A CONVEX OPTIMIZATION APPROACH FOR SOLVING THE ROBUST STRICTLY POSITIVE REAL (SPR) PROBLEM

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ABSTRACT

In system identification and adaptive control, the problem of designing strictly positive real (SPR) transfer functions in the presence of uncertain adaptation parameters is essential for stability and convergence in a group of parameter adaptation algorithms. This paper proposes a convex optimization approach to address the robust SPR problem. Besides achieving the robust SPR condition, the presented solution is optimal in the sense of minimizing the distance from the transfer function to unity. Such consideration is important for parameter convergence in practical applications. New topics such as minimum-order compensator and minimum high-frequency magnitude design are also introduced.

1 INTRODUCTION

The strictly positive real (SPR) condition of a transfer function has substantial importance in adaptive control and system identification [1,2]. An essential problem (see, e.g., [3–6] and the references therein) in a group of recursive parameter adaptation algorithms is: given an uncertain polynomial $A(x)$ ($x = s$ in continuous-time problems or z^{-1} in discrete-time problems), design a polynomial compensator $C(x)$ such that the transfer function

$$\frac{C(x)}{A(x)} - \alpha \qquad (1)$$

is SPR for all possible values of $A(x)$. Here $\alpha (\in [0,1])$ is a fixed scalar that depends on the adaptation algorithm.

For instance, in the output error method with a fixed compensator, the condition that $C(z^{-1})/A(z^{-1}) - 1/2$ being SPR is crucial for the stability and the parameter convergence during adaptation [6]. In this case, $A(z^{-1})$ comes from the transfer function of the plant to be identified. Conventionally, one has to guess or apply another parameter adaptation algorithm to obtain a $C(z^{-1})$ that is hopefully close to $A(z^{-1})$. The same problem occurs in the more general pseudo linear regression algorithm, where the importance of the SPR condition has been remarked in Section 8.6 of [1].

The SPR condition is a strong requirement and is usually not easy to guarantee for an uncertain $A(x)$. Investigation of the above problem has therefore been popular in the control community. Approaches that use: (a) complex polynomial analysis [3, 5, 7–15]; (b) geometrical design [13,14]; and (c) linear matrix inequalities (LMI) [16,17] have been proposed to address the problem. More specifically, [3, 12] and [15] characterized the SPR condition and discussed the case when $A(x)$ belongs to a set of stable and known polynomials. [13] and [14] analyzed the situation when the uncertainty in $A(x)$ comes from its root locations or bounded uncertain frequency responses. An important general classification was discussed in [7–11, 16, 17], where $A(x)$ is assumed to lie in a known polytope, with bounded coefficients in the polynomial. Among the existing results, the majority discussed the case where $\alpha = 0$; [9] and [16] investigated the more difficult situation where $\alpha > 0$. [3, 5, 8–12, 15–17] mainly analyzed the continuous-time version of the problem. The discrete-time robust SPR problem has different characteristics compared to the continuous-time version [9]. Within this category, [7] provided conditions for the existence of a solution but did

not discuss how to construct it; [14] showed a geometrical design approach for systems with disk uncertainties; later in [16], the authors formed linear matrix inequalities to analyze the general SPR condition for an uncertain transfer function $G(x) - \alpha$. The formation and realization of a compensator $C(x)$ was however not discussed, and the equations are slightly more complex than the present approach.

The most natural (and recommended in the related text books [1,2,6]) way of designing $C(x)$ is to make it "close" to $A(x)$, such that $C(x)/A(x) - \alpha$ is approximately $1 - \alpha$ (usually $\alpha \leq 1$). In fact, this is also substantial for parameter convergence in adaptation algorithms (see Section 4.5.4 of [18]). This aspect has however been largely discredited in previous robust SPR design algorithms.

