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Linear Analysis 2010

1.) (a) Let X be a banach space, Y be a norm space.

$\mathcal{F} = \{ F = \{T_\alpha : \alpha \in I\} \in \mathcal{B}(X,Y)$:

If $\forall x \in X$: $\sup\{\|T_\alpha(x)\| : \alpha \in I\} < \infty$, $\exists M < \infty$

s.t $\sup\{\|T_\alpha\| : \alpha \in I\} < M$ (Banach Steinhaus)

(Proof): Let $A_n = \{x \in X : \sup\{\|T_\alpha(x)\| : \alpha \in I\} \leq n\}$

$= \bigcap_{\alpha \in I} T_\alpha^{-1}(\bar{D}(0,n))$ ($\cap$ of closed sets $\Rightarrow$ closed)

(Baire category)

$\therefore X = \bigcup_{n \geq 1} A_n$; X complete $\Rightarrow \exists n \in \mathbb{N}$ s.t $A_n^\circ \neq \phi$

$\therefore \exists D(x, \varepsilon) \subseteq A_n^\circ$ ($x \in X$, $\varepsilon > 0$)

$\therefore \forall z \in D(0,\varepsilon)$: $\|T_\alpha(z)\| \leq \|T_\alpha(z+x)\| + \|T_\alpha(z)\| \leq 2n$.

$\therefore \forall z \in \bar{D}(0,1)$: $\|T_\alpha(z)\| \leq 2/\varepsilon \cdot 2n = 4n/\varepsilon$

$\therefore \|T_\alpha\| \leq 4n/\varepsilon \Rightarrow \sup\{\|T_\alpha\| : \alpha \in I\} \leq 4n/\varepsilon < \infty$

(b) X is a NVS $\Rightarrow$ ($\mathbb{C}$ complete) X' is a Banach space

$\forall x \in S$: $\varphi_x \in X''$, $\varphi_x(f) = f(x)$ ~~is~~

$\therefore \{\varphi_x : x \in S\} \subseteq X''$ is ~~po~~ bounded on each

element of X' $\Rightarrow$ (Banach Steinhaus) Uniformly bounded

$\therefore \exists K \geq 0$ s.t. $\forall x \in S$, $\|\varphi_x\| = \cancel{\|x\|} \leq K$.

Claim : $\|\varphi_x\| = \|x\|$

$\forall f \in X'$, $\|f\| = 1$: $|f(x)| \leq \|f\| \cdot \|x\| \Rightarrow \|\varphi_x\| \leq \|x\|$

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Hahn - Banach $\Rightarrow$ (Fix x) $\exists$ $f \in X'$, $\|f\| = 1$, $f(x) = \|x\|$

$\therefore |\varphi_x(f)| = \|x\| \cdot \underbrace{\|f\|}_{1} \Rightarrow \|\varphi_x\| \geq \|x\| \Rightarrow$ Equality

$\therefore \forall x \in S, \ \|x\| \leq K.$

(c) Suppose otherwise: Since $\| \ \|_1 \neq \| \ \|_2$, $\exists$ X_n, $n \in \mathbb{N}$

s.t. $\begin{cases} \|X_n\|_1 \geq n \cdot \|X_n\|_2, & \|X_n\|_1 = 1 \quad \cancel{\text{WLOG}} \quad \text{OR} \\ \|X_n\|_2 \geq n \ \|X_n\|_1, & \|X_n\|_2 = 1 \end{cases}$

WLOG, we use the 1^{st} case: $S = \{X_n : n \in \mathbb{N}\}$

$\forall f \in X'_{\|\cdot\|_1}$: $\cancel{\|f}$ $|f(x_k)| \leq \|f\| \Rightarrow f(S)$ is bounded.

Since $\cancel{f \in}$ $X'_{\|\cdot\|_1} = X'_{\|\cdot\|_2}$: $\forall f \in X'_{\|\cdot\|_2}$, $\cancel{|f(x_k)}$ $f(S)$ bounded

$\Rightarrow \{\|X_k\|_2 : k \in \mathbb{N}\}$ bounded $\Rightarrow$ Contradiction

$\therefore \exists f$, f is cont. wrt $\|\cdot\|_1$, not bounded (cont.) wrt $\|\cdot\|_2$

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2.) (a) ~~Vector~~ λ^∞ is clearly a vector space ; ~~[illegible]~~ norm:

$\| x \|_\infty = 0$ iff $\sup \{ |x_n| : n \in \mathbb{N} \} = 0$ iff $x_n = 0$ $(\forall n \in \mathbb{N})$

iff $x = 0$.

$\| \lambda x \|_\infty = \sup \{ |\lambda x_n| : n \in \mathbb{N} \} = |\lambda| \cdot \sup \{ |x_n| : n \in \mathbb{N} \} = |\lambda| \cdot \| x \|_\infty$

$\| x + y \|_\infty = \sup \{ |x_n + y_n| : n \in \mathbb{N} \} \leq \sup \{ |x_n| + |y_n| : n \in \mathbb{N} \}$
$\leq \sup \{ |x_n| : n \in \mathbb{N} \} + \sup \{ |y_n| : n \in \mathbb{N} \}$

$\therefore$ Valid norm.

Complete : Let $(x^{(n)})_{n \geq 1}$ be a cauchy sequence:

$\forall k \in \mathbb{N}$: $| x^{(n)}_k - x^{(m)}_k | \leq \| x^{(n)} - x^{(m)} \|_\infty$

$\therefore$ $(x^{(n)}_k)_{n \geq 1}$ is a cauchy sequence in $\mathbb{C}$ $\Rightarrow$ convergent.

Let $y_k = \lim_{n \to \infty} x^{(n)}_k$:

~~$\sup \{ | x^{(n)}_k - y_k | : k \in \mathbb{N} \}$~~

Fix $k \in \mathbb{N}$: $| x^{(n)}_k - y_k | = \lim_{m \to \infty} | x^{(n)}_k - x^{(m)}_k |$.

Given $\varepsilon > 0$, pick N s.t. $n, m \geq N \Rightarrow \| x^{(n)} - x^{(m)} \|_\infty < \varepsilon/2$

$\therefore$ $n \geq N \Rightarrow |x^{(n)}_k - y_k| = \lim_{\substack{m \to \infty \\ m \geq N}} |x^{(n)}_k - x^{(m)}_k| \leq \varepsilon/2$.

(True $\forall k \in \mathbb{N}$) : $\| x^{(n)} - y \| \leq \varepsilon/2 < \varepsilon$

$\therefore$ $x^{(n)} - y \to 0$ in $\| \cdot \|_\infty$

$\therefore$ $\| y \|_\infty \leq \| x^{(n)} - y \| + \| x^{(n)} \| < \infty$ for $n \geq N$.

$\therefore$ $y \in \ell^\infty$, $x^{(n)} \to y$.

$\therefore$ Done.

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(b) If $\underset{\sim}{X} \in \ell^\infty \setminus C_0$: $\lim_n X_n \nrightarrow 0$.

$\therefore \exists\ \varepsilon > 0$, subsequence n_k s.t. $|X_{n_k}| \geq \varepsilon$.

$\forall\ \underset{\sim}{Z} \in D(\underset{\sim}{X}, \varepsilon/2)$: $|X_{n_k} - Z_{n_k}| < \varepsilon/2 \Rightarrow |Z_{n_k}| \geq \varepsilon/2$

$\therefore Z_n \nrightarrow 0 \Rightarrow D(\underset{\sim}{X}, \varepsilon/2) \subseteq \ell^\infty \setminus C_0 \Rightarrow \ell^\infty \setminus C_0$ open

$\Rightarrow C_0$ is closed.

If $\underset{\sim}{X} \in C_0$: $\forall\ \lambda \in \mathbb{C}$, $\lim_{n\to\infty} \lambda \cdot X_n = \lambda \cdot \lim_{n\to\infty} X_n = 0$

$\underset{\sim}{X}, \underset{\sim}{Y} \in C_0$: $\lim_{n\to\infty} X_n + Y_n = \lim_{n\to\infty} X_n + \lim_{n\to\infty} Y_n = 0 + 0 = 0$

$\therefore C_0 \leq \ell^\infty$. $\therefore$ Closed subspace.

Define : $\Phi : \ell^1 \to C_0^*$

$$\Phi(\underset{\sim}{X})(\underset{\sim}{Y}) = \sum_{k\geq 1} X_k Y_k.$$

(Well defined) : $\Phi(\underset{\sim}{X})(\lambda_1 \underset{\sim}{Y}_1 + \lambda_2 \underset{\sim}{Y}_2) = \sum_{k\geq 1} X_k (\lambda_1 (Y_k)_1 + \lambda_2 (Y_k)_2)$

$= \lambda_1 \Phi(\underset{\sim}{X})(Y_1) + \lambda_2 \Phi(\underset{\sim}{X})(Y_2)$

$\therefore$ Linear.

$\forall\ y \in C_0$:

$|\Phi(\underset{\sim}{X})(\underset{\sim}{Y})| \leq \sum_{k\geq 1} |X_k| \cdot |Y_k| \leq \|Y\|_\infty \cdot \|X\|_1$

$\therefore \|\Phi(X)\| \leq \|X\|_1 < \infty$. $\therefore$ Bounded.

Claim: $\|\Phi(X)\| = \|X\|_1$

$z \in \mathbb{C}$: Let $\operatorname{sgn}(z) = \overline{z}/|z|$ $(z \neq 0)$ $\quad (Y_n)_k = 1_{k\leq n} \cdot \overline{\operatorname{sgn}(X_k)}$

$= 1$ $(z = 0)$

$\therefore \Phi(X)(Y_k) = \sum_{r=1}^{k} |X_r|$, $\|Y_k\|_\infty = 1$

$\therefore \|\Phi(X)\| \geq \operatorname{Sup}\{|\Phi(X)(Y_k)| : k \in \mathbb{N}\} = \|X\|_1 \Rightarrow$ Equality

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$\therefore \Phi$ is an isometry $\Rightarrow$ injective, continuous, inverse (if it exists) is continuous

Claim: Φ surjective

Let $\Lambda \in C_0^*$: $x_k = \Lambda(e_k)$, $(e_k)_i = 1_{i=k}$.

$\therefore \Lambda(y) = \sum_{k\geq 1} x_k y_k.$

Let $(y_k)_i = 1_{i \leq k} \operatorname{sgn}(x_k)^{-1}$: $y_k \in C_0$

$\Lambda(y_k) = \sum_{r=1}^{k} |x_r| \leq \|\Lambda\| \Rightarrow (\lim k \to \infty)\ \|x\| \leq \|\Lambda\|$

$\therefore \underset{\sim}{x} \in \lambda^1$

WTP: $\forall y \in C_0$, $\Lambda(y) = \sum_{k=1}^{\infty} x_k y_k.$

$\left|\Lambda(y) - \sum_{k=1}^{N} x_k y_k\right| = |\Lambda(y) - \Lambda(y_1, \dots y_N, 0, \dots)|$

$\leq \|\Lambda\| \ \|(0, \dots 0, y_{N+1}, \dots)\|_\infty$

Since $\lim_{N\to\infty} y_N = 0$: $\lim_{N\to\infty} \|(0, \dots 0, y_{N+1}, \dots)\|_\infty = 0.$

$\therefore \sum_{k=1}^{N} x_k y_k \to \Lambda(y) \Rightarrow \Phi(\underset{\sim}{x}) \cdot \underset{\sim}{y} = \Lambda(\underset{\sim}{x})$

$\therefore$ Surjective $\Rightarrow C_0^* \cong \lambda_1$

(c) ~~Pick~~ $(\underset{\sim}{x})_k = 1/k$: $\underset{\sim}{x} \in C_0$

$(\underset{\sim}{x}^k)_i = (\underset{\sim}{x})_i \cdot 1_{i\leq k}$, $\underset{\sim}{x}^k \in C_{00}$.

$\therefore \|\underset{\sim}{x}^k - \underset{\sim}{x}\|_\infty = 1/(k+1) \to 0$ as $k \to \infty$.

$\therefore \underset{\sim}{x}^k \to \underset{\sim}{x} \notin C_0$ $\therefore$ Not closed.

~~But $(\underset{\sim}{x}^k)_{k\geq 1}$ is Cauchy in λ^∞ (conv) $\Rightarrow$ Cauchy in~~

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$c_{00} \leq c_0 \Rightarrow \overline{c_{00}} \leq c_0$ (closed)

$\forall \underset{\sim}{y} \in c_0$, $(\underset{\sim}{y}^k)_i := (\underset{\sim}{y}^k)_i \cdot 1_{i \leq k} \in c_{00}$, -

$\underset{\sim}{y}^k \to \underset{\sim}{y}$ in λ^∞

$\therefore c_0 \subseteq \overline{c_{00}} \Rightarrow \overline{c_{00}} = c_0$ ※

4) (a) $\forall E$ bounded in X: $T(E)$ is totally bounded.

(b) Subspace: If $T_i \in \mathcal{B}_0$, $\lambda_i \in \mathbb{C}$,

~~Th~~ Let E be bounded: $\exists x_n \in E$ s.t. $T_1(x_n)$ is cauchy

$\{ \cancel{T_1} \in x_n : n \in \mathbb{N} \} \leq E$ bounded $\Rightarrow \exists x_{n_k}$ s.t. $T_2(x_{n_k})$ cauchy.

$\therefore T_i(x_{n_k})$ cauchy $\Rightarrow (\lambda_1 T_1 + \lambda_2 T_2)(x_{n_k})$ cauchy.

$\therefore (\lambda_1 T_1 + \lambda_2 T_2)(E)$ is pre-compact. $\therefore \lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{B}_0$

Closed: If $T_n \in \mathcal{B}_0$, $T_n \to T$.

Fix E bounded, $\varepsilon > 0$: Let $T_n(\{x_1^{(n)}, \dots x_{R_n}^{(n)}\})$ be a ~~ε~~ $\varepsilon/3$-net of $T_n(E)$

$\forall x \in E$: $\|T(x) - T(x_k^{(n)})\| \leq \|T(x) - T_n(\cancel{x_k^{(n)}})\| + \|T(x_k^{(n)}) - T_n(x_k^{(n)})\|$
$+ \|T_n(x) - T(x_k^{(n)})\|$
$\leq 2 \operatorname{Sup}\{\|y\| : y \in E\} \cdot \|T - T_n\|$
$+ \|T_n(x) - T(x_k^{(n)})\|$

Pick N ~~large enough~~ s.t. $n \geq N \Rightarrow \|T_n - T\| < \varepsilon/3 \cdot \operatorname{Sup}\{\|y\| : y \in E\}$

$\therefore \operatorname{Sup}_{x} \operatorname{Max}\{\|T(x) - T(x_k^n)\| : 1 \leq k \leq R_n\} < 2\varepsilon/3 + \operatorname{Sup} \operatorname{Max}\{\|T_n(x) - T_n(x_k^n)\| : 1 \leq k \leq R_n\}$

$< \varepsilon$.

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$\therefore$ ~~$T \in B_0$~~ $\forall\, T(E)$ is totally bounded $\Rightarrow$ (E_1, ε_1 arbitary) $T \in B_0$

If $S \in B$, $T \in B_0$:

$\forall E$ bounded, $S(E)$ bounded $\Rightarrow$ $T \cdot S(E)$ totally bounded

$\therefore T \cdot S \in B_0$

$\exists$ cauchy $(T(x_n))_{n\geq 1}$, $x_n \in E$

$\therefore (S \cdot T(x_n))_{n\geq 1}$ is also cauchy $\Rightarrow$ $S \cdot T \in B_0$

(c) T_n: $(T_n(x))_k = x_{k+1}/k+1 \quad 1_{k\leq n}$.

$$\therefore \| T - T_n(x) \|_2^2 = \sum_{r\geq n+2} |x_{k_r}|^2 / r^2 \leq \frac{1}{r^2} \cdot \|x\|^2$$

(True $\forall x \in \lambda^2$) $\therefore \| T - T_n \| \leq 1/r \Rightarrow T_n \to T$.

T_n is finite rank operator $\Rightarrow$ Compact ($T(E)$ bounded in finite dim. space $\Rightarrow$ Pre-compact)

$\therefore T \in B_0$

$Tx = \lambda x$: $\quad \frac{x_{k+1}}{k+1} = \lambda x_k \Rightarrow x_{k+1} = \lambda (k+1) x_k$.

$\Rightarrow x_{n+1} = \lambda^n \cdot (n+1)!\, x_1$

If $x_1 \neq 0$: $\| \underset{\sim}{x} \|^2 = \left(\sum_{k\geq 1} \lambda^{k-1} k! \right) |x_1|^2 = \infty$ (rej.)

$\therefore \sigma_{p\pi} = \phi \Rightarrow \sigma_\pi(T) = \sigma_p \cup \{0\} = \{0\}$

3.) Let K be compact, $A \subseteq C_{\mathbb{R}}(K)$ be an algebra
(pointwise $+$, $\cdot$ of function, scalar mult.)

$\forall\, x, y \in K,\ x \neq y \Rightarrow \exists\, f \in A$ s.t. $f(x) \neq f(y)$.

If $\exists\, x_0$ s.t. $\forall f \in A$, $f(x_0) = 0$: $\bar{A} = \{ f \in C(K) : f(x_0) = 0 \}$

Else, $\bar{A} = C(K)$. (Stone Weierstrass)

Suppose there is no such x_0. : Let $\cancel{g}\, f \in C(K)$

Fix $x, y \in K$, $x \neq y$:

$\cancel{\exists\, f \text{ s.t. } f(x) \neq f(y),\ g \text{ s.t. } g(x) \neq 0}$

$\cancel{\text{If } f(x), f(y) \neq 0 :\ h_1 = f,\ h_2 = f^2.}$

$\cancel{\text{Else, for } \varepsilon \text{ small, } (f+\varepsilon g)(x) \neq 0,\ (f+\varepsilon g)(y) \neq (f+\varepsilon g)(x) \quad h_1 = f+\varepsilon g,\ h_2 = h_1^2}$

$\cancel{\therefore \begin{pmatrix} h_1(x) & h_2(x) \\ h_1(y) & h_2(y) \end{pmatrix}}$

$\cancel{\text{If } g(y)}$

$\cancel{\text{Claim: } \exists\, h}$

$\cancel{\text{Let } h(x) \neq h(y),\ g(x) \neq 0. \text{ By multiplying with a scalar, WLOG, } g(x) \neq 1, -1}$

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1.) Let K be compact, $A \subseteq C(K)_{\mathbb{R}}$ be an algebra.

If $\forall x, y \in K$, $x \neq y \Rightarrow \exists f_{x,y} \in A$ s.t. $f_{x,y}(x) \neq f_{x,y}(y)$:

If $\exists x_0 \in K$ s.t. $\forall f \in A$, $f(x_0) = 0$; $\bar{A} = \{ f \in C(K)_{\mathbb{R}} : f(x_0) = 0 \}$

Else, $\bar{A} = C(K)_{\mathbb{R}}$. (Stone Weierstrass, $\mathbb{R}$)

(a) $[0,1]$ compact $\Rightarrow [0,1]^2$ compact.

Let $A = \{ (x,y) \to \sum_{n=1}^{N} f_n(x) \cdot g_n(y) : f_n, g_n \in C_{\mathbb{R}}[0,1], N \in \mathbb{N} \}$

Claim: A is an algebra.

$$\left(\sum_{k=1}^{N_1} f_k g_k \right) \cdot \left(\sum_{j=1}^{N_2} f'_j g'_j \right) = \sum_{j=1,k=1}^{N_2, N_1} (f_k f'_j) \cdot (g_k g'_j) \in A.$$

$$\lambda \in \mathbb{R}: \quad \lambda \sum_{k=1}^{N_1} f_k g_k = \sum_{k=1}^{N_1} (\lambda \cdot f_k) \cdot g_k \in A$$

$$\sum_{k=1}^{N_1} f_k g_k + \sum_{k=N_1+1}^{N_2} f_k g_k = \sum_{k=1}^{N_2} f_k g_k \in A.$$

Claim: A is separating.

If $(x_1, y_1) \neq (x_2, y_2)$: WLOG $x_1 \neq x_2$.

$g(y) = 1$, $f(x) = \inf \{ |x - z| : z \in \bar{D}(x_1, \varepsilon) \}$, $\varepsilon < |x_1 - x_2|$

$f(x_1) = \varepsilon$, $f(x_2) = 0$, $|f(x) - f(x')| \leq |x - x'| \Rightarrow f$ cont.

$\therefore f \cdot g(x_1, y_1) \neq f \cdot g(x_2, y_2)$

$\forall$ ~~$x \in$~~ $(x,y) \in [0,1]^2$: $f, g = 1 \Rightarrow f \cdot g(x,y) = 1 \neq 0$

$\therefore \bar{A} = C([0,1]^2)_{\mathbb{R}}$. (SWT)

(b) $\therefore$ Given $\varepsilon > 0$, $k \in C([0,1]^2)_{\mathbb{R}} \Rightarrow \exists \sum_{k=1}^{N} f_k g_k \in A$ s.t. $\| K - \sum_{k=1}^{N} f_k g_k \|_\infty < \varepsilon$

$\therefore K$ can be uniformly approximated

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(b) T is ~~bounded~~ compact if $\overline{T(D(0,1))}$ is compact

T is finite rank if $\mathrm{Im}(T) \leq X$ is finite dimensional

$i : C[0,1] \to C[0,1]$ is not compact.

i is clearly linear, $\|i\| = 1$ (bounded); $\overline{i(D(0,1))} = \overline{D(0,1)}$ is not compact as $C[0,1]$ is ∞ dim $\Rightarrow$ Unit closed ball not compact.

(c) $T_n(D(0,1))$ is bounded (T_n bounded), $T_n(D(0,1))$ in a finite dim sub space (closed in X) $\Rightarrow$ $\overline{T_n(D(0,1))}$ compact (Closure of bounded set in finite dim space is compact)

Claim: T is compact. We will show E bounded $\Rightarrow$ $T(E)$ totally bounded

Fix $\varepsilon > 0$, ~~pick n s.t.~~ let $M = \sup\{\|x\| : x \in E\}$

Pick n s.t. $\|T_n - T\| < \varepsilon/3M$ Let $\{T_n(x_1), \dots T_n(x_k)\}$ be $\varepsilon/3$ net of $T_n(E)$.

$$\forall x \in E: \quad \|T(x) - T(x_r)\| \leq \|T(x) - T_n(x)\| + \|T(x_r) - T_n(x_r)\| + \|T_n(x_r) - T_n(x)\|$$
$$\leq 2\|T - T_n\| \cdot M + \|T_n(x_r) - T_n(x)\|$$

$$\therefore \min_{1 \leq r \leq k}\{\|T(x) - T(x_r)\|\} \leq 2\varepsilon/3 + \min_{1 \leq r \leq k}\{\|T_n(x_r) - T_n(x)\|\} < \varepsilon.$$

$\therefore$ T compact.

(d) Pick $k_n(x,y) = \sum_{r=1}^{N_n} f_r^{(n)}(x)\, g_r^{(n)}(y)$, $\|k - k_n\|_\infty < 1/n$.

~~$T_n(x) = \int_0^1 k(x)k($~~ $T_n(f)(x) = \int_0^1 k_n(x,y)\, f(y)\, dy$

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$T_n(f) = \sum_{r=1}^{N_n} f_r \cdot \int_0^1 g_r(y) f(y)\, dy \in \langle f_1, \dots f_{N_n} \rangle$

$\therefore$ T_n is finite rank.

$\| (T_n - T)(f) \| \leq \| x \mapsto \int_0^1 |f(y)| |k_n - k(x,y)| \, dy \|$

$\leq \|f\|_\infty \cdot \|k - k_n\| \leq 1/n \, \|f\|_\infty$

(Valid $\forall f$): $\therefore \|T_n - T\| \to 0$ as $n \to \infty$

$\therefore$ T is compact.

2.) (a) Baire Category Theorem (BCT): Let (X, d) be a complete metric space. $U_n \subseteq X$ be open, dense. $\bigcap_{n \geq 1} U_n \neq \emptyset$

Pick $x_1 \in U_1$ ($\neq \emptyset$, dense): $\varepsilon_1 \in (0,1)$ s.t. $D(x_1, 2\varepsilon_1) \subseteq U_1$

Suppose we have $x_1, \dots x_n$; $\varepsilon_1, \dots \varepsilon_n$ satisfying: $\varepsilon_k \in (0, 1/k)$,

$$\left[\begin{rcases} x_k \in D(x_k, 2 \cdot \varepsilon_k) \subseteq U_k \cap D(x_{k-1}, \varepsilon_{k-1}) \\ x_1 \in D(x_1, 2\varepsilon_1) \subseteq U_1 \end{rcases} (n \geq k \geq 1) \right]$$

Pick $x_{n+1} \in U_{n+1} \cap D(x_n, \varepsilon_n)$ (open, $\neq \emptyset$ by density of U_{n+1}),

$\therefore$ $\varepsilon_{n+1} \in (0, 1/n+1)$ s.t. $D(x_{n+1}, 2\varepsilon_{n+1}) \subseteq U_{n+1} \cap D(x_n, \varepsilon_n)$

$\therefore$ We can continue inductively (with axiom of choice) to construct sequence $(x_n)_{n \geq 1}$

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$\forall n \in \mathbb{N}$: $\quad x_{n+1} \in D(x_{n+1}, 2\varepsilon_n) \subseteq D(x_n, \varepsilon_n)$

$\therefore \quad x_{n+1} \in \bar{D}(x_{n+1}, \varepsilon_n) \subseteq \bar{D}(x_n, \varepsilon_n)$ 또한

and $\quad \bar{D}(x_n, \varepsilon_n) \subseteq U_n$

$\therefore \quad \forall\ n, m \geq N, \quad x_n, x_m \in \bar{D}(x_N, \varepsilon_N) \Rightarrow d(x_n, x_m) < 1/N$

$\therefore$ $(x_n)_{n\geq 1}$ is cauchy.

But $\forall\ k \geq 1$, $\quad x_{n+k} \in \bar{D}(x_{n+k}, \varepsilon_{n+k}) \subseteq \ldots \subseteq \bar{D}(x_n, \varepsilon_n) \subseteq U_n$

$\therefore$ $(x_{n+k})_{k\geq 1}$ (subsequence of cauchy sequence) is cauchy $\Rightarrow$ converges to x in $\in X$; $\bar{D}(x_n, \varepsilon_n)$ closed $\Rightarrow$

$x \in \bar{D}(x_n, \varepsilon_n) \subseteq U_n$

Since subsequences have the same limit as the cauchy sequence:

$x \in \bigcap_{n\geq 1} U_n \Rightarrow$ RHS $\neq \phi$ $\times$

(b) f is cont. at x: ~~iff $\forall \delta>0$~~, Given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t

$$\sup_{y \in \bar{D}(0,\delta)} \{ |f(y) - f(x)| \} < \varepsilon/2 \Rightarrow \sup_{y_i \in \bar{D}(0,\delta)} \{ |f(y_1) - f(y_2)| \} \leq$$

$$\sup_{y_1 \in \bar{D}(0,\delta)} \{ |f(y_1) - f(x)| \} + \sup_{y_2 \in \bar{D}(0,\delta)} \{ |f(y_2) - f(x)| \}$$

$$< \varepsilon.$$

$$\therefore \underbrace{\inf_{\delta>0} \left[\sup_{|y_i - x| \leq \delta} \{ |f(y_1) - f(y_2)| \} \right]}_{(1)} \leq \varepsilon$$

ε arbitrary $\Rightarrow$ (1) $= 0$.

If $\omega_f(x) = 0$: Given $\varepsilon > 0$, $\quad \omega_f(x) = 0 \Rightarrow \exists\ \delta > 0$ s.t

$$\underbrace{\sup_{|y_i - x| \leq \delta} \{ |f(y_1) - f(y_2)| \}}_{(2)} < \varepsilon \Rightarrow; \quad (2) \geq \sup_{|y_1 - x| \leq \delta} \{ |f(y_1) - f(x)| \}$$

$\therefore \quad |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$

$\therefore \quad f$ cont. at x $\times$

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Let $x_0 \in \{x : \omega_f(x) < \varepsilon\}$: $\exists\ \delta > 0$ s.t. $\sup_{|y_i - x| \leq \delta} \{ |f(y_1) - f(y_2)| \} < \varepsilon$

$\forall\ z \in D(x, \delta/4)$: $\forall\ w_i \in \overline{D}(z, \delta/4)$, $|w_i - x| \leq \delta/2$.

$\therefore \sup_{|w_i - z| \leq \delta/4} \{ |f(w_1) - f(w_2)| \} < \varepsilon \Rightarrow \omega_f(z) < \varepsilon$.

$\therefore D(x, \delta/4) \subseteq \{x : \omega_f(x) < \varepsilon\} \Rightarrow$ RHS is open.

(c) Suppose otherwise: $A_n = \{x \in \mathbb{R} : \omega_f(x) \geq 1/n\}$

$\therefore \{x \in \mathbb{R} : f \text{ not cont. at } x\} = \bigcup_{n \geq 1} A_n$.

$\mathbb{R} = \bigcup_{n \geq 1} \{q_n\} \cup \bigcup_{n \geq 1} A_n$, $\mathbb{R}$ is complete,

* $\{q_n : n \in \mathbb{N}\}$ is an enumeration of $\mathbb{Q}$, $\{q_n\}$ is closed, nowhere dense.

From (b), A_n is closed.

If $A_n^\circ \neq \phi$, $\exists\ D(z, \varepsilon) \subseteq A_n^\circ$. But $\mathbb{Q}$ dense in $\mathbb{R} \Rightarrow$ $\mathbb{Q} \cap D(z, \varepsilon) \neq \phi \Rightarrow \mathbb{Q} \cap A_n \neq \phi$ (reject as f not cont. ~~points~~ on A_n, cont. on $\mathbb{Q}$).

$\therefore A_n^\circ = \phi$ (nowhere dense)

$\therefore \mathbb{R}$ = countable union of nowhere-dense closed sets $\Rightarrow$

ϕ = countable intersection of dense open sets.

$\therefore$ BCT $\Rightarrow$ Contradiction.

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3.) (a) $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\}$

(b) Unique extension:

Define $\tilde{T} : H \to H$, $\tilde{T}(\sum_n x_n e_n) = \sum_n x_n (e_{n-1} + e_{n+1})$

$= \sum_n (x_{n-1} + x_{n+1}) e_n$

$\| \sum_n (x_{n-1} + x_{n+1}) e_n \|_2^2 = \sum_n (x_{n-1} + x_{n+1})^2 \leq 2 \| \underset{\sim}{x} \|^2 < \infty.$

$\therefore \tilde{T}$ is bounded, ($\tilde{T}$ clearly linear) $\Rightarrow \tilde{T} \in B(H)$

$\| T(e_n) \|_2^2 = 2 \| e_n \|_2^2 \Rightarrow \sqrt{2} \geq \| \tilde{T} \| \geq \sqrt{2} \Rightarrow$ Equality

$\tilde{T}(e_n) = T(e_n)$ by def $\Rightarrow$ Extension

If $\tilde{T}_i$ are both extensions: $\langle e_n : n \in \mathbb{Z} \rangle$ is dense $\Rightarrow$

$\forall x \in H$: $\exists\, a_n \in \langle e_k : k \in \mathbb{Z} \rangle$, $a_n \to x$.

$\tilde{T}_i(a_n) \to \tilde{T}_i(x)$ (by continuity)

$\tilde{T}_0(a_n) = T(a_n) \Rightarrow$ ~~Unique~~ $\tilde{T}_i(x)$ equal. $\therefore$ Unique

Self adjoint: $\langle T(\sum_k x_k e_k), \sum_{k_i} y_{k_i} e_{k_i} \rangle = \sum_n \overline{(x_{n-1} + x_{n+1})}\, y_n$

$= \sum_n \cancel{\langle x}\ \overline{x_n}\, y_{n+1} + \overline{x_n}\, y_{n-1} = \langle \sum_i x_k e_k, T(\sum_i y_{k_i} e_i) \rangle$ $\cancel{x}$

Claim: $0 \in \sigma(T)$

Else: T surjective. Let $(T(\underset{\sim}{x}))_k = \delta_{k,0}$

$\therefore k \neq 0$: $x_{k-1} = -x_{k+1}$; $k = 0$ $x_{-1} + x_1 = \cancel{1/2}\ 1 \Rightarrow x_{\pm 1} = 1/2$

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$\therefore$ ~~X_{2k}~~ $|X_{2k+1}| = 1/2 \quad (k \in \mathbb{Z})$

$\therefore \|X\|_2^2 \geq \sum_{k\in\mathbb{Z}} |X_{2k+1}|^2 = \infty \Rightarrow$ contradiction

$\therefore$ T not surjective $\Rightarrow$ T not invertible

(c) Let $TX = \lambda X$ =

$\forall n \in \mathbb{Z}: \quad X_{n+1} + X_{n-1} = \lambda X \quad , \quad \lambda \in \mathbb{R}$ (T self adjoint $\Rightarrow \sigma_p(T) \subseteq \mathbb{R}$)

$\sigma_p(T) \subseteq \sigma(T) \subseteq \overline{D}(0, \sqrt{2})$

If $\lambda = 0$: $X_{n+1} = -X_{n-1} \Rightarrow X_n \equiv 0 \quad (\forall n \in \mathbb{Z})$ else $\|X\| = \infty$.

