Lecture 16: Stability of Parameter Adaptation Algorithms

Big picture

- For
$$\hat{\theta}(k+1) = \hat{\theta}(k) + [\text{correction term}]$$
we haven't talked about whether $\hat{\theta}(k)$ will converge to the true value θ if $k \to \infty$. We haven't even talked about whether $\hat{\theta}(k)$ will stay bounded or not!
- Tools of stability evaluation: Lyapunov-based analysis, or hyperstability theory (topic of this lecture)

Outline

1. Big picture
2. Hyperstability theory
   Passivity
   Main results
   Positive real and strictly positive real
   Understanding the hyperstability theorem
3. Procedure of PAA stability analysis by hyperstability theory
4. Appendix
   Strictly positive real implies strict passivity

Hyperstability theory

history

Vasile M. Popov:

- born in 1928, Romania
- retired from University of Florida in 1993
- developed hyperstability theory independently from Lyapnov theory

Hyperstability theory

Consider a closed-loop system in Fig. 1

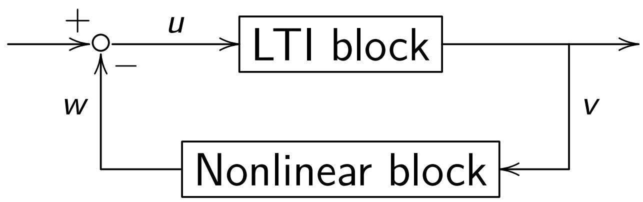


Figure 1: Block diagram for hyperstability analysis

The linear time invariant (LTI) block is realized by

continuous-time case:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$v(t) = Cx(t) + Du(t)$$

discrete-time case:

$$x(k+1) = Ax(k) + Bu(k)$$
$$v(k) = Cx(k) + Du(k)$$

Hyperstability discusses conditions for "nice" behaviors in x.

Passive systems

Definition (Passive system).

The system $v \longrightarrow \boxed{\text{System}} \longrightarrow w$ is called passive if

$$\int_0^{t_1} w^T(t)\, v(t)\, dt \geq -\gamma^2,\ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\, v(k) \geq -\gamma^2,\ \forall k_1 \geq 0$$

where δ and γ depends on the initial conditions.

- intuition: $\int_0^{t_1} w^T(t)\, v(t)\, dt$ is the work/supply done to the system. By conservation of energy,

$$E(t_1) \leq E(0) + \int_0^{t_1} w^T(t)\, v(t)\, dt$$

Strictly passive systems

If the equality is *strict* in the passivity definition, with

$$\int_0^{t_1} w^T(t)\,v(t)\,\mathrm{d}t \geq -\gamma^2 + \delta \int_0^{t_1} v^T(t)\,v(t)\,\mathrm{d}t + \varepsilon \int_0^{t_1} w^T(t)\,w(t)\,\mathrm{d}t,\ \forall t_1 \geq 0$$

or in the discrete-time case

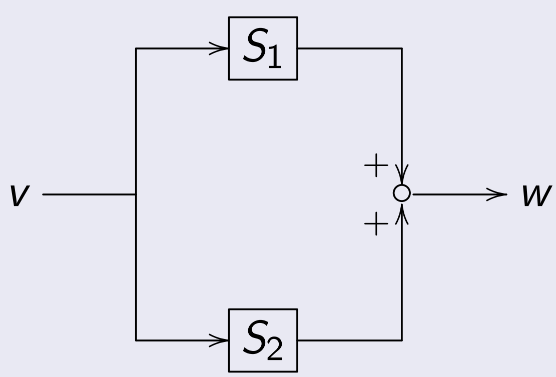
$$\sum_{k=0}^{k_1} w^T(k)\,v(k) \geq -\gamma^2 + \delta \sum_{k=0}^{k_1} v^T(k)\,v(k) + \varepsilon \sum_{k=0}^{k_1} w^T(k)\,w(k),\ \forall k_1 \geq 0$$

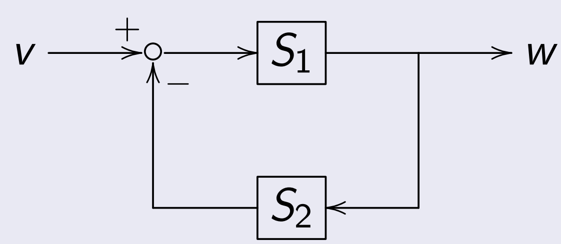
where $\delta \geq 0$, $\varepsilon \geq 0$, but not both zero, the system is *strictly passive.*

Passivity of combined systems

Fact (Passivity of connected systems).

