Lecture 17: PAA with Parallel Predictors

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Big picture: we know now...

$$u(k) \longrightarrow \boxed{\dfrac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)}} \longrightarrow y(k+1)$$

simply means:

$$
\begin{aligned}
y(k+1) &= B\left(z^{-1}\right) u(k) - \left(A\left(z^{-1}\right) - 1\right) y(k+1) \\
&= \theta^T \phi(k)
\end{aligned}
$$

In RLS:

$$\hat{y}^o(k+1) = \hat{\theta}^T(k)\phi(k) = \hat{B}\left(z^{-1},k\right)u(k) - \left(\hat{A}\left(z^{-1},k\right) - 1\right)y(k+1)$$

Understanding the notation: if $B\left(z^{-1}\right) = b_o + b_1 z^{-1} + \cdots + b_m z^{-m}$,
then $\hat{B}\left(z^{-1},k\right) = \hat{b}_o(k) + \hat{b}_1(k) z^{-1} + \cdots + \hat{b}_m(k) z^{-m}$
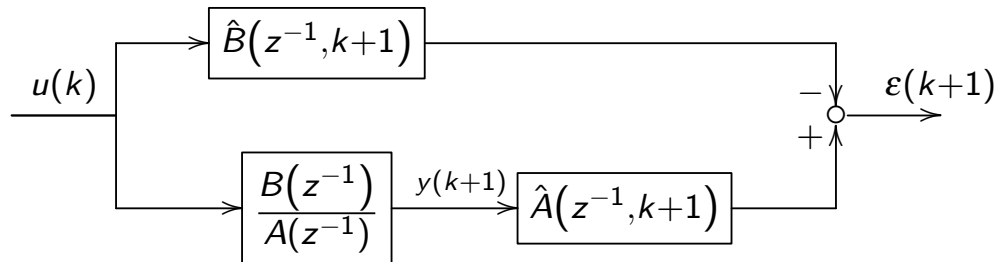Remark: z^{-1}–shift operator; some references use q^{-1} instead

RLS is a series-parallel adjustable system

RLS in *a posteriori* form:

$$\hat{y}(k+1) = \hat{B}(z^{-1}, k+1) u(k) - \left(\hat{A}(z^{-1}, k+1) - 1\right) y(k+1)$$

prediction error:

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \hat{A}(z^{-1}, k+1) y(k+1) - \hat{B}(z^{-1}, k+1) u(k)$$

A series-parallel structure: $\hat{A}(z^{-1}, k+1)$–in series with plant; $\hat{B}(z^{-1}, k+1)$–in parallel with the plant
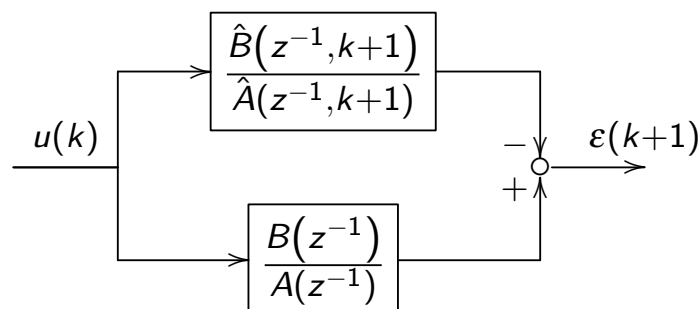
Observation

If hyperstability holds such that $\varepsilon(k+1) \to 0$, $\hat{y}(k+1) \to y(k+1)$, it seems fine to do instead:

$$\hat{y}(k+1) = \hat{B}(z^{-1}, k+1) u(k) - \left(\hat{A}(z^{-1}, k+1) - 1\right) \hat{y}(k+1) \quad (1)$$
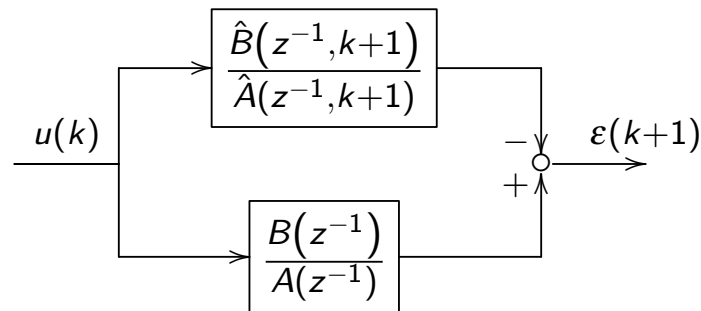
i.e.

$$u(k) \longrightarrow \boxed{\dfrac{\hat{B}(z^{-1}, k+1)}{\hat{A}(z^{-1}, k+1)}} \longrightarrow \hat{y}(k+1)$$
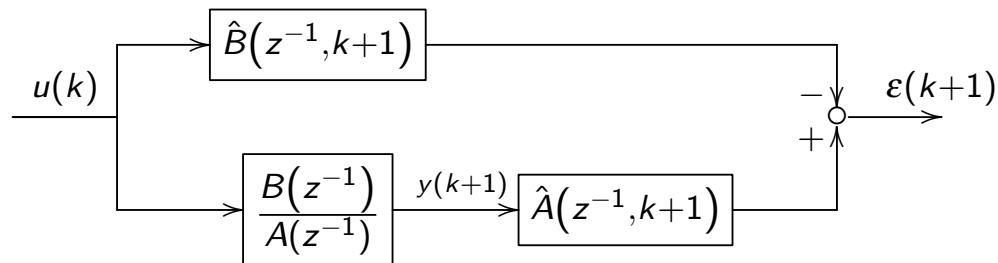
then we have a parallel structure