$\therefore 0 \notin \sigma_p(T)$

We need $\lim_{n\to\infty} X_n = 0$.

If $\lambda \neq 0$: $X_n = (A + nB)\alpha^n$ or $A\alpha^n \sin(\theta_1 \cdot n + \theta_2)$

Case 1: If $|\alpha| < 1$: $\lim_{n\to -\infty} |(A+nB)\alpha^n| = \infty$ if ~~$\forall$~~ $A, B \neq 0$.

$|\alpha| \geq 1 \quad \lim_{n\to\infty} |(A+nB)\alpha^n| \geq 1$ if $A, B \neq 0$.

$\therefore$ ~~Reject~~. No solution

Case 2:

~~Claim: $|\sin(\theta_1 n + \theta_2)| > \varepsilon$ i.o. (as $n \to \infty$) if~~ $\theta_1 \neq 0$.

$\theta_1 \in (0, \pi)$

If $\theta_1 = 0$: $X_n = A \cdot \alpha^n \sin(\theta_2) \equiv 0 \quad (\forall n \in \mathbb{N})$ or

$\sum_n |X_n|^2 = |A \sin\theta_2|^2 \cdot \sum_n |\alpha|^n = \infty$ } rej

Let $\theta_1 \in (0, \pi)$: Pick $\varepsilon < \min\{\theta_1, \pi - \theta_1\}$

$\therefore \Theta_\varepsilon = [0, \varepsilon) \cup (\pi - \varepsilon, \pi]$

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If $\Theta_1 n + \Theta_0 \in \Theta_\varepsilon$: $\Theta_1 (n \pm 1) + \Theta_0 \notin \Theta_\varepsilon$

$\therefore$ ~~infinitely many~~ n

$$\| X \|_2^2 \geq |A|^2 \sum_{\substack{n : \cancel{|\sin(n\Theta_1+\Theta_0)|} \\ (n\Theta_1 + \Theta_0) \notin \Theta_\varepsilon \ (\mathrm{mod}\ \pi)}} |\alpha^n \sin(n\Theta_1 + \Theta_0)|^2$$

$$\geq |A|^2 \cdot |\sin \varepsilon|^2 \cdot \sum_{n_k} |\alpha|^{n_k}$$, n_k is a subsequence ($k \in \mathbb{Z}$)

$$\geq \mathrm{Max} \left\{ |A|^2 |\sin \varepsilon|^2 \sum_{k \geq 0} |\alpha|^{\pm k} , \right\}$$

$$= \infty$$

$\therefore$ No eigenvalue $\Rightarrow$ No eigenvector.

Not compact: Else self adjoint, compact $\Rightarrow$ $\exists$ eigenvalue.

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4.) (a) Let X be a normal space, $A_n \subseteq X$ be disjoint closed.

$\exists f : X \to [0,1]$ cont. s.t. $f|_{A_0} = 0$, $f|_{A_1} = 1$ (Urysohn)

Tietze: Let $f : A \to [0,1]$ be cont., A closed, $A \subseteq X$,

X normal. $\exists \tilde{f} : X \to [0,1]$ cont., $\tilde{f}|_A = f$.

~~If $f : A \to [0, a]$, $\exists g : X \to [0, a]$ cont., $\|f - g\| \leq 2a/3$~~

~~Let $A_1 = f^{-1}[0, a/3]$, $A_2 = f^{-1}[2a/3, a]$; A_i: closed, disjoint.~~

~~$\therefore \exists g : X \to [0, a]$ s.t. $g|_{A_1} = 0$, $g|_{A_2} = a$~~

~~$\therefore \forall x \in f^{-1}(a/3, 2a/3)$, $g(x) \in [0, a] \Rightarrow |g(x) - f(x)| \leq 2a/3$.~~

~~$\therefore \|f - g\| \leq 2a/3$.~~

~~$\therefore$ Let $f_0 = f$; Given $f_n : A \to [0, (2/3)^n]$, $\exists g_n$ s.t. $\|f_n - g_n\|_A \leq (2/3)$~~

~~$\|f_n - g_n|_A\| \leq (2/3)^{n+1}$, g_n cont. on X~~

~~$\therefore$ Let $f_{n+1} = f_n -$~~

If $f : A \to [-a, a]$: Let $A_1 = f^{-1}[-a, -a/3]$, $A_2 = f^{-1}[a/3, a]$.

$\therefore A_i$ disjoint, closed.

Urysohn: $\exists g : X \to [-a/3, a/3]$ s.t. $g|_{A_1} = -a/3$, $g|_{A_2} = a/3$

$\wedge \forall z \in f^{-1}(-a/3, a/3)$, $|g(z) - f(z)| \leq 2a/3$.

$\therefore \|f - g\| \leq 2a/3$

Let $f_0 = f$; ~~Given $f_n : A \to [-(\frac{1}{3})^n, (\frac{1}{3})^n]$~~ pick g_n ~~as above~~ s.t.

~~$\|g_n - f_n\| \leq 2/3$~~

Given $f_n : A \to [-(\frac{2}{3})^n, (\frac{2}{3})^n]$: $\exists g_n : X \to [-(\frac{1}{3})(\frac{2}{3})^n, (\frac{1}{3})(\frac{2}{3})^n]$

cont., $\|f_n - g_n|_A\| \leq (2/3)^{n+1}$, $\|g_n\| \leq \frac{1}{3} \cdot (2/3)^n$.

$\therefore \| f - g_0|_A - \cdots - g_n|_A \| \le (2/3)^{n+1} \to 0$

$g = \sum_{k \ge 0} g_k$; $\| g - \sum_{k=0}^{N} g_k \| \le \cancel{2/9 \frac{1}{1-2/3}} = 2/3$ ~~n~~ $\frac{1}{3} \left(\frac{2}{3}\right)^{N+1} \frac{1}{1-2/3}$

$\to 0$ as $N \to \infty$. (Uniform conv.)

$\sum_{k=0}^{N} g_k$ cont., g uniform limit of cont. function $\Rightarrow$ cont.

$\| f - \sum_{k=0}^{N} g_k|_A \| \le (2/3)^{n+1} \to 0 \Rightarrow$ ~~K~~ $f = \sum_{k=0}^{N} g_k |_A$.

$\therefore$ Done

① Claim : X is normal

Let $A_i \subseteq X$ be closed disjoint.

(4)$_1$ $\forall x \in A_1$, $x \in X \setminus A_2$ (open) $\Rightarrow \exists\, \varepsilon_x > 0$ s.t.
$D(x, \varepsilon_x) \subseteq X \setminus A_2$

(4)$_1$ Similarly, $\forall y \in A_2$, $\exists\, \varepsilon_y > 0$ s.t. $D(y, \varepsilon_y) \subseteq X \setminus A_1$

$U_1 = \bigcup_{x \in A_1} D(x, \varepsilon_x/3)$, $U_2 = \bigcup_{y \in A_2} D(\cancel{x}\, y; \varepsilon_y/3)$

If $z \in U_1 \cap U_2$: $z \in D(x, \varepsilon_x/3) \cap D(y, \varepsilon_y/3)$

$\therefore d(x, y) \le d(x, z) + d(z, y) \le \varepsilon_x/3 + \varepsilon_y/3 \le 2/3\ \varepsilon_x \vee \varepsilon_y$
$< \varepsilon_x$ or ε_y.

$\therefore y \in D(x, \varepsilon_x)$ or $x \in D(y, \varepsilon_y) \Rightarrow$ Contradiction

$\therefore$ Normal; Tietze theorem $\Rightarrow$ Extension property

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(2) X compact hausdorff $\Rightarrow$ Normal

Let $g: \mathbb{R} \to (0,1)$ be a homeomorphism

$\therefore g \cdot f : S \to (-1, 1) \subseteq [-1, 1]$ is cont.

Tietze $\Rightarrow \exists\, h: X \to [-1, 1]$, $h|_S = g \cdot f$.

$A = h^{-1}\{-1, 1\}$ is closed; Urysohn $\Rightarrow \exists\, \phi$ cont., $\phi: X \to [0,1]$,
$\phi|_A = 0$, $\phi|_S = 1$ ($A \cap S = \phi$ as $S \subseteq h^{-1}(-1,1)$)

$\therefore x \mapsto \phi(x) \cdot h(x)$ has im $(-1, 1)$,
$\phi \cdot h|_S = g \cdot f$.

$\therefore g^{-1} \cdot \phi \cdot h$ is a cont. extension of f.

(Valid)

(3) $X = \mathbb{N}$, $\mathcal{T}$ = finite complement

$S = \{1, 2\} \subseteq X$, $\mathbb{N} \setminus \{1\}$ open $\Rightarrow \{2\}$ open in S subspace topology

$\therefore \mathcal{T}_S$ = discrete.

$f: \left.\begin{array}{l} 1 \to 1 \\ 2 \to -1 \end{array}\right\}$ cont. on S

If $g: X \to [-1,1]$ is cont., $g(x) \neq g(y)$: Pick ε s.t.
$D(x, \varepsilon) \cap D(y, \varepsilon) = \phi$

$\therefore g^{-1}(D(x, \varepsilon)) \subseteq \underbrace{X \setminus g^{-1}(D(x, \varepsilon))}_{\text{finite}} \Rightarrow X \setminus g^{-1}(D(x, \varepsilon))$ not finite $\Rightarrow$ not open

$\therefore$ Contradiction

$\therefore g$ constant $\Rightarrow$ No cont. extension of f.

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4.) (a) $\sigma(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ not invertible}\}$,

$\text{Gap}(T) = \{\lambda \in \mathbb{C} : \exists x_n \in X, \|x_n\| = 1, \|(T-\lambda I)(x_n)\| \to 0\}$

Claim: $\text{Gap}(T) \subseteq \sigma(T)$

Suppose $\exists \lambda \in \text{Gap}(T)$, $\lambda \notin \sigma(T)$

$\therefore$ $T-\lambda I$ is invertible. Let $T_\lambda = T - \lambda I$

$\therefore$ ~~T_λ^{-1} is bounded,~~ T_λ is bijective (inv. invertible), bounded map between banach spaces $\Rightarrow$ T_λ ~~has~~ is an open map $\Rightarrow$ T_λ^{-1} is continuous $\Rightarrow$ T_λ^{-1} is bounded.

If $\|x_n\| = 1$, $y_n = (T_\lambda)\cdot(x_n)$, $z_n = y_n / \|y_n\|$;

T_λ invertible, $\|x_n\| \neq 0 \Rightarrow y_n \neq 0 \Rightarrow \|y_n\| \neq 0$.

$\therefore T_\lambda^{-1}(z_n) = x_n / \|y_n\| \Rightarrow \|T_\lambda^{-1}(z_n)\| = 1/\|y_n\| \to \infty$, $\|z_n\| = 1$

$\therefore$ Contradicts T_λ^{-1} bounded.

$\therefore \text{Gap}(T) \subseteq \sigma(T)$

Claim: Invertible operators forms an open set (*: Extra)

~~Pf~~ Let S be invertible: $T = S - (S-T) = S\cdot(I - S^{-1}(S-T))$

If $\|S^{-1}\cdot(S-T)\| < 1$, $I - S^{-1}(S-T)$ is invertible $\Rightarrow$ T invertible

If $\|S-T\| < 1/\|S^{-1}\|$: $\|S^{-1}\cdot(S-T)\| \leq \|S^{-1}\|\cdot\|S-T\| < 1$.

$\therefore D(S, 1/\|S^{-1}\|) \subseteq \{\text{inv. operators}\} \Rightarrow$ open

$\therefore$ If $T-\lambda I$ is invertible ($\lambda \notin \sigma(T)$): $D(\lambda, 1/\|(T-\lambda I)^{-1}\|) \subseteq \sigma(T)^c$.

$\therefore \sigma(T)^c$ open $\Rightarrow \sigma(T)$ is closed.

$T - \lambda I = -\lambda (I - T/\lambda)$ ∴ If $\|T\| < |\lambda|$, $\|T/\lambda\| < 1 \Rightarrow$
$I - T/\lambda$ invertible $\Rightarrow$ $T - \lambda I$ invertible

∴ $\sigma(T) \subseteq \overline{D}(0, \|T\|)$ (Bounded)

(b) Claim: $\|T\| \overset{(1)}{=} \sup_{\|x\|, \|y\| \le 1} \{ |\langle T(x), y \rangle| \} \overset{(2)}{=} \sup_{\|x\| \le 1} \{ |\langle T(x), x \rangle| \}$

$$\|T\|^2 = \sup_{\|x\| \le 1} \{ |\langle T(x), T(x) \rangle| \} = \|T\| \cdot \sup_{\|x\| \le 1} \left\{ \left|\left\langle T(x), \frac{T(x)}{\|T\|} \right\rangle\right| \right\}$$

$$\le \|T\| \sup_{\substack{\|x\| \le 1, \\ \|y\| \le 1}} \{ |\langle T(x), y \rangle| \} \le \|T\| \cdot \|T\| \cdot \underbrace{\|x\| \, \|y\|}_{=1}$$

∴ Equality (1) holds

$$\sup_{\|x\|, \|y\| \le 1} \{ |\langle T(x), y \rangle| \} \ge \sup_{\|x\| \le 1} \{ |\langle T(x), x \rangle| \}$$ by definition

If $\langle T(x), y \rangle$ is real:

$$\langle T(x+y), x+y \rangle - \langle T(x-y), (x-y) \rangle = 2 \langle T(x), y \rangle + 2 \langle T(y), x \rangle$$
$$= 4 \langle T(x), y \rangle$$

$$\therefore |\langle T(x), y \rangle| \le \frac{1}{4} \{ |\langle T(x+y), x+y \rangle| + |\langle T(x-y), (x-y) \rangle| \}$$

$$\le \frac{\cancel{\|T\|}\,(*)}{4} \{ \|x+y\|^2 + \|x-y\|^2 \} = \cancel{\|T\|} \ (*)$$

$$(*) = \sup_{\|z\| \le 1} \{ |\langle T(z), z \rangle| \}$$

$$\therefore \text{LHS} = \sup_{\substack{\|x\|, \|y\| \le 1, \\ \langle T(x), y \rangle \in \mathbb{R}}} \{ |\langle T(x), y \rangle| \} \le \text{RHS} \Rightarrow \text{Done.}$$

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Claim : $\|T\| = \lambda$, λ or $-\lambda$ is an eigenvalue.

~~Pick x_n, $\|x_n\| = 1$ s.t. $\langle x_n, T(x_n)\rangle \to \|T\|$~~

~~$\therefore \exists$ Cauchy subsequence~~ $(T(x_{n_k}))_{k\geq 1}$ is cauchy

$\langle x, T(x)\rangle = \langle T(x), x\rangle = \overline{\langle x, T(x)\rangle} \Rightarrow \in \mathbb{R}$.

Pick x_n, $\|x_n\| = 1$ s.t. $\langle x, T(x)\rangle \to \lambda$ or $-\lambda$ $= \alpha$

By passing to a subsequence: $T(x_{n_k})$ is cauchy, $T(x_{n_k}) \to y$

$0 \leq \|T(x_n) - \alpha x_n\|^2 = \|T(x_n)\|^2 + \alpha^2 - 2\alpha \langle T(x_n), x_n\rangle$

$\leq 2\alpha^2 - 2\alpha \langle T(x_n), x_n\rangle \to 0.$

$\therefore T(x_n) - \alpha x_n \to 0 \Rightarrow x_n \to y/\alpha$

$\therefore T(y) = \alpha \lim_n T(x_n) = \alpha y \Rightarrow$ Eigenvalue.

$\therefore \lambda$ or $-\lambda \in \sigma(T) \Rightarrow \neq \emptyset$

(c) Let $\Lambda = \{q_n : n \in \mathbb{N}\} = K \cap \{$rational coordinates of $\mathbb{C}\}$, Λ is countable, bounded (compact)

Define : $(f(\underline{x}))_n = (q_n \cdot x_n)$

$\|f(x_n)\|_2^2 = \sum_{n\geq 1} |q_n|^2 \cdot |x_n|^2 \leq M \cdot \|x\|^2$, $\Lambda \subseteq \bar{D}(0, M)$

$\therefore f$ is bounded, linear

e_n is eigenvector to f, $f - q_n(e_n) = 0$.

$\therefore \Lambda \subseteq \sigma(f)$

If $\lambda \in K^c$ (open) : $|\lambda - q_n| \geq \varepsilon$, $D(\lambda, \varepsilon) \subseteq K^c$.

$\therefore ((f-\lambda I)^{-1}(\underline{x}))_n = \dfrac{x_n}{q_n - \lambda} \Rightarrow$ Invertible.

$\therefore \Lambda \subseteq \sigma(f) \subseteq \mathbb{C} \setminus K.$

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$G(f)$ closed $\Rightarrow$ $\overline{A} = K \subseteq G(f)$

$\therefore$ $G(f) = K$.

3.) (a) Let K be compact, $C(K) = \{ f: K \to \mathbb{R} : f \text{ continuous} \}$

Arzela Ascoli : Let $F \subseteq C(K)$. F is bounded, equicontinuous iff F is totally bounded.

F bounded: $\sup \{ \| f \|_\infty : f \in F \} < \infty$

F equicontinuous : $\forall x \in K$, $\forall \varepsilon > 0$: $\exists U_x \subseteq K$ open, $x \in U_x$ s.t. $\forall y \in U_x$, $\forall f \in F$, $|f(x) - f(y)| < \varepsilon$.

F totally bounded: $\forall \varepsilon > 0$, $\exists f_1, \dots f_n \in F$ s.t. $F \subseteq \bigcup_{k=1}^{n} D(f_k, \varepsilon)$

Stone - Weierstrass: Let $A \subseteq C(K)$ be closed wrt scalar wrt:

$\forall f \in A$, $\forall \lambda \in \mathbb{R}$: $\lambda \cdot f \in A$

$\forall f_i \in A$: $(f_1 + f_2) : x \to f_1(x) + f_2(x)$, $f_1 + f_2 \in A$

$(f_1 \cdot f_2) : x \to f_1(x) f_2(x)$, $f_1 f_2 \in A$.

If $\forall x, y \in K$, $x \neq y$: $\exists f \in A$ s.t. $f(x) \neq f(y)$:

If $\exists x_0 \in \overset{K}{\cancel{A}}$ s.t. $\forall f \in A$, $f(x_0) = 0$: $\overline{A} = \{ f \in C(K) : f(x_0) = 0 \}$

Else, $\overline{A} = C(K)$

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(ii) $\forall f \in F: \quad |f(x)| \leq \sum_{n\geq 1} |a_n| \cdot 1 \leq \sum_{n \geq 1} 1/n^3 < \infty.$ (independent of x)

$\therefore F$ is bounded.

$$|f(x) - f(y)| \leq \sum_{n\geq 1} |a_n| \cdot \underbrace{|\sin(nx) - \sin(ny)|}_{\text{Mean value theorem}}$$

Mean value theorem : $\leq n|x-y| \cdot |\cos z|$

$\leq n|x-y|$

$$\leq |x-y| \sum_{n\geq 1} n \cdot 1/n^3 = C \cdot |x-y|, \quad C \overset{>0}{,} \text{ is constant}$$

$\therefore$ Given $\varepsilon > 0$, $\forall x \in [0,1]$: Pick $U_x = D(x, \varepsilon/C)$

$y \in U_x$, $f \in F \Rightarrow |f(y) - f(x)| < \varepsilon$.

$\therefore$ Equicontinuous.

Arzela Ascoli : F is totally bounded $\Rightarrow \bar{F}$ is compact.

$\forall$ sequence $(f_n)_{n\geq 1}$ in F:

$\therefore \exists$ subsequence $(f_{n_k})_{k\geq 1}$ s.t. $(f_{n_k})_{k \geq 1} \to f \in \bar{F}$ (wrt $\|\cdot\|$ norm)

$\therefore$ ~~$(f_{n_k})_{k\geq 1}$ is Cauchy.~~

(iii) Let $g(x) = f(x)/x \quad : x \neq 0$
$\qquad = f'(0) \quad : x = 0$ $\Big\}$ $\lim_{x\to 0} g(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(x)$

$\therefore g$ is cont. $\Rightarrow g \in C[0,1]$

Let $A = \left\{ \sum_{k=0}^{n} a_k x^{2k} : \forall n \in \mathbb{N}, a_k \in \mathbb{R} \right\}$: A is an algebra

If $x, y \in [0,1]$, $x \neq y$: $x^2 \neq y^2 \Rightarrow A$ is separating

$\forall x \in [0,1]$: $1(x) = 1 \neq 0 \Rightarrow \bar{A} = C[0,1]$

Given $\varepsilon > 0$, $\exists P_\varepsilon \in A$ s.t. $\|g - P\| < \varepsilon$.

$x \neq 0$: $g(x) \cdot x = f(x)$; $x = 0$: $g(x) \cdot x = 0 = f(x)$

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$$\sup_{x\in[0,1]}\{|f(x)-xP(x)|\} \le \underbrace{\sup_{x\in[0,1]}\{|x|\}}_{\le 1}\cdot\sup_{x\in[0,1]}\{|g(x)-P(x)|\}$$

$$< 1\cdot\varepsilon = \varepsilon.$$

$P(x)$ only has even ~~degree terms~~ powers $\Rightarrow$ $xP(x)$ ~~is only~~ only has odd powers

(iv) Let A_0 be polynomials with 0 constant term. (clearly an algebra)

$x, y \in [0,1]$, $x \neq y \Rightarrow f(z)=z$, $f(x)\neq f(y)$, $f\in A_0$.

$\forall f\in A_0 : f(0)=0 \Rightarrow \bar{A}_0 = \{f : f(0)=0\}$

$\therefore \exists q \in A_0$ s.t. $\|q-f\|_\infty < \varepsilon/2$ as $f(0)=0$.

$q(0)=0$, $q'(0)$ exists (q smooth

$\therefore \exists p \in A$ s.t. $\|q-p\|_\infty < \varepsilon/2$

$\therefore \|f-p\|_\infty < \varepsilon/2+\varepsilon/2 = \varepsilon.$

$\therefore$ Done.

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2.) (a) Closed graph theorem (CGT): Let X, Y be banach spaces, $T: X \to Y$ be linear; $\Gamma_T = \{ (x, T(x)) : x \in X \} \subseteq X \times Y$

T is bounded iff $\Gamma_T \subseteq X \times Y$ is closed.

Claim: Γ_T is closed.

Let $(x_n, T(x_n)) \to (x, y)$: $x_n \to x$, $T(x_n) \to y$.

Pick $z \in X$ s.t. $T(z) = y$.

~~Claim $\| x_n - x \| \leq 1/c \| T(x_n) - T(x) \|$~~

~~$\therefore \lim_{n \to \infty} \| T(x$~~

$\therefore \| x_n - z \| \leq 1/c \, \| T(x_n) - y \| \to 0 \Rightarrow x_n \to z$.

Banach spaces have unique limits: $x = z \Rightarrow y = T(x)$

$\therefore (x, y) \in \Gamma_T$

CGT: T is bounded $\Rightarrow$ Continuous

(b) Let $X = Y = c_{00} = \{ x \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, n \geq N \Rightarrow x_n = 0 \}$, l_∞ norm:

$T: X \to Y$, $(T(x))_n = n \cdot x$

$\forall y \in Y$: $T(y_1/1, \ldots y_N/N, 0 \ldots) = y$ $(y = (y_1, \ldots y_N))$

$\therefore T$ is surjective, T clearly linear.

$\forall x \in X$: $\| T(x) \|_\infty = \sup_{n \geq 1} \{ n \cdot |x_n| \} \geq \sup_{n \geq 1} \{ |x_n| \} = \| x \|$

$\therefore$ Criteria fulfilled

But $\| T \| \geq \sup_{n \geq 1} \{ \| T(0, \ldots 0, \underbrace{1}_{n^{th} \text{ coordinate}}, 0, \ldots) \| \} = \sup_{n \geq 1} \{ n \} = \infty$

$\therefore T$ not ~~bo~~ cont.

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(b) $A_M = \{ x_1 - x_2 : x_i \in C : \|x_i\| \le M \}$; $\overline{A_M}$ is closed.

$\forall x \in X : \exists x_i \in C$ s.t. $x = x_1 - x_2 \Rightarrow x \in A_{\lceil \max\{\|x_i\| : i=1,2\} \rceil}$

$\therefore X \subseteq \bigcup_{M \ge 1} \overline{A_M}$; X is complete.

Baire category theorem: Complete metric space $\ne$ countable union of closed, nowhere dense set.

$\therefore \exists M \in \mathbb{N}$ s.t. $\overline{A_M}^{\circ} \ne \phi \Rightarrow \exists D(\overset{z}{\cancel{a}}, \varepsilon) \subseteq \overline{A_M}$

$\cancel{z = x_1 - x_2}$

$\cancel{\forall \omega \in D(0, \varepsilon) : z + \omega = x_3 - x_4}$ $\Big]$ $\cancel{\omega = \underbrace{x_3 + x_2}_{\in C} - (\underbrace{x_1 + x_4}_{\in C})}$

$\cancel{\therefore \forall \omega \in D(0,1), \ \exists x_i \in C \text{ s.t. } \varepsilon \omega = x_1 - x_2, \ \varepsilon > 0}$

$\cancel{\text{If } \omega = 0 : \omega = x_1 - x_2. \quad \therefore \omega = 1/\varepsilon \cdot x_1 - 1/\varepsilon \, x_2, \ x_i/\varepsilon \in C.}$

$\cancel{\omega \ne 0 :}$

$\cancel{\therefore D \ \|x_3 + x_2\|, \ \|x_1 + x_4\| \le}$

If $a \in \overline{A_M}$: $\exists \overset{x_1^{(n)} - x_2^{(n)}}{\cancel{x_n}} \in A_M$, $x_1^{(n)} - x_2^{(n)} \to a$

$\therefore x_2^{(n)} - x_1^{(n)} \to -a \Rightarrow -a \in \overline{A_M}$

$\therefore D(-z, \varepsilon) \subseteq \overline{A_M}$

$\forall a \in D(0, \overset{1}{\cancel{\varepsilon}}) : \varepsilon a = -z + (z + \overset{\varepsilon}{}a)$: $x_1^{(n)} - x_2^{(n)} \xrightarrow{(n)} z + \varepsilon a$, $x_3^{(n)} - x_4^{(n)} \xrightarrow{(n)} -z$ $\Big]$ $x_i \in C$, $\|x_i\| \le M$

$\therefore (x_1^{(n)} + x_3^{(n)}) - (x_2^{(n)} + x_4^{(n)}) \to \varepsilon a$, $\cancel{\|\varepsilon(x_1^{(n)} + x_3^{(n)})\|, \|x_2^{(n)}}$ $+ \overset{1/\varepsilon}{\{} x_4^{(n)} \}\| \le 2M$

$\cancel{\therefore a \in \overline{A_{2M}} \Rightarrow D(0, \varepsilon) \subset \overline{A_{2M}}}$

$\therefore \underbrace{\left(\dfrac{x_1^{(n)} + x_3^{(n)}}{\varepsilon} \right)}_{x_5} - \underbrace{\left(\dfrac{x_2^{(n)} + x_4^{(n)}}{\varepsilon} \right)}_{x_6} \to a$

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$X_5, X_6 \in C; \quad \|X_5\|, \|X_6\| \leq 1/\varepsilon \cdot 2M$

$\therefore \ a \in \overline{A_N}, \quad N \geq 2M/\varepsilon.$

$\therefore \ D(0,1) \in \overline{A_N}$

(c) ~~If we allow $M \in \mathbb{R}_{>0}$: $D(0,M) \subseteq \overline{A_{N \cdot M}}$ by scaling the approximation~~

~~Fix $\omega \in D(0,1)$: $\exists \ X_1^{(1)} + X_2^{(1)} = y_1^{Z_1}$, $X_i^{(n)} \in C$, st~~
~~$\|y - \omega\| < 1/2$, $\|X_i^{(1)}\| \leq N$.~~

~~Let $y_1 = \omega - Z_1$: Given $y_k \in D(0, 1/2^k)$, and $Z_k = X_1^{(k)} - X_2^{(k)}$,~~
~~$X_i^k \in C$, $\|X_i^k\| \leq 1/2^{k-1} \cdot N$:~~

~~$y_k \in \overline{A_{N/2^k}}$: $\exists \ X_1^{k+1} - X_1^{k+1} = Z_{k+1}$ s.t. $\|y_k - Z_{k+1}\| < 1/2^{k+1}$,~~
~~$\|X_i^{k+1}\| < 1/2^k \cdot N$.~~

~~$\|\omega - \sum_{k=1}^{N} y_k\| \to 0$, $\sum_{k=1}^{\infty} \|y_k\| \leq 1 \Rightarrow \sum_{k=1}^{N} y_k$ is cauchy $\Rightarrow$~~

~~$\Rightarrow \omega = \sum_{k=1}^{\infty} y_k = \sum_{k=1}^{\infty}$~~

$D(0,k) \subseteq \overline{A_{k \cdot N}}$:

Fix $\omega \in D(0,1)$: Let $\omega_0 = \omega$, pick $y_1 = X_1^{(1)} - X_2^{(1)}$ s.t.
$\|y_1 - \omega\| < 1/2$, $X_i^{(1)} \in C$, $\|X_i^{(1)}\| \leq N$.

Let $\omega_1 = \omega - y_1$: $\|\omega_1\| < 1/2$.

Given $\omega_k \in D(0, 1/2^k)$: Pick $y_{k+1} = X_1^{(k+1)} - X_2^{(k+1)}$ s.t. $\|y_{k+1} - \omega_k\| < 1/2^{k+1}$

and $\|X_i^{(k+1)}\| \leq N/2^k$, $X_i^{(k+1)} \in C$.

$\omega_{k+1} = y - \omega_k$, $\|\omega_{k+1}\| < 1/2^{k+1}$

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$\therefore \| \cancel{y} \; \omega - \sum_{k=1}^{N} y_k \| \to 0$ as $N \to \infty \Rightarrow \omega = \sum_{k=1}^{\infty} y_k$

$\omega = (\sum_{k=1}^{\infty} X_1^{(k)}) - (\sum_{k=1}^{\infty} X_2^{(k)})$: $\sum_{k=1}^{M} X_i^{(k)} \in C$,

$\sum_{k>M} \| X_i^{(k)} \| \leq \sum_{k>M} N/2^{k-1} = N/2^M \to 0$ as $M \to \infty$

$\therefore \left(\sum_{k=1}^{M} X_i^{(k)} \right)_{M \geq 1}$ is cauchy $\Rightarrow X_i^{\infty} \in C$ (C closed)

$\therefore \omega \in X_1^{\infty} - X_2^{\infty}$, $\| X_i^{\infty} \| \leq N \cdot 2N$.

$\therefore D(0,1) \subseteq A_{2N}$.

$\therefore \forall x \in X$, $\cancel{x/2}$ $x \neq 0 \Rightarrow x/2\|x\| = X_1 - X_2$, $X_i \in C$, $\| X_i \| \leq 2N$.

$\therefore X = X_1' - X_2'$, $X_i' \in C$, $\| X_i' \| \leq 4N \cdot \| X \|$

(Hold for 0 as well)

$\therefore (k = 4N)$.

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1.) (a) $T: X \to Y$ is an isomorphism if T is bijective; T, T^{-1} linear, bounded.

If T is an isomorphism: T^{-1} bounded $\Rightarrow$ $\forall y \in Y$, $\| T^{-1}(y) \| \leq c \cdot \| y \|$ for some $c > 0$ $\Rightarrow$ ~~$\|x$ $\forall x \in Y$~~

T clearly surjective $\Rightarrow$; $\forall x \in X$, $\| x \| \leq c \cdot \| T(x) \|$] $y = T(x)$

If T is bounded linear: $T(x) = 0 \Rightarrow$ $\| T(x) \| \geq c \cdot \| x \| \Rightarrow \| x \| = 0$

$\therefore$ T is injective; $\| T(\cdot) \| \geq c \| \cdot \| \Rightarrow T^{-1}$ bounded.

$\therefore$ T is an isomorphism

(b) $T: X \to Y$ be isomorphism; $(y_n)_{n \geq 1}$ be cauchy in Y.

$\| T^{-1}(y_n) \| \leq {}^{1}/_{c} \| y_n \| \Rightarrow \{ (T^{-1}(y_n))_{n \geq 1}$ cauchy in $Y \Rightarrow T^{-1}(y_n) \to x$.

T cont. : $y_n \to T(x) \in Y$

$\therefore$ Y complete.

(c) Let $\dim(V) = n$, $\{ e_1, \dots e_n \}$ be a basis;

Consider norm $\| \sum_{k=1}^{n} x_k e_k \|_2^2 = \sum_{k=1}^{n} x_k^2$.