If two systems S_1 and S_2 are both passive, then the following parallel and feedback combination of S_1 and S_2 are also passive





Hyperstability theory

Definition (Hyperstability).

The feedback system in Fig. 1 is hyperstable if and only if there exist positive constants $\delta > 0$ and $\gamma > 0$ such that

$$\|x(t)\| < \delta\left[\|x(0)\| + \gamma\right],\ \forall t > 0 \text{ or } \|x(k)\| < \delta\left[\|x(0)\| + \gamma\right],\ \forall k > 0$$

for all feedback blocks that satisfy the *Popov inequality*

$$\int_0^{t_1} w^T(t)\, v(t)\, \mathrm{d}t \geq -\gamma^2,\ \forall t_1 \geq 0 \text{ or } \sum_{k=0}^{k_1} w^T(k)\, v(k) \geq -\gamma^2,\ \forall k_1 \geq 0$$

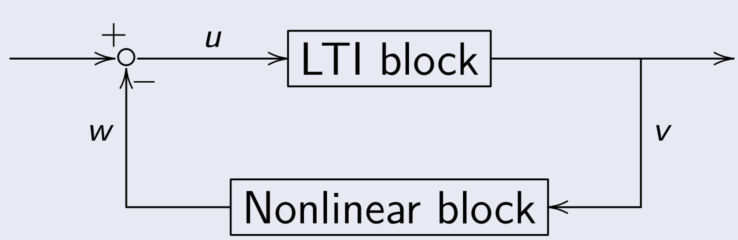
In other words, the LTI block is bounded in states for any initial conditions for any *passive* nonlinear blocks.

Hyperstability theory

Definition (Asymptotic hyperstability).

The feedback system below is asymptotically hyperstable if and only if it is hyperstable and for all *bounded w* satisfying the Popov inequality we have

$$\lim_{k\to\infty} x(k) = 0$$



Hyperstability theory

Theorem (Hyperstability).

The feedback system in Fig. 1 is hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *(i.e., it is passive) and the LTI transfer function is* ***positive real****.*

Theorem (Asymptotical hyperstability).

The feedback system in Fig. 1 is asymptotically hyperstable if and only if the nonlinear block satisfies ***Popov inequality*** *and the LTI transfer function is* ***strictly positive real****.*

intuition: a strictly passive system in feedback connection with a passive system gives an asymptotically stable closed loop.

Positive real and strictly positive real

Positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is called *positive real* (**PR**) if

- $G(s)$ is real for real values of s
- $\mathrm{Re}\{G(s)\} > 0$ for $\mathrm{Re}\{s\} > 0$

The above is intuitive but not practical to evaluate. Equivalently, $G(s)$ is PR if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} > 0$ (no unstable poles)
2. any pole on the imaginary axis $j\omega_0$ does not repeat and the associated residue (i.e., the coefficient appearing in the partial fraction expansion) $\lim_{s\to j\omega_0}(s - j\omega_0)G(s)$ is non-negative
3. $\forall \omega \in \mathbb{R}$ where $s = j\omega$ is not a pole of $G(s)$, $G(j\omega) + G(-j\omega) = 2\mathrm{Re}\{G(j\omega)\} \geq 0$

Positive real and strictly positive real

Strictly positive real transfer function (continuous-time case): a SISO transfer function $G(s)$ is *strictly positive real* (**SPR**) if

1. $G(s)$ does not possess any pole in $\mathrm{Re}\{s\} \geq 0$
1. $\forall \omega \in \mathbb{R},\ G(j\omega) + G(-j\omega) = 2\mathrm{Re}\{G(j\omega)\} > 0$

