Lecture 19: Adaptive Control based on Pole Assignment

Big picture

reasons for adaptive control:

- unknown or time-varying plants
- unknown or time-varying disturbance (with known structure but unknown coefficients)

two main steps:

- decide the controller structure
- design PAA to adjust the controller parameters

two ways of adaptation process:

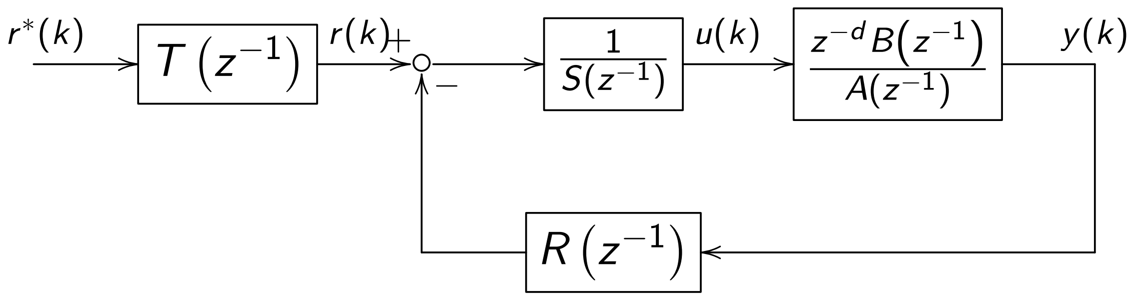
- indirect adaptive control: adapt the plant parameters and use them in the updated controller
- direct adaptive control: directly adapt the controller parameters

RST control structure

Plant:

$$G\left(z^{-1}\right)=\frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})} \qquad \begin{aligned} B\left(z^{-1}\right)&=b_0+b_1z^{-1}+\cdots+b_mz^{-m},\ b_0\neq 0\\ A\left(z^{-1}\right)&=1+a_1z^{-1}+\cdots+a_nz^{-n}\end{aligned}$$

Consider RST type controller:



Closed-loop transfer function:

$$\frac{Y\left(z^{-1}\right)}{R\left(z^{-1}\right)}=\frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})\,S\left(z^{-1}\right)+z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)}$$

Pole placement

Closed-loop pole assignment via:

$$z^{-d}B(z^{-1})R(z^{-1})+S(z^{-1})A(z^{-1})=D(z^{-1})$$

- this is a polynominal (Diophantine) equation
- design $D\left(z^{-1}\right)$, find $S\left(z^{-1}\right)$ and $R\left(z^{-1}\right)$ by coefficient matching

Pole placement for plants with stable zeros

If zeros of plant are all stable, they can be cancelled. We can do

$$S\left(z^{-1}\right) = S^{'}(z^{-1})B(z^{-1})$$
$$D(z^{-1}) = D^{'}(z^{-1})B(z^{-1})$$

yielding

$$z^{-d}R(z^{-1}) + S^{'}(z^{-1})A(z^{-1}) = D^{'}(z^{-1}) \tag{1}$$

where the polynomials should match order:

$$S^{'}\left(z^{-1}\right) = 1 + s_1^{'}z^{-1} + \cdots + s_{d-1}^{'}z^{-(d-1)}$$
$$R\left(z^{-1}\right) = r_0 + r_1 z^{-1} + \cdots + r_{n-1}z^{-(n-1)}$$

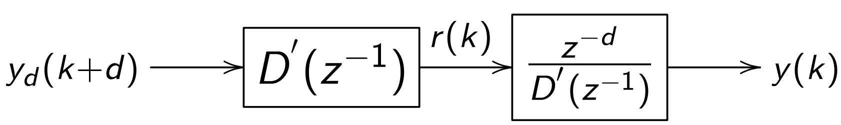
The transfer function from $r(k)$ to $y(k)$ is thus

$$G_{r\rightarrow y}(z^{-1}) = \frac{z^{-d}B(z^{-1})}{S(z^{-1})A(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1})} = \frac{z^{-d}}{D^{'}(z^{-1})}$$

Pole placement for plants with stable zeros

Hence we can let

$$T(z^{-1}) = D^{'}(z^{-1}),\ r^{*}(k) = y_d(k+d)$$



which means

$$D^{'}\left(z^{-1}\right)\left[y\left(k+d\right) - y_d\left(k+d\right)\right] = 0$$

