Your Name: ____________

Harvard University
Computer Science 20

Midterm 2

Wednesday, March 23, 2016

PROBLEM 1

For each of the following, state whether the set is finite, countably infinite, or uncountable. No justification required.

(A) The set of all total functions with domain $\{0, 1\}$ and codomain $\{0, 1\}$.
(B) The set of all total functions with domain $\mathbb{N}$ and codomain $\{1\}$.
(C) The set of all total functions with domain $\mathbb{N}$ and codomain $\{0, 1\}$.
(D) The set of all total functions with domain $\{0, 1\}$ and codomain $\mathbb{N}$.

Solution.

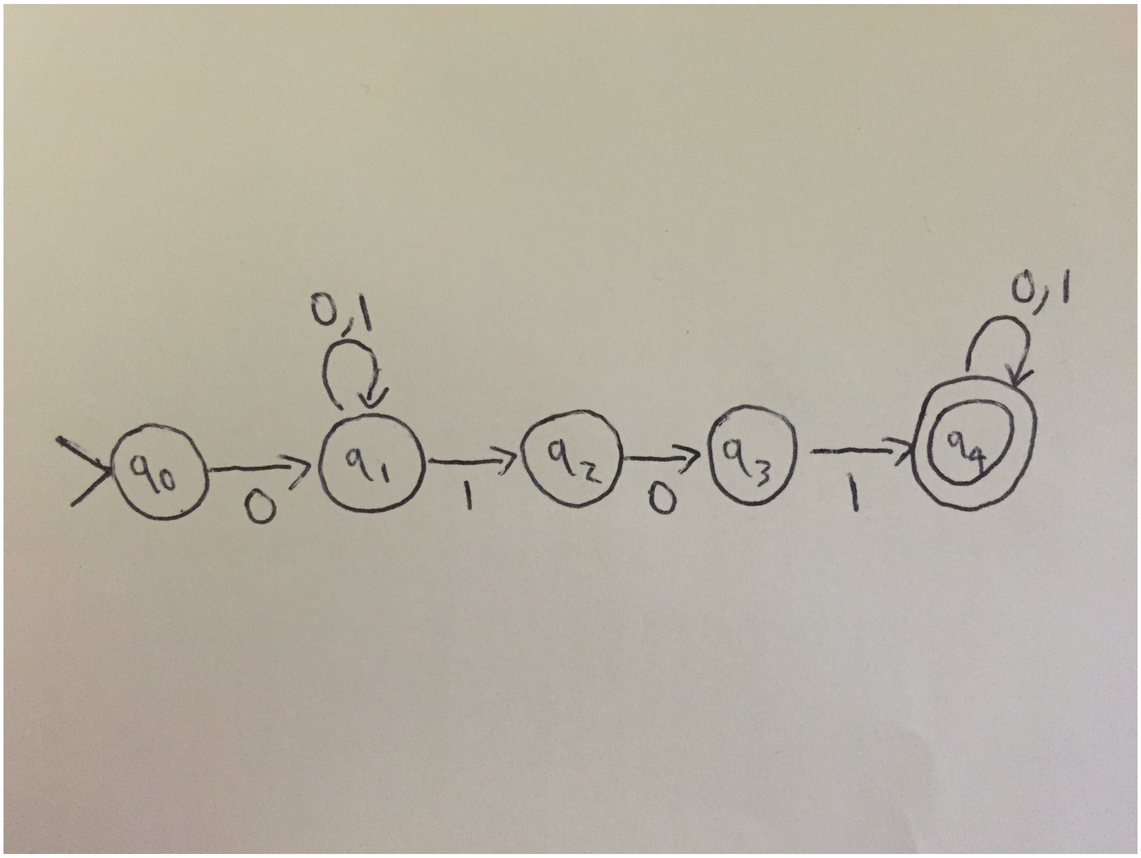
(A) Finite
(B) Finite
(C) Uncountable
(D) Countably Infinite

PROBLEM 2

Draw state machines that only accept strings in the following set. Assume that the alphabet is $\Sigma = \{0, 1\}$; that is, for all possible input strings s we have $s \in \Sigma^*$.

$$\{w : w \text{ starts with 0 and contains the substring 101, i.e. } w = 0x101y \text{ for some } x \text{ and } y\}$$

Solution.



PROBLEM 3

Let G be a directed graph with n vertices. Show that if G has a path of length greater than n, then G has a cycle (a path has length k if it contains k edges).

Solution.

Proof by contradiction: assume there is a path of length $m > n$. The path can be written as: $(v_0, v_1), (v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)$. There are $m + 1 > n + 1$ vertices on the path. By the pigeonhole principle, at least 2 of those vertices are equal as the graph only has n vertices. Let the index of the first instance of the vertex in the path be i and the index of the second instance of the vertex be j, then there is a cycle starting and ending at that vertex with the path $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$.

PROBLEM 4

Let $G = (V, E)$ be a directed acyclic graph. Define a relation R on V by (v_1, v_2) which is an element of R iff there is a path from v_1 to v_2.

(A) Is R reflexive? Prove your answer.
(B) Is R symmetric? Prove your answer.
(C) Is R transitive? Prove your answer.

Solution.

(A) R is not reflexive. A DAG contains no cycles, and therefore there cannot be any paths from a vertex to itself.

