Harvard University
Computer Science 20

In-Class Problems 15

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Executive Summary

1. **Definition of Recursive Data Types:** common way of defining mathematical objects, which says how to construct new data elements from previous ones.
   - *Base Case(s):* specify that some known mathematical elements are in the data type
   - *Constructor Rule(s):* specify how to construct new data elements from previously constructed elements or from base elements.
   - *Nothing else (generally implicit):* the only way you can get whatever is you defining is by starting from the base case(s) and applying the constructor rule(s) one or more times.
2. **The Principle of Structural Induction:** to prove P(x) holds for all x in a recursively defined set S, prove
   - *Basis Step:* P(b) is true for each base case element $b \in S$, and
   - *Recursive Step:* P(c(x_1,...,x_k)) for each constructor c, assuming as the induction hypothesis that P(x_1),..., and P(x_k) all hold.

PROBLEM 1

Recursive Definition:
(A) There's an error in the following definition of the set of even integers (E). Find the error and fix it.

- Base Case: $0 \in E$
- Constructor Rule: For any element x in E, x+2 is in E.
- Nothing else (generally implicit): Nothing is in E unless it is obtained from the base case and constructor rule.

(B) Give a recursive definition of the natural numbers $\mathbb{N}$.
(C) Give a recursive definition of the sequence b_n, $b_n = 2n + 5, n \in \mathbb{N}$

Solution.

(A) It doesn't include negative Even Integer. There should be one more constructor rule x-2 is in EI.
(B)

- Base Cases: $1 \in \mathbb{N}$
- Constructor Rule: If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
- Nothing else (generally implicit): Nothing is in $\mathbb{N}$ unless it is obtained from the base case and constructor rule.

(C)

- Base Cases: $b_1 = 7$
- Constructor Rule: $b_{n+1} = b_n + 2$ for $n \in \mathbb{N}$
- Nothing else (generally implicit): Nothing is in b_n unless it is obtained from the base case and constructor rule.

PROBLEM 2

Let S be the set defined as follows:

- Base Case: $(1, 2) \in S$
- Constructor Rules: If $(x, y) \in S$, then C1: $(x + 2, y) \in S$ and C2: $(y, x) \in S$

(A) Is $(4, 3) \in S$? If it is, how can you derive it from (1,2)?
(B) Use induction to prove that $(2n + 2, 2n + 1) \in S$ for all n $\in \mathbb{N}$.

Solution.

(A) Yes, it is. Apply C1 to (1,2), we can get (3,2); Apply C2 to (3,2), we can get (2,3); Apply C1 to (2,3), we can get (4,3).
(B) Let P(n): $(2n + 2, 2n + 1) \in S$. We must show that for all $n \in \mathbb{N}, P(n)$.

- Base case: When n=1, $(4, 3) \in S$. (Already proved in (A))
- Induction step:
  Assuming P(n): $(2n + 2, 2n + 1) \in S$ holds, we want to prove that $P(n + 1) : (2(n + 1) + 2, 2(n + 1) + 1) = (2n + 4, 2n + 3) \in S$.
  Apply C1 to (2n+2, 2n+1), we then could get $(2n + 4, 2n + 1) \in S$. Then apply C2 to (2n+4, 2n+1), we could get $(2n + 1, 2n + 4) \in S$. Then, apply C1 to (2n+1, 2n+4), then we can get $(2n + 3, 2n + 4) \in S$. Finally, apply C2 again to (2n+3, 2n+4), then we can get $(2n + 4, 2n + 3) \in S$.

PROBLEM 3

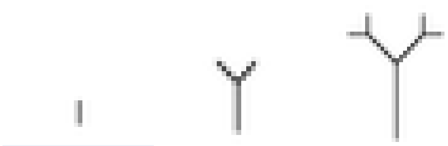
(Bonus) The set of Pythagoras trees:
The shape of Pythagoras tree is built by recursively feeding the axiom through the production rules. Each character of the input string is checked against the rule list to determine which character or string to replace it with in the output string. In this example, a '1' in the input string becomes '11' in the output string, while '[' remains the same.

- variables: 0,1
- constants: [,]
- axiom: 0
- rules: $(1 \rightarrow 11), (0 \rightarrow 1[0]0)$

(A) Give a recursive definition of the Pythagoras tree.
(B) List the first 4 elements.
(C) This string can be drawn as an image by using turtle graphics, where each symbol is assigned a graphical operation for the turtle to perform. For example, in the sample above, the turtle may be given the following instructions:

- 0: draw a line segment ending in a leaf
- 1: draw a line segment
- [: push position and angle, turn left 45 degrees
-]: pop position and angle, turn right 45 degrees

Applying the graphical rules listed above to the earlier recursion, we could get the graphical representation of axiom, 1st and 2nd recursion as the following :



Draw out the graphical representation of the 3rd and 4th recursion.

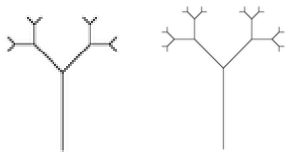
Solution.

(A)

- Base Case: $0 \in E$
- Constructor Rule: $(1 \rightarrow 11), (0 \rightarrow 1[0]0)$
- Nothing else (generally implicit): Nothing is in the set of Pythagoras tree unless it is obtained from the base case and constructor rule.

(B)

- axiom: 0
- 1st recursion: 1[0]0
- 2nd recursion: 11[1[0]0]1[0]0
- 3rd recursion: 1111[11[1[0]0]1[0]0]11[1[0]0]1[0]0

(C) 

PROBLEM 4

(Bonus) Let S be the set defined as follows:

- Base Case: $(0,0) \in S$
- Constructor Rules: If $(a,b) \in S$, then C1: $(a, b+1) \in S$, C2: $(a+1, b+1) \in S$ and C3: $(a+2, b+1) \in S$

(A) List 5 elements.
(B) Use structural induction to prove that for every $(a,b) \in S, a \leq 2b$.

Solution.

(A) (0,1), (1,1), (2,1), (1,2), (2,2)...
(B)

- Basis step: By the base case of the definition of S, $(0,0) \in S$. $0 \leq 2(0)$.
- Recursive step:
  Now consider the constructor rule in the definition of S. Assume elements $a, b \in S$ and $a \leq 2b$. We must show that $a \leq 2(b+1)$, $(a+1) \leq 2(b+1)$ and $(a+2) \leq 2(b+1)$.
  1. prove $a \leq 2(b+1)$: $a \leq 2b \to a \leq (2b+2) = 2(b+1)$
  2. prove $(a+1) \leq 2(b+1)$: $a \leq 2b \to (a+1) \leq (2b+1) \leq (2b+2) = 2(b+1)$
  3. prove $(a+2) \leq 2(b+1)$: $a \leq 2b \to (a+2) \leq (2b+2) = 2(b+1)$