Macro Problem Set 2

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1 Neo-Classical Model

1.1 Competitive equilibrium

Arrow-Debreu competitive equilibrium in Neo-Classical model consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ and the allocations for household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that,

(i) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the firm allocation $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ solves the firm problem,

$$
\begin{aligned}
\Pi = \max_{\{y_t,k_t,l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t) \\
\text{s.t. } & y_t = f(k_t, l_t) = z k_t^{\alpha} l_t^{1-\alpha}, \forall t \geq 0; \\
& y_t, k_t, l_t \geq 0, \forall t \geq 0.
\end{aligned}
\tag{1.1}
$$

(ii) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and the profit of firm Π, the household allocation $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves the household problem,

$$
\begin{aligned}
\max_{\{c_t,i_t,x_{t+1},k_t,l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) \\
\text{s.t. } & \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t) + \Pi; \\
& x_{t+1} = i_t + (1-\delta)x_t, \forall t \geq 0; \\
& 0 \leq l_t \leq 1, 0 \leq k_t \leq x_t, c_t \geq 0, x_{t+1} \geq 0, \forall t \geq 0; \\
& x_0 \text{ is given.}
\end{aligned}
\tag{1.2}
$$

(iii) The market clear conditions,

$$
\begin{aligned}
y_t &= c_t + i_t \text{ (goods market);}\\
l_t^d &= l_t^s \text{ (labour market);}\\
k_t^d &= k_t^s \text{ (capital market).}
\end{aligned}
$$

1.2 Steady state

For firm problem,

$$
\begin{aligned}
r_t =& z\alpha k_t^{\alpha-1} l_t^{1-\alpha},\\
w_t =& z(1-\alpha) k_t^{\alpha} l_t^{-\alpha}.\\
y_t =& r_t k_t + w_t l_t = z k_t^{\alpha} l_t^{1-\alpha}\\
\Pi =& 0.
\end{aligned}
\tag{1.3}
$$

Then for household problem, obviously, it is optimal to choose $k_t = x_t$, combining the production function and goods market clear condition. We can write the Lagrange function and FOC for that(assuming interior solution),

$$
\begin{aligned}
L(\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}; \lambda) =& \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) + \lambda\left(\sum_{t=0}^{\infty} p_t (z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta) k_t\right)\\
\frac{\partial L}{\partial c_t} =& \beta^t c_t^{-\sigma} - \lambda p_t = 0,\\
\frac{\partial L}{\partial l_t} =& -\beta^t \chi l_t^{\eta} + \lambda p_t z (1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0,\\
\frac{\partial L}{\partial k_t} =& p_t (z\alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - p_{t-1} = 0,\\
c_t =& z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t (\text{use goods market clear to replace} \frac{\partial L}{\partial \lambda}).
\end{aligned}
\tag{1.4}
$$

For Steady state, we can normalize $p_0 = 1$,

$$
\frac{\partial L}{\partial c_t} = 0 \Rightarrow p_t = \beta^t \Rightarrow \lambda = c_t^{-\sigma},
$$

then

$$
\frac{\partial L}{\partial l_t} = 0, \frac{\partial L}{\partial k_t} = 0, \frac{\partial L}{\partial \lambda} = 0 \Rightarrow
$$

$$
\begin{aligned}
&c^\sigma l^\eta = z(1-\alpha)k^\alpha l^{-\alpha}/\chi \\
&z\alpha k^{\alpha-1}l^{1-\alpha} = 1/\beta - 1 + \delta \\
&c = zk^\alpha l^{1-\alpha} - \delta k
\end{aligned} \tag{1.5}
$$

Use second one in 1.5, we have

$$
M = \frac{k}{l} = \left(\frac{z\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}},
$$

Plug in third one in 1.5,

$$
N = \frac{c}{l} = z(\frac{k}{l})^\alpha - \delta\frac{k}{l} = zM^\alpha - \delta M,
$$

