~~$\Rightarrow$: proof~~ $\vdash$: proof.

~~Se~~ Sytatically imply

Consistent. $S \nvdash \perp$.

$\models$: valuation

semantically imply

Model. $\exists v$

propositional logic.

$L_1 = \{\perp\} \cup P$, $L_{n+1} = L_n \cup \{(p \Rightarrow q) : p, q \in L_n\}$.

v : model of S if $v(s) = 1$ for all $s \in S$.

Completeness Thm: $S \vdash t \overset{\text{soundness}}{\underset{\text{Adequacy}}{\Longleftrightarrow}} S \models t$.

Soundness: $S \vdash t \Rightarrow S \models t$.

proof: Each line t_i of proof t from S has $v(t_i) = 1$.

Adequacy: $S \models t \Rightarrow S \vdash t$.

claim ~~Lemma~~: Enough to show $S \models \perp \Rightarrow S \vdash \perp$.

[S no model $\Rightarrow$ S inconsistent]

Equivalently, S consistent $\Rightarrow$ S has a model

[Model existing lemma].

Proof of ~~lemma~~ claim: $S \models t$, so $S \cup \{\neg t\}$ has no model.

$S \cup \{\neg t\} \vdash \perp$ (given lemma).

$S \vdash (\neg t) \Rightarrow \perp$ Deduction thm

i.e. $S \vdash (\neg\neg t)$

But $S \vdash ((\neg\neg t) \Rightarrow t)$ (Axiom 3)

So $S \vdash t$ (mp).

Model existence lemma: (~~Completeness Thm~~)

$S \subset L$ consistent $\Rightarrow$ S has a model.

Bullet Proof:

① Either $S \cup \{p\}$ or $S \cup \{\neg p\}$ is consistent

② List L (countable) as $\{t_1, t_2, \dots\}$.

$S_0 = S$. $S_1 = S_0 \cup \{t_1\}$ or $S_0 \cup \{\neg t_1\}$, s.t. S_{i+1} consistent

$\bar{S} = S_0 \cup S_1 \cup S_2 \cup \cdots$.

③ $\bar{S}$ consistent. By proofs are finite.

④ Note $\bar{S}$ deductively closed

⑤ Define $v: L \to \{0,1\}$. $p \mapsto \begin{cases} 1, & p \in \bar{S} \\ 0, & p \notin \bar{S} \end{cases}$.

~~proof~~ prove ✓ valuation:

(i) $v(p)=1, v(q)=0$

(ii) $v(q)=1$

(iii) $v(p)=0$. $\square$.

$S \subset L, t \in L$.

Compactness Thm: If $S \models t$ then some finite $S' \subset S$, $S' \models t$.

2nd form: if Every finite subset of S has a model, then S has a model.

Decidability Thm: For finite $S \subset L, t \in L$, $\exists$ algo to determine if $S \vdash t$.

Cardinals

Motivation: $\operatorname{card} x = \operatorname{card} y \iff (x \overset{\text{bijection}}{\hookrightarrow} y)$.

Let $\operatorname{card} x$ be least ordinal α s.t. $x \leftrightarrow \alpha$.

Initial: α initial if $(\forall \beta < \alpha)(\neg \beta \hookrightarrow \alpha)$.

Define ω_α, $\alpha \in ON$ recursively by:

$$\omega_0 = \omega$$
$$\omega_{\alpha+1} = \gamma(\omega_\alpha)$$
$$\omega_\lambda = \sup\{\omega_\beta : \beta < \lambda\}$$

Each ω_α is initial. Every infinite initial δ is an ω_α.

Let $\aleph_\alpha = \operatorname{card} \omega_\alpha$.

For cardinals m, n, say $m \leq n$ if $\exists$ injection $M \to N$.
card $M = m$, card $N = n$.

Total order in ZFC.