In this paper, we present a convex optimization approach to address the robust SPR problem (with a general non-negative α). The common polytopic uncertainty [7–11, 16, 17] is adopted here. We will be focusing on the discrete-time version of the problem, partially due to its popularity in system identification and adaptive control, and partially because of the fact that results in the more explored continuous-time robust SPR problem do not necessarily generalize to discrete-time systems [9, 16].[1] Additional contributions of the paper are as follows. First, we provide a design approach that not only assures the robust SPR condition but also finds the optimal $C(x)$ that is "closest" to $A(x)$. Second, we discuss the achievement of additional optimal properties to the compensator design. This provides us the possibility to investigate several new issues. For instance, in output error based adaptation algorithms, it is favorable for the compensator to have minimum order and/or small gain in the high-frequency region.

2 SPR ANALYSIS

We start by introducing the definition of SPR transfer functions:

Definition 2.1. *a proper and rational discrete-time transfer function $G(z^{-1})$ is strictly positive real (SPR) if*[2]

1. $G(z^{-1})$ *does not possess any pole outside of or on the unit circle on z-plane;*
2. $\forall |\omega| < \pi$, $G(e^{-j\omega}) + G(e^{j\omega}) = 2Re\left\{G(e^{j\omega})\right\} > 0$.

From the above definition, the following properties can be obtained:

If $G(z^{-1})$ is SPR, then

1. it is asymptotically stable;

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2. the phase response of $G(z^{-1})$, after normalization to $[-\pi, \pi]$, lies inside the region $(-\frac{\pi}{2}, \frac{\pi}{2})$ (see, e.g., [7,15]);
3. the Nyquist plot of $G(z^{-1})$ lies in the closed right-half complex plane [20].

Property 1 and 2 are direct results of the first and the second points in Definition 1. The third property is an equivalent statement of Property 2.

The robust SPR problem discussed in this paper is stated as follows:

Problem 2.1. *Given $\alpha \in [0,1]$ and a monic stable polynomial*

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}, \qquad (2)$$

with n unknown but bounded coefficients

$$\underline{a}_i \leq a_i \leq \bar{a}_i, \ i = 1, 2, \ldots, n, \qquad (3)$$

find a polynomial $C(z^{-1})$ such that $\frac{C(z^{-1})}{A(z^{-1})} - \alpha$ is SPR.

In practical applications, α is usually strictly positive [1,2]. In this case the problem can be normalized as shown in the following proposition.

Proposition 2.1. *For $\alpha > 0$, $\frac{C(z^{-1})}{A(z^{-1})} - \alpha$ is SPR if and only if $\frac{C'(z^{-1})}{A(z^{-1})} - \frac{1}{2}$ is SPR for some polynomial $C'(z^{-1})$.*

Proof. Under the assumption that $\alpha > 0$,

$$\frac{C(z^{-1})}{A(z^{-1})} - \alpha = 2\alpha \left(\frac{C'(z^{-1})}{A(z^{-1})} - \frac{1}{2} \right)$$

where $C'(z^{-1}) = C(z^{-1})/(2\alpha)$. The proof follows by noting the fact that scaling a transfer function by a positive number does not change the SPR property.

For the above normalized problem, [4] has shown that the SPR condition of $Re\left\{C'(e^{j\omega})/A(e^{j\omega}) - 1/2\right\} > 0$ is equivalent to $|A(e^{j\omega})/C'(e^{j\omega}) - 1| < 1$, from which it is clear that letting $C'(z^{-1})/A(z^{-1}) \approx 1$ is a feasible solution. This is the suggested way of designing the compensator $C'(z^{-1})$ in text books of system identification and adaptive control [1, 2, 6, 18], and is also important for the parameter convergence, as discussed in Section 1.

3 POLYTOPIC UNCERTAINTY

In this section, we briefly review the characterization of the polytopic constraint (3) and provide a general result of the robust SPR problem.

