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SFM 2017:

1.) (a) Let $(\Omega, F, \mathbb{P})$ be probability space: $(F_n)_{n\geq 0}$ be a filtration

If X_n is F_n - measurable, $X_n \in L^1(\mathbb{P})$ ~~and~~ $(\forall n \geq 0)$;

$(n \geq 1)$ $\mathbb{E}[X_n \mid F_{n-1}] = X_{n-1}$: $(X_n)_{n\geq 1}$ is a martingale.

(b) Integrable: ~~$\mathbb{E}[X_n$~~ $X_0 = \Delta_0 \in L^1(\mathbb{P})$

(Induct): If $X_0, \dots X_n$ integrable, $\mathbb{E}[|X_{n+1}|] \leq \mathbb{E}[|X_n|] + \sum_{k=0}^{n} \frac{1}{n+1}$ ~~$\mathbb{E}[X_k$~~ $\cdot \mathbb{E}[|X_k| \cdot |\Delta_{n+1}|]$

Since X_n is a function of $\{\Delta_0, \dots \Delta_n\}$, X_k, Δ_{n+1} independent for $k \leq n$

$\therefore \mathbb{E}[|X_{n+1}|] \leq \mathbb{E}[|X_n|] + \mathbb{E}[|\Delta_{n+1}|] / (n+1) \sum_{k=0}^{n} \mathbb{E}[|X_k|] < \infty$ (induction hyp.)

$\therefore X_{n+1} \in L^1 \Rightarrow$ Induction complete

X_n is clearly $\sigma(X_0, X_1, \dots X_n)$ - measurable.

$\mathbb{E}[X_{n+1} - X_n \mid F_n] = \sum_{k=0}^{n} \frac{1}{n+1} \mathbb{E}[\Delta_{n+1} \cdot X_k \mid F_n]$

$= \mathbb{E}[\Delta_{n+1}] \cdot f(X_0, \dots X_n)$ ($X_0, \dots X_n$ is F_n - measurable, Δ_{n+1} indep. of F_n)

$= 0.$

$\therefore$ Valid martingale.

(c) If $T \leq N$ A.S.: $(X_n)_{n\geq 1}$ is a martingale.

Optional stopping: $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$

Let $Y_n = X_{\min\{T, t\}}$:

$Y_n = \sum_{k=0}^{n-1} 1_{T=k} X_k + 1_{T\geq n} \cdot X_n$ is $\sigma(X_1, \dots X_n)$ measurable

$$\mathbb{E}[\,|Y_n|\,] \le \sum_{k=0}^{n} \mathbb{E}[\,|X_k|\,] < \infty$$

$$\mathbb{E}[\,Y_{n+1} \mid F_n\,] = \sum_{k=0}^{n} 1_{T=k}\, X_k + \mathbb{E}[\,X_{n+1}\, 1_{T>n} \mid F_n\,]$$

$$= \sum_{k=0}^{n} 1_{T=k} \cdot X_k + 1_{T>n}\, \mathbb{E}[\,X_{n+1} \mid F_n\,]$$

$$= \sum_{k=0}^{n-1} 1_{T=k} \cdot X_k + (1_{T\ge n} \cdot X_n) = Y_n.$$

$\therefore$ Valid martingale.

Since $T \le N$ A.S. $Y_N = X_T$ (A.S.)

$$\therefore \mathbb{E}[\,X_T\,] = \mathbb{E}[\,Y_N\,] = \mathbb{E}[\,Y_0\,] \quad \text{(martingale)}$$

$$= \mathbb{E}[\,X_0\,]$$

2.) (a) Let $(\Omega, F, \mathbb{P})$ be a probability space:

$(B_t)_{t\ge 0}$ is a Brownian Motion (BM) if:

$$\left.\begin{array}{l} B_0 = 0 \\ t \to B_t \text{ is cont.} \end{array}\right\} \text{A.S.}$$

· $(B_t)_{t\ge 0}$ has independent increments, $\mathbb{P}$ $(B_t - B_s) \sim N(0, t-s)$ for $t \ge s \ge 0$

(b) $\tilde{B}_0 = \frac{1}{c} \cdot B_0 = 0$ iff $B_0 = 0$

$t \to \tilde{B}_t = \frac{1}{c} B_{c^2 t}$ is cont. iff $t \to B_t$ is cont.

Since $(B_t)_{t\ge 0}$ has independent increments, $(B_{c^2 t})_{t \ge 0}$ increments must be independent $\Rightarrow$ $\tilde{B}_t$ has independent increments.

$$(\tilde{B}_t - \tilde{B}_s) = \frac{1}{c} (B_{c^2 t} - B_{c^2 s}) \sim \frac{1}{c} \cdot N(0, c^2(t-s)) = N(0, t-s)$$

(c) $c\cdot Z = \sup_{t\geq 0} c\cdot B_t = \sup_{t\geq 0} c\cdot B_{t/c^2}$

Since $(B_t)_{t\geq 0}$, $(c\cdot B_{t/c^2})_{t\geq 0}$ have the same distribution,
$c\cdot Z \overset{(d)}{=} Z$

$\therefore \mathbb{P}(Z\leq a) = \mathbb{P}(c\cdot Z\leq a) = \mathbb{P}(Z\leq a/c)$

$\therefore \mathbb{P}(Z\leq a)$ is constant on $a\in(0,\infty)$.

Since $Z\in\{0,\infty\}\cup(0,\infty)$: $\mathbb{P}(Z\in(a,b]) = \mathbb{P}(Z\leq b) - \mathbb{P}(Z\leq a) = 0$
for $a, b > 0$.

$\therefore \mathbb{P}(Z\in(0,\infty)) = \mathbb{P}\left(\bigcup_{n\geq 1}\{Z\in(1/n, n]\}\right) \leq \sum_{n\geq 1}\mathbb{P}(Z\in(1/n, n]) = 0$
$= 0$

$\therefore Z\in\{0,\infty\}$ A.S.

(d) If $Z = 0$: Let $Z_1 = \sup\{B_{t+1} - B_1 : t\geq 0\}$.

$Z_1 \overset{(d)}{=} Z$; $Z = 0 \Rightarrow B_1 \leq 0$ and $B_{t+1} \leq 0$ $(t\geq 0)$
$\Rightarrow B_1\leq 0$ and $Z_1 \leq B_1 < \infty$
$\Rightarrow B_1 \leq 0$, $Z_1 \leq 0$.

$(B_{t+1} - B_1)_{t\geq 0}$ indep.

$\therefore \mathbb{P}(Z=0) \leq \mathbb{P}(B_1\leq 0)\cdot \mathbb{P}\left(\sup_{t\geq 0}\{B_{t+1} - B_1\} \leq 0\}\right)$ independent of B_1

$= 1/2 \cdot \mathbb{P}(Z=0)$

$\therefore \mathbb{P}(Z=0) = 0 \Rightarrow Z = \infty$ A.S.

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3.) (a) Let S_t^k be price of asset k at time t, Fix $T > 0$.

Asset 0 be a riskless asset.

~~be~~

There is no arbitrage portfolios (*) iff $\exists$ equivalent martingale measure of $\left(S_t^k / S_t^0 \right)_{0 \le t \le T}$

(*) $(\Theta_k)_{k=0,\dots T}$ is an arbitrage if it is a self-financing pre-visible strategy and $\Theta_0 \cdot S_0 \le 0$, $\Theta_T \cdot S_T \ge 0$ (A.S.)
$\mathbb{P}(\Theta_T \cdot S_N > 0) > 0$

(b) $(S_T - K)^+ - (K - S_T)^+ = S_T - K$

$\therefore$ ~~market, T-t~~ $EC(S_0, K, T)$ be price of european call with strike K, expiry T, current price S_0.

$\therefore EC(S_t^1, K, T-t_0) - EP(S_t^1, K, T-t_0) = S_t^1 - K(1+r)^{-(T-t_0)}$

$\therefore$ ~~EC~~ Pick call option iff $S_t^1 \ge K \cdot (1+r)^{-(T-t_0)}$

$\therefore$ Payout $= (S_T^1 - K)^+ \mathbb{1}_{S_{t_0}^1 > K(1+r)^{t_0 - T}} + (K - S_T^1)^+ \mathbb{1}_{S_{t_0}^1 \le K(1+r)^{t_0-T}}$

(c) $C(K, t_0, T) - \mathbb{E}(S_T^1 - K)^+ = \mathbb{1}_{S_{t_0}^1 \le K(1+r)^{t_0-T}} \left\{ (K - S_T^1)^+ - (S_T^1 - K)^+ \right\}$

$= \mathbb{1}_{S_{t_0}^1 \le K(1+r)^{t_0-T}} \left\{ K - S_T^1 \right\}$

$= (K(1+r)^{t_0-T} - S_{t_0}^1)^+ + \underbrace{\mathbb{1}_{S_{t_0}^1 \le K(1+r)^{t_0-T}} \left\{ S_{t_0}^1 - S_T^1 \right\}}_{\text{Hed}}$

$= (K(1+r)^{t_0-T} - S_{t_0}^1)^+ + \mathbb{1}_{S_{t_0}^1 \le K(1+r)^{t_0-T}} \{ K-$

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$$\cancel{C(K, t_0, T) = (S_T' - K)^+ + 1_{S_{t_0}' \le (1+r)^{t_0-T}\cdot K} \{(K - S_T')^+ - (S_T' - K)^+\}}$$

$$= (S_T' - K)^+ + ((1+r)^{t_0-T} K - S_{t_0}')^+$$

$$\cancel{+ 1_{S_{t_0}' \le K\cdot(1+r)^{t_0-T}} \{K - S_T + S_{t_0}' - (1+r)^{t_0-T}\cdot K\}}$$

$$\cancel{\mathbb{E}^G[\text{line } 2 \mid F_{t_0}] = 1_{S_{t_0}' \le K(1+r)^{t_0-T}} \{K - (1+r)^{t_0-T} K + S_{t_0}' - (1+r)^{T-t_0} S_{t_0}'\}}$$

$$= 1_{S_{t_0}' \le K(1+r)^{t_0-T}} (1 - (1+r)^{t_0-T}) \{K\cancel{a+r} - (1+r)^{T-t_0} S_{t_0}'\}$$

$$\cancel{\text{TVEC}}\ \cancel{(1+r)^{-T}\cdot \mathbb{E}^G[1_{S_{t_0}' \le (1+r)^{t_0-T}\cdot K}\{(K - S_T')^+\} \mid F_{t_0}] = (*)}$$

$$\cancel{= (1+r)^{-T} 1_{S_{t_0}' \le (1+r)^{t_0-T} K}\, \mathbb{E}^G[(K - S_T')^+ \mid S_{t_0}]}$$

Payout ~~arbitrage~~ $= EC(S_t', K, T - t_0) + \cancel{\text{Max}}\ (K(1+r)^{-(T-t_0)} - S_t')^+$

$$= EC(S_0, K, T\cancel{-t}) + EP(S_0, K(1+r)^{t_0-T}, t_0)$$

$$\therefore \Pi = \Pi(EC(K, T)) + \Pi(EP(K(1+r)^{t_0-T}, t_0))$$

4\) (a) $S_t^0 = (1+r)^t$, $S_t' = S_0' \cdot \prod_{k=1}^{t}(1+R_k)$, where $(R_k)_{k\ge1}$ are iid,

$\mathbb{P}(R_k = a) = p$, $\mathbb{P}(R_k = b) = q$, $-1 < a < b$. (CRR model)

Arbitrage free iff $a < r < b$.

G: $ap^* + b(1-p^*) = r \Rightarrow (a-b)p^* = r - b \Rightarrow p^* = \frac{b-r}{b-a}$

$G(R_k = a) = p^*$ is the martingale measure (unique)

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(b) Assuming no orbitrage, market has unique martingale measure, ($\mathbb{Q}$) and all claims replicable

- Let $(\Theta_n)_{n=0,\dots T}$ be replicating strategy of C:

$$C = \Theta_{T} \cdot S_T \quad (A.S.) \quad = (1+r)^T \; \Theta_T \cdot X_T$$

$$\therefore \mathbb{E}^{\mathbb{Q}}[C] = (1+r)^T \cdot (\Theta_0 \cdot S_0) \Rightarrow \Pi = (1+r)^{-T} \mathbb{E}^{\mathbb{Q}}[C]$$

is the fair price.

$$V(n; S_0, \dots S_n) = \Theta_n (S_0, \dots S_n) \cdot S_n = (1+r)^{n-N} \mathbb{E}^{\mathbb{Q}}[C \mid \mathcal{F}_n](S_0, \dots S_n)$$

Hedge : $$\Theta_{n+1} = \frac{V(n+1; S_0, \dots S_n, S_n(1+b)) - V(n+1; S_0, \dots S_n, S_n(1+a))}{(b-a) S_n}$$

(c) $(1+b)^{T-t} \cdot \left(\frac{1+a}{1+b}\right)^n > K$

Let $N = \sup \left\{ n \geq 0 : (1+b)^{T-t} \cdot \left(\frac{1+a}{1+b}\right)^n > K \right\}$

$\therefore$ We can take at most N down-step after t.

$$\mathbb{E}^{\mathbb{Q}}[C] = \sum_{k=0}^{N} \binom{T-t}{k} p^{*k} (1-p^*)^{T-t-k} \left\{ (1+b)^{T-t-k} (1+a)^k - K \right\}$$

$$\therefore \text{price} = (1+r)^{-T} \sum_{k=0}^{N} \binom{T-t}{k} p^{*k} (1-p^*)^{T-t-k} \left\{ (1+ab)^{T-t-k} (1+a)^k - K \right\}$$

$$(N = \left\lfloor \log\left(K (1+b)^{t-T}\right) / \log\left(\tfrac{1+a}{1+b}\right) \right\rfloor)$$

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SFM 2018

1.) (a) ~~Let~~ $\forall n \geq 0$: M_n is F_n - measurable, $M_n \in L^1(\mathbb{P})$ and

$$\mathbb{E}[M_n \mid F_{n-1}] = M_{n-1}$$

(b) $$\mathbb{E}[|M_n|] = \mathbb{E}\left[\prod_{k=1}^{n} |Y_k|\right] = \prod_{k=1}^{n} \mathbb{E}[|Y_k|] = 1 < \infty.$$

M_n is $\sigma(Y_1, \dots Y_n)$ ~~G~~ $= F_n$ - measurable.

$$\mathbb{E}[M_n \mid F_{n-1}] = \prod_{k=1}^{n-1} Y_k \cdot \underbrace{\mathbb{E}[Y_n \mid F_{n-1}]}_{= \mathbb{E}[Y_n] \text{ as } Y_n \text{ indep. of } F_{n-1}} = M_{n-1}$$

(c) Let $A_0 = 0$, $A_{n+1} - A_n = \mathbb{E}[X_{n+1} - X_n \mid F_n]$ (F_n measurable)

~~$\therefore \mathbb{E}[A_{n+1}] = \sum_{k=1}^{n} \mathbb{E}[X_{k+1} - X_k] = \mathbb{E}[X_{n+1}] - X_0$~~

$\therefore A_{n+1} - A_n$ is F_n - measurable.