Cardinal Arithmetic

$m+n$ $\quad \operatorname{car}(M \sqcup N)$

mn $\quad \operatorname{card}(M \times N)$

m^n $\quad \operatorname{card}(M^N)$. $M^N = \{f : f$ a function from N to $M\}$

M, N any set with card m, n.

eg. $\mathbb{R} \leftrightarrow \mathbb{P}\omega \hookrightarrow 2^\omega$, so $\operatorname{card}\mathbb{R} = \operatorname{card}(\mathbb{P}\omega) = \operatorname{card}(2^\omega) = 2^{\aleph_0}$

Note: $\aleph_0 \aleph_0 = \aleph_0$

$m+n = n+m$

$mn = nm$

$(m^n)^p = m^{np}$

Predicate Logic.

Language $L = L(\Omega, \Pi, \alpha)$ is the set of formulas

Variables $x_1, x_2, \cdots$

Terms (i) variable are terms. (ii) if $f \in \Omega$, $\alpha(f) = n$, $t_1, \cdots, t_n$ terms, so is $f t_1 \cdots t_n$.

Atomic formulae:

(i) $\perp$ is

(ii) s, t terms $\Rightarrow (s = t)$ is

(iii) $\phi \in \Pi$, $\alpha(\phi) = n$, $\phi(t_1 t_2 \cdots t_n)$ is.

Formula: def Inductively by: (i) Atomic is
(ii) p, q is then $(p \Rightarrow q)$
(iii) p formula, x variable, then $(\forall x) p$ is.

Closed terms: No variables.

Free / bounded variables: bound if inside $\forall x$.

Sentense: Formula with no free variable.

Substitution. $p[t/x]$, p formula, x variable, t term.

Semantic entailment.

L language. L-structure is non-empty set A with

(i) For Each $f \in \Omega$, a function $f_A : A^n \to A$, $n = \alpha(f)$.

(ii) For each $\phi \in \Pi$, a set $A_\phi \subset A^n$, $n = \alpha(\phi)$

Interpretation: For sentence p in L-stucture A, $p_A \in \{0, 1\}$ by ~~Theory~~ $\cdots$

Theory T: set of sentences.

A model of T if A is model of p $\forall p \in T$.

$T \models p$: every model of T also model of p.

LS. Propositional.

Soundness: $S \vdash t \Rightarrow S \models t$:

Every line of proof t_i ~~has~~ has $v(t_i)=1$:

1. t_i (from S).
2. Axiom (Tautology)
3. M.P

$v(p, (p \Rightarrow q)) = 1 \Rightarrow$ then $v(q)=1$.

q

Adequacy: $S \models t \Rightarrow S \vdash t$.

Claim: Suffice to prove $S \nvdash \bot \Rightarrow S \nvDash \bot$

~~as if (*) holds and $S \models t$, then~~

which is equivalent to prove $S \models \bot \Rightarrow S \vdash \bot$. (*)

Assume (*) holds, then

If $S \models t$, then $S \cup \{\neg t\} \models \bot$,

assumption $S \cup \{\neg t\} \vdash \bot$.

Deduction $S \vdash (\neg t) \Rightarrow \bot$.

$S \vdash (\neg\neg t)$

Axiom $S \vdash ((\neg\neg t) \Rightarrow t)$

MP. $S \vdash t$

LS. ES. 1.

9. $t_1, t_2, \dots$

If n unbounded:

~~If Untrue~~, then $\{\neg t_1, \neg t_2, \dots\}$

has no model.

So $\underset{\text{subsets}}{\text{Finite}} \vdash \bot$. ✓

10. ~~Indep~~ Claim: Every finite set of propositions has an indep subset equivalent to it.

Proof:

Finite: ✓

Infinite: A_1,

$A_1 \wedge A_2$,

$A_1 \wedge A_2 \wedge A_3$,

...

Infinite: φ_1

$\varphi_1 \Rightarrow \varphi_2$

$(\varphi_1 \wedge \varphi_2) \Rightarrow \varphi_3$,

...

Deduction Thm.