- this is the desired control goal, you can compare it with the goal in system identification: $y\left(k+1\right) - \hat{y}\left(k+1\right) = 0$
- next we express $D^{'}\left(z^{-1}\right)y\left(k+d\right)$ and $D^{'}\left(z^{-1}\right)y_d\left(k+d\right)$ in forms similar to "$\theta^{T}\phi\left(k\right)$"

Pole placement for plants with stable zeros

the $D'\left(z^{-1}\right)y\left(k+d\right)$ term

For a tuned pole placement with known plant model:

- $z^{-d}R(z^{-1})+S'(z^{-1})A(z^{-1})=D'(z^{-1})$ yields

  $$A(z^{-1})S'(z^{-1})y(k+d)=D'(z^{-1})y(k+d)-z^{-d}R(z^{-1})y(k+d)$$

- and the plant model

  $$u(k)\longrightarrow\boxed{\frac{z^{-d}B\left(z^{-1}\right)}{A(z^{-1})}}\longrightarrow y(k)$$

  gives

  $$A(z^{-1})y(k+d)=B(z^{-1})u(k)$$

Combining the two gives

$$D'(z^{-1})y(k+d)=B(z^{-1})S'(z^{-1})u(k)+R(z^{-1})y(k) \qquad (2)$$

Pole placement for plants with stable zeros

the $D'\left(z^{-1}\right)y\left(k+d\right)$ term

We will now simplify (2). Note first:

$$S\left(z^{-1}\right)=B\left(z^{-1}\right)S'\left(z^{-1}\right)=s_0+s_1z^{-1}+\cdots+s_{d+m-1}z^{-(d+m-1)}$$

hence

$$\begin{aligned}\underline{D'(z^{-1})y(k+d)}&=\overbrace{B(z^{-1})S'(z^{-1})}^{S(z^{-1})}u(k)+R(z^{-1})y(k)\\&=\underline{\theta_c^T\phi(k)}\end{aligned}$$

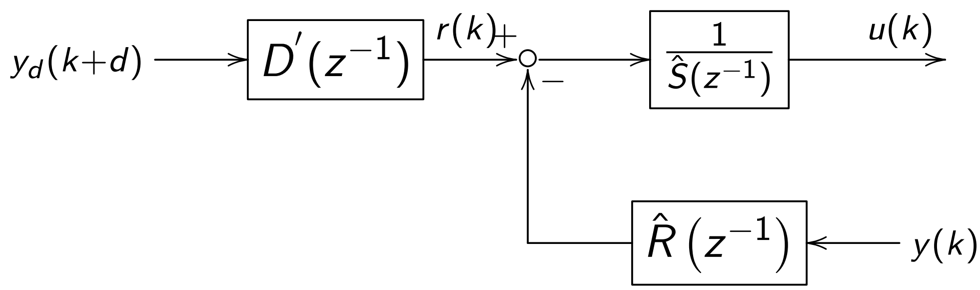
where

$$\theta_c^T=\left[s_0,s_1,\ldots,s_{d+m-1},r_0,\ldots,r_{n-1}\right]$$
$$\phi(k)=\left[u(k),u(k-1),\ldots,u(k-d-m+1),y(k),\ldots,y(k-n+1)\right]^T$$

Pole placement for plants with stable zeros

the $D'(z^{-1})\, y_d(k+d)$ term

For the actual adaptive $S(z^{-1})$ and $R(z^{-1})$, the control law is



i.e. $$u(k) = \frac{1}{\hat{S}(z^{-1})}\left[D'(z^{-1})y_d(k+d) - \hat{R}(z^{-1})y(k)\right]$$

yielding

$$\underline{D'(z^{-1})y_d(k+d)} = \hat{S}(z^{-1})u(k) + \hat{R}(z^{-1})y(k) = \underline{\hat{\theta}_c^T \phi(k)} \qquad (3)$$

This is a direct adaptive control: no explicit $B(z^{-1})$ and $A(z^{-1})$ in $\hat{\theta}_c$

