Graph Theory

1. $\Delta(G), \delta(G)$
2. Tree is a connected, acyclic graph = maximal acyclic graph = minimum connected graph
3. leaf
4. Every tree has a leaf
5. $e(T) = |T| - 1$
6. Spanning tree

1. Bipartite graph
2. A graph G is bipartite if and only if G contains no odd cycles
3. circuit: An odd circuit contains an odd cycle

Matching

21, 19, 17.

1. Hall's theorem (augmenting path, alternating path)
2. cor: A k-regular bipartite graph contains a perfect matching.
3. deficiency; Let G be a bipartite graph. Then G contains a matching saturating $|A| - d$ vertices in A if and only if for all $A_0 \subseteq A$, we have $|N(A_0)| \geq |A_0| - d$ [hint: Find d extra vertices]
4. system of distinct representatives
5. left and right coset

Connectivity

1. cut vertex
2. separator
3. Peterson graph: $\kappa(G) = 3$.
4. 20, 18, 16. Menger's Theorem, First Form: Let $G = (V, E)$ be a graph, with distinct and non-adjacent $a, b \in V$. If every ab separator in G has size at least k then we can find k independent ab paths
5. Let $G = (V, E)$ be a graph. Then G is k-connected if and only for all $u, v \in V$ with $u \neq v$, there exists k independent uv-paths
6. Edge cut
7. Cut edge
8. Edge connectivity: $\lambda(G)$
9. k-edge-connected
10. Edge form: Let $G = (V, E)$ be a graph, and u, v be distinct vertices of G. If every set of edges $F \subseteq E$ that separates u from v has size greater than or equal to k, then there exists k edge disjoint paths from u to v.
11. Line Graph

8. Erdos-Stone: Let H be a graph with $\chi(G) = r$, and $r \geq 2$. Then $lim_{n \to \infty} ex(n, H)/\frac{n}{2} = 1 - \frac{1}{r-1}$

Ramsey

19 1. Colour each of the edges of K_6 red or blue . Then there must be a monochromatic triangle.
2. Define $R(t)$ the tth Ramsey number to be the smallest n for which every 2-colouring of K_n contains a monochromatic K_t. So $R(3) = 6$.

8 2\\18 7 3. $R(t)$ is finite and $R(t) \leq 4^t$ [hint: $R(s, t) \leq R(s-1, t) + R(s, t-1)$, $R(s, t) \leq \binom{s+t-2}{s-1}$]
4. Infinite Ramsey Theorem: for every 2-colouring of complete countable graph $G = (N, N^2)$ there exists an infinite set $X \subseteq N$ so that X^2 is monochromatic [hint: take (x_i, c_i) each step]
5. $R(t) \geq 2^{t/2}$ [hint: probabilistic]

Probabilistic Method

1. Binomial Random Graph $G(n, p)$
2. $Z(n, t) \geq (1/4)n^{2-2/(t+1)}$ [modification method]

18 3. Girth, independent set, $\chi(G) \geq n/\alpha(G)$
4. For every $k, G \in N$, there exists a graph G with girth(G) ≥ g and $\chi(G) \geq k$ $p = n^{-1+\frac{1}{g}}$

21 5. $G(n, p(n))$ contains $K_{\cancel{t}\,t,t}$ condition

22 6. $G(n, p(n))$ connectedness condition

Algebraic Method

19
1. diameter: maximal distance
2. Moore graph is a k-regular graph on $k^2 + 1$ vertices that has diameter 2
3. Adjacency matrix: symmetric, trace = 0, eigenvalue sum = 0
4. $\frac{1}{n} \sum_x x \in Vd(x) \leq \lambda_{max}(G) \leq \Delta(G)$ [hint: $\lambda_{max} = max_{x:|x|^2=1} x^T Ax$, take $w = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1)$; RHS maximal coordinate of Ax_{max}
5. $\lambda_{max}(G) = \Delta(G)$ if and only if G is Δ-regular; $\lambda_{min}(G) = \Delta(G)$ if and only if G is Δ-regular and bipartite
6. Moore graph of degree k: $\frac{1}{2}(k^2 \pm \frac{k^2-2k}{\sqrt{4k-3}})$ are integers.

Hall Thm

For ~~bi~~ $G=G(X\cup Y)$, ~~iff~~ $|N(A)|\geq|A|$ $\forall A\subseteq X$,

if ~~then~~ $\exists$ matching saturating A.

($\Leftarrow$) ✓

($\Rightarrow$). ① If $\forall A\subseteq X$ (while $A\neq\phi, X$) $|N(A)|\geq|A|+1$,

let $G'=G(X-\{x_1\}\cup Y-\{y_1\})$ for $x_1y_1\in E$.

Then for any $A\subseteq X-\{x_1\}$,

$$|N_{G'}(A)|\geq|N_G(A)|-1$$
$$\geq|A|+1-1=|A|.$$

So G' has hall. ✓.

