Harvard University
Computer Science 20
In-Class Problems 16

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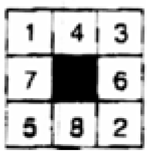
Executive Summary

PROBLEM 1

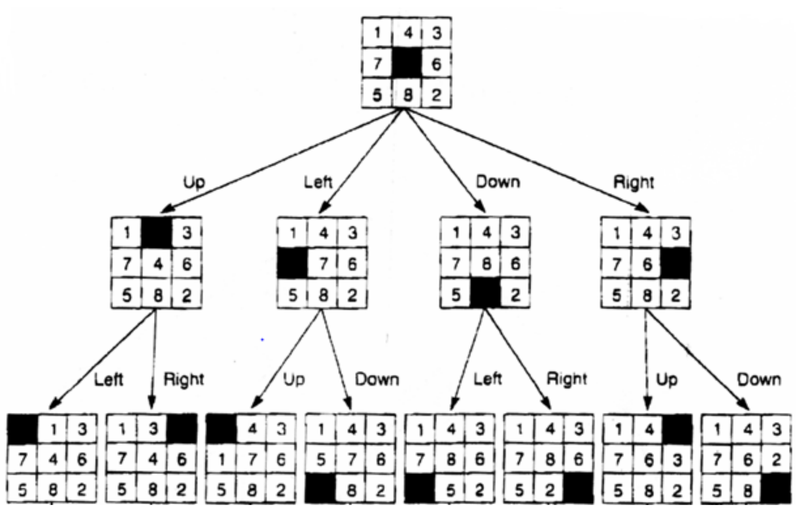
We have a puzzle looks like the following graph.



(A) What's the size of the state space after 1 move?
(B) What's the size of the state space after 2 moves?

Solution.

(A) The size of the state space after 1 move is 4.
(B) The size of the state space after 2 moves is 8.



PROBLEM 2

Use loop invariant to prove correctness property that y=c ($c > 0$)after the following loop terminates.

- x=c; y=0;
- while ($x > 0$):

  $x - -$

  $y + +$

(A) Construct a loop invariant for the proof.
(B) Use induction to prove that the loop preserves the invariant.
(C) Use loop invariant to prove correctness property that y=c ($c > 0$)after loop terminates.

Solution.

(A)

- Let state set S=$N * N * N$: the values of (x, y, c)
- $(x, y, c) \rightarrow (x-1, y+1, c)$ in each iteration of loop.
- Let $P(x, y, n) \equiv$"x+y=c"
- $y++$

(B)

- Base case: loop invariant x+y=c+0=c $\rightarrow$ P(c,0,c) holds.
- Induction step:
  Assume loop invariant holds after k iterations:
  x=c-k; y=k;
  after the (k+1) th iteration, y=k+1, x=c-k-1
  And x+y=k+1+c-k-1=c
  Therefore, the loop preserves the invariant P(x,y,n).

(C) After final iteration: x=0;
we also know our loop invariant holds: x+y=c. Therefore, y=c.

PROBLEM 3

Consider the following piece of code ($n > 0$):

- y=0; i=0
- while ($i < n$):

  $y+=2^i$

  $i++$

(A) Compute the value of y after 0th,1th,2nd,3rd iteration and guess what would be the value of y after the loop termination?
(B) Take use of what you get in (A) to construct a loop invariant.
(C) Use induction to prove that the loop preserves the invariant.
(D) Use loop invariant to prove correctness property that y=c ($c > 0$)after loop terminates.

Solution.

(A)

- iteration 0: $y_0 = 0 = 2^0 - 1$
- iteration 1: $y_1 = 2^0 = 1 = 2^1 - 1$
- iteration 2: $y_2 = 2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$
- iteration 3: $y_3 = 2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$
- iteration n: $y_n = 2^0 + 2^1 + 2^2 + ... + 2^{n-1} = 2^n - 1$

(B)

- Let state set S=$N * N * N$: the values of (y, i, n)
- $(y, i, n) \to (y + 2^i, i + 1, n)$ in each iteration of loop.
- Let $P(y, i, n) \equiv y = 2^i - 1$

(C)

- Base case: i=0: $y_0 = 0 = 2^0 - 1 \to$ P(0,0,n) holds.
- Induction step:
  Assume that at the start of the k-th iteration $y_k = 2^k - 1$
  Then, at the start of the (k+1)-th iteration we will have:
  $y_{k+1} = y_k + 2^k = 2^k - 1 + 2^k = 2 * 2^k - 1 = 2^{k+1} - 1$ Q.E.D.

(D) When the loop terminates i=n. Thus after the loop execution we have: $y = 2^n - 1$