Pole placement for plants with stable zeros

Hence we can define

$$\varepsilon(k+d) = D'(z^{-1})y(k+d) - \hat{\theta}_c^T(k+d)\phi(k)$$

or equivalently

$$\text{a posteriori:} \qquad \varepsilon(k) = D'(z^{-1})y(k) - \hat{\theta}_c^T(k)\phi(k-d)$$
$$\text{a priori:} \qquad \varepsilon^o(k) = D'(z^{-1})y(k) - \hat{\theta}_c^T(k-1)\phi(k-d)$$

and apply parameter adaptation for θ_c, e.g., using series-parallel predictors

$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + \frac{F(k-1)\phi(k-d)}{1+\phi(k-d)^T F(k-1)\phi(k-d)}\varepsilon^o(k)$$
$$F^{-1}(k) = \lambda_1(k)F^{-1}(k-1) + \lambda_2(k)\phi(k-d)\phi^T(k-d)$$

Comparison with system identification

Comparison:
standard system identification:

$$y(k+1) = \theta^T \phi(k)$$
$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$
$$\varepsilon(k+1) = \frac{\varepsilon^o(k+1)}{1+\phi^T(k)F(k)\phi(k)}$$

adaptive pole placement:

$$D'\left(z^{-1}\right)y(k) = \theta_c^T\phi(k-d)$$
$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$
$$\varepsilon(k) = \frac{\varepsilon^o(k)}{1+\phi^T(k-d)F(k-1)\phi(k-d)}$$

Pole placement for plants with stable zeros

PAA Stability

First obtain the *a posteriori* dynamics of the parameter error:

$$\hat{\theta}_c(k) = \hat{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$
$$\Rightarrow \tilde{\theta}_c(k) = \tilde{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$

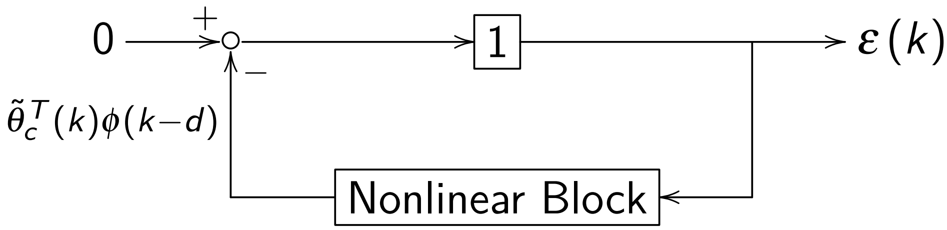
In the mean time

$$\begin{aligned}\varepsilon(k) &= D'(z^{-1})y(k) - \hat{\theta}_c^T(k)\phi(k-d)\\ &\Downarrow \text{recall } D'\left(z^{-1}\right)y(k+d) = \theta_c^T\phi(k)\\ &= \theta_c^T\phi(k-d) - \hat{\theta}_c^T(k)\phi(k-d)\\ &= -\tilde{\theta}_c(k)^T\phi(k-d)\end{aligned}$$

Pole placement for plants with stable zeros

PAA Stability

$$\varepsilon(k) = -\tilde{\theta}_c(k)^T \phi(k-d)$$
$$\tilde{\theta}_c(k) = \tilde{\theta}_c(k-1) + F(k-1)\phi(k-d)\varepsilon(k)$$



The PAA thus is in a standard series-parallel structure with the LTI block being 1. Hyperstability easily follows, which gives

$$\lim_{k\to\infty} \varepsilon(k) = \frac{D'(z^{-1})y(k) - \hat{\theta}_c^T(k-1)\phi(k-d)}{1+\phi^T(k-d)F(k-1)\phi(k-d)} \to 0$$

Similar as before, to prove $\varepsilon^o(k) = D'(z^{-1})(y(k) - y_d(k)) \to 0$, we need to show that $\phi(k-d)$ is bounded, which can be shown to be true (see ME233 reader).

Pole placement for plants with stable zeros

Design procedure:

Step 1: choose desired $D'(z^{-1})$ ($\deg D'(z^{-1}) \le n+d-1$). The overall closed-loop characteristic polynomial is $D'(z^{-1})B(z^{-1})$.

Step 2: determine orders in the Diophantine equation $S'(z^{-1})$ ($\deg S'(z^{-1}) = d-1$) and $R(z^{-1})$ ($\deg R(z^{-1}) = n-1$).