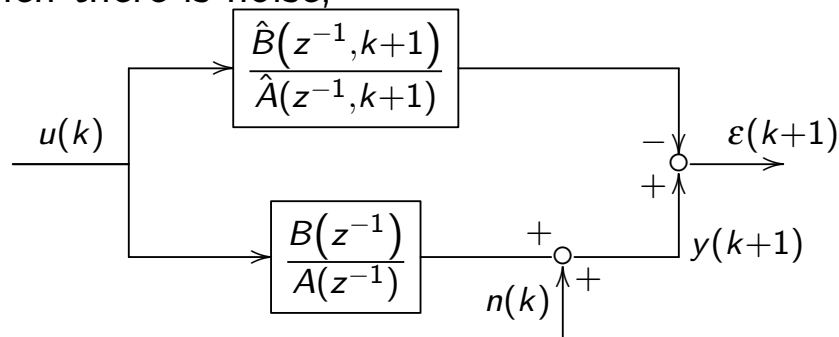▶ it turns out this brings certain advantages

Other names

$$\dfrac{\hat{B}(z^{-1},k+1)}{\hat{A}(z^{-1},k+1)}$$

$u(k)$

$\dfrac{B(z^{-1})}{A(z^{-1})}$

$-$ $\varepsilon(k+1)$
$+$

is also called an output-error method

$\hat{B}(z^{-1},k+1)$

$u(k)$

$\dfrac{B(z^{-1})}{A(z^{-1})}$ $y(k+1)$ $\hat{A}(z^{-1},k+1)$

$-$ $\varepsilon(k+1)$
$+$

is also called an equation-error method

Benefits of parallel algorithms

Intuition: when there is noise,

$$\dfrac{\hat{B}(z^{-1},k+1)}{\hat{A}(z^{-1},k+1)}$$

$u(k)$

$\dfrac{B(z^{-1})}{A(z^{-1})}$

$-$ $\varepsilon(k+1)$
$+$

$+$
$n(k)$ $+$ $y(k+1)$

provides better convergence of $\hat{\theta}$ than

$\hat{B}(z^{-1},k+1)$

$u(k)$

$\dfrac{B(z^{-1})}{A(z^{-1})}$ $+$ $y(k+1)$ $\hat{A}(z^{-1},k+1)$

$n(k)$ $+$

$-$ $\varepsilon(k+1)$
$+$
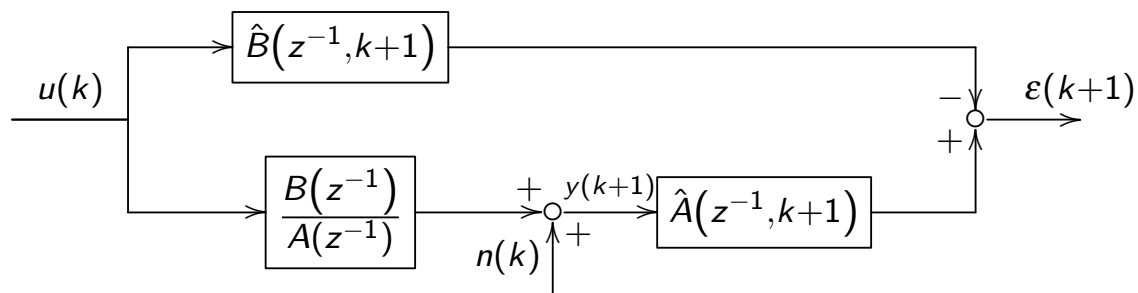
We will talk about the PAA convergence in a few more lectures.

Outline

1. Big picture

   Series-parallel adjustable system (equation-error method)

   Parallel adjustable system (output-error method)
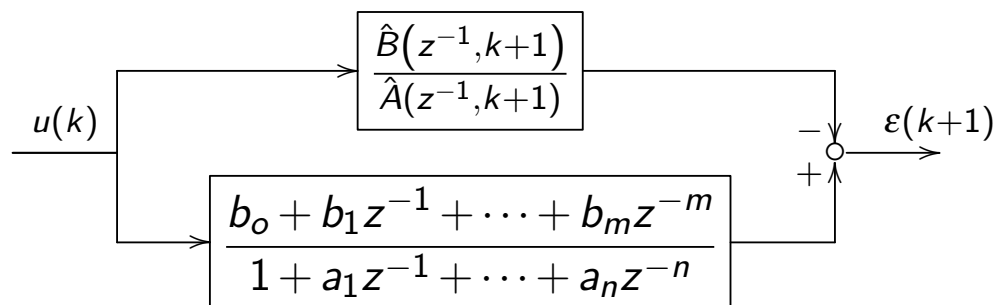
2. RLS-based parallel PAA

   Formulas

   Stability requirement for PAAs with fixed adaptation gain

   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)

RLS based parallel PAA

PAA summary:

- *a priori*
$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1 + \phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

- *a posteriori*
$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

$$\boxed{\phi^T(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k+1-n), u(k), \ldots, u(k-m)]}$$

Stability of RLS based parallel PAA

parameter estimation error :

