[\delta_1 - \theta])^2$~~

~~Coordinate wise : $Var(\delta_{1,k}) \geq [(\theta + B_k(\theta))']^2 / I(\theta) = \frac{(1+B_k'(\theta))^2}{1}$~~

~~$(B_k(\theta) = \mathbb{E}[\delta_{1,k} - \theta])$~~

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~~Suppose $\hat{\Theta}$ is not minimax:~~ Let δ be any estimator:

~~But~~ $R(\delta)_{\pi_n} \geq R(\delta_{\pi_n})_{\pi_n}$, $\pi_n \sim N(0, n^2 I_p)$

$$\therefore \sup_{\theta \in \mathbb{R}^p} \left\{ R(\delta, \theta) \right\} \geq \limsup_{n \to \infty} R(\delta)_{\pi_n} \geq \lim_{n \to \infty} R(\delta_{\pi_n})_{\pi_n}$$

$$= p$$

$\therefore$ $\hat{\Theta}$ is minimax.

4.) (a)

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Principle of Statistics 2019

1.) (a) $Y \sim N_n(X \cdot \theta_0, \sigma^2 I_n)$

$\therefore f(Y) = \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\frac{1}{2\sigma^2} \sum_{k=1}^{n} |Y_k - (X \cdot \theta_0)_k|^2}$

$\therefore$ Maximised by minimising $\sum_{k=1}^{n} |Y_k - (X \cdot \theta_0)_k|^2$:

Diff. wrt θ_j : $2 \sum_{k=1}^{n} (Y_k - (X \cdot \theta)_k) X_{kj} = 0$

$\therefore$ ~~$Y \ldots (X \cdot \theta)$~~ ~~$Y^T X$~~ =

$\therefore X^T \cdot Y = (X^T \cdot X) \cdot \theta$

X rank $p \Rightarrow (X^T \cdot X)$ inv. $\Rightarrow \hat{\theta}_{MLE} = (X^T X)^{-1} \cdot X^T Y$

(unique MLE)

If $p > n$: ~~Rk~~ If $r(X) \geq n$, $\exists\ \infty$ exact solutions that give 0 square error.

$\therefore$ Results could be over-fitted.

(b) $\sum_{k=1}^{n} (Y_k - (X \cdot \theta)_k)^2 + \lambda \sum_{k=1}^{p} |\theta_k|^2$

For $j = 1, \ldots p$: Diff. wrt θ_j

$2 \sum_{k=1}^{n} (Y_k - (X \cdot \theta)_k)(-X_{kj}) + 2\lambda \sum_{k=1}^{p} \theta_j \delta_{jk} = 0$

$\therefore \lambda \theta_j = \sum_{k=1}^{n} (X^T_{jk} \cdot Y_k) - X^T_{jk} \cdot (\cancel{Y_k}\ (X \cdot \theta)_k)$

$\therefore \lambda \underset{\sim}{\theta} = X^T \cdot Y - (X^T X) \cdot \theta \Rightarrow (\lambda + X^T X) \cdot \theta = X^T \cdot Y$

$\forall t \in \mathbb{R}^p$: $t^T \cdot (\lambda + X^T X) t = \lambda \|t\|^2 + \|X \cdot t\|^2 \geq 0$ ($= 0$ iff $t = 0$)

$\therefore \lambda + X^T X$ is invertible $\Rightarrow \theta = (\lambda + X^T X)^{-1} \cdot (X^T Y)$

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$\therefore \exists$ unique maximiser.

(c) $X^T \cdot X = V \Lambda V^T \Rightarrow (X \cdot V)^T \cdot (X \cdot V) = \Lambda$

* Question is weird. X is given $\Rightarrow$ Deterministic?

$Cov(U_i, U_j) = \frac{1}{n} U_i^T \cdot U_j = \frac{1}{n} (XV)^T_{(i^{th} \text{ row})} \cdot (XV)_{(j^{th} \text{ row})}$

$\therefore Cov = \frac{1}{n} (V^T \underbrace{X^T \cdot X}_{V^T \Lambda V^T} \cdot V)_{ij}$

$= \frac{1}{n} \left((V^T V) \Lambda (V^T \cdot V) \right)_{ij} = \frac{1}{n} \Lambda_{ij}$

(d) $\hat{Y}_{MLE} = X \cdot \underbrace{(X^T \cdot X)^{-1}}_{V \Lambda^{-1} V^T} \cdot X^T \cdot Y = (X \cdot V) \Lambda^{-1} (X \cdot V)^T \cdot Y$

$= U \Lambda^{-1} U^T \cdot Y$

$\hat{\Theta}_{MLE}$ minimises $\| Y - X(*) \|^2 \Rightarrow X \cdot \hat{\Theta}_{MLE}$ is the closest point to Y, that is in subspace generated by span on X columns

Since V is orthogonal $X \cdot V$ columns have the same span as X.

(e) $\hat{Y}_\lambda = X \cdot \hat{\Theta}_\lambda = X \cdot (\lambda I + X^T X)^{-1} \cdot X^T Y = X \cdot (\lambda I + V \Lambda V^T)^{-1} X^T Y$

$= (X \cdot V) \; (\lambda I + \Lambda)^{-1} (X \cdot V)^T \cdot Y$

$= U \cdot (\lambda I + \Lambda)^{-1} U^T \cdot Y$

Write $U = (U_1 \;\; U_2)$, $\Lambda = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$:

Projection to U_i: $U_i (D_i^{-1}) U_i^T \cdot Y$

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$$\hat{Y}_\lambda = \underbrace{U_1 (\lambda I + \Lambda_1)^{-1} U_1^T \cdot Y}_{\cong \ U_1 \Lambda_1^{-1} U_1^T \cdot Y} + \underbrace{U_2 (\lambda \cdot I + \Lambda_1)^{-1} U_2^T \cdot Y}_{\geq \frac{\lambda + \Lambda_{q+1}}{\Lambda_{q+1}} \text{ times smaller } \cong 0}$$

$\therefore$ Projection to 1^{st} p coordinates

2) (a)

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4.) (a) $\hat{\Theta}$ is a MLE if $\hat{\Theta}$ maximises: $\Theta \mapsto f(X, \theta)$

$$I(\theta) = \mathbb{E}_{\theta}\left[\nabla \log(f(X,\theta)) \cdot \nabla \log(f(X,\theta))^T \right]$$

(b) Strong law of large #: $\hat{\mu}_n \to \mathbb{E}[h(X)] = \mu(\theta)$ A.S.

Let $\hat{\Theta}_n = \mu^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(X_k)\right)$: μ is 1 to 1 $\Rightarrow$ invertible

Claim: μ^{-1} is cont.

~~Suppose $\exists y_n \in \mathbb{R}$ s.t. $y_n \to y$, $\mu^{-1}(y_n) \nrightarrow y$~~

~~But $y_n = \mu(x_n) \Rightarrow \mu(x_n) \to \mu(x)$, $x_n \nrightarrow x$.~~

~~$\therefore \exists \delta > 0$ s.t. $x_n \notin D(x, \delta) \Rightarrow \mu\, y_n \notin \mu(D(x,\delta))$~~

$\forall \theta \in \mathbb{R}$: $\mu\big|_{\bar{D}(\theta,1)} : \bar{D}(\theta,1) \to \mu(\bar{D}(\theta,1))$ is bijective cont. map

from compact set to hausdorff set ; $\Rightarrow \mu\big|^{-1}_{\bar{D}(\theta,1)}$ is cont.

$\therefore \mu^{-1}$ cont. at $\mu(\theta)$, $\theta \in \mathbb{R} \Rightarrow \mu^{-1}$ cont.

Cont. map theorem: $\mu^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \mu^{-1}(h(X_k))\right) \to \underbrace{\mu^{-1}(\mu(\theta))}_{\theta}$ A.S

$\therefore \hat{\Theta}_n$ is consistent

(Delta):

(ii) $\sqrt{n}\left(\mu^{-1}(\hat{\mu}_n) - \mu^{-1}(\mu(\theta_0))\right) \xrightarrow{(d)} (\mu^{-1})'(\mu(\theta_0)) \cdot \lim_{n\to\infty} (\sqrt{n}(\hat{\mu}_n - \mu(\theta_0)))$

Note: μ cont. diff., strictly monotone $\Rightarrow \mu^{-1}$ is cont. diff when $\mu' \neq 0$.

(CLT):

~~Case 1: 2 $\theta_0 \neq$~~ $(\mu^{-1})'$, $\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} h(X_k) - \mathbb{E}[h(X)]\right) \to N(0, \mathrm{Var}(h(X_k)))$

If $\mu'(\theta_0) \neq 0$: $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{(d)} \frac{1}{\mu'(\theta_0)} \cdot N(0, \underbrace{\mathrm{Var}(h(X))}_{\mathbb{E}[h(X)^2] - \mu(\theta_0)^2})$

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$\mu'(\theta_0) = 0$: ???

