# 1 Leibniz rule

Named after Gottfried Leibniz (German, 1646-1716). Often regarded as the most important rule for calculus.

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f\left(x,z\right)\mathrm{d}x = \frac{\mathrm{d}b\left(z\right)}{\mathrm{d}z} f\left(b\left(z\right),z\right) - \frac{\mathrm{d}a\left(z\right)}{\mathrm{d}z} f\left(a\left(z\right),z\right) + \int_{a(z)}^{b(z)} \frac{\partial f\left(x,z\right)}{\partial z}\mathrm{d}x$$

*Proof.* Let

$$\psi\left(z\right) = \int_{a(z)}^{b(z)} f\left(x,z\right)\mathrm{d}x$$

Then

$$\begin{aligned}\psi\left(z+\Delta z\right) &= \int_{a(z+\Delta z)}^{b(z+\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x \\ &= \int_{a(z)+\Delta z\frac{\mathrm{d}a}{\mathrm{d}z}+H.O.T.(\Delta z)}^{b(z)+\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x \\ &= \int_{b(z)}^{b(z)+\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x \\ &\quad + \int_{a(z)}^{b(z)} f\left(x,z+\Delta z\right)\mathrm{d}x - \int_{a(z)}^{a(z)+\Delta z\frac{\mathrm{d}a}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x\end{aligned}$$

where $H.O.T.\left(\Delta z\right)$ are the higher-order terms of $\Delta z$.

So

$$\begin{aligned}\frac{\psi\left(z+\Delta z\right)-\psi\left(z\right)}{\Delta z} &= \frac{1}{\Delta z}\int_{b(z)}^{b(z)+\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x \\ &\quad + \int_{a(z)}^{b(z)} \frac{f\left(x,z+\Delta z\right)-f\left(x,z\right)}{\Delta z}\mathrm{d}x - \frac{1}{\Delta z}\int_{a(z)}^{a(z)+\Delta z\frac{\mathrm{d}a}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x\end{aligned}$$

Take the limit of $\Delta z\to 0$. $\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.\left(\Delta z\right)$ and $\Delta z\frac{\mathrm{d}a}{\mathrm{d}z}+H.O.T.\left(\Delta z\right)$ are both going to zero. The integral $\int_{b(z)}^{b(z)+\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x$ approximates $f\left(b\left(z\right),z\right)\left(\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.\left(\Delta z\right)\right)$. Hence

$$\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{b(z)}^{b(z)+\Delta z\frac{\mathrm{d}b}{\mathrm{d}z}+H.O.T.(\Delta z)} f\left(x,z+\Delta z\right)\mathrm{d}x = \lim_{\Delta z\to 0} f\left(b\left(z\right),z\right)\left(\frac{\mathrm{d}b}{\mathrm{d}z}+\frac{1}{\Delta z}H.O.T.\left(\Delta z\right)\right)$$

The higher-order terms are going to zero in the order of $\left(\Delta z\right)^2$, making $\lim_{\Delta z\to 0}\frac{1}{\Delta z}H.O.T.\left(\Delta z\right)=0$.

Finally, we have

$$\begin{aligned}\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f\left(x,z\right)\mathrm{d}x &= \lim_{\Delta z\to 0}\frac{\psi\left(z+\Delta z\right)-\psi\left(z\right)}{\Delta z} \\ &= \frac{\mathrm{d}b}{\mathrm{d}z}f\left(b\left(z\right),z\right) + \lim_{\Delta z\to 0}\int_{a(z)}^{b(z)}\frac{f\left(x,z+\Delta z\right)-f\left(x,z\right)}{\Delta z}\mathrm{d}x - \frac{\mathrm{d}a}{\mathrm{d}z}f\left(a\left(z\right),z\right) \\ &= \frac{\mathrm{d}b}{\mathrm{d}z}f\left(b\left(z\right),z\right) + \int_{a(z)}^{b(z)}\frac{\partial f\left(x,z\right)}{\partial z}\mathrm{d}x - \frac{\mathrm{d}a}{\mathrm{d}z}f\left(a\left(z\right),z\right)\end{aligned}$$

□