Let $S \subset L$, $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ iff $S \cup \{p\} \vdash q$.

proof: $(\Rightarrow)$ ✓.

$(\Leftarrow)$ Let $t_1, \ldots, t_n$ be proof from $S \cup \{p\}$ to q.

~~Then~~ W.T.S. $S \vdash (p \Rightarrow t_i)$ $\forall i$.

1. t_i axiom:

$$t_i \Rightarrow (p \Rightarrow t_i)$$
$$t_i$$
$$p \Rightarrow t_i$$

2. $t_i \in S$: same.

3. $t_i = p$, $\emptyset \vdash (p \Rightarrow p)$

4. t_i is by MP,

Early t_j, t_k, $t_k = (t_j \Rightarrow t_i)$.

Induction $S \vdash (p \Rightarrow t_j)$, $S \vdash (p \Rightarrow (t_j \Rightarrow t_i))$.

$$[p \Rightarrow (t_j \Rightarrow t_i)] \Rightarrow [(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)]$$
$$(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)$$
$$p \Rightarrow t_i.$$

W-O & Ordinals

Subset Collapse.

X W-O, $Y \subset X$, then $Y \overset{iso}{\simeq}$ unique initial segment of X.

proof: Y has least element.

$Y: Y \to X$

$f(x) = \min X \setminus \{f(y) : y \in Y, y < x\}$.

Cannot have $\emptyset = \emptyset$, because $x \notin \{f(y) : y < x, y \in Y\}$.

Done by existence and uniqueness of recursion

Thm: X, Y be W-O. Then $X \leq Y$ or $Y \leq X$.

$X \leq Y$ & $Y \leq X \Rightarrow X \overset{iso}{\simeq} Y$.

proof: $g \circ f$ from X to an initial segment of X.

$f: X \to Y$, $g: Y \to X$.

$g \circ f: X \to X$ is identity on V (by uniqueness of initial segment)

$f \circ g$ is identity of Y.

so $f: X \to Y$ bijection.

C2: Well-Orderings & Ordinals.

Total Order / Line order: irreflexive, transitive, tricho.

Well-ordering: $\forall$ non-empty subset of X has least element.

proof by induction: X well-order, let $S \subset X$ s.t. $\forall x \in X$, if $y \in S$ for all $y < x$, then $x \in S$. Then $S = X$.

X, Y be iso well-O. Then there is unique iso from X to Y.
[Given $f(y) = g(y)$ $\forall y < x$, $f(x) = \cancel{\text{least}}$ int $Y \setminus \{f(y) : y < x\}$ otherwise # to order-preserving of iso. f].

Recursion.

Subset Collapse. $Y \subset X$. X is WO. Then Y is iso to an unique initial segment of X. ~~Unique~~.

Thm: Let X, Y be W-O. Then $X \leq Y$ or $Y \leq X$.

propo: X, Y WO. $X \leq Y$, $Y \leq X \Rightarrow$ X, Y are iso.

New from old: Nested $\{X_i : i \in I\}$, $\exists$ W-O X, $X \geq X_i$ $\forall i$.

Ordinal.

Def: W-O set, two regarded same if iso.

Thm. Let α be an ordinal. Then ordinals $< \alpha$ form W-O set of order type α.

Let S be non-empty set of ordinals, then S has least element.

Thm: Burali-Forti.

LS Day 3 Zorn.

Poset: ~~$(X, <)$~~ $(X <)$:

1. $\neg x < x$

2. $x<y, y<z \Rightarrow x<z$. $(\forall x,y,z \in X)$.

No trichoty.

Chain: S subset of X, S is a total order.

Antichain: no two elements of S related.

Upperbound: $y \leq x$ $\forall y \in S$.

Complete: Every $S \subset X$ has a supremum.

f: order preserving if $x \leq y \Rightarrow f(x) \leq f(y)$.

Knaster-Tarski: X be complete

f be order-preserving.

Then f has a fixed point.