(Induct): A_0 is ~~F_0 measurable constant~~ constant

~~A_k~~ is $A_1 - A_0$ is F_0 measurable $\Rightarrow A_1$ is F_0 measurable

If A_n is F_{n-1} measurable: $A_{n+1} = A_{n+1} - A_n + A_n$ is F_n - measurable

Let $M_n = X_n - A_n$: $\mathbb{E}[|M_n|] \leq \mathbb{E}[|X_n|] + \mathbb{E}[|A_n|]$

$\mathbb{E}[|X_n|] < \infty$ (by def.)

$$\mathbb{E}[|A_n|] \leq \sum_{k=1}^{n} \mathbb{E}[\mathbb{E}[|X_k - X_{k-1}| \mid F_{k-1}]]$$

$$= \sum_{k=1}^{n} \mathbb{E}[|X_k - X_{k-1}|] < \infty$$

$\therefore M_n \in L^1$

M_n is clearly F_n - measurable

$$\mathbb{E}[M_{n+1} - M_n \mid F_n] = \mathbb{E}[X_{n+1} - X_n \mid F_n] - (A_{n+1} - A_n) = 0$$

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$\therefore (M_n)_{n\geq 0}$ is a martingale.

(Unique): ~~If~~ Since $A_0 = 0$, $A_{n+1} - A_n = \mathbb{E}[X_{n+1} - X_n \mid F_n]$,

$(A_n)_{n\geq 0}$ is uniquely defined.

$\therefore M_n = X_n - A_n$ is uniquely defined.

* $X_n = M_n + A_n = M_n' + A_n'$, $A_0 = A_0' = 0$.

$M_n - M_n' = A_n' - A_n = Y_n$, ~~$(A_n)_{n}$ is a martingale.~~

$\mathbb{E}[Y_n \mid F_{n-1}] = Y_{n-1}$ (martingale property of $M_n - M_n'$)

$= Y_n$ ($A_n' - A_n$ is F_{n-1} measurable)

$\therefore Y_n = Y_0$ (A.S) $= 0 \Rightarrow$ Unique (A.S.) *

(Question badly phrased!!!)

(d) If $(X_n)_{n\geq 0}$ is a martingale:

$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n \mid F_{n-1}]]$ (Tower)

$= \mathbb{E}[X_{n-1}]$

$\therefore \mathbb{E}[X_0] = \mathbb{E}[X_1] = \dots = \mathbb{E}[X_n]$

(ii) If $\mathbb{E}[X_n] = \mathbb{E}[X_0]$

Let $X_n = M_n + A_n$: $\therefore$ ~~$\mathbb{E}[A_n] = \mathbb{E}[A_0]$~~ $= 0$.

$\mathbb{E}[A_n] = \mathbb{E}[X_n] - \mathbb{E}[M_n] = \mathbb{E}[X_0] - \mathbb{E}[M_0] = 0$.

But $(X_n)_{n\geq 0}$ supermartingale $\Rightarrow \mathbb{E}[X_{n+1} - X_n \mid F_n] \leq 0$ A.S.

$\therefore A_{n+1} - A_n \leq 0$ A.S.

But $\mathbb{E}[A_{n+1} - A_n] = 0 \Rightarrow A_{n+1} = A_n$ (A.S.)

$\therefore A_0 = \dots = A_n$ A.S. $(=0) \Rightarrow X_n = M_n \Rightarrow$ Martingale

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2.) (a) $S_t = S_0 e^{\sigma B_t + \mu t}$, $(B_t)_{t\geq 0}$ is a $\mathbb{P}$- brownian motion.

$\therefore X_t = S_0 e^{\sigma B_t + (\mu - r)t}$ is discounted process.

(wrt $\mathbb{G}$: $d\mathbb{G}/d\mathbb{P} = e^{cB_T - c^2T/2}$) $(B_t)_{t\geq 0}$ is a c- drift B.M.

(Cameron Martin)

$\therefore X_t = S_0 e^{\sigma W_t + \sigma c t + (\mu - r)t}$

Pick c s.t. $(\sigma c + \mu - r) = -\sigma^2/2 \Rightarrow c = (-\sigma^2/2 + r - \mu)/\sigma$

$\therefore X_t = S_0 e^{\sigma W_t - \sigma^2 t/2}$; W_t is a $\mathbb{G}$ - B.M.

$\mathbb{G}$ is the equivalent martingale measure

(b) Price: $\mathbb{E}^{\mathbb{G}}[e^{-rT} f(S_T)]$ $\begin{cases} S_T = S_0 e^{\sigma W_T + (r - \sigma^2/2)T}, \\ W_T \sim N(0, T) \end{cases}$

$\therefore \cancel{\mathbb{E}^{\mathbb{G}}}$ $\Pi = e^{-rT} \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi y}} \cancel{f(S_0} f(S_0 e^{\sigma\sqrt{T} y + (r - \sigma^2/2)T}) dy.$

(c) $\Pi_c = e^{-rT} \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi y}} \cdot S_0^{\gamma} e^{\gamma\sigma\sqrt{T}\cdot y} e^{\gamma(r - \sigma^2/2)T} dy$

$= S_0^{\gamma} e^{[(\gamma - 1)r - \gamma\sigma^2/2]T} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \gamma\sigma\sqrt{T})^2} e^{\gamma^2\sigma^2 T} dy$

$= S_0^{\gamma} e^{-rT + \gamma(r - \sigma^2/2)T + \gamma^2\sigma^2 T}$

(d) At time t: Replace $S_0 \to S_t$, $T \to T - t$

$\therefore \cancel{G}\; S_t^{\gamma} e^{-r(T-t) + \gamma(r - \sigma^2/2)(T-t) + \gamma^2\sigma^2(T-t)} = V(t)$

$\Theta_t =$

Hedge: $\frac{\partial V}{\partial S_t} = \gamma S_t^{\gamma - 1} e^{-r(T-t) + \gamma(r - \sigma^2/2)(T-t) + \gamma^2\sigma^2(T-t)}$ units of stock

; replicates strategy.

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- Hold $-\theta_t$ at time t to hedge.

4) (a) $$\mathbb{E}[U'(X)(X-\mu)] = \int_{\mathbb{R}} U'(x) \frac{e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}{\sqrt{2\pi\sigma^2}} (x-\mu)\, dx$$

$$= \underbrace{\left[-U'(x) \frac{e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}{\sqrt{2\pi\sigma^2}} \cdot \sigma^2\right]_{\mathbb{R}}}_{\to 0 \text{ as } U' \text{ has polynomial growth}} + \sigma^2 \int_{\mathbb{R}} U''(x) \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx$$

$$= \mathbb{E}[U''(X)] \cdot \sigma^2$$

(b) $X = \mu + \sigma \cdot Z$, $Z \sim N(0,1)$.

$$\frac{\partial f}{\partial \mu} = \mathbb{E}\left[\frac{\partial U(\mu + \sigma Z)}{\partial \mu}\right] = \mathbb{E}[U'(\mu + \sigma Z)] \geq 0.$$

Since $U' \geq 0$: Integral ≥ 0 as well $\Rightarrow \partial f / \partial \mu \geq 0$.

$$\frac{\partial f}{\partial \sigma} = \mathbb{E}[\underbrace{U'(\mu + \sigma Z)}_{V'(Z)} \cdot Z] = \mathbb{E}[V''(Z)] = \sigma\, \mathbb{E}[U''(\mu + \sigma Z)]$$

~~$V(Z) = U'(\mu+\sigma Z)$~~

$V'(Z)$, $V(Z) = \frac{1}{\sigma} U(\mu + \sigma Z)$

≤ 0 as U is concave.

~~If X maximises expected utility, X must have the largest mean of all~~

~~If $X(\mu_0, \sigma_0^2)$ maximises $\mathbb{E}[U(X)]$: μ_0 For all normal~~

f is fully determined by μ, σ^2: For fixed μ_A, ~~σ_A, σ^2~~ we can maximise $\mathbb{E}[U(X)]$ by minimising σ^2 (mean - var efficient);

Also for fixed σ^2, maximise $\mathbb{E}[U(X)]$ by maximising μ.

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(c) $\dfrac{\partial^2 f}{\partial \mu^2} = \mathbb{E}[U''(\mu + \sigma Z)] \leq 0.$

$\dfrac{\partial^2 f}{\partial \sigma^2} = \sigma^2 \, \mathbb{E}[U''(\mu + \sigma Z) \cdot Z^2] \leq 0.$

$\dfrac{\partial^2 f}{\partial \sigma \cdot \partial \mu} = \cancel{\sigma \, \mathbb{E}[U''(\mu+\sigma Z)]} \; \sigma \, \mathbb{E}[U''(\mu + \sigma Z) \cdot Z]$

$\cancel{|\nabla^2 f|}$ $\det(\nabla^2 f) = \sigma^2 \, \mathbb{E}[U''(\mu + \sigma Z)] \cdot \mathbb{E}[U''(\mu + \sigma Z) \cdot Z^2]$
$- \sigma^2 \, \mathbb{E}[U''(\mu + \sigma Z) \cdot Z]^2$ — (1)

~~Cauchy~~ Let $\alpha = \mathbb{E}[-U''(\mu + \sigma Z)]$.

Note : $-U''(\mu + \sigma Z) \geq 0 \Rightarrow \dfrac{dG}{dP} = -U''(\mu + \sigma Z) \geq 0.$

$\therefore$ ~~$\alpha \sigma^2$~~ $(1)/\sigma^2 = (\mathbb{E}^G[1] \cdot \mathbb{E}^G[Z^2] - \mathbb{E}[Z]^2) \cdot \alpha^2.$

$\therefore (1)/\sigma^2 \cdot \alpha^2 = \mathbb{E}^G[Z^2] - \mathbb{E}[Z]^2 \geq 0$ (Jensen's)

$\therefore (1) \geq 0.$

Hint: Negative semi definite 2^{nd} derivative.

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4.) (a) $(\Theta_n)_{n=0,\dots T}$ is an arbitrage if $\Theta_T \cdot \overset{X_T}{\cancel{S_T}} - \Theta_0 \cdot S_0 \geq 0$ (A.S)

$\mathbb{P}(\Theta_T \cdot \overset{X_T}{\cancel{S_T}} - \Theta_0 \cdot S_0 > 0) > 0.$

(b) FTAP 1: There is no arbitrage opportunity iff discounted process has an equivalent martingale measure

(c) If $\mathbb{G}$ is a martingale measure:

$$V_n = \Theta_n \cdot S_n = \Theta_0 \cdot S_0 + \sum_{k=1}^{n} \Theta_n (S_n - S_{n-1}) \quad \text{(self financing)}$$

$$\therefore \mathbb{E}^{\mathbb{G}}[|V_n|] \leq |C| \cdot (|S_0| + \sum_{k=1}^{n} \mathbb{E}^{\mathbb{G}}[|S_n| + |S_{n-1}|]) < \infty$$

Since Θ_n is F_{n-1} measurable, $V_n = \Theta_n S_n$ ⇒ F_n - measurable.

$$\mathbb{E}^{\mathbb{G}}[V_n - V_{n-1} | F_{n-1}] = \mathbb{E}^{\mathbb{G}}[\Theta_n \cdot (S_n - S_{n-1}) | F_{n-1}]$$
$$= \Theta_n \cdot \mathbb{E}^{\mathbb{G}}[S_n - S_{n-1} | F_{n-1}] = 0$$

$\therefore (V_n)_{n\geq 0}$ is a $\mathbb{G}$ - martingale

If $(V_n)_{n\geq 0}$ is a $\mathbb{G}$ - martingale: $\mathbb{E}[V_T] = \mathbb{E}[\mathbb{E}[V_T | F_{T-1}]] = \mathbb{E}[V_{T-1}]$

$\therefore \mathbb{E}[V_0] = \dots = \mathbb{E}[V_T]$

Claim: (iii) ⇒ (i)

WTP: $\mathbb{E}[\cancel{S} X_n | F_{n-1}] = \cancel{\mathbb{E}[} X_{n-1}$.

