Harvard University
Computer Science 20

Problem Set 5

Due Wednesday, March 9, 2016 at 9:59am. All students should submit an electronic copy.

Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE
(See the syllabus for information)**

PART A (Graded by Ben)

PROBLEM 1 (4 points, suggested length of 1/3 page)

Show by giving an example for each case that the intersection of two uncountable sets can be: empty, finite, countably infinite, or uncountably infinite.

Solution.
An example of the empty set resulting from the intersection of two uncountable sets is $\boxed{[0,1] \cap [2,3] = \emptyset}$.

An example of a finite set resulting from the intersection of two uncountable sets is $\boxed{[0,1] \cap [1,2] = \{1\}}$.

An example of a countably infinite set resulting from the intersection of two uncountable sets is $\boxed{(\mathbb{Z} \cup [0,1]) \cap (\mathbb{Z} \cup [2,3]) = \mathbb{Z}}$.

An example of an uncountable set resulting from the intersection of two uncountable sets is $\boxed{\mathbb{R} \cap \mathbb{R} = \mathbb{R}}$.

PROBLEM 2 (4+2 points, suggested length of 1 page)

In the large city of Hundred-land, there is a house for every real number between 1 and 100, inclusive. Let's assume for the sake of the problem that every house in Hundred-land is occupied by at least one citizen. In addition, every house is labelled with a unique real number between 1 and 100.

(A) The Schröder-Bernstein Theorem states that for sets S and T, if there exist injective functions $f : S \to T$ and $g : T \to S$, then S and T have the same cardinality. The neighboring city of Hundred-tropolis has a house for every real number between, but not including, 1 and 100. Using the Schröder-Bernstein Theorem, show that Hundred-land and Hundred-tropolis have the same number of houses.

(B) The band One Direction pays a visit to Hundred-land as part of their world tour and, absolutely enamored, the mayor declares the city to now be named One-land. As such, each house will now have to be labelled with a unique real number from 0 to 1, inclusive. Is it possible to accomplish this? Devise a system so that each citizen in Hundred-land can re-label their house or, if such a labelling system does not exist, explain why.

Solution.

(A) The solution to the problem is the same as showing that the cardinality of $\left[0, 100\right]$ is the same as $(0, 100)$. Using the Schröder-Bernstein Theorem, we want to construct an injective function from each set to the other. We can set the injective function $f : (0, 100) \to \left[0, 100\right]$ to be $f(x) = x$ for all $x \in (0, 100)$. If $f(x_1) = f(x_2)$, then $x_1 = x_2$, so f is injective.

Next, we want to construct an injective function $g : \left[0, 100\right] \to (0, 100)$. The idea is to find a "copy" of $\left[0, 100\right]$ in $(0, 100)$, then do some scaling and translation to map $\left[0, 100\right]$ onto the copy. We find that $g(x) = \frac{x}{2} + 25$ is an appropriate function for $0 \le x \le 100$. If $0 \le x \le 100$, then $0 \le \frac{x}{2} \le 50$, so $25 \le \frac{x}{2} + 25 \le 75$. This proves that g is a function from $\left[0, 100\right]$ to $\left[25, 75\right]$, which is a subset of $(0, 100)$.

Next, we want to show that g is injective. It suffices to show that if $g(x_1) = g(x_2)$, then $x_1 = x_2$. For $g(x_1) = g(x_2)$, we have:

$$\begin{aligned} \frac{x_1}{2} + 25 &= \frac{x_2}{2} + 25 \\ \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Having found satisfactory injective functions f and g, we know by the Schröder-Bernstein Theorem that the cardinality of $\left[0, 100\right]$ is the same as $(0, 100)$ and the two cities have the same number of houses.