$\| \underline{x} \|_2 = 0$ iff $\underline{x} = \sum_{k=1}^{n} 0 \cdot e_k = 0$

$\| \lambda \underline{x} \|_2 = |\lambda| \cdot \| \underline{x} \|$

~~Cauchy Schwartz~~ : $\| \sum_{k=1}^{n} (x_k + y_k) e_k \|_2^2 = \sum_{k=1}^{n} x_k^2 + y_k^2 + 2 x_k y_k$

$\leq \left(\sqrt{\sum_{k=1}^{n} x_k^2} + \sqrt{\sum_{k=1}^{n} y_k^2} \right)^2$

$\left(\sum_{k=1}^{n} x_k y_k \leq \sum_{k=1}^{n} |x_k| \cdot |y_k| \leq \sqrt{\sum_{k=1}^{n} x_k^2} \cdot \sqrt{\sum_{k=1}^{n} y_k^2} \right)$

Cauchy Schwartz:

$\therefore$ $\| \cdot \|_2$ is a norm $\Rightarrow$ $\| \cdot \|_2, \| \cdot \|$ equivalent

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$\therefore \Phi : V \to \cancel{\mathbb{F}}\ \mathbb{C}^n$, $\Phi(\underline{x}) = (x_1, \ldots x_n)$, $\underline{x} = \sum_{k=1}^{n} x_k e_k$

$\|.x\|_1 = \|\Phi(x)\|$

Since $\exists\ 0 < A < B$ s.t. $A\|x\| \leq \|\Phi(x)\| \leq B\|x\|$,

Φ clearly linear, bijective : $V \cong (\mathbb{C}^n, \|.\|_2)$

$\therefore$ All dim n spaces $\cong (\mathbb{C}^n, \|.\|_2) \Rightarrow$ Isomorphic.

Since $\mathbb{C}^n$ is complete, V must be complete (a)

Let $\dim(V) = n$:

If $A \subseteq V$ is closed, bounded: $\Phi(A)$ must be bounded

(Φ cont. $\Rightarrow$ bounded), closed (Φ^{-1} cont $\Rightarrow$ Φ closed map)

$\Phi(A) \subseteq \mathbb{C}^n$ compact iff closed, bounded $\Rightarrow \Phi(A)$ compact.

Φ^{-1} cont. $\Rightarrow \Phi^{-1}(\Phi(A)) = A$ is compact.

(c) ~~Let $(z_n+ f_n)$ not be convergent~~

$|d(x, Z) - d(y, Z)|$: If $\|x - z_1\| \leq d(x, Z) + \varepsilon$, ~~$\|y - z_1\| \leq$~~

~~$d(y, Z) \leq d(x, z) + d(y, z)\ \|x-y$~~

$\therefore$ ~~$d(y, Z) - d(x, Z) \leq \|\ (d(y, Z) \leq d(x, z) + \|y-x\|$~~

$\therefore d(y, Z) \leq \|x - y\| + \|x - z_1\| \Rightarrow d(y, Z) - d(x, Z) \leq \varepsilon + \|x - y\|$

$\varepsilon > 0$ arbitrary $\Rightarrow d(y, Z) - d(x, Z) \leq \|x - y\|$

Symmetry $\Rightarrow |d(x, Z) - d(y, Z)| \leq \|x - y\|$

$\therefore \Phi : F \to [0, \infty)$, $\Phi(x) = d(x, Z)$ is cont.

$\therefore \Phi$ cont., ~~$\overline{B}_F(0,1)$ compact $\Rightarrow \Phi|_{\overline{B}_F(0,1)}$ attains~~

$S = \{x \in F : \|x\| = 1\}$ compact $\Rightarrow \Phi|_S$ attain its inf.

Let $d = \inf(\Phi|_S)$, $d > 0$ as $\exists f$ $d = d(f, Z)$

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Z closed $\Rightarrow$ $\exists$ $D(d, \varepsilon) \subseteq Z^c \Rightarrow d \geq \varepsilon > 0.$

If $(z_n + f_n)_{n \geq 1}$ is ~~cauchy~~: conv. :

$$\cancel{(1): \| z_n - z_m + f_n - f_m \| \geq \inf_{z \in Z} d(f_n - f_m, Z)}$$

$$\cancel{\text{If } \|f\| = 1, \; d(f, Z) = a : d(k \cdot f, Z) \; (\forall k \in \mathbb{C})}$$

$$\cancel{(1) \geq \inf_{x \in S_{\|f_n - f_m\|}} \{ \Phi(x) \}}$$

If $f_n \neq f_m$:

Fix n, m : $\| z_n - z_m + f_n - f_m \| \geq \| f_n - f_m \| \inf_{z \in Z} \left\{ \left\| z - \frac{f_n - f_m}{\| f_n - f_m \|} \right\| \right\}$

$= d \cdot \| f_n - f_m \|$ (Valid if $f_n = f_m$ as well)

$\therefore$ $(f_n)_{n \geq 1}$ is cauchy ; F finite dim. $\Rightarrow$ complete $\Rightarrow$ $f_n \to f$.

$\therefore$ $z_n \to z \in X$; Z closed $\Rightarrow z \in Z$.

$\therefore$ $z_n + f_n \to z + f \in Z + F$

$Z + F$ closed.

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Linear Analysis 2015

1.) (a) Let $X_n = \langle e_k : 1 \leq k \leq n \rangle$; X_n is a finite dim. subspace

Given $x \in X$, : P_n be projection map to X_n.

$$P_n(x) = \sum_{k=1}^{n} \langle x, e_k \rangle \cdot e_k \qquad (*)$$

Since $\{e_n : n \in \mathbb{N}\}$ forms a basis, : Given $\varepsilon > 0$, $\exists$

$$\sum_{k=1}^{N} a_k e_k = y_N \quad \text{s.t.} \quad \| y_N - x \| \cancel{b} < \varepsilon.$$

$\forall n \geq N$: $y_N \in X_n \Rightarrow \| x - y_N \| \geq \| x - P_n(x) \|$

$\therefore P_n(x) \to x$

$$\therefore x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \qquad \text{(existence)}$$

If $x = \sum_{k=1}^{\infty} a_k \cancel{x_k}^{e_k} = \sum_{k=1}^{\infty} b_k \cancel{x_k}^{e_k}$:

$$\forall \cancel{k}^{r} \in \mathbb{N} : \langle x, e_{\cancel{k}r} \rangle = \left\langle \sum_{k=1}^{\infty} a_k \cancel{x_k}^{e_k}, e_{\cancel{k}r} \right\rangle = \sum_{\cancel{k}\, k=1}^{\infty} a_{\cancel{r}k} \langle e_k, e_r \rangle \quad \text{(cont. of IP)}$$

$$= \sum_{k \geq 1} a_{\cancel{r}k} \cdot \delta_{rk} = a_r \ldots$$

$\therefore a_k = b_k = \langle x, e_k \rangle \Rightarrow$ Unique.

$\forall x \in X$: x uniquely written as $\overline{\sum_{k \geq 1}} \langle x, e_k \rangle \cdot e_k$

Define : $U(x) = \overline{\sum_{k \geq 1}} \langle x, e_k \rangle \, U(e_k)$

$\therefore U(e_n) = e_n \quad (\forall n \in \mathbb{N})$

U is clearly linear : $\| U(x) \|^2 = \left\langle \overline{\sum_{i \geq 1}} \langle x, e_i \rangle f_i , \overline{\sum_{j \geq 1}} \langle x, e_j \rangle f_j \right\rangle$

$$= \sum_{i \geq 1} |\langle x, e_i \rangle|^2 = \| x \|^2$$

$\therefore \| U \| = 1$ (bounded), U is unitary.

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Unique: If $U(e_n) = f_n$, continuity $\Rightarrow U(\sum_{k\geq 1} a_k e_k) = \sum_{k\geq 1} a_k f_k$

Since $\forall x \in X$, x uniquely written as $\sum_{k\geq 1} a_k e_k$, U is unique.

(b) Claim: ~~$\forall n \geq 1$~~, $\exists\, d_{ij} \in \{0,1\}$, $i,j \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$:

$$\sum_{i=1}^{2^n} \left(\| \sum_{j=1}^{2^n} d_{ij} X_j \|^2 \right) = 2^n \left(\sum_{k=1}^{2^n} \| X_k \|^2 \right)$$

$(\forall X_k \in \lambda^2)$

(Proof): $n=1$: parallelogram law & $\begin{pmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{pmatrix} = A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Suppose true for some n:

$(A_n)_{ij} = (d_{ij})_{1 \leq i,j \leq 2^n}$: $A_{n+1} = \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix}$

$$\therefore \text{LHS} = \sum_{i=1}^{2^{n+1}} \left\{ \| \sum_{j=1}^{2^n} d_{ij} X_j + \sum_{j=2^n+1}^{2^{n+1}} d_{ij} X_j \|^2 \right\}$$

$$= \sum_{i=1}^{2^n} \left\{ \| \sum_{j=1}^{2^n} A_{ij} X_j + A_{ij} X_{2^n+j} \|^2 + \| \sum_{j=1}^{2^n} A_{ij} X_j - A_{ij} X_{2^n+j} \|^2 \right\}$$

$$= 2 \sum_{i=1}^{2^n} \| \sum_{j=1}^{2^n} A_{ij} X_j \|^2 + \| \sum_{j=1}^{2^n} A_{ij} X_{2^n+j} \|^2$$

$$= \text{(induction) } 2 \cdot 2^n \left(\sum_{j=1}^{2^{n+1}} \| X_j \|^2 \right)$$

~~$\therefore$ If $\lambda_2 \overset{\Phi}{\cong} \lambda_p$, $p \neq 2$: $\exists$ ~~$\exists\, a$~~ Φ, Φ^{-1} bounded $\Rightarrow \exists\, A, B \in (0,\infty)$~~

~~s.t. ~~$\forall x \in \lambda_2$~~ ~~$A \| x \|^2 \leq \| \Phi(x) \|^2 \leq B \| x \|^2$~~~~

~~$\forall x \in \lambda_2$, $A \| \Phi(x) \|_p^2 \leq \| x \|_2^2 \leq B \| \Phi(x) \|_p^2$~~

$\therefore$ Let $X_j = e_j$

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If $\Phi : \lambda_p \to \lambda_2$ is an isomorphism: $\exists\ 0 < A < B$ s.t.:

$\forall x \in \lambda_p$

$$A \| \Phi(x) \|_2 \leq \| x \|_p \leq B \cdot \| \Phi(x) \|_2$$

$$\therefore \sum_{k=1}^{2^n} \left(\sum_{l=1}^{2^n} \| A_{kl} \cdot u_l \|^2 \right) \geq \frac{1}{B^2} \sum_{k=1}^{2^n} \left(\sum_{l=1}^{2^n} \| A_{k,l}\, \Phi^{-1}(u_l) \|_p^2 \right)$$

$$\leq \frac{1}{A^2} \sum_{k=1}^{2^n} \left(\sum_{l=1}^{2^n} \| A_{k,l}\, \Phi^{-1}(u_l) \|_p^2 \right)$$

But $\text{LHS} = 2^n \sum_{k=1}^{2^n} \| u_k \|^2 \geq 1/B^2 \sum_{k=1}^{2^n} 2^n \| \Phi^{-1}(u_k) \|_p^2$

$$\leq 1/A^2 \sum_{k=1}^{2^n} 2^n \| \Phi^{-1}(u_k) \|_p^2$$

$$\therefore A^2/B^2 \sum_{l,k=1}^{2^n} \| A_{k,l}\, \Phi^{-1}(u_l) \|_p^2 \leq 2^n \sum_{k=1}^{2^n} \| \Phi^{-1}(u_k) \|_p^2 \leq B^2/A^2 \sum_{k=1}^{2^n} \left(\sum_{l=1}^{2^n} \| A_{k,l}\, \Phi^{-1}(u_l) \|_p^2 \right)$$

Pick $\Phi^{-1}(u_l) = e_l$:

$$\sum_{k=1}^{2^n} \| e_l \|_p^2 = 2^n;$$

$$\sum_{l,k=1}^{2^n} \| A_{k,l}\, e_l \|_p^2 = \sum_{k=1}^{2^n} 2^{n(2/p)}$$

$$\therefore A^2/B^2 \cdot 2^{n(1+2/p)} \leq 2^{n(1+1)} \leq B^2/A^2\, 2^{n(1+2/p)}$$

$\therefore$ Contradiction

$\therefore \lambda^2 \not\cong \lambda^p$

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2.) (a) T is bounded if : $\exists\, k \geq 0$ s.t $\forall\, x \in X$, $\|T(x)\| \leq k \cdot \|x\|$

If T is bounded: If $V \subseteq Y$ is open, $x \in T^{-1}(V) \Rightarrow \exists\, \varepsilon > 0$ s.t. $D(T(x), \varepsilon) \subseteq T^{-1}(V)$

$(\forall x)$ $\|T(x)\| \leq k \cdot \|x\|$: $T(D(0, \varepsilon/k)_x) \subseteq D(0, \varepsilon)$

$\therefore D(x, \varepsilon/k)_x \subseteq T^{-1}(V) \Rightarrow T^{-1}(V)$ open $\Rightarrow$ cont.

T cont. ~~[illegible]~~ : $T^{-1}(D(0,1)_Y)$ is open

$0 \in T^{-1}(D(0,1)) \Rightarrow \exists\, \varepsilon > 0$ s.t $D(0, \varepsilon) \subseteq T^{-1}(D(0,1))$

$\therefore$ ~~$\forall x, \|x\|$~~ If $x \neq 0$: $T\left(\frac{\varepsilon \cdot x}{2\|x\|}\right) \in D(0,1)$

$\therefore \|T(x)\| \leq 2/\varepsilon \, \|x\| \Rightarrow$ Bounded.

$$\|T\| = \sup_{\|x\| \leq 1} \{ \|T(x)\| \}$$

$\mathcal{B}(X, Y)$ is clearly a vector space :

$\|T\| = 0$ iff $\sup_{\|x\| \leq 1} \{ \|T(x)\| \} = 0$ iff $\forall\, x$, $\|x\| \leq 1 \Rightarrow T(x) = 0$

iff $T = 0$

$$\|\lambda \cdot T\| = \sup_{\|x\| \leq 1} \{ |\lambda| \cdot \|T(x)\| \} = |\lambda| \cdot \|T\|$$

$$\|T_1 + T_2\| = \sup_{\|x\| \leq 1} \{ \|T_1(x) + T_2(x)\| \} \leq \sup_{\|x\| \leq 1} \{ \|T_1(x)\| + \|T_2(x)\| \}$$

$$\leq \sup_{\|x\| \leq 1} \{ \|T_1(x)\| \} + \sup_{\|x\| \leq 1} \{ \|T_2(x)\| \} = \|T_1\| + \|T_2\|$$

$\therefore$ Valid norm.

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(b) (i) Not bounded.

~~Then $\sin(n\pi x)$~~ $f_n(x) = \sin(2n\pi x)$; $f_n'(x) = 2n\pi \cos(2n\pi x)$

$\therefore \|f_n\|_\infty = 1$, $\|f_n\| = 1 + 2\pi n$.

$\therefore$ No such $A > 0$ s.t. $\|f_n\| \le A \cdot \|f_n\|_\infty$ $(\forall n \in \mathbb{N})$

$\therefore$ Not compact.

(ii) Bounded: $\forall f \in C'[0,1]$, $\|f\|_\infty \le \|f\| \Rightarrow \|T\| \le 1$

$\|T(x \to 1)\| = 1 = \|x \mapsto 1\|_\infty$

$\therefore \|T\| = 1$

~~Not~~ Compact: Let $F \subseteq C'[0,1]$ be $\|\cdot\|_\infty$ bounded $(\le M)$

$\therefore \forall f \in F$, $\|f\|_\infty \le M$ (Bounded F)

$\forall f \in F$, $\|f'\|_\infty \le M$.

$\therefore$ Given ε, pick $\delta \in (0, \varepsilon/2M)$: ~~$\forall \varepsilon \exists$~~ ($\forall x \in X$, $\forall y \in D(x,\delta)$,

$\forall f \in F$: $\exists z \in (x,y)$: $|f(x) - f(y)| \le |f'(z)| \cdot |x-y| \le M \cdot |x-y| \le \varepsilon/2 < \varepsilon$.

$\therefore$ Equicontinuous

Arzela Ascoli: (K compact) F totally bounded $\Rightarrow$ compact

(iii) $\|f'\|_\infty \le \|f\| \Rightarrow$ Bounded, $\|T\| \le 1$

$f_n' = n\cos nx$

$f_n(x) = 1/n \sin(n^2 x)$, $\therefore$ (For $n \ge \sqrt{2\pi}$) ~~$\|f_n\|$~~ $\|f_n'\|_\infty = n$.

$\therefore \|f_n\| = \|f_n\|_\infty + \|f_n'\|_\infty = 1/n + n$.

$\therefore \|T\| \ge \sup_{n \in \mathbb{N}} \left\{ \frac{n}{n + 1/n} \right\} = 1 \Rightarrow \|T\| = 1$

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Not compact: $\forall g \in C[0,1]$, let $f(x) = \int_0^x g(y)\,dy$.

Fundamental theorem of calc: g cont. $\Rightarrow f' = g$.

$|f(x)| \leq \int_0^x |g| \leq \|g\|_\infty$

$\therefore \bar{D}(0,1)_{C[0,1]} \subseteq T(\bar{D}(0,2)_{C'[0,1]})$

$\bar{D}(0,2)_{C'[0,1]}$ bounded ; ~~But $\bar{D}(0,1)$~~ ~~If RHS totally compact,~~

~~$\bar{D}(0,1)_{C[0,1]} \subseteq$ compact set $\Rightarrow$ LHS compact~~

~~$\therefore$ Contradiction~~

Since $D(0,1)_{C[0,1]}$ is not totally bounded, $T(D(0,2)_{C'[0,1]})$ not totally bounded $\Rightarrow$ T not compact.

(iv) T ~~is~~ has finite dim image $\Rightarrow$ Compact

$|T(f)| \leq \|f\|_\infty \cdot \|h\|_1 \Rightarrow \|T\| \leq \|h\|_1 \Rightarrow T$ bounded.

Claim: $\|T\| = \|h\|_1$

~~Measure theory method: f_n~~

~~Fix $\varepsilon>0$: $f_n(x) = \operatorname{Min}\{1, n\,d(x, h^{-1}[\varepsilon,\infty))\}$~~

~~$\forall x>0$: $x^{1/n} \to 1$ as $n \to \infty$~~

Let $\|h\|_\infty = M$ (cont. h on compact $[0,1] \Rightarrow$ Bounded)

f_n ~~$(x) = h(x)^{1/n}$, when : $h(x) \geq h(x)^{1/n}$: $x \geq 0$~~

~~$-(-h(x))^{1/n}$: $x \leq 0$.~~

$\therefore$ ~~$f_n(x) = \operatorname{Max}$~~ $\{$

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Fix $\varepsilon > 0$: $U = f^{-1}(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ is open $\Rightarrow$ countable union of disjoint open sets: $\bigcup_{n \geq 1} (a_n, b_n)$

Let $\| h \|_\infty = M$ (h cont. on compact set $\Rightarrow$ Bounded)

Define f_n: $f_n(x) = 0$ outside of $\bigcup_{n \geq 1} (a_n, b_n)$,

$f_n(x) = \operatorname{sgn}(h(x))$ on $[a_n - \varepsilon/2^n, b_n + \varepsilon/2^n]$ (*),

linear interpolation between $[a_n, a_n - \varepsilon/2^n]$, ~~$[b_n, b_n -$~~ $[b_n - \varepsilon/2^n, b_n]$

$\therefore \| \int_0^1 f_n h - \int_0^1 |h| 1_{|h| \geq \varepsilon} dx \| \leq \| h \|_\infty \cdot \sum_{n \geq 1} 2 \cdot (\varepsilon/2^n)$

$= 2M \cdot \varepsilon.$

$\| \int_0^1 |h| 1_{|h| \geq \varepsilon} - \int_0^1 |h| \| \leq \varepsilon.$

$\therefore | \int_0^1 f_n h - \| h \|_1 | \leq \varepsilon (1 + 2M)$

$\| f \| = 1$, $|T(f)| = \| h \|_1 - \varepsilon (1 + 2M)$

ε arbitrary $\Rightarrow \| T \| = \| h \|_1$

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3.) (a) Baire Category Theorem (BCT): Let (X, d) be a complete metric space.

~~$U_n \subseteq X$ be dense, open: $\bigcap_{n\geq 1} U_n \neq \phi$~~

Pick $x_1 \in U_1$, $\varepsilon_1 \in (0,1)$ s.t. $\bar{D}(x_1, 2\varepsilon_1) \subseteq U_1$;

Given $x_1, \dots x_n$; $\varepsilon_1, \dots \varepsilon_n$ satisfying: $\varepsilon_k \in (0, 1/k)$, $(k \geq 1)$

~~$x_k \in \bar{D}(x_k, 2\varepsilon_k) \subseteq D(x_{k-1}, \varepsilon_k) \cap U_k$~~ $(k \geq 2)$

Pick $x_{n+1} \in U_{n+1} \cap D(x_n, \varepsilon_n)$ (open, $\neq \phi$), $\varepsilon_{n+1} \in (0, 1/n+1)$,

~~$x_{n+1} \in \bar{D}(x_{n+1}, 2\varepsilon_{n+1}) \subseteq D(x_n, \varepsilon_n) \cap U_n$~~

$\therefore$ Inductively construct $(x_n)_{n\geq 1}$

$\forall n \in \mathbb{N}$: $x_{n+1} \in \bar{D}(x_{n+1}, \varepsilon_{n+1}) \subseteq D(x_n, \varepsilon_n) \subseteq \bar{D}(x_n, \varepsilon_n) \subseteq U_n$.

$\therefore (x_{n+k})_{k\geq 1}$ is a sequence in $\bar{D}(x_n, \varepsilon_n)$, in U_n

$\forall m, n \geq N$: $d(x_m, x_n) \leq 2/N$ as $x_m, x_n \in \bar{D}(x_N, \varepsilon_N) \subseteq \bar{D}(x_N, 1/N)$

$\therefore (x_n)_{n\geq 1}$ cauchy

X complete $\Rightarrow x_n \to x \in X$; $\bar{D}(x_n, \varepsilon_n)$ closed ~~set~~, $\Rightarrow$ ~~$x_n \to x \in X$~~

$x \in \bar{D}(x_n, \varepsilon_n) \subseteq U_n$

$\therefore x \in \bigcap_{n\geq 1} U_n \Rightarrow RHS \neq \phi$

(b) Let $\varphi: \lambda_2^2 \to C_0$ be an isometry.

λ_2^2 complete $\Rightarrow \lambda_2^2 \times \lambda_2^2$ complete; Let $S_1 \subseteq \lambda_2^2$, ~~$S_1$ unit~~ sp

$S_1 = \{\underline{x} \in \lambda_2^2 : \|\underline{x}\| = 1\}$ is closed.

$\therefore S \times S$ is complete.

Let $A_n = \{(\underline{x}, \underline{y}) \in S^2 : \|\varphi(x) - \varphi(y)\|_\infty = |\varphi_n(x) - \varphi_n(y)|\}$

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Since $\forall z \in c_0$, $z_n \to 0$: $\|z\|_\infty = |z_n|$ for some n.

$(\|z\| = 0 \Rightarrow \|z\| = |z_1|$; $\|z\| > 0 \Rightarrow \exists N$ s.t. $n \geq N \Rightarrow |z_n| < \|z\| \Rightarrow \|z\| = |z_k|$, $k < N)$

$\|\varphi(x)\|_\infty = |\varphi_n(x)|$ for some n.

$\therefore \bigcup_n A_n = S \times S$.

Claim: A_n is closed.

$$(x, y) \xrightarrow{\Phi_1} \left(\|\varphi(x) - \varphi(y)\|_\infty,\ |\varphi_n(x) - \varphi_n(y)|\right) \xrightarrow{\Phi_2} \left|\, \|\varphi(x) - \varphi(y)\|_\infty - |\varphi_n(x) - \varphi_n(y)| \,\right|$$

is a composition of cont. function

$A_n = (\Phi_2 \circ \Phi_1)^{-1}\{0\}$ is closed.

BCT $\Rightarrow \exists A_N$ s.t. A_N contain open ball U.

By applying a translation (isometric) we can WLOG $\varphi(0) = 0$.

$\therefore \forall x \in S$, $\|x\| = 1 \Rightarrow \|\varphi(x) - 0\|_\infty = \|x\|_1 = 1$

I think hint is broken.

Claim: $\forall z \in c_0$, $\|z_\infty\| = z_n$ for some $n \in \mathbb{N}$

If $\|z\| = 0$: $\|z\| = z_1$

Else: $\exists N \in \mathbb{N}$ s.t. $n \geq N$, $|z_n| < \|z\|_\infty / 2$ $(\|z\|_\infty > 0)$

$\therefore \|z\|_\infty = \mathrm{Max}\{|z_k| : 1 \leq k \leq N\}$

$\therefore$ Done.

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Let $\varphi : \ell_2 \to c_0$ be an isometry:

ℓ_2^2 is a banach space; $F_n = \{ (x,y) \in \ell_2^2 : \|x-y\|_2 = \|\varphi(x)-\varphi(y)\|_\infty = |\varphi_n(x) - \varphi_n(y)| \}$

Since $\varphi(x) - \varphi(y) \in c_0$, $\|\varphi(x)-\varphi(y)\| = |\varphi_n(x) - \varphi_n(y)|$ for some n

$\therefore \bigcup_n F_n = \ell_2^2$

$(x,y) \xrightarrow{\Phi_1} (\|x-y\|_2, |\varphi_n(x) - \varphi_n(y)|) \xrightarrow{\Phi_2} \big| \|x-y\|_2 - |\varphi_n(x)-\varphi_n(y)| \big|$

is ~~cont~~ composition of cont. maps.

$\therefore F_n = \Phi_2 \circ \Phi_1^{-1}\{0\}$ is closed.

$\therefore \exists N \in \mathbb{N}$ s.t. $F_N^\circ \neq \phi$

$\therefore \exists D(x_0, \varepsilon) \times D(y_0, \varepsilon) \subseteq F_n$

$\therefore \Lambda_1 = D(y_0, \varepsilon) \cap \{ z : \|z - x_0\| = \|x_0 - y_0\| \}$ contain infinite point,

continuity ~~$\Rightarrow \varphi_N(.)$ constant here.~~

~~Similarly~~, $\Lambda_2 = D(x_0, \varepsilon) \cap \{ z : \|z - y_0\| = \|x_0 - y_0\| \}$

contain infinite point; $\varphi_N(.)$ constant

$\forall z \in \Lambda_1$, $|\varphi_n(z) - \varphi_n(x_0)|$ is constant.

$\therefore \varphi_n(z) = \varphi_n(x_0) \pm \|x_0 - y_0\|$

Since Λ is connected ($\cap$ of connected sets), φ_n constant on Λ_1

Similarly, ~~Λ_2~~ φ_n constant on Λ_2

$\therefore \forall z_i \in \Lambda_i$, ~~d~~ $\|z_1 - z_2\| = |\varphi_n(z_1) - \varphi_n(z_2)|$ is constant

(reject).

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$\therefore$ No such isometry.

(c) Fix $n \in \mathbb{N}$:

$\Lambda : \lambda_1^n \to C_0$; Let $\{A_1, \dots A_{2^n}\}$ be enumeration of $P(\{1, \dots n\})$

For $1 \leq k \leq 2^n$: $\Lambda_k(x) = \sum_{r=1}^{n} (-1)^{1_{r \in A_k}} x_r$.

$k > 2^n$: $\Lambda_k(x) = 0$.

$\therefore \Lambda$ is pointwise linear $\Rightarrow$ linear

$|\Lambda_k(x)| \leq \sum_{k=1}^{n} |x_k| = \|x\|_1$, Equality attained as we included all possible $\pm$ signs

$\therefore \|\Lambda_k(x)\|_\infty = \|x\|_1 \Rightarrow$ Linear isometry.

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$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

4) (a) Adjoint T^*: $\forall x, y \in H$, $\cancel{\langle T(x), y\rangle = \langle x, T^*(y)\rangle}$

Let $\Phi : H \to H^*$, $\Phi(x) = \langle \cdot, x\rangle$: (Riesz theorem $\Rightarrow$ H is isometric isomorphism)

~~We want~~ : Let T' be dual of T^*, $T^*: H^{*'} \to H'$

$T^*(f) = f(T(\cdot))$

$\cancel{\Phi(T'(x))\ T'(\Phi(y))(x) =}$

We want $S: H \to H$ satisfying: $\cancel{\Phi(\Phi(y)) = \Phi(T(y))}$

$\therefore$ $\cancel{\$}$ $\Phi(S(y)) = T'(\Phi(y))$ $(\forall y \in H)$

$\therefore$ $S = \Phi^{-1} \cdot T' \cdot \Phi$ $\Rightarrow$ S exists. $(= T^*)$

(Composition of bounded, linear maps $\Rightarrow$ linear, Bounded)

Unique: If $\langle T(x), y\rangle = \langle x, S_i(y)\rangle$ $(i=1,2)$

$\forall x, y \in H$: $\langle x, (S_1 - S_2)(y)\rangle = 0$

$\therefore$ $\forall y \in H$: $\langle (S_1 - S_2)(y), (S_1 - S_2)(y)\rangle = 0$ $\Rightarrow$ $S_1 - S_2 \equiv 0$

$\therefore$ $S_1 = S_2$.

(b) T is normal if $T \cdot T^* = T^* \cdot T$

$T: \lambda_2 \to \lambda_2$, $T(\underline{x})_k = x_k$ for $k \geq 2$

T acts on 1^{st} 2 coordinates by : $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\left.\begin{matrix} e_1 \to 0 \\ e_2 \to e_1 \end{matrix}\right]$

T^*: $e_1 \to e_2$, $e_2 \to 0$, $e_k \to e_k$ $(k \geq 3)$

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$T \cdot T^*$: $e_k \to e_k$ $(k \geq 2)$
$e_1 \to e_1$
$e_2 \to 0$

$T^* \cdot T$: $e_k \to e_k$ $(k \geq 2)$
$e_1 \to 0$
$e_2 \to e_2$

$\therefore$ Not equal $\Rightarrow$ Not normal

(c) If T normal: $\forall x \in H$, $\underbrace{\langle T(x), T(x) \rangle}_{\|T(x)\|} = \langle x, T^*T(x) \rangle$

$= \langle x, TT^*(x) \rangle = \langle T^*(x), T^*(x) \rangle = \|T^*(x)\|$

$\cancel{\langle x, T(x) \rangle} = \langle T(x), y \rangle = \langle x, T^*(y) \rangle = \overline{\langle T^*(y), x \rangle} = \overline{\langle y, T^{**}(x) \rangle}$
$= \langle T(x), y \rangle$

(True $\forall x, y \Rightarrow T = T^{**}$)

If $\|T(x)\| = \|T^*(x)\|$:

$\|T(x)\|^2 = \|T^*(x)\|^2 \Rightarrow \langle x, T^*T(x) \rangle = \langle x, TT^*(x) \rangle$

Let $S = TT^* - T^*T$: $\langle x, S(x) \rangle = 0$ $(\forall x \in H)$

$\therefore S = 0. \Rightarrow T$ normal

(d) We know $\sigma_{ap}(T) \subseteq \sigma(T)$:

Let $\lambda \in \sigma(T)$: If $T - \lambda I$ not injective, $\lambda \in \sigma_p(T) \subseteq \sigma_{ap}(T)$

If $T - \lambda I$ injective: $\ker(T) = \ker(T^*) = 0$

Suppose T_λ bounded below: $\forall y_n$, $(y_n)_{n \geq 1}$ cauchy, $y_n = T(x_n)$:
$\Rightarrow$ x_n is also cauchy, $\Rightarrow x_n \to x \Rightarrow T_\lambda(y_n) \to T_\lambda(x)$
$\therefore \operatorname{Im}(T_\lambda)$ closed ($T_\lambda = T - \lambda I$)

But Claim: $\ker(T^*) = \mathrm{Im}(T)^\perp$ ~~for T~~ ~~normal~~

$x \in \mathrm{Im}(T)^\perp$ iff $\forall y \in H, \ \langle x, T(y) \rangle = 0$

iff $\forall y \in H, \ \langle T^*(x), y \rangle = 0$

iff $T^*(y) = 0.$

T normal $\Rightarrow \overline{\mathrm{Im}(T)} = \ker(T^*)^\perp = \ker(T)^\perp$

Since $T - \lambda I$ normal : $\ker(T - \lambda I) = \{0\} \Rightarrow \mathrm{Im}(T)$ dense.

But $\mathrm{Im}(T - \lambda I)$ closed $\Rightarrow$ $T - \lambda I$ surjective $\Rightarrow \lambda \notin \sigma(T)$

$\therefore$ Contradiction $\Rightarrow$ $T - \lambda I$ not bounded below $\Rightarrow$ $\lambda \in \sigma_{Ap}(T)$ $\times$

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4) (a) Let X, Y be banach spaces, $f: X \to Y$ be linear.

$\Gamma_f \subseteq X \times Y$, $\Gamma_f = \{(x, f(x)) : x \in X\}$

Closed graph theorem: Γ_f is closed set iff f is continuous

(b) If f is continuous: Y is a metric space $\Rightarrow$ Hausdorff.

If $(x, y) \in \Gamma_f^c$: $f(x) \neq y \Rightarrow \exists$ disjoint open $U_i \subseteq Y$ s.t. $f(x) \in U_1$, $y \in U_2$

$\forall x \in f^{-1}(U_1)$, $\forall y \in U_2$:

f cont. $\Rightarrow$ $f^{-1}(U_1)$ is open

$U_1 \cap U_2 = \phi \Rightarrow f(x) \neq f(y)$

$\therefore f^{-1}(U_1) \times U_2 \subseteq \Gamma_f^c$, s.t. $f^{-1}(U_1) \times U_2$ open in $X \times Y$.