Since X_{n-1} is F_{n-1} measurable, suffice to show:

$\forall A \in F_{n-1}$: $\mathbb{E}[X_n \cdot 1_A] = \mathbb{E}[X_{n-1} 1_A]$

Consider the following strategy: (for $\cancel{S_k}$ asset k, $k \in \{1, \dots d\}$)

(k is fixed):

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Stochastic Financial Models 2019

1.) (a) $(M_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale if:

$\forall n \geq 0$: $M_n \in L^1(\mathbb{P})$, M_n is $\mathcal{F}_n$ measurable

(3/3) $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$

(b) $|X_n| \leq \text{Max}\{1, n \cdot |X_{n-1}|\}$

$\therefore \mathbb{E}[|X_n|] \leq 1 + n\,\mathbb{E}[|X_{n-1}|]$ (*)

Since $X_0 = 0 \Rightarrow X_0 \in L^1$: If $X_{k-1} \in L^1$, $\Rightarrow X_k \in L^1$ (from *)

$\therefore$ (Induct): $X_n \in L^1$ $(\forall n \geq 0)$

(3/3) ~~Xn is a function of~~ X_n is $\sigma(X_1, \dots X_n)$ measurable.

$\mathbb{E}[X_n | \mathcal{F}_{n-1}] = \mathbb{E}[X_n | X_0, \dots X_{n-1}] = \mathbb{E}[X_n | X_{n-1}]$

If $X_{n-1} = 0$: $\mathbb{E}[X_n | \mathcal{F}_{n-1}] = \frac{1}{2n}(1 - 1) = 0 = X_{n-1}$

If $X_{n-1} \neq 0$: $\mathbb{E}[X_n | \mathcal{F}_{n-1}] = \frac{1}{n}(n \cdot X_{n-1}) + 0 = X_{n-1}$

$\Big\}$ $\mathbb{E}[X_n | \mathcal{F}_{n-1}] = X_{n-1}$

(4/4) $\therefore$ Valid martingale

(c) (i) $\Rightarrow$ (ii)

$M_n^T = \sum_{k=0}^{n-1} 1_{T=k} \cdot M_k + 1_{T\geq n} M_n$ $\Big|$ $1_{T=k}$ is $\mathcal{F}_k$ measurable, $1_{T\geq n}$ is $\mathcal{F}_{n-1}$ measurable.

$\therefore$ As M_n^T is a sum/product of $\mathcal{F}_n$-measurable R.V. $\Rightarrow$ M_n^T is $\mathcal{F}_n$-measurable.

$\mathbb{E}[|M_n^T|] \leq \sum_{k=0}^{n-1} \mathbb{E}[|M_k|] + \mathbb{E}[|M_n|] < \infty$ $(M_n \in L^1)$

$\mathbb{E}[M_n^T | \mathcal{F}_{n-1}] = \underbrace{\sum_{k=0}^{n-1} 1_{T=k} M_k}_{\mathcal{F}_{n-1}\text{-measurable}} + \underbrace{1_{T\geq n}} \mathbb{E}[M_n | \mathcal{F}_{n-1}]$

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$= \sum_{k=0}^{n-1} 1_{\tau=k} M_k + 1_{\tau>n-1} \cdot M_{n-1} = \sum_{k=0}^{n-2} 1_{\tau=k} M_k + 1_{\tau \geq n-1} M_{n-1}$

$= M_{n-1}^{\tau}$

(4/4)

$\therefore$ Martingale.

(ii) $\Rightarrow$ (iii) : $\mathbb{E}[M_n^2] = \mathbb{E}[\mathbb{E}[M_n^2 | \mathcal{F}_{n-1}]]$ (Tower) $= \mathbb{E}[M_{n-1}^2]$.

(2/2)

$\therefore$ Iterating : $\mathbb{E}[M_n^2] = \cancel{M_1^2} \cdot \mathbb{E}[M_0^2] = \mathbb{E}[M_0]$

(iii) $\Rightarrow$ (i) Fix $A \in \mathcal{F}_{n}$; $\tau = n 1_{\omega A} + (n+1) 1_{A^c}$

(Stopping time) : If $k < n$, $\{\tau \leq k\} = \cancel{\Omega}\ \phi \in \mathcal{F}_k$.

$\{\tau \leq n\} = A \in \mathcal{F}_n$

$\{\tau \leq k\}$ ~~for k ≥ n+1~~ $= \Omega \in \mathcal{F}_k$ for $k \geq n+1$

$\therefore \mathbb{E}[M_{n+1}^{\tau}] = \mathbb{E}[M_n 1_A + M_{n+1} 1_{A^c}]$

$\mathbb{E}[M_n^2] = \mathbb{E}[M_n]$

$\mathbb{E}[(M_{n+1} - M_n) 1_{A^c}] = \mathbb{E}[M_{n+1}^2 - M_n^2] = 0.$

Since this is true $\forall A \in \mathcal{F}_n$: $\forall B \in \mathcal{F}_n$, $\mathbb{E}[M_{n+1} 1_B] = \mathbb{E}[M_n 1_B]$.

$\mathbb{E}$ ~~[M_{n+1} |~~ $\mathcal{F}_n$ M_n is $\mathcal{F}_n$ measurable.

$\therefore$ (Uniqueness of conditional expectation): $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$

~~Since $\mathbb{E}[|M_n|] \leq \sum_{k=0}^{n} \mathbb{E}[|M_k|] < \infty$, $(M_n$~~

M_n integrable is given $\Rightarrow$ $(M_n)_{n \geq 0}$ is a martingale wrt $(\mathcal{F}_n)_{n \geq 0}$

(4/4)

(20/20)

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2.) (a) Let $(X_n)_{n\geq 0}$ be the discounted price process:

$\mathbb{G}$ is an equivalent martingale measure if: $\mathbb{P}$, $\mathbb{G}$ have the same null sets and

(3/3) $(X_n)_{n\geq 0}$ is a $\mathbb{G}$-martingale.

(b) FTAP: $\exists$ equivalent martingale measure iff there is no arbitrage opportunity.

(2/2)

(c) We want $\tilde{\sigma}$, $\tilde{m}$ s.t.:

$$\mathbb{E}\left[e^{\tilde{\sigma}Z+\tilde{m}} \cdot e^{\sigma Z+m}\right] = \mathbb{E}\left[e^{\tilde{\sigma}Z+\tilde{m}}\right]$$

$$LHS = e^{(m+\tilde{m})+\frac{1}{2}(\sigma+\tilde{\sigma})^2}, \quad RHS = e^{\tilde{m}+\frac{1}{2}\tilde{\sigma}^2}$$

Pick: $m+\frac{1}{2}(2\sigma\cdot\tilde{\sigma})+\frac{1}{2}\sigma^2 = 0$

$$\therefore \sigma\cdot\tilde{\sigma} = -m - \sigma^2/2 \Rightarrow \tilde{\sigma} = -m/\sigma - \sigma/2.$$

Let $\tilde{m}=0$, $\tilde{\sigma} = -m/\sigma - \sigma/2$:

$$d\mathbb{G}/d\mathbb{P} = e^{\tilde{\sigma}Z} \Big/ \mathbb{E}[e^{\tilde{\sigma}Z}]$$ is an equivalent measure

$\mathbb{E}^{\mathbb{G}}[S_1] = 1 = S_0 \Rightarrow$ Equivalent martingale measure

(9/9) $\therefore$ FTAP $\Rightarrow$ No arbitrage.

(d) Let $\tilde{\sigma}_k = -m_k/\sigma_k - \sigma_k/2$

$$d\mathbb{G}/d\mathbb{P} = \prod_{k=1}^{T} e^{\tilde{\sigma}_k Z_k} \Big/ \mathbb{E}[e^{\tilde{\sigma}_k Z}]$$: $\mathbb{Q}$, $\mathbb{P}$ are equivalent

say it is a measure as Z_k -ind ...

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$$\mathbb{E}^{G}[S'_{t+1} \mid F_t] = S'_t \cdot \mathbb{E}^{G}[e^{G_{t+1} Z_{t+1} + M_{t+1}} \mid F_t] \quad -(1)$$

~~$= S'_t$. ~~ Since changing measures preserve measurable sets, Z_{t+1} remains independent of F_t.~~ (*) Z_{t+1} independent of F_t.

$$\therefore (1) = S'_t \cdot \mathbb{E}^{G}[e^{G_{t+1} Z_{t+1} + M_{t+1}}]$$

$$= S'_t \cdot \mathbb{E}\Big[\prod_{\substack{k=1,\\ k\neq t+1}}^{T} e^{\tilde{G}_k Z_k} / \mathbb{E}[e^{\tilde{G}_k Z_k}] \cdot \cancel{e^{\tilde{G}_{t+1} Z + G_{t+1} Z + M_{t+1}} / \mathbb{E}[e^{\tilde{G}_{t+1} Z_{t+1}}]}$$

$$\cdot\, e^{\tilde{G}_{t+1} Z_{t+1} + G_{t+1} Z_{t+1} + M_{t+1}} / \mathbb{E}[e^{\tilde{G}_{t+1} Z_{t+1}}]\Big]$$

$$= S'_t \quad (\text{independence of } Z_k)$$

(*) : Z_{t+1} & $(Z_n)_{n\geq 1}^{T}$ is G - independent.

$$\mathbb{P}^{G}(Z_1 \in A_1, \ldots Z_{n_T} \in A_{n_T}) = \mathbb{E}\Big[\prod_{k=1}^{n_T} \frac{1_{Z_k \in A_k} \cdot e^{\tilde{G}_k Z_k}}{\mathbb{E}[e^{\tilde{G}_k Z_k}]}\Big]$$

$$= \prod_{k=1}^{n_T} \Big(\mathbb{E}\Big[\frac{1_{Z_k \in A_k}}{\mathbb{E}[e^{\tilde{G}_k Z_k}]} \cdot e^{\tilde{G}_k \tilde{Z}_k}\Big] \cdot \cancel{\prod_{j=1}^{n}} \cancel{\prod_{\substack{j=1,\\ j\neq k}}^{T} \mathbb{E}[e^{\tilde{G}_k Z_k}}$$

$$\cdot \prod_{j\neq k} \mathbb{E}\big[e^{\tilde{G}_{kj} Z_j} / \mathbb{E}[e^{\tilde{G}_j Z_j}]\big]\Big)$$

$$= \prod_{k=1}^{T} \mathbb{E}^{G}[1_{Z_k \in A_k}] \qquad \therefore \text{Independent}$$

Why is S'_t integrable?

$\therefore$ Q is an equivalent martingale measure on (discounted) price process

FTAP $\Rightarrow$ No arbitrage.

(5/6)

(19/2)

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$$e^{-y^2/2}/\sqrt{2\pi}$$

3.) (a) $\Pi(EC) = e^{-rT} \int_{\mathbb{R}} S_0 e^{(r-\sigma^2/2)T} \left(e^{\sigma\sqrt{T}y} - K/S_0 \, e^{-(r-\sigma^2/2)T} \right)^+ dy$

$$= S_0 e^{-\sigma^2 T/2} \int_{1/\sigma\sqrt{T}\,(\log(K/S_0) - (r-\sigma^2/2)T)} \frac{1}{\sqrt{2\pi}} \left(e^{-y^2/2} e^{\sigma\sqrt{T}y} - e^{-y^2/2} \cdot \frac{K e^{-(r-\sigma^2/2)T}}{S_0} \right) dy$$

$$= S_0 \cdot \Phi\Big(\underbrace{\sigma\sqrt{T} - 1/\sigma\sqrt{T}\,(\log(K/S_0) - (r-\sigma^2/2)T)}_{d^+}\Big) - K e^{-rT} \Phi\Big(\underbrace{\frac{(r-\sigma^2/2)T - \log(K/S_0)}{\sigma\sqrt{T}}}_{d^-}\Big)$$

Payout : $(S_T - K)^+ - (K - S_T)^+ = S_T - K$

We need 1 stock, $-Ke^{-rT}$ to replicate RHS

$\therefore \Pi(EC) - \Pi(EP) = S_0 - Ke^{-rT}$ (put-call parity)

$\therefore \Pi(EP) = Ke^{-rT}\Phi(-d^-) - S_0 \Phi(-d^+)$

(10/10)

(b) At $t=0$: $\Pi = e^{-rT} \int_{\mathbb{R}} 1_{S_0 e^{\sigma\sqrt{T}y + (r-\sigma^2/2)T} \geq K} \, e^{-y^2/2}/\sqrt{2\pi} \, dy$

$$= e^{-rT} \int_{\frac{1}{\sigma\sqrt{T}}[\log(K/S_0) - (r-\sigma^2/2)T]}^{\infty} e^{-y^2/2}/\sqrt{2\pi} \, dy$$

$$= e^{-rT}(\Phi(d^-))$$

At time t: $\left.\begin{array}{l} S_0 \to S_t \\ T \to T-t \end{array}\right\}$ $\Pi = e^{-r(T-t)} \Phi\left(\frac{-\log(K/S_t) + (r-\sigma^2/2)(T-t)}{\sqrt{T-t} \cdot \sigma} \right)$

Hedge at t: $\frac{\partial \Pi}{\partial S_t} = e^{-r(T-t)} \Phi'\left(\frac{-\log(K/S_0) + (r-\sigma^2/2)(T-t)}{\sigma \cdot \sqrt{T-t}} \right) \cdot \left(\frac{1}{S_t \cdot \sigma\sqrt{T-t}} \right)$

replicates delta of option

(7/7) $\therefore$ Hold $\partial\Pi/\partial S_t$ units of stock to hedge.