(B) In this case the cardinality of the set that we are mapping from is equal to the cardinality of the set that we are mapping to (every set of real numbers over a non-zero interval has the same cardinality by definition). An example of a successful system (a correct injective function) would be to have a citizen currently living in a house with label value n to re-label their house to have value $\frac{n-1}{99}$. In this fashion, every house number between 1 and 100 can be re-labelled as a unique house number between 0 and 1, inclusive.

PROBLEM 3 (2+2 points, suggested length of 1/2 page)

A robot named Wall-E wanders around a two-dimensional grid. He starts at (0,0) and is allowed to take four different types of steps:

1. (-2, +2)
2. (-4, +4)
3. (+1, -1)
4. (+3, -3)

For example, Wall-E might take the following stroll. The types of his steps are denoted by each arrow's subscript:

$$(0, 0) \to_1 (-2, 2) \to_3 (-1, 1) \to_2 (-5, 5) \to_4 (-2, 2) \to \ldots$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at (1, 2).
(A) Describe a state machine model of this problem.
(B) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.

Solution.

(A) In our state machine model, every state takes the form (x, y). The start state is $(0, 0)$, and the possible transitions out of a state (x, y) are to $(x-2, y+2), (x-4, y+4), (x+1, y-1), (x+3, y-3)$.
(B) He will not. We propose the following invariant: for any state (x, y)

$$x = -y$$

We see that it is true for the start state $(0, 0)$, and all of the transitions preserve this invariant, as they add or subtract the same amount to both the x and y coordinates. However, our desired final state $(1, 2)$ does not satisfy this invariant, and so it must be impossible to reach.

PART B (Graded by Crystal)

PROBLEM 4 (3+1 points, suggested length of 1/2 page)

(A) Give a recursive definition of the set S of bit strings with no more than a single 1 in them (e.g. 00010, 010, or 000)
(B) Is $0010 \in S$? How can you derive it from your base case?

Solution.

(A)

- Base cases: the empty string ϵ, 0, $1 \in S$
- Constructor Rule: If w is in S, then

  C1: $0w \in S$

  C2: $w0 \in S$
- Nothing else (generally implicit): Nothing is in S unless it is obtained from the base case and constructor rule.

(B) Yes, because

- $1 \in S$ (Base case)
- $10 \in S$ by C2
- $010 \in S$ by C1
- $0010 \in S$ by C1

PROBLEM 5 (2+2 points, suggested length of 1/2 page)

Let $A = \{5n | n \in \mathbb{N}\}$ and let S be the set defined as follows:

- Base Case: $5 \in S$
- Constructor Rule: If $x \in S$ and $y \in S$, $(x + y) \in S$

(A) Use induction to prove that $A \subseteq S$.
(B) Use structural induction to prove that $S \subseteq A$.

Solution.

(A) Proving that $A \subseteq S$ is equivalent to proving that $\forall x[x \in A \to x \in S]$

- Let P(n): $5n \in S$. We must show that for all $n \in \mathbb{N}, P(n)$.
- Base case: When n=1, $5(1) = 5 \in S$ by base case in the definition of S.
- Induction step:
  Assuming for some $n \in \mathbb{N}$ P(n) is true, i.e. $5n \in S$, we want to prove that $P(n+1) : 5(n+1) = 5n + 5 \in S$.
  Since 5, 5n$\in S$, $5n + 5 \in S$ by the constructor rule of definition of S.

(B) Proving that $S \subseteq A$ is equivalent to proving that $\forall x[x \in S \to x \in A]$

- Basis step: By the base case of the definition of S, $5 \in S$. Since $5 = 5(1)$, $5 \in A$
- Recursive step:
  Now consider the constructor rule in the definition of S. Assume elements $x, y \in S$ are also in A. We must show that $x + y \in A$.
  Since $x, y \in A$, $x = 5i$ and $y = 5j$ for some natural numbers i and j. So $x + y = 5i + 5j = 5(i + j)$, where $i + j \in \mathbb{N}$ since $i, j \in \mathbb{N}$. Thus $x + y \in A$.