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

a posteriori prediction error : $y(k+1) = \frac{B(z^{-1})}{A(z^{-1})}u(k)$ gives

$$B(z^{-1})u(k) = A(z^{-1})y(k+1)$$
$$\hat{B}(z^{-1},k+1)u(k) = \hat{A}(z^{-1},k+1)\hat{y}(k+1)$$

hence

$$A(z^{-1})y(k+1) - \hat{A}(z^{-1},k+1)\hat{y}(k+1)\boxed{\pm A(z^{-1})\hat{y}(k+1)}$$
$$= B(z^{-1})u(k) - \hat{B}(z^{-1},k+1)u(k)$$

i.e. $A(z^{-1})\varepsilon(k+1) = [B(z^{-1}) - \hat{B}(z^{-1},k+1)]u(k)$
$$- [A(z^{-1}) - \hat{A}(z^{-1},k+1)]\hat{y}(k+1)$$

Stability of RLS based parallel PAA

a posteriori prediction error (cont'd):

$$A(z^{-1})\varepsilon(k+1) = \overbrace{\left[B(z^{-1}) - \hat{B}(z^{-1},k+1)\right]u(k)}^{[\star]}$$
$$- \left[A(z^{-1}) - \hat{A}(z^{-1},k+1)\right]\hat{y}(k+1)$$

Look at $[\star]$: $B(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}$ gives

$$\left[B(z^{-1}) - \hat{B}(z^{-1},k+1)\right]u(k)$$

$$= \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-m} \end{bmatrix} u(k) = \begin{bmatrix} b_0 - \hat{b}_0(k+1) \\ b_1 - \hat{b}_1(k+1) \\ \vdots \\ b_m - \hat{b}_m(k+1) \end{bmatrix}^T \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-m) \end{bmatrix}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

Similarly, for $A\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$

$$\left[\hat{A}\left(z^{-1}, k+1\right) - A\left(z^{-1}\right)\right] \hat{y}\left(k+1\right) = \begin{bmatrix} a_1 - \hat{a}_1\left(k+1\right) \\ a_2 - \hat{a}_2\left(k+1\right) \\ \vdots \\ a_n - \hat{a}_n\left(k+1\right) \end{bmatrix}^T \begin{bmatrix} -\hat{y}\left(k\right) \\ -\hat{y}\left(k-1\right) \\ \vdots \\ -\hat{y}\left(k+1-n\right) \end{bmatrix}$$

Recall:
$$\theta^T = [a_1, a_2, \cdots a_n, b_0, b_1, \cdots, b_m]^T$$

$$\phi\left(k\right) = [-\hat{y}\left(k\right), -\hat{y}\left(k-1\right), \ldots, -\hat{y}\left(k+1-n\right), u\left(k\right), \ldots, u\left(k-m\right)]$$

hence

$$A\left(z^{-1}\right)\varepsilon\left(k+1\right) = \left[B\left(z^{-1}\right) - \hat{B}\left(z^{-1}, k+1\right)\right]u\left(k\right)$$

$$- \left[A\left(z^{-1}\right) - \hat{A}\left(z^{-1}, k+1\right)\right]\hat{y}\left(k+1\right) = \underline{-\tilde{\theta}^T\left(k+1\right)\phi\left(k\right)}$$

Stability of RLS based parallel PAA

step 1: transformation to a feedback structure

PAA equations:

$$\tilde{\theta}\left(k+1\right) = \tilde{\theta}\left(k\right) + F\left(k\right)\phi\left(k\right)\varepsilon\left(k+1\right)$$

$$A\left(z^{-1}\right)\varepsilon\left(k+1\right) = -\tilde{\theta}^T\left(k+1\right)\phi\left(k\right)$$
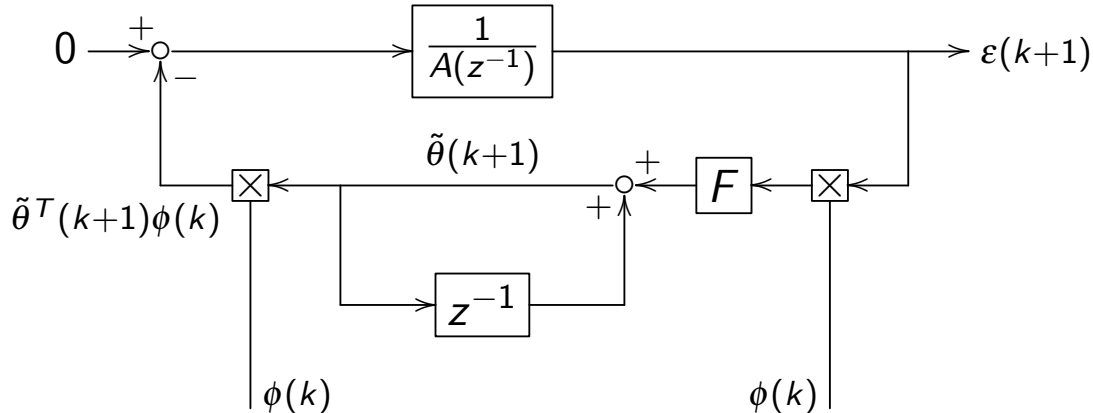
equivalent block diagram:
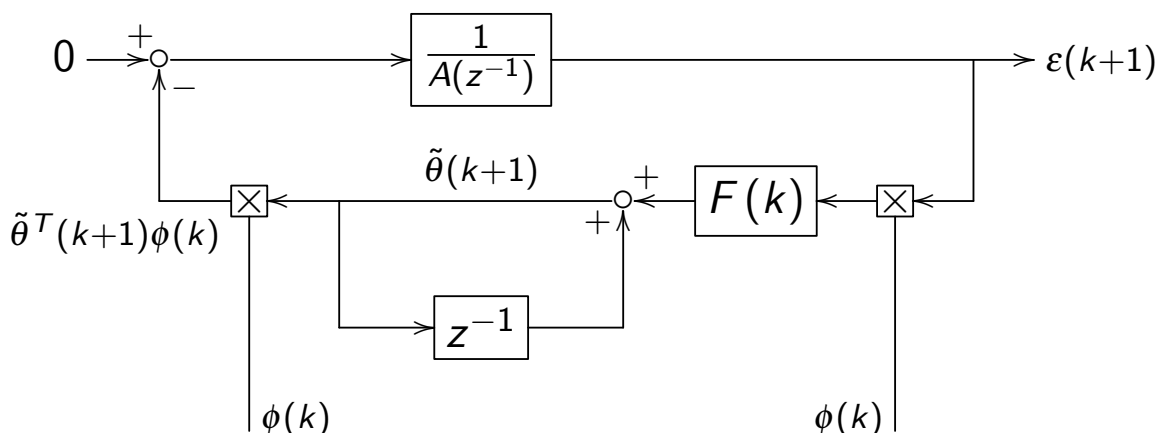
Stability of RLS based parallel PAA