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(d) Payouts: $\mathbb{1}_{S_T \geq K} + \mathbb{1}_{S_T < T} = 1$

1 can be replicated with e^{-rT}

$\therefore \ \Pi(C_{\text{dig Call}}) + \Pi(C_{\text{dig Put}}) = e^{-rT}$ (put - call parity)

(3/3) $\therefore \ \Pi(C_{\text{dig Put}}) = e^{-rT} \ \Phi(-d^-)$

|20 d|

4.) (a) Let $S_t^0 = (1+r)^t$, fix $-1 < a < b$:

$(R_n)_{n \geq 1}$ be iid; $\mathbb{P}(R_n = a) = p$, $\mathbb{P}(R_n = b) = 1-p = q.$

(4/4) $S_t^1 = S_0^1 \cdot \prod_{k=1}^{t} (1 + R_k)$ (CRR model)

Arbitrage free iff $a < r < b$:

$X_t^1 = S_0^1 \cdot \prod_{k=1}^{t} \left(\frac{1+R_k}{1+r} \right)$

$\therefore \ \mathbb{E}^{G}[X_{t+1}^1 \mid \cancel{X_t^1}\ \mathcal{F}_t] = X_t^1 \cdot \mathbb{E}^{G}\left[\frac{1+R_{t+1}}{1+r} \mid \mathcal{F}_t\right] = X_t^1$

iff : $\cancel{\mathbb{P}(} \ p^* = \cancel{\mathbb{P}(} G(R_{t+1} = a \mid \mathcal{F}_t)$, $q^* = G(R_{t+1} = b \mid \mathcal{F}_t)$

$r = p^* a + (1-p^*) b$

iff $p^* = \frac{b-r}{b-a}$, ($\in (0,1)$ iff $b > r > a$)

$\therefore \ \exists$ equivalent martingale measure iff $a < r < b$,

(3/3) $G\{R_1 = a_1, \ldots R_n = a_n\} = \left(\frac{b-r}{b-a}\right)^{\sum_{i=1}^{n} \mathbb{1}_{a_i = a}} \cdot \left(\frac{r-a}{b-a}\right)^{\sum_{i=1}^{n} \mathbb{1}_{a_i = b}}$

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(b) Let $(\Theta_n)_{n=0,\dots T}$ be replication strategy:

$$H = \Theta_T \cdot X_T \quad (A.S.)$$

At time t: $\Theta_t \cdot X_t = \mathbb{E}^G[\Theta_T \cdot X_T \mid F_t] = \mathbb{E}[H \mid F_t]$

$$\therefore V_t(x_0, \dots x_t) = \mathbb{E}^G[H \mid S_0 = x_0, \dots S_t = x_t]$$

$$= \mathbb{E}^G\left[h\left(x_0, \dots x_t;\ x_t \cdot \frac{(1+R_{t+1})}{(1+r)}, \dots\ x_t \prod_{k=t+1}^{T} \frac{(1+R_k)}{1+r}\right)\right]$$

But $\left(\frac{(1+R_k)}{1+r}\right)_{k=t+1,\dots T}$ have the same distribution as $\left(\frac{(1+R_k)}{1+r}\right)_{k=1,\dots T-t}$

$$\therefore V_t(x_0, \dots x_t) = \mathbb{E}^G\left[h\left(x_0, \dots x_t;\ x_t\left(\frac{1+R_1}{1+r}\right), \dots\ x_t \prod_{k=1}^{T-t} \frac{(1+R_k)}{1+r}\right)\right]$$

$$= \mathbb{E}^G\left[h\left(x_0, \dots x_t;\ x_t \cdot \left(S_1'/S_0'\right), \dots\ x_t\left(S_{T-t}'/S_0'\right)\right)\right]$$

(5/5) $\left(S_k' = S_0' \cdot \prod_{i=1}^{k} (1+R_i)/_{1+r}\right)$

(c) $V_t(x_0, \dots x_t) = \mathbb{E}_G^G\left[h\left(x_t \cdot S_{T-t}'/S_0'\right)\right]$ (from (b))

(2/3) $= \sum \dots$

$\therefore V_t$ depend only on x_t: $V_t(x_t) = \mathbb{E}^G\left[h\left(x_t \cdot S_{T-t}'/S_0'\right)\right]$

(d) $V_t(x_0, \dots x_t) = \mathbb{E}^G\left[h\left(x_t\left(S_{T-t}'/S_0'\right),\ \operatorname{Max}\left\{x_0, \dots x_t,\ x_t\left(S_1'/S_0'\right), \dots\ x_t\left(S_{T-t}'/S_0'\right)\right\}\right)\right]$

$= \mathbb{E}^G\left[h\left(x_t\left(S_{T-t}'/S_0'\right),\ \operatorname{Max}\left\{\operatorname{Max}\{x_0, \dots x_t\},\ x_t/S_0' \cdot \operatorname{Max}\{S_1', \dots S_{T-t}'\}\right\}\right)\right]$

$= \mathbb{E}^G\left[h\left(x_t\left(S_{T-t}'/S_0'\right),\ \operatorname{Max}\left\{\operatorname{Max}\{x_0, \dots x_t\},\ x_t/S_0' \cdot \operatorname{Max}\{S_0', \dots S_{T-t}'\}\right\}\right)\right]$

$\left(x_t/S_0' \cdot S_0' = x_t\right)$

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$\therefore$ Function of $\cancel{X_T}\ S'_{T-t}$, $x = \cancel{\ }\ X_t$, $m = \text{Max}\{X_0, \ldots X_t\}$, S'_0, M_{T-t}

$$g(x, m, S'_0, S'_{T-t}, M_{T-t}) = h\left(x \cdot S'_{T-t}/S'_0,\ \text{Max}\{m, x/S'_0 \cdot M_{T-t}\}\right)$$

(5/5)

(19 ✓)

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SFM 2020

1.) (a) $\theta \cdot S_1 = \theta \cdot N(\mu, V) + (1+r)(\omega_0 - \theta \cdot S_0)$

$= N\left((1+r)\omega_0 + \theta(\mu - (1+r)\cdot S_0),\ \theta^T V \theta\right)$

$\therefore$ Min. $\frac{1}{2}\theta^T V\theta$ subject to $(1+r)\omega_0 + \theta^T \cdot \alpha = \omega_1$,

$\alpha = \mu - (1+r)\cdot S_0$.

Lagrangian : $\frac{1}{2}\theta^T V\theta - \lambda\left((1+r)\omega_0 + \alpha^T \cdot \theta - \omega_1\right)$

$V \cdot \theta = \lambda \cdot \alpha \Rightarrow \theta^* = \lambda \cdot (V^{-1} \cdot \alpha)$

Constraint: $\omega_1 - (1+r)\omega_0 = \theta^T \cdot \alpha = \lambda(\alpha^T V^{-1} \cdot \alpha)$

$\therefore \lambda = \dfrac{\omega_1 - (1+r)\omega_0}{\alpha^T V^{-1} \alpha}$

$\therefore$ Portfolio : $\theta = \lambda \cdot V^{-1}(\mu - (1+r)S_0)$, $\lambda = \dfrac{\omega_1 - (1+r)\omega_0}{(\mu-(1+r)S_0)^T V^{-1}(\mu - (1+r)S_0)}$.

(12/12)

(b) $\mathbb{E}\left[e^{s \cdot N(\mu, \sigma^2)}\right] = e^{s\mu + s^2\sigma^2/2}$ (Gaussian MGF)

$\therefore \mathbb{E}[U(\bar{\theta}\cdot \bar{S}_1)] = -\mathbb{E}\left[e^{-\gamma N((1+r)\omega_0 + \theta\cdot\alpha,\ \theta^T V \theta)}\right]$

$= -e^{-\gamma[(1+r)\omega_0 + \theta\cdot\alpha] + \gamma^2/2\ \theta^T V\theta}$

is maximised by maximising $(1+r)\omega_0 + \theta\cdot\alpha - \gamma/2\ \theta^T V\theta$ - (1)

(1) is a quadratic in $\theta \Rightarrow$ Unique global maximiser at

$\nabla(1) = 0$.

$\therefore \alpha = \gamma V \cdot \theta$

Portfolio $= \frac{1}{\gamma} V^{-1} \alpha$; Pick $\gamma = \dfrac{\alpha^T V^{-1} \alpha}{\omega_1 - (1+r)\omega_0}$ for the same portfolio

as in (a)

(8/8)

(20/20)

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2.) (a) Let $X_n = S_n / S_n^0$:

$(\Theta_n)_{n=1,\dots T}$ is an arbitrage if:

(2/2) $\Theta_T \cdot X_T - \Theta_0 X_0 \geq 0$ (A-S), $\mathbb{P}(\Theta_T \cdot X_T - \Theta_0 X_0 > 0) > 0$,

(ii) Market is complete if all claims can be replicated
(1/2) in the market. explain what this means?

(iii) Let $\mathbb{Q}$ be any equivalent measure:

$\cancel{\mathbb{E}^{\mathbb{Q}}}$ Since F is not changed, conditional expectation

$$\mathbb{E}^{\mathbb{Q}}[S_{n+1}/S_{n+1}^0 \mid F_n] = S_n/(1+r)^{n+1} \cdot \mathbb{E}^{\mathbb{Q}}[e^{\log(Z_{n+1})} \mid F_n]$$

Pick: $$d\mathbb{Q}/d\mathbb{P} = \prod_{k=1}^{n} \frac{e^{\tilde{\sigma}_k \cdot N_k + \tilde{\mu}}}{\mathbb{E}[e^{\tilde{\sigma}_k \cdot N_k}]},$$ $(\log(Z_i) = \mu + \sigma N_n)$

Claim: $(Z_n)_{n\geq 1}$ remains independent.

$$\mathbb{Q}(Z_1 \in A_1, \dots Z_T \in A_T) = \mathbb{E}\left[\prod_{k=1}^{T} \frac{1_{Z_k \in A_k} \cdot e^{\tilde{\sigma}_k N_k + \tilde{\mu}}}{\mathbb{E}[e^{\tilde{\sigma}_k N_k}]}\right]$$

$$= \prod_{k=1}^{T} \mathbb{E}\left[1_{Z_k \in A_k} \cdot \frac{e^{\tilde{\sigma}_k N_k + \tilde{\mu}}}{\mathbb{E}[e^{\tilde{\sigma}_k N_k}]}\right] = \prod_{k=1}^{T} \left\{\mathbb{E}\left[1_{Z_k \in A_k} \frac{e^{\tilde{\sigma}_k N_k + \tilde{\mu}}}{\mathbb{E}[e^{\tilde{\sigma}_k N_k}]}\right] \prod_{j \neq k} \mathbb{E}\left[e^{\tilde{\sigma}_j N_j} / \mathbb{E}[e^{\tilde{\sigma}_j N_j}]\right]\right\}$$

$$= \prod_{k=1}^{T} \mathbb{E}\left[1_{Z_k \in A_k} \, d\mathbb{Q}/d\mathbb{P}\right] = \prod_{k=1}^{T} \mathbb{E}^{\mathbb{Q}}[1_{Z_k \in A_k}]$$ $\cancel{x}$

$$\therefore\ \mathbb{E}^{G}\left[S_{n+1}/S^{0}_{n+1} \mid F_n \right] = S_n/S_n^{0} \cdot \frac{1}{(1+r)} \cdot \mathbb{E}^{G}\left[e^{\log(Z_{n+1})} \right]$$

We want: $\mathbb{E}^{G}\left[e^{\log(Z_{n+1})} \right] = (1+r)$

$$\text{LHS} = \mathbb{E}^{G}\left[e^{\mu + \sigma N_{n+1}} \cdot e^{\tilde{\sigma}_{n+1} N_{n+1}} \right] \Big/ \mathbb{E}\left[e^{\tilde{\sigma}_{n+1} N_{n+1}} \right]$$

$$= e^{\mu + \frac{1}{2}(\sigma + \tilde{\sigma}_{n+1})^2} \Big/ e^{\frac{1}{2}\tilde{\sigma}_{n+1}^2} = (1+r)$$

$$\therefore\ \log(1+r) = \mu + \sigma^2/2 + \sigma \cdot \tilde{\sigma}_{n+1}$$

$\therefore$ Pick

Is $Z_k = e^{\sigma N_k + \mu}$?