Step 3: at each time index, do the following:

- apply an appropriate PAA to estimate the coefficients of $S(z^{-1}) = S'(z^{-1})B(z^{-1})$ and $R(z^{-1})$, based on the reparameterized plant model

$$D'(z^{-1})y(k) = \theta_c^T \phi(k-d)$$

- use the estimated parameter vector, $\hat{\theta}_c(k)$, to compute the control signal $u(k)$ according to

$$u(k) = \frac{1}{\hat{S}(z^{-1})}\left[D'(z^{-1})y_d(k+d) - \hat{R}(z^{-1})y(k)\right]$$

Example

Consider a plant (discrete-time model of $1/(ms+b)$ with an extra delay)

$$G_p\left(z^{-1}\right)=\frac{z^{-2}b_0}{1+a_1z^{-1}}$$

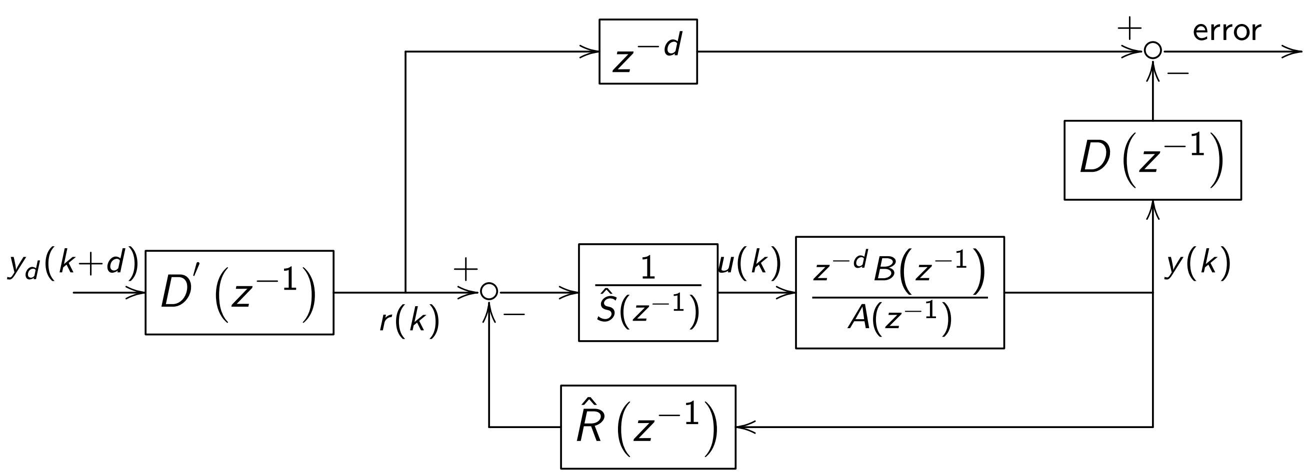
We have $B\left(z^{-1}\right)=b_0$ ($m=0$ here); $A\left(z^{-1}\right)=1+a_1z^{-1}$ ($n=1$ here); $d=2$. The pole placement equation is

$$(1+a_1z^{-1})(1+s_1^{'}z^{-1})+z^{-2}r_0=1+d_1^{'}z^{-1}+d_2^{'}z^{-2}$$
$$\Rightarrow s_1^{'}=d_1^{'}-a_1, r_0=d_2^{'}-a_1(d_1^{'}-a_1)$$

and $S(z^{-1})=S^{'}(z^{-1})B(z^{-1})=s_0+s_1z^{-1}$; $R(z^{-1})=r_o$

$$u(k)=\frac{1}{\hat{S}(z^{-1})}\left[D^{'}(z^{-1})y_d(k+d)-\hat{R}(z^{-1})y(k)\right]$$
$$=\frac{1}{\hat{s}_0(k)}\left[D^{'}(z^{-1})y_d(k+2)-\hat{r}_0(k)y(k)-\hat{s}_1(k)u(k-1)\right]$$

Remark



Parameter convergence is achieved if the excitation y_d is rich in frequency (which may not be assured in practice). Yet the performance goal of making $D^{'}\left(z^{-1}\right)[y(k)-y_d(k)]$ small can still be achieved even if y_d is not rich in frequency.