(B) R is not symmetric. Proof by contradiction: suppose R is symmetric, and that there is a path from $(v_1, v_2) \in R$. Then there must be a path from v_2 to v_1, which implies that $(v_1, v_1) \in R$. We have already shown that this cannot be the case in a DAG, so R cannot be symmetric.

(C) R is transitive. If there is a path from v_1 to v_2 and a path from v_2 to v_3, we can concatenate the two paths to construct a path from v_1 to v_3.

PROBLEM 5

(A) Using set-builder notation, give a formal description of the union of two sets A and B.
(B) Using set-builder notation, give a formal description of the complement of a set A.
(C) Let $|A| = n$ and $|B| = m$. If $A \subseteq B$, what is $|A \cap B|$?
(D) Let $|A| = n$ and $|B| = m$. If $A \subseteq B$, what is $|A - B|$?
(E) What is the power set of $\{h, a, i\}$?

Solution.

(A) $\{x : x \in A \vee x \in B\}$
(B) $\{x : x \notin A\}$
(C) n
(D) 0
(E) $\{\emptyset, \{h\}, \{a\}, \{i\}, \{h, a\}, \{h, i\}, \{a, i\}, \{h, a, i\}\}$

PROBLEM 6

Let S be the set defined as follows:

- Base Case: $(1, 2) \in S$
- Constructor Rules: If $(x, y) \in S$, then C1: $(x + 2, y) \in S$ and C2: $(y, x) \in S$

Use structural induction to prove that for any pair (x,y) in S, x and y can not both be odd or both be even.

Solution.

- Basis step: By the base case of the definition of S, $(1, 2) \in S$. 1 is even and 0 is odd.
- Recursive step:
  Assume elements (x,y) in S and P(x,y): *x and y are not both be odd or both be even* holds. Now consider the constructor rule in the definition of S. We want to prove P(x+2,y) and P(y,x) also holds.

  Case 1: x is even and y is odd,

  x+2 would be even, y would be odd $\rightarrow$ P(x+2,y) holds.

  y would be odd, x would be even $\rightarrow$ P(y,x) holds.

  Case 2: x is odd and y is even,

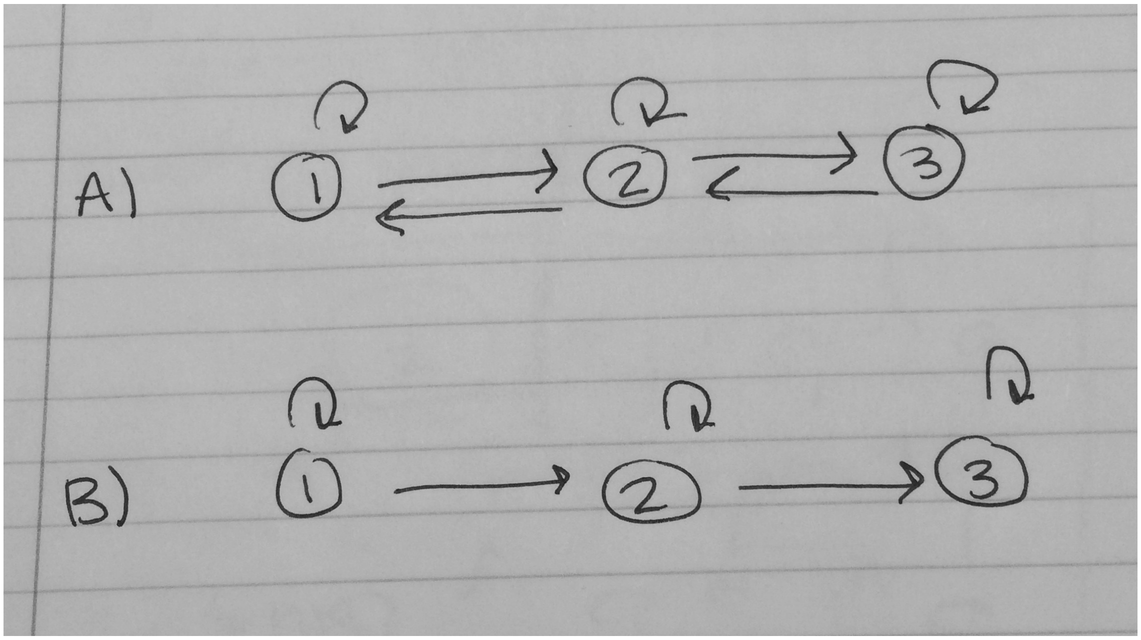
  x+2 would be odd, y would be even $\rightarrow$ P(x+2,y) holds.

  y would be even, x would be odd $\rightarrow$ P(y,x) holds.

PROBLEM 7

Please provide a graph of binary relations over the set $\{1, 2, 3\}$ that fulfils the following properties, or explain why it would be impossible to construct one.
(A) Symmetric transitive
(B) Asymmetric, reflexive.
(C) Transitive, symmetric, irreflexive.

Solution.



C) It would be impossible to provide a graph of relations that are transitive, symmetric, and irreflexive. If the relation between two nodes in the graph is symmetric and transitive, it must also be reflexive. Thus, we have have a contradiction since it cannot be reflexive if it is irreflexive.