Let $dG/d\mathbb{P} = \prod_{k=1}^{T} e^{\tilde{\sigma} N_k + \mu} \Big/ \mathbb{E}\left[e^{\tilde{\sigma} N_k + \mu} \right]$:

Since $Z_1, \dots Z_T$ is independent, we can write its measure

as $P_1 \otimes \dots P_T = \mathbb{P}$.

$$\therefore\ G = \bigotimes_{k=1}^{T} P_k\left(e^{\tilde{\sigma} N_k + \mu} \Big/ \mathbb{E}\left[e^{\tilde{\sigma} N_k + \mu} \right] \cdot \right) \Rightarrow (Z_1, \dots Z_T)$$

remains independent

Martingale: $\mathbb{E}^{G}\left[e^{\mu + \sigma N_{n+1}} \right] = (1+r)$

$$\therefore\ (1+r) \cdot \mathbb{E}^{G}\left[e^{\tilde{\mu} + \tilde{\sigma} N_{n+1}} \right] = \mathbb{E}\left[e^{\mu + \tilde{\mu} + (\sigma + \tilde{\sigma}) N_{n+1}} \right]$$

$$\therefore\ (1+r)\, e^{\tilde{\mu} + \tilde{\sigma}^2/2} = e^{(\mu + \tilde{\mu}) + \frac{1}{2}(\sigma + \tilde{\sigma})^2}$$

$$\therefore\ \log(1+r) = \cancel{\tilde{\mu}+}\ \mu + \sigma \cdot \tilde{\sigma} + \sigma^2/2$$

$$\therefore\ \tilde{\sigma} = 1/\sigma \left[\log(1+r) - \mu - \sigma^2/2 \right]$$

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$\therefore \; \mathbb{E}^{G}\left[S^1_{n+1}/S^0_{n+1} \;\middle|\; \mathcal{F}_n \right] = S^1_n / S^0_n$

S^1_{n+1}/S^0_{n+1} is clearly $\mathcal{F}_n$ measurable ;

$\mathbb{E}^{G}\left[\, \left| S^1_{n+1}/S^0_{n+1} \right| \,\right] = (1+r)^{-n}\, \mathbb{E}^{G}[S_n] = \cancel{\ldots} < \infty.$

(mgf of a gaussian $< \infty$)

(8/8) $\therefore$ Martingale.

(iv) FTAP : No arbitrage iff $\exists$ equivalent martingale measure.

(2/2) $\therefore$ No arbitrage.

(v) Let $T = 1$: If market is complete, all claims are replicable

Suppose :

$\wedge \; (\theta_1^0, \theta_1)$ be replication of S_1^2

$\therefore \; \theta_1^0 + \theta_1 \cdot S_1 = S_1^2 \quad (A.S.)$

$\theta_1^0 + S_0(\theta_1 Z_1) = S_0^2 (\cancel{\ldots}\, Z_1)^2 \quad (A.S.)$

$\therefore \; \theta_1 \cancel{= S_0 Z_1 - \theta_1^0 / S_0 Z_1 = S_0 e^{\mu + \sigma N_1} - \theta_1^0 / S_0 \cdot e^{-\mu - \sigma N_1}}$

~~Since $S_0 Z_1 \in (0, \infty)$~~ : Let $\alpha = S_0 Z_1$

$\alpha^2 - \theta_1 \alpha + \theta_1^0 = 0$ (θ_1, θ_1^0 are deterministic as $\mathcal{F}_0$ - measurable)

$\therefore \; \alpha \in \dfrac{\theta_1 \pm \sqrt{\theta_1^2 - 4\theta_1^0}}{2}$ with $\mathbb{P} = 1$

$\therefore$ Contradiction as $\mathbb{P}(\log(N(\mu, \sigma^2)) \in \{A\}) = 0$ if $|A| < \infty$.

(4/4) $\therefore$ Market not complete.

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3.) (i) $B_0 = 0$ A.S.

$t \to B_t$ is cont. (A.S)

(2/2) $(B_t)_{t\geq 0}$ has independent increments, $t \geq s \Rightarrow B_t - B_s \sim N(0, t-s)$

(ii) Let $W_t = \underset{0 \leq s \leq t}{\text{Max}} \{ B_s \}$.

(4/4) If $a, b \geq 0$: $\mathbb{P}(W_t \geq a, B_t \leq a-b) = \mathbb{P}(B_t \geq a+b)$

(reflection principle)

~~better~~ say $\tilde{B}_t = \begin{cases} B_t, & t \leq T_a \\ 2a - B_t, & t > T_a \end{cases}$ is BM.

(b) (i) $\mathbb{P}(M_t \leq a, B_t \leq b) = \mathbb{P}(B_t \leq b) - \mathbb{P}(M_t > a, B_t \leq b)$

$= \mathbb{P}(B_t \leq b) - \mathbb{P}(M_t \geq a, B_t \leq a - (a-b))$

$= \mathbb{P}(B_t \leq b) - \mathbb{P}(B_t \geq 2a-b)$

$(0 \leq a, \quad b \leq a.)$

$$\therefore f_{M,B}(a,b) = \partial^2 \mathbb{P}(M_t \leq a, B_t \leq b) / \partial a \partial b = -2 f_{N(0,t)}(2a-b)$$

$$= (-2) \cdot \left(-\frac{1}{2t} \cdot 2(2a-b) \right) \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(2a-b)^2}$$

(4/4)
$$= \frac{2(2a-b)}{t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(2a-b)^2}$$

(ii) ~~$\mathbb{P}(M_t \leq y, M_t - B_t \leq z)$~~ $M_t, Z_t \in [0, \infty)$

$$\begin{pmatrix} M_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} M_t \\ B_t \end{pmatrix}$$

$$\therefore f_{M_t, Z_t}(x, x-y) = f_{M_t, B_t}(x, y) \quad 1 \quad \left(1 \left| \begin{matrix} 1 & 0 \\ 1 & -1 \end{matrix} \right| 1 = 1 \right)$$

$$\therefore \mathbb{P}(M_t \in dy, Z_t \in dz) = f_{M_t, B_t}(y, y-z) \, dy \, dz$$

(6/6)
$$= \frac{2}{\sqrt{2\pi t}} \cdot \frac{2y - (y-z)}{t} e^{-\frac{1}{2t}(2y-(y-z))^2} \, dy\,dz = \frac{2(y+z)}{\sqrt{2\pi t} \cdot t} e^{-\frac{1}{2t}(y+z)^2} \, dy\,dz$$

POP bazic

(iii) ~~Claim $(M_t)_{t\geq 0}$~~ ,

$$\mathbb{P}(Z_1 \leq x) = \int_0^\infty \int_0^x \frac{2}{\sqrt{2\pi}} \frac{(y+z)}{1} e^{-(y+z)^2/2} \, dy \, dz$$

$$= \int_0^\infty -\frac{2}{\sqrt{2\pi}} \left[e^{-(y+z)^2/2} \right]_0^x dz$$

$$= \int_0^\infty \frac{2}{\sqrt{2\pi}} \left[e^{-z^2/2} - e^{-(x+z)^2/2} \right] dz$$

$$= 1 - \int_x^\infty \frac{2}{\sqrt{2\pi}} e^{-(x+z)^2/2} dz$$

$\mathbb{P}(M_1 \geq 2, M_2 \leq 1) = 0$ as M_t is increasing.

But $\mathbb{P}(|B_1| \geq 2, |B_2| \leq 1) \geq \mathbb{P}(B_1 \in [2,3], B_2 - B_1 \in [-3,-2]) > 0$.

$\mathbb{P}(M_1 - B_1 \geq 2, M_2 - B_2 \geq 1) \geq \mathbb{P}(\cancel{M_1 \geq 2, B_1 \leq 0}, M_2 \in [2,3], B_1 \leq 0, B_2 \in [2,3])$

> 0.

$\therefore$ ~~(M_t) cannot have~~ $(M_t)_{t\geq 0}$ cannot have the same distribution as $(|B_t|)_{t\geq 0}$, $(\{Z_t\})_{t\geq 0}$.

4(4) $\therefore$ $(|B_t|)_{t\geq 0}$, $(Z_t)_{t\geq 0}$ have the same dist.

20 2

* How to actually show this ? #

No.: Date:

4) (a) Praise $S_t = S_0 \cdot e^{\sigma \cancel{W_t} B_t + \mu t}$ is the price of risky asset.

(1/1) risk free asset : $S_t^0 = e^{-rt}$ B_t is?

(ii) $\Pi(C) = e^{-rT} \cdot \mathbb{E}^{\mathbb{G}}\left[C\left(S_0\, e^{\sigma W_t + (r-\sigma^2/2)t} : 0 \le t \le T \right)\right]$,

when $(W_t)_{t\ge0}$ is a $\mathbb{G}$ - brownian motion

(2/2)

* C is a function of $(S_t)_{0\le t\le T}$.

(iii) Payout at time T: $(S_T - k)^+ \geq S_T - k$.

To replicate $S_T - k$: We hold 1 unit of risky asset, $-e^{-rT}\cdot k$ riskless asset at time 0.

$\therefore \Pi(S_T - k \text{ payout}) = S_0 - e^{-rT}\cdot k$

Since $(S_T - k)^+ \geq S_T - k$: $\Pi\left((S_T - k)^+ \text{ payout} \right) \geq S_0 - e^{-rT}\cdot k$

$\therefore \Pi(EC) \geq S_0 - e^{-rT}\cdot k$

2(2)

(iv) $e^{-rT}\, \mathbb{E}^{\mathbb{G}}\left[\left(S_0\, e^{\sigma W_t + (r-\sigma^2/2)t} - k \right)^+ \right] =$

$e^{-rT} \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \cdot S_0\, e^{(r-\sigma^2/2)t} \cdot \left(e^{\sigma\sqrt{T}\cdot y} - \tilde{k} \right) dy$, $\tilde{k} = k/S_0\, e^{-(r-\sigma^2/2)T}$

$= S_0\, e^{-\sigma^2 T/2} \int_{d^-}^{\infty} e^{-y^2/2}/\sqrt{2\pi}\, \left(e^{\sigma\sqrt{T}\cdot y} - \tilde{k} \right) dy$, $d^- = \log(\tilde{k})/\sigma\sqrt{T}$

$= S_0 \cdot \int_{d^-}^{\infty} \frac{e^{-(y-\sigma\sqrt{T})^2/2}}{\sqrt{2\pi}}\, dy - S_0\, \tilde{k}\, e^{-\sigma^2 T/2} \int_{d^-}^{\infty} e^{-y^2/2}/\sqrt{2\pi}$ $~~dd^+~~$ =

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$$= S_0 \cdot \Phi(\sigma\sqrt{T} - d^-) - e^{-rT} \cdot K\, \Phi(-d^-) \qquad (\text{expand } \tilde{K})$$

$$= S_0 \cdot \Phi\Bigg(\underbrace{\frac{(r+\sigma^2/2)\cdot T - \log(K/S_0)}{\sigma\sqrt{T}}}_{d^+}\Bigg) - K e^{-rT} \cdot \Phi\Bigg(\underbrace{\frac{(r-\sigma^2/2)T - \log(K/S_0)}{\sigma\sqrt{T}}}_{d^-}\Bigg)$$

(10/10) (relabel d^- here)

(v) At time t:

If $K > S_t$, we do not activate the contract.

~~Valuation of activating~~ : $\cancel{(S_t - K)^+}$

If $K \leq S_t$: If activate, payout $= S_t - K$

But value of keeping option to expiry

$$= EC(S_t, K, \sigma, r, T-t) \geq S_t - e^{-r(T-t)} \cdot K \quad (\text{iii})$$

$$\geq S_t - K \quad (r \geq 0)$$

$$= \text{Value of current activation}$$

$\therefore$ Do not activate.

$\therefore$ Do not activate at time t in all scenarios

(5/5)

(20/20)

No.: Date: Zayne / Queens'

SFM 2021

1.) $\forall n \geq 0$: X_n is integrable, $\mathcal{F}_n$ - measurable, $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$

$\Rightarrow (X_n)_{n\geq 0}$ is a martingale

(2/2)

(b) Distribution of ε_n: $\mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] = 0 \Rightarrow \mathbb{E}[X_n - X_{n-1}] = 0$ (Tower)

$\therefore \mathbb{P}(\varepsilon_n = 1) \cdot - \mathbb{P}(\varepsilon_n = -1) = 0 \Rightarrow \frac{1}{2} = \mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1)$

Claim: $(\varepsilon_n)_{n\geq 1}$ is independent.

~~$\mathbb{P}(\varepsilon_1 = 1, 2, \ldots, \varepsilon_n = 1) = \mathbb{P}($~~

Since $\varepsilon_n = X_n - X_{n-1}$: ε_n is $\mathcal{F}_n$ measurable.

$\therefore \mathbb{E}[\varepsilon_n | \mathcal{F}_{n-1}] = \mathbb{P}(\varepsilon_n = 1 | \mathcal{F}_{n-1}) \cdot - \mathbb{P}(\varepsilon_n = 0 | \mathcal{F}_{n-1}) = 0.$

$\Rightarrow \frac{1}{2} = \mathbb{P}(\varepsilon_n = \pm 1 | \mathcal{F}_{n-1})$

Since $\varepsilon_1, \ldots \varepsilon_{n-1}$ are $\mathcal{F}_{n-1}$ measurable: $\mathbb{P}(\varepsilon_n = \pm | \varepsilon_1, \ldots \varepsilon_{n-1}) = \mathbb{P}(\varepsilon_n = \pm 1)$

$\therefore \mathbb{P}(\varepsilon_n = a_n, \ldots \varepsilon_1 = a_1) =$ ~~$\mathbb{P}(\varepsilon_n = a_n) \mathbb{P}(\varepsilon_{n-1}$~~ $\frac{1}{2} \cdot \mathbb{P}(\varepsilon_1 = a_1, \ldots$ and $\varepsilon_{n-1} = a_{n-1})$

Proceeding iteratively: $\mathbb{P}(\varepsilon_1 = a_1, \ldots \varepsilon_n = a_n) = (\frac{1}{2})^n$

$\therefore$ ~~Inde~~ (True $\forall n \geq 1$) $(\varepsilon_n)_{n\geq 1}$ is independent

(4/4) $\therefore$ Random walk.