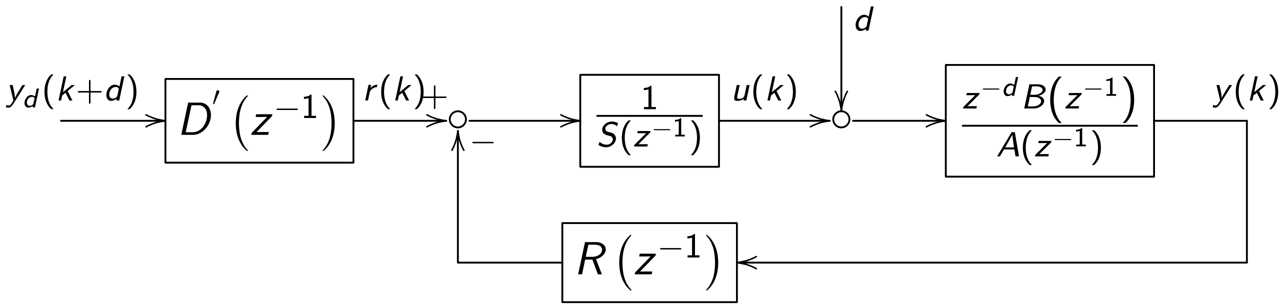
Add now disturbance cancellation

If the disturbance structure is known, we can estimate its parameters for disturbance cancellation. Consider, e.g.,

$$y(k) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}[u(k)+d(k)]$$

where $B(z^{-1})$ is cancallable and the disturbance satisfies

$$W(z^{-1})\,d(k) = (1-z^{-1})\,d(k) = 0$$



the deterministic control law should be:

$$u(k) = \frac{1}{S(z^{-1})}\left[-R(z^{-1})\,y(k) + D^{'}(z^{-1})y_d(k+d)\right] - d$$

Disturbance cancellation

$$u(k) = \frac{1}{S(z^{-1})}\left[-R(z^{-1})\,y(k) + D^{'}(z^{-1})y_d(k+d)\right] - d$$

can be equivalently represented as

$$\begin{aligned} D^{'}(z^{-1})\,y_d(k+d) &= \theta_c^T\phi(k) + d^*,\ d^* = S(z^{-1})\,d \\ &= \theta_{ce}^T\phi_e(k),\ \theta_{ce} = \left[\theta_c^T, d^*\right]^T,\ \phi_e(k) = \left[\phi^T(k), 1\right]^T \end{aligned}$$

In the adaptive case:

$$D^{'}(z^{-1})\,y_d(k+d) = \hat{\theta}_{ce}^T(k+d)\,\phi_e(k)$$

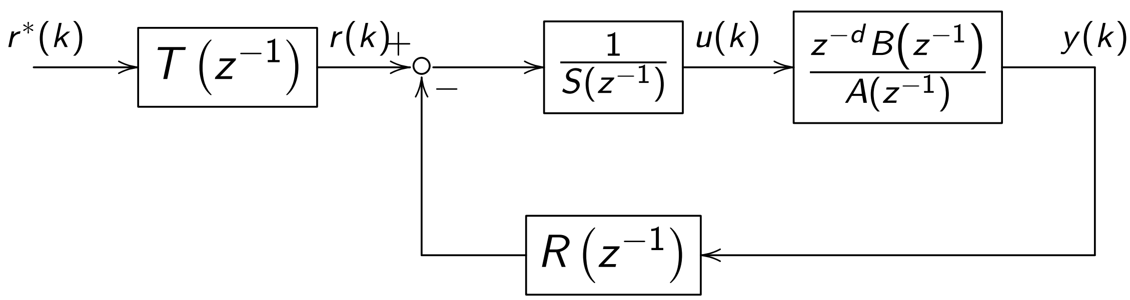
where $\hat{\theta}_{ce}(k)$ is updated via a PAA, e.g.

$$\hat{\theta}_{ce}(k) = \hat{\theta}_{ce}(k-1) + \frac{F(k-1)\,\phi_e(k-d)\left[D^{'}(z^{-1})y(k) - \hat{\theta}_{ce}^T(k-1)\,\phi_e(k-d)\right]}{1+\phi_e^T(k-d)\,F(k-1)\,\phi_e(k-d)}$$

