Harvard University
Computer Science 20

In-Class Problems 15

Wednesday, March 02, 2016

Author: Crystal Chang

Executive Summary

1. **Definition of Recursive Data Types:** common way of defining mathematical objects, which says how to construct new data elements from previous ones.
   - *Base Case(s):* specify that some known mathematical elements are in the data type
   - *Constructor Rule(s):* specify how to construct new data elements from previously constructed elements or from base elements.
   - *Nothing else (generally implicit):* the only way you can get whatever is you defining is by starting from the base case(s) and applying the constructor rule(s) one or more times.
2. **The Principle of Structural Induction:** to prove P(x) holds for all x in a recursively defined set S, prove
   - *Basis Step:* P(b) is true for each base case element $b \in S$, and
   - *Recursive Step:* P(c(x_1,...,x_k)) for each constructor c, assuming as the induction hypothesis that P(x_1),..., and P(x_k) all hold.

PROBLEM 1

Recursive Definition:
(A) There's an error in the following definition of the set of even integers (E). Find the error and fix it.

- Base Case: $0 \in E$
- Constructor Rule: For any element x in E, x+2 is in E.
- Nothing else (generally implicit): Nothing is in E unless it is obtained from the base case and constructor rule.

(B) Give a recursive definition of the natural numbers $\mathbb{N}$.
(C) Give a recursive definition of the sequence b_n, $b_n = 2n + 5, n \in \mathbb{N}$

PROBLEM 2

Let S be the set defined as follows:

- Base Case: $(1, 2) \in S$
- Constructor Rules: If $(x, y) \in S$, then C1: $(x + 2, y) \in S$ and C2: $(y, x) \in S$

(A) Is $(4, 3) \in S$? If it is, how can you derive it from (1,2)?
(B) Use induction to prove that $(2n + 2, 2n + 1) \in S$ for all n $\in \mathbb{N}$.

PROBLEM 3

(Bonus) A palindrome is a sequence of characters (do not need to be a word) which reads the same backward or forward e.g. dad, mom, abba, moom... Let's define the set $\sum$ as the set of all letters $\{a, b, c, ..., z\}$, λ as the empty string and P as the set of all palindromes(excepting empty string). Give a recursive definition for the set P.

PROBLEM 4

(Bonus) Let S be the set defined as follows:

- Base Case: $(0, 0) \in S$
- Constructor Rules: If $(a, b) \in S$, then C1: $(a, b + 1) \in S$, C2: $(a + 1, b + 1) \in S$ and C3: $(a + 2, b + 1) \in S$

(A) List 5 elements in set S.
(B) Use structural induction to prove that for every $(a, b) \in S, a \leq 2b$.