(c) $\hat{X}_n$ is a function of $H_1, \ldots H_n$; $X_0, \ldots X_n$ (all $\mathcal{F}_n$ - measurable!

$\therefore \hat{X}_n$ is $\mathcal{F}_n$ - measurable.

$\mathbb{E}[|\hat{X}_n|] \leq \sum_{k=1}^{n}$ ~~$\|H_n\|_{n\geq 1}$, $\sup\{|H_k| \ldots$~~ $M \cdot \mathbb{E}[|X_k| + |X_{k-1}|]$

$< \infty$ $(M = \sup\{|H_k| : 1 \leq k \leq n\})$

$\therefore \hat{X}_n$ is integrable.

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$\mathbb{E}[\hat{X}_{n+1} - \hat{X}_n \mid F_n] = \mathbb{E}[H_{n+1}(X_{n+1} - X_n) \mid F_n] = H_{n+1} \cdot \mathbb{E}[X_{n+1} - X_n \mid F_n]$

(H_{n+1} is F_n - measurable)

$= H_{n+1} \cdot 0$ (martingale property of $(X_n)_{n\geq 0}$) $= 0$.

$\therefore$ $(\hat{X}_n)_{n\geq 0}$ is a martingale.

(3/3)

(d) Claim: $(X_n)_{n\geq 0}$ is a martingale wrt $(F_n)_{n\geq 0}$, $F_n = \sigma(X_0 \ldots X_n)$

$\mathbb{E}[|X_n|] \leq n < \infty$, . X_n is F_n - measurable

$\mathbb{E}[X_{n+1} - X_n \mid F_n] = 0$ (symmetric) $\therefore$ Done.

$\cancel{X_n}$ Define: $H_k = 1$ if $k \leq T_a$] $H_k = 1 - 2\,\mathbb{1}_{k > T_a}$

$H_k = -1$ if $k > T_a$]

Since $\{T_a < k n\} = \bigcap_{r=0}^{n-1} \{X_r \neq a\} \in F_{n-1}$: T_a is a stopping time,

H_k is F_{k-1} measurable.

$\cancel{\forall}$ $|H_k| \leq 1 \Rightarrow$ Bounded.

$\therefore$ (d c) $\Rightarrow$ $(\hat{X}_n)_{n\geq 0}$ $(\cancel{\hat{X}_0} = 0)$ is a martingale.

$n \leq T_a$: $\hat{X}_n = \cancel{X_0}\ 0 + \sum_{k=1}(X_k - X_{k-1}) = X_n$

$n > T_a$: $\hat{X}_n = \underbrace{\sum_{k=1}^{\wedge T_a} (X_k - X_{k-1})}_{a} \mp \underbrace{\sum_{r=T_a+1}^{n} (X_k - X_{k-1})}_{X_n - a}$

$= 2a - X_n$.

$\therefore$ $(\hat{X}_n)_{n\geq 0}$ is a martingale

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$\forall n \geq 1: \hat{X}_n - \hat{X}_{n-1} = 1 \quad (\mathbb{P} = 1/2)$] Symmetry of $(X_n)_{n\geq 0}$

$= -1 \quad (\mathbb{P} = 1/2)$]

$\therefore$ (b) $\Rightarrow$ ~~Mar~~ Random Walk.

[4/4]

(c) ~~$\mathbb{P}(M_n \geq a) = \mathbb{P}($~~ $(\hat{M}_n = \text{Max}\{\hat{X}_0, \dots \hat{X}_n\})$

$\mathbb{P}(M_n \geq a, X_n \leq a-b) = \mathbb{P}(\hat{M}_n \geq a, \hat{X}_n \leq a-b)$ ~~$\Leftrightarrow \text{Max}\{\hat{X}_n\} \geq a)$~~

$= \mathbb{P}(M_n \geq a, X_n \geq a+b) \quad (M_n \geq a \Leftrightarrow \hat{M}_n \geq a)$

$= \mathbb{P}(X_n \geq a+b) \quad (b \geq 0)$

$\therefore \mathbb{P}(M_n \geq a) = \mathbb{P}(X_n \geq a+0) + \mathbb{P}(M_n \geq a, X_n > a)$

$= \mathbb{P}(X_n \geq a) + \mathbb{P}(X_n \geq a+1)$

$\therefore \mathbb{P}(M_n = a) =$ ~~$\mathbb{P}(X_n \geq a) + \mathbb{P}(X_n \geq a+1) - \mathbb{P}(X_n \geq a+1) - \mathbb{P}(X_n \geq a)$~~

$\mathbb{P}(M_n \geq a) - \mathbb{P}(M_n \geq a+1) = \mathbb{P}(X_n \geq a) - \mathbb{P}(X_n \geq a+2)$

$= \mathbb{P}(X_n \in \{a, a+1\})$

= ?

(6/7)

H 9, 2

2.) (a) If $\varphi \cdot (S_1 - (1+r) S_0) \geq 0$ (A.S), $\mathbb{P}(\varphi \cdot (S_1 - (1+r) S_0) > 0) > 0$:

(2/2) φ is an arbitrage.

(b) If Θ is maximising portfolio, $t=1$ payout X_1:

~~$\mathbb{E}[U(\Theta \cdot (S_1 - (1+r) S_0))]$~~ ~~maxi~~

~~$\mathbb{E}[U(\Theta \cdot (S_1 - (1+r) S_0) + (1+r) X_0)]$~~ ~~Maximised~~

Let φ be an arbitrage portfolio:

~~$\mathbb{E}[U(X_2 + \varphi)] > 2$~~

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$U(X_1) \leq U(X_1 + \varphi\cdot(S_1 - (1+r)S_0))$ (A.S.)

$U(X_1) < U(X_1 + \varphi\cdot(S_1 - (1+r)S_0))$ with $\mathbb{P} > 0$

$\therefore \mathbb{E}[U(X_1)] < \mathbb{E}[U(X_1 + \varphi\cdot(S_1 - (1+r)S_0))]$

$\therefore$ Contradict optimal solution

[4/4] $\therefore$ No arbitrage exists

(c) Let X^* maximise $\mathbb{E}[U(X^*)]$, $\Lambda = \{$ attainable $t=1$ strategies $\}$

Λ is a vector space determined by $\underline{\theta}$ (# of risky shares held)

Pred:

$\pi(Y)$ is the solution to: $\mathbb{E}[U(X^*)] = \sup_{A\in\Lambda}\{\mathbb{E}[U(A + Y - \overset{(1+r)}{\wedge}\pi(Y))]\}$

Fix $t \in (0,1)$; Let Y_1, Y_2 be claims, A^i optimises

$A \mapsto \mathbb{E}[U(A + Y_i - \overset{(1+r)}{\wedge}\pi(Y_i))]$.

$\mathbb{E}[U(X^*)] = t\,\mathbb{E}[U(A_1 + Y_1 - \pi(Y_1))] + (1-t)\,\mathbb{E}[U(A_2 + Y_2 - \pi(Y_2))]$

$\leq \mathbb{E}[U(tA_1 + (1-t)A_2 + tY_1 + (1-t)Y_2 - (t\pi(Y_1) + (1-t)\pi(Y_2)))]$

$\leq \sup_{A\in\Lambda}\{\mathbb{E}[U(A + tY_1 + (1-t)Y_2 - (t\pi(Y_1) + (1-t)\pi(Y_2)))]\}$

But LHS $= \sup_{A\in\Lambda}\{\mathbb{E}[U(A + tY_1 + (1-t)Y_2 - (\pi(tY_1 + (1-t)Y_2))]\}$

$\therefore \pi(tY_1 + (1-t)Y_2) \geq t\pi(Y_1) + (1-t)\pi(Y_2)$

[5/7] $\therefore$ Concave.

POP bazic

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If $\pi(Y) > \sup\{p \in I\}$: Buying Y at $\pi(Y)$ is an (neg.) arbitrage.

why?

$\therefore$ ~~$\mathbb{E}$~~ $u(X^*) \gneq u(X^* + Y - \pi(Y))$ (A.S.)

$u(X^*) \gneq u(X^* + Y - \pi(Y))$ with $\mathbb{P} > 0$

$\therefore$ ~~$\mathbb{E}[u(X^*)] \gneq \mathbb{E}[u(X^* + Y - \pi(Y))$~~

Let A maximise $\mathbb{E}[u(A + Y - \pi(Y))]$

$\therefore$ $\mathbb{E}[u(A + Y - \pi(Y))] < \mathbb{E}[u(A)] \leq \mathbb{E}[u(X^*)]$

$\mathbb{E}[u(X^*)] \leq$

$\therefore$ Contradiction $\Rightarrow$ $\pi(Y) \leq \sup\{p \in I\}$

(5/7)

(18)

3.) (a) ~~Claim~~: $\mathbb{E}[M_{n+1}] = \mathbb{E}[\cancel{M_{n+1}}\ \mathbb{E}[M_{n+1} | F_n]] = \mathbb{E}[M_n]$

$\therefore$ $\mathbb{E}[M_0] = \mathbb{E}[M_1] = \ldots = \mathbb{E}[M_n]$

Since $\mathbb{E}[\hat{M}_{n+1} | F_n] \leq \hat{M}_n$ (supermartingale):

$\mathbb{E}[\hat{M}_{n+1} - \hat{M}_n] = \mathbb{E}[\underbrace{\mathbb{E}[\hat{M}_{n+1} - \hat{M}_n | F_n]}_{\leq 0 \text{ A.S.}}] \leq 0.$

$\therefore$ $(\mathbb{E}[\hat{M}_n])_{n \geq 0}$ is a decreasing sequence $\Rightarrow$ ~~$\mathbb{E}[\hat{M}_n] \leq \mathbb{E}[\hat{M}_0]$~~ $= \mathbb{E}[M_0]$ $= \mathbb{E}[M_n]$

~~$M_T = \sum$~~ Let $T \leq N$ A.S.:

$M_T = \sum_{k=0}^{N} 1_{T=k} \cdot M_k$

$\therefore$ $\mathbb{E}[M_T - \hat{M}_T] = \sum_{k=0}^{N} \mathbb{E}[1_{T=k} \cdot (\underbrace{M_{T_k} - \hat{M}_{T_k}}_{\bar{M}_k})]$

(3/7)

$= \sum_{k=0} \mathbb{E}[1_{\{k \leq T\}} (\bar{M}_k - M_{k-1})]$

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We have $M_0 - \hat{M}_0 = 0$: $\Rightarrow$ ~~$\mathbb{E}[M_0 - \hat{M}_0]$~~ $= 0$

If $\mathbb{E}[M_k - \hat{M}_k + F_k] \geq 0$:

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4.) (a) $(B_t)_{t\geq 0}$ is a brownian motion (BM) if:

$B_0 = 0$ } A.S.

$t \to B_t$ is cont.

$(B_t)_{t\geq 0}$ has stationary incrmats, $(s\leq t)$ $B_t - B_s \sim N(0, t-s)$

If $(W_t)_{t\geq 0}$ is gaussian: $W_0 = 0$, $t \to W_t$ is cont. (A.S.)

$(\mathbb{E}[W_0] = 0,\ \mathbb{E}[W_0^2] = 0 \Rightarrow W_0 = 0$ A.S.)

