ex1 :

1. $$\|(H-H_0)Y\|^2 = \|((I-H_0)-(I-H))Y\|^2$$

$$= \|(I-H_0)Y\|^2 + \|(I-H)Y\|^2 - 2Y^T(I-H_0)^T(I-H)Y$$

$$\cancel{= \|(I-H_0)Y\|^2 + (Y^T(I-H)^T - 2Y^T(I-H_0)^T)(I-H)Y}$$

$$\cancel{= \|(I-H_0)Y\|^2 + Y^T(2H_0 - I - H)^T(I-H)Y}$$

$$\cancel{= \|(I-H_0)Y\|^2 + 2Y^T H_0 Y - 2Y^T H_0 H Y}$$

$$= \|(I-H_0)Y\|^2 + \|(I-H)Y\|^2 - 2Y^T(I - H_0 - H + \underbrace{H_0H}_{=H_0})Y$$

$$= \|(I-H_0)Y\|^2 + \|(I-H)Y\|^2 - 2Y^T(I-H)Y$$

$$= \|(I-H_0)Y\|^2 - \|(I-H)Y\|^2$$

$$= \|Y\|^2 + \|H_0Y\|^2 - 2Y^TH_0Y - \|Y\|^2 - \|HY\|^2 + 2Y^THY$$

$$= Y^THY - Y^TH_0Y = \|HY\|^2 - \|H_0Y\|^2 \quad \square$$

2. $$Y \mid X \sim N(X\beta, \sigma^2\Sigma)$$

$$f(y) = \frac{1}{(2\pi)^{n/2}|\sigma^2\Sigma|^{1/2}} \exp(-(Y-X\beta)^T\Sigma^{-1}\sigma^{-2}(Y-X\beta)/2)$$

$$\text{log-likelihood} = -\frac{n}{2}\log(\sigma^2) - (Y-X\beta)^T\Sigma^{-1}\sigma^{-2}(Y-X\beta)/2$$

$$\Rightarrow \hat{\beta}_\Sigma = \arg\min_\beta (Y-X\beta)^T\Sigma^{-1}(Y-X\beta)$$

objective function $L(\beta) = Y^T\Sigma^{-1}Y - 2\beta^TX^T\Sigma^{-1}Y + \beta^TX^T\Sigma^{-1}X\beta$

$$\frac{\partial L(\beta)}{\partial \beta} = -2X^T\Sigma^{-1}Y + 2X^T\Sigma^{-1}X\beta \quad \text{set to } 0$$

$$\Rightarrow \hat{\beta}_\Sigma = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y \quad \text{as long as } X^T\Sigma^{-1}X \text{ noncingular}$$

$$E[\hat{\beta}_\Sigma] = \beta + E[(X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}\varepsilon)]$$

$$= \beta + E[(X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}E[\varepsilon \mid X]]$$

$$= \beta \quad \text{so unbiased}$$

let $\tilde{\beta} = \tilde{X}Y$ unbiased $\Rightarrow \tilde{X}X = I$

$$\mathrm{Var}(\hat{\beta}_\Sigma \mid X) = \sigma^2(X^T\Sigma^{-1}X)^{-1}$$

$$\mathrm{Var}(\tilde{\beta} \mid X) = \sigma^2\tilde{X}\Sigma\tilde{X}^T$$

$$\Rightarrow \sigma^2\tilde{X}\Sigma\tilde{X}^T - \sigma^2(X^T\Sigma^{-1}X)^{-1}$$

$$= \sigma^2(\tilde{X}\Sigma\tilde{X}^T - \tilde{X}X(X^T\Sigma^{-1}X)^{-1}X^T\tilde{X}^T)$$

$$= \sigma^2\tilde{X}(\Sigma - X(X^T\Sigma^{-1}X)^{-1}X^T)\tilde{X}^T \quad \text{let } \Sigma^{-1/2}X = X^*$$

$$= \sigma^2\tilde{X}(\Sigma - \Sigma^{1/2}X^*(X^{*T}X^*)^{-1}X^{*T}\Sigma^{1/2})\tilde{X}^T$$

$$= \sigma^2\tilde{X}\Sigma^{1/2}\underbrace{(I - X^*(X^{*T}X^*)^{-1}X^{*T})}_{=M}\Sigma^{1/2}\tilde{X}^T$$

$$= \sigma^2\tilde{X}\Sigma^{1/2}M\underbrace{M\Sigma^{1/2}\tilde{X}^T}_{=D} \qquad M \text{ idempotent}$$

$$= \sigma^2 D^TD \geq 0 \qquad \square.$$

3. $\frac{1}{n} E(\|HY - Y^*\|^2) = \frac{1}{n} E(\| H(\mu + \varepsilon) - (\mu + \varepsilon^*)\|^2)$