$\hat{\theta}_n$ may not be unbiased $\Rightarrow$ Not necessary for $Var(\hat{\theta}_n) \geq (nI(\theta_0))^{-1}$

(iii) $T_{(-i)} = \hat{\theta}_{n-1}(\underbrace{X_1, \dots X_n}_{\text{without } i^{th} \text{ sample.}})$

$\hat{B}_n = (n-1)\left(\frac{1}{n}\sum_{k=1}^{n} T_{(-i)} - \hat{\theta}_n\right)$: $\tilde{T}_{Jack} = \hat{\theta}_n - \hat{B}_n$

has smaller bias.

(iv) CLT: $\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} h(X_k) - \mathbb{E}[h(X)]\right) \to N(0, Var(h(X)))$

Asymptotic normality: $\sqrt{n}(\underbrace{\hat{\theta}_{n_{MLE}} - \theta_0}) \xrightarrow{(d)} N(0, (nI(\theta_0))^{-1})$

$\hat{\theta}_{MLE} = \frac{1}{n}\sum_{k=1}^{n} h(X_k) \Rightarrow$ Converges A.S. to some limit

$\Rightarrow \theta_0 = \mathbb{E}[h(X)]$

$\therefore (nI(\theta_0))^{-1} = Var(h(X))$

Estimator is asymptotically efficient as variance of estimator $\to$ lower bound of given by cramer rao as $n \to \infty$.

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3.) (a) $f(x,\theta) = \theta e^{-\theta x}$ is maximised by:

Max $\log f = \log(\theta) - \theta x$

$1/\theta = x \Rightarrow$ Pick $\hat{\theta} = 1/x$

$f(x_1, \dots x_n; \theta) = \theta^n e^{-\theta(x_1 + \dots + x_n)}$.

$\therefore n \log \theta - \theta(x_1 + \dots + x_n)$: Diff wrt θ : $n/\theta - (x_1 + \dots + x_n)$

$$\therefore \hat{\theta}_n = 1 / \left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)$$

$\mathbb{E}[|X_k|], \mathbb{E}[|X_k|^2] < \infty$: CLT

Central limit theorem: $\frac{\sqrt{n}}{n}\sum_{k=1}^{n}(X_k - 1/\theta) \to N(0, Var(X_1))$

$= N(0, 1/\theta^2)$

Delta: $1/\theta$ smooth on $(0,\infty)$

$$\therefore \sqrt{n}\left(\frac{1}{\frac{1}{n}\sum_{k=1}^{n} X_k} - \frac{1}{\frac{1}{\theta_0}}\right) \to \left(-\frac{1}{\theta^2}\right)\Big|_{\theta = \frac{1}{\theta_0}} \cdot N(0, 1/\theta_0^2)$$

$$= (-\theta_0^2) \cdot N(0, 1/\theta_0^2)$$

$$= N(0, \theta_0^2)$$

$\therefore \sqrt{n}(\hat{\theta}_n - \theta_0) \to N(0, \theta_0^2)$

(b) $\pi(\theta | x_1, \dots x_n) \propto \theta^{\alpha-1} e^{-\beta\theta} \cdot \theta^n e^{-\theta\sum_{k=1}^{n} X_k}$

$$\sim \Gamma\left(\alpha + n, \beta + \sum_{k=1}^{n} X_k\right)$$

(Square loss): Posterior estimator = Mean of π

$$\therefore \int_0^\infty \frac{(\beta + n\bar{X}_n)^{\alpha+n}}{\Gamma(\alpha+n)} e^{-(\beta+n\bar{X}_n)\cdot\theta}\, \theta^{\alpha+n-1} \cdot \theta \, d\theta$$

$$= \underbrace{\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)}}_{\frac{1}{(\beta+n\bar{X}_n)}} \int_0^\infty f_{\Gamma(\alpha+n+1, \beta+n\bar{X}_n)}(\theta)\, d\theta = \frac{\alpha+n}{\beta + \sum_{k=1}^{n} X_k}$$

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Since posterior optimal is a Bayes rule:

$$\underset{\sim}{X} \mapsto \frac{\alpha+n}{\beta+\sum_{k=1}^{n} X_k} \quad \text{is a bayes rule.}$$

(a) $$\frac{1}{\hat{\delta}_n} = \frac{\beta+\sum_{k=1}^{n} X_k}{\alpha+n} = \underbrace{\left(\frac{\beta}{\alpha+n}\right)}_{\to 0} + \underbrace{\left(\frac{n}{\alpha+n}\right)}_{\to 1} \cdot \underbrace{\frac{1}{n}\sum_{k=1}^{n} X_k}_{\to 1/\theta \text{ A.S., } \mathbb{P}}$$

$$\to 1/\theta_0 \text{ in } \mathbb{P}$$

$1/x$ cont. on $(0,\infty)$: $\hat{\delta}_n \to \theta_0$ in $\mathbb{P}$ $\Rightarrow$ Consistent.

$$\frac{\alpha+n}{\beta+n\bar{X}_n} - \frac{1}{\bar{X}_n} = \frac{\alpha\bar{X}_n - \beta}{\bar{X}_n(\beta+n\bar{X}_n)}$$

$$= \frac{\alpha}{\beta+n\bar{X}_n} - \frac{\beta}{\bar{X}_n(\beta+n\bar{X}_n)} \quad - (1)$$

Law of large # (Strong): $\bar{X}_n \to 1/\theta$, $\sqrt{n}\,\bar{X}_n \to \infty$ A.S.

$$\sqrt{n}\,|1| \le \frac{\alpha}{\sqrt{n}\,\bar{X}_n} + \frac{\beta}{\bar{X}_n \cdot \sqrt{n}\,\bar{X}_n} \to 0 \quad \text{A.S.}$$

$$\therefore \sqrt{n}(\hat{\delta}_n - \theta) = \sqrt{n}(\hat{\theta}_n - \theta) + \underbrace{\sqrt{n}(\hat{\delta}_n - \hat{\delta}_n)}_{\to 0 \text{ in } \mathbb{P}}$$

$$\to N(0, \theta^2) \text{ in } \mathcal{L} \text{ dist.}$$

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2.) (a) $\mathbb{E}[s^X] = \sum_{k\geq 0} e^{-\theta}\frac{\theta^k}{k!}s^k = e^{\theta(s-1)} = f(s)$

$$\left.\begin{aligned} f'(s) &= \theta e^{\theta(s-1)} \\ f''(s) &= \theta^2 e^{\theta(s-1)} \end{aligned}\right] \quad \begin{aligned} f'(1) &= \theta = \mathbb{E}[X] \\ f''(1) &= \theta^2 = \mathbb{E}[X^2 - X] \end{aligned}$$

$\therefore \mathrm{Var}(X) = \mathbb{E}[X^2 - X] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \theta^2 + \theta - \theta^2 = \theta$

$\therefore \mathbb{E}[X] = \mathrm{Var}(X) = \theta$

(b) Max. $\sum_{k=1}^{n} \log(e^{-\theta}\theta^{X_1}/X_1!) \equiv \text{Max} -n\theta + (X_1 + \dots + X_n)\log(\theta)$

$\therefore +n = (X_1 + \dots X_n)/\theta \Rightarrow \hat{\theta}_n = \frac{1}{n} X_1 + \dots + X_n$

Central limit theorem: $\mathbb{E}[|X_1|] = \mathbb{E}[X_1^2] = \theta < \infty$.

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}(X_k - \theta)\right) \xrightarrow{(d)} N(0, \mathrm{Var}(X_1)) = N(0, \theta)$$

(c) $(1-\pi) + \pi e^{-\lambda} + \sum_{k\geq 1} \pi e^{-\lambda}\frac{\lambda^k}{k!} = (1-\pi) + \pi \underbrace{\sum_{k\geq 0} e^{-\lambda}\lambda^k/k!}_{=1} = 1$

$$\left.\begin{aligned} &\text{Since } (1-\pi) + \pi e^{-\lambda} \geq 1 - \pi \geq 0, \\ &\pi e^{-\lambda}\lambda^k/k! \geq 0 \end{aligned}\right] \text{ Valid distribution}$$

Let $Z \sim \mathrm{Ber}(\pi)_{(0,1)}$, $\therefore$ π is equivalent to $Z \cdot X$, $X \sim \mathrm{Poi}(\lambda)$, Z, X independent

$\mathbb{E}[Z \cdot X] = \mathbb{E}[Z] \cdot \mathbb{E}[X] = \pi \cdot \lambda$

$\mathbb{E}[Z^2 X^2] = \mathbb{E}[Z^2] \cdot \mathbb{E}[X^2] = \pi \cdot (\lambda + \lambda^2)$

$\therefore \mathrm{Var}(Z \cdot X) = \pi(\lambda + \lambda^2) - \pi^2\lambda^2 = \lambda\pi + \pi\lambda^2\underbrace{(1-\pi)}_{\geq 0}$

$\geq \lambda \cdot \pi = \cancel{\pi\theta} = \mathbb{E}[Z \cdot X]$

$\therefore$ Models dispersion for $\pi \neq 1$.

