

Harvard University  
Computer Science 20

In-Class Problems 15

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**Executive Summary**

1. **Definition of Recursive Data Types:** common way of defining mathematical objects, which says how to construct new data elements from previous ones.
  - *Base Case(s):* specify that some known mathematical elements are in the data type
  - *Constructor Rule(s):* specify how to construct new data elements from previously constructed elements or from base elements.
  - *Nothing else (generally implicit):* the only way you can get whatever is you defining is by starting from the base case(s) and applying the constructor rule(s) one or more times.
2. **The Principle of Structural Induction:** to prove  $P(x)$  holds for all  $x$  in a recursively defined set  $S$ , prove
  - *Basis Step:*  $P(b)$  is true for each base case element  $b \in S$ , and
  - *Recursive Step:*  $P(c(x_1, \dots, x_k))$  for each constructor  $c$ , assuming as the induction hypothesis that  $P(x_1), \dots$ , and  $P(x_k)$  all hold.

PROBLEM 1

Recursive Definition:

(A) There's an error in the following definition of the set of even integers (EI). Find the error and fix it.

- Base Case:  $0 \in EI$
- Constructor Rule: For any element  $x$  in EI,  $x+2$  is in EI.
- Nothing else (generally implicit): Nothing is in EI unless it is obtained from the base case and constructor rule.

(B) Give a recursive definition of the natural numbers  $\mathbb{N}$ .

(C) Give a recursive definition of the sequence  $b_n$ ,  $b_n = 2n + 5, n \in \mathbb{N}$

**Solution.**

(A) It doesn't include negative Even Integer. There should be one more constructor rule x-2 is in EI.

(B)

- Base Cases:  $0 \in \mathbb{N}$
- Constructor Rule: If  $n \in \mathbb{N}$ , then  $n + 1 \in \mathbb{N}$ .
- Nothing else (generally implicit): Nothing is in  $\mathbb{N}$  unless it is obtained from the base case and constructor rule.

(C)

- Base Cases:  $b_1 = 7$
- Constructor Rule:  $b_{n+1} = b_n + 2$  for  $n \in \mathbb{N}$
- Nothing else (generally implicit): Nothing is in  $b_n$  unless it is obtained from the base case and constructor rule.

**PROBLEM 2**

Let  $S$  be the set defined as follows:

- Base Case:  $(1, 2) \in S$
- Constructor Rules: If  $(x, y) \in S$ , then C1:  $(x + 2, y) \in S$ , C2:  $(y, x) \in S$

(A) Is  $(4, 3) \in S$ ? If it is, how can you derive it from  $(1, 2)$ ?

(B) Use induction to prove that  $(2n + 2, 2n + 1) \in S$  for all  $n \in \mathbb{N}$ .

**Solution.**

(A) Yes, it is. Apply C1 to  $(1, 2)$ , we can get  $(3, 2)$ ; Apply C2 to  $(3, 2)$ , we can get  $(2, 3)$ ; Apply C1 to  $(2, 3)$ , we can get  $(4, 3)$ .

(B) Let  $P(n)$ :  $(2n + 2, 2n + 1) \in S$ . We must show that for all  $n \in \mathbb{N}$ ,  $P(n)$ .

- Base case: When  $n=1$ ,  $(4, 3) \in S$ . (Already proved in (A))
- Induction step:  
Assuming  $P(n)$ :  $(2n + 2, 2n + 1) \in S$  holds, we want to prove that  $P(n + 1)$ :  $(2(n + 1) + 2, 2(n + 1) + 1) = (2n + 4, 2n + 3) \in S$ .  
Apply C1 to  $(2n+2, 2n+1)$ , we then could get  $(2n + 4, 2n + 1) \in S$ . Then apply C2 to  $(2n+4, 2n+1)$ , we could get  $(2n + 1, 2n + 4) \in S$ . Finally, apply C1 to  $(2n+1, 2n+4)$ , then we can get  $(2n + 3, 2n + 4) \in S$ .

**PROBLEM 3**

Let  $S$  be the set defined as follows:

- Base Case:  $(0, 0) \in S$
- Constructor Rules: If  $(a, b) \in S$ , then C1:  $(a, b + 1) \in S$ , C2:  $(a + 1, b + 1) \in S$  and C3:  $(a + 2, b + 1) \in S$

(A) List 5 elements.

(B) Use structural induction to prove that for every  $(a, b) \in S, a \leq 2b$ .

**Solution.**

(A)  $(0, 1), (1, 1), (2, 1), (1, 2), (2, 2) \dots$

(B)

- Basis step: By the base case of the definition of  $S$ ,  $(0, 0) \in S$ .  $0 \leq 2(0)$ .
- Recursive step:  
Now consider the constructor rule in the definition of  $S$ . Assume elements  $a, b \in S$  and  $a \leq 2b$ . We must show that  $a \leq 2(b + 1)$ ,  $(a + 1) \leq 2(b + 1)$  and  $(a + 2) \leq 2(b + 1)$ .
  1. prove  $a \leq 2(b + 1)$ :  $a \leq 2b \rightarrow a \leq (2b + 2) = 2(b + 1)$
  2. prove  $(a + 1) \leq 2(b + 1)$ :  $a \leq 2b \rightarrow (a + 1) \leq (2b + 1) \leq (2b + 2) = 2(b + 1)$
  3. prove  $(a + 2) \leq 2(b + 1)$ :  $a \leq 2b \rightarrow (a + 2) \leq (2b + 2) = 2(b + 1)$

#### PROBLEM 4

(Bonus) Construct a recursive definition for the set of strings  $S$  over the alphabet  $a, b$  excepting empty string, i.e. set of string consisting of  $a$ 's and  $b$ 's such as  $abbab$ ,  $bbabaa$ , etc.

**Solution.**

- Base Cases:  $a \in S$ , and  $b \in S$
- Constructor Rule: For any element  $x$  in  $S$ ,  $ax \in S$ , and  $bx \in S$
- Nothing else (generally implicit): Nothing is in  $S$  unless it is obtained from the base case and constructor rule.