$= \frac{1}{n} E(\| (H - I)\mu + H\varepsilon - \varepsilon^*\|^2)$

$= \frac{1}{n} E\Big( \|(H-I)\mu\|^2 + \|H\varepsilon\|^2 + \|\varepsilon^*\|^2 + \overset{=0}{2\varepsilon^T H^T (H-I)\mu}$

$\overset{0 \text{ expectation}}{- 2\varepsilon^{*T}(H-I)\mu} \quad \overset{0 \text{ expection}}{- 2\varepsilon^{*T} H\varepsilon} \Big)$

$= \frac{1}{n}\|(I-H)\mu\|^2 + \frac{1}{n} E\|H\varepsilon\|^2 + \frac{1}{n} E\|\varepsilon^*\|^2$

$= \frac{1}{n}\|(I-H)\mu\|^2 + \frac{1}{n}\sigma^2 \cdot tr(H) + \frac{1}{n}\cdot n\sigma^2$

$= \sigma^2 + \frac{1}{n}\|(I-H)\mu\|^2 + \frac{p}{n}\sigma^2.$ $\square$.

4. a). $\|X\|^2 = \|\Pi_W X + (I - \Pi_W)X\|^2 = \|\Pi_W X\|^2 + \|(I - \Pi_W)X\|^2$

$\geq \|\Pi_W X\|^2$

$\|\Pi_W X\|^2 = \|\Pi_V X + (\Pi_W - \Pi_V)X\|^2 = \|\Pi_V X\|^2 + \|(\Pi_W - \Pi_V)X\|^2$

$+ \overset{=0}{2X^T \Pi_V (\Pi_W - \Pi_V) X} \geq \|\Pi_V X\|^2$. $\square$.

b). $\hat{\beta}_j = (X_j^T M_{-j} X_j)^{-1} X_j^T M_{-j} Y = \dfrac{(X_j^{\perp})^T Y}{\|X_j^{\perp}\|^2}$

~~$Var(\hat{\beta}_j) =$~~ $= (X_j^T M_{-j} X_j)^{-1} X_j^T M_{-j}(X_j\beta_j + X_{-j}\beta_{-j} + \varepsilon)$

$= \beta_j + (X_j^{\perp})^T \varepsilon / \|X_j^{\perp}\|^2$

$\Rightarrow Var(\hat{\beta}_j) = \sigma^2 / \|X_j^{\perp}\|^2$

similarly $\mathrm{Var}(\hat{\beta}_{0,j}) = \sigma^2 / \|X_{0,j}^{\perp}\|^2$

$\|X_{0,j}^{\perp}\|^2 \geq \|X_j^{\perp}\|^2$ by a). $\square$.

5. $\text{log-likelihood}^{L(\sigma^2)} = -\frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(Y - X\beta)^T(Y - X\beta)$

$$\frac{\partial L(\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(Y - X\beta)^T(Y - X\beta)$$

plug $\hat{\beta}_{MLE}$ and set to 0

$\Rightarrow \hat{\sigma}^2_{MLE} = (Y - X\hat{\beta})^T(Y - X\hat{\beta})/n = \|(I-H)Y\|^2/n$

$E[\hat{\sigma}^2_{MLE}] = \frac{1}{n} E[\|(I-H)\varepsilon\|^2] = \frac{1}{n} E\left[E[\varepsilon^T(I-H)\varepsilon \mid X]\right]$

$= \frac{1}{n} E[\sigma^2 \cdot \mathrm{tr}(I-H)] = \frac{n-p}{n} \cdot \sigma^2$. $\square$.

6. $P(\beta \in C) = P\left(\bigcap_{j=1}^{p} \beta_j \in C_j(\alpha/p)\right)$

$= 1 - P\left(\bigcup_{j=1}^{p} \beta_j \notin C_j(\alpha/p)\right) \geq 1 - \sum_{j=1}^{p} P(\beta_j \notin C_j(\alpha/p))$

$\geq 1 - \sum_{j=1}^{p} \alpha/p = 1 - \alpha$. $\square$.