We will consider a simplified case with $\underline{F(k) = F \succ 0}$:
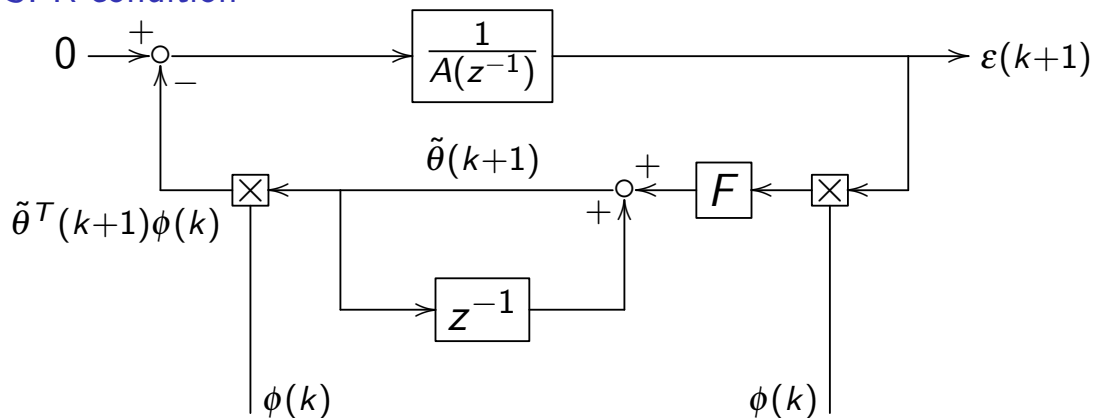

The nonlinear block is exactly the same as that in RLS, hence satisfying Popov inequality:

$$\sum_{k=0}^{k_1} \tilde{\theta}^T (k+1) \phi (k) \varepsilon (k+1) \geq -\frac{1}{2} \tilde{\theta}^T (0) F^{-1} \tilde{\theta} (0)$$

Stability of RLS based parallel PAA

step 3: SPR condition

If $G\left(z^{-1}\right) = \frac{1}{A(z^{-1})}$ is SPR, then the PAA is asmptotically hyperstable
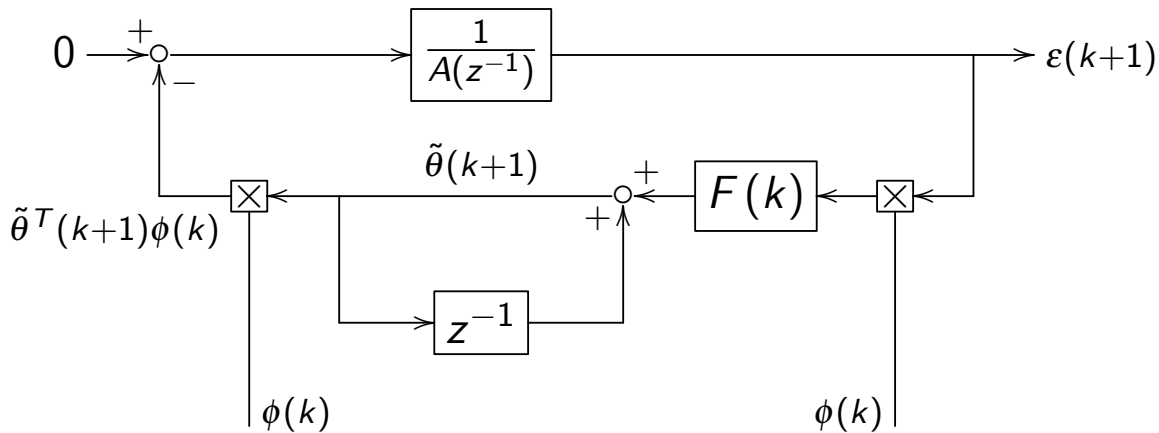
Remarks:

- ► RLS has an identity block: $G\left(z^{-1}\right) = 1$ which is independent of the plant
- ► $1/A\left(z^{-1}\right)$ depends on the plant (usually not SPR)
- ► several other PAAs are developed to relax the SPR condition

Stability of RLS based parallel PAA: extension

For the case of a time-varying $F(k)$ with

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

the nonlinear block is more involved; we'll prove later, that it requires

$$\frac{1}{A(z^{-1})} - \frac{1}{2}\lambda, \text{ where } \lambda = \max_k \lambda_2(k) < 2, \text{ to be SPR}$$

