Harvard University
Computer Science 20

Problem Set 5

Due Wednesday, March 9, 2016 at 9:59am. All students should submit an electronic copy.

Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**

PART A (Graded by Ben)

PROBLEM 1 (4 points, suggested length of 1/3 page)

Show that the intersection of two uncountable sets can be empty, finite, countably infinite, or uncountably infinite.

PROBLEM 2 (2+3 points, suggested length of 2/3 page)

In the city of Hundred-land, the citizens all live in houses, each of which is labelled with a unique number. Specifically, the set of all house numbers is equal to the set of all real numbers from 1 to 100, inclusive. Let's assume for the sake of the problem that every house in Hundred-land is occupied by at least one citizen.

(A) Exasperated with the unfathomable costs associated with painting all of the house numbers, the mayor of Hundred-land declares that the numbering system be changed such that each house is labelled with a unique positive integer instead. Ignoring questions of time and resource-consumption, explain why it is impossible to label all of the houses in Hundred-land this way.

(B) Fortunately, before the mayor can enact his plan, the band One Direction pays a visit to Hundred-land as part of their world tour and, absolutely enamored, the mayor declares the city to now be named One-land. As such, each house will now have to be labelled with a unique real number from 0 to 1, inclusive. Is it possible to accomplish this? Devise a system so that each citizen in Hundred-land can re-label their house or, if such a labelling system does not exist, explain why.

PROBLEM 3 (2+4 points, suggested length of 1/3 page)

A robot named Wall-E wanders around a two-dimensional grid. He starts at (0,0) and is allowed to take four different types of steps:

1. (+1,-2)
2. (+5, -1)

3. (+0, +3)
4. (-2, -2)

For example, Wall-E might take the following stroll. The types of his steps are denoted by each arrow's subscript:

$$(0,0) \rightarrow_1 (1,-2) \rightarrow_3 (1,1) \rightarrow_2 (6,0) \rightarrow_4 (4,-2) \rightarrow \ldots$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at (1, 2).
(A) Describe a state machine model of this problem.
(B) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.

PART B (Graded by Crystal)

PROBLEM 4 (3+1 points, suggested length of 1/2 page)

(A) Give a recursive definition of the set S of bit strings with no more than a single 1 in them (e.g. 00010, 010, or 000)
(B) Is $0010 \in S$? How can you derive it from your base case?

PROBLEM 5 (2+2 points, suggested length of 1/2 page)

Let $A = \{5n : n \in \mathbb{N}\}$ and let S be the set defined as follows:

- Base Case: $5 \in S$
- Constructor Rule: If $x \in S$ and $y \in S$, $(x + y) \in S$

(A) Use induction to prove that $A \subseteq S$.
(B) Use structural induction to prove that $S \subseteq A$.