7. $(n-p)\hat{\sigma}^2/\sigma^2 = \|(I-H)Y\|^2/\sigma^2 = \|(I-H)\varepsilon\|^2/\sigma^2$

$= \|(I-H)\varepsilon/\sigma\|^2$ $\quad \varepsilon/\sigma \sim N(0, I)$

$\sim \chi^2_{n-p} \Rightarrow [(n-p)\hat{\sigma}^2/\chi^2_{\alpha/2, n-p},\ (n-p)\hat{\sigma}^2/\chi^2_{1-\alpha/2, n-p}]$

~~consider ||P||~~ consider $||Pz||^2 = z^T P z$, $z \sim N(0, I)$

$P$ being any projection matrix

diagonalize $P = C^T D C$, where eigenvalues are only 0 and 1

$\Rightarrow z^T P z = (Cz)^T D (Cz)$ $\quad E[Cz] = 0$, $Var[Cz] = CC^T = I$

$\Rightarrow Cz \sim N(0, I)$ $\quad \sim \chi^2_{tr(D)}$, where $tr(D) = tr(P) = rank(P)$

$\Rightarrow ||Pz||^2 \sim \chi^2_{tr(P)}$. $\square$

8. $\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$ $\Rightarrow$ $\frac{\hat{\beta} - \beta}{\sigma} \sim N(0, (X^T X)^{-1})$

$$\frac{\hat{\beta} - \beta}{\hat{\sigma}} = \frac{\hat{\beta} - \beta}{\sigma} \Big/ \frac{\hat{\sigma}}{\sigma} \qquad (n-p)\hat{\sigma}^2/\sigma^2 \sim \chi^2_{n-p}$$

$$\sim \frac{N(0, (X^T X)^{-1})}{\sqrt{\chi^2_{n-p}/(n-p)}}$$

$\Rightarrow$

$$\frac{(x^*)^T (\hat{\beta} - \beta)}{\sigma} \sim N(0, (x^*)^T (X^T X)^{-1} x^*)$$

$$\frac{(x^*)^T (\hat{\beta} - \beta)}{\sqrt{\hat{\sigma}^2 (x^*)^T (X^T X)^{-1} x^*}} \sim t_{n-p}$$

$$\left[ (x^*)^T \hat{\beta} \pm \sqrt{\hat{\sigma}^2 (x^*)^T (X^T X)^{-1} x^*} \cdot t_{n-p, \alpha/2} \right]$$

similarly, $(x^*)^T \hat{\beta} - Y^* = (x^*)^T (\hat{\beta} - \beta) - \varepsilon^* \sim N(0, \sigma^2 (x^*)^T (X^T X)^{-1} x^* + \sigma^2)$

$$\Rightarrow \frac{(x^*)^T\hat\beta - Y^*}{\sqrt{\hat\sigma^2((x^*)^T(X^TX)^{-1}x^*+1)}} \sim t_{n-p}$$

$$\left[ (x^*)^T\hat\beta \pm \sqrt{\hat\sigma^2((x^*)^T(X^TX)^{-1}x^*+1} \cdot t_{n-p,\alpha/2} \right]$$

$\hookrightarrow$ longer. $\square$

9. test statistic $= 2\log\left[ \sup_{\beta,\sigma^2} f(y;\beta,\sigma^2) \Big/ \sup_{\beta_1=0,\sigma^2} f(y;\beta,\sigma^2) \right]$

$$= \frac{\sup_{\beta,\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp(-(Y-X\beta)^T(Y-X\beta)/2\sigma^2)}{\sup_{\beta_1=0,\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp(-(Y-X\beta)^T(Y-X\beta)/2\sigma^2)}$$

Full model: $\hat\beta_{MLE} = (X^TX)^{-1}X^TY$, $\hat\sigma^2_{MLE} = \|(I-H)Y\|^2/n$

null model: likelihood becomes $\frac{1}{(2\pi\sigma^2)^{n/2}} \exp(-(Y-X_0\beta_0)^T(Y-X_0\beta_0)/2\sigma^2)$

$$= \frac{\sup_{\beta,\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp(-(Y-X\beta)^T(Y-X\beta)/2\sigma^2)}{\sup_{\beta_0,\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp(-(Y-X_0\beta_0)^T(Y-X_0\beta_0)/2\sigma^2)}$$

$$= \frac{\frac{1}{(\|(I-H)Y\|^2/n)^{n/2}} \exp\left(-\|(I-H)Y\|^2 / 2\|(I-H)Y\|^2 \cdot n\right)}{\frac{1}{(\|(I-H_0)Y\|^2/n)^{n/2}} \exp\left(-\|(I-H_0)Y\|^2 / 2\|(I-H_0)Y\|^2 \cdot n\right)}$$

$$= \left( \|(I-H_0)Y\|^2 / \|(I-H)Y\|^2 \right)^{n/2} \quad \square.$$

10. $Y_i - X_i^T \hat{\beta}$ is the ith element of $Y - X\hat{\beta} = (I-H)Y$

$= (I-H)\varepsilon \qquad Var[(I-H)\varepsilon \mid X] = \sigma^2 (I-H)$

$\Rightarrow Var(Y_i - X_i^T \hat{\beta} \mid X) = \sigma^2 (1 - H_{ii})$, i,ith element of $\sigma^2 (I-H)$. $\square$.