Outline

1. Big picture
   Series-parallel adjustable system (equation-error method)
   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA
   Formulas
   Stability requirement for PAAs with fixed adaptation gain
   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)

Parallel algorithm with a fixed compensator

Instead of:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1 + \phi^T(k)F(k)\phi(k)}\varepsilon^o(k+1)$$

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$$

$$\phi^T(k) = [-\hat{y}(k), -\hat{y}(k-1), \ldots, -\hat{y}(k+1-n), u(k), \ldots, u(k-m)]$$

do: $\quad \hat{\theta}(k+1) = \hat{\theta}(k) + \frac{F(k)\phi(k)}{1 + \phi^T(k)F(k)\phi(k)}v^o(k+1)$

where

$$v(k+1) = C(z^{-1})\varepsilon(k+1) = \left(c_0 + c_1 z^{-1} + \ldots c_n z^{-n}\right)\varepsilon(k+1)$$
$$v^o(k+1) = c_0\varepsilon^o(k+1) + c_1\varepsilon(k) + \ldots c_n\varepsilon(k-n+1)$$

Parallel algorithm with a fixed compensator

The SPR requirement becomes

$$\frac{C\left(z^{-1}\right)}{A(z^{-1})} - \frac{\lambda}{2}, \quad \lambda = \max_k \lambda_2(k) < 2 \qquad (2)$$

should be SPR.

Remark:

- if c_i's are close to a_i's, (2) approximates $1 - \lambda/2 > 0$, and hence is likely to be SPR
- problem: $A\left(z^{-1}\right)$ is unknown *a priori* for the assigning of $C\left(z^{-1}\right)$
- solution: make $C\left(z^{-1}\right)$ to be adjustable as well

Parallel algorithm with an adjustable compensator

If $A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$, let $\hat{C}(z^{-1}) = 1 + \hat{c}_1 z^{-1} + \cdots + \hat{c}_n z^{-n}$ and

$$v(k+1) = \hat{C}(z^{-1}, k+1)\,\varepsilon(k+1)$$

$$v^o(k+1) = \varepsilon^o(k+1) + \sum_{i=1}^{n} \hat{c}_i(k)\,\varepsilon(k+1-i)$$

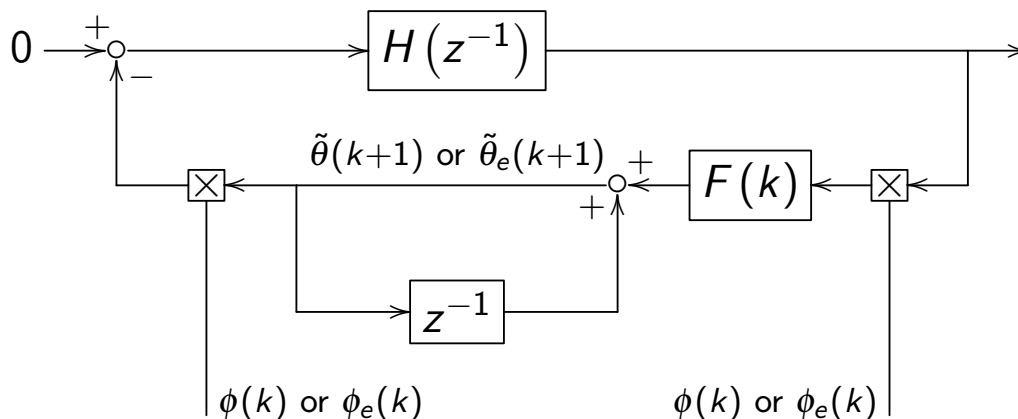do $\qquad \hat{\theta}_e(k+1) = \hat{\theta}_e(k) + \dfrac{F_e(k)\,\phi_e(k)}{1 + \phi_e^T(k)\,F_e(k)\,\phi_e(k)}\,v^o(k+1)$

$$\hat{\theta}_e^T(k) = \left[\hat{\theta}^T(k), \hat{c}_1(k), \ldots, \hat{c}_n(k)\right]$$

$$\phi_e^T(k) = \left[\phi^T(k), -\varepsilon(k), \ldots, -\varepsilon(k+1-n)\right]$$

$$F_e^{-1}(k+1) = \lambda_1(k)\,F_e^{-1}(k) + \lambda_2(k)\,\phi_e(k)\,\phi_e^T(k)$$
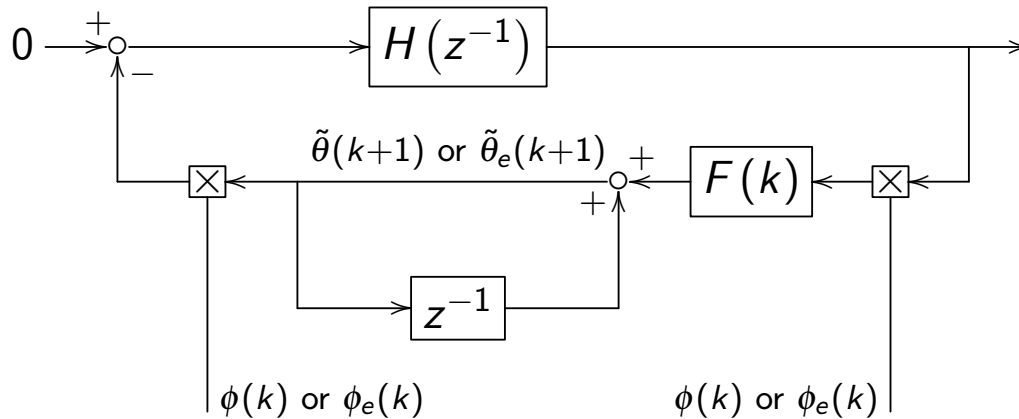
which has guaranteed **asymptotical stablility.**