Use $c = lN$ in first one in 1.5,

$$
\begin{aligned}
(lN)^\sigma l^\eta =& z(1-\alpha)(\frac{k}{l})^\alpha/\chi \Rightarrow \\
l^{\sigma+\eta} =& \frac{z(1-\alpha)M^\alpha}{\chi N^\sigma} \Rightarrow \\
l =& \left(\frac{z(1-\alpha)M^\alpha}{\chi N^\sigma}\right)^{\frac{1}{\sigma+\eta}},
\end{aligned}
$$

Then we can have all steady state variable,

$$
\begin{aligned}
k =& Ml \\
c =& Nl \\
y =& zk^\alpha l^{1-\alpha} = zM^\alpha l \\
r =& \alpha zk^{\alpha-1}l^{1-\alpha} = \alpha zM^{\alpha-1} \\
w =& (1-\alpha)zk^\alpha l^\alpha = (1-\alpha)zM^\alpha.
\end{aligned}
$$

1.3 Social planner problem

The problem of the social planner is that, given the initial capital k_0,

$$\begin{aligned} w(k_0) = &\max_{\{c_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) \\ s.t.\ \ z k_t^{\alpha} l_t^{1-\alpha} &= c_t + k_{t+1} - (1-\delta)k_t, \quad \forall t \geq 0 \\ &c_t \geq 0,\ k_t \geq 0,\ 0 \leq l_t \leq 1, \quad \forall t \geq 0 \\ &k_0 \text{ is given.} \end{aligned} \tag{1.6}$$