Notice that (2) can be equivalently represented as

$$A(z^{-1}) = 1 + \left[z^{-1}, z^{-2}, \ldots, z^{-n}\right][a_1, a_2, \ldots, a_n]^T. \qquad (4)$$

Consider an n dimensional vector space that contains the coefficient vector $[a_1, a_2, \ldots, a_n]^T$. An alternative representation of (3) is to use the concept of convex hull (see, e.g., [21]), which states that $[a_1, a_2, \ldots, a_n]^T$ can be characterized by the extreme edge vectors that are defined by lower and upper bounds of a_i's:

$$[a_1, a_2, \ldots, a_n]^T = \sum_{j=1}^{2^n} \theta_j \left[b_{j,1}, b_{j,2}, \ldots, b_{j,n}\right]^T, \ \theta_j \geq 0, \ \sum_{j=1}^{2^n} \theta_j = 1,$$

where $b_{j,i} = \underline{a}_i$ or $\bar{a}_i$. There are 2^n edge vectors. This number can be reduced if some parameters are known a prior. Applying the above result to (4) yields

$$A(z^{-1}) = \sum_{j=1}^{2^n} \theta_j A_j(z^{-1}), \ \theta_j \geq 0, \ \sum_{j=1}^{2^n} \theta_j = 1, \qquad (5)$$

where $A_j(z^{-1})$'s are the edge polynomials defined by $b_{j,i}$'s. Note that since $A(z^{-1})$ is stable by assumption,[3] all the $A_j(z^{-1})$'s therefore are also stable.

Equation (5) provides a convenient interpretation of the uncertainty in $A(z^{-1})$. Instead of a polynomial of unknown coefficients, we now have a convex combination of a finite number of fixed polynomials. Moreover, the following result holds:

Lemma 3.1. *If $A(z^{-1})$ is given by (5), then $\frac{C(z^{-1})}{A(z^{-1})} - \alpha$ is SPR if and only if $\frac{C(z^{-1})}{A_i(z^{-1})} - \alpha$ is SPR $\forall i = 1, 2, \ldots, 2^n$.*

Remark: this is an enhanced version of Lemma 3.1 in [9]. The "only if" part of the proof is trivial since $C(z^{-1})/A_i(z^{-1})$ corresponds to $\theta_j = 1$ for $j = i$ and $\theta_j = 0$ for $j \neq i$ in (5). The "if" part of the proof is similar to that in [9] and omitted here.

4 ACHIEVING THE ROBUST SPR CONDITION

From the positive-real lemma [26–28], a square discrete-time system $G_p(z^{-1}) = C_p\left(zI - A_p\right)^{-1} B_p + D_p$ is SPR

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[3]To obtain the robust stability condition of such 'interval polynomials' with bounded coefficients, the Kharitonov's Theorem [22, 23] provides a complete solution to the continuous-time domain problem and has motivated various research towards its discrete-time equivalences (see, e.g., [24, 25] and the references contained therein.)

if and only if there exists a positive definite matrix $P = P^T > 0$ such that the following matrix inequality holds

$$\begin{bmatrix} P - A_p^T P A_p & C_p^T - A_p^T P B_p \\ C_p - B_p^T P A_p & D_p^T + D_p - B_p^T P B_p \end{bmatrix} > 0. \qquad (6)$$

The SPR requirement at infinite frequencies in Definition 2.1 is now translated to a single matrix inequality.

Given the SPR problem, we have first shown that the uncertain $A(z^{-1})$ has the equivalent representation in (5). Lemma 3.1 then leads to investigation of the SPR condition for each edge transfer function $C(z^{-1})/A_j(z^{-1}) - \alpha$. It is now proposed to apply the positive-real lemma to form the conditions for $C(z^{-1})/A_j(z^{-1}) - \alpha$ to be SPR.

Let $G(z^{-1}) = C(z^{-1})/K(z^{-1}) - \alpha$, where $K(z^{-1})$ represents an edge polynomial $A_j(z^{-1})$. Define

$$C(z^{-1}) = c_0 + c_1 z^{-1} + \cdots + c_m z^{-m}$$
$$K(z^{-1}) = 1 + k_1 z^{-1} + \cdots + k_n z^{-n}, \qquad (7)$$

where m, the order of $C(z^{-1})$, is chosen by the designer.