11. a).

$$\begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{pmatrix} = \cancel{\begin{pmatrix} \cancel{X_{11}^T} \\ \cancel{X_{12}^T} \\ \cancel{X_{13}^T} \\ \cancel{X_{21}^T} \\ \cancel{X_{22}^T} \\ \cancel{X_{23}^T} \end{pmatrix} \beta}$$

$$= \begin{pmatrix} 1 & X_{11} & 0 & 0 \\ 1 & X_{12} & 0 & 0 \\ 1 & X_{13} & 0 & 0 \\ 0 & 0 & 1 & X_{21} \\ 0 & 0 & 1 & X_{22} \\ 0 & 0 & 1 & X_{23} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \delta_{11} \\ \delta_{12} \\ \delta_{13} \\ \delta_{21} \\ \delta_{22} \\ \delta_{23} \end{pmatrix}$$

b). i). $\beta_1 = \beta_2$ ii). $\alpha_1 = \alpha_2$ iii). $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$

$\Rightarrow$ reparametrize $\tilde{\beta} = \beta_1 - \beta_2$, $\tilde{\alpha} = \alpha_1 - \alpha_2$

$\alpha_1 + \beta_1 X_{1k} = \tilde{\alpha} + \tilde{\beta} X_{1k} + \alpha_2 + \beta_2 X_{1k}$

$$\Rightarrow \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{pmatrix} = \begin{pmatrix} 1 & X_{11} & 1 & X_{11} \\ 1 & X_{12} & 1 & X_{12} \\ 1 & X_{13} & 1 & X_{13} \\ 0 & 0 & 1 & X_{21} \\ 0 & 0 & 1 & X_{22} \\ 0 & 0 & 1 & X_{23} \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \alpha_2 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \delta_{11} \\ \delta_{12} \\ \delta_{13} \\ \delta_{21} \\ \delta_{22} \\ \delta_{23} \end{pmatrix}$$

i). $\tilde{\beta} = 0$ drop column 2 ii). $\tilde{\alpha} = 0$, drop column 1

iii). $\tilde{\alpha} = \tilde{\beta} = 0$, drop both ~~+ & 2~~ 1 and 2

12. note $\|(H - H_0)Y\|^2 / \|(I - H_0)Y\|^2$ $\hat{\sigma}^2 = \|(I-H)Y\|^2/(n-p)$

$$= \|(I - H_0)Y\|^2 / \|(I-H)Y\|^2 - 1$$

$$\hat{\beta}_p = ((I - H_0)X_p)^T Y / \|(I - H_0)X_p\|^2$$

$$\sigma^2 (X^TX)^{-1}_{pp} = \mathrm{Var}[\hat{\beta}_p] = \sigma^2 / \|(I - H_0)X_p\|^2$$

$$\Rightarrow \text{need } [((I - H_0)X_p)^T Y]^2 / \|(I - H_0)X_p\|^2 = \|(H - H_0)Y\|^2$$

note $(I - H_0)X_p = (I - H_0)HX_p = (H - H_0)X_p$

and $H - H_0$ is a projection matrix with rank 1

rank $(H - H_0) = \mathrm{tr}(H - H_0) = \mathrm{tr}(H) - \mathrm{tr}(H_0) = p - p_0 = 1$

$\Rightarrow$ $(H - H_0)X_p$ and $(H - H_0)Y$ are parallel vectors

$$(H - H_0)X_p = \frac{\pm \|(H - H_0)X_p\|}{\|(H - H_0)Y\|}(H - H_0)Y$$

$$\Rightarrow \left[ \left( (I - H_0) X_p \right)^T Y \right]^2$$

$$= \left[ \left( (H - H_0) X_p \right)^T (H - H_0) Y \right]^2$$

$$= \frac{\| (H - H_0) X_p \|^2}{\| (H - H_0) Y \|^2} \| (H - H_0) Y \|^4$$

$$= \| (\overset{I}{\cancel{H}} \overset{= H - H_0}{-} H_0) X_p \|^2 \cdot \| (H - H_0) Y \|^2$$

combine these above the proof is complete. $\square$