Harvard University
Computer Science 20

In-Class Problems 10

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Executive Summary

1. Propositions and predicates
   - A proposition P is like a boolean variable. Its value is either "true" or "false."
   - A predicate $P(x)$ is like a boolean-valued function. It may have the value "true" for some values of the x and the value "false" for others.
   - A predicate can have more than one argument; e.g. Enrolled(x, y) might mean "student x is enrolled in course y."
   - In principle it is important to know the (possibly infinite) set of values D that x and y might assume (the domain of the function).
2. Quantifiers
   - The existential quantifier: $\exists x P(x)$ or $\exists x.P(x)$ or $\exists x$ s.t. $P(x)$ means "there exists at least one x in the domain D such that $P(x)$ is true.
   - The universal quantifier: $\forall x P(x)$ or $\forall x.P(x)$ means "for every x in the domain D, $P(x)$ is true.
3. Multiple quantifiers
   - $\exists x P(x)$ and $\forall x P(x)$ are both propositions, subject to the rules of logic that you already know.
   - $\exists x P(x, y)$ and $\forall x P(x, y)$ are both predicates of the form $Q(y)$, subject to the rules of quantificational logic that you are learning.
   - $\exists x \exists y P(x, y)$ and $\forall x \forall y P(x, y)$ are both propositions. The order of the quanitifiers is irrelevant.
   - $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ are both propositions, but they are different! The order of the quantifiers is important.
4. Negation and quantifiers
   - $\neg(\exists x.P(x)) \leftrightarrow \forall x.(\neg P(x))$.
   - $\neg(\forall x.P(x)) \leftrightarrow \exists x.(\neg P(x))$.

PROBLEM 1

Recall from Problem Set 1: Let A be the set of your pigeons, and let B be the set of pigeonholes in which they live. The *Generalized Pigeonhole Principle* states that for a natural number k, if $|A| > k|B|$, then there is a pigeonhole in which more than k pigeons live. Restate the GPP using quantifiers. Then negate the GPP (as you would for a proof by contradiction).

Solution.
Restatement: $\forall k \in \mathbb{N}, \forall A, B, |A| > k|B| \implies \exists b \in B$ s.t. b contains more than k pigeons.
Negation: $\exists k \in \mathbb{N}, \exists A, B$ s.t. $|A| > k|B|$ and $\forall b \in B$, b contains less than or equal to k pigeons.

PROBLEM 2

The domain of discourse is the set of all finite-length binary strings. The predicates $\mathrm{Sub}(x, y)$ (meaning x is a substring of y) and $\mathrm{Pre}(x, y)$ (meaning x is a prefix of y) are available.

(A) Write an expression that means x consists of alternating 0s and 1s, e.g 01010 or 101010.
(B) Write two different expressions that mean x consists of one or more 1s, and no 0s. Important caveat: neither of your expressions may contain "0".
(C) (BONUS) Write two additional expression for (c) under the same constraint.

Solution.

(A) $Alternating(x) = \neg Sub(00, x) \wedge \neg Sub(11, x)$
(B) 1: $AllOnes(x) = \forall y\ Sub(y, x) \implies Sub(1, y)$.
2: $AllOnes(x) = \forall y\ Sub(y, x) \implies Pre(1, y)$.

(C) 3. $AllOnes(x) = Pre(1, x) \wedge \forall y\ (Pre(y, x) \implies Pre(1, y))$ 4. $AllOnes(x) = Pre(1, x) \wedge \forall y\ (Pre(y, x) \implies AllOnes(y))$ (or Sub may be used)

PROBLEM 3

(BONUS) We define a committee to be a subset of senators $S = \{s_1, s_2, \cdots, s_n\}$. The predicate $M(s, C)$ means "Senator s is a member of committee C." Rewrite the following in terms of predicate logic. You may use "=" and "$\in$" in your expressions.

(A) Every committee has at least two senators serving on it.
(B) No two senators serve on more than one committee together.

Solution.

(A) $\forall C, \exists s_i, s_j \in S$ s.t. $i \neq j$ and $s_i, s_j \in C$.
(B) $\forall s_i, s_j,\ (s_i, s_j \in C_1) \wedge (s_i, s_j \in C_2) \implies C_1 = C_2$.

PROBLEM 4

The domain of discourse is the set of integers. Let $S(x, y, z)$ mean that "z is the sum of x and y."

(A) Write a formula that means x is an even integer.
(B) Write a formula that symbolizes the commutative property for addition ($x + y = y + x$) of integers.
(C) Write a formula that symbolizes the associative law for addition of integers:
$x + (y + z) = (x + y) + z$.

Solution.

(A) $Even(z) = \exists y.S(y, y, z)$
(B) $\forall x, y \exists z.S(x, y, z) \wedge S(y, x, z)$
(C) $\forall x, y, z, \exists u, v, w.S(y, z, u) \wedge S(x, y, v) \wedge S(x, u, w) \wedge S(v, z, w)$