Outline

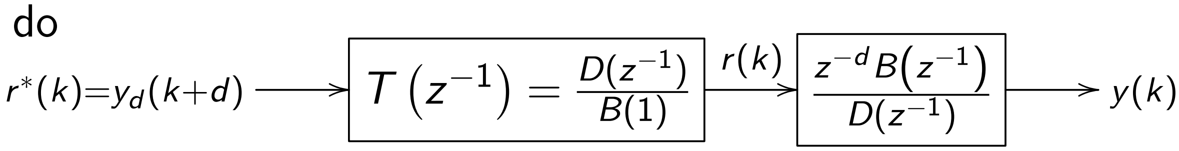
1. Big picture
2. Adaptive pole placement
   Cancellable $B\left(z^{-1}\right)$
   Remark
3. Extension: adaptive pole placement with disturbance cancellation
4. Pole placement with no cancellation of $B\left(z^{-1}\right)$
5. Indirect adaptive pole placement

Uncancellable $B\left(z^{-1}\right)$



$$\frac{Y\left(z^{-1}\right)}{R\left(z^{-1}\right)}=\frac{z^{-d}B\left(z^{-1}\right)}{A\left(z^{-1}\right)S\left(z^{-1}\right)+z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)}=\frac{z^{-d}B\left(z^{-1}\right)}{D\left(z^{-1}\right)}$$

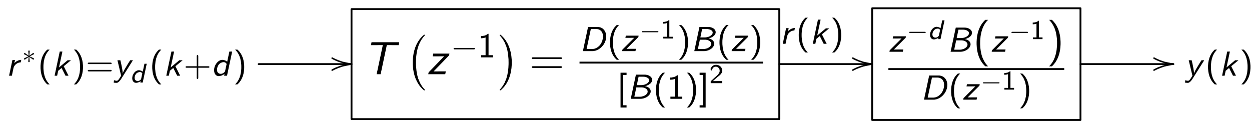
If $B\left(z^{-1}\right)$ contains unstable roots or if we don't want to cancel it, we can do



$$\Rightarrow D\left(z^{-1}\right)\left[y\left(k+d\right)-\frac{B\left(z^{-1}\right)}{B(1)}y_d\left(k+d\right)\right]=0$$

Uncancellable $B\left(z^{-1}\right)$

or

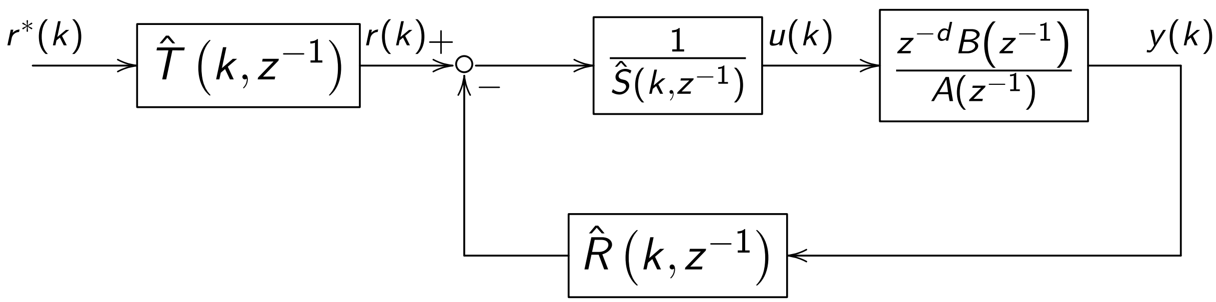


$$\Rightarrow D\left(z^{-1}\right)\left[y\left(k+d\right)-\frac{B\left(z^{-1}\right)B\left(z\right)}{\left[B\left(1\right)\right]^2}y_d\left(k+d\right)\right]=0$$

which gives zero phase error tracking.
Remark: can also partially cancel the stable parts of $B\left(z^{-1}\right)$
Note: now we explicitly need $B\left(1\right)$ and/or $B\left(z\right)$ in $T\left(z^{-1}\right)$ $\Rightarrow$ need adaptation to find the plant parameters $\Rightarrow$ indirect adaptive control

Indirect adaptive pole placement: big picture

Consider the plant $z^{-d}B\left(z^{-1}\right)/A\left(z^{-1}\right)$.