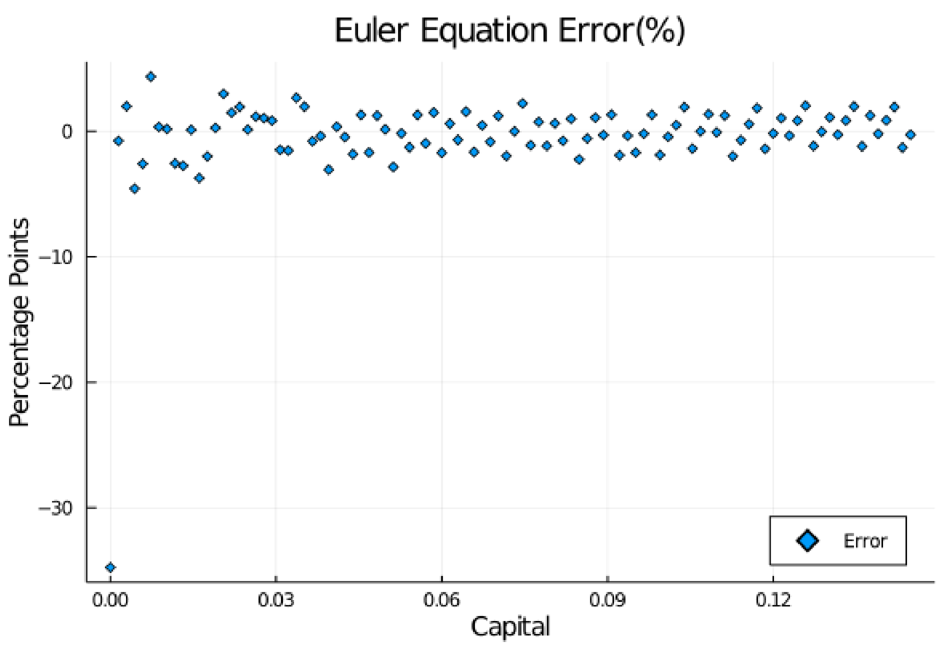
Bellman equation,

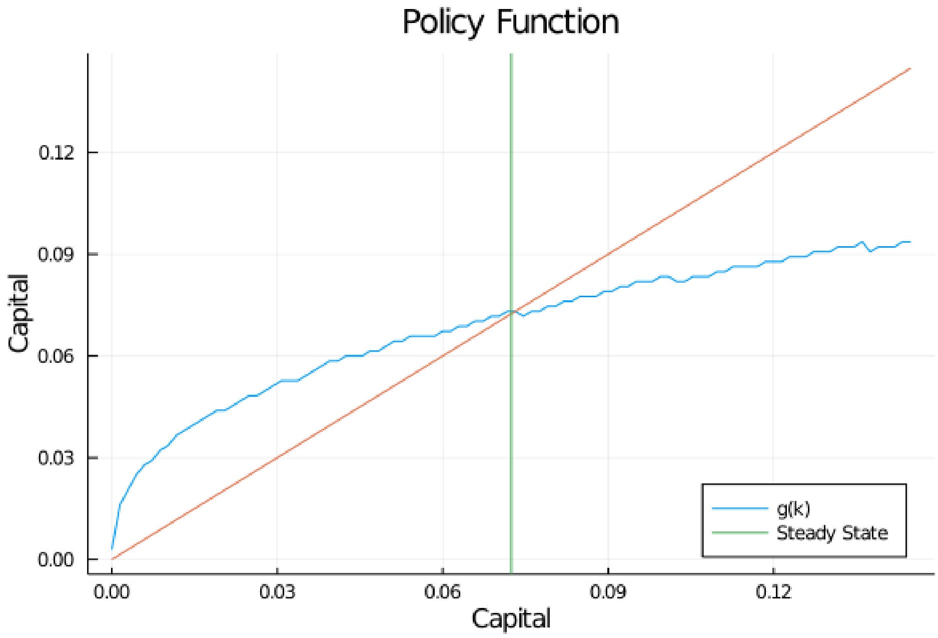
$$V(k) = \max_{\substack{0 \leq l \leq 1 \\ 0 \leq k' \leq z k^{\alpha} l^{1-\alpha} + (1-\delta)k}} \left\{ \frac{(z k^{\alpha} l^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta} + \beta \mathbb{E} V(k') \right\} \tag{1.7}$$