Depending on the values of m and n, different situations exist for the design of (6):

Case 1: if $m \geq n$, it is straightforward to show that

$$\frac{C(z^{-1})}{K(z^{-1})} - \alpha = (c_0 - \alpha) +$$

$$\frac{(c_1 - c_0 k_1) z^{m-1} + \cdots + (c_n - c_0 k_n) z^{m-n} + c_{n+1} z^{m-n-1} + \cdots + c_m}{z^m + k_1 z^{m-1} + \cdots + k_n z^{m-n}}$$

which has the following state-space realization:

$$A_p = \begin{bmatrix} 0_{m-1,1} & I_{m-1,m-1} \\ 0 & * \end{bmatrix}_{m \times m}, \ B_p = \begin{bmatrix} 0_{m-1,1} \\ 1 \end{bmatrix} \qquad (8)$$
$$* = [0_{1,m-n-1}, -k_n, \ldots, -k_1]$$
$$C_p = \left[c_m, \ldots, c_{n+1}, c_n, \ldots, c_1 \right] - c_0 \left[0_{1,m-n}, k_n, \ldots, k_1 \right]$$
$$D_p = c_0 - \alpha.$$

Here the controllable canonical form is proposed, so that when we form (6), the matrix on the left-hand side is linear in the decision variables $[c_0, c_1, \ldots, c_m]$.

Case 2: if $n > m$, similar analysis gives

$$A_p = \begin{bmatrix} 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \\ -k_n & \ldots & -k_2 & -k_1 \end{bmatrix}_{n \times n} , \quad B_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1} \qquad (9)$$

$$C_p = -c_0 \left[k_n \ \ldots \ k_{m+1} , k_m \ \ldots \ , k_1 \right] + \left[0_{1,n-m}, c_m, \ldots, c_1 \right]$$

$$D_p = c_0 - \alpha.$$

Equation (8) or (9) can now be applied to construct (6). Such constructions are repeated for each edge transfer function. We can now formulate the following feasibility problem ('s.t.' in the following equation denotes 'subject to'):

$$\text{find } c_0, \ldots c_m \in \mathbb{R} \text{ and } P_j = P_j^T > 0 \qquad (10)$$

$$\text{s.t.} \begin{bmatrix} P_j - A_{p,j}^T P_j A_{p,j} & C_{p,j}^T - A_{p,j}^T P_j B_{p,j} \\ C_{p,j} - B_{p,j}^T P_j A_{p,j} & D_{p,j}^T + D_{p,j} - B_{p,j}^T P_j B_{p,j} \end{bmatrix} > 0,$$

$$j = 1, 2, \ldots 2^n,$$

where for each j, $\left(A_{p,j}, B_{p,j}, C_{p,j}, D_{p,j} \right)$ is defined by (8) or (9), with $K(z^{-1}) = A_j(z^{-1})$. By construction, $C_{p,j}$ and $D_{p,j}$ depend affinely on c_i's. Problem (10) is thus a convex, actually semidefinite programming (SDP) problem, and can be solved by efficient interior-point methods (using, e.g., [29]) in convex optimization.[4]

5 OPTIMAL PROPERTIES

Section 4 provides a tool to obtain a feasible solution. Another main result of the present article is the ability to obtain coefficients $c_0, \ldots, c_m$ in (10) with designer-assigned optimal properties, one of which is to keep $C(z^{-1})/A(z^{-1})$ close to 1. In this section, together with the "close-to-1" condition, we provide a few examples to obtain the optimal compensator $C(z^{-1})$. The discussions are separated into subsections but they can be combined by a weighted sum or the minimum of the weighted costs, to satisfy multiple design objectives. All the results in this section are subject to the baseline SPR requirement in Section 4.

5.1 $C(z^{-1})/A(z^{-1})$ **Being Close to 1**

The intuition and the importance of this objective has been discussed in Section 2. For the z-domain transfer function $C(z^{-1})/A(z^{-1})$ to be close to 1, we aim at minimizing

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[4]When the problem is formulated for computer solvers, the positive definite constraint $P = P^T > 0$ is transformed to $P - \epsilon I \geq 0$, where ϵ is a small positive number chosen as the lower bound of all the eigenvalues of P.

the maximum value of $|C(e^{-j\omega})/A(e^{-j\omega}) - 1|$ over the entire frequency region, i.e.,

$$\min_{c_0, \ldots c_m \in \mathbb{R}} \left\| \frac{C(z^{-1})}{A(z^{-1})} - 1 \right\|_\infty. \qquad (11)$$

