Lecture 2: Discrete-time Linear Quadratic Optimal Control

Big picture
Example
Convergence of finite-time LQ solutions

Big picture

- previously: dynamic programming and **finite-horizon** discrete-time LQ
- this lecture: infinite-horizon discrete-time LQ and its properties

Review: solution of the general discrete-time LQ problem

- system dynamics:

$$x(k+1) = A(k)x(k) + B(k)u(k), \; x(k_0) = x_o \tag{1}$$

- performance index:

$$J = \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{k=k_0}^{N-1}\left\{x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)\right\}$$

$$Q(k) = C^T(k)C(k) \succeq 0, \; S = S^T \succeq 0, \; R(k) = R^T(k) \succ 0$$

- optimal $J^o = \frac{1}{2}x_o^T P(0) x_o$ achieved by the state-feedback control law:

$$u^o(k) = -\left[R(k) + B^T(k)P(k+1)B(k)\right]^{-1} B^T(k)P(k+1)A(k)x(k)$$

- Riccati equation:

$$P(k) = A^T(k)P(k+1)A(k) + Q(k)$$
$$-A^T(k)P(k+1)B(k)\left[R(k) + B^T(k)P(k+1)B(k)\right]^{-1} B^T(k)P(k+1)A(k)$$

  with the boundary condition $P(N) = S$.

Example: double integrator

- plant dynamics:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k), \; T = 1$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- performance index:

$$J_N(x_0) = \frac{1}{2}x^T(N)Sx(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left\{x^T(k)Qx(k) + Ru^2(k)\right\}$$

  where

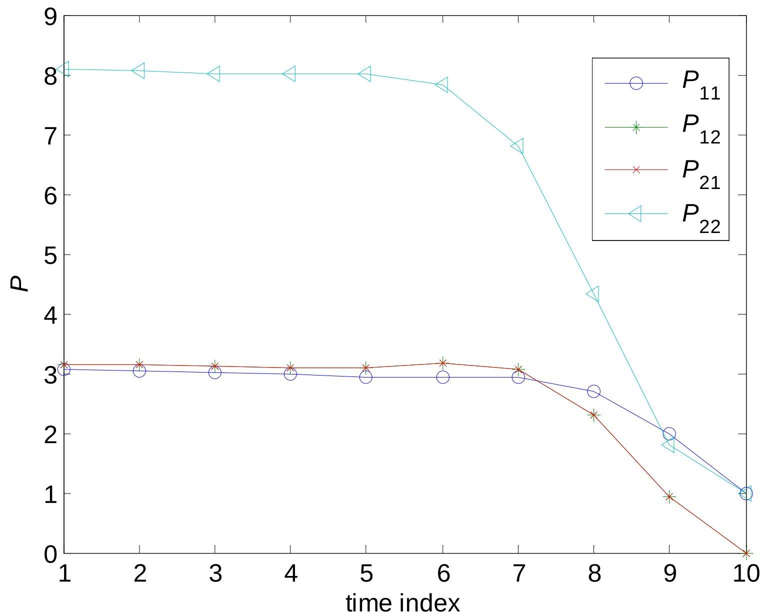
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; R > 0$$

- next: examine the convergence of $P(k)$ with different $P(N) = S\,(\succeq 0)$

$$P(k) = A^T P(k+1)A + Q - A^T P(k+1)B\left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A$$

Example: double integrator

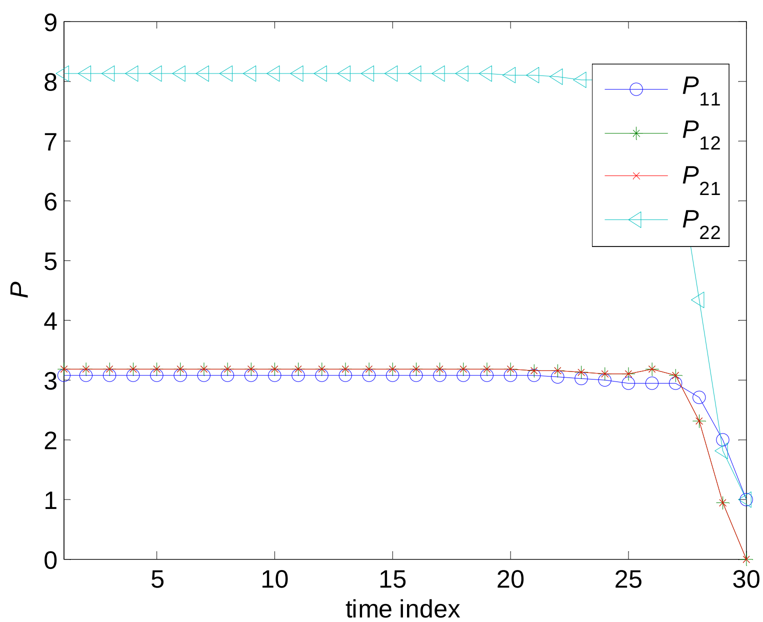
- $N = 10,\ P(N) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



$$P(k) = A^T P(k+1) A + Q - A^T P(k+1) B \left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) A$$