$\therefore \Gamma_f^c$ open $\Rightarrow \Gamma_f$ closed

If Γ_f is closed: X, Y banach space $\Rightarrow X \times Y$ is ~~a banach space~~ complete.

~~Let $A \subseteq Y$ be closed; $\pi : X \times Y \to X$ be projection~~ map ~~$f^{-1}(A) = \pi(\{X \times A\} \cap \Gamma_f)$~~

Γ_f is a subspace (closed) of $X \times Y \Rightarrow \Gamma_f$ is complete (Banach)

$\varphi : \Gamma_f \to X$ be projection map, $\varphi(x, y) = x$.

$\therefore \|\varphi(x, y)\| = \|x\| \leq \|(x, y)\| \Rightarrow \varphi$ bounded.

φ is clearly linear, surjective.

If $\varphi(x, y) = \varphi(x_1, y_1)$: $x = x_1 \Rightarrow y = f(x) = f(x_1) = y_1$

$\therefore \varphi$ injective.

Inverse mapping theorem $\Rightarrow$ ~~M_p~~ φ is a homeomorphism $\Rightarrow$ closed map?

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$\therefore$ If $A \subseteq Y$ closed: $f^{-1}(A) = \varphi(\underbrace{\Gamma_f \cap X \times A}_{\text{closed}})$ is closed

$\therefore$ f is cont.

(c) Let $X_n, x \in X$, $X_n \to x$; $y_n \in Y$, $y_n = T(X_n)$ and $y_n \to y$.

$S \cdot T$ cont.: $S \cdot T(X_n) \to S \cdot T(x)$

Since $T(X_n) \to y$: S ~~be~~ cont. $\Rightarrow S \cdot T(X_n) \to S(y)$

S injective $\Rightarrow T(x) = y \Rightarrow (x, y) \in \Gamma_T$

Since (X_n, y_n) ~~w~~ sequence was arbitrary, Γ_T must be closed.

$\therefore$ Closed graph theorem $\Rightarrow T$ is cont. (bounded)

2.) (a) Let $S \subseteq C_{\mathbb{R}}(K)$:

S is uniformly bounded if: $\exists M \geq 0$ s.t $\sup\{\|f\|_\infty : f \in S\} \leq M$

Equicontinuous: Given $\varepsilon > 0$, $x \in K$, $\exists U$ open in K, $x \in U$ s.t $\forall y \in U$, $\forall f \in M$, $|f(y) - f(x)| < \varepsilon$

Relatively compact: Closure of S in $C_{\mathbb{R}}(K)$ is compact

(b) If K not compact:

$K = \mathbb{R}$, $f_n(x) = 1_{x \in [n, n+1]} \cdot \min\{|x-n|, |x-(n+1)|\}$

$S = \{f_n : n \in \mathbb{N}\}$

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f_n is ~~piecewise~~ ~~cont.~~ equicont.: Given $\varepsilon>0$, ~~$\varepsilon>0$~~ $x \in \mathbb{R}$, pick $U = D(x, \varepsilon/2)$

$\forall y \in U$, $\forall n \in \mathbb{N}$ $|f_n(x) - f_n(y)| \leq |x-y| < \varepsilon$.

$\|f_n\| = 1$ $(\forall n \in \mathbb{N})$ $\Rightarrow$ Bounded.

But $n \neq m \Rightarrow \|f_n - f_m\| = 1 \Rightarrow$ No cauchy subsequence in S.

$\therefore$ Not relatively compact.

Bounded: $K = [0,1]$, $f_n = n$ (cont., equicont. as f_n are all constant. $\therefore$ Given $\varepsilon>0$, $x \in K$, $D(x,1)$ works)

$n \neq m \Rightarrow \|f_n - f_m\| \geq 1 \Rightarrow$ No cauchy subsequence in $\{f_n : n \in \mathbb{N}\}$

$\therefore$ Not relatively compact

Equicont.: $K = [0,1]$, $f_n(x) = 1_{[0, 1/n]}(1-nx)$

$\|f_n\| \leq 1 \Rightarrow$ Bounded.

If $\{f_n : n \in \mathbb{N}\}$ is relatively compact, $\exists$ cauchy subsequence $(f_{n_k})_{k \geq 1}$

But $m_1 > m_2$: $\|f_{m_1} - f_{m_2}\| \geq f_{m_2}(1/m_1) = 1 - \frac{m_1}{m_2} = \frac{m_2 - m_1}{m_2} \to 1$ as $m_2 \to \infty$.

$\therefore \forall N \in \mathbb{N}$: For k suff. large, $\|f_N - f_k\| \geq 1/2$

$\therefore$ Not cauchy $\Rightarrow$ Not relatively compact.

(c) We will show a finite ε-net exists $\forall \varepsilon>0$:

Given $\varepsilon>0$, pick K_n s.t. $\forall f \in S$, $\forall x \in L \setminus K_n$, $|f(x)| < \varepsilon/3$.

S bounded, equicontinuous on L ~~$\Rightarrow$~~:

$\forall f \in S$, $\sup\{|f(x)| : x \in L\} \geq \sup\{|f(x)| : x \in K_n\}$

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$\therefore S|_{K_n}$ is bounded;

Given $\delta > 0$, $\forall x \in K_n$, $\exists U_x$ open, $x \in U_x$ s.t. $|f(y) - f(x)| < \delta$ $\forall y \in U_x$ (S equicont.) $\Rightarrow S|_{K_n}$ is equicontinuous

$\therefore S|_{K_n}$ is relatively compact $\Rightarrow \exists\ \varepsilon$-net $\{f_1|_{K_n}, \ldots f_N|_{K_n}\}$

~~$\forall x \in K_n$~~: $\forall f \in S$: $\exists f_r$ s.t. $\| f - f_r \|_{K_n} < \varepsilon$.

$\forall x \notin K_n$: $|f(x) - f_r(x)| \leq |f(x)| + |f_r(x)| < 2\varepsilon/3$

$\therefore \| f_r - f \|_L < \varepsilon \Rightarrow \{f_1, \ldots f_n\}$ is an ε-net for S.

$\therefore S$ relatively compact.

3.) (a) Euclidean Space: $\mathbb{R}$ Vector space with inner product

Banach space : Complete $\mathbb{R}$. vector space with norm.

(b) No norm for Banach space :

Let $A_N = \{ \underset{\sim}{x} \in X : \forall k > N, x_k = 0 \}$. $\therefore X = \bigcup_N A_N$.

Suppose $(X, \|\cdot\|)$ is complete : A_N is a finite dimensional space and is hence complete $\Rightarrow A_N \subseteq X$ is closed.

Baire category theorem $\Rightarrow \exists N \in \mathbb{N}$ s.t. $A_N^\circ \neq \phi$

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$\therefore \exists\ D(\underline{x}, \varepsilon) \subseteq A_N$

~~let~~ ~~$y_n=$~~ Let $(y_n)_n = 1_{n=N+1}$. $\therefore \frac{\varepsilon}{2\|y\|} \cdot y \in D(0, \varepsilon)$

$\Rightarrow x + y \in A_N$ (contradiction; $N+1^{th}$ coordinate $\neq 0$)

$\therefore$ No such norm

Define $\langle \underline{x}, \underline{y} \rangle = \sum_{k=1}^{\infty} x_k y_k$ (Since finite coordinates $\neq 0$, sum is finite $\Rightarrow$ Well defined)

$\langle x, x \rangle = 0$ iff $\sum_{k \geq 1} |x_k|^2 = 0$ iff $\forall k \geq 1$, $|x_k|^2 = 0$ iff $x = 0$

$\langle x, y \rangle = \langle y, x \rangle$ (symmetry)

$\langle \lambda_1 x_1 + \lambda_2 x_2, y \rangle = \sum_{k \geq 1} (\lambda_1 x_1 + \lambda_2 x_2)_k y_k = \lambda_1 \langle x_1, y \rangle + \lambda_2 \langle x_2, y \rangle$

$\therefore$ Valid inner product #

Since ~~norm~~ ~~is~~ inner product is determined by norm,
pick $\|x\| = \sqrt{\langle x, x \rangle}$: $(X, \|\cdot\|)$ is an I.P. space.

(c) $\langle x, y \rangle = \langle y, x \rangle$ by definition;
~~$\|x\|$~~ $\langle x, x \rangle = \|x\|^2 = 0$ iff $x = 0$

Fix $a, b, c \in X$:

$$\langle a+b, c \rangle = \frac{1}{4} \left\{ \|a+b+c\|^2 - \|a+b-c\|^2 \right\}$$

$$\left.\begin{array}{ll} (1) & \|a+b+c\|^2 + \|a-b+c\|^2 = 2\|a+c\|^2 + 2\|b\|^2 \\ (2) & \|a+b-c\|^2 + \|a-b-c\|^2 = 2\|a-c\|^2 + 2\|b\|^2 \\ (3) & \|b+a+c\|^2 + \|b-a+c\|^2 = 2\|b+c\|^2 + 2\|a\|^2 \\ (4) & \|b+a-c\|^2 + \|b-a-c\|^2 = 2\|b-c\|^2 - 2\|a\|^2 \end{array}\right\} \text{Parallelogram law}$$

~~$(1)+(3)-2(2)-(4)$: $2\|a+b+c\| + 2\|a+b-a\|$~~

$(1) - (2) + (3) - (4)$: $2(\|a+b+c\|^2 - \|a+b-c\|^2) = 2(\|a+c\|^2 - \|a-c\|^2) +$
$2(\|b+c\|^2 - \|b-c\|^2)$

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Divide by δ: $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$

Claim: $\langle \lambda a, b\rangle = \lambda \langle a, b\rangle$ — $(*)$

If $\lambda \in \mathbb{N}$: $\langle \underbrace{a}_{\lambda}, b\rangle = \langle a + \dots + a, b\rangle = \langle a, b\rangle + \dots + \langle a, b\rangle$

$= \lambda \langle a, b\rangle$

$\langle (\lambda - \lambda) a, b\rangle = 0$ (definition) $\Rightarrow \langle \lambda a, b\rangle = -\langle -\lambda a, b\rangle$

$\therefore \langle -\lambda a, b\rangle = -\lambda \langle a, b\rangle$ $\therefore$ $(*)$ Valid for $\lambda \in \mathbb{Z}$

If $\lambda \in p/q$: $\sum_{i=1}^{q} \langle \lambda a, b\rangle = \langle pa, b\rangle = p \cdot \langle a, b\rangle$

$\therefore \langle \lambda a, b\rangle = \lambda \langle a, b\rangle$ $\therefore$ $*$ valid for $\lambda \in \mathbb{Q}$

If $\lambda \in \mathbb{R}$: $\exists \lambda_n \in \mathbb{Q}$, $\lambda_n \to \lambda$

Cont. of norm: $\| \lambda_n x \pm y\|^2 \to \|\lambda x \pm y\|$

$\therefore \langle \lambda_n x, y\rangle \to \langle \lambda x, y\rangle$

LHS $= \lambda_n \cdot \langle x, y\rangle \to \lambda \langle x, y\rangle \Rightarrow \langle \lambda x, y\rangle = \lambda \langle x, y\rangle$

$\therefore$ Valid inner product.

(d) X^* is a banach space: Let $(f_n)_{n\geq 1}$ be cauchy.

$\forall x \in X$: $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| . \|x\| \to 0 \Rightarrow$
$(f_n(x))_{n\geq 1}$ is cauchy.

$\mathbb{R}$ complete: $f(x) = \lim_{n\to\infty} f_n(x)$ is well defined.

Claim: f is linear.

$f(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n\to\infty} f_n(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n\to\infty} \{\lambda_1 f_n(x_1) + \lambda_2 f_n(x_2)\}$

$= \lambda_1 f(x_1) + \lambda_2 f(x_2)$

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$\cancel{|f(x)| = \lim_{n\to\infty} |f_n(x)|}$

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \sup_{m_i \geq n} \{ |f_{m_1}(x) - f_{m_2}(x)| \}$$

$$\leq \|x\| \cdot \sup_{m_i \geq n} \{ \|f_{m_1} - f_{m_2}\| \} \to 0 \quad \text{as} \quad n \to \infty.$$

$\therefore$ $\cancel{\|f_n - f\| \to}$ 0

& $|f(x)| \leq |f_n(x)| + |f_n(x) - f(x)| \Rightarrow \|f\| \leq \|f_n\| + \|f_n - f\|$ $< \infty$ for n large.

$\therefore$ f bounded $\Rightarrow f \in X^*$

$\|f_n - f\| \to 0 \Rightarrow f_n \to f \in X^*$ $\therefore$ X^* complete.

If X^* is euclidean, X^* is hilbert space $\Rightarrow$ X'' is hilbert space.

$\varphi : X \to X''$, $\varphi(x)(f) = f(x)$: φ is isometric $\Rightarrow$

$$\|\varphi(x)\| = \|x\| \Rightarrow \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 =$$

$$\|\varphi(x) + \varphi(y)\|^2 + \|\varphi(x) - \varphi(y)\|^2 - 2\|\varphi(x)\|^2 - 2\|\varphi(y)\|^2$$

$$= 0$$

$\therefore$ X is euclidean $\Rightarrow$ Contradiction

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4.) (a) If $|\lambda| > \|T\|$:

$$(T-\lambda I) = -\lambda\,(I - T/\lambda), \quad \|T/\lambda\| < 1$$

$\therefore$ $S_n = \sum_{k=0}^{n} (T/\lambda)^k$, $\quad \|S_n - S_m\| \leq \sum_{k>n} \|T/\lambda\|^k \to 0$ as $n \to \infty$.

$\therefore$ $(S_n)_{n\geq 1}$ is Cauchy sequence in $\mathcal{B}(H)$ (complete)

$\therefore$ $S = \sum_{k\geq 0} (T/\lambda)^k$ is well defined.

$S(I - T/\lambda) = (I - T/\lambda)\cdot S = 0 \Rightarrow (I - T/\lambda)$ invertible.

$(T-\lambda I)^{-1} = -1/\lambda \cdot S$ $\Rightarrow$ $T - \lambda I$ invertible $\Rightarrow$ $\lambda \notin \sigma(T)$

$\therefore$ $\sigma(T) \subseteq \overline{D}_{\mathbb{C}}(0, \|T\|)$

(b) $\forall x \in E_\lambda$, $\forall y \in E_\mu$:

$$\langle T(x), y\rangle = \lambda \langle x, y\rangle = \langle x, T(y)\rangle = \bar{\mu}^* \langle x, y\rangle$$

Claim: $\lambda \in \mathbb{R}$.

$$\langle T(x), x\rangle = \lambda\langle x, x\rangle = \langle x, T(x)\rangle = \bar{\lambda}^* \langle x, x\rangle$$

$\langle x, x\rangle \neq 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

$\therefore$ $(\lambda - \mu)\cdot\langle x, y\rangle = 0$; $\lambda - \mu \neq 0 \Rightarrow \langle x, y\rangle = 0$.

$\therefore$ $E_\lambda \perp E_\mu$.

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(c) ~~Claim~~ Suppose $\dim(E_\lambda) = \infty$: $\exists$ lin. indp. $\{x_n : n \in \mathbb{N}\} \subseteq E_\lambda$

Let $X_n = \langle x_1, \dots x_n \rangle$: $X_n \subsetneq X_{n+1}$

Pick $y_n \in X_{n+1} \cap X_n^{\perp}$ ($\neq \{0\}$ as $X_{n+1} \neq X_n$), $\|y_n\| = 1$

Note: $T(y_n) = y_n$

If T is compact : $\{T(y_n) : n \in \mathbb{N}\}$ is pre-compact.

But $(n>m)$: ~~$\|T_n(y_n) - T_m(y_m)\|^2$~~ $= |\lambda|^2 \|y_n - y_m\|^2$

$= |\lambda|^2 (2 - 2 \underbrace{\langle y_n, y_m \rangle}_{y_m \in X_m \subseteq X_{n-1},\ y_n \perp X_{n-1}}) = 2|\lambda|^2$

$\therefore$ No cauchy subsequence $\Rightarrow$ Contradiction

$\therefore \dim(E_\lambda) < \infty$

(d) $H = H_1 \oplus H_1^{\perp}$; $\forall z \in H$, z uniquely written as $x + y$, $x \in H_1$, $y \in H_1^{\perp}$

$P(z) = x$

Let $z_i = x_i + y_i$:

$\langle P(z_1), z_2 \rangle = \langle x_1, x_2 \rangle + \underbrace{\langle x_1, y_2 \rangle}_{x_1 \in H_1 \Rightarrow \langle x_1, y_2 \rangle = 0} = \langle x_1, x_2 \rangle$

$\langle z_1, P(z_2) \rangle = \langle x_1 + y_1, x_2 \rangle = \langle x_1, x_2 \rangle$

$\therefore P^* = P$ (uniqueness of adjoint)

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(ii) Let $\lambda \neq 0, 1$:

$(P - \lambda I)(x + y) = (1-\lambda)x + (-\lambda)y$

$P(P-\lambda I)(x+y) = (1-\lambda)x$

$(I-P)(P-\lambda I)(x+y) = -\lambda y$

$\Big\} \; x + y = \left(\frac{P}{1-\lambda} - \frac{I-P}{\lambda}\right)(P-\lambda I)(x+y)$

$P/_{1-\lambda} + {}^{P-I}/_{\lambda}$ bounded, right inverse of $P - \lambda I$.

$(P - \lambda I) \cdot \left(P/_{1-\lambda} + {}^{P-I}/_{\lambda}\right) = P/_{1-\lambda} + 0 - \lambda P/_{1-\lambda} - (P - I) = I$

$\therefore$ $P - \lambda I$ invertible

$\therefore \; \sigma(P) \subseteq \{0, 1\}$

Pick $H_1 \neq H \Rightarrow H_1^{\perp} \neq \{0\}$. Pick $z \in H_1^{\perp} \setminus \{0\}$

$\therefore \; P(z) = 0 \Rightarrow 0 \in \sigma_p(P)$

Pick $x \in H_1 \setminus \{0\}$, $(P - I)(x) = 0 \Rightarrow 1 \in \sigma_p(P)$

$\therefore \; \{0, 1\} \subseteq \sigma_p(P) \subseteq \sigma(P)$ $\therefore \; \sigma_p(P) = \sigma(P) = \{0,1\}$

(iii) E_1 : $P(\cancel{x}\; z) = z$ iff $z \in H_1$

$\therefore$ If P compact, $\dim(E_1) < \infty \Rightarrow \dim(H_1) < \infty$.

If $\dim(H_1) < \infty$: $\operatorname{Im}(P) \subseteq H_1$ $\Rightarrow$ (finite dim) $\Rightarrow$ P is finite rank

$\therefore$ Compact

Condition : $\dim(H_1) < \infty$

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Linear Analysis 2018

1.) * 2 ways: Hint $\Rightarrow$ ~~1~~ 1^{st} way.

Let $D = \{ x_n : n \in \mathbb{N} \} \subseteq X$: Given $(f_n)_{n\geq 1}$, $\exists$ subsequence $(f_{\varphi(n)})_{n\geq 1}$ s.t. $\forall r \in \mathbb{N}$, $(f_{\varphi(n)}(x_r))_{n\geq 1}$ is convergent.

(Proof): Let $\varphi_0 : \mathbb{N} \to \mathbb{N}$, $\varphi_0(n) = n$.

Given subsequence φ_r, $(f_{\varphi_r(n)}(x_k))_{n\geq 1}$ is cauchy:

$\forall n \in \mathbb{N}$: $| f_n(x_{r+1}) | \leq \| x_{r+1} \| \Rightarrow \{ f_{\varphi_r(n)}(x_{r+1}) : n \geq 1 \}$ is bounded

$\therefore \exists$ cauchy subsequence $\varphi_{r+1}(n)$ of $\varphi_r(n)$ s.t.

$(f_{\varphi_{r+1}(n)}(x_{r+1}))_{n\geq 1}$ is cauchy.

Let $\varphi(n) = \varphi_n(n)$: $(f_{\varphi(n)})_{n\geq 1}$ is a subsequence of $(f_n)_{n\geq 1}$

$\forall N \in \mathbb{N}$: $(f_{\varphi(n)})_{n\geq N}$ is a subsequence of $(f_{\varphi_N(n)})_{n\geq 1}$

$\therefore (f_{\varphi(n)}(x_N))_{n\geq 1}$ is cauchy $\Rightarrow$ convergent.

Let $f : D \to \mathbb{F}$: $f(x_N) = \lim_{n\to\infty} f_{\varphi(n)}(x_N)$:

* Method 1: Since X is separable, $\exists$ countable ~~Hamel~~ basi linearly independent set $\{ x_n : n \in \mathbb{N} \} = \Lambda$,

$X = \overline{\text{span}(\Lambda)}$

Pick $D = \Lambda$: Define $f : \text{span}(D) \to \mathbb{F}$,

$f\left(\sum_{k=1}^{N} a_k x_k \right) = \sum_{k=1}^{N} a_k f(x_k) \Rightarrow f$ is linear on $\langle D \rangle$

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f is bounded: Fix $x \in \langle D \rangle$, $x = \sum_{k=1}^{N} a_k x_k$.

$$|f(x)| = \cancel{\lim_{n\to\infty} |f_{e(n)}(x)|} \; \left|\sum_{k=1}^{N} a_k \cancel{x} f(x_k)\right|$$

$$= \lim_{n\to\infty} \left|\sum_{k=1}^{N} a_k f_{e(n)}(x_k)\right| = \lim_{n\to\infty} |f_{e(n)}(x)|$$

$$\leq \lim_{n\to\infty} \|x\|.$$

$\therefore$ f bounded.

$\therefore$ f extends uniquely to $\tilde{f} : X \to \mathbb{F}$, $\tilde{f} \in X^*$

Method 2: Banach Alaoglu Technique.

Pick D to be a dense set in X, and derive

$f : D \to \mathbb{F}$, $D \subseteq X$ is dense.

If $z_i \in D$: $$|f(z_1) - f(z_2)| \leq \sum_{i=1}^{2} |f_{e(n)}(z_i) - f(z_i)| + |f_{e(n)}(z_1) - f_{e(n)}(z_2)|$$

(True $\forall n$) $$\therefore |f(z_1) - f(z_2)| \leq \sum_{i=1}^{2} |f_{e(n)}(z_i) - f(z_i)| + \underbrace{\|f_{e(n)}\|}_{1} \cdot \|z_1 - z_2\|$$

$$\to \|z_1 - z_2\|$$

$\therefore$ f is cont. on D wrt subspace topology from X.

$\therefore$ f extends uniquely to: $f : X \to \mathbb{F}$, $\cancel{f_D}$.

f is linear: $$|f(\lambda_1 y_1 + \lambda_2 y_2) - \lambda_1 f(y_1) - \lambda_2 f(y_2)| \leq$$
$$\|f - f_{e(n)}\| \cdot \left[\|\lambda_1 y_1 + \lambda_2 y_2\| + |\lambda_1| \|y_1\| + |\lambda_2| \|y_2\|\right]$$

$\to 0$ as $n \to \infty$ $\therefore = 0$.

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f ~~bounded~~ linear: Fix $\lambda \in \mathbb{F}$, $x, y \in X$, $z = x + \lambda y$.

$$|f(z) - f(x) - \lambda f(y)| \leq |f(z) - f(z')| + |f(x) - f(x')| + |\lambda|\,|f(y) - f(y')| + |f(z') - f_{e(n)}(z')| + |f(x') - f_{e(n)}(x')| + |\lambda|\,|f_{e(n)}(y) - f(y')| + 0$$

For x', y', z' suff. close to x, y, z; n suff. large: LHS is arbitrarily small $\Rightarrow = 0$. $\therefore$ f is linear.

~~fi~~ ~~bo~~ $\therefore$ $f \in X^*$

(b) Claim: ~~$I^\#$~~ $W^\perp = I$

$\forall x \in H$: $x \in W^\perp$ iff $\forall z \in H$, $\langle x, U(z) - z \rangle = 0$

iff $\forall z \in H$, $\langle x, U(z) \rangle = \langle x, z \rangle$

iff $\forall z \in H$, $\langle U^*(x), z \rangle = \langle x, z \rangle$

iff $x \in \ker(U^* - I)$

But $U^* - I\,(x) = 0$ iff $\|U^* - I\,(x)\| = 0$

iff $\|U \cdot (U^* - I)(x)\| = 0$

iff $x \in \ker(U - I)$

Since H complete: $H \cong \overline{W} \oplus W^\perp = \overline{W} \oplus I$

(c) $\forall$ ~~z~~ ~~w~~ $z \in I$: $U(z) = z$

$$\therefore \underbrace{1/n \sum_{k=0}^{n-1} U^{(k)}(z)}_{P_n =} = z \Rightarrow P_n\big|_I = \text{identity map}$$

$\therefore$ $P_n|_I \to id$ map.

If $z \in \overline{W}$: For $\varepsilon > 0$, pick $x \in W$ s.t. $\|z - x\| < \varepsilon$.

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$$\| P_n(Z) \| \le \| P_n(Z-x) \| + \underline{\| P_n(x) \|}$$

$$\le \frac{1}{n} \sum_{k=0}^{n-1} \underbrace{\| U^{(k)}(Z-x) \|}_{= \| Z - x \|}$$

If $x \in W$: $x = U(y) - y$

$$\therefore P_n(x) = \frac{1}{n} \left\{ \sum_{k=1}^{n} U^{(k)}(y) - \sum_{k=0}^{n-1} U^{(k)}(y) \right\} = \frac{U^{(n)}(y) - y}{n}$$

$$\therefore \| P_n(x) \| \le \frac{1}{n} \left\{ \| U^{(n)}(y) \| + \| y \| \right\} \to 0.$$

$\therefore \lim_{n \to \infty} \| P_n(Z) \| \le \varepsilon \Rightarrow \lim_{n\to\infty} \| P_n(Z) \| = 0$ (ε arbitary) $\Rightarrow$ ~~$P_n(Z$~~

$\therefore \forall h \in H$: $h = \omega + Z$, $\omega \in I$, $Z \in \bar{W}$

$\lim_{n \to \infty} P_n(h) = \omega \Rightarrow$ Projection map to I.

$I = \ker(U - I) \Rightarrow$ Closed.

(c) Let $\mu(f) = f(1)$: Equip $C(S')$ with sup norm.

Let $\mu_n(f) = 1/n \sum_{k=0}^{n-1} \mu(T^{(k)}(f))$

$\| \mu_n$ If $\| f \|_\infty = 1$: $| \mu_n(f) | \le 1/n \sum_{k=0}^{n-1} | f(\tau^{(k)}(1)) | \le 1$

$\therefore \| \mu_n \| \le 1 \Rightarrow$ (a) $\exists$ convergent subsequence $\mu_{n_k} \to \mu_\infty$.

$$| \mu_\infty(T(f) - f) | = \lim_{k \to \infty} \left[| \mu_{n_k}(T(f)) - f) | \right] \le \lim_{k \to \infty} \frac{1}{n_k} \left\{ \left| \sum_{i=0}^{n_k - 1} f(\tau^{(i+1)}(1)) - f(\tau^{(i)}(1)) \right| \right\}$$

$$\le \lim_{k \to \infty} 1/n_k \left\{ | f(\tau^{n_k}(1)) - f(1) | \right\}$$

$$\le \lim_{k \to \infty} 2 \| f \| / n_k \to 0 \quad (\forall f \in C(S'))$$

$\therefore \mu_\infty(T(f)) = \mu_\infty(f)$ ∗

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$\mu(1_S) = 1 \Rightarrow$ Done.

(d) ~~Using μ as before~~ Let $\mathcal{I}$ be complex conjugation

$$\left.\begin{aligned} \mu_1 &: f \to f(i) \\ \mu_2 &: f \to \tfrac{1}{2} f(i) + \tfrac{1}{2} f(-i) \end{aligned}\right\} \quad \begin{aligned} \mu_1(\mathcal{I}\cdot f) &= f(i) = \mu_1(f) \\ \mu_2(\mathcal{I}\cdot f) &= \tfrac{1}{2} f(\bar{i}) + \tfrac{1}{2} f(-\bar{i}) \\ &= \mu_2(f) \end{aligned}$$

But $\mu_1 \neq \mu_2$: $f: z \to z^2$ $i \cdot z$

$\mu_1(f) = f(i) = i$

$\mu_2(f) = \frac{1}{2}(f(i) + f(-i)) = \frac{1}{2}(-1 + (--1)) = 0$

2.) (a) T is compact if $\forall E \subseteq X$, E bounded : $T(E)$ is totally bounded

$\mathcal{G}$ T_i: bounded $\Rightarrow \forall E \subseteq X$ bounded, $T_i(E)$ bounded.

But $T_i(E) \subseteq$ Finite dim space; $\Rightarrow$ Totally bounded $\Leftrightarrow$ Bounded $\Rightarrow$ $T_i(E)$ is totally bounded

If $T_n \to T$: let $E \subseteq X$ be bounded. Fix $\varepsilon > 0$.

Let $M \geq \operatorname{Sup}\{\|x\| : x \in E\}$

Pick $N \in \mathbb{N}$ s.t. $\|T_N - T\| < \varepsilon/3M$; let $\{x_1, \dots x_m\} \subseteq E$ be a ε-net of $T_n(E)$.

$\forall x \in E$, $\exists x_r$ s.t. $\|T_n(x - x_r)\| < \varepsilon/3$.

$$\begin{aligned} \|T(x) - T(x_r)\| &\leq \|T_n(x - x_r)\| + \|T_n - T(x)\| + \|T_n - T(x_r)\| \\ &< \varepsilon/3 + \underbrace{\{\|x\| + \|x_r\|\}}_{\leq 2M} \cdot \underbrace{\|T_n - T\|}_{< \varepsilon/3M} \\ &< \varepsilon \end{aligned}$$

$$\therefore T(E) \subseteq \bigcup_{k=1}^{m} D_Y(T(x_k), \varepsilon)$$

$\therefore$ $T(E)$ totally bounded $\Rightarrow$ T compact

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(b) ~~Let $E \subseteq Y^*$ be a bounded set: $(f_n)_{n\geq 1}$ be any sequence in E. $\exists M>0$ s.t. $\forall n \in \mathbb{N}$, $\|f_n\| \leq M$~~

~~$T(f_n) = f_n(T(\cdot))$. Let $S = \{x \in X : \|x\| = 1$~~

~~Claim: $F = \{T(f_n) : n \in \mathbb{N}\}$ is bounded.~~

~~$\forall x \in X$: $|f_n(T(x))| \leq \|f_n\| \cdot \|T(x)\| \leq \|f_n\| \, M \cdot \|T\| \cdot \|x\|$~~

~~$\therefore \|f_n(T(\cdot))\| \leq M \cdot \|T\| \Rightarrow F$ bounded~~

~~Claim: F equicontinuous.~~

~~Given $x \in X$~~

~~Let $D = B_X(0,1)$, $A = T(D) \subseteq Y$: A is pre-compact $\Rightarrow \bar{A}$ compact~~

~~Consider $F = \{ f_n|_{\bar{A}} : n \in \mathbb{N}\}$;~~

~~F bounded: $\forall x \in \bar{A}$, $\forall n \in \mathbb{N}$, $|f_n(x)| = |f_n(T(y))|$ $(T(y)=x)$~~

~~$\leq \|f_n\| \cdot \|T\| \cdot 1$~~

~~$\leq \|T\| \cdot M$~~

~~$\therefore F$ bounded.~~

~~F equicontinuous: Given $T(y) = x \in A$~~

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Let $A = D(0,1)_{Y^*}$: $E = T(\bar{D}(0,1)_X) \subseteq Y$

T compact $\Rightarrow$ $\bar{E}$ is compact.

Let $\mathcal{F} = \{ f|_E : f \in A \}$:

If $z \in E$: $\exists$ $x_n \in \bar{D}(0,1)_X$ s.t $T(x_n) \to z$.

$\therefore$ $|f(z)| = \lim_{n\to\infty} |f(T(x_n))| \leq \underbrace{\|f\|}_{\leq 1} \cdot \|T\| \cdot \underbrace{\|x_n\|}_{\leq 1} \leq \|T\|$

$\therefore$ $\sup_{f \in A} \{ \|f|_E\|_\infty \} \leq \|T\|$ $\Rightarrow$ $\mathcal{F}$ bounded.

If $z_i \in \bar{E}$, : $\|f(z_1 - z_2)\| \leq \underbrace{\|f\|}_{L} \cdot \|z_1 - z_2\|$

$\therefore$ $\mathcal{F}$ is uniformly Lipschitz $\Rightarrow$ Equicontinuous

Arzela Ascoli $\Rightarrow$ $\mathcal{F}$ uniformly bounded. - (1)

Let $(f_n)_{n\geq 1}$ be a sequence in A:

Let $(f_{n_k}|_E)_{k\geq 1}$ be cauchy ($(n_k)_{k\geq 1}$ exists by (1))

$$\|T^* f_{n_{k_1}} - T^* f_{n_{k_2}}\| = \sup_{x \in \bar{D}(0,1)} \{ |f_{n_{k_1}} - f_{n_{k_2}}(T(x))| \}$$
$$\leq \|f_{n_{k_1}} - f_{n_{k_2}}|_E\|_\infty$$

$\therefore$ $(f_{n_k})_{k\geq 1}$ cauchy $\Rightarrow$ $T^*(A)$ is pre-compact

$\therefore$ T^* compact.

