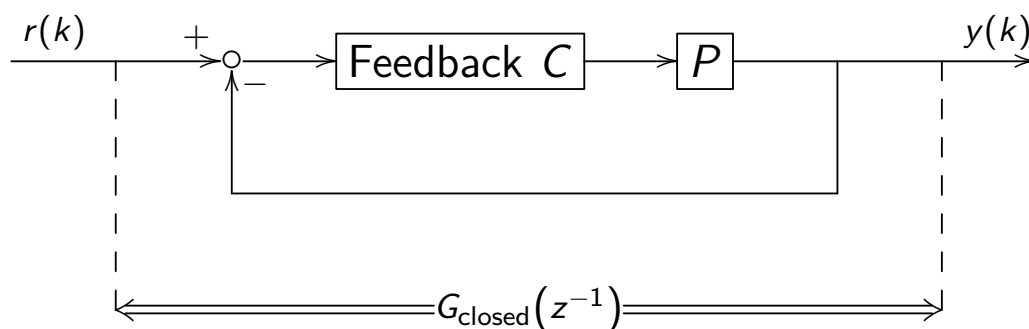# Lecture 11: Feedforward Control Zero Phase Error Tracking

Big picture
Stable pole-zero cancellation
Phase error
Zero phase error tracking

Big picture
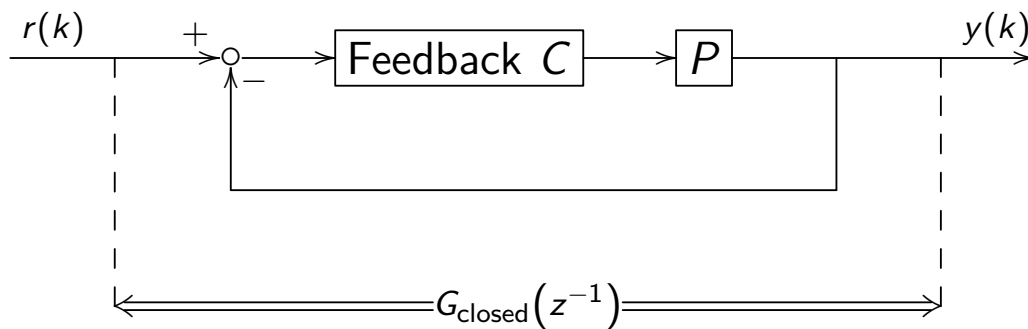
why are we learning this:

- ▶ two basic control problems: tracking (the reference) and regulation (against disturbances)
- ▶ feedback control has performance limitations
- ▶ For tracking $r(k)$, ideally we want

$$G_{\text{closed}}\left(z^{-1}\right) = 1$$

which is **not attainable** by feedback. We thus need **feedforward** control.

Big picture

- notation:

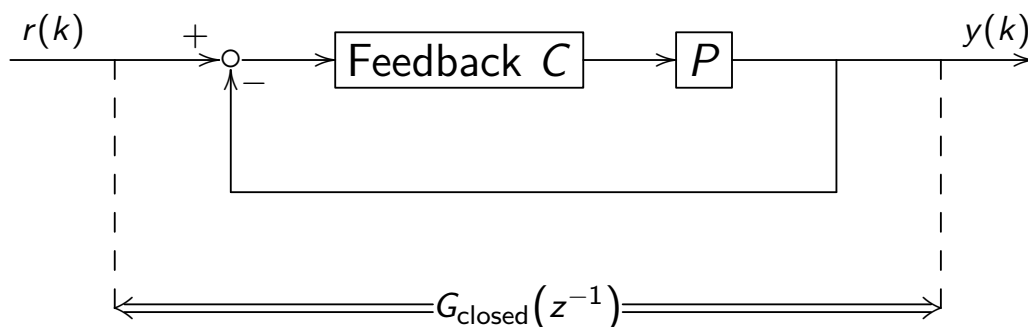$$G_{\text{closed}} \left(z^{-1} \right) = \frac{z^{-d} B_c \left(z^{-1} \right)}{A_c \left(z^{-1} \right)}$$

where

$$B_c \left(z^{-1} \right) = b_{c0} + b_{c1} z^{-1} + \cdots + b_{cm} z^{-m}, \ \ b_{co} \neq 0$$
$$A_c \left(z^{-1} \right) = 1 + a_{c1} z^{-1} + \cdots + a_{cn} z^{-n}$$