Cor: Schröder-Bernstein: $f: A \to B$; $g: B \to A$ be injections.

$\exists$ bijection from A to B.

Zorn's lemma

If every chain of X has an upper bound. X be poset. Then X has a ~~max~~ maximal element.

Proof: If X no maximal element.

$\forall x \in X \ \exists x' \in X$ with $x' > x$.

Let $\gamma = \gamma(X)$, by Hartogs' lemma.

Define x_α, $\alpha < \gamma$ recursively by:

$x_0 = x$

$x_{\alpha^+} = x_\alpha'$ ← AOC.

$x_\lambda = u(\{x_\alpha : \alpha < \lambda\})$ for λ a non-zero-limit.

↑ exist as every chain has upper bound. $\{x_\alpha : \alpha < \lambda\}$ is a chain.

Then x_α, $\alpha < \gamma$ are distinct, so inject γ into X #.

App 1. Every vector space has a basis.

proof: Let $X = \{A \subset V : A \text{ is LI}\}$, ordered by $\subset$.

Seek maximal $A \in X$. Then done: if A not span, $x \notin \langle A \rangle$, then $A \cup \{x\}$ is LI.

Given $\{A_i : i \in I\}$, Let $A = \bigcup_{i \in I} A_i$, $A_i \subset A$ $\forall i$. ~~A is upper~~

To prove A is upper bound, need $A \in X$, i.e. A is LI.

Say $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$, $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n$ scalars, not all 0.

Then $x_1 \in A_{i_1}, \ldots, x_n \in A_{i_n}$, some $i_1, \ldots, i_n \in I$.

Since $\{A_i\}$ is a chain, some A_{i_k} has $A_{i_1}, \ldots, A_{i_n} \subset A_{i_k}$, # to A_{i_k} LI.

Model existing lemma. Uncountable case.

seek max consistent $\bar{S} > S$ ~~s.t. $\forall t \in L(P)$, $t \in \bar{S}$ or $\neg t \in \bar{S}$~~.

~~Then~~ If exist, done, as if $t \notin \bar{S}$, then $\bar{S} \cup \{t\} \vdash \bot$,

so $\bar{S} \vdash \neg t$, $\neg t \in \bar{S}$ by maximality of $\bar{S}$.

Let $X = \{T \subset L(P) : T \text{ consistent}, S \subset T\}$, ordered by $\subset$.

$X \neq \phi$ as $S \in X$.

Given non-empty chain $\{T_i : i \in I\}$ in X, let $T = \bigcup_{i \in I} T_i$.

Then $T_i \subset T \; \forall i$, so just need $T \in X$.

Have $S \subset T$. If T inconsistent, as proofs are finite,

have $t_1, \dots, t_n \in T$ with $\{t_1, \dots, t_n\} \vdash \bot$.

$t_j \in T_{i_j}$ for some $i_1, \dots, i_n$.

So $t_1, \dots, t_n \in T_{i_k}$ for some T_{i_k} (as T_i form a chain)

T_{i_k} consistent. ✓

Crux: ~~Proofs are finite~~.

W-O Principle: Every set can be well-ordered.

proof: ~~Or~~ Let $X = \{(A, R) : A \subset S, R \text{ is a W-O of } A\}$.

Order X by extension: $(A, R) \geqslant (A', R')$ if $A' \subset A$

and R and R' agree on A'

and A' is an initial segment of A in ordering R.

$X \neq \phi$ since ϕ is W-O.

Given chain $\{(A_i, R_i): i \in I\}$, (A_i, R_i) are nested family

so $(\bigcup_{i\in I} A_i, \bigcup_{i \in I} R_i)$ is an upper bound.

By Zorn, X has maximal element, say (A, R)

Claim: $A = S$.

proof: If $A \neq S$. Choose $x \in S \setminus A$,

define W-O on $A \cup \{x\}$ by $x > y$ $\forall y \in A$,

to maximality.

