Harvard University
Computer Science 20

In-Class Problems 6

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Executive Summary

1. Properties of binary relations
   - *Transitive*: A binary relation R on the set A is transitive iff $uRv \wedge vRw \implies uRw$ for all $u, v, w \in A$.
   - *Reflexive*: uRu for all $u \in A$.
   - *Irreflexive*: $\neg(uRu)$ for all $u \in A$
   - *Symmetric*: $uRw \implies wRu$ for all $u, w \in A$.
   - *Antisymmetric*: $uRw \implies \neg(wRu)$ for all $u, w \in A$, $u \neq w$.
   - *Asymmetric*: $uRw \implies \neg(wRu)$ for all $u, w \in A$.
2. Recall that G is a binary relation on V, where uGw means that there is an edge from u to w.
   - G^+ is transitive and is the *transitive closure* of G. This means that G^+ is the minimal transitive relation that includes G (i.e. $G \subseteq G^+$).
   - G^* is reflexive, transitive, and the *reflexive transitive closure* of G.
3. The vertices $u, v \in V$ are *strongly connected* iff $uG^*v \wedge vG^*u$. That is, if there exists a walk from u to v and a walk back from v to u.
4. Special types of relations
   - *Strict partial orders*: transitive and irreflexive
   - *Weak partial orders*: transitive, reflexive, and antisymmetric
   - *Equivalence relations*: transitive, reflexive, and symmetric
   - A relation R is a weak partial order iff $R = D^*$ for some DAG D
   - A relation R is a equivalence relation iff R is the strongly connected relation of some digraph
5. An equivalence relation R decomposes the domain into subsets called *equivalence classes*, where aRb iff a and b are in the same equivalence class.

PROBLEM 1

Draw one directed graph with 3 vertices A, B, C for each of the following relationships
(A) Reflexive
(B) Symmetric

(C) Antisymmetric
(D) Transitive

Solution.

PROBLEM 2

Prove that if a relation R is transitive and irreflexive, then it is asymmetric.

Solution.

PROBLEM 3

Say that a string x overlaps a string y if there exist strings p, q, r such that $x = pq$ and $y = qr$, with $q \neq \epsilon$. For example, $abcde$ overlaps $cdefg$, but does not overlap bcd or $cdab$. Answer each of the following questions and prove your answer, or provide a counterexample.

(A) Is the overlap relation reflexive?
(B) Is it symmetric?
(C) Is it transitive?

Solution.

PROBLEM 4

Determine what properties each of the following relations have. For those that are equivalence relations, briefly describe what the equivalence classes are in the relation.

(A) The relation "shares a class with", where two people share a class if there is a class they are both enrolled in this semester.
(B) The relation R on $\mathbb{Z}$, where aRb if b is a multiple of a.
(C) The relation R on $\mathbb{Z} \times \mathbb{Z}$, with $(a, b)\ R\ (c, d)$ if $ad = bc$.

Solution.