Harvard University
Computer Science 20

In-Class Problems 12

Wednesday, February 24, 2016

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Executive Summary

- A set is simply a "bunch of objects." The objects that comprise a set are called its *elements* or *members*.
- A set is determined by its elements: two sets are equal if and only if they have exactly the same elements. For this reason, sets are not inherently ordered and elements of a set cannot "appear more than once" in the set.
- A set A is a *subset* of a set B, denoted by $A \subseteq B$, if every object in A is also in B.
- The *union* of two sets A and B, denoted by $A \cup B$, is the set of all objects that are either in A or in B (or both).
- The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of all objects that are in both A and B.
- The *product* of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (x, y) such that x is an element of A (written $x \in A$) and y is an element of B (written $y \in B$).
- The *difference* of A and B, denoted by $A - B$ or $A \setminus B$, is the set of elements that are in A but are not in B.
- The *complement* of a set X (in a domain D) is the set $\overline{X} = \{y \in D : y \notin X\}$ or $\overline{X} = \{y \in D : \neg(y \in X)\}$ containing all objects that are not elements of X.
- The *power set* of A, denoted by $P(A)$, is the set of all subsets of A.
- We often use *set-builder notation* as a concise way to describe what it means for a particular element x to be a member of a set S, denoting the set of all $x \in D$ satisfying the property P by $\{x \in D \mid P(x)\}$.
- A set is *finite* if it can be put in one-to-one correspondence with a bounded sequence of natural numbers $(1, \dots, n)$.
- The *cardinality* of a finite set A, written $|A|$, is the number of elements in that set.

PROBLEM 1

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, 4, 5\}$.

(A) What is $A \cap B$?
(B) What is $A \cup B$?
(C) What is $|A \times B|$?
(D) What is $|P(A)|$?

Solution.

(A) $\{4, 5\}$
(B) $\{1, 2, 3, 4, 5, a, b, c, d\}$
(C) 30
(D) 2^5

PROBLEM 2

Using set-builder notation, give formal descriptions of the following sets:

(A) The product of two sets X and Y.
(B) The difference between two sets X and Y.
(C) The power set of a set X, denoted $P(X)$.

Solution.

(A) $XY = \{(x, y) \mid x \in X \wedge y \in Y\}$
(B) $X - Y = \{z \mid z \in X \wedge z \notin Y\}$
(C) $P(X) = \{A \mid A \subseteq X\}$

PROBLEM 3

(A) Explain why $|X \cup Y| \neq |X| + |Y|$.
(B) Provide a formula for $|X \cup Y|$ in terms of $|X|$, $|Y|$, and $|X \cap Y|$.
(C) (BONUS) Generalize this formula to unions of more than two sets.

Solution.

(A) $|X \cup Y| \neq |X| + |Y|$ if X and Y have any elements in common. Each element that is in both X and Y is counted twice in $|X| + |Y|$, but only counted once in $|X \cup Y|$.
(B) $|X \cup Y| = |X| + |Y| - |X \cap Y|$
(C)

$$\bigcup_{i=1}^{n} S_i = (\sum_{i=1}^{n} |S_i|) - (\sum_{i \leq i < j \leq n} |S_i \cap S_j|) + (\sum_{i \leq i < j < k \leq n} |S_i \cap S_j \cap S_k|) - \cdots + ((-1)^{n-1} | \bigcap_{i=1}^{n} S_i|)$$

PROBLEM 4

(BONUS) There are 100 students enrolled in at least one of the following classes: CS20, CS51, and Math 21b. There are 60 students enrolled in CS20, 70 students enrolled in CS51, 30 students enrolled in Math 21b, and 10 students enrolled in all three classes.

(A) Let A, B, and C represent the set of all students in CS20, CS51, and Math 21b respectively. Represent the information given above using set union, intersection, and cardinality.
(B) How many students are enrolled in exactly two of the classes?

Solution.

(A) $|A \cup B \cup C| = 100$, $|A| = 60$, $|B| = 70$, $|C| = 30$, $|A \cap B \cap C| = 10$
(B) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
Let t be the number of students enrolled in exactly two classes.
$100 = 160 - t + 10$
$t = 70$