AOC: Every $\{A_i : i \in I\}$ of non-empty sets

has a choice function $f: I \to \bigcup_{i \in I} A_i$

s.t. $f(i) \in A_i$ $\forall i$.

Hence AOC $\Rightarrow$ Zorn: ✓.

Zorn $\Rightarrow$ AC:

$\{A_i : i \in I\}$, $A_i \neq \emptyset$, a partial choice func is $f: J \to \bigcup_{i \in I} A_i$,

some $J \subset I$ s.t. $f(j) \in A_j$ $\forall j \in J$.

$X = \{(J, f): J \subset I, f$ a partial choice func $J \to \bigcup_{i \in I} A_i\}$,

order by extention.

$X \neq \emptyset$ since $(\emptyset, \emptyset) \in X$.

Given $\{(J_q, f_q): q \in Q\}$ has upper bound $(\bigcup_{q\in Q} J_q, \bigcup_{q \in Q} f_q)$.

By Zorn, have maximal $(J, f) \in X$

If $J \neq I$, choose $i \in I \setminus J$, choose $x \in A_i$,

$(J \cup \{i\}, f \cup \{(i, x)\})$, #.

WO $\Rightarrow$ AC: well-order $\bigcup_{i \in I} A_i$,

Let $f(i)$ = least element of A_i.

AC $\Rightarrow$ WO:

Let f be choice function for $\{A \subset X : A \neq \emptyset\}$,

Def x_α, $\alpha < \gamma(X)$ recursively by:

if $\{x_\beta : \beta < \alpha\} = X$, stop, o-w set

$x_\alpha = f(X - \{x_\beta : \beta < \alpha\})$.

Must stop, else inject $\gamma(X)$ into X, $\#$

So have injection from X to w-o set,

an initial segment of $\gamma(X)$.

Upward L-S Thm:

Let S be a theory with an infinite model. Then S has an uncountable model.

✓

① Downward L-S Thm:

Let S be a theory ~~..~~ in a countable langue

If S has a model, then S has a countable model.

Cardinality:

$\mathrm{Card}(x)$ is the least ordinal α s.t. $x \leftrightarrow \alpha$.

By W-O, every infinite set can be well-orded.

We say ordinal α is initial if

$$(\forall \beta < \alpha)(\neg \beta \leftrightarrow \alpha).$$

Define ω_α for each $\alpha \in ON$ by:

$$\omega_0 = \omega$$
$$\omega_{\alpha+1} = \gamma(\omega_\alpha)$$
$$\omega_\lambda = \sup\{\omega_\alpha : \alpha < \lambda\} \text{ for non-zero limit } \lambda.$$

Then ω_α is unbounded in ordinals,

since $\omega_\alpha \geq \alpha$ $\forall \alpha$. By induction

~~Then~~ Also, every infinite initial δ is an ω_α:

since unbounded, take least α s.t. $\omega_\alpha \geq \delta$.

(1) If $\alpha = \beta^+$. Then $\omega_\beta < \delta \leq \omega_\alpha$.

But there's no initial ordinal between ω_β and $\omega_\alpha = \gamma(\omega_\beta)$ by definition.

So $\delta = \omega_\alpha$.

(2) If α limit, by definition can't have $\delta < \omega_\alpha$, else there is some $\beta < \alpha$ with $\delta \leq \omega_\beta$

So $\omega_\alpha = \delta$.

Thus every initial is an ω_α.

Write $\aleph_\alpha$ to be ~~cardinality of~~ $card(\omega_\alpha)$

Thus $\aleph_\alpha$ are the cardinalities of all infinite sets.

Thm. For $\alpha \in ON$, we have $\aleph_\alpha \aleph_\alpha = \aleph_\alpha$.

proof: Induction on α. $\aleph_0 \aleph_0 = \aleph_0$ as $\omega \times \omega \hookrightarrow \omega$.