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(c) Max: $\sum_{k=1}^{n} \log\left(\pi e^{-\lambda} \frac{\lambda^{Y_k}}{Y_k!} + (1-\pi) 1_{Y_k=0}\right)$

$= a \log(\pi) + (n-a) \log((1-\pi) + \pi e^{-\lambda})$, $a = |\{k=1,\dots n \mid Y_k \neq 0\}|$

Diff. wrt π: $a/\pi + \frac{(n-a)(e^{-\lambda}-1)}{(1-\pi)+\pi e^{-\lambda}} = 0$

$\therefore (1-\pi)\cdot a + a\pi e^{-\lambda} = (n-a)\pi(1-e^{-\lambda})$

$a - a\pi(1-e^{-\lambda}) = (n-a)\pi(1-e^{-\lambda})$

$\therefore a = n\pi(1-e^{-\lambda}) \Rightarrow \pi = \frac{a}{n(1-e^{-\lambda})}$

But $\pi \in [0,1]$: $e^{-\lambda} \in (0,1) \Rightarrow \frac{1}{1-e^{-\lambda}} \in (1,\infty)$

$\therefore \pi = \mathrm{Min}\left(\frac{a}{n(1-e^{-\lambda})}, 1\right)$ (Derivative $\geq 0 \Rightarrow$ Pick largest point possible)

(d) No. $\Theta = [0,1]$ is not open. $\Rightarrow$ If $\pi_0 = 1$, asymptotic regularity may fail. (limiting normal dist. could be on either side of 1, but will be forced to 1)

A $(A_n)_{n\geq 1}$ iid, $A_n \sim \mathrm{Ber}(\pi)$

$\pi_N = (1-e^{-\lambda})^{-1} \mathrm{Min}\left(1-e^{-\lambda}, \frac{1}{n}\sum_{n=1}^{N} A_n\right)$

We know: $\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} A_n - \pi\right) \xrightarrow{d} N(0, \pi(1-p))$

(Central limit theorem)

$\hat{\pi}_{MLE} - \pi = 1 + 1\left(\frac{\sum_{k=1}^{n} A_k}{n} \leq 1-e^{-\lambda}\right)\left(\frac{a}{n(1-e^{-\lambda})}\right)$

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(a): $\pi = 1 \in \partial[0,1]$.

$\therefore$ Asymptotic ~~norm.~~ convergence ~~w~~ may not be valid

$\mathbb{P}(Y_k = 0) = (1-\pi) + \pi e^{-\lambda} = \alpha$

If $(A_n)_{n\geq 1}$ iid, $A_n \sim \text{Ber}(\alpha)$: $S_n = \frac{1}{n}\sum_{k=1}^{n} A_k$

$\therefore$ $\hat{\pi}_{MLE}$ have the same distribution as $\text{Min}\left(1, \frac{S_n}{(1-e^{-\lambda})}\right)$

$\mathbb{E}[A_k] = {}_{\wedge}^{1-}(1 - \pi(1-e^{-\lambda})) = \pi(1-e^{-\lambda})$

$\therefore$ $\mathbb{E}\left[\frac{A_k}{1-e^{-\lambda}}\right] = \pi$.

$\therefore$ CLT: $\sqrt{n}\left(\frac{S_n}{1-e^{-\lambda}} - \underbrace{1}_{H_0}\right) \xrightarrow{(d)}$ ~~Var~~ $N(0, Var(A_1))$

$= N(0, \alpha(1-\alpha))$

$\therefore$ $\sqrt{n}\left(\text{Min}\left(\frac{S_n}{1-e^{-\lambda}}, 1\right) - 1\right) = \text{Min}\left(\sqrt{n}\left(\frac{S_n}{1-e^{-\lambda}} - 1\right), 0\right)$ - (1)

Since $\text{Min}(\cdot)$ is cont. $\therefore$

~~CMT~~: Cont. map theorem: (1) $\to$ $\text{Min}(N(0,$ ~~Var(~~ $\alpha(1-\alpha)), 0)$

$\therefore$ Not normal (+ve part truncated

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1.) (a) Let $X \sim f(\cdot, \theta)$: $\tilde{\theta}$ be an unbiased estimator of θ

$$\mathrm{Var}(\tilde{\theta}) \geq I(\theta)^{-1}, \qquad I(\theta) = \mathbb{E}[\nabla \log(f(X,\theta)) \cdot \nabla \log(f(X,\theta))]$$

~~Let~~ Let $Z = \frac{d}{d\theta}\log(f(X,\theta))$:

$$\mathbb{E}[Z] = \frac{d}{d\theta}\int_{\mathcal{X}} \frac{\frac{d}{d\theta}f(X,\theta)}{f(X,\theta)} f(X,\theta)\,dx = \frac{d}{d\theta}\int_{\mathcal{X}} f(x,\theta)\,dx$$

$$= 0$$

$$\therefore \mathrm{Var}(Z) = \mathbb{E}[Z^2] = I(\theta)$$

Cauchy Schwarz: $\{\mathrm{Var}(\tilde{\theta}) \cdot \mathrm{Var}(Z) \geq \mathrm{Cov}(\tilde{\theta}, Z)^2$

$$\mathrm{Cov}(\tilde{\theta} Z) = \mathbb{E}[\tilde{\theta} Z] - \mathbb{E}[\tilde{\theta}]\cdot\mathbb{E}[Z] = \int_{\mathcal{X}} \frac{f(x,\theta)\ \tilde{\theta}(X),\ \frac{d}{d\theta}f(x,\theta)}{f(x,\theta)}\,d\theta$$

$$= \frac{d}{d\theta}\int_{\mathcal{X}} f(x,\theta)\ \tilde{\theta}(X)\,dx = \frac{d}{d\theta}(\theta) = 1$$

$$\therefore \mathrm{Var}(\tilde{\theta})\cdot\mathrm{Var}(Z) \geq 1 \Rightarrow \mathrm{Var}(\tilde{\theta}) \geq {}^{1}/I(\theta)$$

* Waffle: Equality case

$$\mathrm{Var}(X)\cdot\mathrm{Var}(Y) = \mathrm{Cov}(X,Y)^2 \quad \text{iff} \quad Y = c\cdot X \quad (c \text{ constant})$$

$$\therefore \frac{{}^{d}/_{d\theta}\ \log(f(X,\theta))}{f(X,\theta)} = \underbrace{c\cdot\tilde{\theta}(X)}_{\text{constant in } \theta}$$

$$\therefore f(X,\theta) = A(x)\cdot e^{c\tilde{\theta}(X)\cdot\theta}, \quad c \text{ is a constant}$$

$\therefore$ Exponential family.

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(b) δ is minimax (wrt loss function L) if $\forall\ \delta'$:

$$\sup_{\theta \in \Theta} \{ R(\delta, \theta) \} \le \sup_{\theta \in \Theta} \{ R(\delta', \theta) \}$$

(c) $\forall \theta \in [0, \infty)$: $R(\bar{X}_n, \theta) = \mathbb{E}_\theta[(\bar{X}_n - \theta)^2] = Var(\bar{X}_n) = 1/n$

Let δ be any estimator: $B(\theta) = \mathbb{E}_\theta[\delta - \theta]$

$$\therefore \mathbb{E}_\theta[(\delta-\theta)^2] = Var(\delta) + B(\theta)^2 \geq B(\theta)^2 + \frac{[(\theta + B(\theta))']^2}{I(\theta)}$$

$$= B(\theta)^2 + \frac{(1+B'(\theta))^2}{\cancel{I}\, n}$$

If $B'(\theta) > 0$: $(1+B')^2/n > 1/n = R(\bar{X}_n, \theta) \Rightarrow \sup_{\theta} \{R(\delta,\theta)\} > 1/n = \sup_{\theta} \{R(\bar{X}_n, \theta)\}$.

~~WLOG~~ $B'(\theta) \le 0$: $\Rightarrow \lim_{\theta\to\infty} B'(\theta)$ exists (monotone convergence)

If $\lim_{\theta\to\infty} B'(\theta) \neq 0$: $|B(\theta)| \to \infty$ as $\theta \to \infty$ ~~case~~

$$\therefore \sup_{\theta} \{ R(\delta,\theta) \} > \sup_{\theta} \{ R(\bar{X}_n, \theta) \}$$

If $\lim_{\theta\to\infty} B'(\theta) = 0$: $\lim_{\theta\to\infty} \mathbb{E}_\theta[(\delta-\theta)^2] \geq \lim_{\theta\to\infty} \frac{(1+B'(\theta))^2}{n} = 1/n$

$$\therefore \sup_{\theta} \{ R(\bar{X}_n, \theta) \} = 1/n \le \sup_{\theta} \{ R(\delta, \theta) \}$$

$\therefore \bar{X}_n$ is minimax.