In a more general and flexible form, we consider

$$\min_{c_0, \ldots c_m \in \mathbb{R}} \left\| R(z^{-1}) \right\|_\infty \triangleq \left\| \frac{W(z^{-1})}{V(z^{-1})} C(z^{-1}) - \eta \right\|_\infty \qquad (12)$$

with $W(z^{-1}) = w_0 + w_1 z^{-1} + \cdots + w_m z^{-m}$, $V(z^{-1}) = 1 + v_1 z^{-1} + \cdots + v_m z^{-m}$, and $C(z^{-1}) = c_0 + c_1 z^{-1} + \cdots + c_m z^{-m}$. For simplicity, it is assumed that $W(z^{-1})$, $V(z^{-1})$ and $C(z^{-1})$ have the same order. If not, one can classify different situations in a way similar to that in Section 4, or simply constraint the coefficients of the excessive high-order terms to be zero. Using (12), we can essentially constraint $C(z^{-1})$ to have an arbitrary desired (if feasible) frequency response.

(12) can be transformed to a tractable optimization problem in a form similar to (10), by utilizing the bounded-real lemma (see, e.g., [30]). To do that, we need the following system construction: notice that $H(z^{-1}) \triangleq W(z^{-1})C(z^{-1}) = h_0 + h_1 z^{-1} + \cdots + h_{2m} z^{-2m}$ is given by the convolution

$$\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ \vdots \\ h_{2m} \end{bmatrix} = \underbrace{\begin{bmatrix} w_0 & 0 & \cdots & \cdots & 0 \\ w_1 & w_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ w_m & w_{m-1} & \cdots & w_1 & w_0 \\ 0 & w_m & \ddots & \ddots & w_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & w_{m-1} \\ 0 & \cdots & \cdots & 0 & w_m \end{bmatrix}}_{(2m+1) \times (m+1)} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

from which $R(z^{-1}) = H(z^{-1})/V(z^{-1}) - \eta$ has the following state-space realization:

$$A_r = \begin{bmatrix} 0_{2m-1,1} & I_{2m-1,2m-1} \\ 0 & * \end{bmatrix}_{2m \times 2m}, \quad B_r = \begin{bmatrix} 0_{2m-1,1} \\ 1 \end{bmatrix} \qquad (13)$$

$$* = [0_{1,m-1}, -v_m, -v_{m-1}, \ldots, -v_1], \quad D_r = h_0 - \eta$$

$$C_r = [h_{2m}, h_{2m-1}, \ldots, h_1] - h_0 [0_{1,m}, v_m, \ldots, v_1].$$

We can now apply the bounded-real lemma, which says that (12) can be achieved if and only the following problem can be solved:

$$\min_{\gamma, P_r, c_i} : \gamma \geq 0 \qquad (14)$$

$$s.t. : -\begin{bmatrix} A_r^T P_r A_r - P & A_r^T P_r B_r & C_r^T \\ B_r^T P_r A_r & B_r^T P_r B_r - \gamma I & D_r^T \\ C_r & D_r & -\gamma I \end{bmatrix} \geq 0 \qquad (15)$$

$$P_r = P_r^T > 0$$

Again, we used the controllable canonical form in (13), and the matrix in inequality (15) is thus affine in γ, P_r and c_i's. After adding the SPR constraint (10), (14) remains a convex optimization problem.

A candidate polynomial $A(z^{-1})$ is needed in (11). For the specific polytopic uncertainty (3), the center of the polytope can be used.

5.2 Minimum-Order Compensator

The order of the compensator is directly related to the required computation complexity in the related system identification or adaptive control problems. The common practice in system identification is to apply $m = n$ in (7). With the proposed algorithms, it is however possible to find the compensator $C(z^{-1})$ with the minimum number of coefficients. This can be rapidly achieved through the optimization formulation, by starting the feasibility problem (10) with $m = n$, and iteratively reducing m until (10) becomes infeasible.