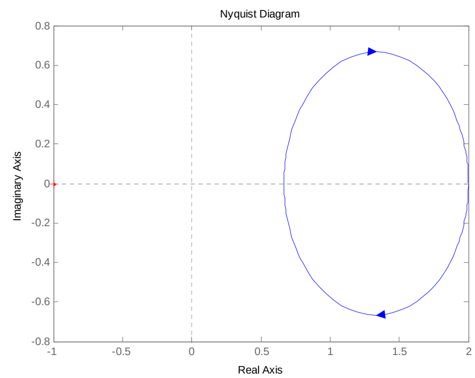


Figure: example Nyquist plot of a SPR transfer function

Positive real and strictly positive real

discrete-time case

A SISO discrete-time transfer function $G(z)$ is positive real (**PR**) if:

1. it does not possess any pole outside of the unit circle
2. any pole on the unit circle does not repeat and the associated redsidue is non-negative
3. $\forall |\omega| \leq \pi$ where $z = e^{j\omega}$ is not a pole of $G(z)$, $G(e^{-j\omega}) + G(e^{j\omega}) = 2\mathrm{Re}\{G(e^{j\omega})\} \geq 0$

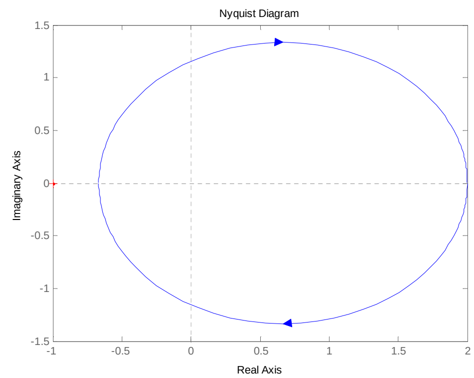
$G(z)$ is strictly positive real (**SPR**) if:

1. $G(z)$ does not possess any pole outside of or on the unit circle on z-plane
2. $\forall |\omega| < \pi,\ G(e^{-j\omega}) + G(e^{j\omega}) = 2\mathrm{Re}\{G(e^{j\omega})\} > 0$

Examples of PR and SPR transfer functions

- $G(z) = c$ is SPR if $c > 0$
- $G(z) = \frac{1}{z-a}$, $|a| < 1$ is asymptotically stable but not PR:

$$2\mathrm{Re}\left\{G\left(e^{j\omega}\right)\right\} = \frac{1}{e^{j\omega}-a} + \frac{1}{e^{-j\omega}-a}$$
$$= 2\frac{\cos\omega - a}{1+a^2-2a\cos\omega}$$



- $G(z) = \frac{z}{z-a}$, $|a| < 1$ is asymptotically stable and SPR

Strictly positive real implies strict passivity

It turns out [see Appendix (to prove on board at the end of class)]:
Lemma: the LTI system $G(s) = C(sI - A)^{-1}B + D$ (in minimal realization)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Analogous results hold for discrete-time systems.

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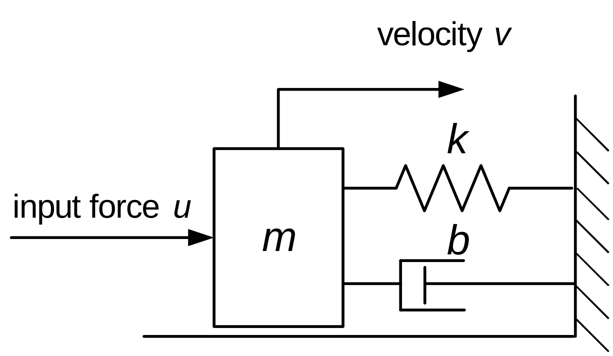
Understanding the hyperstability theorem

