Harvard University
Computer Science 20

In-Class Problems 14

Monday, February 29, 2016

Author: Ben Zheng

Executive Summary

1. **Some set notation**

   Given a set $S = \{0, 1\}$, we have that:

   - $\{0, 1\}^n$ is the set of strings of exactly length n: e.g. $01001 \in \{0, 1\}^5$.
   - $\{0, 1\}^*$ is the set of strings of finite length, including the empty string: e.g. $010010001 \in \{0, 1\}^*$.
   - $\{0, 1\}^\omega$ is the set of sequences of infinite length: e.g. $010010001 \cdots$.
     NOTE: We say "sequence" because strings are defined to have finite length (i.e. they are finite sequences).
   - The collection of all subsets of S is its power set, denoted $\mathcal{P}(S)$. Note that $\emptyset \in \mathcal{P}(S)$ for all S.

2. **Countable sets**

   - Two finite sets A and B have the same cardinality if there is a bijection between them: i.e. A bij B.
   - An infinite set A is called *countably infinite* if A bij $\mathbb{N}$.
   - The set of all integers $\mathbb{Z}$ is countably infinite.
   - For finite sets A and B, A is a proper subset of B if $A \subseteq B$ and $|A| < |B|$. For countably infinite sets this is not necessarily so!
   - Countably infinite sets are closed under the following operations: subset, intersection, Cartesian product and countably infinite union.
   - We use "countable" to refer to sets that are finite or countably infinite.

3. **Uncountable sets**

   - **Cantor's Theorem:** For any set A, the cardinality of $\mathcal{P}(A)$ is greater than that of A, i.e. a bijection f does not exist between A and $\mathcal{P}(A)$.
   - Proof approach: Given a bijection f, consider the set W consisting of elements in A that are matched to elements in $\mathcal{P}(A)$ that do not contain them (remember, an element in $\mathcal{P}(A)$ is a subset!). By the definition of f, some element in A must match to W since W is a subset of A and thus an element of $\mathcal{P}(A)$, but by the definition of W no element in A can match to W, which is a contradiction.
   - Uncountable sets: S^ω for any set S such that $|S| > 1$, $\mathcal{P}(\mathbb{N})$, and the set of real numbers within any interval.

PROBLEM 1

Suppose $S = \{0,1\}^*$. Which of the following sets are countable?
(A) The union of two finite sets
(B) The powerset of a countably infinite set
(C) The union of a finite set and a countably infinite set
(D) The powerset of a finite set
(E) $\bigcup_{i\geq 0} S_i$, where $S_i = \{s \ : \ s \in S, \ |s| = i\}$
(F) $S \times S$
(G) The set of all functions mapping from $\mathbb{N}$ to $\{0,1\}$

PROBLEM 2

Show that for any uncountable set A and countable set B, the set $A - B$ is uncountable.

PROBLEM 3

Show that the Cartesian product $\mathbb{N} \times \mathbb{N} = \{(a,b) : a, b \in \mathbb{N}\}$ is countably infinite by creating a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

PROBLEM 4

(BONUS) In the Infinity Inn (a Hilton brand) there is a countably infinite number of rooms available for booking. Attracted to the novelty of the building's architecture, a countably infinite number of people arrive on vacation, and quickly occupy all of the rooms in the hotel.
(A) A new celebrity guest arrives at the hotel and demands a room. Devise a method to move each current hotel resident to a new room to open up a room for the incoming guest.
(B) News of the hotel spreads to a parallel universe, and another countably infinite number of people arrive at the already-booked hotel. Figure out a new method to move each current hotel resident to a new room to make space for all of the new guests. (To learn more about this problem, search for "Hilbert's Hotel" online!)