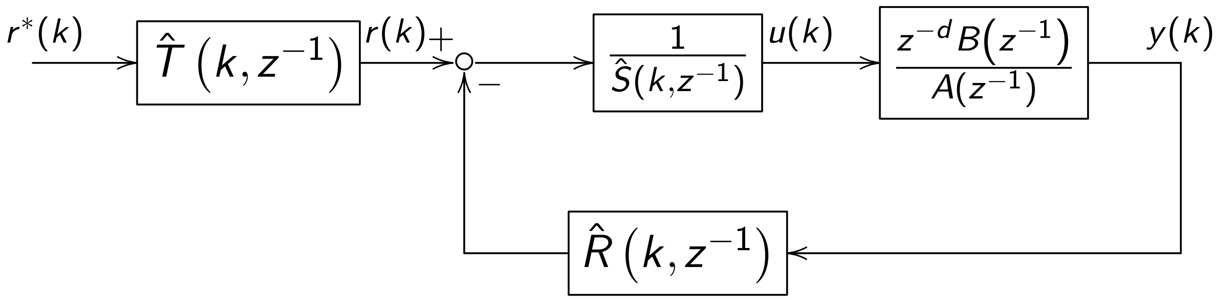
Pole placement with known plant parameters:

$$A\left(z^{-1}\right)S\left(z^{-1}\right)+z^{-d}B\left(z^{-1}\right)R\left(z^{-1}\right)=D\left(z^{-1}\right)$$

Assumptions:

- we know n, m, and d;
- the plant is irreducible.

Indirect adaptive pole placement: big picture



- At time k, identify $\hat{B}\left(k,z^{-1}\right)$ and $\hat{A}\left(k,z^{-1}\right)$ (using a suitable PAA); design $\hat{T}\left(k,z^{-1}\right)$ based on methods previously discussed.
- Solve Diophantine equation

$$\hat{A}\left(k,z^{-1}\right)\hat{S}\left(k,z^{-1}\right)+z^{-1}\hat{B}\left(k,z^{-1}\right)\hat{R}\left(k,z^{-1}\right)=D\left(z^{-1}\right)$$

for $\hat{S}\left(k,z^{-1}\right)$ and $\hat{R}\left(k,z^{-1}\right)$.

Indirect adaptive pole placement: details

- Controller order:

$$\underbrace{\hat{A}\left(k,z^{-1}\right)}_{\text{order: } n}\underbrace{\hat{S}\left(k,z^{-1}\right)}_{\text{order: } d+m-1}+\underbrace{z^{-d}\hat{B}\left(k,z^{-1}\right)}_{\text{order: } d+m}\underbrace{\hat{R}\left(k,z^{-1}\right)}_{\text{order: } n-1}=\underbrace{D\left(z^{-1}\right)}_{\text{order}\leq n+m+d-1}$$

- Controller parameters:

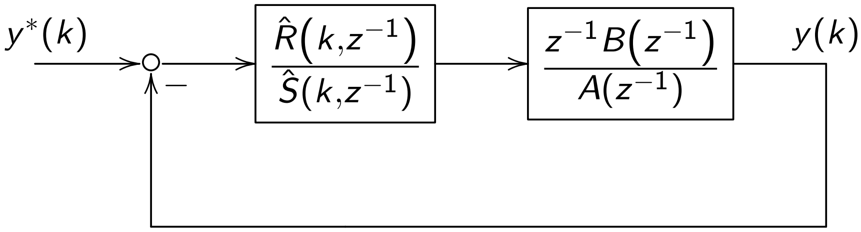
$$\hat{S}\left(k,z^{-1}\right)=\hat{s}_0(k)+\hat{s}_1(k)z^{-1}+\cdots+\hat{s}_{r-1}(k)z^{-d-m+1}$$
$$\hat{R}\left(k,z^{-1}\right)=\hat{r}_0(k)+\hat{r}_1(k)z^{-1}+\cdots+\hat{r}_{r-1}(k)z^{-n+1}$$