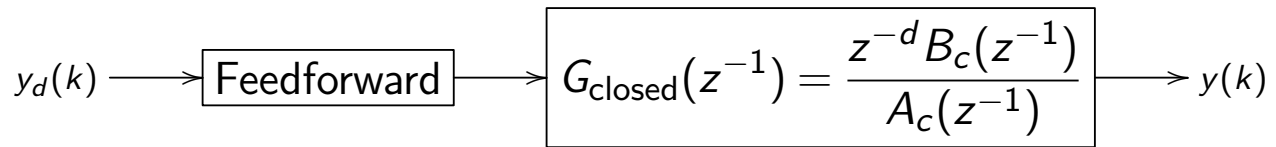- z^{-1}: one-step delay operator. $z^{-1} r(k) = r(k-1)$

Big picture

one naive approach: to let $y(k)$ track $y_d(k)$, we can do

$$r(k) = G_{\text{closed}}^{-1} \left(z^{-1} \right) y_d(k) = \frac{z^d A_c \left(z^{-1} \right)}{B_c \left(z^{-1} \right)} y_d(k) = \frac{A_c \left(z^{-1} \right)}{B_c \left(z^{-1} \right)} y_d(k+d) \quad (1)$$

- **causality**: (1) requires knowledge of $y_d(k)$ for at least d steps ahead (usually not an issue)
- **stability**: poles of $G_{\text{closed}}^{-1} \left(z^{-1} \right)$, i.e., zeros of $G_{\text{closed}} \left(z^{-1} \right)$, must be all stable (usually an issue)
- **robustness**: the model $G_{\text{closed}} \left(z^{-1} \right)$ needs to be accurate

The cancellable parts in $G_{\text{closed}}\left(z^{-1}\right)$

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

- $G_{\text{closed}}\left(z^{-1}\right)$ is always stable $\Rightarrow A_c\left(z^{-1}\right)$ can be fully canceled
- $B_c\left(z^{-1}\right)$ may contain *uncancellable parts* (zeros on or outside the unit circle)
- partition $G_{\text{closed}}\left(z^{-1}\right)$ as

$$G_{\text{closed}}\left(z^{-1}\right) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})} = \frac{z^{-d}\overbrace{B_c^+(z^{-1})}^{\text{cancellable}}\overbrace{B_c^-(z^{-1})}^{\text{uncancellable}}}{A_c(z^{-1})} \quad (2)$$

Stable pole-zero cancellation

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$
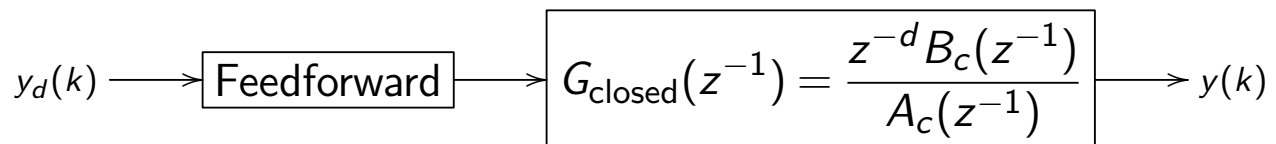
feedforward via stable pole-zero cancellation:

$$G_{\text{spz}}\left(z^{-1}\right) = \frac{z^d A_c\left(z^{-1}\right)}{B_c^+\left(z^{-1}\right)} \frac{1}{B_c^-(1)} \quad (3)$$

where $B_c^-(1) = B_c^-\left(z^{-1}\right)\big|_{z^{-1}=1}$
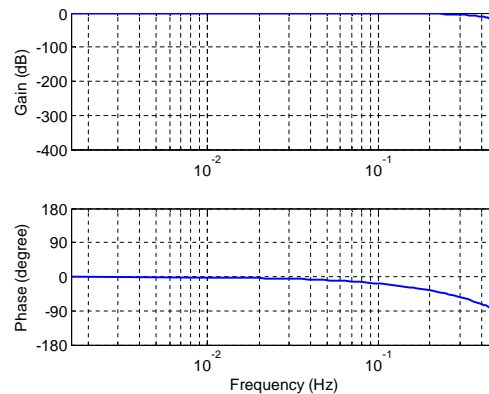- $B_c^-(1)$ makes the overall DC gain from $y_d(k)$ to $y(k)$ to be one:

$$G_{y_d \to y}\left(z^{-1}\right) = G_{\text{spz}}\left(z^{-1}\right) G_{\text{closed}}\left(z^{-1}\right) = \frac{B_c^-(z^{-1})}{B_c^-(1)}$$