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3.) (a) We will show detailed balance: Let $\theta_1, \theta_2 \in \Theta$,

(WLOG) $\dfrac{\mu(\theta_1)\ q(\theta_2 \mid \theta_1)}{\mu(\theta_2)\ q(\theta_1 \mid \theta_2)} \leq 1$

(WTP) (1): $\mu(\theta_1) \cdot q(\theta_2 \mid \theta_1)\ P(\theta_1, \theta_2) = \mu(\theta_2)\ q(\theta_1 \mid \theta_2)\ P(\theta_2, \theta_1)$

$P(\theta_1, \theta_2)$: $\text{Min}(1, \mu(\theta_2)\ q(\theta_1 \mid \theta_2) / \mu(\theta_1)\ q(\theta_2 \mid \theta_1)) = 1$

$P(\theta_2, \theta_1)$: $\mu(\theta_1)\ q(\theta_2 \mid \theta_1) / \mu(\theta_2) \cdot q(\theta_1 \mid \theta_2)$

(1) is valid.

$\forall$ measurable B: $\displaystyle\int_{\Theta} \mu(t)\ \mathbb{P}(V_{m+1} \in B \mid V_m = t)\ dt$

$$= \int_B \int_\Theta \mu(t)\ q(x \mid t)\ P(t, x)\ dt\ dx$$

$$= \int_B \int_\Theta \mu(x)\ \underbrace{q(t \mid x)\ P(x, t)}_{\mathbb{P}(V_{m+1} = t \mid V_m = x)}\ dt\ dx \Rightarrow \int \text{ over } t \in \Theta = 1$$

$$= \int_B \mu(x)\ dx$$

(b) $\mu(\theta) \propto \pi(\theta)\ f(x, \theta)$

$$P(t, s) = \text{Min}\left\{1, \frac{\pi(s)\ f(x, s)\ q(t \mid s)}{\pi(t)\ f(x, t)\ q(s \mid t)}\right\}$$

$\therefore$ We want: $\underbrace{\dfrac{\pi(s)}{\pi(t)}}_{e^{-\frac{1}{2}(\|s\|^2 - \|t\|^2)}}\ \dfrac{q(t \mid s)}{q(s \mid t)}$ to be constant

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Pick : $q(t|s) = C \cdot e^{-\frac{1}{2\delta_1}(t-as)^2}$

$\therefore \pi(s) \cdot q(t|s) = C_1 \cdot e^{-\frac{1}{2\delta}[\delta_1|*s|^2 + |t|^2 + a^2|s|^2 - 2a\langle s,t\rangle]}$

Pick $a = \sqrt{1-\delta_1}$: $\frac{\pi(s) \cdot q(t|s)}{\pi(t) \cdot q(s|t)} = 1$ eindep of t, s)

$q(\cdot|s) \sim N(as, \delta_1 I)$

$\therefore$ Pick $\delta_1 = 2\delta$:

$q|_s \sim N(\sqrt{1-2\delta}\, s, 2\delta \cdot I)$.

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4) (a) $Z(h) = \log\left(e^{-\frac{1}{2}(X-h)^T \cdot I(\theta_0)^{*} \cdot (X-h)} \Big/ e^{-\frac{1}{2} X^T \cdot I(\theta_0)^{*} \cdot X} \right)$

$= \underset{\sim}{h}^T \cdot I(\theta_0)^{*} \cdot \underset{\sim}{X} - \frac{1}{2} \underset{\sim}{h}^T \cdot I(\theta_0)^{*} \cdot \underset{\sim}{h}$

~~$Z_n(h)$~~ $q(X, \theta) = \log(f(X, \theta))$:

$Z_n(h) = \sum_{k=1}^{n} q(X_k, \theta + h/\sqrt{n}) - q(X_k, \theta^*)$ (Taylor series)

$= \frac{1}{\sqrt{n}} \underset{\sim}{h} \cdot \left(\sum_{k=1}^{n} \nabla q(X_k, \theta) \right) \cdot \underset{\sim}{h} + \frac{1}{2n} \underset{\sim}{h}^T \left(\sum_{k=1}^{n} \nabla^2 q(X_k, \theta) \right) \cdot \underset{\sim}{h}$

$+ \frac{1}{6 n^{3/2}} \left(\sum_{k=1}^{n} \nabla^3 q(X_k, \tilde{\theta}_n) \right) \cdot (h, h, h)$

where $\tilde{\theta}_n \in [\theta, \theta + h/\sqrt{n}]$ segment.

Term 1: $\underset{\sim}{h} \cdot \left[\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^{n} \nabla q(X_k, \theta_0) - \underbrace{\mathbb{E}[\nabla q(X_k, \theta_0)]}_{=0} \right) \right]$

$\xrightarrow{(d)} \underset{\sim}{h} \cdot N(0, \underbrace{Var(\nabla q(X, \theta_0))}_{I(\theta_0)}) = \underset{\sim}{h}^T I(\theta_0) \cdot N(0, I(\theta_0)^{-1})$

Term 2: $\frac{1}{n} \sum_{k=1}^{n} \nabla^2 q(X_k, \theta) \xrightarrow{\mathbb{P}} \mathbb{E}[\nabla^2 q(X_k, \theta)] = -I(\theta_0)$ (Law of large #)

$\therefore \frac{1}{2n} h^T \left(\sum_{k=1}^{n} \nabla^2 q(X_k, \theta) \right) \cdot h \to \frac{1}{2} h^T \cdot I(\theta_0) \cdot h$ in $\mathbb{P}$

Term 3: $\frac{1}{6\sqrt{n}} \left[\frac{1}{n} \sum_{k=1}^{n} \nabla^3 q(X_k, \tilde{\theta}_n) - \mathbb{E}[\nabla^3 q(X_k, \tilde{\theta}_n)] \right]$ — (1)

$+ \frac{1}{6\sqrt{n}} \mathbb{E}[\nabla^3 q(X, \tilde{\theta}_n)]$ — (2)

(1): ~~$\left| \frac{1}{n} \sum_{k=1}^{n} (\nabla^3 q(X_k, \tilde{\theta}_n) - \mathbb{E}[\nabla^3 q(X_k, \tilde{\theta}_n)]) \right|$~~ $\leq$

(2): $1/6\sqrt{n} \to 0$, $\theta \to \mathbb{E}[\nabla^3 q(X, \cdot)]$ is cont.

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$|\tilde{\Theta}_n - \Theta_0| \leq |h/\sqrt{n}| \to 0 \Rightarrow \tilde{\Theta}_n \to \Theta_0$ in $\mathbb{P}$

$\therefore \mathbb{E}[\nabla^3 q(X, \tilde{\Theta}_n)] \to \mathbb{E}[\nabla^3 q(X, \Theta_0)]$ in $\mathbb{P}$

$\therefore (2) \to 0$ in $\mathbb{P}$

(1) $\Lambda_n^{(\tilde{\Theta}_n)} = \frac{1}{n} \sum_{k=1}^{n} \left(\nabla^3 q(X_k, \tilde{\Theta}_n) - \mathbb{E}[\nabla^3 q(X_k, \tilde{\Theta}_n)] \right)$

~~$|\Lambda(\tilde{\Theta}_n)| \leq \sup_{|\Theta-\Theta_0|\leq 1}$~~

$\mathbb{P}(|\Lambda(\tilde{\Theta}_n)| \geq \varepsilon) \leq \mathbb{P}(|\tilde{\Theta}_n - \Theta_0| \geq 1) +$

$\mathbb{P}(|\tilde{\Theta}_n - \Theta_0| \leq 1, \sup_{|\Theta-\Theta_0|\leq 1} |\Lambda(\Theta)| \geq \varepsilon)$

$\lim_{n\to\infty} \mathbb{P}(|\tilde{\Theta}_n - \Theta_0| \geq 1) = 0$

$\mathbb{P}(\sup_{|\Theta-\Theta_0|\leq 1} |\Lambda(\Theta)| \geq \varepsilon) \to 0$ (Uniform law of large #)

$\therefore (1) \to 0$ in $\mathbb{P}$

$\therefore \left| \frac{1}{6n^{3/2}} \sum_{k=1}^{n} \nabla^3 q(X_k, \tilde{\Theta}_n) \cdot \langle h, h, h \rangle \right| \to 0$ in $\mathbb{P}$

Slutsky: $Z_n(h) \to \underbrace{h^T \cdot I(\Theta_0) \cdot N(0, I(\Theta_0)^{-1}) + \frac{1}{2} h^T I(\Theta_0) \cdot h}_{\text{Same dist as } Z(h)}$ (in $\mathbb{P}$)

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2.) (a) ~~Constraints~~ $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$, $S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2$

$$\Lambda(\Theta_0, \Theta_1) = 2\log\left(\frac{(2\pi S^2)^{-n/2}}{(2\pi)^{-n/2}} \frac{e^{-\frac{1}{2S^2}(nS^2)}}{e^{-\frac{n}{2}S^2}}\right)$$

$$= -n\log(S^2) + n(S^2-1) = -n\left(\log(S^2) - (S^2-1)\right)$$

$$\log(S^2) = \log(1+(S^2-1)) = S^2 - 1 + \frac{(S^2-1)^2}{2}\left(-\frac{1}{\omega}\right), \quad \omega \in [1, 1+(S^2-1)]$$

