

Harvard University
Computer Science 20
In-Class Problems 14
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Executive Summary

1. Some set notation

Given a set $S = \{0, 1\}$, we have that:

- $\{0, 1\}^n$ is the set of strings of exactly length n : e.g. $01001 \in \{0, 1\}^5$.
- $\{0, 1\}^*$ is the set of strings of finite length, including the empty string: e.g. $010010001 \in \{0, 1\}^*$.
- $\{0, 1\}^\omega$ is the set of sequences of infinite length: e.g. $010010001 \dots$.
NOTE: We say “sequence” because strings are defined to have finite length (i.e. they are finite sequences).
- The collection of all subsets of S is its power set, denoted $\mathcal{P}(S)$. Note that $\emptyset \in \mathcal{P}(S)$ for all S .

2. Countable sets

- Two finite sets A and B have the same cardinality if there is a bijection between them: i.e. $A \text{ bij } B$.
- An infinite set A is called *countably infinite* if $A \text{ bij } \mathbb{N}$.
- The set of all integers \mathbb{Z} is countably infinite.
- For finite sets A and B , A is a proper subset of B if $A \subseteq B$ and $|A| < |B|$. For countably infinite sets this is not necessarily so!
- Countably infinite sets are closed under the following operations: subset, intersection, Cartesian product and countably infinite union.
- We use “countable” to refer to sets that are finite or countably infinite.

3. Uncountable sets

- **Cantor’s Theorem:** For any set A , the cardinality of $\mathcal{P}(A)$ is greater than that of A , i.e. a bijection f does not exist between A and $\mathcal{P}(A)$.
- Proof approach: Given a bijection f , consider the set W consisting of elements in A that are matched to elements in $\mathcal{P}(A)$ that do not contain them (remember, an element in $\mathcal{P}(A)$ is a subset!). By the definition of f , some element in A must match to W since W is a subset of A and thus an element of $\mathcal{P}(A)$, but by the definition of W no element in A can match to W , which is a contradiction.
- Uncountable sets: S^ω for any set S such that $|S| > 1$, $\mathcal{P}(\mathbb{N})$, and the set of real numbers within any interval.

PROBLEM 1

Suppose $S = \{0, 1\}^*$. Which of the following sets are countable?

- (A) The union of two finite sets
- (B) The powerset of a countably infinite set
- (C) The union of a finite set and a countably infinite set
- (D) The powerset of a finite set
- (E) $\bigcup_{i \geq 0} S_i$, where $S_i = \{s \mid s \in S, |s| = i\}$
- (F) $S \times S$
- (G) The set of all functions from \mathbb{N} to \mathbb{N}

PROBLEM 2

Show that the difference of an uncountable set and a countable set is uncountable.

PROBLEM 3

Show that the Cartesian product $\mathbb{N} \times \mathbb{N} = \{(a, b) \mid a, b \in \mathbb{N}\}$ is countably infinite by creating a bijection between \mathbb{N} and $\mathbb{N} \times \mathbb{N}$.

PROBLEM 4

(BONUS) In the Infinity Inn (a Hilton brand) there is a countably infinite number of rooms available for booking. Attracted to the novelty of the building's architecture, a countably infinite number of people arrive on vacation, and quickly occupy all of the rooms in the hotel.

- (A) A new celebrity guest arrives at the hotel and demands a room. Devise a method to move each current hotel resident to a new room to open up a room for the incoming guest.
- (B) News of the hotel spreads to a parallel universe, and another countably infinite number of people arrive at the already-booked hotel. Figure out a new method to move each current hotel resident to a new room to make space for all of the new guests. (To learn more about this problem, search for "Hilbert's Hotel" online!)