Example: consider a mass-spring-damper system

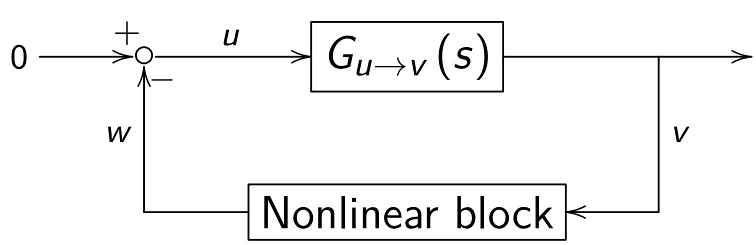


$$m\ddot{x} + b\dot{x} + kx = u \Rightarrow$$

$$G_{u\to x}(s) = \frac{1}{ms^2 + bs + k}$$

$$G_{u\to v}(s) = \frac{s}{ms^2 + bs + k}$$

with a general nonlinear feedback control law



- $\int_0^{t_1} u(t)v(t)dt$ is the total energy supplied to the system

Understanding the hyperstability theorem

- if the nonlinear block satisfies the Popov inequality

$$\int_0^{t_1} w(t)v(t)\,\mathrm{d}t \geq -\gamma_0^2,\ \forall t_1 \geq 0$$

then from $u(t) = -w(t)$, the energy supplied to the system is bounded by

$$\int_0^{t_1} u(t)v(t)\,\mathrm{d}t \leq \gamma_0^2$$

- energy conservation (assuming $v(0) = v_0$ and $x(0) = x_0$):

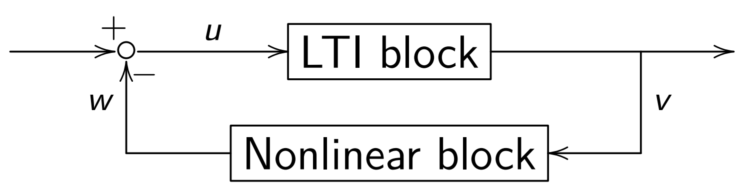
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 - \frac{1}{2}mv_0^2 - \frac{1}{2}kx_0^2 = \int_0^{t_1} u(t)v(t)\,\mathrm{d}t \leq \gamma_0^2$$

- define state vector $x = [x_1, x_2]^T$, $x_1 = \sqrt{\frac{k}{2}}x$, $x_2 = \sqrt{\frac{m}{2}}v$, then

$$||x(t)||_2^2 \leq ||x(0)||_2^2 + \gamma_0^2 \leq (||x(0)||_2 + \gamma_0)^2$$

which is a special case in the hyperstability definition

Understanding the hyperstability theorem

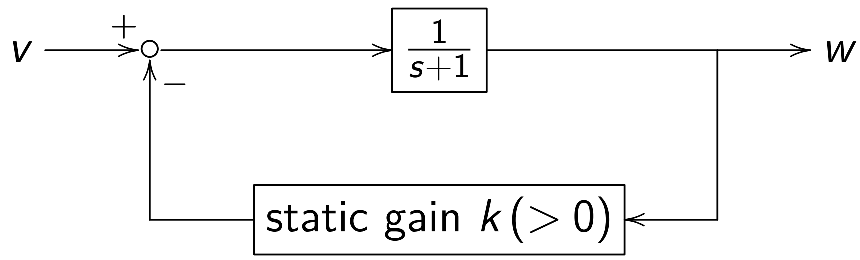


intuition from the example:
The nonlinear block satisfying Popov inequality assures bounded supply to the LTI system. Based on energy conservation, the energy of the LTI system is bounded. If the energy function is positive definite with respect to the states, then the states will be bounded.

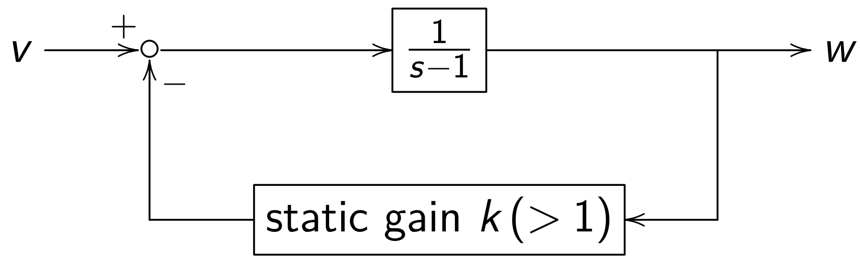
more intuition:
If the LTI system is strictly passive, it consumes energy. The bounded supply will eventually be all consumed, hence the convergence to zero for the states.