General PAA block diagram

$H(z^{-1})$	PAA
1	RLS/parallel predictor with adjustable compensator
$1/A(z^{-1})$	parallel predictor
$C(z^{-1})/A(z^{-1})$	parallel predictor with fixed compensator

General PAA block diagram

- if $F(k) = F$, $H(z^{-1})$ being SPR is sufficient for asymptotic stability
- if $F(k)$ is time-varying, we will show next: $H(z^{-1}) - \frac{1}{2}\lambda$ being SPR is sufficient for asymptotic stability

Outline

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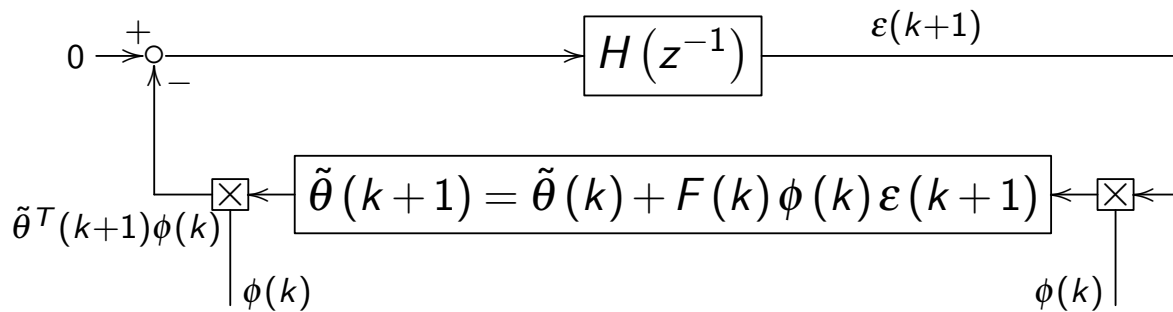2. RLS-based parallel PAA

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4. PAAs with time-varying adaptation gains (revisit)
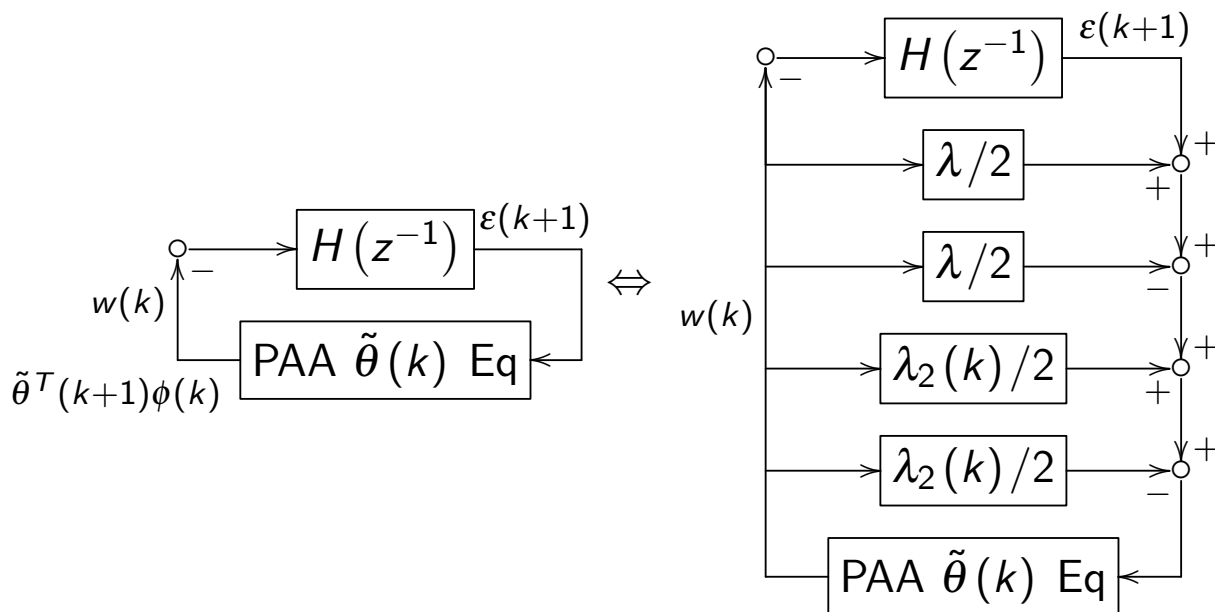
PAA with time-varying adaptation gains