② If $\exists$ $A\neq\phi, X$, $A\subseteq X$, $|N(A)|=|A|$, then

let $G_1=G(A\cup N(A))$, $G_2=G(X\setminus A, Y\setminus N(A))$.

For G_1: $|N_{G_1}(S)|=|N_G(S)|\geq|S|$ for $S\subseteq A$.

~~For~~ G_1 Hall, induction ✓.

For G_2: $B\subseteq X\setminus A$.

$$|N_{G_1}(A)|+|N_{G_2}(B)|=|N_G(A\cup B)|$$
$$\geq|A|+|B|$$

$\therefore$ $|N_{G_2}(B)|\geq|B|$ as $|N_{G_1}(A)|=|A|$.

Done.

Menger's Thm.

Let $G=(V,E)$ be graph, with distinct, non-adjacent $a,b \in V$.
If every ab seperator size $\geq k$, then can find k indep ab paths.

Proof. Suppose not. Then let G be counterexample ~~st that~~ s.t. ~~$\{\kappa_{a,b}(G) \neq k\}$~~

(i) $\kappa_{a,b}(G)$ is minimized.

(ii) Number of edge is minimized.

(0) ~~...~~ a,b not having $\kappa_{a,b}(G)$ independent paths.

$\kappa_{a,b}(G)$ is minimal size of ab seperator.

Write $\kappa_{a,b}(G)=k$.

Claim: $\exists$ ab seperator S. $|S|=k$, $S \nsubseteq N(a)$, $S \nsubseteq N(b)$.

proof: Note ~~$N(A) \cap N(B) = \phi$~~. otherwise if $x \in N(a), x \in N(b)$,
$N(a) \cap N(b) = \phi$

Let $G' = G - x$. By assumption (ii), $\kappa_{a,b}(G') = k-1$.

By (i), G' has $k-1$ indep paths. Then for G,
axb is another one, so have k independent paths. #

~~Then N(a)~~

Then choose a shortest path $ax_1x_2 \cdots x_l b$. Let $G' = G - x_1x_2$.
Then $\kappa_{a,b}(G') = k-1$, by (ii). Let S' be ~~ab~~ ab-seperator in G', and $|S'| = k-1$. ~~Let A be component of a in $G' - S'$, B be component of b in $G' - S'$~~

~~If $S' \subseteq$~~ ~~Then~~ Note $x_1a \in E(G)$, $x_2a \notin E(G)$, as $ax_1x_2 \cdots b$ shortest path.
$x_2 \neq b$ as $N(a) \cap N(b) = \phi$

If $S' \subseteq N(a)$, then $S' \cup \{x_2\}$ is ab seperator of size k in G.
$S' \cup \{x_2\} \nsubseteq N(a).N(b)$. Similarly if $S' \subseteq N(b)$, take $S' \cup \{x_1\}$, ✓. $\square$

Now take such S. Let A, B be components of $G-S$, $a \in A$, $b \in B$.
Let $\tilde{G}_a = G[A \cup S]$, and a vertex joins to all S. Similarly for B.

$\kappa_{a,x}(\hat{G}_a) = k = \kappa_{b,x}(\hat{G}_b)$, so $\hat{G}_a, \hat{G}_b$ satisfy Menger Thm. ~~...~~ $\exists k$ independent ax path, Bx path.

GT 2020

P4, SⅡ. 17G.

Vizing's Thm.

$$\Delta(G) \le \chi'(G) \le \Delta(G)+1.$$

Proof: $\Delta(G) \le \chi'(G)$: This is clear, as for $v \in G$ with $d(v)=\Delta(G)$, we have to use at least $\Delta(G)$ distinct colors.

$\chi'(G) \le \Delta(G)+1$: Induction on # edges.

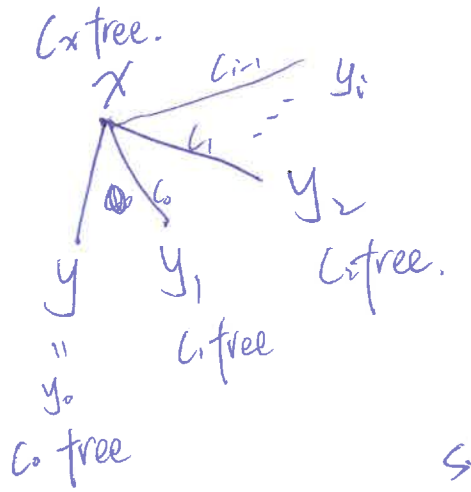
For e, If $G-e$ has $\chi'(G-e)=\Delta(G)$, done by adding new color.

If $G-e$ has $\chi'(G-e)=\Delta(G)+1$,

~~Note~~ Let $e=xy$.