- Solvability of the Diophantine equation: $\hat{A}\left(k,z^{-1}\right)$ and $\hat{B}\left(k,z^{-1}\right)$ need to be coprime. If not, use the previous estimation.
- Control law:

$$u(k)=\frac{1}{\hat{S}\left(k,z^{-1}\right)}\left[\hat{T}\left(k,z^{-1}\right)r^*(k)-\hat{R}\left(k,z^{-1}\right)y(k)\right]$$

Indirect adaptive pole placement: extension

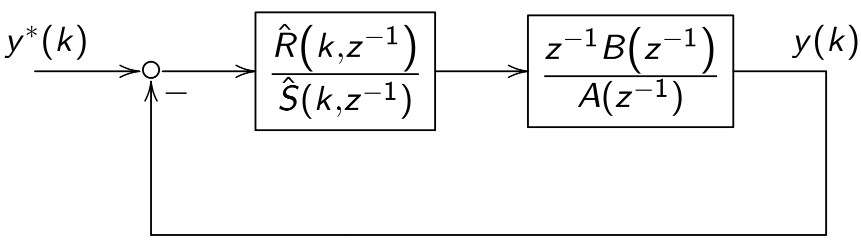
Consider the plant $z^{-1}B\left(z^{-1}\right)/A\left(z^{-1}\right)$ with the general feedback design



Similar as before, but assume we **know only the order of the plant**: $r = \max(n, m+1)$.
Pole placement with known plant parameters:

$$A\left(z^{-1}\right)S\left(z^{-1}\right) + z^{-1}B\left(z^{-1}\right)R\left(z^{-1}\right) = D\left(z^{-1}\right)$$

Indirect adaptive pole placement: extension



- Can write $B\left(z^{-1}\right) = b_0 + b_1 z^{-1} + \cdots + b_{r-1} z^{-r+1}$ and $A\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_r z^{-r}$
- At time k, identify $\hat{B}\left(k, z^{-1}\right)$ and $\hat{A}\left(k, z^{-1}\right)$
- Solve Diophantine equation

$$\hat{A}\left(k,z^{-1}\right)\hat{S}\left(k,z^{-1}\right) + z^{-1}\hat{B}\left(k,z^{-1}\right)\hat{R}\left(k,z^{-1}\right) = D\left(z^{-1}\right)$$

for $\hat{S}\left(k,z^{-1}\right)$ and $\hat{R}\left(k,z^{-1}\right)$

Indirect adaptive pole placement: extension

- Controller order:

$$\underbrace{\hat{A}\left(k,z^{-1}\right)}_{\text{order: } r}\underbrace{\hat{S}\left(k,z^{-1}\right)}_{\text{order: } r-1}+\underbrace{z^{-1}\hat{B}\left(k,z^{-1}\right)}_{\text{order: } r}\underbrace{\hat{R}\left(k,z^{-1}\right)}_{\text{order: } r-1}=\underbrace{D\left(z^{-1}\right)}_{\text{order}\leq 2r-1}$$

- Controller parameters:

$$\hat{S}\left(k,z^{-1}\right)=\hat{s}_0\left(k\right)+\hat{s}_1\left(k\right)z^{-1}+\cdots+\hat{s}_{r-1}\left(k\right)z^{-r+1}$$
$$\hat{R}\left(k,z^{-1}\right)=\hat{r}_0\left(k\right)+\hat{r}_1\left(k\right)z^{-1}+\cdots+\hat{r}_{r-1}\left(k\right)z^{-r+1}$$

- Control law:

$$\begin{aligned}u\left(k\right)&=\frac{\hat{R}\left(k,z^{-1}\right)}{\hat{S}\left(k,z^{-1}\right)}\left[y^*\left(k\right)-y\left(k\right)\right]\\&=\frac{1}{\hat{s}_0\left(k\right)}\left\{-\hat{s}_1\left(k\right)u\left(k-1\right)-\cdots-\hat{s}_{r-1}u\left(k-r+1\right)\right.\\&\left.+\hat{r}_0\left(k\right)\left[y^*\left(k\right)-y\left(k\right)\right]+\cdots+\hat{r}_{r-1}\left(k\right)\left[y^*\left(k-r+1\right)-y\left(k-r+1\right)\right]\right\}\end{aligned}$$

Summary

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   Remark
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References

Goodwin and Sin, "Adaptive Filtering, Prediction and Control," Prentice Hall.