Harvard University
Computer Science 20

Midterm 1

Monday, February 22, 2016

PROBLEM 1

Prove that if you pick 5 integers from $\{1, \ldots, 100\}$, some two differ by at most 24.

Solution.
Divide the integers $\{1, \ldots, 100\}$ into the 4 sets of integers $\{1, \ldots, 25\}$, $\{26, \ldots, 50\}$, $\{51, \ldots, 75\}$, and $\{76, \ldots, 100\}$. Let each of this four sets be a pigeonhole. We are picking 5 integers from $\{1, \ldots, 100\}$. Let each of these five integers be a pigeon. By the pigeonhole principle there must be two integers in the same set.

Since the smallest and largest integers in each set differ by only 24, the 2 integers from the same set can differ by at most 24, and so there must be some 2 of our 5 integers that differ by at most 24.

PROBLEM 2

Let $A = \neg(\neg p \vee q) \rightarrow r$ and $B = \neg r \oplus (\neg p \wedge q)$. For which value(s) of p, q, and r do A and B differ? Use a truth table.

Solution.
They differ only for $p = 1$, $q = 0$, and $r = 0$.

p	q	r	$\neg(\neg p \vee q) \rightarrow r$	$\neg r \oplus (\neg p \wedge q)$
1	1	1	0	0
1	1	0	1	1
1	0	1	1	1
1	0	0	1	0
0	1	1	0	0
0	1	0	1	1
0	0	1	0	0
0	0	0	1	1

PROBLEM 3

Perform the following operations in binary.

(A) $1111_2 + 1111_2$
(B) $1010_2 - 100_2$

Solution.

(A) 11110_2
(B) 110_2

PROBLEM 4

Let the domain of discourse be all Harvard CS courses. The predicate $P(c,d)$ means that course c is a prerequisite for course d. Assume that it is not possible for a course to be a prerequesite of itself. Write the following English sentences using quantificational formulas.

(A) There is a course that is a prerequisite for every other course.
(B) At least one course is a prerequisite for exactly one other course.

Solution.

(A) $\exists c \forall d : (d \neq c) \implies F(c,d)$
(B) $\exists c, a. \forall b : (a \neq c) \wedge F(c,a) \wedge (F(c,b) \implies a = b)$

PROBLEM 5

The Tribonacci numbers are defined by $T_0 = 1, T_1 = 1, T_2 = 2$, and $T_n = T_{n_1} + T_{n-2} + T_{n-3}$ for all $n \geq 3$. The beginning of the Tribonacci sequence is $1, 1, 2, 4, 7, 13, ...$ Use strong induction to prove that $T_n \leq 3^n$ for all natural numbers n.

Solution.
Let $P(n)$ be the predicate $T_n \leq 3^n$. Base cases: $P(0)$ is true, since $T_0 = 1 \leq 3^0 = 1$. $P(1)$ is true, since $T_1 = 1 \leq 3^1 = 3$. $P(2)$ is also true, since $T_2 = 2 \leq 3^2 = 9$. Inductive step: assume $P(1), ..., P(n)$ holds for $n \geq 2$. Then $T_{n+1} = T_n + T_{n-1} + T_{n-2}$
$\leq 3^n + 3^{n-1} + 3^{n-2}$
$= 3^{n+1}(\frac{1}{3} + \frac{1}{9} + \frac{1}{27})$
$= 3^{n+1}\frac{13}{27}$
$\leq 3^{k+1}$

PROBLEM 6

Prove that in any group of six people, at least two of them know the same number of people. Note that you don't know yourself, and that if A knows B then B knows A (thus "knows" is a symmetric relation).

Solution.

There are 6 possible numbers of people a person can know (0 through 5 since you cannot know yourself).

Suppose that each person knows a different number of people. Then someone (person A) knows 0 people and someone (person B) else knows 5 people. Since B knows 5 people and cannot know him or herself, then B knows A. Since knowing is a symmetric relation, A knows B. This is a contradiction because A knows 0 people.

Since we have arrived at a contradiction, we can conclude that in any group of six people, at least two of them know the same number of people.