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(c) Fix $\alpha > 0$: Claim: $\exists$ finite lin. indp. eigenvectors x_n w.r.t. with eigenvalue λ_n, $|\lambda_n| \geq \alpha$.

Suppose otherwise:

$X_n = \langle x_1, \dots x_n \rangle$: $X_{n+1} \neq X_n \Rightarrow X_{n+1} = X_n \oplus X_n^{\perp_{n+1}}$ (Finite dim spaces closed); $\perp_{n+1}$ is w.r.t. X_{n+1}

Pick $z_{n+1} \in \cancel{X_{n+1}} X_n^{\perp_{n+1}}$ $(n \geq 1)$, $z_1 \in X_1$;

$\| z_n \| = 1$

$z_n = x_n + y_n$, $x_n \in X_{n-1}$, $y_n \in \langle x_n \rangle$

$\therefore$ $T - \lambda_n (z_n) \in X_{n-1}$

Consider: $(n > m)$ $\| T(z_n) - T(z_m) \|^2 = \| (T-\lambda_n)(z_n) - (T-\lambda_m) z_m + \lambda_n z_n - \lambda_m z_m \|^2$

$= \| \lambda_n z_n + (\text{term in } X_{n-1}) \|^2 \geq |\lambda_n|^2 \geq \alpha^2$

$\therefore$ No cauchy subsequence in $(T(z_n))_{n \geq 1}$

$\{ z_n : n \in \mathbb{N} \}$ bounded $\Rightarrow$ Contradiction

(L. I.)

$\therefore$ Given $\varepsilon > 0$, $\exists$ finite eigenvector with eigenvalue $|\lambda|$ λ, $|\lambda| \geq \varepsilon$.

$\therefore$ $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |\lambda_n| \leq \varepsilon$.

$\therefore$ $\lambda_i \to 0$.

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1.) (a) Let K be compact, $C_X(K) = \{ f: K \to X : f \text{ cont.} \}$

Arzela Ascoli: $F \subseteq C_{\mathbb{R}/\mathbb{C}}(K)$ is ~~com~~ totally bounded iff

F is bounded, equicontinuous

$C(\mathbb{R})$

Stone Weierstrass: Let $A \subseteq C_{\mathbb{R}}(K)$ be an algebra that is separating ($x \neq y \Rightarrow \exists f \in A,\ f(x) \neq f(y)$)

If $\exists x_0$ s.t. $\forall f \in A : f(x_0) = 0$, $\bar{A} = \{ f \in C_{\mathbb{R}}(K) : f(x_0) = 0 \}$

Else, $\bar{A} = C_{\mathbb{R}}(K)$

$(\mathbb{C})$: If $A \subseteq C_{\mathbb{C}}(K)$ is a separating algebra. i; $f \in A$ iff $\bar{f} \in A$;

If $\exists x_0$ s.t. $\forall f \in A,\ f(x_0) = 0$: $\bar{A} = \{ f \in C_{\mathbb{C}}(K) : f(x_0) = 0 \}$

Else: $\bar{A} = C_{\mathbb{C}}(K)$

$K = [0,1]$, $f(x) = 1_{[0, 1/n]}(1 - nx)$

$\|f\|_\infty = 1 \Rightarrow \{ f_n : n \in \mathbb{N} \} = \Lambda$ is bounded.

If $(f_{n_k})_{k \geq 1}$ is a subsequence: $\| f_{n_k} - f_{n_{k_2}} \| \geq$ $(k_1 \geq k_1)$

$| f_{n_{k_1}}(1/n_{k_2}) - f_{n_{k_2}}(1/n_{k_2}) | = 1 - n_{k_1}/n_{k_2} \to 1$ as $k_2 \to \infty$.

$\therefore$ Not cauchy $\Rightarrow \Lambda$ not pre-compact.

(b) $\forall k \in \mathbb{N}$: $A_k = [-k, k]^n$ is compact.

Let $\Lambda_k = \{ \text{polynomials on } [-k,k]^n \}$, Λ_k is an algebra on ~~of~~ $C(A_k)$

If $\underline{x} \neq \underline{y}$: $x_r \neq y_r$ for some $y \in [1, n]$

$f(\underline{z}) = z_r$ separates $\underline{x}, \underline{y}$

Since constant $1 \in \Lambda_k$: SWT $(\mathbb{R}) \Rightarrow \overline{\Lambda_k} = C_{\mathbb{R}}(A_k)$

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$\exists \; P_k$ s.t. $\| P_k - f|_{A_k} \|_\infty < 1/k$

Given B compact, $B \subseteq A_R$ for some $R \in \mathbb{N}$:

$\therefore \; r \geq R: \quad \| P_r - f|_B \| \leq \| P_r - f|_{A_R} \|_\infty \leq$

$\| P_r - f|_{A_r} \|_\infty < 1/r \to 0.$

$\therefore$ Done.

If f is a polynomial: $P_n = f$, $P_n \to f$ uniformly

If $P_n \to f$ uniformly: Sequence $(P_n)_{n\geq 1}$ must be Cauchy.

$\therefore \; \exists \; N \in \mathbb{N}$ s.t $n \geq N \Rightarrow \| P_n - P_N \|_\infty \leq 1$

~~Given~~:

But $P_n - P_N$ is a polynomial; bounded $\Rightarrow$ $P_n - P_N$ is constant

$\therefore \; P_n = P_N + a_n$, a_n constant.

$f(x) = \lim_{n\to\infty} P_N + a_n \Rightarrow f = P_N + \lim_{n\to\infty} a_n \Rightarrow f$ is a polynomial

$\therefore \; P_n \to f$ uniformly iff f is a polynomial

(c) Fix $x \in K$: Pick U open s.t $x \in U$, s.t $\forall f \in C(K)$,

$\forall y \in U, \quad |f(x) - f(y)| \leq 1/2$

If $\exists$ ~~distinct~~ $y \in U$, $y \neq x$:

Urysohn lemma $\Rightarrow \exists \; f \in C(K)$, s.t $f(x) = 0$, $f(y) = 1$

$(\{x\}, \{y\}$ closed, disjoint$)$

$\therefore \; |f(x) - f(y)| = 1 > 1/2 \Rightarrow$ Contradiction $\Rightarrow U = \{x\}$

$\therefore \; K = \bigcup_{x \in K} \{x\}$, $\{x\}$ open $\Rightarrow$ (K compact) $\exists$ finite subcover

$\therefore \; K = \bigcup_{r=1}^{n} \{x_r\} \Rightarrow K$ is finite.

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Consider $K = \mathbb{N}$, discrete topology.

$\therefore \forall x \in K$, pick $U = \{x\}$. $\therefore \forall y \in U$, $\forall f \in C(K)$,

$|f(y) - f(x)| = 0$. $\therefore$ Equicontinuous

$|K| = \infty$.

$V =$

2.) (a) $\mathcal{B}(X, Y)$ is a vector space.

If $\lambda_i \in \mathbb{F}$, $T_i \in V$: $(\lambda_1 T_1 + \lambda_2 T_2)$ remains linear, bounded

$\therefore V$ is a $\mathbb{F}$-space

Norm: $\| T \| = \sup_{\|x\| \le 1} \| T(x) \|$

$\| T \| \ge 0$; $= 0$ iff $\forall x \in X, \|x\| = 1 \Rightarrow T(x) = 0$ iff $T \equiv 0$

$\|\lambda T\| = \sup_{\|x\| \le 1} \left\{ |\lambda| \cdot \|T(x)\| \right\} = |\lambda| \cdot \| T \|$

$\| T_1 + T_2 \| = \sup_{\|x\| \le 1} \left\{ \| (T_1 + T_2)(x) \| \right\} \le \sup_{\|x\| \le 1} \left\{ \| T_1(x) \| + \| T_2(x) \| \right\}$

$\le \| T_1 \| + \| T_2 \|$.

$\therefore$ Valid norm

Complete: Let $(T_n)_{n \ge 1}$ be a cauchy sequence.

$\forall x \in X$: $\| T_n(x) - T_m(x) \| \le \|x\| \cdot \| T_n - T_m \| \Rightarrow$

$\forall (T_n(x))_{n \ge 1}$ is cauchy in Y.

Y complete $\Rightarrow T : X \to Y$, $T(x) = \lim_{n \to \infty} T_n(x)$ is well defined.

T linear: $T(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n \to \infty} \lambda_1 T_n(x_1) + \lambda_2 T_n(x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$

Consider: $T_n - T$

If $\|x\| \le 1$: $\| T_n - T(x) \| = \lim_{m \to \infty} \| T_n - T_m(x) \|$

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Given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $n, m \geq N \Rightarrow \|T_n - T_m\| < \varepsilon$.

$\therefore$ For $n \geq N$: $\|T - T_n(x)\| \leq \lim_{m \geq n} \|T_m - T_n(x)\| < \varepsilon \cdot \|x\| = \varepsilon$.

(Valid $\forall$ x, $\|x\| = 1$)

$\therefore$ $\|T - T_n\| \leq \varepsilon$ for $n \geq N$. — (1)

$\therefore$ $\|T\| \leq \|T - T_N\| + \|T_N\| < \infty \Rightarrow T \in V$.

(1): $T_n \to T \in V \Rightarrow V$ is complete $\Rightarrow$ Banach space.

(h) * Waffle Alert *

Let X, Y be metric spaces: $D \subseteq X$ be dense.

If $\tilde{f}: D \to Y$ is cont., can $\tilde{f}$ be extended to $f: X \to Y$ cont. ?

Looks trivial, but extension in general is hard. The trivial part is uniqueness.

Consider: $X = [0,1]$, $Y = \mathbb{R}$,

$f: (0,1) \to \mathbb{R}$ be homeomorphism, $f(0^+) \to -\infty$, $f(1^-) \to +\infty$,

$(0,1)$ dense in X, but no extension possible.

(Y not complete: problem obvious)

~~If X, Y both complete extension is possible.~~

$\therefore$ General: Not true

But over ~~linear~~ NV - spaces X, Y:

Define $f: X \to Y$:

Pick $x_n \to x$, $x_n \in D$

f cont., bounded $\Rightarrow$ $|f(x_n) - f(x_m)| \leq \|f\| \cdot \|x_n - x_m\|$

$\therefore$ $(f(x_n))_{n \geq 1}$ is cauchy in $Y \Rightarrow$

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(Y complete) : $f(x_n) \to y \in Y$

Let $f(x) = y$

(Well defined): $x_n \to x,\ z_n \to x \Rightarrow x_n - z_n \to 0$

$\therefore\ |f(x_n) - f(z_n)| \to 0 \Rightarrow \lim_n f(x_n) = \lim_n f(z_n)$

* Metric spaces : We need f to be uniformly cont. *

Claim: f is linear.

$$f(a + \lambda b) = \lim_n f(a_n + \lambda b_n) = f(a) + \lambda f(b) \qquad \left.\begin{matrix} a_n \to a \\ b_n \to b \end{matrix}\right.$$

* f clearly an extension

Claim: f is cont.

$$|f(x)| = \lim_{n\to\infty} |f(x_n)|, \qquad x_n \to x$$

$$\le \|f\| \limsup_n \|x_n\| \qquad |f| \le \|f\| \cdot \|x\|$$

$x_n \to x \Rightarrow \|x_n\| \to \|x\|$

$\therefore$ Same norm

* Metric Space:

$d(f(x), f(y)) = \lim_n d(f(x_n), f(y_n))$

Given $\varepsilon > 0$, $f|_D$ u.cont. $\Rightarrow \exists\ \delta > 0$ s.t. $d(z_1, z_2) < \delta \Rightarrow d(f(z_1), f(z_2)) < \varepsilon/2$

If $d(x, y) < \delta/2$: $\exists N$ s.t. $n \ge N \Rightarrow d(x_n, x), d(y_n, y) < \delta/4$

$\therefore\ d(x_n, y_n) < \delta \Rightarrow d(f(x_n), f(y_n)) < \varepsilon$

$\therefore$ Done

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X not nec. complete.

Y needs to be complete: let C_{00} be $= \{ x \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n = 0 \}$

C_{00} dense in λ_1

Consider $f : C_{00} \to C_{00}$, f is id map

f cannot be extended to $\tilde{f} : \lambda_1 \to C_{00}$

(c) Define : $\Phi : X^* \to \lambda_1$, $\Phi(f) = (f(x_1), \dots f(x_2), \dots)$

$\sum_{k \geq 1} |f(x_k)| < \infty \Rightarrow \Phi(f) \in \lambda_1$

Φ is clearly linear ; λ_1, X^* are both complete.

If $f_n \to f$, $\Phi(f_n) \to y$.

$y_r = \lim_{n \to \infty} f_n(x_r) = f(x_r) \Rightarrow \underset{\sim}{y} = \Phi(f)$

$\therefore$ Graph of Φ is closed.

$\therefore$ Closed graph theorem $\Rightarrow$ Φ is bounded

$\therefore \exists\ C > 0$ s.t. $\underbrace{\| \Phi(f) \|}_{\sum_{k \geq 1} |f(x_k)|} \leq C \| f \|$

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Linear Analysis 2019

1.) (a) F dense in λ^p:

Given $\underset{\sim}{x} \in \lambda^p$: Let $(x^n)_k = x_k \cdot 1_{k \leq n}$

$$\| \underset{\sim}{x} - \underset{\sim}{x}^{(n)} \|_p^p = \sum_{k>n} |x_k|^p \to 0 \text{ as } n \to \infty .$$

$\therefore \underset{\sim}{x}^{(n)} \to \underset{\sim}{x}$ in $\lambda^p \Rightarrow F \leq \lambda^p$ is dense.

Consider $\mathbb{1} \in \lambda^\infty$ $(\mathbb{1} = (1, \ldots\ldots))$

$\forall\ \underset{\sim}{x} \in F: \exists\ N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n = 0.$

$\therefore \| \mathbb{1} - \underset{\sim}{x} \|_\infty \geq | \mathbb{1}_N - x_N | \geq 1$

$\therefore$ No sequence in F converging to $\mathbb{1}$

$\therefore$ Not dense

(b) (Existence): Given $\underset{\sim}{x} \in \lambda^p$, pick $x^n \in F$ s.t. $\| x^{(n)} - x \|_p \to 0$.

~~Let~~ $\therefore (x^{(n)})_{n \geq 1}$ is cauchy in F (wrt $\|\cdot\|_p$ norm)

$\| T(x^{(n)}) - T(x^{(m)}) \| \leq C \cdot \| x^{(n)} - x^{(m)} \| \Rightarrow (T(x^{(n)}))_{n \geq 1}$ is cauchy in λ^{p}

Let: ~~is~~ $\tilde{T}(x) = \lim_{n \to \infty} T(x^{(n)}) \in \lambda_p$ (λ_p complete)

(Well defined) : If $x^{(n)}, y^{(n)} \to x$, $x^{(n)} - y^{(n)} \to 0$.

$\therefore T(x^{(n)}) - T(y^{(n)}) \to 0 \Rightarrow \lim_n T(x^{(n)}) = \lim_n T(y^{(n)})$

$\therefore \tilde{T}$ well defined

~~$T(\lambda_1 x_1' + \lambda_2 x_2') = \lim_n T(\lambda_1 z')$~~

(linear) : $\tilde{T}(z + \lambda x) = \lim_n T(z_n + \lambda x_n)$, $\begin{cases} z_n \to z,\ x_n \to x, \\ z_n,\ x_n \in F \end{cases}$

$= \lim_n T(z_n) + \lambda \lim_n T(x_n)$ $\therefore$ Done

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$\| \tilde{T}(z) \| = \lim_n \| T(z_n) \| \le \limsup_n C \cdot \| z_n \|$

But $z_n \to z$ in $\lambda_p \Rightarrow \|z_n\| \to \|z\|$

$\therefore \| \tilde{T}(z) \| \le \|z\| \cdot C \Rightarrow \|\tilde{T}\| \le C$

$\therefore \tilde{T} \in B(\lambda_p, \lambda_p)$, $\|\tilde{T}\| \le C$.

If $\tilde{S}, \tilde{T}$ both fulfill criteria: $\forall x \in \lambda^p$, let $x_n \in F$, $x_n \to x$

$\tilde{S}(x) = \tilde{T}(x) = \lim_n T(x_n) \Rightarrow \tilde{T}$ unique.

(c) T_1 not bounded: $\|T_1\| \ge \sup \{ \|T_1(e_k)\| : k \in \mathbb{N} \}$

$= \sup \{ \| k e_k \| : k \in \mathbb{N} \} = \sup \infty$

$((e_k)_r = 1_{r=k}$, $e_k \in \lambda_p$, $p \in [1, \infty])$

(ii) ~~T_2 not bounded:~~ $~~\|T_2\| \ge \sup \{ \|T_2(e_n)\| : n \in \mathbb{N} \}~~$

$~~= \sup \{ \| n e_n - (n-1) e_{n-1} \| : n \in \mathbb{N} \}~~$

$~~= \sup \{ (n^p + (n-1)^p)^{1/p} : n \in \mathbb{N} \}~~$

$\ge \sup \{ n : n \in \mathbb{N} \} = \infty.$

(ii) $p \in [1, \infty)$, $T_2(e_n) = \underbrace{n e_n - (n-1) e_{n-1}}_{\|\cdot\| \ge n.}$

$\therefore \|T_2\| \ge n$ $\forall n \in \mathbb{N}$: $\Rightarrow T_2$ not bounded

$p = \infty$: $\forall n \in \mathbb{N}$: $\|T_2(e_n)\|_\infty \ge n \Rightarrow \|T_2\| \ge n \Rightarrow$ Not bounded

~~(iii) p =~~

T_3 or T_4: If $T(e_n) = a_n \cdot e_n$, $|a_n| \le 1$

$p \in [1, \infty)$: $\|T(x)\|_p^p = \sum_{k \ge 1} |a_k|^p \cdot |x_k|^p \le \sum_{k \ge 1} |x_k|^p$

$\therefore \|T\| \le 1 \Rightarrow$ Bounded

$p = \infty$: $\|T(x)\|_\infty = \sup \{ |a_k| \cdot |x_k| : x \in \mathbb{N} \} \le \sup \{ |x_k| : k \in \mathbb{N} \}$

$= \|x\|_\infty \Rightarrow$ Bounded

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$\cancel{\left|\sum_{j=1}^{n} x_j\right|}$

$$\|T(e_n)\|_p^p = \sum_{r\geq n} (1/2^r)^p = \frac{1}{2^{np}} \cdot \sum_{r\geq 0} \frac{1}{(2^p)^r}$$

$$= \frac{1}{2^{np}} \frac{1}{1-2^{-p}} \leq \frac{1}{2^{np}}$$

$$\therefore \|T(e_n)\|_p \leq \frac{1}{2^n}$$

$$\therefore \|T(\sum_{k\geq 1} x_k e_k)\|_p \leq \sum_{k\geq 1} |x_k| \cdot \|T(e_k)\| \leq \|x\|_p \underbrace{\left(\sum_{k\geq 1} \left(\frac{1}{2^k}\right)^q\right)^{1/q}}_{<\infty}$$

$\therefore$ $p \in [1, \infty)$: Bounded

$p = \infty$: $$\left|\sum_{j=1}^{n} x_j / 2^n\right| \leq \frac{n \|x\|_\infty}{2^n} \leq \|x_\infty\|$$

$$\therefore \|T(x)\|_\infty \leq \|x\|_\infty.$$

(d) If $\bar{D}(0,1)$ is compact:

$\bar{D}(0,1) \subseteq \bigcup_{x\in \bar{D}(0,1)} D(x, 1/2) \Rightarrow \exists$ finite sub cover.

$\bar{D}(0,1) \subseteq \bigcup_{k=1}^{n} D(x_k, 1/2) \Rightarrow \bar{D}(0,1) \subseteq \langle x_1, \dots x_n \rangle + D(0, 1/2)$

$\therefore D(0, 1/2^N) \subseteq \langle x_1, \dots x_n \rangle + D(0, 1/2^{N+1})$

$\therefore D(0,1) \subseteq \langle x_1, \dots x_n \rangle + \dots + \langle x_1, \dots x_n \rangle + D(0, 1/2^{N+1})$

$\therefore$ ~~$\langle x_1, \dots x_n \rangle$ dense in $D(0,1)$~~ $\forall z, \in D(0,1)$,

$\exists$ ~~x_n~~ $y_n \in \langle x_1, \dots x_n \rangle$, $y_n \to z$.

But $\langle x_1, \dots x_n \rangle$ finite - dim $\Rightarrow$ ~~closed~~ ~~sp~~ complete space.

$\therefore$ $\langle x_1, \dots x_n \rangle$ is closed

$\therefore$ $D(0,1) \subseteq \langle x_1, \dots x_n \rangle \Rightarrow$ ~~$\forall$~~ $X = \langle x_1, \dots x_n \rangle$

$\therefore$ X is finite dim.

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2.) (a) Let K be compact: $C(K) = \{ f: K \to \mathbb{R} : f \text{ cont.} \}$

Let $A \subseteq K$ be an algebra; $\forall x, y \in K$, $x \neq y \Rightarrow \exists f$ s.t. $f(x) \neq f(y)$

If $\exists x_0 \in K$ s.t. $\forall f \in A$, $f(x_0) = 0$: $\bar{A} = \{ f \in C(K) : f(x_0) = 0 \}$

Else: $\bar{A} = C(K)$ (SWT): Stone Weierstrass Theorem

Tietze Extension: Let X be normal, $S \subseteq X$ be closed

If $f: S \to [0,1]$ is cont., $\exists \tilde{f}: X \to [0,1]$ cont., s.t.

$\tilde{f}|_S = f$.

(b) Claim: $f(x) = \cancel{\text{for}}\ a_1 x + b_1 \quad (x \in [0, 1/2])$ (L_1)

$= a_2 x + b_2 \quad (x \in [1/2, 1])$ (L_2)

~~for iff~~ $\cancel{a_1 > a_2 \text{ iff } a_1 x + b_1 \geq a_2 x + b_2 \text{ on } [1/2, 1]}$

$\therefore$ $\cancel{f(x)}$ ~~[illegible]~~

If $a_1 \geq a_2$: $f(x) = \frac{1}{2}(L_1(x) + L_2(x)) - \frac{1}{2}|L_2(x) - L_1(x)|$

$a_1 < a_2$: $f(x) = \frac{1}{2}(L_1(x) + L_2(x)) + \frac{1}{2}|L_2(x) - L_1(x)|$

$a_1 \geq a_2$: $L_1(x) \geq L_2(x)$ on $[1/2, 1]$

$L_1(x) \leq L_2(x)$ on $[0, 1/2]$

$a_1 < a_2$: $L_1(x) \leq L_2(x)$ on $[1/2, 1]$

$L_1(x) \geq L_2(x)$ on $[0, 1/2]$

Let $p(x) = \frac{L_1 + L_2}{2}$, $q(x) = \frac{L_2 - L_1}{2}$. $\}$ p, q are polynomials

$\therefore f = p(x) + |q(x)|$ or $p(x) - |q(x)|$

$\therefore f = p \pm \{ \sqrt{q^2(x)}$

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Let $M = \| q|_{[0,1]} \| < \infty$

$\therefore \quad \left| q(x)/M \right| \le 1 \quad$ for $x \in [0,1]$

If $\quad \| P_n(x) - \sqrt{x} \|_\infty \le \varepsilon$:

$$\left\| \sqrt{(q(x)/M)^2} - P_n\left(\left(q(x)/M \right)^2 \right) \right\|_\infty \le \varepsilon \quad \text{as} \quad \left| q(x)/M \right| \in [\cancel{0} \; [-1,1]$$

$$\Rightarrow \left| q(x)/M \right|^2 \in [0,1]$$

$$\therefore \left| f(x) - \underbrace{\left(P(x) \pm P_n\left(\left(q^2/M^2 \right) \right) \right)}_{\text{polynomial}} \right| \le \varepsilon \quad \text{on } [0,1]$$

$\therefore \ \exists$ Polynomials Q_n s.t. $\| f - Q_n \|_\infty \to 0$.

(d) Arzela Ascoli: Let K be compact. $\mathcal{F} \subseteq C(K)$ is relatively compact iff $\mathcal{F}$ is bounded and equicontinuous.

F_1: $\quad f_n(x) = \sin(n\pi x)/n \quad ; \quad \| f_n \| \le 1/n \Rightarrow F_1$ bounded.

$$|f_n(x) - f_n(y)| \le |x-y| \cdot |\cos(n\pi z)| \qquad (z \in (x,y))$$
$$\le |x-y|$$

$\therefore$ F_1 is uniformly Lipschitz $\Rightarrow$ Equicontinuous

F_2: Let $\quad f_n(x) = \sin(n\pi x)/\sqrt{n}, \quad F_2 = \{ f_n(x) : n \in \mathbb{N} \}$

~~Given any sequence $(f_{n_k})_{k \ge 1}$ in F_2: By If $\exists$ finite distinct~~
~~elements in sequence, $\Rightarrow \exists$ constant subsequence $\Rightarrow$ Cauchy~~
~~* $(n_k)_{k \ge 1}$ not necessarily increasing~~
~~Else, $\exists \infty$ distinct terms Max. elements. Let $f_{\varphi(1)} = f_{n_1}$~~
~~Given $f_{\varphi(r)}$: Pick $f_{\varphi(r+1)} = f_{n_k}$, $n_k > n_{\varphi(r)}$~~
~~n_k exists else $\{ f_{n_k} : k \ge 1 \} = \{ f_{n_1}, \dots f_{n_{\varphi(r)}} \}$ is finite.~~

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If ~~f_n~~ $f_{n_r} \in F_2$: $\{ f_{n_r} : r \in \mathbb{N} \}$ finite $\Rightarrow \exists$ infinite repeated elements in series $\Rightarrow \exists$ constant ($\Rightarrow$ Cauchy) subsequence

If $\{ f_{n_r} : r \in \mathbb{N} \}$ ~~not~~ not finite : Given f_{n_r}, $\exists\, r' > r$ s.t. $n_{r'} > n_r$. Else, $\{ f_{n_k} : k \in \mathbb{N} \} = \{ f_{n_1}, .. f_{n_r} \}$ is finite.

$\therefore \exists$ subsequence $(f_{\varphi(k)})_{k \geq 1}$, ($(\varphi(k))_{k \geq 1}$ is strictly increasing) $\Rightarrow \varphi(k) \geq k$)

$\therefore$ Given $\varepsilon >$ $\| f_{\varphi(k)} \|_\infty = \| 1/\sqrt{\varphi(k)} \| \leq 1/\sqrt{k} \to 0$. in $C(K)$

$\therefore$ Subsequence is convergent, $\Rightarrow$ Cauchy

$\therefore$ F_2 relatively compact.

F_3: Fix $\varepsilon > 0$; Pick $n \in \mathbb{N}$ s.t. $\frac{1}{2n} < \varepsilon$.

$\therefore$ ~~x~~ $f_n(x) = \sin(n\pi x) \Rightarrow f_n(1/2n) = 1$, $1/2n \in D(0, \varepsilon)$

$\therefore \forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ ~~&~~, $y \in D(0, \varepsilon)$ s.t. $| f_n(y) - f_n(0) | = 1$

$\therefore$ Not equicontinuous at 0.

Arzela–Ascoli $\Rightarrow$ Not relatively compact.

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3.) (a) Since $x \to \alpha x$ is an homeomorphism for $\alpha \neq 0$:

$D(0, \alpha) \subseteq T(D(0, \alpha))$

Let $\alpha \in (0, 1)$

Fix $x_0 \in D(0, \alpha)$: Let $\varepsilon \in (0, 1)$;

$\exists\, y_1 \in D(0, \alpha)$ s.t. $\| x_0 - T(y_1) \| < \varepsilon \cdot \alpha$

Let $x_1 = x_0 - y_1$

Given $x_n \in D(0, \alpha \cdot \varepsilon^n)$: Pick $y_{n+1} \in D(0, \alpha \cdot \varepsilon^n)$ s.t

$\| T(y_{n+1}) - x_n \| < \alpha \cdot \varepsilon^{n+1}$

Let $z_n = \sum_{k=1}^{n} y_k$

If $n, m \geq N$: $\| z_n - z_m \| \leq \sum_{k \geq N} \| y_k \| < \alpha \varepsilon^N / 1-\varepsilon \to 0$ as $N \to \infty$.

$\therefore (z_n)_{n \geq 1}$ is cauchy in $X \Rightarrow z_n \to z \in X$.

$z = \sum_{k=1}^{\infty} y_k$, $\| z \| < \alpha / 1-\varepsilon$

$\| T(z_n) - x_0 \| < \alpha \cdot \varepsilon^n \to 0 \Rightarrow T(z_n) \to x_0$

$T(z_n) \to T(z) \Rightarrow T(z) = x_0$

$\therefore$ $x_0 \in T(D(0, \alpha/1-\varepsilon))$; For ε suff. small,

$x_0 \in T(D(0,1)) \Rightarrow D(0, \alpha) \subseteq T(D(0,1))$

$\therefore \bigcup_{0 \leq \alpha < 1} D(0, \alpha) = D(0,1) \subseteq T(D(0,1))$.

(b) Baire category Theorem: If (X, d) is complete metric space, $A_n \subseteq X$ is closed, nowhere dense.

$X \neq \bigcup_{n \geq 1} U_n$

Let $P_n = \{ f \in P : \deg(f) \leq n \} = \langle 1, x, \dots x^n \rangle$

$\therefore P_n$ is finite dim. $\Rightarrow P_n \leq P$ is closed.

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Since $P_n \leq P$: If $\exists\ D(f,\varepsilon) \subseteq P_n$, $(-1)\cdot f + D(f,\varepsilon) = D(0,\varepsilon) \subseteq P_n$

$\therefore \forall d>0: D(0,d) \in P_n \Rightarrow P_n = P$ (contradiction)

$\therefore P_n^\circ$ ~~is~~ $= \phi$; $P = \bigcup_{n\geq 0} P_n \Rightarrow P$ canot be complete

as P_n is closed, nowhere dense.

(c) $A_n = \{ z \in \mathbb{C} : f^{(n)}(z) = 0 \} = f^{(n)^{-1}}\{0\}$ is closed as

$f^{(n)}$ is cont.

$\mathbb{C}$ is complete, $\mathbb{C} = \bigcup_{n\geq 1} A_n \Rightarrow \exists N \in \mathbb{N}$, $A_N^\circ \neq \phi$

$\therefore \exists\ D(z,\varepsilon) \subseteq A_N \Rightarrow f^{(N)} \equiv 0$ on $D(z,\varepsilon) \Rightarrow f^{(N)} \equiv 0$ on $\mathbb{C}$

$\therefore n \geq N \Rightarrow f^{(n)} \equiv 0$.

f entire $\Rightarrow \exists$ taylor series at 0 ; $f^{(n)} = 0$ for $n \geq N \Rightarrow$

$f(x) = \sum_{k=0}^{N-1} a_k x^k \Rightarrow$ polynomial

(d) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 = 4$

$\therefore 4 \geq \varepsilon^2 + \|x+y\|^2 \Rightarrow \left\|\frac{x+y}{2}\right\|^2 \leq 1 - (\varepsilon/2)^2$

~~Pick $\delta \in (0,$~~ $\therefore \left\|\frac{x+y}{2}\right\| \leq \sqrt{1-(\varepsilon/2)^2} \in [0,1)$

Pick δ s.t. $1-\delta \geq \sqrt{1-(\varepsilon/2)^2}$ $\overset{(\Xi)}{(\text{ex})}$ $0 < \delta \leq 1-\sqrt{1-(\varepsilon/2)^2} \in (0,1]$

$\therefore$ Done.

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4.) $\Phi : H \to H'$, $\Phi(x) = \langle \cdot , x \rangle$ is isometric, ~~antilinear~~, isomorphism (Riesz rep. theory)

Φ well defined: $\Phi(x)(\lambda_1 y_1 + \lambda_2 y_2) = \langle \lambda_1 y_1 + \lambda_2 y_2 , x \rangle = \lambda_1 \Phi(x)(y_1) + \lambda_2 \Phi(x)(y_2)$

$|\Phi(x)(y)| = |\langle y, x\rangle| \leq \|x\| \cdot \|y\|$

$\therefore |\Phi(x)| \leq \|x\|$

$\therefore \Phi(x) \in H'$

Inner product linear in 2nd argument $\Rightarrow$ $\Phi(x)$ is linear.

$\|\Phi(x)\| \geq \left\| \frac{\Phi(x)(x)}{\|x\|} \right\| = \|x\| \quad (x \neq 0)$

$\therefore \|\Phi(x)\| = \|x\|$

$\therefore$ Φ is injective; Φ (and Φ^{-1} if it exists) is cont.

