Harvard University
Computer Science 20

In-Class Problems 11

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Midterm Review This in-class midterm review is a puzzle. Each problem gives you one piece of the puzzle. Solve all the problems and put together the pieces to discover the keyword. Enjoy!

PROBLEM 1

What follows is alleged to be a proof of the "Anti-Friendship Theorem": that for all n, there exists a set of n people for which all subsets of 3 people are neither all friends nor all enemies. Find the *first* line that contains a **logical flaw**, if one exists. (If something is "unnecessary" but logically true, it is not a logical flaw.)

1. Proceeding by induction, we define the predicate $P(n)$ = there exists a set of n people for which all subsets of 3 people are neither all friends nor all enemies.
2. Base cases: Suppose Alice and Bob are friends, Bob and Charlotte are friends, but Alice and Charlotte are not friends. $P(3)$ evidently holds. Add Dianne, a friend of Bob but not of Alice or Charlotte. Thus $P(4)$ holds.
3. Add Eric, a friend of Alice and Charlotte but not Bob or Dianne. Thus $P(5)$ holds as well.
4. Inductive step: Assume for that $n \geq 5$, $P(3), P(4), ..., P(n)$ holds.
5. Let S be a set of $n+1$ people. We can split S into two smaller sets, one with n people and another with 1 person. Call the 1 lonely person $s_0 \in S$ and the set of n people S_n.
6. By the inductive hypothesis, any 3 people in S_n are neither all friends nor all enemies.
7. For a subset of 3 in S_n, we can take those 3 and add s_0 to create a set with 4 people.
8. We showed in the base case that $P(4)$ holds. Therefore this new set of 4 people must contain a subset of 3 who are neither all friends nor all enemies.
9. These 4 people were all from the original set of size $n+1$, so we have found the set we're looking for to prove $P(n+1)$. By induction, the Anti-Friendship Theorem holds.
10. No flaw exists.

1	2	3	4	5	6	7	8	9	10
G	R	S	A	H	L	M	E	O	B

Table 1: Problem 1 Clue

Solution.
The flaw occurs on line 8. Just because we "split" the original set doesn't meant that friendship relationships were erased.

PROBLEM 2

Construct a truth table for $(p \leftrightarrow q) \rightarrow (q \leftrightarrow r)$. How many rows in the truth table are True?

0	1	2	3	4	5	6	7	8	>8
E	N	A	V	B	R	I	Z	X	Q

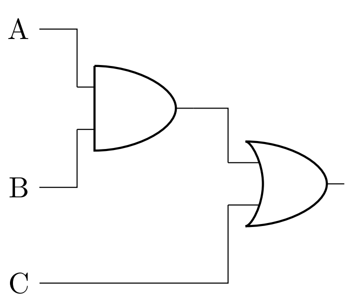
Table 2: Problem 2 Clue

Solution.

6 rows are True:

p	q	r	$(p \leftrightarrow q) \rightarrow (q \leftrightarrow r)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	1

PROBLEM 3



Which of the following does the above logic circuit compute?

1. $A \cdot B + C$
2. $A + B + C$
3. $A + B \cdot C$
4. $A \cdot B \cdot C$
5. $(A + B) \cdot C$

Solution.
(1).

1	2	3	4	5
S	E	O	T	R

Table 3: Problem 3 Clue

PROBLEM 4

Use the quantificational logic statements below to deduce which of the propositions is True.

1. $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}. n \cdot m = 1$
2. $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}. n + m = n$
3. $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n + m = n$
4. $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n \cdot m = n$
5. $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}. n \cdot m = m$

1	2	3	4	5
M	N	R	E	L

Table 4: Problem 4 Clue

Solution.
(5) is true; $n = 1$.

PROBLEM 5

What is the least value of m for which the following is true? "In any set of m propositions, all involving only p, two of the propositions are logically equivalent."

2	5	9	12	17	26	50	> 50
A	W	B	K	U	O	N	T

Table 5: Problem 5 Clue

Solution.
(BONUS) By the pigeonhole principle, 5, since there are only 4 possible truth tables.

PROBLEM 6

Final Answer:

Solution.
LEWIS