Fix $0 = t_0 < t_1 \ldots < t_n$:

$(\cancel{W_{t_1}}, W_{t_1 - t_0}, \ldots W_{t_n - t_{n-1}})$ is a Gaussian vector $\Rightarrow$

determined be mean vector (0), Cov matrix $\cancel{\ldots} t_i t_j$

If $(B_t)_{t\geq 0}$ is BM: $B_{t_k} - B_{t_{k-1}} \sim N(0, t_k - t_{k-1})$

$\mathbb{E}[B_s B_t] = s + \mathbb{E}[\underbrace{B_s (B_{t-s})}_{\text{indepadet}}] = s$ $(s \leq t)$

$$\therefore \begin{pmatrix} B_{t_1} - B_{t_0} \\ \vdots \\ B_{t_n} - B_{t_{n-1}} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ -1 & \ddots & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} B_{t_1} \\ \vdots \\ B_{t_n} \end{pmatrix}, \quad \cancel{(B_{t_1}, \ldots B_{t_n}) \sim N}$$

$(B_{t_1}, \ldots B_{t_n})$ is normal, mean 0, $\cancel{Var} = (Cov)_{ij} = \text{Min}\{t_i, t_j\}$

$\therefore$ Since $\begin{pmatrix} B_{t_1} \\ \vdots \\ B_{t_n} \end{pmatrix}$, $\begin{pmatrix} W_{t_1} \\ \vdots \\ W_{t_n} \end{pmatrix}$ have the same & mean, cov, are are gaussian $\Rightarrow$ Same distribution

$\therefore$ $(W_t)_{t\geq 0}$ has indepadet incrmats, $(s \leq t)$ $B_t - B_s \sim N(0, t-s)$

(6/6) $\therefore$ $(W_t)_{t\geq 0}$ is a BM

(b) $\hat{W}_0$. $\mathbb{E}[\hat{W}_0] = 0$, $\mathbb{E}[\hat{W}_t] = t \cdot 0$ $(t > 0)$

$\mathbb{E}[\hat{W}_0 \hat{W}_t] = 0$, $\mathbb{E}[\hat{W}_s \hat{W}_t] = st\, \mathbb{E}[W_{1/s} W_{1/t}] = s$ $(0 < s \leq t)$

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$(W_{t_1}, \dots W_{t_n})$ gaussian $(\forall\, t_i \in [0,\infty))$ $\Rightarrow$

$(t_1 W_{1/t_1}, \dots t_n W_{1/t_n})$ gaussian

(4/4) (a) $\Rightarrow$ $(\hat{W}_t)_{t\geq 0}$ is ~~gaus~~ BM

(c) ~~Let $A_n \in \mathcal{B}(\mathbb{R})$~~:

~~$\mathbb{P}(W_{t_1}+ct_1 \in A_1, \dots W_{t_n}+ct_n \in A_n)$~~

$$\text{LHS} = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} P_{x_0, x_1}(t_1 - t_0) \dots P_{x_{n-1}, x_n}(t_n - t_{n-1})\, f(x_1 + ct_1, \dots x_n + ct_n)\, dx_n \dots dx_1$$

$$P_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}$$

$$= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f(x_1, \dots x_n)\, P_{x_1 - ct_1}(t_1) \dots P_{x_{n-1}}$$

$$P_{t_n}(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}$$

$$\text{LHS} = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f(x_1 + ct_1, \dots x_n + ct_n)\, P_{t_1}(x_1), \dots P_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_n \dots dx_1$$

$$= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f(x_1, \dots x_n)\, P_{t_1}(x_1 - ct_1) \dots P_{t_n - t_{n-1}}(x_n - x_{n-1} - c(t_n - t_{n-1}))\, dx_n \dots dx_1$$

why? $\circledcirc$ $$= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f(x_1 \dots x_n)\, P_{t_1}(x_1) \dots P_{t_n - t_{n-1}}(x_n - x_{n-1}) \cdot e^{+cx_1} \dots e^{+c(x_n - x_{n-1})} \cdot e^{-c^2/2\, t_1} \dots e^{-c^2/2\,(t_n - t_{n-1})}\, dx_n \dots dx_1$$

$$= \mathbb{E}\left[f((W_t)_{0\leq t\leq T}) \cdot e^{cW_{t_n} - c^2 t_n/2} \right] = \text{RHS}$$

(5/6)

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(d) $f\big((W_t + x)_{t \geq T}\big) = f\Big(\Big\{ t \cdot \big(\tfrac{1}{t} W_t + \tfrac{x}{t}\big)_{t \geq T} \Big\}\Big)$

$= f\Big(\Big\{ \tfrac{1}{t} \cdot (\hat{W}_t + t x) : 0 < t \leq \tfrac{1}{T} \Big\}\Big)$

$\therefore$ Cameron - Martin:

LHS $\mathbb{E}[f] = \mathbb{E}\Big[e^{x W_{1/T} - \frac{x^2}{2}(\frac{1}{T})} \cdot f\Big\{ \tfrac{1}{t} W_t : 0 \leq t \leq \tfrac{1}{T} \Big\}\Big]$

$= \mathbb{E}\Big[e^{-\frac{x^2}{2T} + x(\frac{1}{T} W_T)} f\Big\{ \tfrac{1}{t} \cdot t\, W_{1/t} : 0 < t \leq \tfrac{1}{T} \Big\}\Big]$

$= \mathbb{E}\Big[e^{-\frac{x^2}{2T} + \frac{x}{T} W_T} f\{ W_t : t \geq T \}\Big]$ ※.

(4/4)

(15, d)

3.) (a) Let $T \leq N$ A.S.

$M_T = \sum_{k=0}^{N-1} 1_{T=k} X_k + 1_{T \geq N} X_N$ (similarly for $\hat{M}_T$)

$\mathbb{E}[M_T - \hat{M}_T] = \sum_{k=0}^{N} \mathbb{E}[1_{T=k}(M_k - \hat{M}_k)]$

$1_{T=k}$ is F_k - measurable: (super) - Martingale property

$\mathbb{E}[1_{T=k} M$

Claim: $\mathbb{E}[\hat{M}_T] \leq \mathbb{E}[M_0]$

$\hat{M}_T = \hat{M}_0 + \sum_{k=1}^{N} 1_{T \geq k} (\hat{M}_k - \hat{M}_{k-1})$

$\therefore \mathbb{E}[\hat{M}_T - \hat{M}_0] = \sum_{k=1}^{N} \mathbb{E}[1_{T \geq k} \cdot (\hat{M}_k - \hat{M}_{k-1})]$

$1_{T \geq k}$ is F_{k-1} measurable $\Rightarrow \mathbb{E}[1_{T \geq k}(\hat{M}_k - \hat{M}_{k-1})] \leq 0.$

$\mathbb{E}[1_{T \geq k}(M_k - M_{k-1})] = 0.$

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$\therefore \mathbb{E}[\hat{M}_T] \leq \mathbb{E}[\hat{M}_0] = \mathbb{E}[M_0] = \mathbb{E}[M_T]$

(7/7)

(b) Hold θ_n stock : $X_{n-1} - \theta_n \cdot S_n$ cash left

$X_{n-1} - \theta_n S_{n-1} \to (1+r) X_{n-1} - \theta_n S_n (1+r)$ $\Big\}$ $X_n = (1+r) X_{n-1} + \theta_n (S_n - (1+r) S_{n-1})$

(3/3)

$\theta_n S_{n-1} \to \theta_n S_n$

(c) Let θ_n be trading strategy:

~~We~~ $V_n(X_n)$ is a function in $X_n \Rightarrow F_n$ - measurable.

$\mathbb{E}[V_{n-1}(X_{n-1}) - V_n(X_n) \mid F_{n-1}] = V_{n-1}(X_{n-1}) - ~~V_n~~$
$\mathbb{E}[V_n((1+r) X_{n-1} + \theta_n (S_{n-1})(\xi - (1+r))) \mid F_{n-1}]$

since ξ_n is independent of F_{n-1}

But $\mathbb{E}[V_n((1+r) X_{n-1} + \theta_n S_{n-1} (\xi - (1+r))) \mid F_{n-1}] \leq \sup_{t \in \mathbb{R}} \{ \mathbb{E}[V_n((1+r) X_{n-1} + t(\xi - (1+r))) \mid F_{n-1}] \}$

$= V_{n-1}(X_{n-1})$

(3/3)

$\therefore \mathbb{E}[V_{n-1}(X_{n-1}) - V_n(X_n) \mid F_{n-1}] \leq 0 \Rightarrow$ Super-martingale (*)

~~B~~ $\mathbb{E}[U(X_N)] = \mathbb{E}[V_N(X_N)] \leq \mathbb{E}[V_0(X_0)]$ (super-martingale)
$= V_0(X_0)$ deterministic. - (1)

(True for all strategies).

If θ^* ~~optimises~~ gives martingale process: Equality in (1) is attained

(7/7)

$\therefore$ Upper Bound in (1) is attained $\Rightarrow$ Solution is optimal

(20/20)

(*) Note: did not mention L^1 property. Can we ignore it?

Seems you can assume it here ...

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1.) (a) $V_N(x) = x^2$

$$V_{n-1}(x) = \inf_u \left\{ u^2 + \mathbb{E}\left[V_n(x+u+\xi) \right] \right\}$$

(b) Try $V_n(x) = A_n + B_n \cdot x + C_n \cdot x^2$: $A_N = B_N = 0$, $C_N = 1$

$$u^2 + A_n + B_n(x+u) + C_n \left\{ \mathrm{Var}(x+u+\xi) + \mathbb{E}[x+u+\xi]^2 \right\}$$

$$= u^2 + A_n + B_n(x+u) + C_n \left\{ \sigma^2 + (x+u)^2 \right\}$$

Diff. wrt u: $2u + B_n + 2C_n(x+u) = 0$

$$\therefore 2(1+C_n) \cdot u = -B_n - 2C_n x$$

$$\therefore u = -(B_n + 2C_n x) \Big/ 2(1+C_n)$$

$$\therefore V_{n-1}(x) = (1+C_n)u^2 + (B_n + 2xC_n)u + (xB_n + \sigma^2 C_n + x^2 C_n)$$

$$= \frac{-(B_n + 2C_n x)^2}{4(1+C_n)} + (xB_n + \sigma^2 C_n + x^2 C_n) \quad \Bigg\} \quad B_n = 0 \Rightarrow B_{n-1} = 0$$

Since $B_n = 0 \Rightarrow B_{n-1} = 0$ and $B_N = 0$: $B_n = 0 \quad (1 \le n \le N)$

$$\therefore V_{n-1}(x) = \frac{-C_n^2 x^2}{1+C_n} + C_n x^2 + \sigma^2 C_n$$

$$= x^2 \left\{ \frac{C_n + C_n^2 - C_n^2}{1+C_n} \right\} + \sigma^2 C_n = x^2 \left\{ \frac{C_n}{1+C_n} \right\} + \sigma^2 C_n$$

$\therefore C_{n-1} = C_n / 1+C_n$:

$C_N = 1$, $C_{N-1} = 1/2$, $C_{N-2} = \frac{1/2}{1+1/2} = 1/3 = 1/3$

If $C_n = 1/k$: $C_{n-1} = 1/k+1$ $\Rightarrow$ $C_n = \frac{1}{1+(N-n)}$

$A_{n-1} = \sigma^2 C_n = \sigma^2 / 1+(N-n)$ ✗

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Optimal Control: $u_n^* = -\frac{C_n X}{1+C_n} = -C_{n-1} \cdot X = -\frac{x_{n-1}}{2+N-n}$

$\therefore u_1^* = -x_0/N+1$ (Base case)

(Induct): If $u_k^* = -x_0/N+1 - \dots - \varepsilon_{k-1}/N-k+2$

$u_{k+1}^* = -\frac{1}{2+N-(k+1)} \left[\underbrace{X_{k-1} + u_k^*} + \varepsilon_{k+1} \right]$

~~$X_k = X_{k-1} +$~~ $X_{k-1} \left(1 - \frac{1}{2+N-k}\right) = X_{k-1} \left(\frac{1+N-k}{2+N-k}\right)$

$\therefore u_{k+1}^* = -\frac{1}{2+N-k+1} \left[\frac{X_{k-1}(1+N-k)}{2+N-k} \right] - \frac{\varepsilon_{k+1}}{2+N-(k+1)}$

$= \underbrace{-\frac{X_{k-1}}{2+N-k}}_{u_k^*} - \frac{\varepsilon_{k+1}}{2+N-(k+1)}$

$= u_k^* - \varepsilon_{k+1}/2+N-(k+1)$

$\therefore$ Induction complete.

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2.) (a) Minimise : $\frac{1}{2}\Theta^T V \Theta$ subject to

$$\Theta^T(\mu - (1+r)\cdot S_0) + (1+r)X = m$$

Lagrangian : $V\cdot\Theta - \lambda(\mu - (1+r)\cdot S_0) = 0$

$\therefore \Theta = \lambda V^{-1}\alpha, \quad \alpha = \mu - (1+r)\cdot S_0$

Constraint: $\lambda(\alpha^T V^{-1}\alpha) + (1+r)X = m \Rightarrow \lambda = (m - (1+r)X)/\alpha^T V^{-1}\alpha$

$$\therefore \Theta = \underbrace{\left((m-(1+r)X)/\alpha^T V^{-1}\alpha \right)}_{\lambda} \cdot V^{-1}(\mu - (1+r) S_0)$$

Minimise $\frac{1}{2}\Theta^T V \Theta$: subject to $\mathbb{E}[X_1] \geq m$.

If $\mathbb{E}[X_1] = m$: Min Var : $\Theta^T V \Theta = \lambda^2 \{\alpha^T V^{-1}\cdot V\cdot V^{-1}\alpha^T\}$

$= (m - (1+r)X)^2 / \alpha^T V^{-1}\alpha.$

$\therefore \text{Min}\{\text{Var}(X_1) : \mathbb{E}[X_1] \geq m\} = \text{Min}\left\{ (k - (1+r)X)^2/\alpha^T V^{-1}\alpha : k \geq m \right\}$

$= 0$ if $m \leq (1+r)\cdot X$

$= (m - (1+r))^2 / \alpha^T V^{-1}\alpha$ if $m > (1+r)\cdot X$.