- example: $B_c^-\left(z^{-1}\right) = 1 + z^{-1}$, $B_c^-(1) = 2$, then

$$G_{y_d \to y}\left(z^{-1}\right) = \frac{1 + z^{-1}}{2}: \text{ a moving-average low-pass filter}$$

Stable pole-zero cancellation

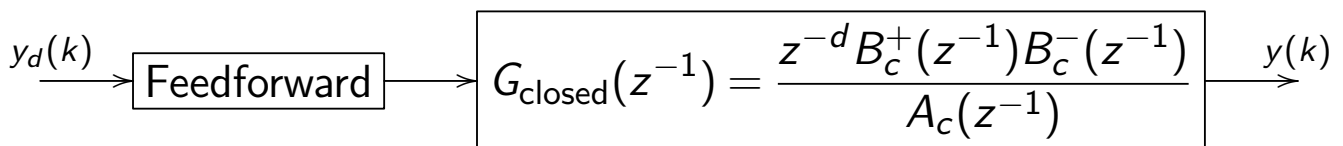properties of $G_{y_d \to y}\left(z^{-1}\right) = \frac{1+z^{-1}}{2}$:

- there is always a **phase error** in tracking
- example: if $y_d(k) = \alpha k$ (a ramp signal)

$$y(k) = G_{y_d \to y}\left(z^{-1}\right) y_d(k) = \alpha k - \frac{\alpha}{2}$$

which is always delayed by a factor of $\alpha/2$

Zero Phase Error Tracking (ZPET)

Zero Phase Error Tracking (ZPET): extend (3) by adding a $B_c^-(z)$ part

$$G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2} \tag{4}$$

where $B_c^-(z) = b_{c0}^- + b_{c1}^- z + \cdots + b_{cs}^- z^s$ if
$B_c^-(z^{-1}) = b_{c0}^- + b_{c1}^- z^{-1} + \cdots + b_{cs}^- z^{-s}$

- overall dynamics between $y(k)$ and $y_d(k)$:

$$G_{y_d \to y}\left(z^{-1}\right) = G_{\text{closed}}\left(z^{-1}\right) G_{\text{ZPET}}\left(z^{-1}\right) = \frac{B_c^-(z) B_c^-(z^{-1})}{\left[B_c^-(1)\right]^2} \tag{5}$$

Zero Phase Error Tracking (ZPET)

understanding (5):

- ▶ the frequency response always has **zero phase error**:

$$B_c^- \left(e^{j\omega} \right) = \overline{B_c^- \left(e^{-j\omega} \right)} \text{ (a complex conjugate pair)}$$

- ▶ $B_c^- (1)^2$ normalizes $Gy_d \to y$ to have unity DC gain:

$$G_{y_d \to y} \left(e^{-j\omega} \right) \Big|_{\omega=0} = \frac{B_c^-(e^{j\omega}) \big|_{\omega=0} \; B_c^-(e^{-j\omega}) \big|_{\omega=0}}{\left[B_c^- (1) \right]^2} = \frac{\left[B_c^- (1) \right]^2}{\left[B_c^- (1) \right]^2} \nearrow 1$$

- ▶ example: $B_c^- \left(z^{-1} \right) = 1 + z^{-1}$, then

$$G_{y_d \to y} \left(z^{-1} \right) = \frac{(1+z)\left(1 + z^{-1} \right)}{4}$$

- ▶ if $y_d (k) = \alpha k$, then $y(k) = \alpha k$!
- ▶ fact: ZPET provides perfect tracking to step and ramp signals at steady state (see ME 233 course reader)

Zero Phase Error Tracking (ZPET)

Example: $B_c^- \left(z^{-1} \right) = 1 + 2z^{-1}$

$$G_{y_d \to y} \left(z^{-1} \right) = \frac{(1+2z)(1+2z^{-1})}{9} = \frac{2z + 5 + 2z^{-1}}{9}$$
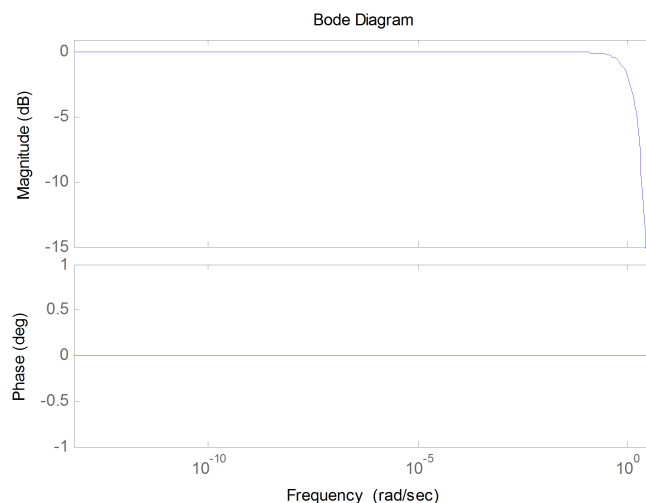
Figure: Bode Plot of $\frac{2z+5+2z^{-1}}{9}$