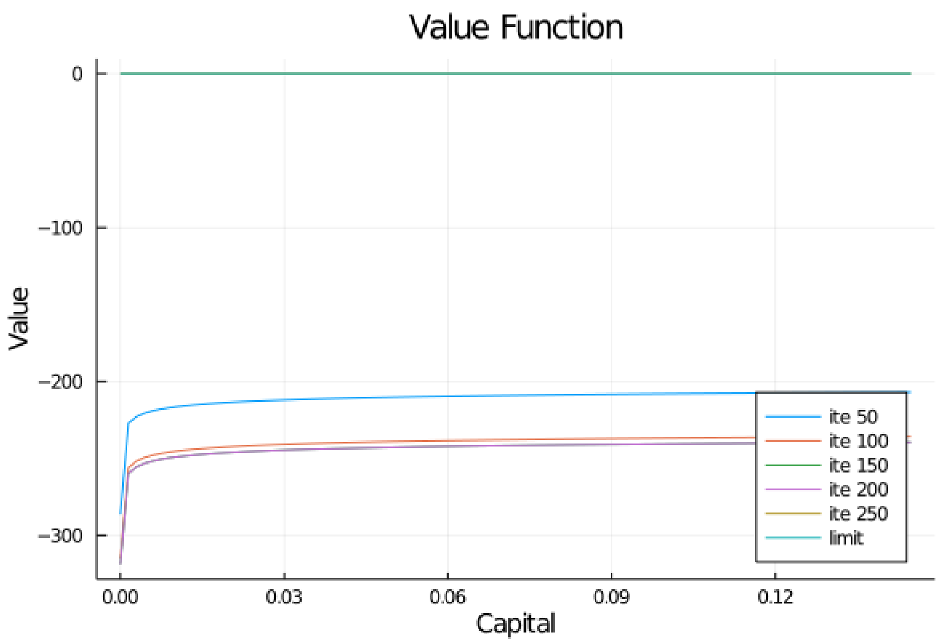
1.4 Chi

1.5 Solve the planner' problem numerically using value function iteration

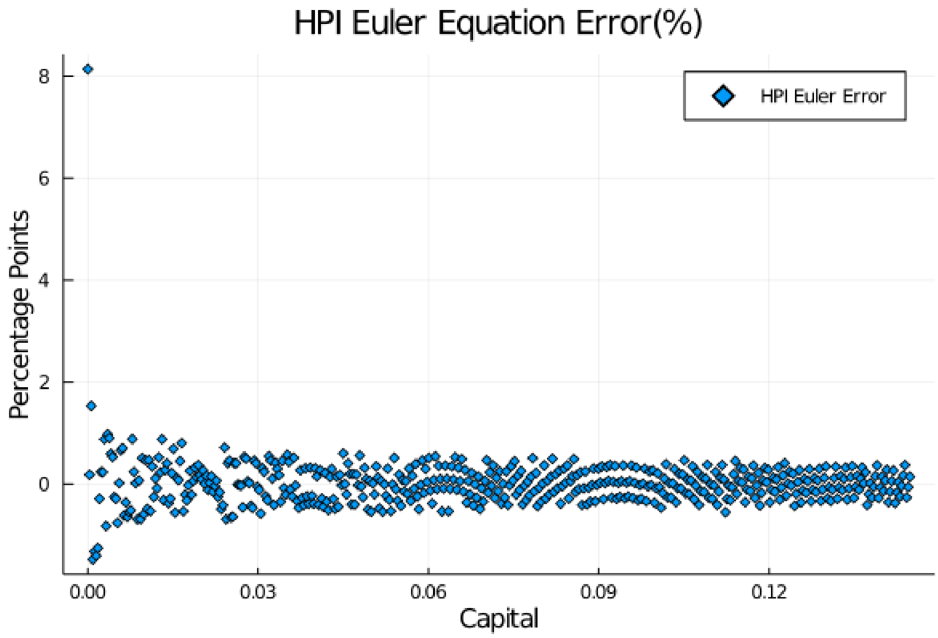
a

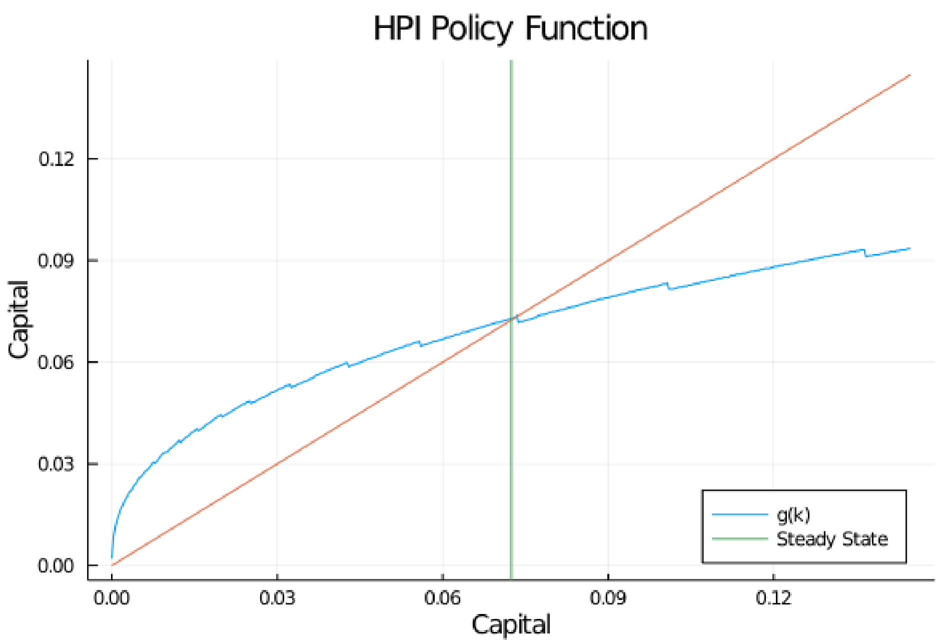