If $X_k \sim N(\mu, 1)$: $S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2 + (\bar{X}_n - \mu)^2$

$$\therefore \Lambda = \frac{n}{2}\left((S^2-1)\left(\frac{1}{\omega}\right)(S^2-1)\right)$$

$$\sqrt{n}(S^2-1) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2 - 1\right) + \underbrace{\underbrace{\sqrt{n}(\bar{X}_n-\mu)}_{\xrightarrow{d} N(0,1)}\underbrace{(\bar{X}_n-\mu)}_{\xrightarrow{\mathbb{P}} 0}}_{\text{Slutsky: } \xrightarrow{\mathbb{P}} 0.}$$

$\mathbb{E}[(X_k-\mu)^2] = 1$, $\mathrm{Var}((X_k-\mu)^2) = 2$ - (*)

CLT: $\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2 - 1\right) \xrightarrow{(d)} N(0,2)$

Slutsky: $\sqrt{n}(S^2-1) \to N(0,2)$ in dist.

$$\Lambda = \left(\frac{\sqrt{n}}{\sqrt{2}}(S^2-1)\cdot\frac{1}{\sqrt{\omega}}\right)^2$$

$\omega \in [1, S^2]$: $S^2 = \underbrace{\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2}_{\xrightarrow{\mathbb{P}} 1 \text{ (Strong law of large \#)}} + \underbrace{(\bar{X}_n-\mu)^2}_{\xrightarrow{\mathbb{P}} 0 \text{ (S law of large \#, cont. map)}}$

Slutsky: $S^2 \to 1$ in $\mathbb{P}$ $\Rightarrow$ $\omega \to 1$ in $\mathbb{P}$

Cont. map: $x \to 1/\sqrt{x}$ cont. at 1 $\Rightarrow$ $1/\sqrt{\omega} \to 1$ in $\mathbb{P}$

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$\therefore \frac{\sqrt{n}}{\sqrt{2}}(S^2-1)\frac{1}{\sqrt{\omega}} \to N(0,1)$ in dist. (Slutsky)

Cont. map theorem: $\Lambda_n \to N(0,1)^2 = \chi_1^2$ in dist.

(*) $Z \sim N(0,1)$, $\mathbb{E}[Z^4] = \frac{d^4}{ds^4}\mathbb{E}[e^{sZ}]\Big|_{s=0}$

$= \frac{d^4}{ds^4}\left(e^{s^2/2}\right)\Big|_{s=0}$

$= \frac{d}{ds}\left((2s+s+s^3)e^{s^2/2}\right)\Big|_{s=0} = 2.$

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(c) $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$; $\underset{\sim}{X} = \theta \begin{pmatrix} 1 \\ 1-\sqrt{r} \\ \vdots \\ 1-\sqrt{r} \end{pmatrix} + \sqrt{r} \begin{pmatrix} 0 & & & \\ 1 & 0 & & 0 \\ & 1 & 0 & \\ 0 & & \ddots & \ddots \\ & & 1 & 0 \end{pmatrix} \underset{\sim}{X} + \begin{pmatrix} 1 & & \\ & \sqrt{1-r} & \\ & & \ddots \\ & & & \sqrt{1-r} \end{pmatrix} \underset{\sim}{\varepsilon}$

$f_\varepsilon(\underset{\sim}{y}) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{1}{2}\|y\|^2}$

$\therefore \exists$ matrix M s.t. $M \cdot \underset{\sim}{X} - \theta \begin{pmatrix} 1 \\ (1-\sqrt{r})/(\sqrt{1-r}) \\ \vdots \\ \vdots \end{pmatrix} = \varepsilon$

$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left| \partial \varepsilon_i / \partial x_j \right| f_\varepsilon \left(M \cdot \underset{\sim}{x} - \theta \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha \end{pmatrix} \right)$, $\alpha = \frac{1-\sqrt{r}}{\sqrt{1-r}}$

$\therefore \log(f_{\underset{\sim}{X}}(\underset{\sim}{x})) = \underbrace{\log \left| \partial \varepsilon_i / \partial x_j \right| - \frac{n}{2}\log(2\pi)}_{\text{indep. of } \theta} - \frac{1}{2} \| M \cdot x - \theta \underbrace{\begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha \end{pmatrix}}_{\underset{\sim}{V}} \|^2$

$\frac{\partial^2 \log(f_X(x))}{\partial^2 \theta} = -\frac{\partial}{\partial \theta}\left[\sum_{k=1}^{n} \left((M \cdot x)_k - \theta V_k \right)(-V_k) \right]$

$= \sum_{k=1}^{n} V_k (-V_k) = -(1 + (n-1)\alpha^2)$

$\therefore I_n(\theta) = 1 + (n-1)\alpha^2 = n \cdot I_1(\theta) = n$ iff $\alpha^2 = 1$

~~iff $\alpha = 1$~~ ~~as $\alpha > 0$~~ ~~iff $\alpha = 1$~~ ~~$(\alpha = \frac{1-\sqrt{r}}{\sqrt{1-r}} \geq 0)$~~

$\therefore 1 + r - 2\sqrt{r} = 1 - r \Rightarrow r = \sqrt{r} \Rightarrow r^2 = r \Rightarrow r = 0$ or 1

$\therefore r = 0.$

{ Cramer - Rao: $Var_\theta(\hat{\theta}) \geq I_n(\theta)^{-1} = \frac{1}{1 + (n-1)\frac{(1-\sqrt{r})^2}{1-r}}$

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(a) $S_n(\theta) = \nabla_{\hat{\theta}} \log f(x_1, \dots x_n ; \theta)$

$I_n(\theta) = \mathbb{E}[\, S_n(\theta) \cdot S_n(\theta)^T \,]$

If $\hat{}$ $I_n(\theta) = n \cdot I_1(\theta)$: I tensorises

(b) $\mathbb{E}[\nabla S_n(\theta)]_{ij} = \mathbb{E}\left[\frac{\partial}{\partial \theta_i} \left(\frac{\cancel{\partial f / \partial \theta_j} \; \partial \log f / \partial \theta_j}{f} \right) \right]$

$= \mathbb{E}\left[- \frac{\partial \log(f) / \partial \theta_i}{f} \quad \frac{\partial \log(f) / \partial \theta_j}{f} + \frac{\partial^2 \log f / \partial \theta_i \partial \theta_j}{f} \right]$

$= -I_n(\theta) + \frac{\partial^2}{\partial \theta_i \partial \theta_j} \mathbb{E}[\, f / f \,] = -I(\theta)$

$\therefore I(\theta) = \mathbb{E}[\, -\nabla S_n(\theta) \,]$

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2.) (a) Koglomorov - Smirnov Theorem: Let F be continuous,

$$\lim_{n\to\infty} \sqrt{n} \| \hat{F}_n - F \|_\infty \xrightarrow{d} \sup_{t\in[0,1]} |B_t| = \beta$$

Hypothesis test: Calculate $\sqrt{n}\| F_n - F \|_\infty = \Lambda$.

If $\mathbb{P}(\beta \geq \beta) \leq \alpha$, reject H_0. (Test of size α)

(c) $Y_k = \mathbb{1}_{X_k \in \bar{D}(x, \frac{1}{2m})}$: $m^3 = n$.

$$\therefore \mathbb{E}[Y_k] = \int_{x-\frac{1}{2m}}^{x+\frac{1}{2m}} f(z)\, dz$$

$$\therefore \left| \mathbb{E}[\cancel{\quad} Y_k - \tfrac{1}{m} f(x)] \right| \doteq \int_{\bar{D}(x, \frac{1}{2m})} | f(x) - f(z) | \, dz$$

$$\leq \int_{\bar{D}(x, \frac{1}{2m})} |x - z| \underbrace{|f(z')|}_{\leq 1} \, dz$$

$$\leq 2 \int_{\cancel{-\frac{1}{2m}}\ 0}^{\frac{1}{2m}} z \, dz = [z^2]_0^{\frac{1}{2m}} = \frac{1}{4m^2}$$

$$\mathrm{Var}(Y_k) = \underbrace{\mathbb{E}[Y_k^2]}_{= \mathbb{E}[Y_k]} - \mathbb{E}[Y_k]^2$$

Let $\mathbb{E}[Y_k] = p$. $p \in \bar{D}(\frac{1}{m} f(x), \frac{1}{4m^2})$

$$\cancel{\therefore \mathbb{E}[(\hat{f}_n(x) - f(x))^2] = \mathbb{E}\left[\left(m^{1/3} \frac{1}{n} \sum_{k=1}^n m (m \cdot Y_k - f(x))\right)^2\right]}$$

$$= m^2 \cancel{\frac{1}{m}} \cancel{\mathbb{E}} \left[\mathrm{Var}\left(\frac{1}{n} \sum_{k=1}^n \cancel{m}(Y_k - \tfrac{1}{m} f(x))\right) + \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^n (Y_k - \tfrac{1}{m} f(x))\right]^2 \right]$$

$$= m^2 \left[\frac{1}{n} \mathrm{Var}(Y_1 - \tfrac{1}{m} f(x)) + \mathbb{E}[Y_1 - \tfrac{1}{m} f]^2 \right]$$

$$\leq \frac{1}{m}(p - p^2) + m^2\left(\frac{1}{16 m^4}\right) \leq \frac{|p|}{m} + \frac{1}{4m^2} \leq \frac{|f(x)/m| + \frac{1}{4m^2}}{m} + \frac{1}{4m^2}$$

$$\leq \frac{1}{m^2} + \frac{1}{4m^2} + \frac{1}{4m^3}$$

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$n \geq 1 \Rightarrow \quad m \geq 1$