A related design is to obtain sparse (having large amounts of zeros) coefficients in $C(z^{-1})$. In that case, if a feasible order m is firstly assigned, we can apply the l_1-norm approximation for cardinality minimization (see, e.g., [21]), and add the cost function $\min_{c_i, P_i} \|[c_0, c_1, \ldots, c_m]^T\|_1$ to (10), which shall provide a sparse $[c_0, c_1, \ldots, c_m]$.

5.3 Minimum High-Frequency Gains

In the output error method with a fixed compensator, the output error is filtered through $C(z^{-1})$ to obtain the adaptation error, denoted as $v(k)$, for parameter identification or adaptive control (see one example in [31]). Excessive high-frequency components in $v(k)$ reduces the signal-to-noise ratio and increases the quantization error. It is therefore favorable to limit the high-frequency magnitude of $C(z^{-1})$ (and hence the high-frequency energy in $v(k)$).

Recall that the compensator is given by $C(z^{-1}) = c_0 + c_1 z^{-1} + \cdots + c_m z^{-m}$, whose frequency response $C(e^{-j\omega})$ at $\omega = \pi$ is $C(e^{-j\pi}) = C(z^{-1})\big|_{z=-1}$. To minimize the high-frequency gain (at the Nyquist frequency) of $C(z^{-1})$, we can add the following objective:

$$\min \left| C(e^{-j\pi}) \right| = \left| [1, -1, 1, -1, \ldots] [c_0, c_1, \ldots, c_m]^T \right| \qquad (16)$$

which is linear in the decision variables c_i's.

6 DESIGN EXAMPLES

We provide now several examples for verification of the proposed algorithm. Consider identification of the following second-order system:

$$\frac{B(z^{-1})}{A(z^{-1})} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - 2\beta a_1 z^{-1} + \beta^2 z^{-2}},$$

where the damping ratio $\beta = 0.98$ and the sampling period $T_s = 1/26400$ sec. Such a transfer function generalizes various rigid-body models of mechanical systems, and can also represent the model of vibrations with a single-frequency component [32]. Assume that we know the resonance of the system is between 700 Hz and 1000 Hz, i.e., the unknown numerator coefficient $a_1 \in [\cos(2\pi T_s \times 700), \cos(2\pi T_s \times 1000)]$. When using the output error identification method with a fixed compensator and a forgetting factor of 1, one will have $\alpha = 1/2$ in $C(z^{-1})/A(z^{-1}) - \alpha$. Fig. 1 demonstrates the frequency response of 7 possible $1/A(z^{-1})$'s uniformly sampled from the uncertainty region. One can observe that in a large frequency region, the phase responses of $1/A(z^{-1})$ is below -90 degree, i.e., $Re\left(1/A(e^{-j\omega})\right) \leq 0$. Therefore, $1/A(z^{-1})$ is not SPR, not to say $1/A(z^{-1}) - 1/2$.
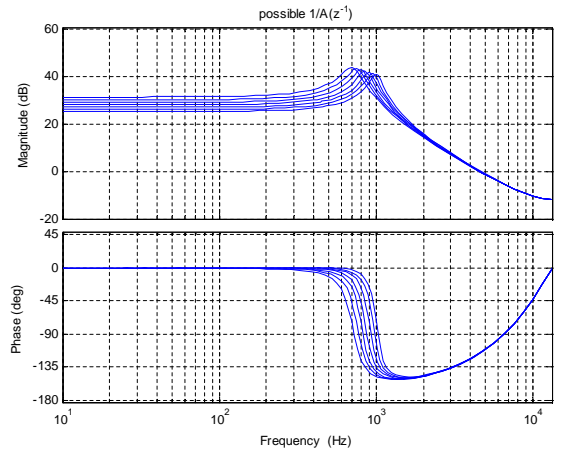

Figure 1. Frequency responses: samples of the uncertain $\frac{1}{A(z^{-1})}$.