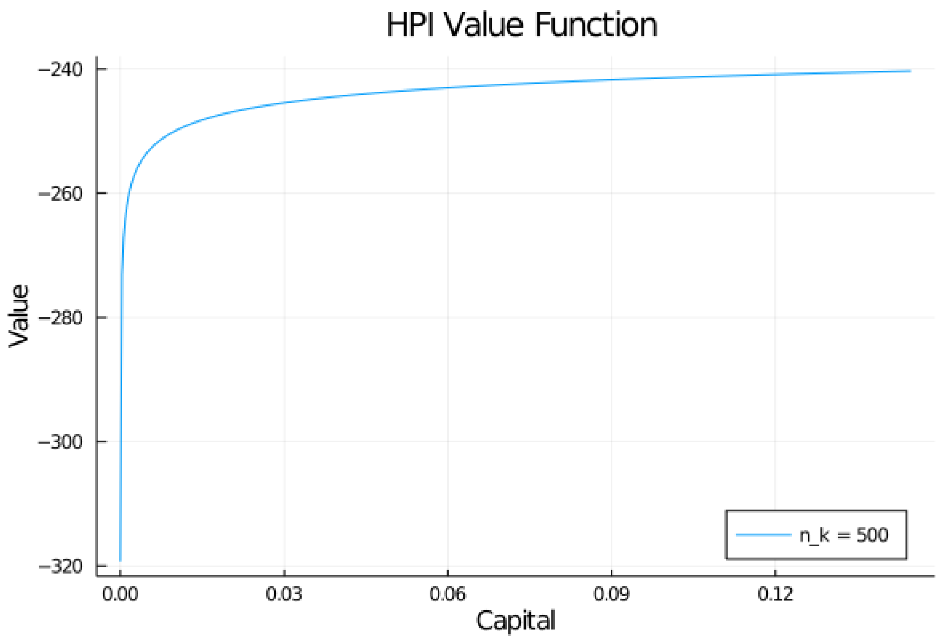

Policy Function
Capital
0.00
0.03
0.06
0.09
0.12
g(k)
Steady State


Value Function
Value
Capital
0
−100
−200
−300
0.00
0.03
0.06
0.09
0.12
ite 50
ite 100
ite 150
ite 200
ite 250
limit

