Harvard University
Computer Science 20

In-Class Problems 14

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Executive Summary

1. **Some set notation**

   Given a set $S = \{0, 1\}$, we have that:

   - $\{0,1\}^n$ is the set of strings of exactly length n: e.g. $01001 \in \{0,1\}^5$.
   - $\{0,1\}^*$ is the set of strings of finite length, including the empty string: e.g. $010010001 \in \{0,1\}^*$.
   - $\{0,1\}^\omega$ is the set of sequences of infinite length: e.g. $010010001 \cdots$.
     NOTE: We say "sequence" because strings are defined to have finite length (i.e. they are finite sequences).
   - The collection of all subsets of S is its power set, denoted $\mathcal{P}(S)$. Note that $\emptyset \in \mathcal{P}(S)$ for all S.

2. **Countable sets**

   - Two finite sets A and B have the same cardinality if there is a bijection between them: i.e. A bij B.
   - An infinite set A is called *countably infinite* if A bij $\mathbb{N}$.
   - The set of all integers $\mathbb{Z}$ is countably infinite.
   - For finite sets A and B, A is a proper subset of B if $A \subseteq B$ and $|A| < |B|$. For countably infinite sets this is not necessarily so!
   - Countably infinite sets are closed under the following operations: subset, intersection, Cartesian product and countably infinite union.
   - We use "countable" to refer to sets that are finite or countably infinite.

3. **Uncountable sets**

   - **Cantor's Theorem:** For any set A, the cardinality of $\mathcal{P}(A)$ is greater than that of A, i.e. a bijection f does not exist between A and $\mathcal{P}(A)$.
   - Proof approach: Given a bijection f, consider the set W consisting of elements in A that are matched to elements in $\mathcal{P}(A)$ that do not contain them (remember, an element in $\mathcal{P}(A)$ is a subset!). By the definition of f, some element in A must match to W since W is a subset of A and thus an element of $\mathcal{P}(A)$, but by the definition of W no element in A can match to W, which is a contradiction.
   - Uncountable sets: S^ω for any set S such that $|S| > 1$, $\mathcal{P}(\mathbb{N})$, and the set of real numbers within any interval.

PROBLEM 1

Suppose $S = \{0, 1\}^*$. Which of the following sets are countable?
(A) The union of two finite sets
(B) The powerset of a countably infinite set
(C) The union of a finite set and a countably infinite set
(D) The powerset of a finite set
(E) $\bigcup_{i \geq 0} S_i$, where $S_i = \{s \mid s \in S,\ |s| = i\}$
(F) $S \times S$
(G) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$

Solution.
A and C are countable since countable sets are closed under union with a finite number of countable sets. B is uncountable by definition. D is countable since it is a finite set. E is equivalent to S, which is countable. F is countable since we know that countable sets are closed under Cartesian product. G is uncountable using Cantor's diagonlization argument.

PROBLEM 2

Given the finite set S with n unique elements, what is the cardinality of $\mathcal{P}(S)$?

Solution.
For every subset of S in $\mathcal{P}(S)$, each element in S is either included in the subset or not included. This gives us a total of $\boxed{2^n}$ unique subsets that are elements in $\mathcal{P}(S)$.

PROBLEM 3

Show that the difference of an uncountable set and a countable set is uncountable.

Solution.
We perform a proof by contradiction. For an arbitrary uncountable set A and an arbitrary countable set B, we assume first that $A - B$ is countable. Next, we note that $A - B = A - A \cap B$. But $A \cap B$ is countable since $|A \cap B| \leq |B|$ and B is countable. Since the union of countable sets is countable, this implies that $(A - A \cap B) \cup (A \cap B) = A$ is countable as well, a contradiction.

PROBLEM 4

(BONUS) Show that the Cartesian product $\mathbb{N} \times \mathbb{N} = \{(a, b) | a, b \in \mathbb{N}\}$ is countably infinite by creating a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

Solution.
To create a bijection f we first note that for any positive integer n, there are exactly n elements (a, b) in $\mathbb{N} \times \mathbb{N}$ such that $a + b = n + 1$. Hence, we can map the first natural number 1 to elements in $\mathbb{N} \times \mathbb{N}$ whose components sum to 2 (in this case $(1, 1)$), the next two natural numbers to elements whose components sum to 3 in order from smallest-to-largest first component ($(1, 2)$ and $(2, 1)$), and so on. We know that f is surjective since every element in $\mathbb{N} \times \mathbb{N}$ can be mapped to in this way, and f is injective because for every positive integer n we can map n unique elements in $\mathbb{N}$ to n distinct elements (a, b) in $\mathbb{N} \times \mathbb{N}$ where $a + b = n + 1$. Therefore, f is a bijection.