Consider color for which each vertex is free.

[must exists such color, as using $\Delta(G)+1$ colors]



Consider y_i connected to x.

If $C_x=C_0$, done by using C_x.

If $C_0 \ne C_x$, $\exists y_1$ s.t. xy_1 colored C_0 (so that C_0 not free in x).

If $C_1 \ne C_x$, $\exists y_2$ s.t. xy_2 colored C_1.

...

Stop either Case (i) $C_k=C_i$ for some $k>i$.

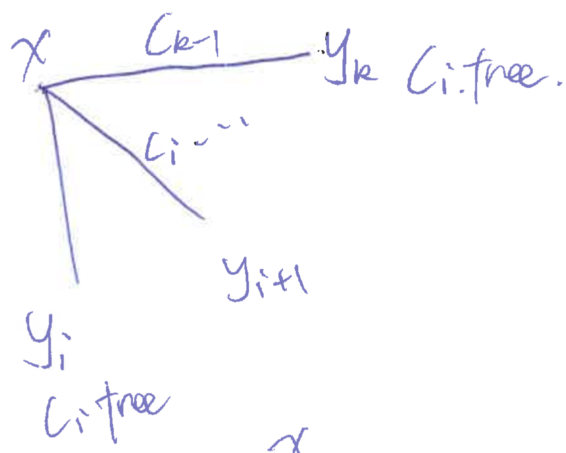
Case (ii) $C_k=C_x$ for some k. ✓

In case (ii), done, as we can switch colors.

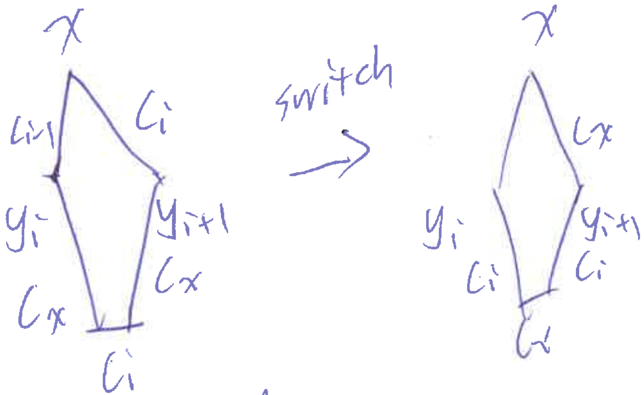
~~In case (i),~~

Switch colors: xy_k colour $c_x = c_k$, xy_{k-1} colour c_{k-1}, $\ldots$, xy_0 color c_0. ✓.

In case (i), take longest path from y_{i+1} of alternating color c_i and c_x.

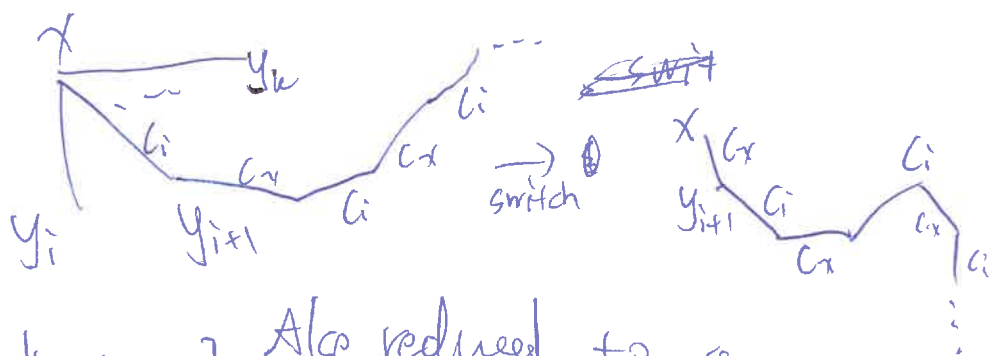


Subcase (a)



Then x with c_i free, to case (ii). done.

Subcase (b).



[Note: switch won't break colorings, as take longest alternating path]. Also reduced to case 1

$\mathbb{E}[\#H] = c\, n^N p^M + o(n^N)$,

$c = \frac{1}{|\mathrm{Aut}(H)|}$

$\Rightarrow$ when $p \ll n^{-\frac{N}{M}}$, $\mathbb{E} \to 0$.

$$\mathrm{Var}[\#H] \le \sum_{\substack{H_1 \cap H_2 = H' \\ H' \text{ has edges.}}} \mathbb{P}[H_1 \,\&\, H_2]$$

$$= \sum_{\substack{H' \\ e(H') \neq 0}} \sum_{\substack{H_1, H_2 \\ H_1 \cap H_2 \cong H'}} p^{2M - e(H')}$$

$$= \sum_{\substack{H' \subset H \\ e(H') \neq 0}} C_{H'}\, n^{2N - |H'|} p^{2M - e(H')}$$

Note $\mathbb{E}^2 = C \cdot n^{2N} p^{2M}$.