Surjective: Let $\Lambda \in H'$; $\ker(\Lambda)$ is closed

$\therefore H \cong \ker(\Lambda) \oplus \ker(\Lambda)^{\perp}$

If $x, y \in \ker(\Lambda)^{\perp}$: $x, y \neq 0 \Rightarrow \Lambda(x), \Lambda(y) \neq 0$

$\Lambda\left(\frac{x}{\Lambda(x)} - \frac{y}{\Lambda(y)} \right) = 0 \Rightarrow \frac{x}{\Lambda(x)} = \frac{y}{\Lambda(y)} \Rightarrow x, y$ are parallel

$\therefore \dim(\ker(\Lambda)^{\perp}) = 1$

Let $\ker(\Lambda)^{\perp} = \langle w \rangle$

We want: $\|\lambda w\|^2 = \Lambda(\lambda w)$

$\therefore \lambda = \frac{\Lambda(w)}{\|w\|^2}$

Claim: $\Lambda = \Phi(\lambda w)$

$\forall x \in H : x = z + k \cdot (\lambda w)$, $k \in \mathbb{R}$, $z \in \ker(\Lambda)$,

$\Lambda(x) = k \Lambda(\lambda w) = k \|\lambda w\|^2$ Equal

$\Phi(\lambda w)(x) = \langle z + k(\lambda w), \lambda w \rangle = \underbrace{\langle z, \lambda w \rangle}_{=0} + k \|\lambda w\|^2$

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$\therefore$ Φ surjective $\Rightarrow$ Inverse exists.

$\therefore$ Done.

(b) If T is invertible ~~thus~~ $\because$ H is complete, T bijective, bounded

$\Rightarrow$ (Inverse mapping theorem) T^{-1} is ~~isom~~ bounded

$\therefore$ $\exists$ $c>0$ s.t $\| T^{-1}(h) \| \le c \|h\| \Rightarrow \frac{1}{c}\|h\| \le \| T T^{-1}(h)\|$

$\therefore$ T bounded below.

$\cancel{\langle T(x), y\rangle = \langle x, T^*(y)\rangle}$

$\cancel{\langle T^{-1}(x), y\rangle = \langle T(T^{-1}(x)), y\rangle} =$

$\cancel{\langle T^{-1}(x), y\rangle = \langle x, T^{*}(y)\rangle}$

T invertible $\Rightarrow$ $\tilde{T}: H^* \to H^*$ invertible.

$\tilde{T}(f) = f(T(\cdot))$, $\tilde{T}^{-1}(f) = f(T^{-1}(\cdot)) \Rightarrow \widetilde{T^{-1}} = \tilde{T}^{-1}$

$T^* = \Phi^{-1} \cdot \tilde{T} \cdot \Phi$ is a composition of invertible maps $\Rightarrow$ invertible

$\therefore$ T^* bounded below as well

If T, T^* ~~invertible~~: bounded below:

$\|T(x)\| \ge c\|x\| \Rightarrow T$ injective

Let $X = \mathrm{Im}(T)$: $H = \overline{X} \oplus X^\perp$

If $z \in X^\perp$: $\forall y \in H, \langle T(y), z\rangle = 0 \Rightarrow \langle y, T^*(z)\rangle = 0$

$\therefore$ $\langle T^*(z), T^*(z)\rangle = 0 \Rightarrow T^*(z) = 0 \Rightarrow z = 0$

$\therefore$ $\overline{\mathrm{Im}(T)}$ ~~dense~~ $= H$.

If $T(x_n) \in \mathrm{Im}(T)$ is cauchy:

$\|T(x_n) - T(x_m)\| \ge c\|x_n - x_m\| \Rightarrow (x_n)_{n\ge 1}$ is cauchy

$\therefore$ $x_n \to x \in H \Rightarrow T(x_n) \to T(x) \in \mathrm{Im}(T)$

$\therefore$ $\overline{\mathrm{Im}(T)} = \mathrm{Im}(T) \Rightarrow T$ surjective.

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$\therefore$ T bijective, bounded linear $\Rightarrow$ (Open map. theorem) T is invertible

(c) $\|T(x)\|^2 = \sum_{k\in\mathbb{Z}} |x_{k+1}|^2 = \|x\|^2 \Rightarrow T$ is unitary

If $T(x) = \lambda x$: $\quad x_{n+1} = \lambda \cdot x_n$

If $\lambda = 0$: $\underset{\sim}{x} = 0$ (reject)

If $\lambda \neq 0$: $x_0 = 0 \Rightarrow x_n = \lambda^n \cdot x_0 = 0 \Rightarrow$ reject

$x_0 \neq 0 \Rightarrow x_n \not\to 0$ as $n \to \infty$ or $x_n \not\to 0$ as $n \to -\infty$] reject

$\therefore \sigma_{pt}(T) = \phi$

Claim: $\sigma(T) \subseteq S^1 = \{z \in \mathbb{C} : |z| = 1\}$

$\|T\| = \|\underbrace{T^*}_{T^{-1}}\| = 1$

$T - \lambda I$ invertible iff $T \cdot (I - \lambda T^{-1})$ is invertible

& If $\|\lambda T^{-1}\| < 1$, invertible

$\therefore T - \lambda I$ not invertible $\Rightarrow |\lambda| \geq 1$.

But $T - \lambda I = -\lambda \cdot (I - \frac{1}{\lambda} T)$ is invertible if $\|\frac{1}{\lambda} T\| < 1$ $(|\lambda| > \|T\|)$

$\therefore |\lambda| \leq \|T\|$ for $T - \lambda I$ not invertible.

$\therefore |\lambda| = 1$.

$\therefore \sigma(T) \subseteq S^1$

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Fix $\lambda \in S^1$:

$$X^{(n)} = \cancel{\sum_{k=-n}} \sum_{k=0}^{n-1} \lambda^{-k} e_k , \qquad \| X^{(n)} \| = \sqrt{n}.$$

$$\| T-\lambda (X^{(n)}) \| = \| \sum_{k=0}^{n-1} \lambda^{-k} e_{k-1} - \lambda^{-(k+1)} e_k \|$$

$$= \| \lambda^{-n} e_n - e_{-1} \| \leq 2.$$

$\therefore$ $T-\lambda$ not bounded below $\Rightarrow$ $\lambda \in \sigma(T)$

$\therefore$ $S^1 \subseteq \sigma(T) \subseteq S^1 \Rightarrow S^1 = \sigma(T)$

But $\underbrace{\partial \sigma(T)}_{= \sigma(T)} \subseteq \sigma_{Ap}(T) \subseteq \sigma(T)$

$\therefore$ $\sigma_{Ap}(T) = S^1$.

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Linear Analysis 2020

1.) (a) $X^* = \{ f: X \to \mathbb{R} : f$ is linear, continuous $\}$

X, Y are isometric, isomorphic if $\exists\ \Phi: X \to Y$ s.t.

Φ is bijective, linear; ~~Φ, Φ^{-1} both~~ bounded and

$\|\cdot\|_X = \|\Phi(\cdot)\|_Y$ $\Big\}$ Φ^{-1} bounded follow from inverse map. theorem.

Define $\Phi: \lambda_\infty \to \lambda_1^*$, $\Phi(\underset{\sim}{x}) \cdot \underset{\sim}{y} = \sum_{k \geq 1} x_k y_k$

$\therefore \Phi(x)$ is clearly a linear map; $|\Phi(\underset{\sim}{x}) y| \leq \sum_{k \geq 1} |x_k| \cdot |y_k|$

$\leq \|x\|_\infty \cdot \|y\|_1$

$\therefore \|\Phi(x)\| \leq \|x\|_\infty \Rightarrow \Phi(x)$ is bounded $\Rightarrow \in \lambda_1^*$

$\|\Phi(x)\| \geq \underbrace{|\Phi(x) \cdot (e_n)|}_{|x_n|}$, $(e_n)_k = 1_{n,k}$ $(\forall n \in \mathbb{N})$

$(\|e_n\|_1 = 1)$

$\therefore \|\Phi(x)\| \geq \sup_n \{|x_n|\} = \|x\|$

$\therefore \|\Phi(x)\| = \|x\|$

$\therefore \Phi$ is an isometry $\Rightarrow$ Injective, bounded.

If Φ^{-1} exists, Φ^{-1} bounded as well.

Φ surjective: Let $\Lambda \in \lambda_1^*$, $(\underset{\sim}{x}) = (\Lambda(e_n))_{n \geq 1}$

$|\Lambda(e_n)| \leq \|\Lambda\| \cdot \|e_n\|_1 = \|\Lambda\| \Rightarrow \underset{\sim}{x} \in \lambda^\infty.$

$\forall y \in \lambda_1$: $\left|\Lambda(\underset{\sim}{y}) - \sum_{k=1}^{N} \Lambda(e_k)\, y_k\right| = \left|\Lambda\left(\sum_{k>N} y_k \cdot e_k\right)\right| \leq \|\Lambda\| \cdot \left\|\sum_{k>N} y_k \cdot e_k\right\|_1$

$= \|\Lambda\| \cdot \sum_{k>N} |y_k| \to 0$ as $N \to \infty$.

$\therefore \left|\Lambda(y) - \underbrace{\sum_{k \geq 1} \Lambda(e_k) \cdot y_k}_{\Phi(\underset{\sim}{x}) \cdot \underset{\sim}{y}}\right| = 0 \Rightarrow \Lambda = \Phi(\underset{\sim}{x})!$

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$\therefore\ \Phi$ is an isomorphism (isometric)

(b) (i) Let $g: [0,\infty) \to [0,\infty)$, $g(t) = t/_{1+t}$

g is continuous, monotone increasing, concave ; $g(0)=0$

$\therefore \forall x, y \in [0,\infty)$, $g(x) = g\left(\frac{x}{x+y}(x+y) + \frac{y}{x+y}\cdot 0\right) \geq$

$\frac{x}{x+y}\, g(x+y) + 0.$

$\therefore g(x) + g(y) \geq g(x+y)$

Metric : $d(x,y) \geq 0$, $d(x,y) = d(y,x)$ (from formula)

If $d(x,y) = 0$: $\forall \phi_m \in \lambda_p^*$, $|\phi_m(x-y)| = 0$

~~Pick~~ If $x_k \neq y_k$: $|e_k(x-y)| \neq 0$,

($e_k(\underline{x}) = x_k$)

Pick $\phi_N \in$ s.t. $\|\phi_N - e_k\| \cdot \|x-y\| < |e_k(x-y)|/_2$

$\therefore |\phi_N(x-y)| \geq |e_k(x-y)| - |\phi_N - e_k(x-y)|$

$\geq |e_k(x-y)| - \|\phi_N - e_k\| \cdot \|x-y\| > 0$

$\therefore$ Contradiction. $x = y$.

$\therefore d(x,y) = 0 \Rightarrow$

If $\overset{x=y}{\cancel{d(x,y)=}}0$: $d(x,y) = 0.$

Fix x, y, z: $d(x,y) + d(y,z) = \sum_{m\geq 1} 2^{-m}\left[g(|\phi_m(x-y)|) + g(|\phi_m(y-z)|)\right]$

$= \sum_{m\geq 1} 2^{-m} g(|\phi_m(x-y)| + |\phi_m(y-z)|)$ (From above)

$\geq \sum_{m\geq 1} 2^{-m} g(|\phi_m(x) - \phi_m(z)|)$ ($\triangle$ inequality)

$= d(x,z)$. $\therefore$ Valid metric.

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Note: $g \in [0,1] \Rightarrow d(x,y) \leq \sum_{m\geq 1} 2^{-m} < \infty$ $\therefore$ Well defined.

Since $\lambda_p^* \cong \lambda_q$: We will treat $\Phi \in \lambda_p^*$ as a sequence in λ_q.

If $\forall \Phi \in \lambda_p^*: \Phi(X^{(n)}) \to \Phi(X)$: $\forall m \in \mathbb{N}$, $\Phi_m(X^{(n)}) \to \Phi(x)$

If $\forall m \in \mathbb{N}: \Phi_m(X^{(n)}) \to \Phi(X)_m$: Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ s.t

$\sum_{n>N} 2^{-n} < \varepsilon/2$.

$\therefore \lim_{n\to\infty} \underbrace{\sum_{k=1}^{N} 2^{-k} g(|\Phi_k(X^{(n)} - X)|)}_{(*)} = 0$ as g is cont.

$\therefore$ Pick N_1 s.t. $n \geq N_1 \Rightarrow |(*)| < \varepsilon/2$.

$$\forall n \geq N_1 : |d(X^{(n)}, X)| \leq \sum_{k=1}^{N} 2^{-k} g(|\Phi_k(X^{(n)} - X)|) + \sum_{k>N} 2^{-k}(1) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\therefore d(X^{(n)}, X) \to 0$.

If $d(X^{(n)}, X) \to X$: Fix $\Phi \in \lambda_p^*$; Given $\varepsilon > 0$, pick Φ_{m_0} s.t. $\|\Phi_{m_0} - \Phi\| < \varepsilon/4$.

$g(t) = t/_{1+t} = y \Rightarrow t = y + yt \Rightarrow t = y/_{1-y}$

$\therefore g^{-1}: [0,1) \to [0,\infty)$ exists, ~~well defined~~ continuous

Since $d(X^{(n)}, X) \to 0$: $\forall m \in \mathbb{N}$, $g(|\Phi_m(X^{(n)}, X)|) \leq 2^m \cdot d(X^{(n)}, X) \to 0$ as $n \to \infty$.

g^{-1} cont. $\Rightarrow \{\Phi_m(X^{(n)}, X) \to 0$

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But $X^{(n)} \to X$ in $B \Rightarrow \|X^{(n)}\|, \|X\| \leq 1$

$\therefore \|\Phi.(X^{(n)} - X)\| \leq \|\Phi_m(X^{(n)} - X)\| + \underbrace{\|\Phi_{m_0} - \Phi\|}_{< \varepsilon/4} \cdot \underbrace{\|X^{(n)} - X\|}_{\leq 2}$ — (*)

Pick $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |\Phi_{m_0}(X^{(n)} - X)| < \varepsilon/2$

$\therefore n \geq N \Rightarrow \|\Phi(X^{(n)} - X)\| < \varepsilon \Rightarrow \Phi(X^{(n)}) \to \Phi(x)$

$\therefore$ Done.

Claim: (B, d) is compact.

Let $(X^{(n)})_{n\geq 1}$ be any sequence in B:

$\varphi_0(n) = n \quad (n \in \mathbb{N})$

~~Given $(X^{\varphi_r(n)})_{n\geq 1}$, $|\Phi_{r+1}(X^{\varphi(n)})| \leq \|\Phi_{r+1}\|$~~

~~$\therefore (\Phi_{r+1}(X^{\varphi_r(n)}))_{n\geq 1}$ is bounded $\Rightarrow \exists$ cauchy subsequence.~~

$l_p \cong l_q^*$, $l_p^* \cong l_q$: Let $X^{(n)}$ act on l_q.

Let $\varphi_0(n) = n \quad (n \in \mathbb{N})$.

Given $(X^{\varphi_r(n)})_{n\geq 1}$: $|\Phi_{r+1}(X^{\varphi_r(n)})| \leq \|\Phi_{r+1}\| \Rightarrow$

$\exists$ cauchy subsequence $(\Phi_{r+1}(X^{\varphi_{r+1}(n)}))_{n\geq 1}$

Let $\varphi(n) = \varphi_n(n)$: $\forall m \in \mathbb{N}$, $(\Phi_m(X^{\varphi(n)}))_{n\geq 1}$ is cauchy.

Define : $X : \{\Phi_m : m \in \mathbb{N}\} \to \mathbb{R}$, $X(\Phi_m) = \lim_{n\to\infty} \Phi_m(X^{\varphi(n)})_{n\geq 1}$

$|X(\Phi_m)| \leq \limsup_n \|\Phi_m\| \cdot \|X^{\varphi(n)}\| = \|\Phi_m\| \Rightarrow$

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X is defined on dense set $\{\Phi_m : m \in \mathbb{N}\}$,

$\left.|X(\Phi_m)| \leq \|\Phi_m\|\right\}$ $m \in \mathbb{N}$.

$\therefore$ We can uniquely extend X to a function on λ_q

(linear): $|X(\lambda_1 \Phi + \tilde{\Phi}) \overset{- \lambda_1 X(\Phi) - X(\tilde{\Phi})}{\wedge} | \leq \cancel{\lambda X} \; \|X - X^{\varphi(N)}\| \cdot \{\|\lambda_1 \Phi\| + \|\tilde{\Phi}\| + \|\lambda_1 \Phi + \tilde{\Phi}\|\}$

$+ \|X^{\varphi(N)}\| \cdot \{|\lambda_1| \cdot \|\Phi - \Phi'\| + \|\tilde{\Phi} - \tilde{\Phi}'\| + \|\lambda_1 \Phi + \tilde{\Phi} - \Phi''\|\}$

~~Where~~ $\cancel{X}$ By picking N s.t. $\|X - X^{\varphi(N)}\|$ is small,

Φ', $\tilde{\Phi}'$, $\Phi'' \in \{\Phi_m : m \in \mathbb{N}\}$ s.t. $\|\Phi - \Phi'\|$, $\|\tilde{\Phi} - \tilde{\Phi}'\|$, $\|\lambda_1 \Phi + \tilde{\Phi} - \Phi''\|$ is small,

LHS can be arbitrarily small $\Rightarrow = 0 \Rightarrow X$ is linear.

$$|X(\Phi)| \leq \underbrace{\|X - X^{\varphi(n)}\|}_{\to 0} \cdot \|\Phi\| + \underbrace{\|X^{\varphi(n)}\|}_{\leq 1} \cdot \|\Phi\|$$

$\therefore \|X\| \leq 1. \Rightarrow X \in B.$

$\therefore X^{\varphi(n)} \to X \in B \Rightarrow$ $\cancel{B}$ B is sequentially compact

(ii) Let $X^{(n)} = e_n$, $X = 0$:

$\forall \Phi \in \lambda_q$: $|\Phi(X^{(n)})| = |\Phi_n| \to 0 = |\Phi(X)|$.

But $(e_n)_{n \geq 1}$ not norm convergent.

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2.) (a) Let (X, d) be a complete metric space: Let $U_n \subseteq X$ be open, dense.

Baire Category Theorem (BCT): $\bigcap_{n \geq 1} U_n \neq \phi$

Pick $x_1 \in U_1$; $\varepsilon_1 \in (0, 1)$ s.t. $D(x_1, 2\varepsilon_1) \subseteq U_1$

Suppose we have found $x_1, \dots x_n$; $\varepsilon_1, \dots \varepsilon_n$ satisfying:
$\varepsilon_k \in (0, 1/k)$ $(1 \leq k \leq n)$

$x_{r+1} \in D(x_{r+1}, 2\varepsilon_{r+1}) \subseteq D(x_r, \varepsilon_r) \cap U_{r+1}$
for $1 \leq r < n$. :

Pick $x_{n+1} \in U_{n+1} \cap D(x_n, \varepsilon_n)$ ($\neq \phi$ as U_{n+1} dense)
Pick $\varepsilon_{n+1} \in (0, 1/n+1)$ s.t. $D(x_{n+1}, 2\varepsilon_{n+1}) \subseteq U_{n+1} \cap D(x_n, \varepsilon_n)$
(ε_{n+1} exists as RHS is open)

$\therefore$ We can inductively find $(x_k, \varepsilon_k)_{k \geq 1}$

For $N \in \mathbb{N}$: $D(x_{N+1}, 2\varepsilon_{N+1}) \subseteq D(x_N, \varepsilon_N) \Rightarrow$
$\bar{D}(x_{N+1}, \varepsilon_{N+1}) \subseteq \bar{D}(x_N, \varepsilon_N)$

$\therefore \forall k \geq 1$: $\bar{D}(x_{N+k}, \varepsilon_{N+k}) \subseteq \bar{D}(x_N, \varepsilon_N) \in U_N$.

$\therefore$ $(x_{k+N})_{k \geq 0}$ is a sequence in $\bar{D}(x_N, \varepsilon_N) \subseteq U_N$

But $(x_k)_{k \geq 1}$ is cauchy as $n, m \geq N \Rightarrow$ ~~$d(x_n, x_m)$ $x_n, x_m \in$~~
$x_n, x_m \in \bar{D}(x_N, \varepsilon_N) \Rightarrow d(x_n, x_m) \leq 2/N \to 0$ as $N \to \infty$

& X complete: $x_k \to x \in X$

$\bar{D}(x_N, \varepsilon_N)$ closed $\Rightarrow$ $(x_{N+k})_{k \geq 0} \to x \in \bar{D}(x_N, \varepsilon_N) \subseteq U_N$.

$\therefore x \in \bigcap_{N \geq 1} U_N \Rightarrow \bigcap_{N \geq 1} U_N \neq \phi$ &

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We know: If $1 \leq q < p$, $\cancel{\|x\|_q \leq \|x\|_p}$ $\lambda_q \subseteq \lambda_p$

$\therefore \bigcup_{1 \leq q < p} \lambda_q = \bigcup_{\substack{1 \leq q < p, \\ q \text{ rational}}} \lambda_q$ -(1)

λ_q complete $\Rightarrow$ $\lambda_q \leq \lambda_p$ is closed subspace.

If $\lambda_q^{\circ} \neq \phi$: $\exists\, D(\underset{\sim}{x}, \varepsilon) \subseteq \lambda_q \Rightarrow D(0, \varepsilon) \subseteq \lambda_q$ & (λ_q vector subspace)

$\Rightarrow \lambda_p = \lambda_q$ (reject)

$\therefore$ (1) is a countable union of closed, nowhere dense sets;

λ_p is complete $\Rightarrow$ Not equal

Consider: $\sum_{n \geq 1} \frac{1}{n(\log(n))^2}$: This is convergent (cauchy condensation test)

$\sum_{n \geq 1} \frac{1}{n^{\alpha} (\log(n))^{2\alpha}}$ is divergent for $\alpha < 1$

$\underset{\sim}{x}$: $x_n = \left(\frac{1}{n(\log(n))^2}\right)^{1/p} \Rightarrow \underset{\sim}{x} \in \lambda^p$

If $q < p$: $x_n^q = \frac{1}{n^{q/p} (\log(n)^2)^{q/p}} \Rightarrow \sum_{n \geq 1} x_n^q = \infty$.

$\therefore \underset{\sim}{x}$ is an example.

(b) Let $E_{m,n} = \left\{ f \in C[0,1] : \exists\, x \in [0,1],\ \forall y,\ |x-y| \leq 1/m \Rightarrow |f(y) - f(x)| \leq n|y-x| \right\}$

$E_{m,n}$ closed: Let $(f_k)_{k \geq 1}$ be a sequence in $E_{m,n}$, $f_k \to f \in C[0,1]$

Pick x_k s.t. $|y - x_k| < 1/m \Rightarrow |f_k(y) - f_k(x_k)| \leq n |y - x_k|$

$[0,1]$ compact $\Rightarrow$ $\exists$ convergent subsequence $x_{n_k} \to x$.

(Pass x_{n_k} to subsequence): WLOG, $x_k \to x$.

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$|\cancel{f(y)}$ If $|x-y| \le 1/m$ =

$$+|f(x_k) - f_k(x_k)|$$

$$|f(x) - f(y)| \le \cancel{|f(x) - f_k(x)|} + |f_k(x) - f_k(x_k)| + |f_k(x_k) - f_k(y)|$$
$$+ |f_k(y) - f(y)|$$

$f(x_k) \to f(x)$ (f cont.)

$|f(x_k) - f_k(x_k)|, \ |f_k(y) - f(y)| \le \|f_k - f\| \to 0$

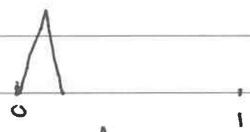
$$|f_k(y) - f_k(x_k)| \le M \cdot |y - x_k| \to M \cdot |y - x|.$$

$$\therefore |f(x) - f(y)| \le M \cdot |x - y|$$

$\therefore f \in E_{n,m} \Rightarrow E_{n,m}$ closed.

If $\exists D(f, \varepsilon) \subseteq E_{n,m}$:

Let $\tilde{\phi}_R = 1_{x \in [0, 1/R]} \operatorname{Min}\{2Rx, \ 1 - 2Rx\}$:



$$\phi_R(x) = \frac{\varepsilon}{2} \sum_{r=0}^{R-1} \tilde{\phi}_R(x - r/R) \quad :$$

$\therefore \forall R \in \mathbb{N}: \quad f + \phi_R \in D(f, \varepsilon) \subseteq E_{n,m}$

$\forall x \in [0,1]: \ |f + \phi_R(x) - f + \phi_R(y)| \ge |\phi_R(x) - \phi_R(y)| - |f(x) - f(y)|$

$\exists y \in \bar{D}(x, 1/n)$ s.t.

$$\ge \varepsilon R/2 \, |x - y| - M|x - y| > M|x - y| \quad \text{for}$$

R suff. large.

$\therefore$ Contradiction $\Rightarrow E_{n,m}$ nowhere dense.

But $C[0,1]$ complete ; If f is diff. at x,

$$f \in \bigcup_{n, m \in \mathbb{N}} E_{n,m}.$$

BCT $\Rightarrow C[0,1] \neq \bigcup_{n, m \in \mathbb{N}} E_{n,m} \Rightarrow \exists$ nowhere diff, cont. function

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(c) $\Phi : X \to X''$,

$e_{x+\lambda y}(\phi) = \phi(x+\lambda y) = e_x(\phi) + (\lambda \cdot \cancel{\phi(y)}\, e_y)(\phi)$ $\therefore \Phi$ linear

$\|\Phi(x)\| = \sup_{\|\phi\| \leq 1} \{|\phi(x)|\} \leq \sup_{\|\phi\| \leq 1} \{\|x\| \cdot \|\phi\|\}$

$= \|\overset{x}{\cancel{\phi}}\|$

$\therefore \|\Phi\| \leq 1 \Rightarrow$ Bounded

If $\exists\, x_n \in X$ s.t. $\|x_n\| = 1$, $\|x_n\|' \geq n$:

~~$\|(X, \|\cdot\|)'\| = (X, \|\cdot\|')'$ as a set~~

For $\phi \in (X, \|\cdot\|')'$: If $\phi \in (X^*, \|\cdot\|)'$ as well,

~~$\{|x_n \cdot (\phi)| : n \in \mathbb{N}\}$~~ is bounded as $|x_n \cdot \phi| \leq \|x_n\| \cdot \|\phi\|$
$\leq \|\phi\|$.

Uniform Bounded Principle $\Rightarrow \{\Phi(x_n) : n \in \mathbb{N}\}$ is bounded as a set in $(X, \|\cdot\|')''$ $\Rightarrow \{x_n : n \in \mathbb{N}\}$ bounded

$\therefore$ Contradiction

$\therefore (X, \|\cdot\|)'$, $(X, \|\cdot\|')'$ not equal

$\therefore$ If $\|\cdot\|$, $\|\cdot\|'$ not equivalent, we can WLOG: $\exists\, x_n$ s.t.
$\|x_n\| = 1$, $\|x_n\|' \geq n$ (Else, $\|x_n\|' = 1$, $\|x_n\| \geq n$)

$\therefore$ Not equivalent $\Rightarrow$ Dual space set is different

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3.) (a) Spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\}$

Point Spectrum: $\sigma_{pr}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not injective}\}$

Compact Operator: ~~T(D(0,1))~~ T maps bounded sets to totally bounded sets

Self adjoint: If $\langle T(.), . \rangle = \langle ., T^*(.) \rangle$: $T = T^*$

Finite rank: $\mathrm{Im}(T)$ is finite dim.

(b) T compact $\Rightarrow$ $T(\tilde{D}(0,1))$ is totally bounded.

$\therefore \exists\ x_1, \dots x_n \in \tilde{D}(0,1)$ s.t. $\{T(x_k) : 1 \le k \le n\}$ forms a $\delta/2$ net of $T(\tilde{D}(0,1))$

Let $E = \langle x_1, \dots x_n \rangle$ (Finite dim): $\forall r \in \mathbb{N}$, $\exists\ k(r) \in \{1, \dots n\}$ s.t. $\| T(x(k(r)) - T(e_r) \| < \delta/2$.

$\therefore \forall r \in \mathbb{N}$: $\| T(e_r) - P_E(T.(e_n)) \| \le \| T(e_r) - y \|$

$(\forall y \in E)$

$\therefore \| T(e_r) - P_E(T(e_r)) \| \le \| T(e_r) - T(x(k(r))) \| < \delta/2 < \delta$.

$\therefore$ Done

Let this subspace be $E(\delta)$

Let $T_n = P_{E(1/n)} \cdot T$: T_n is finite rank ~~com~~

~~Since $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis: $\forall x \in H$, x is uniquely written as $\sum_{k \ge 1} x_k \cdot \underset{\sim}{e_k}$~~

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$\therefore \| T(x) - T_n(x) \|_2 \leq \cancel{\sum_{r=1}} \| (T - T_n) \sum_{r=1}^{R} x_r \cdot e_r \|_2$

$+ \| T - T_n \| \cdot \| \sum_{r>R} x_r e_r \|$

Given $\varepsilon > 0$, pick R s.t. $2\|T\| \cdot \sqrt{\sum_{r>R} |x_r|^2} < \varepsilon/2$.

$(\forall n \in \mathbb{N})\ T_n = P_{E(1/n)} \cdot T \Rightarrow \|T_n\| \leq \|T\| \Rightarrow \|T - T_n\| \cdot \| \sum_{r>R} x_r \cdot e_r \| < \varepsilon/2$.

Since $\lim_{n\to\infty} \| T - T_n (\sum_{r=1}^{R} x_r \cdot e_r) \|_2 \leq \sum_{r=1}^{R} x_r \lim_{n\to\infty} \| (T - T_n) e_r \|$

$\leq \sum_{r=1}^{R} x_r \lim_{n\to\infty} 1/n = 0$:

$\therefore T_n(x) \to T(x)$

From the previous argument: If $\|x\| \leq 1$,

$\| T(x) - P_{E(1/n)} T(x) \| < 1/n$

$\Rightarrow \| T - T_n \| \leq 1/n \Rightarrow T_n \to T$ in norm

(e) $H = \ker(S) \oplus \ker(S)^\perp$

$\forall z \in \ker(S)$: $S - \lambda I (z/{-\lambda}) = 0 + z = z$.

$\therefore \ker(S) \subseteq \operatorname{Im}(S - \lambda I)$

Restrict to $\ker(S)^\perp$: If $S(S-\lambda I)\omega = 0$:

$\omega \in \ker(S)^\perp$, $\omega \neq 0 \Rightarrow S(\omega) \neq 0$; $S - \lambda I$ injective $\Rightarrow S(S-\lambda I)\omega \neq 0$

$\therefore$ Injective

$\operatorname{Im}(H) \cong H / \ker(S) \cong \ker(S)^\perp \Rightarrow$ Finite dim.

$\therefore r(S - \lambda I) = \dim(\ker(S)^\perp)$

If $z \in \operatorname{Im}(S - \lambda I)$, $S(z) = 0 \Rightarrow z$ ~~[illegible]~~ $= 0$.

$S \cdot (S - \lambda I)(\omega) = 0 \Rightarrow \omega = 0 \Rightarrow$ 6 $z = (S - \lambda I)\omega = 0$

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~~$\therefore \operatorname{Im}(S-\lambda I) \cap \operatorname{Ker}(S) = \{0\}$~~

~~$\therefore H = \operatorname{Ker}(S) \oplus \operatorname{Im}(S-\lambda I) = H$~~

$\therefore \operatorname{Im}(S-\lambda I|_{\operatorname{Ker}(S)^\perp}) \cap \operatorname{Ker}(S) = \{0\}$

If $\{z_n : 1 \le n \le N\}$ be basis of $\operatorname{Im}(S-\lambda I|_{\operatorname{Ker}(S)^\perp})$

$z_n = w_n + x_n$, $w_n \in \operatorname{Ker}(S)$, $x_n \in \operatorname{Ker}(S)^\perp$

Since $\{\underbrace{S(x_n)}_{S(z_n)=} : n \in \mathbb{N}\}$ i span $\operatorname{Im}(S\cdot(S-\lambda I))$ (same dim as $r(H)$),

$\{x_n : n \in \mathbb{N}\}$ must be lin. indep., spans $\operatorname{Ker}(S)^\perp$

$\therefore \operatorname{Ker}(S)^\perp \le \operatorname{Im}(S-\lambda I)$

$\therefore S-\lambda I$ surjective.

(d) ~~Fix $x \in H$. Given $\varepsilon > 0$, pick n s.t. $\|T - T_n\| \cdot \|x\| < \varepsilon$ (T_n finite rank, $T_n \to T$).~~

~~Given $\lambda \ne 0$: T has finite dim Eigenspace of~~

Let $T_n \to T$, T_n finite rank:

Pick $\lambda_n \in D(\lambda, 1/n) \setminus \{0\}$: $S_n = T_n - \lambda_n I$ ~~$\to T - \lambda I$~~,

S_n is injective ($\exists$ countable eigenvalues of S_n only $\Rightarrow \lambda_n$ exists)

If $\exists x_n$, $\|x_n\| = 1$, $S_n(x_n) \to 0$

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Claim: $T-\lambda I$ bounded below

Else: $\exists\, X_n$, $\|X_n\|=1$, $(T-\lambda I)X_n \to 0$

T compact $\Rightarrow \exists\, X_{n_k}$ s.t. $T(X_{n_k}) \to y \Rightarrow \lambda X_{n_k} \to y.$

$\therefore X_{n_k} \to y/\lambda \Rightarrow T(X_{n_k}) \to \frac{1}{\lambda} T(y) \Rightarrow T(y) = \lambda y$

$\|y\| = 1 \Rightarrow T-\lambda I$ not injective. $\therefore$ Contradiction

Claim: $\mathrm{Im}(T-\lambda I)$ closed.