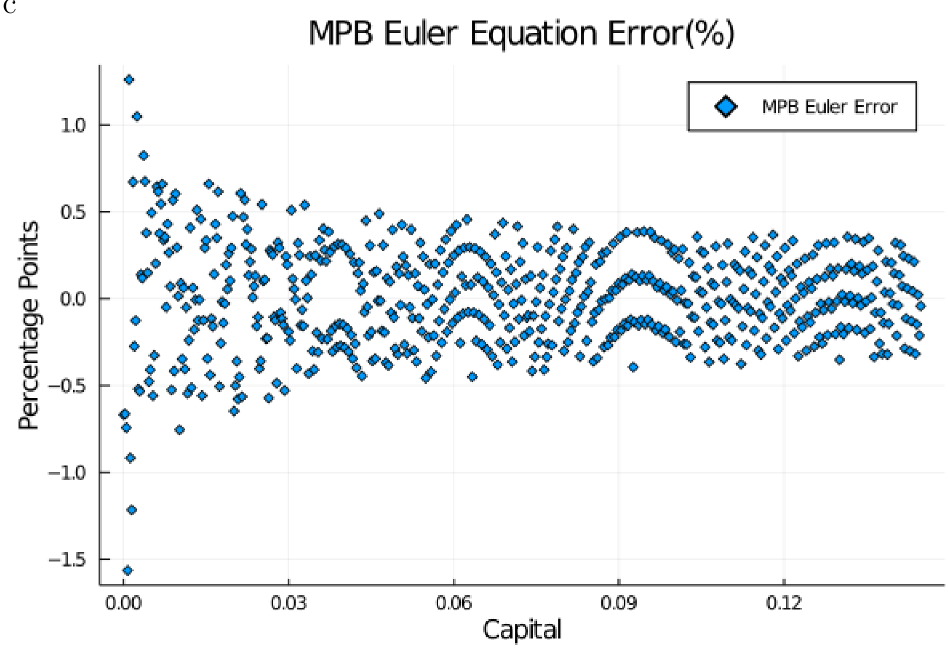
b

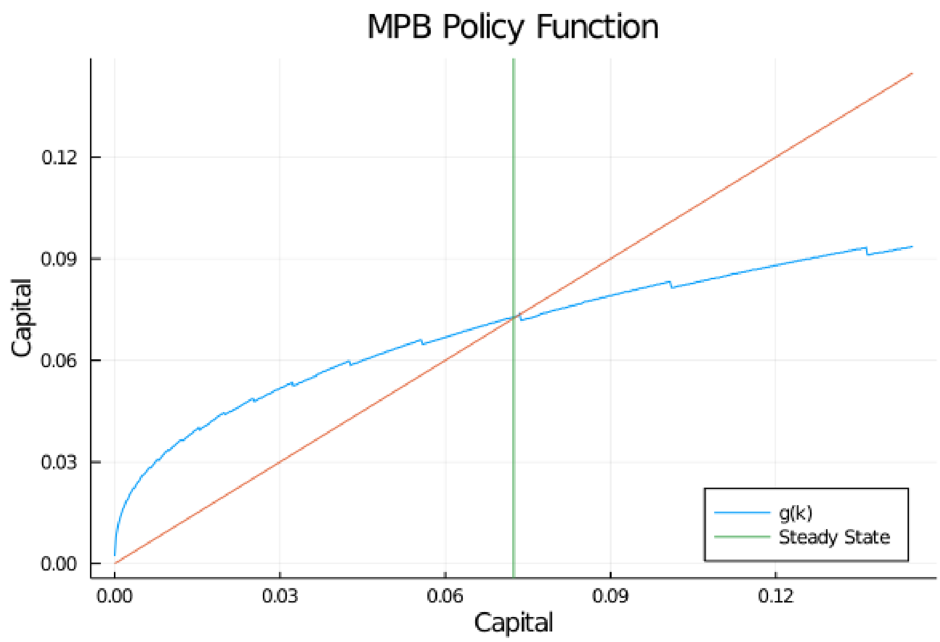

HPI Euler Equation Error(%)
HPI Euler Error
Percentage Points
Capital
8
6
4
2
0
0.00
0.03
0.06
0.09
0.12