$\therefore \quad \text{Expression} \leq \frac{3/2}{m^2} = \frac{1.5}{n^{2/3}} \leq \frac{2}{n^{2/3}}$

(b)

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3.) (a) $f(X_1, \dots X_n ; \beta) = \prod_{k=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} X_k^{\alpha-1} e^{-\beta X_k}$ is maximised

by maximising $\sum_{k=1}^{n} [\alpha \log(\beta) + (\alpha-1)\log(X_k) - \beta X_k - \log(\Gamma(\alpha))$

$= \alpha n \log(\beta) - \beta(X_1 + \dots + X_n) + C$ (constant in β)

Diff wrt β: $\alpha n / \beta = (X_1 + \dots + X_n)$

$\therefore \hat{\beta} = \alpha / S_n$, $S_n = \frac{1}{n} \sum_{k=1}^{n} X_k$.

Consider: $\sqrt{n}(S_n - \alpha/\beta)$, $\mathbb{E}[X_k^2] < \infty \Rightarrow$

(Central limit theorem): $\sqrt{n}(S_n - \alpha/\beta) \to N(0, Var(X_1))$ in dist.

$= N(0, \alpha/\beta^2)$

$\Phi(z) = 1/z$, $dom(\Phi) = (0, \infty) \Rightarrow \Phi$ is smooth

$\therefore \sqrt{n}(\Phi(S_n) - \Phi(\alpha/\beta)) = \sqrt{n}\, \Phi'(Z_n) \cdot (S_n - \alpha/\beta)$

$Z_n \in [S_n, \alpha/\beta]$: Strong law of large #: $S_n \to \alpha/\beta$ in $\mathbb{P} \Rightarrow$

$Z_n \to \alpha/\beta$ in $\mathbb{P}$

$\therefore$ Cont. map theorem: $\Phi'(Z_n) \to \Phi'(\alpha/\beta)$ ~~a.s.~~ in $\mathbb{P}$.

$\therefore$ Slutsky: $\sqrt{n}(\Phi(S_n) - \Phi(\alpha/\beta)) \xrightarrow{(d)} \Phi'(\alpha/\beta) \cdot N(0, \alpha/\beta^2)$

$\therefore \sqrt{n}(\hat{\beta}_n - \beta) \to \alpha\left(-\frac{1}{\alpha^2/\beta^2}\right) \cdot N(0, \alpha/\beta^2)$

$= N(0, \beta^2/\alpha)$ #.

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(b) Pick a s.t. $\mathbb{P}(|N(0, \beta^2/\alpha)| \leq a) = 1-\alpha$.

$C_n = \{ z > 0 : |\hat{\beta}_n - z| \leq a/\sqrt{n} \}$ is our confidence interval.

$\mathbb{P}(\beta \in C_n) = \mathbb{P}(|\sqrt{n}(\hat{\beta}_n - \beta)| \leq a) \to \mathbb{P}(|N(0, \beta^2/\alpha)| \leq a) = 1-\alpha$.

$\therefore$ Asymptotically valid

(c) From $X_1, \dots X_n$: ~~Sam~~ Let $\mathbb{P}_n$ be the measure on $\{X_1, \dots X_n\}$, $\mathbb{P}_n(\{X_k\}) = 1/n$.

Find $\cancel{R_n \text{ s.t. } \mathbb{P}(|\overline{X_{n,b}} - \overline{X}_n| \sqrt{n} \leq R_n) = 1-\alpha,}$

$\cancel{\overline{X}_{n,b} = \frac{1}{n} \sum_{k=1}^{n} X_{k,b}, \quad (X_{k,b})_{k=1,\dots n} \text{ iid} \sim \mathbb{P}_n}$

$\cancel{C_n = \{}$

Let $\hat{\theta}_{n,b}$ = MLE of sample $\{X_{n,1}, \dots X_{n,n}\}$, where $(X_{n,k})_{k \geq 1} \sim \mathbb{P}_n$, independent — (*)

Pick R_n s.t. $\mathbb{P}(\sqrt{n}\,|\hat{\theta}_{n,b} - \hat{\beta}_n| \leq R_n) = 1-\alpha$

$C_n = \{ \beta > 0 : \sqrt{n}\,|\beta - \hat{\beta}_n| \leq R_n \}$ is asymptotic $1-\alpha$ confidence set.

$\hat{\theta}_{n,b} = \underset{\beta > 0}{\operatorname{argmax}} \prod_{k=1}^{n} f(X_{n,k}, \beta)$.

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4.) (a) Posterior: $\pi(\theta|X) \propto \theta^{a-1} e^{-\beta\theta} \frac{e^{-\theta}\theta^{X}}{X!}$

$\propto \theta^{X+a-1} e^{-(\beta+1)\theta}$

$\therefore \Gamma(a+X, \beta+1)$ distribution

$\therefore$ Quadratic loss optimal: $\mathbb{E}[\Gamma(a+X, \beta+1)] = \frac{a+X}{\beta+1}$

Since posterior optimals are bayes rule: $\delta(X) = \frac{a+X}{\beta+1}$

$\mathbb{E}_\theta\left[\left(\frac{a+X}{\beta+1} - \theta\right)^2\right] = (\beta+1)^{-2}\,\mathbb{E}_\theta\left[(a-\beta\theta + (X-\theta))^2\right]$

$= (\beta+1)^{-2}\left[(a-\beta\theta)^2 + \theta\right]$

(b) $\mathbb{P}(e^{-\theta} \leq x) = \mathbb{P}(-\theta \leq \log x) = \mathbb{P}(\theta \geq -\log x)$

$\therefore f_{e^{-\theta}}(x) = f(-\log x)/x$

$\therefore \pi(\mu) \propto f(-\log(\mu))/\mu \cdot \mu(-\log\mu)^{x}$

$\propto f_{\alpha+x,\beta}(-\log\mu)$

$\int_0^1 f_{\alpha+x,\beta}(\underbrace{-\log\mu}_{z})\, \mu\, d\mu = \int_0^\infty f_{\alpha+x,\beta}(z)\, e^{-2z}\, dz$

$dz/d\mu = -1/\mu$

$= \frac{\beta^{\alpha+x}}{(\beta+2)^{\alpha+x}} \int_0^\infty f_{\alpha+x,\beta+2}(z)\, dz = \left(\frac{\beta}{\beta+2}\right)^{\alpha+x}$

$\therefore$ Bayes rule: $\delta_\mu(x) = \left(\beta/_{\beta+2}\right)^{\alpha+x}$

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$$\mathbb{E}_\theta\left[\left(\left(\frac{\beta}{\beta+2}\right)^{d+x} - e^{-\theta}\right)^2\right] = \mathbb{E}\left[\left(\beta/_{\beta+2}\right)^{2d}\cdot\left(\frac{\beta^2}{(\beta+2)^2}\right)^x\right] -$$

$$2e^{-\theta}\left(\beta/_{\beta+2}\right)^d \mathbb{E}\left[\left(\beta/_{\beta+2}\right)^x\right] + e^{-2\theta}$$

$$= \left(\beta/_{\beta+2}\right)^{2d} e^{\theta\left(\beta^2/(\beta+2)^2 - 1\right)} - 2e^{-\theta}\left(\beta/_{\beta+2}\right)^d e^{\theta\left(\beta/_{\beta+2} - 1\right)} + e^{-2\theta}$$

(c) It has constant risk.

$\forall$ estimator δ' : ~~$\sup_\theta \{\delta' R(\delta', \theta)\} \geq R(\delta')_\pi$ (any prior)~~

~~$= R($ $R($~~

If $\sup_\theta \{R(\delta', \theta)\} < \sup_\theta \{R(\delta, \theta)\} = c$

$\therefore \forall\theta,\ R(\delta', \theta) \leq \sup_\theta \{R(\delta', \theta)\} < c \Rightarrow$ Contradict admissible

(d) (a) Quadratic with $\beta\theta^2$ coefficient $\Rightarrow (\beta \neq 0)$ Not constant

Write $S = (\beta/_{\beta+2})$:

Risk: $S^{2d} e^{\theta(S^2-1)} - 2S^d e^{\theta(S-2)} + e^{-2\theta}$

$S < 1$: $\lim \theta \to \infty$: Risk $= 0$.

$\theta = 0$: $(S^d - 1)^2 = (1 - S^d)^2 > 0$

$\therefore$ Not constant.