So $\mathrm{Var}(\#H)$ is $o(\mathbb{E}^2)$ iff $n^{-|H'|} p^{-e(H')} = o(1)$

i.e. $p \gg n^{-\frac{|H'|}{e(H')}}$

GT Quick Run.

Tree: Connected, acyclic graph

= max acyclic graph

= minimum connected graph.

Leaf: Longest path

Every tree has a

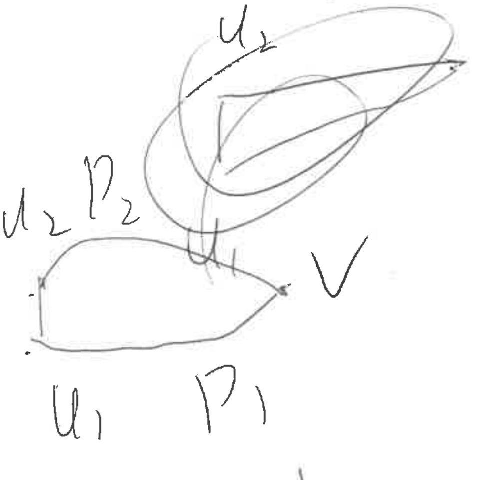
$e(T) = |T| - 1$: induction,

~~Cut~~ Circuit : Induction.

G bipartite $\Leftarrow\Rightarrow$ no odd cycles;

$(\Rightarrow)$, ez

$(\Leftarrow)$.



If $u_1 \sim u_2$, $|P_1| \equiv |P_2|$.

$u_2 \to v$ $u_1 \to v$,

$V P_1 u_1 u_2 P_2$ odd circuit

Hall's Thm.

Let $G = G(X \cup Y)$ be a bipartite graph.

~~If~~ ~~$|N(X)|$~~ $|N(A)| \geq |A| \quad \forall A \subseteq X$,

iff ~~then~~ there's a matching saturating X.

proof. By induction on $|A|$. When $|A| = 1$, True.

Case 1: For all ~~Not exist~~ $A \subset X$, $A \neq X$, $A \neq \phi$, ~~such that~~

we have $|N(A)| \geq |A| + 1$.

Then let $xy \in E$, $x \in X$, $y \in Y$.

Let $G' = G(X - \{x\} \cup Y - \{y\})$.

For all $A \subseteq G'$, we have

$$|N_{G'}(A)| \geq |N_G(A)| - 1 \geq |A|.$$

So Hall holds in G'. ~~Also Hall~~ ✓.

Case 2: $\exists A_0 \subset X$, $A_0 \neq X$, $A_0 \neq \phi$, $|N(A_0)| = |A_0|$.

Let $G_1 = G(A_0 \cup N(A_0))$

$G_2 = G((X - A_0) \cup (Y - N(A_0)))$.

Then for G_1: for any $A \subseteq A_0$,

$$|N_{G_1}(A)| = |N_G(A)| \geq |A| \checkmark$$

For G_2: ~~$|N_{G_2}(A)|$~~ ~~$|N_G(A \cup$~~ $|N_G(A_0 \cup B)|$

$B \subseteq X - A_0$: $\geq |A_0 \cup B| = |A_0| + |B|$

$$|N_G(A_0 \cup B)| = |N_{G_1}(B)| + |N(A_0)| = |A_0| + |N_{G_1}(B)|$$

So $|N_{G_1}(B)| \geq |B|$. ✓

Menger: If every a-b separator in G has size $\geq k$, then we can find k ~~indep~~ vertex disjoint ab paths.

Proof: ~~Claim:~~ If not, let G be the counter-ex such that G
minimize $\kappa_{a,b}(G)$.
conditional on this, minimize # edges.

Claim: exist an a-b seperator S, $S \neq N(a)$, $S \neq N(b)$.

Proof: Note $N(a) \cap N(b) = \phi$ ~~, o.w. we can~~
Because if $\exists x$, $x \in N(a)$, $x \in N(b)$.
Then $G - x$ has $\kappa_{a,b}(G) = k-1$.
Have $k-1$ indep paths.
Then with axb, we have k indep paths.
Done ✓

Take shortest path $a x_1 x_2 \cdots x_c b$.
For $G - x_1 x_2$, by assumption have a-b separator S' of size $k-1$ ✓
Either $S' \cup \{x_1\}$ or $S' \cup \{x_2\}$ satisfy condition.
Done.

Thus, consider components A, B of $G - S$, $a \in A$, $b \in B$.
Let $G_1 = G[A \cup S \cup \{v\}]$, where v is a vertex connecting to all s.
By $S \neq N(a), N(b)$, have $|e(G_1)| < |e(G)|$, and $\kappa_{a,v}(G_1) \geq k$
So have k indep a-v paths.
Similarly for v', b.
Then concatenating and join these k paths, get k indep a-b paths ✓.