Example: double integrator

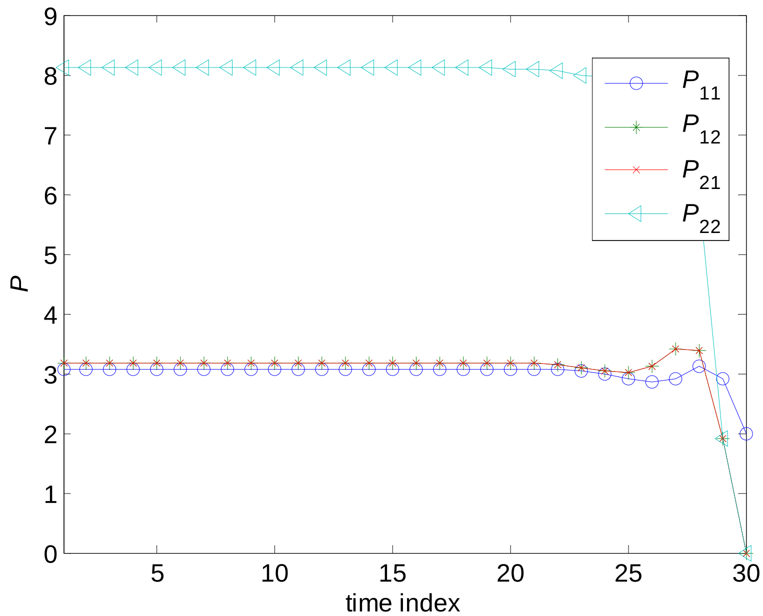
- $N = 30,\ P(N) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



$$P(k) = A^T P(k+1) A + Q - A^T P(k+1) B \left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) A$$

Example: double integrator

- $N = 30$, $P(N) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$



$$P(k) = A^T P(k+1) A + Q - A^T P(k+1) B \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A$$

Example: double integrator

observations:

- $P(k)$ is indeed always symmetric
- regardless of the boundary condition $P(N) (\succeq 0)$, the solution of the Riccati equation converges to the same steady state P_s
- the control law

$$u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A x(k)$$

thus converges (backwards) to

$$u^o(k) = -\underbrace{\left[R + B^T P_s B\right]^{-1} B^T P_s A}_{K_s} x(k)$$

From finite-horizon to infinite-horizon LQ

in the case of $N \to \infty$, it turns out that

- (A, B) is controllable or stabilizable $\Rightarrow$ guaranteed convergence of $P(k)$ to a bounded P_s
- intuition: if (A, B) is unstabilizable, then there are unstable uncontrollable modes that may cause

$$\lim_{N\to\infty} J_N(x_0) = \lim_{N\to\infty} \left\{ \frac{1}{2} x^T(N) S x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left\{ x^T(k) Q x(k) + u^T(k) R(k) u(k) \right\} \right\} = \infty$$

yielding $J_N^o(x_0) = \frac{1}{2} x_o^T P(0) x_0 = \infty \Rightarrow \lim_{N\to\infty} ||P(0)|| = \infty$

Infinite-horizon discrete-time LQ for LTI systems

- system dynamics:

$$x(k+1) = Ax(k) + Bu(k), \; x(k_0) = x_o \tag{2}$$

- performance index:

$$J = \frac{1}{2} \sum_{k=k_0}^{\infty} \left\{ x^T(k) Q x(k) + u^T(k) R u(k) \right\}, \; Q \succeq 0, \; R \succ 0$$

- optimal state-feedback control law:

$$u^o(k) = -\underbrace{\left[R + B^T P_s B \right]^{-1} B^T P_s A}_{K_s} x(k)$$

- Algebraic Riccati equation:

$$P_s = A^T P_s A + Q - A^T P_s B \left[R + B^T P_s B \right]^{-1} B^T P_s A$$

Infinite-horizon discrete-time LQ for LTI systems

conditions for a meaningful solution:

$$\boxed{\begin{array}{l}(A,B) \text{ controllable/stabilizable and } (A,C) \text{ observable/detectable} \\ (Q=C^TC) \Rightarrow \text{ guaranteed closed-loop asymptotic stability for} \\ x(k+1)=(A-BK_s)x(k) \triangleq A_{cl}x(k)\end{array}}$$

"(A,B) controllable/stabilizable $\Rightarrow P_s$ and K_s bounded": already shown