Implementation

$$y_d(k) \longrightarrow \boxed{G_{\text{ZPET}}\left(z^{-1}\right)} \xrightarrow{r(k)} \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c^+(z^{-1})B_c^-(z^{-1})}{A_c(z^{-1})}} \xrightarrow{y(k)}$$
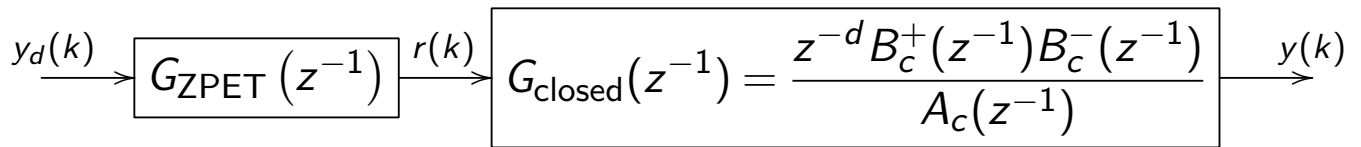
$$r(k) = \left[\frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k)$$

- z^d is not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d)$$

- $B_c^-(z) = b_{c0}^- + b_{c1}^- z + \cdots + b_{cs}^- z^s$ is also not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})} \frac{z^{-s}B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d+s)$$
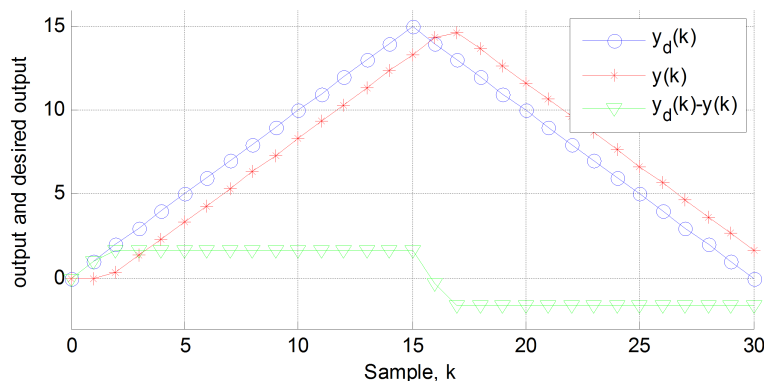
- at time k, requires $y_d(k+d+s)$: $d+s$ steps preview of the desired output

Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$
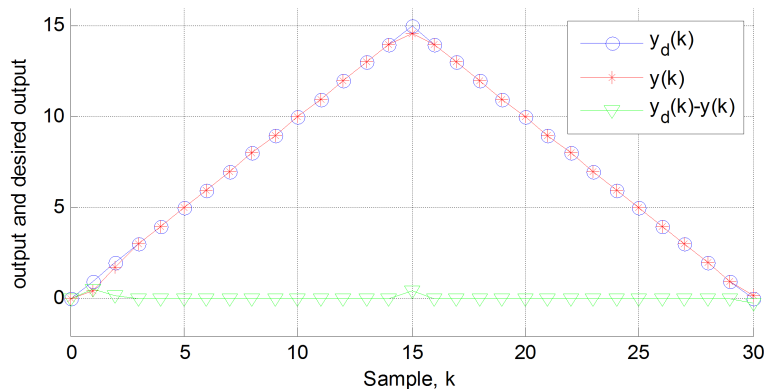
- without feedforward control:

Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$

▶ with ZPET feedforward:

Implementation

ZPET extensions:

▶ standard form:

$$G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}$$

▶ extended bandwidth (ref: B. Haack and M. Tomizuka, "The effect of adding zeros to feedforward controllers," *ASME J. Dyn. Syst. Meas. Control*, vol. 113, pp. 6-10, 1991):

$$G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2} \frac{(z^{-1}-b)(z-b)}{(1-b)^2}, \ 0 < b < 1$$

▶ remark: $(z^{-1}-b)(z-b)/(1-b)^2$, $0 < b < 1$ is a high-pass filter with unity DC gain