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Principle of Statistics 2022

1.) (a) $\Phi(\hat{\theta}_n) - \Phi(\theta_0) = (\hat{\theta}_n - \theta_0) \cdot \nabla\Phi(\tilde{\theta}_n)$, $\tilde{\theta}_n \in [\theta_0, \hat{\theta}_n]$ segment.

Since $\nabla\Phi$ is cont., $|\tilde{\theta}_n - \theta_0| \le |\hat{\theta}_n - \theta_0|$ $\cancel{\Rightarrow}$:

$\sqrt{n}(\hat{\theta}_n - \theta_0) \to Z \Rightarrow \mathbb{P}(\hat{\theta}_n - \theta_0 \le t/\sqrt{n}) \to \mathbb{P}(Z \le t) \Rightarrow$

$\mathbb{P}(\hat{\theta}_n - \theta_0 \le t) \to \mathbb{P}(Z \le t \cdot \sqrt{n}) \to 0$ if $t < 0$

$\to 1$ if $t > 0$.

$\therefore \hat{\theta}_n - \theta_0 \to 0$ in dist. $\Rightarrow \to 0$ in $\mathbb{P}$ (Slutsky)

$\therefore |\tilde{\theta}_n - \theta_0| \le |\hat{\theta}_n - \theta_0| \Rightarrow \tilde{\theta}_n \to \theta_0$ in $\mathbb{P}$.

Cont. map theorem: $\nabla\Phi(\tilde{\theta}_n) \to \nabla\Phi(\theta_0)$ in $\mathbb{P}$

CLT: $\sqrt{n}(\hat{\theta}_n - \theta_0) \to Z$ in dist.

$\therefore$ Slutsky: $\sqrt{n}(\cancel{\theta_n}\ \Phi(\hat{\theta}_n) - \Phi(\theta_0)) \to \nabla\Phi(\theta_0) \cdot Z$ in dist.

(b) $f(X_1, \dots X_n; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n} X_k^2}$, $S^2 = \frac{1}{n}\sum_{k=1}^{n} X_k^2$:

Max: $-\frac{n}{2}\log(\sigma^2) - \frac{n}{2\sigma^2} \cancel{\sum} S^2$

Diff. wrt σ^2: $-\frac{n}{2\sigma^2} + \frac{n}{2\sigma^4} S^2 = 0 \Rightarrow$ Pick $\hat{\sigma}^2 = S^2$ (MLE)

$\mathbb{E}[X_k^2] = \sigma^2$, $\mathbb{E}[X_k^4] = \sigma^4\,\mathbb{E}[\underbrace{(X_k/\sigma)}_{\sim Z}{}^4] = 3\sigma^4$

$\therefore \mathrm{Var}(X_k^2) = 3\sigma^4 - \sigma^4 = 2\sigma^4 < \infty$

$\therefore$ Central limit theorem: $\sqrt{n}(\cancel{\hat{\theta}_n - \theta_0})\ \hat{\sigma}_n^2 - \sigma^2) \to N(0, 2\sigma^4)$

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(b) Since f max. at $\sigma^2 = S$: Pick $\hat{\sigma}_n = \sqrt{S}$.

Since $x \mapsto \sqrt{x}$ is smooth on $(0, \infty)$:

$$\sqrt{n}\left(\sqrt{\hat{\sigma}^2_n} - \sqrt{\sigma^2}\right) \to \frac{d}{dx}(\sqrt{x})\Big|_{x=\sigma^2} \cdot N(0, 2\sigma^4)$$

$$= \frac{1}{2\sigma} N(0, 2\sigma^4) = N(0, \sigma^2/2).$$

(c) $I(\sigma) := \mathbb{E}\left[-\frac{d^2}{d\sigma^2}\left(\log(f(X;\sigma))\right)\right]$

$\log(f(X,\sigma)) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2} X^2$

$\therefore$ Diff wrt σ: $-\frac{1}{\sigma} + \frac{X^2}{\sigma^3}$

Diff wrt σ: $\frac{1}{\sigma^2} - \frac{3X^2}{\sigma^4}$

$\therefore I(\sigma) = 3/\sigma^4\, \mathbb{E}[X^2] - 1/\sigma^2 = 2/\sigma^2$.

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2.) (a) $\mathbb{E}_\theta\left[(a\bar{X}+b-\theta)^2\right] = \mathbb{E}_\theta\left[\left(a\underbrace{(\bar{X}-\theta)}_{0\text{ E.V.}} + b - (1-a)\theta\right)^2\right]$

$= a^2\,\mathbb{E}_\theta\left[(\bar{X}-\theta)^2\right] + (b-(1-a)\theta)^2$

$= \frac{a^2}{n} + (b-(1-a)\theta)^2$

(b) If $a>1$: $R(\hat{\theta}_{a,b},\theta) = \frac{a^2}{n} + (a-1)\theta^2 + 2b(a+1)\theta + b^2$

$R(\hat{\theta}_{1,0},\theta) = 1/n < \frac{a^2}{n} \leq a^2/n + (b-(1-a)\theta)^2 = R(\hat{\theta}_{a,b},\theta)$

$\therefore$ $\hat{\theta}_{1,0}$ strictly dominates $\hat{\theta}_{a,b}$ $(a>1)$ $\Rightarrow$ $\hat{\theta}_{a,b}$ inadmissible.

(c) $R(\hat{\theta}_{0,-b/a-1}) = \left(\left\{-b/(a-1)\right\} - \theta\right)^2 = \frac{1}{(a-1)^2}(b-(1-a)\theta)^2$

$< (b-(1-a)\theta)^2$ $\quad(a<0 \Rightarrow |a-1|>1)$

$\leq a^2/n + (b-(1-a)\theta)^2 = R(\hat{\theta}_{a,b})$ $\quad(a<0)$

$\therefore$ $\hat{\theta}_{0,-b/a+1}$ dominates $\hat{\theta}_{a,b}$ (strictly) $\Rightarrow$ $\hat{\theta}_{a,b}$ inadmissible

(d) Let δ be any estimator: $B(\theta) = \underbrace{\mathbb{E}_\theta[\delta]}_{\mu} - \theta$ is diff.

$R(\bar{X}_n,\theta) = \mathbb{E}_\theta\left[(\bar{X}_n-\theta)^2\right] = Var(\bar{X}_n) = 1/n.$

$R(\delta,\theta) = \mathbb{E}_\theta\left[(\delta-\theta)^2\right] = \mathbb{E}\left[(\delta-\mu)^2 + (\mu-\theta)^2\right] = Var(\delta) + B(\theta)^2$

Cramer Rao: $Var(\delta)\cdot I_n(\theta) \geq Cov(\delta, \log(f(x,\theta))')^2$

$\geq \left[\frac{d}{d\theta}\left(\mathbb{E}_\theta[\delta]\right)\right]^2 = (1+B'(\theta))^2$

$\therefore R(\delta,\theta) \geq B(\theta)^2 + \frac{(1+B'(\theta))^2}{\cancel{4}\, n\cdot I(\theta_0)}$

If $B'(\theta)>0$: $\frac{(1+B'(\theta))^2}{\cancel{4}\; n} > \frac{1}{n} = R(\delta,\theta) \Rightarrow \delta$ will not dominate $\bar{X}$.

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$\therefore B'(\theta) \leq 0.$

But $|B(\theta)|^2 \leq 1/n \Rightarrow B$ bounded $\Rightarrow B' \to 0$ as $\theta \to \pm\infty$.

If $B'(\theta) \neq 0$: $B'(\theta) > 0 \Rightarrow B'(-\infty) > 0$; $B(\theta) < 0 \Rightarrow B'(\infty) < 0$ $\Big\}$ rej.

$\therefore B' = 0 \Rightarrow B(\theta)^2 + 1/n \leq 1/n \Rightarrow B \equiv 0.$

$\therefore R(\delta, \theta) = R(\bar{X}, \theta) \Rightarrow \theta$ is admissible.

(e) $\bar{X_n}$ is admissible with constant risk:

$\forall$ estimators δ in (b), (c), $R(\delta, \theta) > R(\bar{X_n}, \theta)$

$\therefore \sup\{R(\delta, \theta) : \theta \in \mathbb{R}\} > \sup\{R(\bar{X_n}, \theta) : \theta \in \mathbb{R}\}$

$\therefore$ δ not minimax.

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3.) (a) $\mathbb{E}[\hat{B}_n] = (n-1)\left[\mathbb{E}_\theta[T_{n-1}] - \theta \overset{+\theta -}{_\wedge} \mathbb{E}_\theta[T_n]\right]$

$= (n-1)\left[\frac{a}{(n-1)} + \frac{b}{(n-1)^2} - \frac{a}{n} - \frac{b}{n^2} + O(\frac{1}{n^3})\right]$

$= (n-1)\left[\frac{a}{(n-1)\cdot n} + \frac{b\,(2n-1)}{n^2(n-1)^2} + O(\frac{1}{n^3})\right] = \frac{a}{n} + \underbrace{\frac{b\,(2n-1)}{n^2\,(n-1)}}_{O(1/n^2)} + O(\frac{1}{n^3})$

$= \frac{a}{n} + O(1/n^2)$

$\therefore \mathbb{E}_\theta[\tilde{T}_{Jack}] - \theta = \mathbb{E}_\theta[T_n] - \theta - \mathbb{E}[\hat{B}_n] = \frac{a}{n} + \frac{b}{n^2} + O(\frac{1}{n^3}) - \frac{a}{n}$
$+ O(1/n^2)$
$= O(1/n^2)$ *.