(b) Max. : $\dfrac{(1+r)\cdot X + \Theta^T(\mu - (1+r)S_0) - (1+r)\cdot X}{\sqrt{\Theta^T V \Theta}} = f(\Theta).$

& If $\lambda > 0$: $f(\lambda\Theta) = f(\Theta)$

So : Max : $\Theta^T\cdot(\mu - (1+r)S_0)$, subj. $\sqrt{\Theta^T V \Theta} = 1$

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$L: \ (\mu - C(1+r) S_0) - 2\lambda (V \cdot \theta) = 0$ $(\lambda \geq 0!$, else we can go to $-\infty)$

$$\therefore \theta = V^{-1} \left(\frac{\alpha}{2\lambda} \right)$$

Constraint: $\frac{1}{4\lambda^2} \left\{ \alpha^T V^{-1} \cdot V V^{-1} \alpha \right\} = \alpha^T V^{-1} \alpha / 4\lambda^2 = 1$

$$\therefore \lambda = \frac{\sqrt{\alpha^T V^{-1} \alpha}}{2}$$

$\therefore \theta = c \cdot (V^{-1} \cdot \alpha)$, $c > 0$ is any constant

$$\mathbb{E}[X_1 | \theta] = c(\alpha^T V^{-1} \alpha) + C(1+r) x. = M$$

$$\mathrm{Var}(\theta) = c^2 (\alpha^T V^{-1} \alpha) = (M - C(1+r) x)^2 / \alpha^T V^{-1} \alpha$$

$$= \inf \{ \mathrm{Var}(X_1) \mid \mathbb{E}[X_1] = m \}$$

$\therefore$ Mean - Var efficient. $(\forall \ c > 0)$

(c) V is symmetric $\Rightarrow \exists$ orthogonal P s.t. $V = P^T D P$,

$D = \begin{pmatrix} \lambda_1 \ddots \lambda_k & \\ & 0 \end{pmatrix}$, $\lambda_i \neq 0$; $\alpha = \mu - C(1+r) S_0 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$,

α_1 is 1^{st} k coordinate

No solution to

$P^T D P \cdot \theta = \alpha \Rightarrow D \cdot (P \cdot \theta) = \underbrace{(P \alpha)}_{\beta}$ has no solution

$\begin{pmatrix} \lambda_1 \ddots \lambda_k & \\ & 0 \end{pmatrix} \cdot (P \cdot \theta) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$: β_1 is 1^{st} k coordinate

If $\beta_2 = 0$, $\exists$ solution : $\left(\begin{pmatrix} \lambda_1 \ddots \lambda_k \end{pmatrix}^{-1} \beta_1 \atop 0 \right)$

$\therefore \beta_2 \neq 0$

Pick $P \cdot \theta = \begin{pmatrix} \underset{\sim}{0} \end{pmatrix}$

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~~Diagonalise V:~~

$$V = P^T \begin{pmatrix} \lambda_1 \ddots \lambda_k & \\ & 0 \ddots 0 \end{pmatrix} P, \quad P \text{ orthogonal}$$

~~Let~~

Min: $\theta^T V \theta$ subj. $\theta^T \xi \cdot d + (1+r)x = M$

$\equiv$ Min: $\theta^T \cdot D \cdot \theta$ subj. $\theta^T \cdot (Pd) = M - (1+r)x$, $V = P^T D P$

$\therefore$ No sln to $V \cdot \theta = \mu - (1+r) S$ iff

No sln to: $\cancel{V}\ D \cdot \theta = P \cdot (\mu - (1+r) S_0)$

$\therefore$ WLOG, V diagonal $\Rightarrow V = \begin{pmatrix} \lambda_1 \ddots \lambda_k & \\ & 0 \end{pmatrix}$, $P \cdot (\mu - (1+r) S_0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

(a_1 is 1st k coordinates)

If $a_2 = 0$: $\begin{pmatrix} \begin{pmatrix} \lambda_1 \ddots \lambda_k \end{pmatrix}^{-1} \cdot a_1 \\ 0 \end{pmatrix}$ is a solution $\Rightarrow$ reject.

WLOG, $(a_2)_1 \neq 0$:

$\therefore \theta = {}^{m}/_{(a_2)_1}\ \delta_{i,(k+1)}$ is a mean m strat.

with 0 var.

$\therefore \text{Min} \{ Var(X_1) \mid \mathbb{E}[X_1] = m \} = 0$

Pick $m > (1+r) \cdot S_0$: θ be a 0-var strat.

$\therefore S_1 = m > (1+r) S_0$ (A.S.)

$\therefore$ Arbitrage.

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3.) (a) $M_t \geq 0$: $\mathbb{E}[M_t \mid F_s] = M_s \cdot \mathbb{E}\left[e^{c(W_t - W_s) - c^2/2 \, (t-s)} \,\middle|\, F_s \right]$

$= M_s \cdot \mathbb{E}\left[e^{c \cdot N(0, t-s) - c^2 (t-s)/2} \right] = M_s \cdot e^{-c^2(t-s)/2} \cdot e^{0 + c^2/2 (t-s)}$

$= M_s$

$\therefore \mathbb{E}[M_t] = \mathbb{E}[M_t \mid F_0] = M_0 = 1 \Rightarrow$ integrable, measurable in F_t.

$\therefore$ Martingale

(b) Change measure : $\frac{dG}{d\mathbb{P}} = e^{cW_T - c^2T/2}$

$(W_t)_{t \geq 0}$ is a c-drift Brownian motion (Cameron-Martin)

$\therefore$ (Discounted price): $X_t = S_0 \, e^{\sigma W_t + (\mu - r)t}$

$= S_0 \, e^{\sigma B_t + (\sigma c + \mu - r)t}$

Pick c s.t. $\sigma c + \mu - r = -\sigma^2/2 \Rightarrow c = 1/\sigma \, (r - \mu - \sigma^2/2)$

X_t is a G-martingale (risk neutral measure G)

(c) $\Pi = e^{-rT} \cdot \mathbb{E}^G[S_T^p] = e^{-rT} \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} S_0^p \, e^{\sigma p \sqrt{T} y + p(r - \sigma^2/2)T} \, dy$

$= e^{-rT} \cdot S_0^p \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, e^{\sigma p \sqrt{T} \cdot y} \, e^{p(r - \sigma^2/2) \cdot T} \, dy$

$= e^{-rT} S_0^p \, e^{p(r - \sigma^2/2)T} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \sigma p \sqrt{T})^2} \, e^{\frac{1}{2}\sigma^2 p^2 T} \, dy$

$= \cancel{e^{-rT} \cdot S_0^p} \, e \;\; e^{-(1-p) \cdot rT} S_0^p \, e^{\sigma^2 T/2 \, [-p + p^2]}$

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$$Y_1 = \max_{0 \le t \le T} \left[S_0 \, e^{\sigma B_t + (r-\sigma^2/2)t} \right] = S_0 \, e^{\sigma \max_{0 \le t \le T} \left\{ B_t + \frac{(r-\sigma^2/2)}{\sigma} t \right\}}$$

$$Y_2 = S_0^{2-p} \, S_0^{p} \, e^{(\sigma B_T + (r-\sigma^2/2)T)p} \Big/ \min \left\{ S_0 \, e^{\sigma B_t + (r-\sigma^2/2)t} \right\}$$

$$= S_0 \, e^{(\sigma B_T + (r-\sigma^2/2)T)\cdot p} \cdot e^{\max \left\{ -\sigma B_t - (r-\sigma^2/2)t \right\}}$$

$\therefore$ Y_2 has the same distribution as:

$$S_0 \, e^{p(\sigma B_T + (r-\sigma^2/2)T)\cdot p} \, e^{\sigma \cdot \max_{0 \le t \le T} \left\{ B_t - \frac{(r-\sigma^2/2)}{\sigma} t \right\}}$$

Let G_1 be measure: $\frac{dG_1}{dG} = e^{\cancel{(r-\sigma^2/2)}\,\alpha \cdot B_T - \alpha^2 T/2}$, $\alpha = 2(r-\sigma^2/2)/\sigma$

$$\therefore \mathbb{E}^{G_1}[Y_1] = \mathbb{E}^{G}\left[S_0 \, e^{\sigma \max_{0 \le t \le T} \left\{ B_t + (\alpha)t - \alpha t/2 \right\}} \right]$$

$$= \mathbb{E}^{G_1}\left[S_0 \, e^{\sigma \max_{0 \le t \le T} \left\{ B_t - \alpha t/2 \right\}} \right]$$

$$= \mathbb{E}^{G}\left[S_0 \, e^{\alpha B_T - \alpha^2 T/2} \, e^{\sigma \max_{0 \le t \le T} \left\{ B_t - \alpha t/2 \right\}} \right]$$

$$= \mathbb{E}^{G}\left[S_0 \, e^{p(\sigma B_T + \sigma\alpha/2\, T)} \, e^{\sigma \max_{0 \le t \le T} \left\{ B_t - \alpha t/2 \right\}} \right]$$

iff $\sigma p = \alpha$.

$\therefore$ Pick $p = \alpha/\sigma = 2(r-\sigma^2/2)/\sigma^2 \Rightarrow$ Claims have the same $t=0$ price.

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4.) (a) Cost of buying Θ_n shares: $\Theta_n \cdot S_{n-1}$

$\therefore$ Cash held from $n-1 \to n$: $(X_{n-1} - \Theta_n S_{n-1}) \to (1+r)(X_{n-1} - \Theta_n S_{n-1})$

Stock value: $\Theta_n S_{n-1} \to \Theta_n S_n$

$$\therefore X_n = (1+r)(X_{n-1} - \cancel{\quad}\,\Theta_n S_{n-1}) + \Theta_n S_n$$

$$= \Theta_n (S_n - (1+r) S_{n-1}) + (1+r) X_{n-1}$$

(b) ~~$S_1 = S_1 \cdot S_0 \ldots S_n \ldots S_0$~~

Risk neutral: $\mathbb{E}^G [S_{n+1} / (1+r)^{n+1} \mid S_1, \ldots S_n] = S_n / (1+r)^n$

$$\therefore \mathbb{E}^G [\xi_n \mid S_1, \ldots S_n] = (1+r)$$

$$\therefore (1+r) = \mathbb{P}^G(\xi_n = b \mid F_{n-1})(1+b) + G(\xi_n = a \mid F_{n-1})(1+a)$$

$\therefore G(\xi_n = b \mid F_{n-1}) = \frac{r-a}{b-a}$ is independent of n, F_{n-1}

Since $G(\xi_n \mid F_{n-1})$ is uniquely determined, G is a unique measure, $(\xi_n)_{n \geq 1}$ remain independent.

$$G(S_N = S_0 (1+b)^i (1+a)^{N-i}) = \left(\frac{r-a}{b-a}\right)^i \cdot \left(\frac{b-r}{b-a}\right)^{N-i} \cdot \binom{N}{i}$$

$+ q = r-a / b-a$.

(c) Market is complete $\Rightarrow \exists$ replicating strategy $(\Theta_n)_{n=1,\ldots N}$.

$(\Theta_n \cdot X_n)_{n=1,\ldots N}$ is a G martingale $(X_n = S_n/(1+r)^n)$, $\Theta_N \cdot \cancel{X_N}\,S_N = g(S_N)$ (A.S.)

$$V_n = (S_0, \ldots S_n) = \Theta_n S_n = (1-r)^{-n}\,\Theta_n X_n$$

$$= (1-r)^{-n} \cdot \mathbb{E}[\underbrace{\Theta_N \cdot X_N}_{g(S_N)\cdot(1+r)^{N-n}} \mid F_n] = (1-r)^{N-n}\,\mathbb{E}[\cancel{\Theta_N X_N}\; g(S_N) \mid F_n]$$

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$\therefore \Theta_n \cdot S_n (1+b) = V_{n+1} (S_0, \dots S_n ; S_n (1+b))$

$\Theta_n \cdot S_n (1+a) = V_{n+1} (S_0, \dots S_n ; S_n (1+a))$

Either scenario, our ~~Θ_n~~ replicating strategy must match valuation

$$\therefore \underset{(S_0, \dots S_n)}{\Theta_n} = \frac{V_{n+1} (S_0, \dots S_n ; S_n (1+b)) - V_{n+1} (S_0, \dots S_n ; S_n (1+a))}{S_n (b-a)}$$

(d) Payout $= (S_N - K)^+$

Pick $r = \inf \{ k \geq 0 : S_0 (1+b)^k (1-a)^{N-k} > K \}$

$$\mathbb{E}^{G} [(S_N - K)^+] = \sum_{n=r}^{N} \binom{N}{n} q^n (1-q)^{N-n} \{ S_0 (1+b)^{\hat{n}} (1-a)^{N-n} - K \}$$

$$= \sum_{n=r}^{N} \binom{N}{n} (q (1+b))^n ((1-q)(1+a))^{N-n} - K \, G (S_N > K)$$

$$\therefore \Pi = (1+r)^{-N} \cdot \mathbb{E}^{G} [(S_N - K)^+]$$

$$= \sum_{n=r}^{N} \binom{N}{n} \Big[\underbrace{\frac{q(1+b)}{(1+r)}}_{q^*} \Big]^n \Big[\underbrace{\frac{(1-q)(1+a)}{(1+r)}}_{p^*} \Big]^{N-n} - K \, G (S_N > K)$$

$q^* + p^* = (1+r)^{-1} \{ q(1+b) + (1-q)(1+a) \} = 1$ (def of G)

$\hat{Q} (\xi_n = (1+b)) = q^*$

$\hat{Q} (\xi_n = (1+a)) = (1-q^*)$

$(\xi_n)_{n=1,\dots N}$ iid wrt $\hat{Q}$

$$\therefore \Pi = \hat{Q} [S_N > K] - K (1+r)^{-N} Q [S_N > K]$$

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