Define a well-ordering of $\omega_\alpha \times \omega_\alpha$ by:

$(x,y) < (z,t)$ if:

① $\max(x,y) < \max(z,t)$ or

② $\max(x,y) = \max(z,t) = \beta$.

and $y < \beta, t = \beta$

or $y = t = \beta, x < z$

or $x = z = \beta, y < t$.

Then for any $\delta \in \omega_\alpha \times \omega_\alpha$, consider initial segment of I_δ. $I_\delta \subset \beta \times \beta$ for some $\beta < \omega_\alpha$.

But ω_α is initial, so $\text{card}(\beta) < \text{card}(\omega_\alpha)$.

So ~~$\beta\times\beta$~~ ~~to~~ by induction ~~to~~ hypothesis

$\text{card}(I_\delta)$ ~~$\text{card}(\beta)$~~ $\leq \text{card}(\beta \times \beta) = \text{card}(\beta)$

$< \text{card}(\omega_\alpha)$.

So every proper initial segment of δ has order type $< \omega_\alpha$, so well-ordering has order type $\leq \omega_\alpha$.

Thus $\omega_\alpha \times \omega_\alpha \hookrightarrow \omega_\alpha$, $\aleph_\alpha \aleph_\alpha \leq \aleph_\alpha$

We have $\aleph_\alpha \leq \aleph_\alpha \aleph_\alpha$, so $\aleph_\alpha \aleph_\alpha = \aleph_\alpha$.

Quick run of LS.

Completeness Thm:

Let ~~S be a set~~ $S \subset L(P)$.

Then $S \vdash p$ iff $S \models p$.

Proof: $(\Rightarrow)$ Every line of the proof has valuation $=1$:

Say proofs have statements $t_1, t_2, \dots, t_n$.

t_i either from:

① Axioms: tautology, so $V(t_i)=1$

② M-P, so have $t_j, (t_j \Rightarrow t_k)$ for previous lines.

$V(t_j)=1, V(t_j \Rightarrow t_k)=1$.

If $V(t_k)=0$, then $V(t_j \Rightarrow t_k) \neq 1$, can't be true.

So $V(t_k)=1$.

③ t_i in S. Then $V(t_i)=1$.

Thus t_n is p. $V(t_n)=1$

$V(p)=1$.

$(\Leftarrow)$ Claim: Enough to prove that

S consistent $\Rightarrow$ S has a model. $(*)$

If $(*)$ holds, then if S has no model $\Rightarrow S \vdash \perp$.

Since $S \models P$, we know $S \cup \{\neg P\}$ has no model;

~~Indeed, as $S \models (P \Rightarrow \bot)$ (Deduction, $S \models P$, so $S \models \bot$ (MP).~~

~~Thus, $S \cup \{\neg P\} \vdash \bot$. by lemma (*).~~

as valuation can't have $v(P) = 1$ and $v(P \Rightarrow \bot) = 1$ same time.

Thus by lemma (*), $S \cup \{\neg P\}$ inconsistent.

So $S \cup \{\neg P\} \vdash \bot$.

$$S \vdash (P \Rightarrow \bot) \Rightarrow \bot$$

$$S \vdash (\neg\neg P).$$

But by axiom $\vdash (\neg\neg P) \Rightarrow P$.

So $S \vdash P$ (MP). ✓.

Now we prove (*): Model existing lemma.

~~First assume~~

① we want to find deductively closed extension of S: $\bar{S}$ ~~such that~~

For every S, either $S \cup \{P\}$ is true or $S \cup \{\neg P\}$ is true. [...].

~~Define~~ If $L(P)$ countable: $\{l_1, l_2, \dots\}$.

Let $S_0 = S$, $S_{i+1} = S_i \cup \{P\}$ or $S_i \cup \{\neg P\}$ depending on which one consistent.

Then each S_i is consistent.

~~Also~~ Also, $\bar{S}$ is consistent:

~~Then~~ If $\bar{S} \vdash \perp$, then proofs finite

$\Rightarrow S_n \vdash \perp$ for some n, ⨳.

