Harvard University
Computer Science 20

In-Class Problems 15

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Executive Summary

1. **Definition of Recursive Data Types:** common way of defining mathematical objects, which says how to construct new data elements from previous ones.
   - *Base Case(s):* specify that some known mathematical elements are in the data type
   - *Constructor Rule(s):* specify how to construct new data elements from previously constructed elements or from base elements.
   - *Nothing else (generally implicit):* the only way you can get whatever is you defining is by starting from the base case(s) and applying the constructor rule(s) one or more times.
2. **The Principle of Structural Induction:** to prove P(x) holds for all x in a recursively defined set S, prove
   - *Basis Step:* P(b) is true for each base case element $b \in S$, and
   - *Recursive Step:* P(c(x_1,...,x_k)) for each constructor c, assuming as the induction hypothesis that P(x_1),..., and P(x_k) all hold.

PROBLEM 1

Recursive Definition:
(A) There's an error in the following definition of the set of even integers (E). Find the error and fix it.

- Base Case: $0 \in E$
- Constructor Rule: For any element x in E, x+2 is in E.
- Nothing else (generally implicit): Nothing is in E unless it is obtained from the base case and constructor rule.

(B) Give a recursive definition of the natural numbers $\mathbb{N}$.
(C) Give a recursive definition of the sequence b_n, $b_n = 2n + 5, n \in \mathbb{N}$

Solution.

(A) It doesn't include negative Even Integer. There should be one more constructor rule x-2 is in EI.
(B)

- Base Cases: $1 \in \mathbb{N}$
- Constructor Rule: If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
- Nothing else (generally implicit): Nothing is in $\mathbb{N}$ unless it is obtained from the base case and constructor rule.

(C)

- Base Cases: $b_1 = 7$
- Constructor Rule: $b_{n+1} = b_n + 2$ for $n \in \mathbb{N}$
- Nothing else (generally implicit): Nothing is in b_n unless it is obtained from the base case and constructor rule.

PROBLEM 2

Let S be the set defined as follows:

- Base Case: $(1, 2) \in S$
- Constructor Rules: If $(x, y) \in S$, then C1: $(x + 2, y) \in S$ and C2: $(y, x) \in S$

(A) Is $(4, 3) \in S$? If it is, how can you derive it from (1,2)?
(B) Use induction to prove that $(2n + 2, 2n + 1) \in S$ for all n $\in \mathbb{N}$.

Solution.

(A) Yes, it is. Apply C1 to (1,2), we can get (3,2); Apply C2 to (3,2), we can get (2,3); Apply C1 to (2,3), we can get (4,3).
(B) Let P(n): $(2n + 2, 2n + 1) \in S$. We must show that for all $n \in \mathbb{N}, P(n)$.

- Base case: When n=1, $(4, 3) \in S$. (Already proved in (A))
- Induction step:
  Assuming P(n): $(2n + 2, 2n + 1) \in S$ holds, we want to prove that $P(n + 1) : (2(n + 1) + 2, 2(n + 1) + 1) = (2n + 4, 2n + 3) \in S$.
  Apply C1 to (2n+2, 2n+1), we then could get $(2n + 4, 2n + 1) \in S$. Then apply C2 to (2n+4, 2n+1), we could get $(2n + 1, 2n + 4) \in S$. Then, apply C1 to (2n+1, 2n+4), then we can get $(2n + 3, 2n + 4) \in S$. Finally, apply C2 again to (2n+3, 2n+4), then we can get $(2n + 4, 2n + 3) \in S$.

PROBLEM 3

(Bonus) Construct a recursive definition for the set of strings **S** over the alphabet a,b excepting empty string, i.e. set of string consisting of a's and b's such as abbab, bbabaa, etc.

Solution.

- Base Cases: $a \in S$, and $b \in S$
- Constructor Rule: For any element x in S, $as \in S$, and $bx \in S$
- Nothing else (generally implicit): Nothing is in S unless it is obtained from the base case and constructor rule.

PROBLEM 4

(Bonus) Let S be the set defined as follows:

- Base Case: $(0,0) \in S$
- Constructor Rules: If $(a,b) \in S$, then C1: $(a,b+1) \in S$, C2: $(a+1,b+1) \in S$ and C3: $(a+2,b+1) \in S$

(A) List 5 elements.
(B) Use structural induction to prove that for every $(a,b) \in S, a \leq 2b$.

Solution.

(A) (0,1), (1,1), (2,1), (1,2), (2,2)...
(B)

- Basis step: By the base case of the definition of S, $(0,0) \in S$. $0 \leq 2(0)$.
- Recursive step:
  Now consider the constructor rule in the definition of S. Assume elements $a, b \in S$ and $a \leq 2b$. We must show that $a \leq 2(b+1)$, $(a+1) \leq 2(b+1)$ and $(a+2) \leq 2(b+1)$.
  1. prove $a \leq 2(b+1)$: $a \leq 2b \rightarrow a \leq (2b+2) = 2(b+1)$
  2. prove $(a+1) \leq 2(b+1)$: $a \leq 2b \rightarrow (a+1) \leq (2b+1) \leq (2b+2) = 2(b+1)$
  3. prove $(a+2) \leq 2(b+1)$: $a \leq 2b \rightarrow (a+2) \leq (2b+2) = 2(b+1)$