(b) $\mathbb{E}[(\bar{X}_n)^2] = \underbrace{Var(\bar{X}_n)}_{\frac{1}{n}} + \underbrace{\mathbb{E}[\bar{X}_n]}_{\mu}{}^2 = \mu^2 + 1/n.$

$\therefore \mathbb{E}[T_n] = \mu^2 + 1/n.$; $Bias(T_n) = 1/n$

$\mathbb{E}[\hat{B}_n] = (n-1)\left[(\mu^2 + 1/n-1) - (\mu^2 + 1/n)\right] = \frac{1}{n}.$

$\therefore \mathbb{E}[\tilde{T}_{Jack}] = \mu^2 + 1/n - 1/n = \mu^2$ * ; $Bias(\tilde{T}_{Jack}) = 0.$

(c) Central limit theorem: $\sqrt{n}\,(\bar{X}_n - \mu) \to N(0,1)$ (in dist.)

~~Strong~~ Weak law of Large #: $\bar{X}_n \to \mu \Rightarrow \bar{X}_n + \mu \to 2\mu$ (in $\mathbb{P}$)

Slutsky $\sqrt{n}\,(T_n - \mu^2) = \sqrt{n}\,(\bar{X}_n - \mu)\cdot(\bar{X}_n + \mu) \to 2\mu \cdot N(0,1)$
in distribution

$\therefore N(0, 4\mu^2).$

(d) $\sqrt{n}\,(\tilde{T}_{Jack} - \mu^2) = \sqrt{n}\,(T_n - \mu^2) + \sqrt{n}\,(\hat{B}_n)$

$$\hat{B}_n = \frac{n-1}{n} \sum_{k=1}^{n} (T_{(-k)} - T_n) = \frac{n-1}{n} \sum_{k=1}^{n} (\overline{X}_{n-1,k} - \overline{X}_n)(\overline{X}_{n-1,k} + \overline{X}_n)$$

$$\cancel{= \frac{1}{n} \sum_{k=1}^{n} (\overline{X}_n - X_k)(\overline{X}_{n-1,k} + \overline{X}_n)}$$

$$\cancel{= \frac{1}{n} \Big[\overline{X}_n \sum_{k=1}^{n} (\overline{X}_n - X}$$

$$= \frac{1}{n} \sum_{k=1}^{n} (\overline{X}_n - X_k)(\overline{X}_{n-1,k} + \overline{X}_n)$$

$\overline{X}_{n-1,k} = \frac{n}{n-1}\overline{X}_n - \frac{X_k}{n-1}$

$$= \frac{1}{n} \sum_{k=1}^{n} (\overline{X}_n - X_k)\left(\frac{2n-1}{n-1}\overline{X}_n - \frac{X_k}{n-1}\right) = \frac{1}{n(n-1)} \Big[\sum_{k=1}^{n} \overline{X}_n X_k + \sum_{k=1}^{n} X_k^2 \Big]$$

$$= \frac{1}{(n-1)} \overline{X}_n^2 + \underbrace{\frac{1}{(n-1)} \frac{1}{n} \sum_{k=1}^{n} X_k^2}$$

Law of large # : $\rightarrow \mathbb{E}[X_k^2] = 1 + \mu^2$

$\therefore \underbrace{\frac{\sqrt{n}}{n-1}}_{\rightarrow 0} \cdot \frac{1}{n} \sum_{k=1}^{n} X_k^2 \rightarrow 0$ in $\mathbb{P}$.

$\therefore$ $\cancel{\sqrt{n}(\tilde{T}_{Jack} - \mu^2) = \sqrt{n}\left(\frac{n}{n-1} T_n - \mu^2\right)}$

$$\cancel{\frac{n}{n-1}}\ \underbrace{\frac{\sqrt{n}}{(n-1)} (\overline{X}_n^2 - \mu^2)}_{\rightarrow 0 \text{ in dist}} + \underbrace{\frac{\sqrt{n}}{n-1}}_{\rightarrow 0} \mu^2 \rightarrow 0 \text{ in } \cancel{\mathbb{P}} \text{ dist} \Rightarrow \rightarrow 0 \text{ in } \mathbb{P}$$

$\therefore \sqrt{n}\,\hat{B}_n \rightarrow 0$ in $\mathbb{P}$ (Slutsky)

$\therefore \sqrt{n}\,(\tilde{T}_{Jack} - \mu^2)$, $\sqrt{n}\,(T_n - \mu^2)$ have the same asymptotic distribution

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4.) (a) $X_k = -\cancel{\frac{1}{\lambda}} \log(1-U_k)$; (U_k) independent $\Rightarrow$ (X_k) independent

$$\mathbb{P}(X_k \le t) = \mathbb{P}(\log(1-U_k) \ge -t) = \mathbb{P}(1-U_k \ge e^{-t})$$
$$= \mathbb{P}(U_k \le 1-e^{-t}) = 1-e^{-t} \quad \text{(cdf of Exp(1))}$$

(b) $$\mathbb{P}(Y \le y) = \sum_{k=1}^{\infty} \mathbb{P}(\text{succeed on } k^{th} \text{ try } \cancel{3}, Y \le y)$$

$$= \sum_{k=1}^{\infty} p^{k-1} \cdot \int_{-\infty}^{y} h(z) \frac{f(z)}{M \cdot \cancel{\$} h(z)} dz,$$

$$p = \int_{-\infty}^{\infty} h(z)\left(1 - \frac{f(z)}{M h(z)}\right) dz = 1 - \int_0^{\infty} \frac{f(z)}{M} dz = 1 - \frac{1}{M}$$

$$\therefore \mathbb{P}(Y \le y) = \int_{-\infty}^{y} f(z)\, dz \sum_{k=1}^{\infty} p^{k-1} \left(1-\frac{1}{M}\right)^{k-1} \cdot \frac{1}{M} = \int_{-\infty}^{y} f(z)\, dz$$

$\cancel{= \mathbb{P}(X \le y)}$ $\therefore$ pdf $= f$.

(c) $$\frac{\frac{2}{\sqrt{2\pi}} e^{-x^2/2}}{e^{-x}} = \frac{2e^{1/2}}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} \le \frac{2}{\sqrt{2\pi}} \sqrt{e}.$$

$\therefore$ Pick $M = \frac{2}{\sqrt{2\pi}}$

~~Let~~ Generate uniform [0,1] R.V. pairs $U_{k,1}$, $U_{k,2}$ (indep.), $X_k = -\log(1-U_{k,1})$. Start with $k=1$

If $U_{k,2} \le f(X_k) / M(1-U_{k,1})$ $\cancel{\frac{2}{\sqrt{2\pi}} e}$, accept it and output X_k. Else, continue with $k+1$.

(d) Expected trials $= \mathbb{E}[\text{Geo}(1/M)] \overset{=}{\cancel{\circ}} M = \frac{2}{\sqrt{2\pi}} \sqrt{e}.$

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(e) $\mathbb{P}(Z \leq t)$: If $t \leq 0$: $V > 1/2$, $Y \geq -t$

$$\therefore \int_{-t}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-z^2/2} dx \cdot \frac{1}{2} = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

If $t > 0$: ($V \leq 1/2$, $Y \leq t$) or $V > 1/2$

$$\therefore \int_{0}^{t} \frac{2}{\sqrt{2\pi}} e^{-z^2/2} dz \cdot \frac{1}{2} + \frac{1}{2} = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

$\therefore Z \sim N(0,1)$ distribution

(f) $\mathbb{E}$ # of samples $= M \Rightarrow$ Minimise M.

$$M \geq \sup_x \left\{ f(x)/h(x) \right\} = \frac{2}{\sqrt{2\pi} \cdot \lambda} \sup_x \left\{ e^{-x^2/2} e^{\lambda x} \right\}$$

$$= \frac{2}{\sqrt{2\pi}\lambda} \sup_x \left\{ e^{-\frac{1}{2}(x-\lambda)^2} \right\} e^{\lambda^2/2} = \frac{2}{\sqrt{2\pi}} e^{\lambda^2/2}/\lambda.$$

$$\frac{d}{d\lambda}\left(e^{\lambda^2/2}/\lambda\right) = \frac{\lambda^2 e^{\lambda^2/2} - e^{\lambda^2/2}}{\lambda^2} = 0 \Rightarrow \lambda = 1 \quad (\lambda > 0)$$

$\therefore$ Unable to improve efficiency with Exp changing exponential parameter