where $F^{-1}(k+1) = \lambda_1(k) F^{-1}(k) + \lambda_2(k) \phi^T(k) \phi(k)$

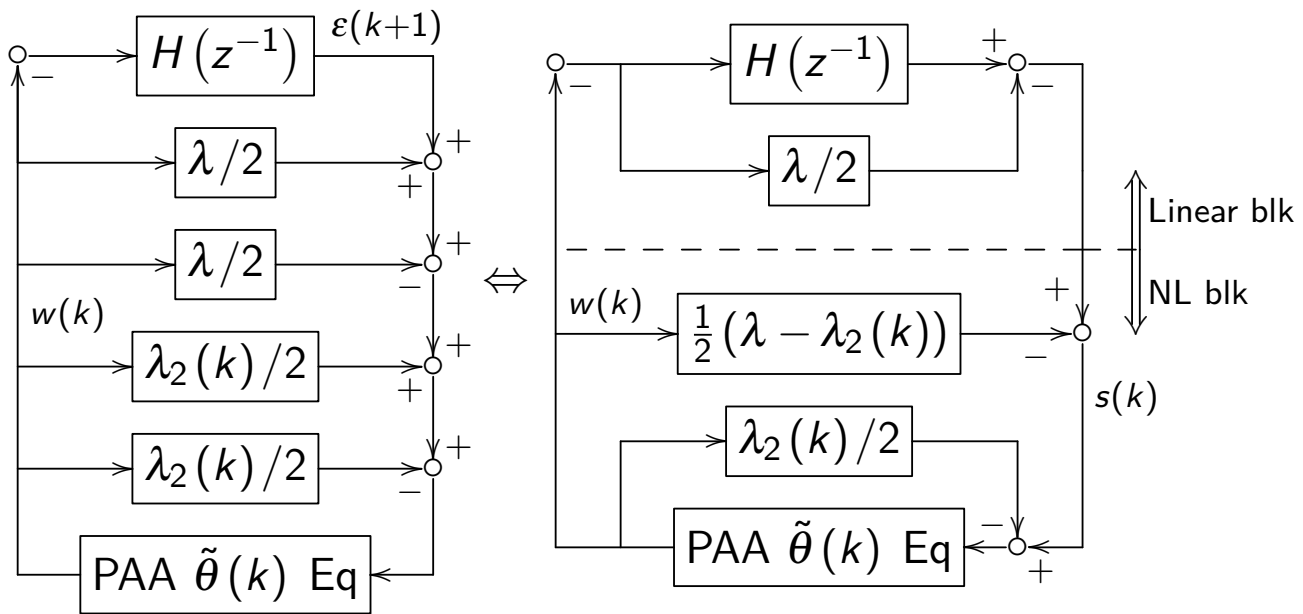- ▶ unfortunately, the nonlinear block does not satisfy Popov inequality (not passive)

PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block

PAA with time-varying adaptation gains

a modification can re-gain the passivity of the feedback block
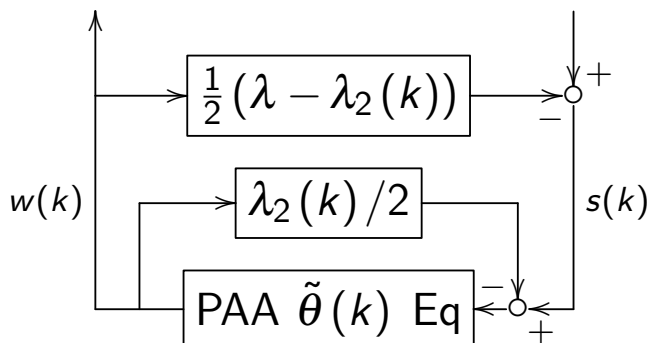
PAA with time-varying adaptation gains

step 1: show that the following is passive
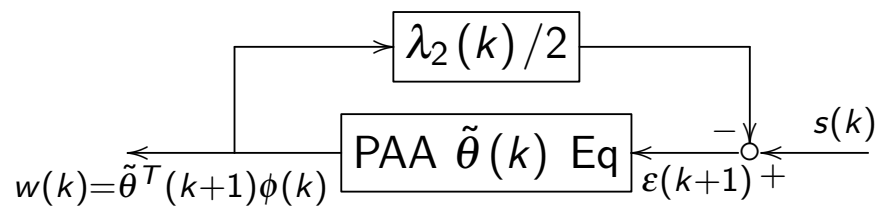
step 2: the following is then passive

note that it is a feedback connection of a passive block with $\frac{1}{2}(\lambda - \lambda_2(k)) \geq 0$
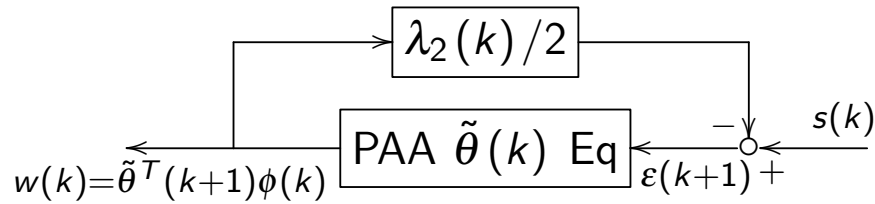
step 3: SPR condition for the linear block $H(z^{-1}) - \frac{\lambda}{2}$

Passivity of the sub nonlinear block

Consider:

$$s(k) = \varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k) \text{ gives}$$

$$\sum_{k=0}^{k_1} w(k)s(k)$$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\left[\varepsilon(k+1) + \frac{\lambda_2(k)}{2}\tilde{\theta}^T(k+1)\phi(k)\right]$$

$\Downarrow$note that $F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\phi(k)\phi^T(k)$

$$= \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) + \frac{1}{2}\tilde{\theta}^T(k+1)\left[F^{-1}(k+1) - \lambda_1(k)F^{-1}(k)\right]\tilde{\theta}(k+1)$$

which is no less than $-\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)$ as shown next.