A remark about hyperstability

An example of a system that is asymptotically hyperstable are stable:



Stable systems may however not be hyperstable: for instance



is stable but not hyperstable ($\frac{1}{s-1}$ is unstable and hence not SPR)

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PAA stability analysis by hperstability theory

- step 1: translate the adaptation algorithm to a feedback combination of a LTI block and a nonlinear block, as shown in Fig. 1
- step 2: verify that the feedback block satisfies the Popov inequality
- step 3: check that the LTI block is strictly positive real
- step 4: show that the output of the feedback block is bounded. Then from the definition of asymptotic hyperstability, we conclude that the state x converges to zero

Example: hyperstability of RLS with constant adaptation gain

Recall PAA with recursive least squares:

- *a priori*

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

- *a posteriori*

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

We use the *a posteriori* form to prove that the RLS with $F(k) = F \succ 0$ is **always asymptotically hyperstable**

Example cont'd

step 1: transformation to a feedback structure

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

parameter estimation error (vector) $\tilde{\theta}(k) = \hat{\theta}(k) - \theta$:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)$$

a posteriori prediction error $\varepsilon(k+1) = y(k+1) - \hat{\theta}^T(k+1)\phi(k)$:

$$\begin{aligned}\varepsilon(k+1) &= \theta^T\phi(k) - \hat{\theta}^T(k+1)\phi(k)\\ &= -\tilde{\theta}^T(k+1)\phi(k)\end{aligned}$$

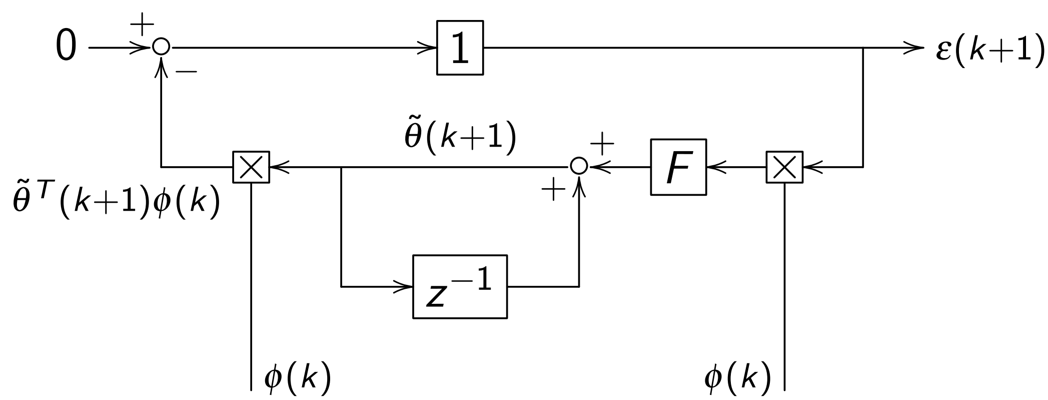
Example cont'd

step 1: transformation to a feedback structure

PAA equations:

$$\begin{aligned}\tilde{\theta}(k+1) &= \tilde{\theta}(k) + F\phi(k)\varepsilon(k+1)\\ \varepsilon(k+1) &= -\tilde{\theta}^T(k+1)\phi(k)\end{aligned}$$

equivalent block diagram:



Example cont'd

step 2: Popov inequality

for the feedback nonlinear block, need to prove

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\gamma_0^2,\ \forall k_1 \geq 0$$

$\tilde{\theta}(k+1) - \tilde{\theta}(k) = F\phi(k)\varepsilon(k+1)$ gives

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)$$
$$= \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$

Example cont'd

step 2: Popov inequality

"adding and substracting terms" gives

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)$$
$$= \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) - \tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$
$$= \sum_{k=0}^{k_1} \Big(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1) \pm \tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)$$
$$-\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\Big)$$

Example cont'd

step 2: Popov inequality

Combining terms yields

$$\sum_{k=0}^{k_1}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)\pm\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)-\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)\right)$$

$$=\underline{\sum_{k=0}^{k_1}\frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}$$

$$+\sum_{k=0}^{k_1}\frac{1}{2}\underbrace{\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-2\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k)+\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)}_{[\bigstar]}$$

- $[\bigstar]$ is equivalent to

$$\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)^T\left(F^{-1/2}\tilde{\theta}(k+1)-F^{-1/2}\tilde{\theta}(k)\right)\geq 0$$