"observability $\Rightarrow P_s \succ 0$": with $u^o(k)=-K_sx(k)$ and $Q=C^TC$

$$x_o^TP_sx_o=\sum_{k=0}^{\infty}\left\{x^T(k)Qx(k)+u^T(k)Ru(k)\right\}=\sum_{k=0}^{\infty}\left\{x^T(k)\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}^T\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}x(k)\right\}$$

$$=\sum_{k=0}^{\infty}\left\{x_o^T\left(A_{cl}^k\right)^T\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}^T\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}A_{cl}^kx_o\right\}=x_o^TW_{cl}x_o$$

where W_{cl} is the observability gramian for

$$x(k+1)=(A-BK_s)x(k)=A_{cl}x(k) \tag{3}$$

$$\tilde{y}(k)=\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}x(k)$$

Infinite-horizon discrete-time LQ for LTI systems

"observability $\Rightarrow P_s \succ 0$" (continued):

- observability is invariant under static output feedback control $\Rightarrow$

$$x(k+1)=(A-BK_s)x(k)=A_{cl}x(k)$$

$$\tilde{y}(k)=\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}x(k)$$

is observable if the open-loop system

$$x(k+1)=Ax(k)+Bu(k)$$

$$\tilde{y}(k)=\begin{bmatrix}C\\R^{1/2}K_s\end{bmatrix}x(k)$$

is observable (which holds as (A,C) is observable). Hence $P_s=W_{cl}$ is positive definite under observability. Analogous analysis can be applied to the detectability case.

Infinite-horizon discrete-time LQ for LTI systems

closed-loop stability of

$$\begin{cases} x(k+1) &= (A - BK_s)x(k) = A_{cl}x(k) \\ \tilde{y}(k) &= \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix} x(k) \end{cases} \tag{4}$$

comes from a transformation from Riccati equation to Lyapunov equation:

$$\begin{aligned}
P_s &= A^T P_s A + Q - \underline{\underline{A^T P_s B \left[R + B^T P_s B\right]^{-1} B^T P_s A}} \\
&= \underline{A^T P_s A + Q} - \underbrace{A^T P_s B \left[R + B^T P_s B\right]^{-1}}_{K_s^T} \left[R + B^T P_s B\right] \underbrace{\left[R + B^T P_s B\right]^{-1} B^T P_s A}_{K_s} \\
&= \underline{(A - BK_s)^T P_s (A - BK_s) + 2\underline{A^T P_s BK_s} - K_s^T B^T P_s BK_s + C^T C} - K_s^T \left(R + B^T P_s B\right) K_s \\
&= (A - BK_s)^T P_s (A - BK_s) + C^T C + K_s^T R K_s
\end{aligned}$$

$$\Longleftrightarrow \boxed{(A - BK_s)^T P_s (A - BK_s) - P_s = -\begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix} \preceq 0} \tag{5}$$

observability of (4) plus $P_s \succ 0 \Rightarrow$ closed-loop stability from (5)

Remark

Theorem (An extension of Lyapunov theory based on observability).

if we find from a Lyapunov equation $A^T PA - P = -Q$ where $P \succ 0$, $Q = C^T C \succeq 0$, and (A, C) is observable, then the system $x(k+1) = Ax(k)$ is asymptotically stable.

Proof: Since $P \succ 0$ and $Q \succeq 0$, the system is stable in the sense of Lyapunov. All eigenvalues of A are hence on or inside the unit circle. Pick any eigenvalue-eigenvector pair (λ, v) where λ is on the unit circle. Then $||Cv||_2^2 = v^* Q v = -v^* \left(A^T PA - P\right) v = -\left(|\lambda|^2 - 1\right) v^* P v = 0$, which implies $Cv = 0$. We thus have

$$\begin{cases} Av &= \lambda v \\ Cv &= 0 \\ CAv &= \lambda Cv = 0 \\ \vdots & \\ CA^{n-1}v &= 0 \end{cases} \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0 \nLeftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has to be full-column rank}$$

Infinite-horizon discrete-time LQ for LTI systems

an example of closed-loop stability requirement: consider

$$x(k+1) = 2x(k) + u(k),\ Q = 0,\ R = 1$$

- the state constraint Q is zero, the optimal control is $u^o(k) = 0$
- the closed-loop system is thus unstable
- on the other hand, (A, C) is clearly unobservable
- the Riccati equation however still converges, as (A, B) is controllable

Infinite-horizon discrete-time LQ for LTI systems

Excellent closed-loop properties

- guaranteed closed-loop stability (just shown moments ago)
- good margins for single-input systems (see ME232 class reader):

$$\text{Phase margin} > 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{R}{R+B^TP_sB}}\right)$$

$$\text{Gain margin} > \frac{1}{1-\sqrt{\frac{R}{R+B^TP_sB}}}$$

- guaranteed stability for "% loop gain change" (see ME232 class reader):

$$\frac{100}{1+\sqrt{\frac{R}{R+B^TP_sB}}} < \text{loop gain change} < \frac{100}{1-\sqrt{\frac{R}{R+B^TP_sB}}}$$

Summary

1. Review: general disccrete-time LQ problem
2. Example: double integrator
3. Convergence of the Riccati equation solution
4. Infinite-horizon discrete-time LQ and its properties