Planar Graph.

2. Euler: $V - E + F = 2$.

3. $|E| \le 3|V| - 6$:

Consider (e,f) pairs $\quad 3F \le |\{(e,f)\}: e \in E$, f face, e on boundary of $f\}|$

$\le 2E$.

$\le 2E$.

$2E \ge 3F$.

$\Rightarrow 2E \ge 3(2+E-V)$

$E \le 3(V-2)$.

4. K_5: v

$K(3,3)$ each face

$2E \ge 4F$.

5.

Graph Colouring.

4. If $\delta(G) < \Delta(G)$ then $\chi(G) \leq \Delta(G)$ ✓.

5. Six-colour: $\frac{d(x)}{n}$

~~6. Five-colour:~~

$$= \frac{2E}{V} \leq \frac{2 \cdot 3(V-2)}{V} < 6.$$

$V - E + F = 2.$

$E \leq 3V - 6$

So $\exists x$ with degree 5,

6. 5-color thm ✓.

E-S: $\dfrac{ex(n,H)}{\binom{n}{2}} = 1 - \dfrac{1}{r-1}$.

$Z(n,t) \geq \frac{1}{4} n^{2-\frac{2}{t+1}}$:

$\overset{||}{\max}\{e(G^{[A\cup B]}) : |A|=|B|=n, \text{ no } K_{t,t} \text{ in } G\}$

For each $K_{t,t}$ we delete an edge.

$G(n,p)$

$\mathbb{E}[e(G)] - \mathbb{E}[\# K_{t,t} \text{ in } G]$

$= n^2 p - \binom{n}{t}^2 p^{t^2}$. ✓.

Girth: $g(G)$ = shortest ~~path~~ cycle in G.

Choose $p = n^{-1+\frac{1}{g}}$.

Claim: ① $\mathbb{P}(\{\# \text{ cycles} \leq g\} < \frac{n}{2}\}) > \frac{1}{2}$

② and $\mathbb{P}(\text{largest } \text{~~independent~~ independent set } \alpha(G) \leq \frac{n}{2k}) = 0$.

$\chi(G) \geq \dfrac{n}{\alpha(G)}$. ⟨Then removing a vertex in each short cycle to obtain H with girth$(H) \geq g$, $\chi(H) \geq k$.

---

① $\mathbb{E}(X) = \sum_{i=3}^{g-1} (np)^i = \sum_{i=3}^{g-1} n^{\frac{i}{g}} \leq g n^{\frac{g-1}{g}} < \frac{n}{4}$

for large n

So ~~$\mathbb{E}[X] < \frac{n}{2}$~~ $\mathbb{P}(X \leq \frac{n}{2}) > \frac{1}{2}$.

② $\mathbb{E} = \binom{n}{t}(1-p)^{\binom{t}{2}}$ <let $t = \frac{n}{2k}$

$\leq n^t e^{-p\binom{t}{2}}$

$\to 0$

as $n \to \infty$. ✓.

~~$Var(\#H \cap G)$~~

$$Var(\#H)$$

$$\le \sum_{\substack{H_1, H_2 \cong H \\ H_1 \cap H_2 = H' \\ H' \text{ has edges}}} \mathbb{P}(H_1 \& H_2 \text{ in } G)$$

$$p^{M} p^{M-e(H')}$$

$$= \sum_{\substack{H' \\ e(H')\neq 0}} \sum_{H_1 \cap H_2 \cong H'} p^{2M-e(H')}$$

$$= \sum_{\substack{H' \subset H \\ e(H')\neq 0}} C_{H'} \cdot n^{2N-|H'|} p^{2M-e(H')}.$$

Vizing:

~~Choose~~ ~~Have~~

Want to color xy_1.

Choose maximal sequence of $y_1, \ldots, y_k$ as follow:

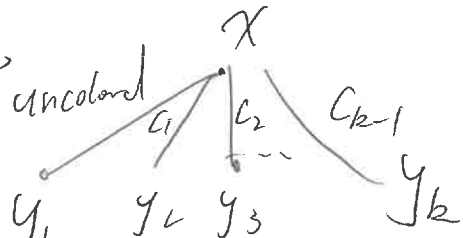
Given xy_i, choose c_i missing at y_i.

If $\exists$ new edge from x ~~to~~ with colour c_i,

Let xy_{i+1} be this edge.

Stop either c_k not used at x,

or $c_k = c_j$ some $j<k$.



① c_k not used at x: recolor by giving xy_i colour c_i for all $1 \le i \le k$.

② $c_k = c_j$ some $j<k$, wlog $j=1$,

recolor xy_i with c_i for $1 \le i \le j-1$, leaving xy_j as uncolored.

Let c be color not used at x.

