

Harvard University  
Computer Science 20  
In-Class Problems 11  
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**Midterm Review** This in-class midterm review is a puzzle. Each problem gives you one piece of the puzzle. Solve all the problems and put together the pieces to discover the keyword. Enjoy!

PROBLEM 1

What follows is alleged to be a proof of the “Anti-Friendship Theorem”: that for all  $n$ , there exists a set of  $n$  people for which all subsets of 3 people are neither all friends nor all enemies. Find the *first* line that contains a **logical flaw**, if one exists. (If something is “unnecessary” but logically true, it is not a logical flaw.)

1. Proceeding by induction, we define the predicate  $P(n)$  = there exists a set of  $n$  people for which all subsets of 3 people are neither all friends nor all enemies.
2. Base cases: Suppose Alice and Bob are friends, Bob and Charlotte are friends, but Alice and Charlotte are not friends.  $P(3)$  evidently holds. Add Dianne, a friend of Bob but not of Alice or Charlotte. Thus  $P(4)$  holds.
3. Add Eric, a friend of Alice and Charlotte but not Bob or Dianne. Thus  $P(5)$  holds as well.
4. Inductive step: Assume for that  $n \geq 5$ ,  $P(3), P(4), \dots, P(n)$  holds.
5. Let  $S$  be a set of  $n + 1$  people. We can split  $S$  into two smaller sets, one with  $n$  people and another with 1 person. Call the 1 lonely person  $s_0 \in S$  and the set of  $n$  people  $S_n$ .
6. By the inductive hypothesis, any 3 people in  $S_n$  are neither all friends nor all enemies.
7. For a subset of 3 in  $S_n$ , we can take those 3 and add  $s_0$  to create a set with 4 people.
8. We showed in the base case that  $P(4)$  holds. Therefore this new set of 4 people must contain a subset of 3 who are neither all friends nor all enemies.
9. These 4 people were all from the original set of size  $n + 1$ , so we have found the set we’re looking for to prove  $P(n + 1)$ . By induction, the Anti-Friendship Theorem holds.
10. No flaw exists.

1	2	3	4	5	6	7	8	9	10
G	R	S	A	H	L	M	E	O	B

Table 1: Problem 1 Clue

**Solution.**

## PROBLEM 2

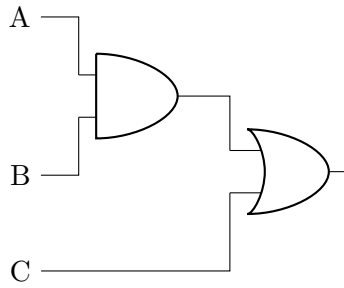
Construct a truth table for  $(p \leftrightarrow q) \rightarrow (q \leftrightarrow r)$ . How many rows in the truth table are True?

0	1	2	3	4	5	6	7	8	> 8
E	N	A	V	B	R	I	Z	X	Q

Table 2: Problem 2 Clue

**Solution.**

## PROBLEM 3



Which of the following does the above logic circuit compute?

1.  $A \cdot B + C$
2.  $A + B + C$
3.  $A + B \cdot C$
4.  $A \cdot B \cdot C$
5.  $(A + B) \cdot C$

1	2	3	4	5
S	E	O	T	R

Table 3: Problem 3 Clue

**Solution.**

## PROBLEM 4

Use the quantificational logic statements below to deduce which of the propositions is True.

1.  $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}. n \cdot m = 1$

2.  $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}. n + m = n$
3.  $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n + m = n$
4.  $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n \cdot m = n$
5.  $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n \cdot m = m$

1	2	3	4	5
M	N	R	E	L

Table 4: Problem 4 Clue

**Solution.**

### PROBLEM 5

(BONUS) What is the least value of  $m$  for which the following is true? “In any set of  $m$  propositions, all involving only  $p$ , two of the propositions are logically equivalent.”

2	5	9	12	17	26	50	> 50
A	W	B	K	U	O	N	T

Table 5: Problem 5 Clue

**Solution.**

### PROBLEM 6

**Final Answer:**

**Solution.**