Your Name: [   ]

Harvard University
Computer Science 20

Midterm 2

Wednesday, March 23, 2016

PROBLEM 1

For each of the following, state whether the set is finite, countably infinite, or uncountable. No justification required.

(A) The set of all total functions with domain $\{0,1\}$ and codomain $\{0,1\}$.
(B) The set of all total functions with domain $\mathbb{N}$ and codomain $\{1\}$.
(C) The set of all total functions with domain $\mathbb{N}$ and codomain $\{0,1\}$.
(D) The set of all total functions with domain $\{0,1\}$ and codomain $\mathbb{N}$.

Solution.

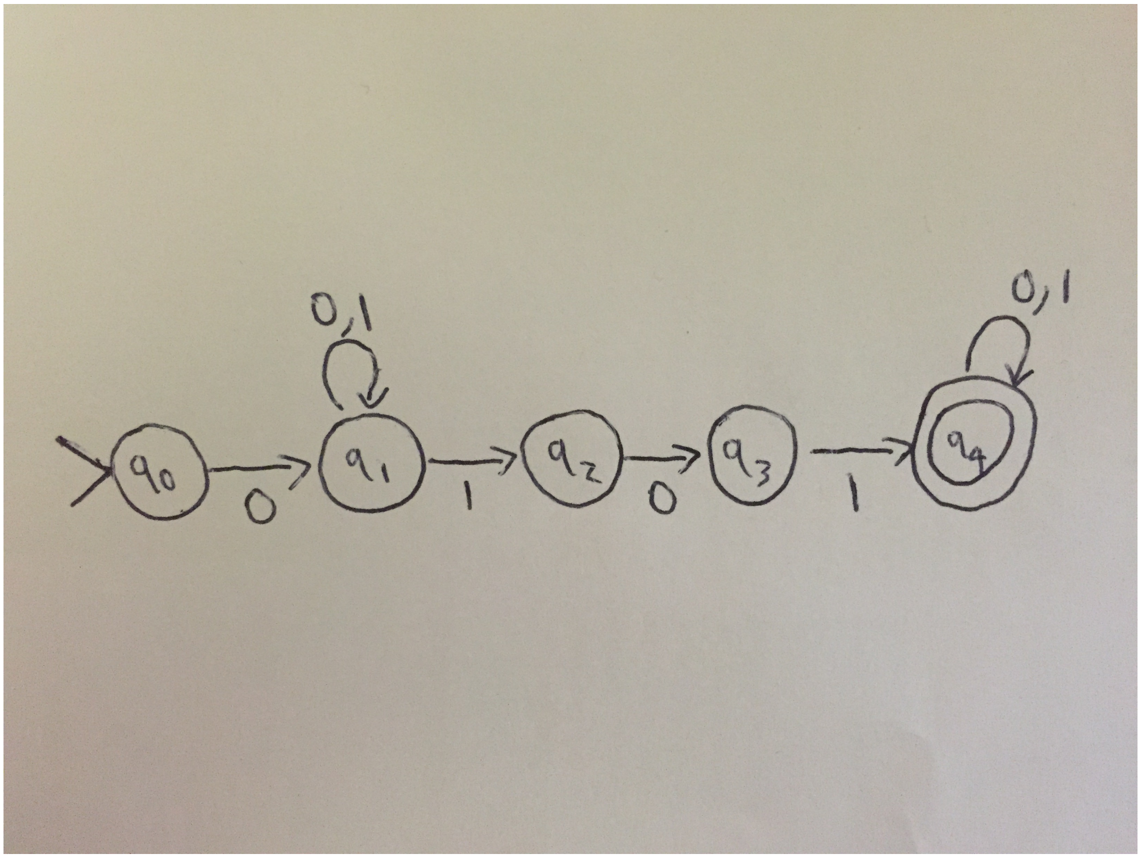
(A) Finite
(B) Finite
(C) Uncountable
(D) Countably Infinite

PROBLEM 2

Draw state machines that only accept strings in the following set. Assume that the alphabet is $\Sigma = \{0,1\}$; that is, for all possible input strings s we have $s \in \Sigma^*$.

$$\{w : w \text{ starts with 0 and contains the substring 101, i.e. } w = 0x101y \text{ for some } x \text{ and } y\}$$

Solution.



PROBLEM 3

Let G be a directed graph with n vertices. Show that if G has a path of length greater than n, then G has a cycle (a path has length k if it contains k edges).

Solution.

Proof by contradiction: assume there is a path of length $m > n$. The path can be written as: $(v_0, v_1), (v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)$. There are $m + 1 > n + 1$ vertices on the path. By the pigeonhole principle, at least 2 of those vertices are equal as the graph only has n vertices. Let the index of the first instance of the vertex in the path be i and the index of the second instance of the vertex be j, then there is a cycle starting and ending at that vertex with the path $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$.

PROBLEM 4

Let $G = (V, E)$ be a directed acyclic graph. Define a relation R on V by (v_1, v_2) which is an element of R iff there is a path from v_1 to v_2.

(A) Is R reflexive? Prove your answer.
(B) Is R symmetric? Prove your answer.
(C) Is R transitive? Prove your answer.

Solution.

(A) R is not reflexive. A DAG contains no cycles, and therefore there cannot be any paths from a vertex to itself.

(B) R is not symmetric. Proof by contradiction: suppose R is symmetric, and that there is a path from $(v_1, v_2) \in R$. Then there must be a path from v_2 to v_1, which implies that $(v_1, v_1) \in R$. We have already shown that this cannot be the case in a DAG, so R cannot be symmetric.

(C) R is transitive. If there is a path from v_1 to v_2 and a path from v_2 to v_3, we can concatenate the two paths to construct a path from v_1 to v_3.

PROBLEM 5

(A) Using set-builder notation, give a formal description of the union of two sets A and B.
(B) Using set-builder notation, give a formal description of the complement of a set A.
(C) Let $|A| = n$ and $|B| = m$. If $A \subseteq B$, what is $|A \cap B|$?
(D) Let $|A| = n$ and $|B| = m$. If $A \subseteq B$, what is $|A - B|$?
(E) What is the power set of $\{h, a, i\}$?

Solution.

(A) $\{x : x \in A \vee x \in B\}$
(B) $\{x : x \notin A\}$
(C) n
(D) 0
(E) $\{\emptyset, \{h\}, \{a\}, \{i\}, \{h, a\}, \{h, i\}, \{a, i\}, \{h, a, i\}\}$

PROBLEM 6

Let S be the set defined as follows:

- Base Case: $(1, 2) \in S$
- Constructor Rules: If $(x, y) \in S$, then C1: $(x + 2, y) \in S$ and C2: $(y, x) \in S$

(A) Use structural induction to prove that for any pair (x,y) in S, x and y can not both be odd or both be even.

Solution.

(A)

- Basis step: By the base case of the definition of S, $(1, 2) \in S$. 1 is even and 0 is odd.
- Recursive step:
  Assume elements (x,y) in S and P(x,y): *x and y are not both be odd or both be even* holds. Now consider the constructor rule in the definition of S. We want to prove P(x+2,y) and P(y,x) also holds.

  Case 1: x is even and y is odd,

  x+2 would be even, y would be odd $\rightarrow$ P(x+2,y) holds.

  y would be odd, x would be even $\rightarrow$ P(y,x) holds.

  Case 2: x is odd and y is even,

  x+2 would be odd, y would be even $\rightarrow$ P(x+2,y) holds.

  y would be even, x would be odd $\rightarrow$ P(y,x) holds.