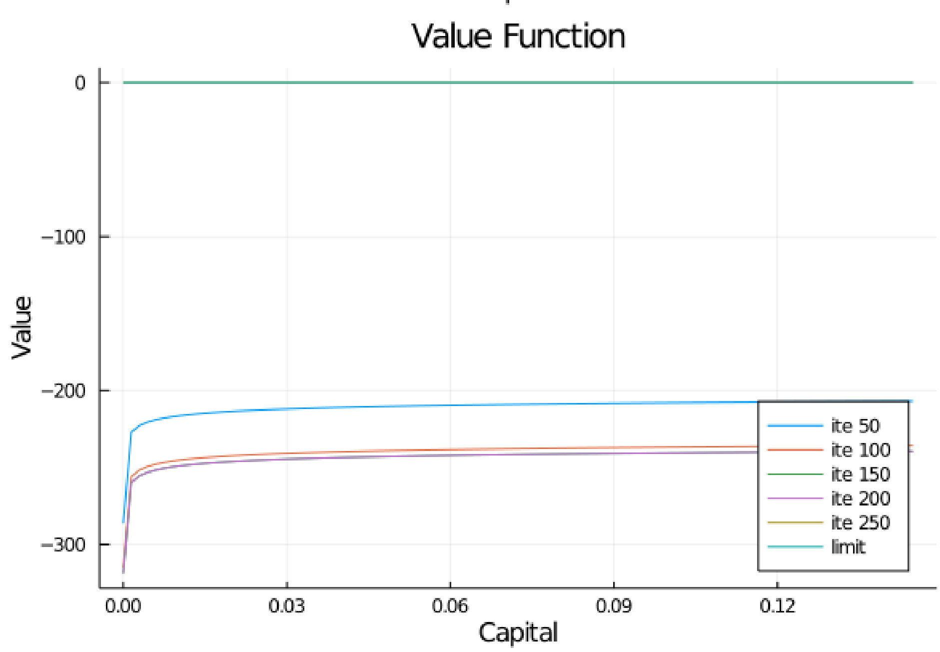

HPI Policy Function
g(k)
Steady State
Capital
Capital
0.12
0.09
0.06
0.03
0.00
0.00
0.03
0.06
0.09
0.12


HPI Value Function
Value
Capital
n_k = 500
−240
−260
−280
−300
−320
0.00
0.03
0.06
0.09
0.12

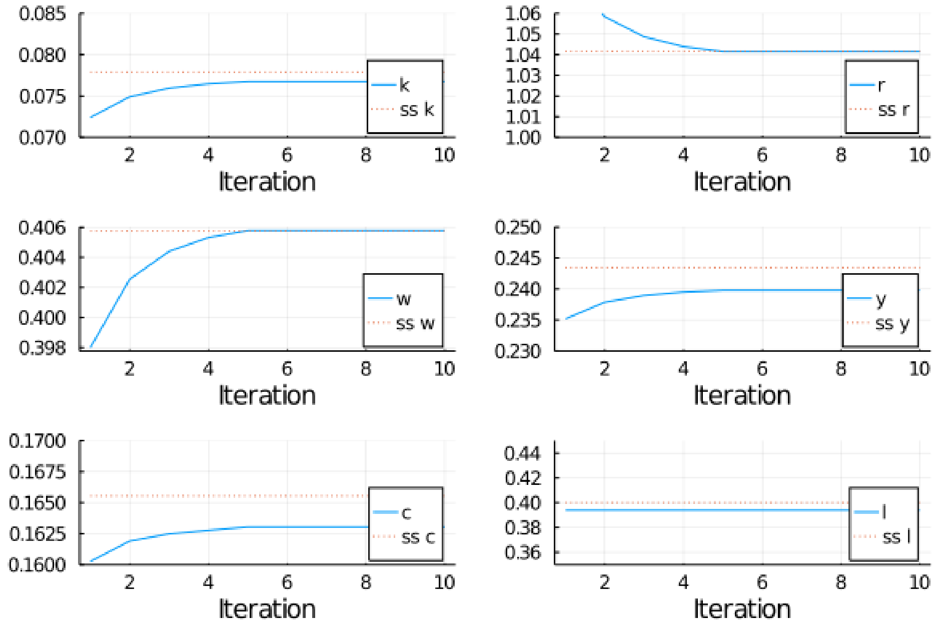
c


MPB Euler Equation Error(%)
Percentage Points
Capital
MPB Euler Error
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.00
0.03
0.06
0.09
0.12


MPB Policy Function
Capital
Capital
0.00
0.03
0.06
0.09
0.12
g(k)
Steady State


Value Function
Value
Capital
0
−100
−200
−300
0.00
0.03
0.06
0.09
0.12
ite 50
ite 100
ite 150
ite 200
ite 250
limit

6


0.085
0.080
0.075
0.070
k
ss k
2 4 6 8 10
Iteration
1.06
1.05
1.04
1.03
1.02
1.01
1.00
r
ss r
2 4 6 8 10
Iteration
0.406
0.404
0.402
0.400
0.398
w
ss w
2 4 6 8 10
Iteration
0.250
0.245
0.240
0.235
0.230
y
ss y
2 4 6 8 10
Iteration
0.1700
0.1675
0.1650
0.1625
0.1600
c
ss c
2 4 6 8 10
Iteration
0.44
0.42
0.40
0.38
0.36
l
ss l
2 4 6 8 10
Iteration