If there is $c-c_1$ path from y_1 to x,

swap c & c_1 on all edges of $c-c_1$ component of y_1.

Then colour xy_1 colour c, done.

If no $c-c_1$ path from y_k to x,

swap c and c_1 on $c-c_1$ components of y_k,

color c for xy_k, c_i for xy_i, done.

O.W., $c-c_1$ component contain y_1 & y_k,

but H connected ~~or~~ and $\Delta(H) \le 2$ (2-edge-colored),

so path or cycle, but $d_H(x) = d_H(y_1) = d_H(y_k) = 1$

Extremal GT.

Eulerian: every edge once.

Euler's Thm: Connected graph has an Eulerian circuit iff every vertex has even degree.

($\Rightarrow$) ✓

($\Leftarrow$) Induction.

If $e(G) > \frac{n(k-1)}{2}$, then G path of len

~~If G has no path of length k, and $e(G) > \frac{n(k-1)}{2}$,~~

~~Then $e(G)$ has path of length $k-1$.~~

G unconnected, then induction,

$e(G) \le \sum \frac{n_i(k-1)}{2} = \frac{n}{2}(k-1)$.

~~G connected, if all vertices degree $\le \frac{1}{2}(k-1)$~~

~~$e(G) \le \frac{n}{2}(k-1)$.~~

~~Else,~~

If G connected, and G has not vertex of degree $\leq \frac{k-1}{2}$, then ~~G has a path~~ $\delta(G) \geq \frac{k}{2}$, so G has a path of length k. #.

~~If G~~ Else, say x. $e(G-x)$ ~~x~~ has degree $\sim$.

✓

GT. 2021. P1. SI.

17G. (a). E_t: G_n contains K_t.

~~$\mathbb{E}$[copies of K_t~~

Let $X = \#$ copies of K_t.

$$\mathbb{E}[X] = \binom{n}{t} \cdot p^{\frac{t(t-1)}{2}}$$

$$\leq n^t p^{\frac{t(t-1)}{2}}.$$

By Markov inequality, $\mathbb{P}(X \geq 1) \leq \mathbb{E}[X]$

$$\leq n^t p^{\frac{t(t-1)}{2}}$$

$$= \left(p \cdot n^{\frac{2}{t-1}}\right)^{\frac{t(t-1)}{2}}$$

$\to 0$ when $p \cdot n^{\frac{2}{t-1}} \to 0$.

So $\mathbb{P}(E_t) = \mathbb{P}(X \geq 1) \to 0$ as $n \to \infty$.

(b) Chebyshev's inequality: $\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \frac{Var(X)}{t^2}$.

~~$\mathbb{E}[E_3]$~~ $X = \#$ copies of K_3.

$$\mathbb{E}[X] = \binom{n}{3} p^3$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \sum_{H_1, H_2 \cong K_3} \mathbb{P}(H_1 K_3, H_2 K_3) - \mathbb{P}(H_1 K_3)\mathbb{P}(H_2 K_3)$$

$$= \sum_{H_1 \cong K_3} \mathbb{P}(H_1 \text{ is } K_3) - \mathbb{P}(H_1 \text{ is } K_3)^2$$

$$+ \sum_{\substack{H_1 \cap H_2 = \text{edge} \\ H_1, H_2 \cong K_3}} \mathbb{P}(H_1 K_3, H_2 K_3) - \mathbb{P}(H_1 K_3)^2.$$

$$= \binom{n}{3}p^3 - \binom{n}{3}p^6 + \binom{n}{5}\binom{5}{3}(p^5 - p^6)$$

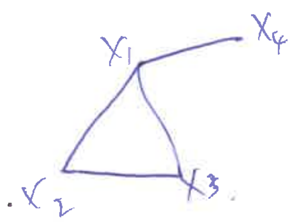
$$\cancel{+ \binom{n}{5}p^5 - \binom{n}{6}\binom{n}{5}p^5 - \binom{n}{3}^2 p^6}$$

$$\frac{Var(X)}{\mathbb{E}[X]^2}$$

$$\leq \frac{n^3p^3 + n^5p^5}{\left(\frac{1}{10}n^3p^3\right)^2}$$

$$= O\left(\frac{1}{n^3p^3}\right) + O\left(\frac{1}{np}\right) \to 0 \quad \text{as } pn \to \infty.$$

(c)



$X = \#$ copies of H.

