Macro problem set 1

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September 23, 2020

1 Problem 1

1.1 Arrow-Debreu comparative equilibrium

An Arrow-Debreu comparative equilibrium (hereafter, ADCE) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that

(1) Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm solves

(1) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the firm allocation $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ solves the firm problem,

$$\begin{aligned} \P\pi = & \max_{\{y_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t) \\ \text{s.t. } & y_t = f(k_t, l_t), \forall t \geq 0; \\ & y_t, k_t, l_t \geq 0, \forall t \geq 0. \end{aligned} \tag{1.1}$$

(2) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and the profit of firm Π, the household

allocation $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves the household problem,

$$
\begin{aligned}
&\max_{\{c_t, i_t, x_{t+1}, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t. } &\sum_{t=0}^{\infty} p_t(c_t + i_t) \le \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t) + \pi; \\
&x_{t+1} = i_t, \forall t \ge 0; \\
&0 \le l_t \le 1, 0 \le k_t \le x_t, c_t \ge 0, x_{t+1} \ge 0, \forall t \ge 0; \\
&x_0 \textit{ is given.}
\end{aligned} \tag{1.2}
$$

(3) The market clear conditions are given by

$$
\begin{aligned}
y_t &= c_t + i_t \quad (\textit{goods market}); \\
l_t^d &= l_t^s \qquad (\textit{labour market}); \\
k_t^d &= k_t^s \qquad (\textit{capital market}).
\end{aligned}
$$

1.2 Social planner's problem

The social planner's problem (hereafter, SPP) is

$$
\begin{aligned}
w(\bar{k}_0) = &\max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t.\ \ &f(k_t, l_t) = c_t + k_{t+1} - (1-\delta)k_t, \ \ \forall t \ge 0 \\
&c_t \ge 0, \ k_t \ge 0, \ 0 \le l_t \le 1, \ \ \forall t \ge 0 \\
&k_0 \textit{ is given.}
\end{aligned} \tag{1.3}
$$

1.3 The proof of welfare theorem

To show the Welfare theorem, we could prove it by showing that the Euler equations and TVCs in the SPP and optimal problem in ADCE are the same.

With full depreciation, denote $f(k_t) := f(k_t, 1) + (1-\delta)k_t = f(k_t, 1)$**.**

SPP

Since there is no utility gained from leisure, it is easy to see $l_t = 1$.

Thus the SPP can be rewritten as

$$
\begin{aligned}
w(k_0) = \max_{\{k_{t+1},\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
0 \geq k_{t+1} \geq f(k_t), \quad \forall t \geq 0 \\
k_0 \textit{ is given.}
\end{aligned}
$$

FOC of for SPP gives the Euler equation

$$u'(h(k_t) - k_{t+1}) = \beta u'(f(k_t) - k_{t+1}) f'(k_t). \tag{1.4}$$

The transversality condition (hereafter, TVC) for SPP is

$$\lim_{t \to \infty} \beta^t u'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0, \tag{1.5}$$

or, equivalently,

$$\lim_{t \to \infty} \ \lambda_t k_{t+1} = 0, \tag{1.6}$$

where λ_t is the Lagrange multiplier for time t in the original SPP 1.3.

ADCE

FOC for firm's problem, yields

$$r_t = f'(k_t)$$

FOC for household's problem yields

$$
\begin{aligned}
\beta^t u'(c_t) = \mu p_t \\
\beta^{t+1} u'(c_{t+1}) = \mu p_t r_{t+1}
\end{aligned}
$$

With market clearing

$$c_t = f(k_t) - k_{t+1},$$

we have the Euler equation for ADCE

$$\lim_{t \to \infty} \beta^t u'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0. \tag{1.7}$$

TVC for household problem is given by

$$\begin{aligned}\lim_{t\to\infty} p_t k_{t+1} &= \frac{1}{\mu}\lim_{t\to\infty}\beta^t u'(c_t)k_{t+1} \\ &= \frac{1}{\mu}\lim_{t\to\infty}\beta^{t-1}u'(c_{t-1})k_t \\ &= \frac{1}{\mu}\lim_{t\to\infty}\beta^{t-1}\beta u'(c_{t-1})r_t k_t \\ &= \frac{1}{\mu}\lim_{t\to\infty}\beta^{t-1}\beta u'(f(k_t)-k_{t+1})k_t.\end{aligned} \tag{1.8}$$

Equivalence

With $p_t = \lambda_t$, the optimal allocation in SPP and that in ADCE has the same Euler equations (see 1.5 and 1.7) and TVCs (see 1.6 and 1.8), then the solutions in two optimality problems are the same. Hence the desired result is obtained.

1.4 Social planner dynamic programming problem

Due to $l_t = 1, c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$, we can get the dynamic programming problem,

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u[f(k_t) - k_{t+1}]$$
$$s.t.\ \ 0 \le k_{t+1} \le f(k_t);$$
$$k_0 \ is \ given.$$

Define the value function $V(k)$ as the value of the lifetime social planner problem given the initial capital as k, then we can get the Bellman equation,

$$V(k) = \max_{0\le k'\le f(k)}\{u(f(k) - k' + \beta V(k')\}$$

1.5 The example of log utility

By that $u(c) = log(c), f(k,l) = zk^\alpha l^{1-\alpha}$, the social planner dynamic programming problem becomes,

$$V(k) = \max_{0\le k'\le zk^\alpha}\{log(zk^\alpha - k') + \beta V(k')\}$$

Guess a solution for V is that $V(k) = Alog(k) + B$, then the FOC for the problem is

$$-\frac{1}{zk^\alpha - k'} + \frac{\beta A}{k'} = 0,$$

thus $k' = \frac{\beta A z k^\alpha}{1+\beta A}$, then we can solve A and B,

$$\begin{aligned}
A =& \frac{\alpha}{1-\alpha\beta}, \\
B =& \frac{1}{(1-\beta)(1-\alpha\beta)}[(\alpha\beta)log(\alpha\beta) + (1-\alpha\beta)log(1-\alpha\beta) + log(z)],
\end{aligned}$$

Then the policy function is $g(k) = k'|_{m=\frac{\alpha}{1-\alpha\beta}} = \alpha\beta k^\alpha$

1.6 Steady state

For steady state, we could let $g(k) = k$ and then get k_s. Therefore,

$$\begin{aligned}
k_s &= (\alpha\beta z)^{\frac{1}{1-\alpha}}; \\
c_s &= f(k_s) - g(k_s) = zk_s^\alpha - \alpha\beta z k_s^\alpha = (1-\alpha\beta)z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}; \\
r_s &= f_k(k_s) = \alpha z k_s^{\alpha-1} = \frac{1}{\beta}; \\
w_s &= f_l(k_s) = (1-\alpha)zk_s^\alpha = (1-\alpha)z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}; \\
y_s &= f(k_s) = zk_s^\alpha = z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}.
\end{aligned}$$