② $\bar{S}$ is ~~de~~ deductively closed:

If $\bar{S} \vdash P$, then if $P \notin S$, then

$\neg P \in S$, so $\bar{S} \vdash \perp$, ⨳

③ ~~We~~ Let valuation v be:

$$v: \bar{S} \to \{0,1\}$$
$$P \mapsto \begin{cases} 1, & P \in \bar{S} \\ 0, & P \notin \bar{S} \end{cases}$$

~~It's~~ It's a legit valuation as:

(i) If $v(P)=1$, $v(q)=0$, $v((P \Rightarrow q)) = 0$

$P \in S$, if $(P \Rightarrow q) \in \bar{S}$, then $q \in S$, ⨳.

(i') $v(q)=1$, then $v((P \Rightarrow q)) = 1$

~~(ii) $v(P)$~~ ~~If $v(P \Rightarrow q) \notin S$,~~

as $\vdash q \Rightarrow (P \Rightarrow q)$

$S \vdash q$

so $S \vdash P \Rightarrow q$.

(iii) $v(P)=0$. Then $\neg P \in \bar{S}$.

$S \vdash (P \Rightarrow \perp)$.

~~Have~~ enough to show $P \Rightarrow \perp \vdash P \Rightarrow q$.

By deduction, $\{P \Rightarrow \perp, P\} \vdash q$

$\perp \vdash q$ true, as

$\vdash \perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp)$ (axiom)

$\perp \Rightarrow \neg\neg q$, $\perp \Rightarrow q$.

Compactness: If every finite subset of S has a model, then S has a model.

proof: ~

Decidability: Given $S \subset L$, $t \in L$,
$\exists$ algo to determine in finite time
if $S \vdash t$

proof: $S \vDash t$: Truth table.

Note: In the proofs if not assume L is finite, then need to use Zorn's lemma;

~~Define~~ Let X be ~~all w-o~~ the ~~se~~ poset, ~~with all se~~ with all consistent subset $S \subset L$, ~~such~~ with $S_1 \leq S_2$ iff $S_1 \subset S_2$.

Then every chain has an upper bound;

$U(X_1, X_2, \ldots) = \bigcup_{i \in I} X_i$.

Note: $\bigcup_{i \in I} X_i \in X$ as, if not, have finite $t_1, \ldots, t_n \vdash \bot$, Nested $\Rightarrow$ some $X_n \vdash \bot$, #.

Thus X has maximal element $\bar{S}$.

Claim: $\bar{S}$ deduction closed.

If not, $\exists P$ s.t. $P \notin S$; $\neg P \notin S$,

then $\bar{S} \cup \{P\}$ or $\bar{S} \cup \{\neg P\}$ larger, # to maximal.

Hartog's Lemma:

Given set X, can find ordinal α s.t. α ~~no~~ does not inject into X.

Proof: Take $W = \{A \in \mathbb{P}(X \times X) : A$ is W-O of a subset of $X\}$

Then let $B = \{\text{order-type}(R) : R \in A\}$.

B is precisely set of all ordinals $\leq$ order-type(X)

Let $w' = \sup B$.

If $w' \hookrightarrow X$, then it's greatest one inject into X, but ~~$w' \hookrightarrow w$~~

then $w'^{+} \hookrightarrow X$.

$w'^{+} > w'$, #.

Zorn's lemma:

Let X be a poset s.t. every chain has sup.
Then X has maximal element.

Proof: Start from $x_0 \in X$. < If no maximal, any element $x \in X$, $\exists y$ s.t. $y > x$.

recursively define:

$x_{\alpha+1} = y$. such that $y > x_\alpha$

AOC.

$x_\beta = u(\{x_\alpha : \alpha < \beta\})$ as $\longrightarrow$ is a chain,

~~Then take ∃ it must stop,~~

o.w. ~~have~~ have injection $\nu(X) \to X$, ↯