$$\mathbb{E}[X] = \binom{n}{4} \cdot 4 \cdot p^4 \doteq \frac{1}{6}n^4p^4 \cancel{= O(n^4)}.$$

$$Var[X] = \sum_{H_1, H_2 \cong H} \mathbb{P}(H_1, H_2 \cong H) - \mathbb{P}(H_1 \cong H)^2.$$

$$\cancel{= \sum_{H_1 \cong H} \mathbb{P}(H_1 \cong H) - \mathbb{P}(H_1 \cong H)^2.}$$

$$\leq \sum_{\substack{H_1, H_2 \cong H \\ H_1 \cap H_2 = H' \\ H' \subseteq H}} \mathbb{P}(H_1, H_2) = \sum_{\substack{H' \\ e(H') \neq 0}} \sum_{\substack{H_1, H_2 \\ H_1 \cap H_2 \cong H'}} p^{8 - e(H')}$$

$$= \sum_{\substack{H' \subseteq H \\ e(H') \neq 0}} C_{H'} \cdot n^{8 - |H'|} p^{8 - e(H')}$$

So $$\frac{Var(X)}{(E[X])^2} \le \frac{\sum_{\substack{H' \subset H \\ e(H') \neq 0}} C_{H'}\, n^{8-|H'|} p^{8-e(H')}}{n^8 p^8}$$

$$= \sum_{\substack{H' \subset H \\ e(H') \neq 0}} \frac{1}{n^{|H'|} p^{e(H')}} \to 0 \quad \text{as } p = n^{-0.9} \text{ when } n \to \infty.$$

since $e(H') \le |H'|$ for all $H' \subset H$.

Graph Theory

Matching.

1. $\Delta(G) = \max \deg$
   $\delta(G) = \min \deg$.

2. Tree is a connected, acyclic graph.

   = maximal acyclic graph:

   ~~If add edge ⇒ cycle.~~

   ~~If not maximal,~~

   max acyclic ⇒ connected: ✓.

   connected $\overset{\text{acyclic}}{\Rightarrow}$ maximal: add edge → cyclic #.

   = min connected graph ✓.

3. Leaf

4. ✓

5. $e(T) = |T| - 1$: Induction on leaf

$G - v$

↑

leaf.

6. ✓.

1. Bipartice:

1.

3. Curcuit: An odd circuit contains odd cycle: ✓

$x_1 x_2 \cdots x_l$, $x_l = x_1$.

If $x_2, \ldots, x_{l-1}$ distinct ✓.

1 Else: $x_1 x_2 \cdots \underbrace{x_i \cdots x_j}_{=} \cdots x_l$

Then either: $x_i \cdots x_j$ odd ✓

Or: $x_1 x_2 \cdots x_i x_{j+1} \cdots x_l$ odd ✓.

2. G bipartite iff G has no odd cycles:

If G ~~has~~ bipartite: clearly no odd cycles.

If G has no odd cycles:

G has no odd circuits.

Let $x \in G$. ~~If deg~~ ~~$d(x,$~~

Let $y \to A$: $d(x,y)$ even

$y \to B$: $d(x,y)$ odd.

If $y_1, y_2 \in A$, $y_1 y_2 \in E$: #. as $d(y_1,x) = d(y_2,x) \pm 1$.

Similarly for $y_1, y_2 \in B$. ✓.

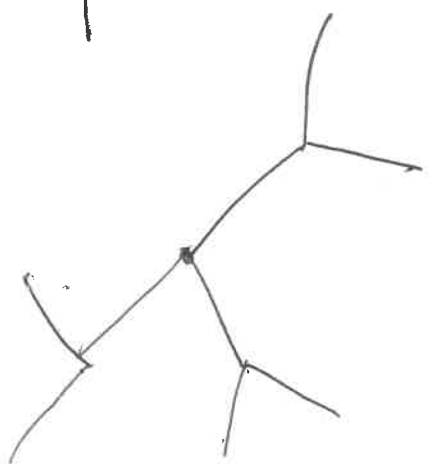
k-connected ⇒ ✓

GT 2021.

P2, SII. 17a.

(a) Acyclic.

(b)



$1+3+3\cdot2+3\cdot2\cdot2+\cdots=n.$

$3k < n$

$k < \log n$

GT Day 2. Planar.

Planar Graph:

o to a graph that can be drawn on $\mathbb{R}^2$ without edge crossing.

Euler's Formula:

$$V-E+F=2.$$

Proof: Induction,

Counting (f, e) pairs:

Each face ≥ 3 edges

Each edge 2 faces.

So $2E \geq 3F$.

$F \leq \frac{2}{3}E$.

$E-F=V-2 \geq \frac{E}{3}$.

$E \leq 3(|V|-2)$.

4. $K(3,3)$ Non-planar.

~

Colouring.

1. $\chi(G)$: Say G is k-colorable s.t. $c(x) \in [k]$ $\forall x \in G$ and $(xy \in E) \Rightarrow (c(x) \neq c(y))$.

Then $\chi(G) = \min_k (G \text{ is } k \text{ colorable})$.

2. $\chi(G) \leq \Delta(G) + 1$:

Greedy algorithm.

