Harvard University
Computer Science 20

In-Class Problems 13

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Executive Summary

1. A *binary relation* describes relationships from one set A, called the *domain*, to another set B, called the *codomain* through a subset of $A \times B$ called the *relation graph*. This is often depicted as a diagram with arrows from elements of the *domain* to elements of the *codomain*. You can also represent a binary relation as a set of ordered pairs.

    - A *function* is a special case of a relation in which each member of the domain has at most one arrow coming out of it. Most of the time when we see functions, they can be described in a concise way, such as $f(x) = x^2$.
    - A relation is *surjective* when every item in the codomain has at least one arrow coming in – that is, every element in the codomain is covered.
    - A relation is *total* when every item in the domain has at least one arrow coming out of it – that is, every element in the domain participates in the relation.
    - A relation is *injective* when every element of the codomain has at most one arrow coming in – that is, if you start with an element in the codomain that has an arrow, there's no ambiguity where in the domain it came from.
    - A relation is *bijective* when every element of the domain has exactly one arrow pointing out, and every element of the codomain has exactly one arrow pointing in. A bijection between 2 sets exist if the 2 sets have the same cardinality, or size.

PROBLEM 1

Determine which labels apply to the following relations: function, total, injective, surjective, bijective. Also, identify the domain and co-domain of each relation.
(A) The relation that associates a Harvard undergraduate student with a residential House
(B) The relation that associates every natural number with its square (from natural numbers to natural numbers)
(C) The relation that associates every integer with its square (from integers to natural numbers)
(D) The relation that associates every student in the class with every class he or she is taking this current semester. Assume there are no empty classes, and no students with empty schedules.
(E) The relation that associates each class with the students enrolled in that class. There might be empty classes, but there are no students with empty schedules.

Solution.

PROBLEM 2

Determine which labels apply to the following functions: total, injective, surjective, bijective.
(A) $f(x) = x + 5$
(B) $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = 2x$
(C) $f : \mathbb{N} \to \mathbb{N}$ where $f(x) =$

$$\begin{cases} x - 1 & x \geq 2 \\ 1 & x = 1 \end{cases}$$

(D) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $f(x, y) = x - y$

Solution.

PROBLEM 3

Define f, g such that f is a total, injective function from set A to set B and g is a total, surjective function from B to C. Let $g \circ f$ denote function composition, i.e., $g \circ f = g(f(x))$. Prove or disprove the following claim: If f is total and injective and g is total and surjective, then $g \circ f$ is injective.

Solution.

PROBLEM 4

(BONUS) Show that if two finite sets A and B are the same size, and r is a total injective function from A to B, then r is also surjective; i.e. r is a bijection.

Solution.

PROBLEM 5

(BONUS) Give a counterexample showing that the conclusion of problem 2 does not necessarily hold if A and B are two infinite sets that have the same cardinality. Hint: $\mathbb{N}$ (the set of natural numbers) and $\mathbb{Z}$ (the set of integers) have the same cardinality.

Solution.