Example cont'd

step 2: Popov inequality

- the underlined term is also lower bounded:

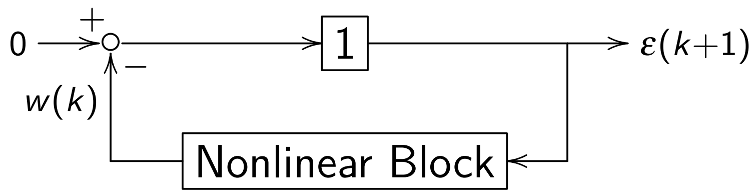
$$\sum_{k=0}^{k_1}\frac{1}{2}\left(\tilde{\theta}^T(k+1)F^{-1}\tilde{\theta}(k+1)-\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k)\right)$$
$$=\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}\tilde{\theta}(k_1+1)-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$
$$\geq-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

hence

$$\boxed{\sum_{k=0}^{k_1}\tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1)\geq-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)}$$

Example cont'd

step 3: SPR condition

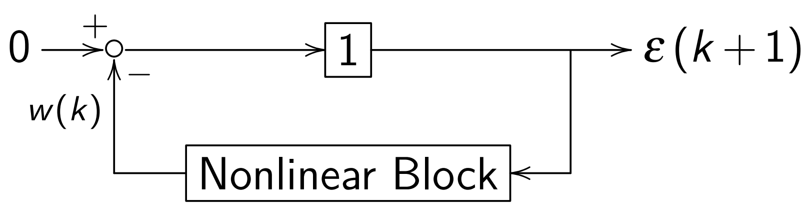


the identity block $G\left(z^{-1}\right) = 1$ is always SPR

- from steps 1-3, we conclude the adaptation system is asymptotically hyperstable
- this means $\varepsilon(k+1)$ will be bounded, and if $w(k)$ is further shown to be bounded, $\varepsilon(k+1)$ converge to zero

Example cont'd

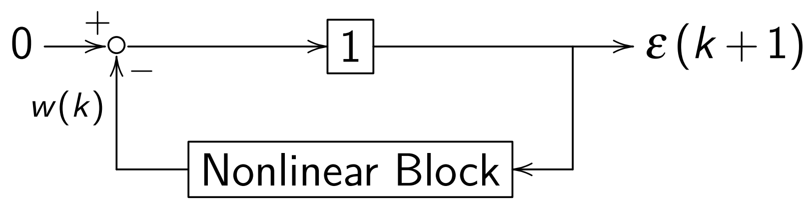
step 4: boundedness of the signal



- $\varepsilon(k+1) = -w(k)$, so $w(k)$ is bounded if $\varepsilon(k+1)$ is bounded
- thus hyperstability theorem gives that $\varepsilon(k+1)$ converges to zero

Example cont'd

intuition



For this simple case, we can intuitively see why $\varepsilon(k+1) \to 0$: Popov inequality gives $\sum_{k=0}^{k_1} \varepsilon(k+1) w(k) \geq -\gamma_0^2$; as $w(k) = -\varepsilon(k+1)$, so

$$\sum_{k=0}^{k_1} \varepsilon^2(k+1) \leq \gamma_0^2$$

Let $k_1 \to \infty$. $\varepsilon(k+1)$ must converge to 0 to ensure the boundedness.

One remark

Recall

$$\varepsilon(k+1) = \frac{\varepsilon^o(k+1)}{1+\phi^T(k) F \phi(k)}$$

- $\varepsilon(k+1) \to 0$ does not necessarily mean $\varepsilon^o(k+1) \to 0$
- need to show $\phi(k)$ is bounded: for instance, the plant needs to be input-output stable for $y(k)$ to be bounded
- see details in ME 233 course reader