It is easy to check that $\forall a_1 \in [\cos(2\pi T_s \times 700), \cos(2\pi T_s \times 1000)]$, $A(z^{-1})$ is stable. The first condition for SPR transfer functions is thus satisfied. The two edge polynomials in this case are $A_1(z^{-1}) = 1 - 2\beta \underline{a}_1 z^{-1} + \beta^2 z^{-2}$ and $A_2(z^{-1}) = 1 - 2\beta \bar{a}_1 z^{-1} + \beta^2 z^{-2}$, with $\underline{a}_1 = \cos(2\pi T_s \times 700)$ and $\bar{a}_1 = \cos(2\pi T_s \times 1000)$. Formulating and solving (via [29]) the feasibility design in Section 4, with $m = n = 2$, we obtain

$$C(z^{-1}) = 12.36 - 10.71z^{-1} - 1.162z^{-2}. \tag{17}$$

Plotting the frequency responses of the sampled $1/A(z^{-1})$ and $C(z^{-1})/A(z^{-1}) - 1/2$ in Fig. 2, one observes that $\forall \omega$, $C(z^{-1})$ is capable of providing robust compensation such that $C(e^{-j\omega})/A(e^{-j\omega}) - 1/2$ stays strictly in the open right-half complex plane (phase$\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$), which, combined with the condition that $A(z^{-1})$ is always stable, indicates the success of the robust SPR design.
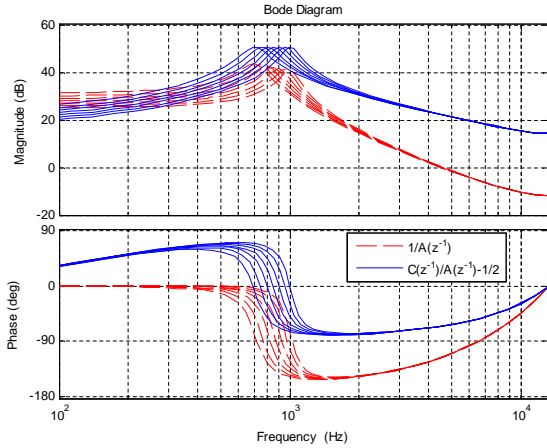

Figure 2.   Frequency responses of the sampled $\frac{1}{A(z^{-1})}$ and $\frac{C(z^{-1})}{A(z^{-1})} - \frac{1}{2}$.

Notice however in Fig. 2, that large gain variations exist in $C(z^{-1})/A(z^{-1}) - 1/2$ and that $C(z^{-1})/A(z^{-1})$ is far away from the unity function. Indeed, the roots of $C(z^{-1})$ are $\{0.9637, -0.0976\}$ while the roots of $A(z^{-1})$ always appear in complex pairs and range from $\beta e^{\pm j2\pi T_s \times 700}$ $(0.9664 \pm 0.1625i)$ to $\beta e^{\pm j2\pi T_s \times 1000}$ $(0.9524 \pm 0.2310i)$. It is thus seen that the obtained $C(z^{-1})$ makes $C(z^{-1})/A(z^{-1}) - 1/2$ robustly SPR, but does not reflect intuitive information about the system $A(z^{-1})$ and additionally may negatively influence convergence of the adaptation process. The algorithm in Section 5.1 is then applied to improve the result. The objective of $\min \|C(z^{-1})/A^*(z^{-1}) - 1\|_\infty$ is enforced, with $A^*(z^{-1})$

being the center of the polytope. The resulted solution is $C(z^{-1}) = 1.027 - 1.932z^{-1} + 0.9464z^{-2}$. Fig. 3 shows the sampled frequency responses of the newly obtained $C(z^{-1})/A(z^{-1}) - 1/2$. Besides the achievement of the robust SPR requirement, $C(e^{-j\omega})/A(e^{-j\omega}) - 1/2$ is significantly confined to be in a smaller region: compared to Fig. 2, the phase of $C(e^{-j\omega})/A(e^{-j\omega}) - 1/2$ is close to 0 degree at the majority of frequencies; the magnitude response is condensed to be within 0.2503 (-12.03 dB) and 2.6931 (8.605 dB); specifically at high frequencies, $C(e^{-j\omega})/A(e^{-j\omega}) - 1/2$ is approximately 0.5107 (-5.836 dB). Therefore, $C(z^{-1})/A(z^{-1})$ is much closer to 1 from the optimal design. The roots of $C(z^{-1})$ in this case are $\{0.9402 \pm 0.1928i\}$, which reflect much more information about $A(z^{-1})$ compared to (17).
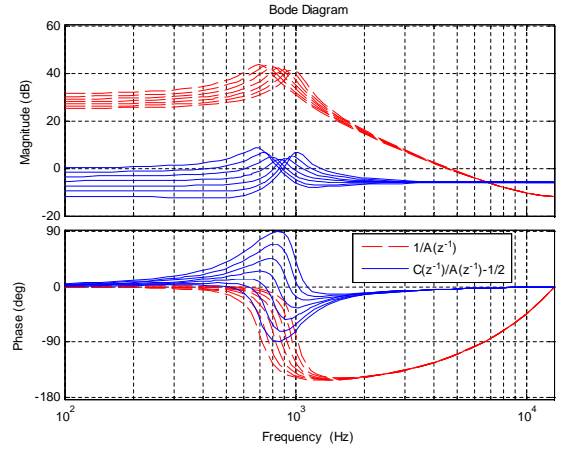