If $y_n = (T-\lambda I)(x_n)$, $(y_n)_{n\geq 1}$ is convergel in H:

$(y_n)_{n\geq1}$ is cauchy

$\| X_n - X_m \| \leq \frac{1}{c} \cdot \| T(x_n - x_m) \| \Rightarrow (X_n)_{n\geq1}$ is cauchy

$\therefore X_n \to X \Rightarrow \cancel{T}(\ T-\lambda I\,(X_n) \to T-\lambda I\,(X) = y \in \mathrm{Im}(T-\lambda I)$

$\therefore \mathrm{Im}(T-\lambda I)$ closed.

If $\mathrm{Im}(T-\lambda I)$ dense: $\mathrm{Im}(T-\lambda I) = \overline{\mathrm{Im}(T-\lambda I)} = H \Rightarrow T-\lambda I$ surjective.

Claim: $\mathrm{Im}(T-\lambda I)$ dense.

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$\forall \varepsilon > 0$:

4.) (a) Equicontinuous: $\forall x \in K, \exists$ open $U \ni x$ s.t $\forall y \in U, \forall f \in S,$
$|f(x) - f(y)| < \varepsilon$

(AAT) Arzela - Ascoli Theorem: Let K be compact; $\mathcal{F} \subseteq C(K)$ is totally bounded iff $\mathcal{F}$ is bounded, equicontinuous

($\Rightarrow$): $\mathcal{F}$ totally bounded $\Rightarrow$ Bounded

Fix $x \in K$, $\varepsilon > 0$: Let $\{f_1, \dots f_n\}$ be a $\varepsilon/3$ cover of $\mathcal{F}$. $U = \bigcap_{r=1}^{n} f_r^{-1}(D(f_r(x), \varepsilon/3))$ is open, contains x.

$\forall y \in U, \forall f \in \mathcal{F}$: Pick f_k s.t $\|f_k - f\| < \varepsilon/3$

$|f(x) - f(y)| \leq 2\|f - f_k\| + \|f_k(x) - f_k(y)\| < \varepsilon$

$\therefore$ Equicontinuous

($\Leftarrow$) Given $\varepsilon > 0$: Pick U_x as described in definition of equicontinuity ($\varepsilon \to \varepsilon/3$)

$K = \bigcup_x U_x \Rightarrow \exists$ finite subcover: $K = \bigcup_{k=1}^{m} U_{x_k}$

Let $A = \{x_1, \dots x_m\}$: $\{\Lambda = \{f|_\Lambda : f \in \mathcal{F}\}$ is isomorphic to subset of $\mathbb{C}^m$

$\mathcal{F}$ bounded $\Rightarrow$ Λ bounded $\Rightarrow$ Λ totally bounded.

Let $\{f_1|_\Lambda, \dots f_n|_\Lambda\}$ be ~~representatives~~ $\varepsilon/3$ net

$\forall f \in \mathcal{F}$: Pick f_i s.t $\|f - f_i|_\Lambda\| < \varepsilon/3$;

$\forall x \in K$: Pick x_j s.t $x \in U_{x_j}$:

$|f(x) - f_i(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)|$
$< \varepsilon$.

$\therefore$

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$\therefore \{ f_i : 1 \le i \le m \}$ is an ε-net of $\mathcal{F} \Rightarrow \mathcal{F}$ totally bounded.

Suppose otherwise:

(b) Let $C(K) = \bigcup_{n \ge 1} E_n$, E_n equicontinuous

$E_n = \bigcup_m E_{n,m}$, $E_{n,m} = \{ f \in E_n : \|f\|_\infty \le m \}$

$C(K)$ complete $\Rightarrow \exists\ n, m$ s.t. $\overline{E_{n,m}}$ has non-ϕ interior. (Baire-category)

Let $D(f, \varepsilon) \subseteq \Lambda = \overline{E_{n,m}}$ ~~closed in compact~~

$E_{n,m}$ bounded, equicontinuous $\Rightarrow$ (AAT) pre-compact $\Rightarrow \overline{E_{n,m}}$ is compact.

~~Let $\{ \delta_{x,y} \}$ ... $\in \{C(K)\}$ $\{ \delta_x \ge \varepsilon/2$ $\delta_y = a$~~

~~Urysohn~~: Let $\{ x_n : n \in \mathbb{N} \}$ be distinct elements

$\{ x_{n+1} \}$, $\{ x_1, \dots x_n \}$ are disjoint closed sets

Urysohn: $\exists\ \delta_n$: $\{ x_1, \dots x_n \} \to 0$; $\{ x_{n+1} \} \to \varepsilon/2$; $\| \delta_n \| = \varepsilon/2$.

$\therefore f + \delta_n \in \Lambda$

Consider: $\overline{E_{n,m}} = \bigcup_{g \in \overline{E_{n,m}}} D(g, \varepsilon/8)$

$\exists$ a finite subcover : $D(g_1, \varepsilon/8), \dots D(g_n ; \varepsilon/8)$

$\therefore \exists$ ~~f_{r_1}, f_r~~ $f + \delta_{r_1}, f + \delta_{r_2} \in D(g_*, \varepsilon/8)$

$\Rightarrow \| \delta_{r_1} - \delta_{r_2} \| < \varepsilon/4$ (reject)

$\therefore |K| < \infty$

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(c) (i) ~~A~~ Well defined: To define $X_k(t)$, we only need values of: ~~$X_k(0)$ ... $X_k(\lfloor kt \rfloor / k)$, $\lfloor kt \rfloor$~~

$X_k(0) = X_0$, $X_k(1/k)$, .. $X_k(r/k)$, $r/k < t \le r+1/k$.

Since $X_k(0)$ is well defined (Base); ~~X_k~~ X_k well defined on $[0, j/k] \Rightarrow$ well defined on $[0, j+1/k]$

$\therefore$ Induction $\Rightarrow$ X_k well defined.

(*) $|X_k(t_1) - X_k(t_2)| \le |t_1 - t_2| \, \|F\|_\infty \Rightarrow$ Lipschitz $\Rightarrow$ Cont.

(ii) X_k is uniformly Lipschitz $\Rightarrow \{X_k : k \in \mathbb{N}\} = \mathcal{F}$ is equicont.

$|X_k(t)| \le |X_0| + M \Rightarrow$ Uniformly Bounded.

(AAT): ~~$\exists$ X_{n_k} subsequence, X_{n_k} cauchy~~

~~For each coordinate: We can extr~~

~~$C([0,1])$ comp~~

$\Downarrow$ $\exists$ subsequence s.t. $((X_{n_k})_1)_{k\ge1}$ is cauchy $\Rightarrow$ ~~convergent~~ ~~to X~~

From the sequence, we can extract another cauchy sequence $((X_{n_k})_2)_{k\ge1}$

After finite extractions: $(X_{n_k})_{k\ge1}$, s.t. all coordinates are cauchy.

$\therefore X_{n_k} \to X \in C([0,1], \mathbb{R}^n)$

X_{n_k} cont. $\Rightarrow$ uniformly cont. on $[0,1]$

~~Pf~~ Given $\delta>0$: Pick N_1 s.t. $|t_1 - t_2| < N_1 \Rightarrow \|X(t_1) - X(t_2)\|_\infty < \delta/2$

Pick N_2 s.t. $\|X_{n_k} - X\| < \delta/2$ for $k \ge N_2$,

$n_{N_2} \ge N_1$

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Pick δ st $|z_1 - z_2| < \delta \Rightarrow |F(z_1) - F(z_2)| < \varepsilon/2$.

$\therefore$ For $k \geq N_2$: $\left| \int_0^t F(y_k(s))\,ds - \int_0^t F(x(s))\,ds \right|$

$\leq \int_0^1 |F(y_k(s)) - F(x(\alpha(s)))|\,ds + \int_0^1 |F(x(\alpha(s))) - F(x(s))|\,ds$

$\alpha(s) = j/k$ for $s \in (j/k, j+1/k]$

$\therefore |y_k(s) - x(\alpha(s))| < \delta$, $|x(\alpha(s)) - x(s)| < \delta$.

$\therefore$ Both integrals $< \|F\|_\infty . \varepsilon$.

$\therefore x_0 + \int_0^t F(y_k(s))\,ds \to x_0 + \int_0^t F(x(s))\,ds$ uniformly

$\therefore x(t) = x_0 + \int_0^t F(x(s))\,ds$.

$\therefore x$ is diff., solves $(*)$

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Linear Analysis 2021:

1.) (a) (i) $\Rightarrow$ (ii): Since $e_i \in H$, $\lim_{n\to\infty} (\langle x_n, e_i \rangle) = 0$ (by def. of (i))

(True $\forall\, i \in \mathbb{N}$)

Claim $\{ x_n : n \in \mathbb{N} \}$ is bounded

We can view x_n as an element of H'', $x_n(f) = f(x_n)$

$\therefore \forall\, f \in H'$: ~~fex~~ Riesz rep. theory $\Rightarrow$ $f = \langle \cdot\,, z_f \rangle$ for some $z \in H$

$\therefore \{ |f(x_n)| : n \in \mathbb{N} \} = \{ |\langle x_n, z_f \rangle| : n \in \mathbb{N} \}$

is bounded as $\lim_{n\to\infty} |\langle x_n, z_f\rangle| = 0$

Banach Steinhaus $\Rightarrow$ $\{ x_n : n \in \mathbb{N} \}$ is bounded.

(ii) $\Rightarrow$ (i): Let $b_k = \lim_{n\to\infty} \langle x_n, e_k \rangle$.

Since $\{ e_n : n \in \mathbb{N} \}$ is Hilbertian basis: $\forall\, x \in H$,

$x = \sum_{n \geq 1} \langle x, e_n \rangle \cdot e_n$, $\quad \|x\|^2 = \sum_{n\geq 1} |\langle x, e_n \rangle|^2$

~~Ex LNE IN~~ $\cancel{\|\sum_{k=1}^{N} b_k e_k\|}$ $\quad \sum_{k=1}^{N} |b_k|^2 = \lim_{n\to\infty} \left[\sum_{k=1}^{N} |\langle x_n, e_k \rangle|^2 \right]$

$\leq \lim_{n\to\infty} \left[\|x_n\|^2 \right] \leq \sup \left\{ \|x_n\|^2 : n \in \mathbb{N} \right\} = M$ (bounded)

$\therefore \left(\sum_{k=1}^{N} b_k e_k \right)_{N \geq 1}$ is cauchy (in H), $x_\infty = \sum_{k=1}^{\infty} b_k e_k \in H$

Claim: $x_n \rightharpoonup x_\infty$

Given $h \in H$, $\varepsilon > 0$: K' $\langle e_n : n \in \mathbb{N} \rangle$ is dense in H.

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$\therefore\ \exists\ \tilde{h} = \sum_{k=1}^{M} a_k e_k$ s.t. $\| \tilde{h} - h \|_2 < \varepsilon / 3 . \sup\{ \|X_n\| : n \in \mathbb{N} \}$

By construction: $\forall i \in \mathbb{N}$, $\lim_{n\to\infty} \langle X_n, e_i \rangle = b_i = \langle X_\infty, e_i \rangle$

$\therefore \lim_{n\to\infty} \langle X_n, \tilde{h} \rangle = \langle X_\infty, \tilde{h} \rangle$

$\$\ \exists\ N \in \mathbb{N}$ s.t $n \geq N \Rightarrow | \langle X_n, \tilde{h} \rangle - \langle X_\infty, \tilde{h} \rangle | < \varepsilon/3.$

$\therefore\ \$\ (n \geq N)$: $| \langle X_n, h \rangle - \langle X_\infty, h \rangle | \leq | \langle X_n, h - \tilde{h} \rangle | + | \langle X_\infty, h - \tilde{h} \rangle |$
$+ | \langle X_n - X_\infty, \tilde{h} \rangle |$
$< \varepsilon.$

$\therefore\ X_n \to X_\infty.$

(b) If $(X_n)_{n\geq 1}$ is bounded:

$\forall i \in \mathbb{N}$: $| \langle X_n, e_i \rangle | \leq \| X_n \| \Rightarrow \{ \langle X_n, e_i \rangle : n \geq 1 \}$ bounded in $\mathbb{R}$

$\Rightarrow \exists$ conv. subsequence.

If $(X^{(n)}_{m_k})_{k\geq 1}$ is a subsequence satisfying:

$\langle X^{(n)}_k, e_i \rangle$ convergent for $i \leq n$: We can extract a subsequence of $(X^{(n)}_k)_{k \geq 1}$ s.t. : $(X^{(n+1)}_k)_{k\geq 1}$ s.t $\langle X^{(n+1)}_k, e_{n+1} \rangle$ convergent.

Let $y_n = X^{(n)}_n$: $(y_n)_{n \geq 1}$ is a subsequence of $(X_n)_{n\geq 1} \Rightarrow$ Bounded.

$\forall i \in \mathbb{N}$: $(y_n)_{n \geq i}$ is a subsequence of $(X^{(i)}_k)_{k \geq 1} \Rightarrow$
$\langle y_n, e_i \rangle$ is convergent.

(a): $\exists\ X_\infty \in H$ s.t. $y_n \to X_\infty$.

(c) If (i) ⇒ (ii): If $X_n \to X_\infty$, $\big|\, \|X_n\| - \|X_\infty\| \,\big| \le \|X_n - X_\infty\| \to 0$ as $n \to \infty$. $\therefore$ Done.

(ii) ⇒ (iii): Fix $\varepsilon > 0$, pick $N \in \mathbb{N}$ s.t. $\sum_{k \ge N} |\langle X_\infty, e_k \rangle|^2 < \varepsilon/2$,

~~$M \in \mathbb{N}$ s.t. $n \ge M \Rightarrow (1 \le i \le N)$, $|\langle X_\infty - X_n, e_i \rangle| < \varepsilon/2N$.~~

~~(M exists as $X_n \to X_\infty$)~~

~~For $n \ge M$: $\sum_{k=1}^{\infty} |\langle X_n, e_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle X_\infty, e_k \rangle|^2$~~

~~$\sum_{k=1}^{N-1} |\langle X_n, e_k \rangle|^2 - \langle X_\infty, e_k \rangle$~~

~~$M \in \mathbb{N}$ s.t. $n \ge M \Rightarrow$ $\sum_{k=1}^{N-1} \big| |\langle X_\infty, e_k \rangle|^2 - |\langle X_n, e_k \rangle|^2 \big| < \varepsilon/2$~~

~~(M exists as $\langle X_n, e_i \rangle \to \langle X_\infty, e_i \rangle$)~~

~~Since $\|X_n\|^2 = \|X_\infty\|^2$: $\sum_{k \ge N} \big| |\langle X_\infty, e_k \rangle|^2 - |\langle X_n, e_k \rangle|^2 \big|$~~

Pick $M \in \mathbb{N}$ s.t. $n \ge M \Rightarrow \left| \sum_{k=1}^{N-1} |\langle X_n, e_k \rangle|^2 - \sum_{k=1}^{N-1} |\langle X_\infty, e_k \rangle|^2 \right| < \varepsilon/2$.

$\|X_n\|^2 = \|X_\infty\|^2 \Rightarrow \left| \sum_{k \ge N} |\langle X_n, e_k \rangle|^2 - \sum_{k \ge N} |\langle X_\infty, e_k \rangle|^2 \right| < \varepsilon/2$

$$\therefore \sum_{k \ge N} |\langle X_n, e_k \rangle|^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

For $1 \le j < M$: $\exists\, N_j$ s.t. $n \ge N_j \Rightarrow \sum_{k \ge n} |\langle X_j, e_k \rangle|^2 < \varepsilon$.

Pick $I(\varepsilon) = \operatorname{Max}\{N_1, \dots N_j, N\}$:

$$\therefore \forall j \ge 1 : \sum_{k \ge I(\varepsilon)} |\langle X_j, e_k \rangle|^2 < \varepsilon.$$

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(iii) $\Rightarrow$ (i)

$$\| X_n - X_\infty \|^2 = \sum_{k=1}^{\infty} |\langle X_n - X_\infty, e_k \rangle|^2$$

Given $\varepsilon > 0$, pick : $\sum_{i \geq I(\varepsilon)} |\langle X_n, e_i \rangle|^2 < \varepsilon \Rightarrow$

$$\forall N \in \mathbb{N}, \quad \sum_{i=I(\varepsilon)}^{N} |\langle X_n, e_i \rangle|^2 < \varepsilon \Rightarrow (\lim_{n \to \infty}) \sum_{i=I(\varepsilon)}^{N} |\langle X_\infty, e_i \rangle|^2 \leq \varepsilon$$

$$\therefore \sum_{i \geq I(\varepsilon)} |\langle X_\infty, e_i \rangle|^2 \leq \varepsilon.$$

Let $N = I(\varepsilon/3)$: $\exists M \in \mathbb{N}$ s.t. $n \geq M \Rightarrow$

$$|\langle X_n - X_\infty, e_i \rangle|^2 < \varepsilon/3N \quad \text{for } 1 \leq i < N$$

$$\therefore n \geq M: \| X_n - X_\infty \|^2 = \sum_{k=1}^{N-1} |\langle X_n - X_\infty, e_k \rangle|^2 + \sum_{k \geq N} |\langle X_n, e_k \rangle|^2 + \sum_{k \geq N} |\langle X_\infty, e_k \rangle|^2$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\therefore X_n \to X_\infty$.

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2.) (a) If $W \leq V$, $\nexists\ W^{\circ} \neq \phi$: $\exists\ D(x, \varepsilon) \subseteq W$

$\therefore \forall z \in D(0,\varepsilon)$: $x,\ z + x \in W \Rightarrow z \in W \Rightarrow D(0,\varepsilon) \subseteq W$.

$\therefore \forall v \in V \setminus \{0\}$: $\varepsilon V / 2\|V\| \in W \Rightarrow V \in W$.

$\therefore W = V \Rightarrow$ If W is proper, $W^{\circ} = \phi$

(b) If $\{x_n : n \in \mathbb{N}\}$ is an countable algebraic basis:

Let $A_n = \langle x_k : 1 \leq k \leq n \rangle$: A_n is a finite-dim vector space $\Rightarrow A_n$ complete $\Rightarrow A_n \subseteq V$ is closed.

$A_n \leq V$ is proper (V ∞-dim, A_n finite dim) $\Rightarrow A_n^{\circ} = \phi$.

But $\bigcup_{n\geq 1} A_n = V \Rightarrow$ Contradicts BCT. (1) $\Rightarrow$ Algebraic basis uncountable.

(1) Baire category theory (BCT): If (X, d) is a complete metric space, $A_n \subseteq X$ is closed, ~~nowhere den~~ empty interior:

$$X \neq \bigcup_{n\geq 1} A_n$$

Let P be polynomials: $\{x^k : k \geq 0\}$ forms an countable algebraic basis $\Rightarrow P$ cannot be complete wrt any norm

(b) Open mapping theorem: Let X, Y be banach space, $T \in \mathcal{B}(X, Y)$ be surjective. T is an open map.

If T is bijective : T is a ~~homeo~~ isomorphism

T is id-map

$T: (V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$; $\|\cdot\|_2 \leq C_1 \|\cdot\|_1 \Rightarrow \|T(x)\|_2 \leq C_1 \cdot \|x\|_1$

$\therefore T$ is bounded ; T is clearly bijective $\Rightarrow T^{-1}$ is bounded.

$\therefore \exists\ C_2 > 0$ s.t. $\forall x \in V$, $\|T^{-1}(x)\|_1 \leq C_2 \|x\|_2$.

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$\therefore \forall x \in V, \quad \|x\|_1 \leq C_2 \|x\|_2.$

Claim: ~~V complete~~ $(V, \|\cdot\|_i)$ complete is necessary.

Let $V = c_{00} \subseteq \mathbb{R}^{\mathbb{N}}$ (finite non-0 coordinates)

$(V, \|\cdot\|_p)$ is incomplete ~~Exercise~~ for $1 \leq p \leq \infty$.

~~Pick $1 \leq p < q \leq \infty$:~~

~~$id : (V, \|\cdot\|_q) \to (V, \|\cdot\|_p)$ is bounded~~

$id : (V, \|\cdot\|_1) \to (V, \|\cdot\|_\infty)$ is bounded as

$$\sum_{n \geq 1} |x_n| \geq \sup \{ |x_n| : n \in \mathbb{N} \}$$

But $id : (V, \|\cdot\|_\infty) \to (V, \|\cdot\|_1)$ is not bounded.

$$x_n = (\underbrace{1, \dots 1}_{n \text{ entries}}, 0, \dots)$$

$\therefore \|x_n\|_1 = n, \quad \|x_n\|_\infty = 1 \Rightarrow$ No such C_2

$\therefore$ Done.

(c) Define $\Phi : V^* \to \ell^1 \quad (V^*, \ell^1$ both complete)

$(\Phi(f))_n = f(x_n)$; $\sum_{n \geq 1} |f(x_n)| < \infty \Rightarrow$ Well defined,

map is clearly linear.

Closed graph theorem: X, Y banach spaces, $T : X \to Y$ is bounded iff $\Gamma_T = \{ (x, T(x)) : x \in X \} \subseteq X \times Y$ is closed

If $f_n \to f$: $\forall r \in \mathbb{N}$, $f_n(x_r) \to f(x_r)$

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~~Claim: $\|\Phi(f_n)\| \to \|\Phi(f)\| \to 0$~~

~~By viewing $X_n \in X''$:~~

If $f_n \to f$, $T(f_n) \to z$:

$\forall k \in \mathbb{N}$: $(T(f_n))_k \to z_k$.

$\therefore f_n(x_k) \to z_k \Rightarrow z_k = f(x_k)$

$\therefore z = T(f)$

$\therefore$ Graph is closed $\Rightarrow$ T is bounded.

$$\therefore \sup_{\|f\| \leq 1} \{ \| \Phi(f) \|_1 \} < \infty.$$

3.) (a) Let K be compact, $C_{\mathbb{R}}(K) = \{ f : K \to \mathbb{R} : f \text{ continuous} \}$

$\mathcal{F} \subseteq C_{\mathbb{R}}(K)$ is equicont. if: Given $x \in K$, $\varepsilon > 0$, $\exists\, U$ open, $x \in U$ s.t. $\forall y \in U$, $\forall f \in \mathcal{F}$: $|f(x) - f(y)| < \varepsilon$

Arzela-Ascoli: $\mathcal{F}$ is pre-compact iff $\mathcal{F}$ is bounded, equicont.

(b) Let $K_n = [-n, n]$: K_n is compact.

$\mathcal{F} = \{ f_n : n \in \mathbb{N} \}$, $\mathcal{F}^m = \{ f_n|_{K_m} : n \in \mathbb{N} \}$

Since ~~f_n~~ $\mathcal{F}$ is bounded, equicont.: $\forall m \in \mathbb{N}$, all subsets of $\mathcal{F}^m$ is bounded, equicont.

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Given sequence f : F^m is totally bounded

If $(f^{(k)}_n)_{n\geq1}$ is a subsequence, $(f^{(k)}_n)_{n\geq1}$ is cauchy on K_k, $\exists$ subsequence $(f^{(k+1)}_n)_{n\geq1}$ cauchy on K_{k+1}

Let $f_{\phi(k)} = f^{(k)}_k$: $(f_{\phi(k)})_{k\geq1}$ is a subsequence, $(f_{\phi(k)})_{k\geq i}$ is cauchy on K_i

$\therefore$ ~~$f_{\phi(k)}|_{K_i} \to$~~ Let $f(x) = \lim_{k\to\infty} f_{\phi(k)}(x)$

$(x \in K_n$ for some $n \Rightarrow (f^{(k)}_{\phi(k)}(x))_{k\geq1}$ cauchy.

Since continuity is local : $\forall x \in \mathbb{R}$, pick n s.t. $D(x,1) \subseteq K_n$

$f_{\phi(k)} \to f$ uniformly on K_n $f_{\phi(k)}$ cont. $\Rightarrow f$ cont. on K_n

$\therefore$ cont. at x . .

$\therefore f$ is cont.

$\forall$ closed, bounded $A \subseteq \mathbb{R}$: $\exists n \in \mathbb{N}$ s.t. $A \subseteq K_n$

$f_{\phi_k}|_{K_n} \to f|_{K_n}$ uniformly $\Rightarrow f_{\phi_k}|_A \to f|_A$ uniformly

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3.) (a) Let $C(X) = \{ f: X \to \mathbb{R} : f \text{ continuous} \}$

$\mathcal{F} \subseteq C(X)$ is equicontinuous if: $\forall x \in X$, $\forall \varepsilon > 0$: $\exists U \subseteq X$
• $x \in U$, U open and $\forall y \in U$, $\forall f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$

Arzela-Ascoli : Let K be compact, $\mathcal{F} \subseteq C(K)$. ($\|\cdot\|_\infty$ norm on $C(K)$)

$\mathcal{F}$ is totally bounded iff $\mathcal{F}$ is equicontinuous, bounded.

(b) Let $A_n = [-n, n]$:

If $(f_{\varphi_n(m)})_{m\geq 1}$ is a subsequence that is uniformly cauchy on A_n :

$\mathcal{F} = \{ f_{\varphi_n(m)}|_{A_{n+1}} : m \in \mathbb{N} \} \subseteq C(A_{n+1})$, A_{n+1} is compact.

$\sup \{ \| f_{\varphi_n(m)}|_{A_{n+1}} \|_\infty : m \in \mathbb{N} \} \leq \sup \{ \| f_m \| : m \in \mathbb{N} \} < \infty$.

$\therefore$ $\mathcal{F}$ is bounded;

$\forall x \in A_{n+1}$: Given $\varepsilon > 0$, $\exists U_x$ open s.t. $\forall n \in \mathbb{N}$, $\forall y \in U_x$ $|f_{\varphi_n}(x) - f_{\varphi_n}(y)| < \varepsilon$
$\therefore$ $\mathcal{F}$ is equicontinuous

(AAT):
Arzela-Ascoli : $\exists$ ~~unif~~ ~~cauchy~~ subsequence $(f_{\varphi_{n+1}(m)})_{m\geq 1}$ s.t.
$(f_{\varphi_{m+1}}|_{A_{n+1}})_{m\geq 1}$ is cauchy.

Let $f_{\varphi(m)} = f_{\varphi_m(m)}$: $(f_{\varphi(m)})_{m\geq 1}$ is a subsequence of $(f_m)_{m\geq 1}$ and $(f_{\varphi(m)})_{m\geq N}$ is a subsequence of $(f_{\varphi_N(m)})_{m\geq 1}$

$\therefore$ $(f_{\varphi(m)})_{m\geq 1}$ is cauchy of A_n $(\forall n \in \mathbb{N})$

$\forall x \in \mathbb{R}$: $|f_{\varphi(m)}(x) - f_{\varphi(n)}(x)| \leq \| f_{\varphi(m)}|_{A_N} - f_{\varphi(n)}|_{A_N} \|$
for $N \geq |x|$

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$\therefore$ $(f_{\varphi(m)}(x))_{m\geq 1}$ is cauchy. Let $f(x) = \lim_{n\to\infty} f_{\varphi(n)}(x)$

On A_N: $f_{\varphi(m)}|_{A_N} \to f|_{A_N}$ uniformly, $f_{\varphi(m)}|_{A_N}$ cont. $\Rightarrow$ $f|_{A_N}$ cont.

& (True $\forall N \in \mathbb{N}$): f is cont.

Given closed, bounded I: Pick N s.t. $I \subseteq A_N$.

$f_{\varphi(m)}|_{A_N} \to f|_{A_N}$ uniformly $\Rightarrow$ $f_{\varphi(m)}|_I \to f|_I$ uniformly

(c) If $\mathcal{K}$ is compact: $\mathcal{K}$ is totally bounded. Since K is compact, AAT $\Rightarrow$ $\mathcal{K}$ is bounded, equicontinuous

* Direct proof: Given $x \in K$, $\varepsilon > 0$, $\mathcal{K} = \bigcup_{f\in\mathcal{K}} D(f, \varepsilon/3)$

$\mathcal{K}$ compact $\Rightarrow$ $\exists$ finite subcover $\{f_1, \dots f_N\}$

Let $U = \bigcap_{k=1}^{N} f_k^{-1}(D(x, \varepsilon/3))$ (open in K):

$\forall f \in \mathcal{K}$: Pick f_i s.t. $\|f - f_i\| < \varepsilon/3$

$\forall y \in U$: $|f(y) - f(x)| \leq |f(y) - f_i(y)| + |f_i(x) - f(x)| + |f_i(y) - f_i(x)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

$\therefore$ Equicontinuous

(d) Fix $x \in K$: Let A be compact neighborhood of x, $x \subseteq W \subseteq A$, W is open

$A \cap U^c$ is closed, $\subseteq$ compact A $\Rightarrow$ compact.

$\forall y \in A \cap U^c$: X hausdorff $\Rightarrow$ $\exists$ $O_{x,y}$, $B_{x,y}$ disjoint s.t. $y \in O_{x,y}$, $x \in B_{x,y}$ ~~AA~~

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$A \cap U^c \subseteq \bigcup_{y \in A \cap U^c} O_{x,y} \Rightarrow \exists$ finite subcover $O_{x,y_1}, \ldots O_{x,y_n}$

$\therefore B_x = \bigcap_{i=1}^{n} B_{x,y_i} \cap W$ is open neighborhood of x, $\subseteq \cancel{K}\, W \subseteq K$,

$\subseteq (\bigcup_{i=1}^{n} O_{x,y_i})^c$

$\overline{B_x} \subseteq \overline{W} \subseteq \cancel{K} A$ (X hausdorff; $\cancel{K} A$ compact $\Rightarrow$ closed)

$\subseteq (\bigcup_{i=1}^{n} O_{x,y_i})^c$

$\therefore \overline{B_x} \subseteq \cancel{K} A \cap (\cancel{K} A \cap U^c)^c = \cancel{K} A \cap [A^c \cup U] = A \cap U \subseteq U$

Since $\cancel{K}$ K is compact : $K \subseteq \bigcup_{x \in K} B_x \Rightarrow \exists$ finite subcover $B_{x_1}, \ldots B_{x_m}$

$B = \bigcup_{i=1}^{m} B_{x_i}$: $K \subseteq B$, B open ; $\overline{B_{x_i}} \subseteq A \cap U \Rightarrow \bigcup_{i=1}^{m} \overline{B_{x_i}} \subseteq A \cap U$

Since $\bigcup_{i=1}^{m} \overline{B_{x_i}}$ closed, $\overline{B} \subseteq \bigcup_{i=1}^{m} \overline{B_{x_i}} \subseteq A \cap U$; $\overline{B} \subseteq A \Rightarrow$ $\overline{B}$ compact.

$\therefore K \subseteq B \subseteq \overline{B} \subseteq U$

$\exists f : \cancel{X} \overline{B} \to [0,1]$, $f|_K = 1$, $f|_{\overline{B} \setminus B} = 0$, f cont.

(Urysohn : $\overline{B}$ compact hausdorff $\Rightarrow$ $\overline{B}$ normal)

Extend $\tilde{f} : X \to [0,1]$; $x \notin \overline{B} \Rightarrow f(x) = 0$.

$\tilde{f}^{-1}[0, a)) = U \setminus \overline{B} \bigcup f^{-1}([0,a))$

$f^{-1}([0,a)) = \overline{B} \cap V$, V open $\Rightarrow$ ~~$V \ \overline{B} \cap V \cup \overline{B}^c \cap V$~~,

~~$\overline{B} \cap W$~~

$\therefore \tilde{f}^{-1}[0,a) = U \setminus \overline{B} \bigcup \overline{B} \cap V = U \cap (\overline{B}^c \cup \overline{B} \cap V) = U \cap (\overline{B}^c \cup V)$

$\therefore$ open

(~~b~~): $\tilde{f}^{-1}[b,1] = f^{-1}[b,1] = V_1 \cap \overline{B}$, V_1 open in X

$f^{-1}[b,1] \subseteq B \Rightarrow = B \cap V_1 \Rightarrow$ open.

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$\therefore$ f is continuous, f supported on $\bar{B} \subseteq V$ and $f = 1$ on K &

4) (a) Let $\Phi_i : H_i \to H_i^*$ be canonical map, $\Phi_i(x)(y) = \langle y, x \rangle$, $\tilde{T}$ be dual map, $\tilde{T}: H_2^* \to H_1^*$

We want T^* s.t. $\tilde{T} \cdot \Phi_2(x_2) = \Phi_1(T^*(x_2))$ $(\forall x_2 \in H_2)$

$\therefore$ Pick $T^* : \Phi_1^{-1} \cdot \tilde{T} \cdot \Phi_2$

T^* composition of bounded linear maps $\Rightarrow$ Bounded linear; T^* satisfies equation

If S satisfies : $\langle T(\cdot), \cdot \rangle = \langle \cdot, S(\cdot) \rangle$, $\Phi_1 \cdot S = \tilde{T} \cdot \Phi_2$.
$\therefore$ $S = T^* \Rightarrow T^*$ uniquely defined.