There are different PAAs with different stability and convergence requirements

Summary

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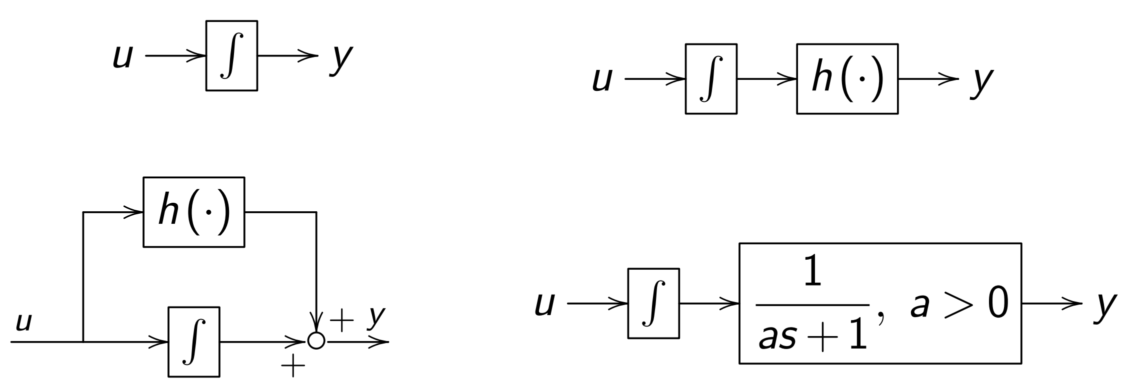
Exercise

In the following block diagrams, u and y are respectively the input and output of the overall system; $h(\cdot)$ is a sector nonlinearity satisfying

$$2|x| < |h(x)| < 5|x|$$

Check whether they satisfy the Popov inequality.



*Kalman Yakubovich Popov Lemma

Kalman Yakubovich Popov (KYP) Lemma connects frequency-domain SPR conditions and time-domain system matrices:
Lemma: Consider $G(s) = C(sI - A)^{-1}B + D$ where (A, B) is controllable and (A, C) is observable. $G(s)$ is **strictly positive real** if and only if there exist matrices $P = P^T \succ 0$, L, and W, and a positive constant ε such that

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}
$$

Proof: see H. Khalil, "Nonlinear Systems", Prentice Hall

*Kalman Yakubovich Popov Lemma

Discrete-time version of KYP lemma: replace s with z and replace the matrix equalities with

$$
\begin{aligned}
A^T PA - P &= -L^T L - \varepsilon P \\
B^T PA - C &= -K^T L \\
D + D^T - B^T PB &= K^T K
\end{aligned}
$$

*Strictly positive real implies strict passivity

From KYP lemma, the following result can be shown:
Lemma: the LTI system $G(s) = C(sI - A)^{-1} B + D$ (in minimal realization)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Analogous results hold for discrete-time systems.

*Strictly positive real implies strict passivity

Proof: Consider $V = \frac{1}{2}x^T Px$:

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}\mathrm{d}t = \int_0^T \left[\frac{1}{2}x^T\left(A^T P + PA\right)x + u^T B^T Px\right]\mathrm{d}t$$

Let u and y be the input and the output of $G(s)$. KYP lemma gives

$$V(x(T)) - V(x(0)) = \int_0^T \left[-\frac{1}{2}x^T\left(L^T L + \varepsilon P\right)x + u^T B^T Px\right]\mathrm{d}t$$

$$\int_0^T u^T y\mathrm{d}t = \int_0^T u^T(Cx + Du)\mathrm{d}t = \int_0^T \left[u^T\left(B^T P + W^T L\right)x + u^T Du\right]\mathrm{d}t$$
$$= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T\left(D + D^T\right)u\right]\mathrm{d}t$$
$$= \int_0^T \left[u^T\left(B^T P + W^T L\right)x + \frac{1}{2}u^T W^T Wu\right]\mathrm{d}t$$

*Strictly positive real implies strict passivity

hence

$$
\begin{aligned}
&\int_0^T u^T y \mathrm{d}t - V(x(T)) + V(x(0)) \\
=&\int_0^T \left[u^T\left(B^TP + W^TL\right)x + \frac{1}{2}u^TW^TWu + \frac{1}{2}x^T\left(L^TL + \varepsilon P\right)x - u^TB^TPx\right]\mathrm{d}t \\
=&\frac{1}{2}\int_0^T (Lx + Wu)^T(Lx + Wu)\mathrm{d}t + \frac{1}{2}\varepsilon x^TPx \geq \frac{1}{2}\varepsilon x^TPx > 0
\end{aligned}
$$