Harvard University
Computer Science 20

Problem Set 5

Due Wednesday, March 9, 2016 at 9:59am. All students should submit an electronic copy.

Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**

PART A (Graded by Ben)

PROBLEM 1 (4 points, suggested length of 1/3 page)

Show, by giving an example for each case, that the intersection of two uncountable sets can be: empty, finite, countably infinite, or uncountably infinite.

Solution.
An example of the empty set resulting from the intersection of two uncountable sets is $\boxed{[0,1] \cap [2,3] = \emptyset}$.

An example of a finite set resulting from the intersection of two uncountable sets is $\boxed{[0,1] \cap [1,2] = \{1\}}$.

An example of a countably infinite set resulting from the intersection of two uncountable sets is $\boxed{(\mathbb{Z} \cup [0,1]) \cap (\mathbb{Z} \cup [2,3]) = \mathbb{Z}}$.

An example of an uncountable set resulting from the intersection of two uncountable sets is $\boxed{\mathbb{R} \cap \mathbb{R} = \mathbb{R}}$.

PROBLEM 2 (4+2 points, suggested length of 1 page)

(A) The Schröder-Bernstein Theorem states that for sets S and T, if there exist injective functions $f : S \to T$ and $g : T \to S$, then S and T have the same cardinality. Using the Schröder-Bernstein Theorem, show that the cardinality of the set of all real numbers between 0 and 100, inclusive, is the same as the cardinality of the set of all real numbers between, but not including, 0 and 100.

(B) Prove the finite case for the Schröder-Bernstein Theorem. That is, prove that for finite sets S and T, if there exist injective functions $f : S \to T$ and $g : T \to S$, then sets S and T have the same cardinality. (HINT: Check out the warm-up proof from the Relations and Functions lesson for some inspiration!)

Solution.

(A) We want to show that the cardinality of $[0, 100]$ is the same as $(0, 100)$. Using the Schröder-Bernstein Theorem, we want to construct an injective function from each set to the other. We can set the injective function $f : (0, 100) \to [0, 100]$ to be $f(x) = x$ for all $x \in (0, 100)$. If $f(x_1) = f(x_2)$, then $x_1 = x_2$, so f is injective.

Next, we want to construct an injective function $g : [0, 100] \to (0, 100)$. The idea is to find a "copy" of $[0, 100]$ in $(0, 100)$, then do some scaling and translation to map $[0, 100]$ onto the copy. We find that $g(x) = \frac{x}{2} + 25$ is an appropriate function for $0 \leq x \leq 100$. If $0 \leq x \leq 100$, then $0 \leq \frac{x}{2} \leq 50$, so $25 \leq \frac{x}{2} + 25 \leq 75$. This proves that g is a function from $[0, 100]$ to $[25, 75]$, which is a subset of $(0, 100)$.

Next, we want to show that g is injective. It suffices to show that if $g(x_1) = g(x_2)$, then $x_1 = x_2$. For $g(x_1) = g(x_2)$, we have:

$$\begin{aligned} \frac{x_1}{2} + 25 &= \frac{x_2}{2} + 25 \\ \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Having found satisfactory injective functions f and g, we know by the Schröder-Bernstein Theorem that the cardinality of $[0, 100]$ is the same as $(0, 100)$.

(B) We use proof by contradiction. Suppose that sets S and T do not have the same cardinality. Without loss of generality, assume that $|S| < |T|$. Now consider the injective function g. By the definition of an injective function, for any a and b in T, $a \neq b \Rightarrow g(a) \neq g(b)$. However, since g is mapping elements from T to elements in S and there are more elements in T than in S, by the Pigeonhole Principle there must be at least two unique elements in T that map to the same element in S. In formal terms, $\exists a, b \in T$ s.t. $a \neq b \wedge g(a) = g(b)$. Then g must not be an injective function, which is a contradiction, so we conclude that sets S and T must have the same cardinality (a symmetrical argument applies if $|S| > |T|$ by examining the injective function f).

PROBLEM 3 (2+2 points, suggested length of 1/2 page)

A robot wanders around a two-dimensional grid. He starts at (0,0) and is allowed to take four different types of steps:

1. (-2, +2)
2. (-4, +4)
3. (+1, -1)
4. (+3, -3)

For example, the robot might take the following stroll. The types of his steps are denoted by each arrow's subscript:

$$(0, 0) \to_1 (-2, 2) \to_3 (-1, 1) \to_2 (-5, 5) \to_4 (-2, 2) \to \ldots$$

(A) Describe a state machine model of this problem.

(B) Will the robot ever reach (1, 2)? Either find an appropriate path for the robot or use the Invariant Principle to prove that no such path exists.

Solution.

(A) In our state machine model, every state takes the form (x, y). The start state is $(0, 0)$, and the possible transitions out of a state (x, y) are to $(x-2, y+2), (x-4, y+4), (x+1, y-1), (x+3, y-3)$.
(B) He will not. We propose the following invariant: for any state (x, y)

$$x = -y$$

We see that it is true for the start state $(0, 0)$, and all of the transitions preserve this invariant, as they add or subtract the same amount to both the x and y coordinates. However, our desired final state $(1, 2)$ does not satisfy this invariant, and so it must be impossible to reach.

PART B (Graded by Crystal)

PROBLEM 4 (3+1 points, suggested length of 1/2 page)

(A) Using string concatenation as the sole constructor, give a recursive definition of the set S of bit strings with no more than a single 1 in them
(e.g. 00010, 010, or 000)
(B) Is $0010 \in S$? How can you derive it from your base case?

Solution.

(A)

- Base cases: the empty string ϵ, 0, $1 \in S$
- Constructor Rule: If w is in S, then
  C1: $0w \in S$
  C2: $w0 \in S$
- Nothing else (generally implicit): Nothing is in S unless it is obtained from the base case and constructor rule.

(B) Yes, because

- $1 \in S$ (Base case)
- $10 \in S$ by C2
- $010 \in S$ by C1
- $0010 \in S$ by C1

PROBLEM 5 (2+2 points, suggested length of 1/2 page)

Let $A = \{5n : n \in \mathbb{N}\}$ and let S be the set defined as follows:

- Base Case: $5 \in S$
- Constructor Rule: If $x \in S$ and $y \in S$, $x + y \in S$

(A) Use induction to prove that $A \subseteq S$.
(B) Use structural induction to prove that $S \subseteq A$.

Solution.

(A) Proving that $A \subseteq S$ is equivalent to proving that $\forall x[x \in A \rightarrow x \in S]$

- Let P(n): $5n \in S$. We must show that for all $n \in \mathbb{N}, P(n)$.
- Base case: When n=1, $5(1) = 5 \in S$ by base case in the definition of S.
- Induction step:
  Assuming for some $n \in \mathbb{N}$ P(n) is true, i.e. $5n \in S$, we want to prove that $P(n+1) : 5(n+1) = 5n + 5 \in S$.
  Since 5, 5n$\in S$, $5n + 5 \in S$ by the constructor rule of definition of S.

(B) Proving that $S \subseteq A$ is equivalent to proving that $\forall x[x \in S \rightarrow x \in A]$

- Basis step: By the base case of the definition of S, $5 \in S$. Since $5 = 5(1)$, $5 \in A$
- Recursive step:
  Now consider the constructor rule in the definition of S. Assume elements $x, y \in S$ are also in A. We must show that $x + y \in A$.
  Since $x, y \in A$, $x = 5i$ and $y = 5j$ for some natural numbers i and j. So $x + y = 5i + 5j = 5(i + j)$, where $i + j \in \mathbb{N}$ since $i, j \in \mathbb{N}$. Thus $x + y \in A$.