(b) Let $D = \{ a_n : n \in \mathbb{N} \}$ be a dense set of H.
We can use gram schmidt orthogonalisation to generate an orthogonal subset $\{ a_{n_k} : k \in \mathbb{N} \}$ with the same linear span as $D \Rightarrow$ & $\{ a_{n_k} : k \in \mathbb{N} \}$ is an orthogonal basis $\Rightarrow$ Frame, $(A = B = 1)$

Let $A = \langle e_n : n \in \mathbb{N} \rangle$: $H = \bar{A} \oplus A^{\perp}$

If $x \in A^{\perp}$: $\langle x, e_n \rangle = 0$ $(\forall n \in \mathbb{N})$

$\therefore$ $A \cdot \|x\|^2 \leq \sum_{n \geq 1} |\langle x, e_n \rangle|^2 = 0 \Rightarrow x = 0.$

$\therefore$ $A^{\perp} = \{0\} \Rightarrow \bar{A} = H \Rightarrow$ ~~A~~ A is dense in H.

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$$\cancel{\langle U(x), \underset{\sim}{y} \rangle = \left\langle \sum_{k=1}^{\infty} \langle x, e_k \rangle f_k , \sum_{k=1}^{\infty} y_k f_k \right\rangle} \quad (f_k \text{ standard } \text{~~basis~~ of } \ell^2)$$

$$\cancel{= \sum_{k=1}^{\infty} \overline{\langle x, e_k \rangle}\, y_k = \left\langle \sum_{k=1}^{\infty} y_k e_k , x \right\rangle}$$

$$\cancel{\sum_{k\geq 1} \langle e_k, x \rangle \cdot y_k}$$

$$\| U(x) \|^2 = \sum_{k=1} |\langle x, e_k \rangle|^2 \leq B \|x\|^2 \Rightarrow \text{Bounded}$$

Adjoint : Let $(f_k)_{k\geq 1}$ be standard basis of ℓ^2

$$\langle U(x), \underset{\sim}{y} \rangle = \sum_{k\geq 1} \langle x, e_k \rangle_H \overline{y_k} = \left\langle x, \sum_{k\geq 1} y_k e_k \right\rangle$$

$$\therefore U^*(\underset{\sim}{y}) = \sum_{k\geq 1} y_k e_k \quad \#$$

(d) H is a metric space. Compactness, sequential compactness equivalent.

Let $(\underset{\sim}{X}_n)_{n\geq 1}$ be any sequence in C_a

Let $\varphi_0(k) = k \quad (k \in \mathbb{N})$:

Given $(\underset{\sim}{X}_{\varphi_r(k)})_{k\geq 1}$ subsequence of $(\underset{\sim}{X}_n)_{n\geq 1}$:

$\cancel{(\underset{\sim}{X}_{\varphi_r(k), r+1})_{k\geq 1}}$ is a sequence in $\overline{D}(0, |\langle a, e_i \rangle|)$

$\therefore \exists$ ~~cauchy subsequence~~ $\cancel{(\underset{\sim}{X}_{\varphi_r(k+1), k+1})}$

$(\underset{\sim}{X}_{\varphi_r(k), i})_{k\geq 1}$ is cauchy for $1 \leq i \leq r$:

$(\underset{\sim}{X}_{\varphi_r(k), r+1})_{k\geq 1}$ is a bounded sequence $\Rightarrow \exists$ cauchy subsequence $(\underset{\sim}{X}_{\varphi_{r+1}(k)})_{k\geq 1}$, cauchy of $1, \ldots r+1$ coordinates.

Let $\varphi(k) = \varphi_k(k)$: $(X_{\varphi(k)})_{k\geq n}$ subsequence of $(X_{\varphi_n(k)})_{k\geq 1}$

$\therefore \forall i \in \mathbb{N} \quad (X_{\varphi(k), i})_{k\geq n}$ is cauchy $\Rightarrow$ convergent.

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$(x_\infty)_i := \lim_{n\to\infty} x_{\varphi(n),i} \in C_a.$

Given $\varepsilon > 0$: Pick $N \in \mathbb{N}$ s.t. $\sum_{k>N} |\langle a, e_k\rangle|^2 < (\varepsilon/3)^2$

$$\therefore \| x_\infty - x_{\varphi(m)} \|^2 \leq \sum_{k \geq N} |\langle x_\infty, e_k\rangle|^2 + |\langle x_{\varphi(m)}, e_k\rangle|^2 + \sum_{k=1}^{N} |\langle x_\infty - x_{\varphi(m)}, e_k\rangle|^2$$

$$\leq 2(\varepsilon/3)^2 + \sum_{k=1}^{N} |\langle x_\infty - x_{\varphi(m)}, e_k\rangle|^2$$

$$\to 2(\varepsilon/3)^2 \quad \text{as} \quad M \to \infty$$

Pick $M \in \mathbb{N}$. s.t. $m \geq M \Rightarrow \sum_{k=1}^{N} |\langle x_\infty - x_{\varphi(m)}, e_k\rangle|^2 < \varepsilon^2/9.$

$\therefore m \geq M \Rightarrow \| x_\infty - x_{\varphi(m)} \| < \sqrt{3}\, \varepsilon/\sqrt{3}... < \varepsilon.$

$\therefore x_{\varphi(m)} \to x_\infty \in C_a \Rightarrow \exists$ conv. subsequence $\Rightarrow C_a$ cauchy.

* Need help with:

2020 / Q 3

2022 / Q 4 (Unable to prove the hint)

2015 / Q 3 (~~My proof is~~ stuck since last year :c)

Q 2, part (iv) : I ~~said~~ wrote " sgn function" can be approximated with simple function (~~Monotone convergence~~ Dominated Convergence Theorem), which is approximated by a cont. function (DCT again)

Sadly, this is not prob & measure.

(Need help)

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Linear Analysis 2022

1.) (a) Let $(\tau(\underset{\sim}{x}))_n = x_{n+1}$: $\varphi(\underset{\sim}{x}) = \varphi(\tau(\underset{\sim}{x}))$

~~Since φ~~ Fix $N \in \mathbb{N}$: $t = \inf_{m \geq N} x_m$, $s = \sup_{m \geq N} x_m$. $\mathbb{1} = (1, \ldots\ldots)$

$\underset{\sim}{x} \in \ell^\infty \Rightarrow |s|, |t| < \infty$;

$\therefore \tau^N(\underset{\sim}{x}) - \cancel{6} \cdot t\mathbb{1} \geq 0 \Rightarrow \varphi(\tau^N(\underset{\sim}{x})) - t \geq 0$

$s\mathbb{1} - \tau^N(\underset{\sim}{x}) \geq 0 \Rightarrow \varphi(\tau^N(\underset{\sim}{x})) \leq s$

$\therefore \inf_{m \geq N} x_m \leq \varphi(\underset{\sim}{x}) \leq \sup_{m \geq N} x_m$

Take limit $N \to \infty$: $\liminf x_n \leq \varphi(\underset{\sim}{x}) \leq \limsup x_n$

(b) No. Suppose $\varphi(\underset{\sim}{x}) = \sum_{n \geq 1} x_n y_n$; ~~that~~ ~~then~~

~~Let $x_n = 1/2^n$ $\underset{\sim}{x} \in \ell^\infty$: $\varphi(\underset{\sim}{x}) = \sum_{n \geq 1} y_n/2^n$~~

~~But $\varphi(\tau^{(k)}(\underset{\sim}{x})) = \sum_{n \geq 1} y_n/2^{n+k}$~~

Let $(x^{(n)})_k = \mathbb{1}_{k \geq n}$; $\therefore \varphi \circ \tau^{(n)}(x^{(n)}) = 1$.

$\therefore \varphi(x^{(n)}) = 1$; $\left| \sum_{k \geq 1} x^{(n)}_k y_k \right| \leq \sum_{k \geq n} |y_k| \to 0$.

$\therefore$ Contradiction

$\therefore$ No such $\underset{\sim}{y} \in \ell^1$

(c) Let $(\underset{\sim}{x})_n = \mathbb{1}_{n \equiv 0 \pmod 2}$

$\therefore \underset{\sim}{x} + \tau(\underset{\sim}{x}) = \mathbb{1}$

$\therefore \varphi(\mathbb{1}) = \varphi(\underset{\sim}{x}) + \varphi(\tau(\underset{\sim}{x})) = 2\varphi(\underset{\sim}{x}) \Rightarrow \varphi(\underset{\sim}{x}) = 1/2$

$\liminf x_n = 0$, $\limsup x_n = 1 \Rightarrow$ Not convergent, $\varphi(\underset{\sim}{x})$ must be $1/2$

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(d) Claim: If $\underset{\sim}{x} \in \ell^\infty$, $\|x\|_\infty \leq \varepsilon \Rightarrow |\varphi(\underset{\sim}{x})| \leq \varepsilon$.

$$\left.\begin{aligned} &\varepsilon \cdot \mathbb{1} - \underset{\sim}{x} \geq 0 \\ &\underset{\sim}{x} + \varepsilon \cdot \mathbb{1} \geq 0 \end{aligned}\right] \quad \left.\begin{aligned} &\varphi(\varepsilon\cdot\mathbb{1} - \underset{\sim}{x}) \geq 0 \Rightarrow \varepsilon \geq \varphi(\underset{\sim}{x}) \\ &\varphi(\underset{\sim}{x} + \varepsilon\mathbb{1}) \geq 0 \Rightarrow \varphi(\underset{\sim}{x}) \geq -\varepsilon \end{aligned}\right] \text{ Done.}$$

$$\frac{1}{p}\sum_{k=0}^{p-1} \varphi(\cancel{\underset{\sim}{x}}\, \tau^{(k)}(\underset{\sim}{x})) = \varphi(\underset{\sim}{x}) \cancel{= y^{(p)}}$$

$$\cancel{y^{(p)}} \quad \text{But} \quad \left(\frac{1}{p}\sum_{k=0}^{p-1} \tau^{(k)}(\underset{\sim}{x})\right)_n = \frac{1}{p}\sum_{i=1}^{p} x_{n+i}$$

$\therefore$ Given $\varepsilon > 0$, pick p s.t. $\left\| \frac{1}{p}\sum_{i=1}^{p} x_{n+i} - y \right\|_{n \in \mathbb{N}} < \varepsilon/2$.

$$\therefore \cancel{\left\| \frac{1}{p}\sum_{k=0}^{p-1} \varphi^{(k)}(\underset{\sim}{x}) - \varphi \right\| < \varepsilon/2 \Rightarrow \left|\varphi\left(\frac{1}{p}\sum_{k=0}^{p-1} \varphi^{(k)}(\underset{\sim}{x})\right)\right|} \leq \varepsilon/2 < \varepsilon$$

$$\therefore \cancel{\varphi\left(\frac{1}{p}\sum_{i=1}^{p}\tau^{i}\right.}$$

$$|\varphi(\underset{\sim}{x}) - y| = \left|\frac{1}{p}\sum_{k=0}^{p-1}\varphi(\tau^{(k)}(\underset{\sim}{x})) - y\right|$$

$$= \Bigg|\varphi\Bigg(\underbrace{\frac{1}{p}\sum_{k=0}^{p-1}\tau^{(k)}(\underset{\sim}{x}) - y}_{\text{Each coordinate has magnitude} \leq \varepsilon/2}\Bigg)\Bigg| \leq \varepsilon/2 < \varepsilon.$$

ε arbitrary $\Rightarrow \varphi(\underset{\sim}{x}) = y$ ☒

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2.) (a) If $T_i \in \mathcal{B}(V,V)$, $\lambda_i \in \mathbb{C}$: $\lambda_1 T_1 + \lambda_2 T_2$ is clearly linear, bounded.

$\therefore$ $\mathcal{B}(V,V)$ is a vector space.

Since norm is given, it suffices to show completeness:

Let $(T_n)_{n\geq 1}$ be a cauchy sequence:

$\forall x \in V$: $\| T_n(x) - T_m(x) \| \leq \|x\| . \| T_n - T_m \|$

$\therefore$ $(T_n(x))_{n\geq 1}$ is cauchy in V (complete)

$\therefore$ Define $T(x) = \lim_{n\to\infty} T_n(x)$

$T(\lambda_1 x_1 + \lambda_2 x_2) = \lim_{n\to\infty} \left[\lambda_1 T_n(x_1) + \lambda_2 T_n(x_2) \right] = \lambda_1 T(x_1) + \lambda_2 T(x_2)$

$\therefore$ T is linear.

$\forall x \in V, \|x\| = 1$: $\| T_n(x) - T(x) \| = \lim_{m\to\infty} \| T_n(x) - T_m(x) \|$ } continuity of norm.

$\therefore$ Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ st $n, m \geq N \Rightarrow \| T_n - T_m \| < \varepsilon/2$.

$\therefore$ $\| T_n(x) - T_m(x) \| < \varepsilon/2 \Rightarrow \| T_n - T(x) \| \leq \varepsilon/2$

$\therefore$ $n \geq N \Rightarrow \forall x \in V, \|x\| = 1$: $\| T_n(x) - T(x) \| \leq \varepsilon/2$.

$\therefore$ $\| T_n - T \| < \varepsilon \Rightarrow \|T\| \leq \varepsilon + \|T_n\| < \infty \Rightarrow T \in \mathcal{B}(V,V)$;

$T_n \to T \in \mathcal{B}(V,V)$!

$\therefore$ Complete.

(b) T is compact if $T(D(0,1)_V)$ is totally bounded.

& Λ is a Hilbertian basis if $\overline{\text{span}(\Lambda)} = V$; $\forall v_i \in \Lambda$,

$v_1 \neq v_2 \Rightarrow \langle v_1, v_2 \rangle = 0$

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If $\lambda_n \to 0$: Let $T_n \in \mathcal{B}(V, V)$

$T_n : e_k \to \lambda_k e_k$ for $1 \le k \le n$
$\to 0$ for $k > n$] T_n is finite rank $\Rightarrow$ compact

If $x \in V$: $\|x\|^2 = \sum_{k \ge 1} |\langle x, e_k \rangle|^2$, $x = \sum_{k \ge 1} \langle x, e_k \rangle \cdot e_k$

Let $\|x\| = 1$: $\|T_N - T(x)\|^2 = \|\sum_{k > N} \langle x, e_k \rangle \cdot \lambda_k \cdot e_k\|^2$

$= \sum_{k > N} |\lambda_k|^2 \cdot |\langle x, e_k \rangle|^2$

$\therefore$ Given $\varepsilon > 0$, pick $N_1 \in \mathbb{N}$ s.t. $n \ge N_1 \Rightarrow |\lambda_n| < \varepsilon$.

$\therefore$ If $N \ge N_1$: $\|T_N - T(x)\|^2 \le \varepsilon^2 \sum_{k > N} |\langle x, e_k \rangle|^2$

$\le \varepsilon^2$.

$\therefore \|T_N - T\| \le \varepsilon \Rightarrow T_N \to T$

Since compact operator set is closed in $\mathcal{B}(V, V)$: $T_n \to T \Rightarrow$
T is compact.

Suppose $\lambda_n \not\to 0$: $\exists\, \varepsilon > 0$, ~~s.t. $|\lambda_n| \ge \varepsilon$~~ subsequence
$(n_k)_{k \ge 1}$ s.t. $|\lambda_{n_k}| \ge \varepsilon$.

Consider $\Lambda = \{ e_{n_k} : k \in \mathbb{N} \}$ (Bounded)

$T(\Lambda) = \{ \lambda_{n_k} e_{n_k} : k \in \mathbb{N} \}$

$\|\lambda_{n_k} e_{n_k} - \lambda_{n_{k'}} e_{n_{k'}}\|^2 = |\lambda_{n_k}|^2 + |\lambda_{n_{k'}}|^2 + 0 \ge 2\varepsilon^2$
(True $\forall\, k, k'$, $k \ne k'$)

$\therefore$ No cauchy subsequence in $(\lambda_{n_k} e_{n_k})_{k \ge 1}$

$\therefore T(\Lambda)$ not totally bounded $\Rightarrow$ T not compact.

$\therefore$ T compact iff $\lambda_n \to 0$

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(c) $$\| T(x) \| = \Big\| \sum_{k \geq 1} T(e_k) \cdot \langle x_k, e_k \rangle \Big\| \leq \sum_{k\geq 1} |\langle x, e_k\rangle| \cdot \| T(e_k) \|$$

$$\leq \Big(\sum_{k \geq 1} \| T(e_k) \|^2 \Big)^{1/2} \underbrace{\Big(\sum_{k\geq 1} |\langle x, e_k \rangle|^2 \Big)^{1/2}}_{= \|x\|} \quad \text{(Cauchy Schwarz)}$$

$\therefore$ T is bounded.

Fix $T \in \mathcal{H}(V,V)$: Let T_n be defined as

$$T_n(e_k) = T(e_k) \quad (k \leq n)$$

$$\therefore T_n \Big(\sum_{k\geq 1} \langle x, e_k \rangle e_k \Big) = \sum_{k=1}^{n} \langle x, e_k \rangle T(e_k)$$

$\operatorname{Im}(T_n) \subseteq \langle T(e_1), \dots T(e_n) \rangle \Rightarrow$ (Finite dim) T_n compact.

$$\| (T - T_N)(x) \|^2 = \Big\| \sum_{k > N} \langle x_k, e_k \rangle T(e_k) \Big\|^2 \leq \Big(\sum_{k>N} |\langle x_k, e_k \rangle| \, \| T(e_k) \| \Big)^2$$

$$\leq \underbrace{\Big(\sum_{k>N} |\langle x_k, e_k \rangle|^2 \Big)}_{\leq \|x\|^2} \cdot \underbrace{\Big(\sum_{k>N} \| T(e_k) \|^2 \Big)}_{\to 0 \text{ as } N \to \infty} - (1) \ \cancel{\to 0 \text{ as } N \to \infty}$$

If $\|x\| \leq 1$: $(1) \leq \sum_{k>N} \| T(e_k) \|^2 \to 0$ as $N \to \infty$

$\therefore$ $\| T - T_N \| \to 0$ wrt $\mathcal{B}(V,V)$ norm.

Compact operator set closed $\Rightarrow$ T is compact.

~~Hilbert space: We know~~ $\cancel{\ell^2}$ ~~is a Hilbert space.~~

$$\cancel{\Phi : \mathcal{H}(V,V) \to \ell^2, \ (\Phi(T))_k = \| T(e_k) \|}$$

$$\cancel{\therefore \Phi(T) \in \ell^2 \text{ (by def.)}}$$

Define inner product: $\langle T_1, T_2 \rangle_{\mathcal{H}} = \sum_{n \geq 1} \langle T_1(e_n), T_2(e_n) \rangle$

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$\therefore \| T \|_*^2 = \langle T, T \rangle_{\mathcal{H}}$

(Valid I.P.) : $\overline{\langle T_1, T_2 \rangle_{\mathcal{H}}} = \overline{\sum_{n\geq 1} \langle T_1(e_n), T_2(e_n) \rangle}$

$= \sum_{n \geq 1} \langle T_2(e_n), T_1(e_n) \rangle = \langle T_2, T_1 \rangle_{\mathcal{H}}$

$\langle \lambda_1 T_1 + \lambda_2 T_2, T_3 \rangle_{\mathcal{H}} = \lambda_1 \langle T_1, T_3 \rangle_{\mathcal{H}} + \lambda_2 \langle T_2, T_3 \rangle_{\mathcal{H}}$

$\langle T, T \rangle_{\mathcal{H}} = \| T \|^2 \geq 0$; $= 0$ iff $T(e_n), \ldots = 0$ $(\forall n \in \mathbb{N})$

iff $T \equiv 0$.

$\therefore$ Valid Inner product.

Complete: If $(T_n)_{n\geq 1}$ is cauchy

$\forall m \in \mathbb{N}$: $(T_n(e_m))_{n \geq 1}$ is cauchy in V.

$\therefore$ Let $T(e_m) = \lim_{n\to\infty} T_n(e_m)$

$$\sum_{j=1}^{R} \| T_n(e_j) - T(e_j) \|^2 = \lim_{m\to\infty} \sum_{j=1}^{R} \| T_n(e_j) - T_m(e_j) \|^2$$

$$\leq \sup_{m \geq n} \left\{ \sum_{j\geq 1} \| T_n(e_j) - T_m(e_j) \|^2 \right\}$$

$\therefore \| T_n - T \|^2 \leq \sup_{m\geq n} \left\{ \| T_m - T_n \|^2 \right\} \to 0$ as $n \to \infty$.

$\therefore \| T \|_* < \infty$, $T_n \to T$ $\therefore$ Complete.

We have shown : $\| T \| \leq \| T \|_*$

Consider map : $\Phi : \mathcal{H} \to \mathcal{B}$ (Identity map)

$\mathcal{H}, \mathcal{B}$ are both complete ; If $T_n \xrightarrow{\|\cdot\|} T$, $T_n \xrightarrow{\|\cdot\|_*} S$,

$\| T_n - S \| \leq \| T_n - S \|_* \Rightarrow$ ~~The~~ $T_n \xrightarrow{\|\cdot\|} S \Rightarrow$ ~~T_n~~ $\Rightarrow S = T$

$\therefore$ Graph is closed $\Rightarrow \Phi$ bounded $\Rightarrow$ Norm equivalent

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3.) (a) (X, d) is normal: Let $A, B \subseteq X$ be closed, disjoint.

$\forall x \in A$: $A \subseteq B^c$ (open) $\Rightarrow \exists\, \varepsilon_x > 0$ s.t. $D(x, \varepsilon_x) \subseteq B^c$.

$\forall y \in B$: $B \subseteq A^c \Rightarrow \exists\, \delta_y > 0$ s.t. $D(y, \delta_y) \subseteq A^c$.

Let $U = \bigcup_{x \in A} D(x, \varepsilon_x/3)$, $V = \bigcup_{y \in B} D(y, \delta_y/3)$

$\therefore$ U, V open, $A \subseteq U$, $B \subseteq V$.

If $z \in U \cap V$: $\exists\, x \in A$, $y \in B$ s.t. $d(x, z) < \varepsilon_x/3$, $d(y, z) < \delta_y/3$.

$\therefore d(x, y) \leq d(x, z) + d(y, z) \leq \frac{2}{3} \operatorname{Max} \{ \varepsilon_x, \delta_y \}$

$\therefore x \in D(y, \delta_y)$ or $y \in D(x, \varepsilon_x) \Rightarrow$ Contradiction

$\therefore U \cap V = \phi \Rightarrow (X, d)$ is normal

(b) Urysohn: Let (X, τ) be normal, $A, B \subseteq X$ be closed, disjoint.

$\exists\, f: X \to [0, 1]$ continuous, $f|_A = 0$, $f|_B = 1$

Let $S \subseteq X$, X is normal;

Tietze: If $f: S \to [0, 1]$ is cont. wrt subspace topology, S is closed; $\exists\, \tilde{f}: X \to [0, 1]$ cont. s.t. $\tilde{f}|_S = f$.

(c) If (X, d) is compact: Given $f: X \to \mathbb{R}$, f continuous

Suppose $\exists\, x_n \in X$ s.t. $|f(x_n)| \geq n$; (X, d) metric space $\Rightarrow$ Compact, sequential compactness equivalent

$\therefore \exists\, x_{n_k} \to x \in X$ (subsequence)

f cont. $\Rightarrow f(x_{n_k}) \to f(x)$; But $n_k \geq k \Rightarrow |f(x_{n_k})| \geq n_k$.

$\therefore (f(x_{n_k}))_{k \geq 1}$ is not convergent $\Rightarrow$ Contradiction

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$\therefore$ No such $(X_n)_{n\geq 1}$ sequence $\Rightarrow$ f is bounded.

If (X, d) is not compact: $\exists$ sequence $(X_n)_{n\geq 1}$ ~~be~~ with no ~~con~~ convergent ~~[illegible]~~ subsequence.

Let $\Lambda = \{X_n : n \in \mathbb{N}\}$ (WLOG: X_n distinct): Fix $N \in \mathbb{N}$: ~~If $\forall \varepsilon > 0$, $D(X_N, \varepsilon) \cap \Lambda$ [illegible]~~

~~$\exists$ subsequence~~ $\forall N \in \mathbb{N}$: $\exists\, \varepsilon_N > 0$ s.t. there are finite X_n terms in $D(X_N, \varepsilon_N)$. Else, $\exists$ subsequence $\to X_N$.

If $D(X_N, \varepsilon_N) \cap \Lambda = \{X_{d_1}, \dots X_{d_r}, X_N\}$, $d_i \neq N$:

~~[illegible] $\Rightarrow \exists$~~ Pick $\tilde{\varepsilon}_N < \{\operatorname{Min}\{\varepsilon_N; d(X_{d_i}, X_N) : X_{d_i} \neq X_N\}\}$

$\therefore \exists\, U_{X_N} \ni X_N$, U_N open, $U_N \cap \Lambda = \{X_N\}$

$\therefore \Lambda$ is sequentially closed $\Rightarrow$ Closed ($\subset$ Metric space X)

Subspace topology of Λ is discrete. $\Rightarrow$ all functions continuous

Define : $f(X_n) = n$

Tietze $\Rightarrow \exists$ cont. extension in X $(\tilde{f})$

$\|\tilde{f}\|_\infty = \infty \Rightarrow$ Not bounded.

$\therefore X$ compact iff $(\forall f : X \to \mathbb{R},\ f \text{ cont} \Rightarrow f \text{ bounded})$

4.) (a) $\forall x \in [0,1]$, $\forall \varepsilon > 0$: $\exists \delta > 0$ s.t. $|y-x| < \delta$, $y \in [0,1]$

$\Rightarrow \forall n \in \mathbb{N}$, $|f_n(y) - f_n(x)| < \varepsilon$ (Equicontinuous)

Arzela – Ascoli : Let $\mathcal{F} \subseteq C(K)$, K is compact.

$\mathcal{F}$ is totally bounded iff $\mathcal{F}$ is bounded, equicontinuous

(b) Consider $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(z) = \int_0^z |\varphi(x)|\, dx$

$\therefore$ $\Phi \geq 0$, continuous increasing. Pick T_1 s.t. $\Phi(T_1) \leq 1$

Claim: $\{f_n : n \geq 1\}$ is bounded, $\|f_n\|_\infty \leq 1$

(Induct): $\|f_0\| = 0 \leq 1$

If $\|f_k\| \leq 1$: $|f_{k+1}(t)| \leq \int_0^t |\varphi$

Let $M = \|\varphi|_{[-1,1]}\|_\infty$; Pick T_1 s.t. $T_1 M \leq 1$.

If $\sup\{|f_n(x)| : x \in [0,T_1]\} \leq 1$:

$$|f_{n+1}(x)| \leq \int_0^x |\varphi(f_n(z))|\, dz \leq \int_0^x M \leq M \cdot T_1 \leq 1$$

$\therefore$ $\|f_{n+1}|_{[0,T_1]}\|_\infty \leq 1$

Since $\|f_0\| = 0 \leq 1$: Induction is complete $\Rightarrow$ $\{f_n : n \in \mathbb{N}\}$ is bounded

Equicont. : $|f_n(x) - f_n(y)| \leq \int_{[x,y]} |\varphi(f_n(z))|\, dz \leq M \cdot |x-y|$

(Works $\forall n \in \mathbb{N}$)

$\therefore$ Given $\varepsilon > 0$ pick $\delta > 0$ s.t. $\delta \cdot M < \varepsilon$:

$|y-x| < \delta \Rightarrow \forall n \in \mathbb{N}$, $|f_n(x) - f_n(y)| < \varepsilon$. $\therefore$ Done.

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Since $[0, T_1]$ is compact, Arzela-Ascoli $\Rightarrow \exists$ cauchy subsequence f_{n_k}

Since $C(C[0, T_1], \| \|_\infty)$ is complete, $f_{n_k} \to f$ uniformly on $[0, T_1]$, f cont.

~~Consider~~ $\cancel{|f_{n+1}(t) - f_n(t)| \leq \int_0^t |\varphi(f_{n+1}(z)) - \varphi(f_{n-1}(z))| \, dz}$

$\leq t \cdot \sup \{ |\varphi(a) - \varphi(b)| : a, b \in [-1, 1] \}$

~~Suppose~~ $\cancel{T_2 \in (0, T_1] : \| f_{n+1} - f_n|_{[0,T_2]} \| \to 0 \text{ as } n \to \infty.}$

$\cancel{\left| \int_0^t \varphi(f_{n_{k+1}}(z)) \, dz - \int_0^t \varphi(f(z)) \, dz \right| \leq}$

$\cancel{\left| \underbrace{\int_0^t \varphi(f_{n_{k-1}}(z))}_{f_{n_k}(t)} - \varphi(f_{n_k}(z)) \, dz \right| + \left| \int_0^t f_{n_k}(t) - \int_0^t \varphi \right.}$

~~Suppose~~ $\cancel{\exists T_2 \in [0, T_1] \text{ s.t. } \| f_{n+1} - f_n|_{[0,T_2]} \|_\infty \to 0 \text{ as } n \to \infty}$

$\cancel{\therefore \left| \int_0^t \varphi(f_{n_{k-1}}(z)) \, dz - f(t) \right| = | f_{n_{k+1}}(t) - f(t) |}$

$\cancel{\leq \| f_{n_{k+1}} - f_{n_{k-1}}|_{[0,T_2]} \|_\infty + \| f_{n_{k-1}}|_{[0,T_2]} - f(t)|_{[0,T_2]} \|_\infty}$

$\cancel{\to 0 \text{ as } k \to \infty}$

$\cancel{\therefore f_{n_k} \to f \text{ uniformly}}$

$\cancel{\int_0^t \varphi(f_{n_{k-1}}) \, ds \to \int}$

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If $\exists\, T_2 \in (0, T_1]$ s.t. $\| f_{n+1} - f_n |_{[0,T_2]} \|_\infty \to 0$ as $n \to \infty$

$f_{n_k} \to f$ wrt $\|\cdot\|_\infty$ on $[0, T_2]$

Claim: $\int_0^t \varphi(f_{n_k - 1}(z))\, dz \to \int_0^t \varphi(f(z))\, dz$ uniformly

$$\left| \int_0^t \varphi(f_{n_k-1}(z)) - \varphi(f(z))\, dz \right| \le$$

$$\le \underbrace{\int_0^t |\varphi(f_{n_k}(z)) - \varphi(f(z))|\, dz}_{(1)} + \underbrace{\int_0^t |\varphi(f_{n_k}(z)) - \varphi(f_{n_k-1}(z))|\, dz}_{(2)}$$

Since φ is uniformly cont. on T_2 compact domain:

Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|x_1 - x_2| < \delta \Rightarrow |\varphi(x_1) - \varphi(x_2)| < \varepsilon$
$(x_i \in [-1, 1])$

$\forall n$: $f_n(z) \in [-1, 1] \Rightarrow f(z) \in [-1, 1]$

Pick K s.t. $k \ge K \Rightarrow \| f_{n_k} - f |_{[0,T_2]} \| < \delta$.

$\therefore |\varphi(f_{n_k}(z)) - \varphi(f(z))| < \varepsilon \Rightarrow (1) \le \varepsilon \cdot T_2$ for $k \ge K$

$\therefore \lim_{k\to\infty} (1) = 0$.

$\lim_{k\to\infty} (2) \le \lim_{k\to\infty} T_2 \| f_{n_k} - f_{n_k-1} |_{[0,T_2]} \|_\infty = 0$

$\therefore \int_0^t \varphi(f_{n_k-1}(z))\, dz \to \int_0^t \varphi(f(z))\, dz$ uniformly for $t \in [0, T_2]$

$\therefore \forall t \in [0, T_2]$, $f(t) = \int_0^t \varphi(f(z))\, dz \Rightarrow f(0) = 0$; f is
$f' = \varphi(f)$ $f \in C^1[0, T_2]$

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Proof of hint: Let $R_n(t) = f_{n+1}(t) - f_n(t)$ // Not sure.

* (3): Mistake. Norms not equivalent

If $T: X \to Y$, ~~The surjective~~ X banach

$$\left.\begin{matrix} x_n \to x \\ T(x_n) \to y \end{matrix}\right\} \Rightarrow x = y \Bigg\} \Rightarrow T \text{ is bounded}$$

We got the direction wrong.

* P_n be projection to 1st n coordinate

$\|\frac{1}{\sqrt{n}} P_n\|_0 \le 1/\sqrt{n} \to 0.$

But $\|1/\sqrt{n}\, P_n\|_*^2 = \frac{1}{n}\sum_{k=1}^{n} \|P_n(e_k)\| = 1$

$\therefore$ Not equivalent.

* Does Y need to be complete, Extend $T: X \to \bar{Y}$,

$\|\cdot\|_{\bar{Y}}\big|_Y = \|\cdot\|_Y$

$\Gamma_T \subseteq X \times \bar{Y}$

* $\left.\begin{matrix} x_n \to x \\ T(x_n) \to y \end{matrix}\right\}$ $y \in \bar{Y} \not\Rightarrow y \in Y$; Fail

* Classic application ; $\nexists\, T: X \to Y$, $\|\cdot\|_X \geq c \|\cdot\|_Y$

If Y complete: $T(x_n) \to y = T(z)$ $\therefore \|x_n - z\| \le 1/c\, \|T(x_n) - T(z)\|$

BUT// Y not complete, $\exists\, y \notin Y \Rightarrow$ Arg. fails.