~~list~~ list vertices $v_1, v_2, \ldots, v_n$

~~Set~~ ~~$v_2 = m_1$~~

$c(v_i) = \min_k (k \neq c(y) \ \forall xy \in E$ and $y \in \{v_1, \ldots, v_{n-1}\})$

3. If $\delta(G) < \Delta(G)$, Then $\chi(G) \leq \Delta(G)$:

~~Order~~ Say $d(x) < \Delta(G)$.

Then order $v_1, \ldots, v_{n-1}$ s.t.

$d(v_1, x) \geq d(v_2, x) \geq \cdots \geq d(v_{n-1}, x)$

~~Then~~ Apply greedy:

~~$c(v_i) = \min_k (k \neq \{c(v_1), c(v_2), \ldots c(v$~~

$c(v_i) = \min_k (k \neq c(v_j)$ for any $v_j v_i \in E(G)$ and $j < i$.

~~For any~~

There are $\leq \Delta(G)-1$ such v_j for each step.

as on the shortest path from v_i to x,

there is some v_k s.t. $v_kv_i \in E$, and $d(v_k) < d(v_i)$.

so $k > i$.

~~Thus~~ [We take $d(v_n, x)=0$ by letting $x=v_n$]

In last step we color v_n, which has $\delta(G) < \Delta(G)$ neighbours.

We only need $\Delta(G)$ colors. So $\chi(G) \leq \Delta(G)$.

4. Brook's Thm:

Let G be a connected graph that;

1st case:

wlog assume G is k-regular.

O.W. can just using above thm.

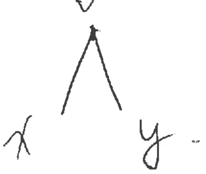
Since not complete, $\exists\, x \in V$ s.t. ~~$G-x$ is not complete~~

$G[N(x)]$ is not complete

Let $x, y \in N(v)$ s.t. $x \nsim y$.

~~Then~~ color x, y same color.

Consider $G-x-y$.



Order $(v_1, \dots, v_{n-3}, v_{n-2}=v)$ by $d(v_1,v) \geq d(v_2,v) \geq \dots$

Since ~~deg~~ $d(v) = k-2$.

and those connecting to x & y is degree k-1.

We can apply Greedy to colour them.

5. 6-colour Thm: for planar graph

Any planar graph is 6-colorable.

Induction. $|E| \le 3(|V|-2)$.

$$\frac{\sum_{i=1}^{n} d(x)}{n} = \frac{2E}{n} \le \frac{6(n-2)}{n} < 6.$$

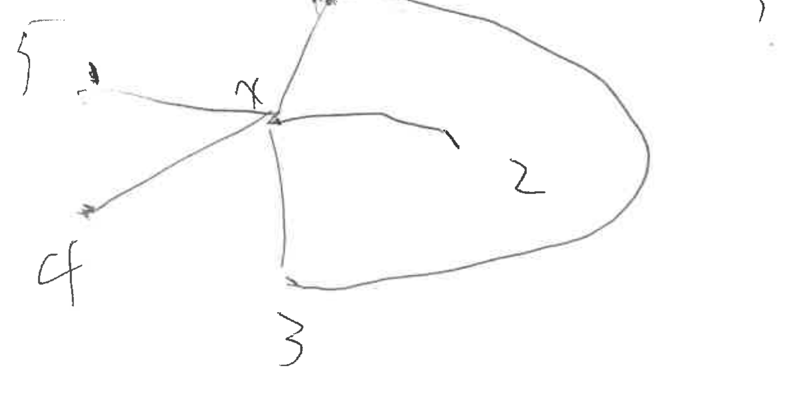
Thus exists x s.t. $d(x) \le 5$.

Do Induction on $G - x$ ✓.

6. 5-colour thmo for planar graph:

x be $d(x) \le 5$.

have ~~d(x)~~ 5-color of $G-x$.
by induction



$c(x_i) = i$

Consider $\{1,3\}$ - component contaiy x_1.

If x_3 not in, swap, ✓

If not, have path $1 \to 3$ ~~consider~~ similarly $2 \to 4$,

non-planar, #.

7. Heawood's Thm.

$$V - E + F \geq 2 - 2g.$$

$$\Rightarrow |E| \leq 3(|G| - (2-2g)).$$

$$\frac{1}{n}\left[\sum_{v \in V} d(v)\right] = \frac{2E}{n} \leq 6\left(1 - \frac{(2-2g)}{n}\right).$$

write as C.

$$= 6\left(1 - \frac{C}{n}\right)$$

Let G have $\chi(G) = k$.

Assume G has min number of edges s.t. $\chi(G) = k$.

$\delta(G) \geq k-1$, else we can remove a vertex and colour the graph with $k-1$ colours.

So $k-1 \leq \delta(G) \leq \frac{2(3(|G| - C))}{|G|} = 6 - 6\frac{C}{k}$.

$C \leq 0$, $k^2 - 7k + 6C \leq 0$.