Figure 3.   Sampled frequency responses of $\frac{1}{A(z^{-1})}$ and $\frac{C(z^{-1})}{A(z^{-1})} - \frac{1}{2}$: with the objective of $\min \|C(z^{-1})/A(z^{-1}) - 1\|_\infty$.

Finally we explore the optimal $C(z^{-1})$ that has the minimum high-frequency magnitude. Applying the algorithms in Section 5.3, we obtain the new optimal solution

$$C^*(z^{-1}) = 6.7331 - 0.7938z^{-1} - 5.5802z^{-2}. \tag{18}$$

Comparing the frequency responses of (17) and (18) in Fig. 4, and noticing the deep notch in the solid line of the magnitude responses, we can see that (18) indeed provides strong high-frequency gain attenuation due to the cost function design in (16).

7   CONCLUSIONS AND DISCUSSIONS

In this paper, a convex optimization approach is proposed to address the design of robust strictly positive real
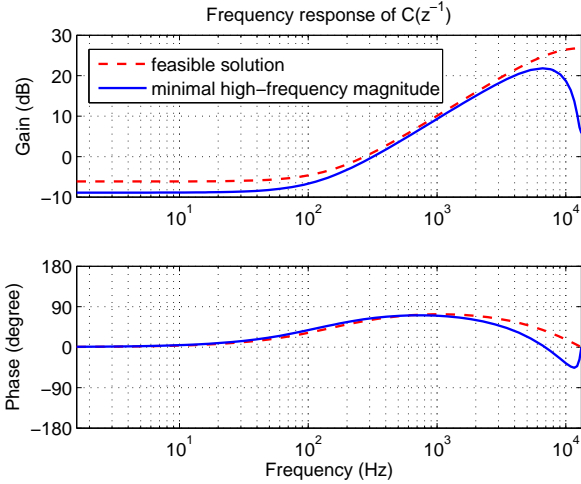
Figure 4. Frequency responses of the optimal $C^*(z^{-1})$ and the $C(z^{-1})$ from the feasibility design.

transfer functions. It is shown that a feasibility SDP formulation can be used to provide the compensator that achieves the desired robust SPR condition. Moreover, the important issue of maintaining the designed transfer function to be close to 1 is addressed, by adding an infinity-norm minimization in the optimization. Additional concepts of cost function design are introduced, which lead to solutions of several new problems. All the formulated optimization problems can be efficiently solved by interior-point methods in convex optimization.

Notice that although the focus has been placed on the discrete-time SPR analysis. By applying the continuous-time positive-real lemma and bounded-real lemma, the present work can be easily extended to solve the continuous-time version of the problem. There is yet one additional condition that makes the latter problem easier to solve: in the continuous-time case, for the SPR condition to hold, the relative degree of $G_p(s)$ equals zero or one [7, 20], hence greatly simplifying the formulation of matrix inequalities.

As for the required computations, mature theories for solving the SDP program are available from the optimization community and various software can efficiently solve the formulated optimization problem. [33] provides benchmark tests for various free optimization software, on the computation time and accuracy for relatively large-size problems. Low-size problems such as the one in the case study can be solved (offline) within one second of time on the majority of recent personal computers.

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