Proof of passivity of the sub nonlinear block

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

hence

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) = \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)F^{-1}(k)\left(\tilde{\theta}(k+1) - \tilde{\theta}(k)\right)$$

Combining terms and after some algebra (see appendix), we get

$$\sum_{k=0}^{k_1} w(k)s(k) = \sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T(k+1)(1-\lambda_1(k))F^{-1}(k)\tilde{\theta}(k+1)$$

$$+ \sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]^T F^{-1}(k)\left[\tilde{\theta}(k+1) - \tilde{\theta}(k)\right]$$

$$+ \underbrace{\sum_{k=0}^{k_1} \frac{1}{2}\left[\tilde{\theta}^T(k+1)F^{-1}(k)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)F^{-1}(k)\tilde{\theta}(k)\right]}_{\frac{1}{2}\tilde{\theta}^T(k_1+1)F^{-1}(k_1)\tilde{\theta}(k_1+1) - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0)} \quad (3)$$

Summary

In summary, the NL block indeed satisfies Popov inequality. For stability of PAA, it is sufficient that

$$H\left(z^{-1}\right) - \frac{\lambda}{2} \text{ is SPR}$$

Appendix: derivation of (3)

$$
\sum_{k=0}^{k_1} \tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\left(\tilde{\theta}\left(k+1\right) - \tilde{\theta}\left(k\right)\right) + \frac{1}{2}\tilde{\theta}^T\left(k+1\right)\left[F^{-1}\left(k+1\right) - \lambda_1\left(k\right) F^{-1}\left(k\right)\right]\tilde{\theta}\left(k+1\right)
$$

$$
= \sum_{k=0}^{k_1} \tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right) - \tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k\right) + \frac{1}{2}\tilde{\theta}^T\left(k+1\right)\left[F^{-1}\left(k+1\right) - \lambda_1\left(k\right) F^{-1}\left(k\right)\right]\tilde{\theta}\left(k+1\right)
$$

$$
= \sum_{k=0}^{k_1} \tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right) - \boxed{\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k\right)} + \frac{1}{2}\tilde{\theta}^T\left(k+1\right)\left[F^{-1}\left(k+1\right) - \underline{\lambda_1\left(k\right)} F^{-1}\left(k\right)\right]\tilde{\theta}\left(k+1\right)
$$

$$
= \sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T\left(k+1\right)\left(1 - \lambda_1\left(k\right)\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right) + \frac{1}{2}\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right) - \boxed{\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k\right)}
$$

$$
+ \frac{1}{2}\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k+1\right)\tilde{\theta}\left(k+1\right) \tag{4}
$$

The term $\frac{1}{2}\tilde{\theta}^T\left(k+1\right)\left(1 - \lambda_1\left(k\right)\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right)$ is always none-negative if $1 - \lambda_1\left(k\right) \geq 0$, which is the assumption in the forgetting factor definition. We only need to worry about

$$
\sum_{k=0}^{k_1} \frac{1}{2}\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k+1\right) - \tilde{\theta}^T\left(k+1\right) F^{-1}\left(k\right)\tilde{\theta}\left(k\right) + \frac{1}{2}\tilde{\theta}^T\left(k+1\right) F^{-1}\left(k+1\right)\tilde{\theta}\left(k+1\right)
$$

$$\tag{5}$$

Appendix: derivation of (3)

The underlined terms are already available in (5). Adding and substracting terms in (5) gives

$$\sum_{k=0}^{k_1} \frac{1}{2} \tilde{\theta}^T (k+1) F^{-1}(k) \tilde{\theta}(k+1) - \frac{1}{2} \underline{\tilde{\theta}^T(k) F^{-1}(k) \tilde{\theta}(k)}$$

$$+ \frac{1}{2} \tilde{\theta}^T (k+1) F^{-1}(k+1) \tilde{\theta}(k+1) - \tilde{\theta}^T(k+1) F^{-1}(k) \tilde{\theta}(k) + \underline{\frac{1}{2} \tilde{\theta}^T(k) F^{-1}(k) \tilde{\theta}(k)}$$

$$= \sum_{k=0}^{k_1} \frac{1}{2} \tilde{\theta}^T (k+1) F^{-1}(k) \tilde{\theta}(k+1) - \frac{1}{2} \tilde{\theta}^T(k) F^{-1}(k) \tilde{\theta}(k)$$

$$+ \underbrace{\frac{1}{2} \left[\tilde{\theta}(k+1) - \tilde{\theta}(k) \right]^T F^{-1}(k) \left[\tilde{\theta}(k+1) - \tilde{\theta}(k) \right]}_{\geq 0}$$

Appendix: derivation of (3)

Summarizing, we get

$$\sum_{k=0}^{k_1} w(k) s(k) = \sum_{k=0}^{k_1} \frac{1}{2} \tilde{\theta}^T (k+1) (1 - \lambda_1(k)) F^{-1}(k) \tilde{\theta}(k+1)$$

$$+ \sum_{k=0}^{k_1} \frac{1}{2} \left[\tilde{\theta}(k+1) - \tilde{\theta}(k) \right]^T F^{-1}(k) \left[\tilde{\theta}(k+1) - \tilde{\theta}(k) \right]$$

$$+ \underbrace{\sum_{k=0}^{k_1} \frac{1}{2} \left[\tilde{\theta}^T (k+1) F^{-1}(k) \tilde{\theta}(k+1) - \tilde{\theta}^T(k) F^{-1}(k) \tilde{\theta}(k) \right]}_{\frac{1}{2} \tilde{\theta}^T(k_1+1) F^{-1}(k_1) \tilde{\theta}(k_1+1) - \frac{1}{2} \tilde{\theta}^T(0) F^{-1}(0) \tilde{\theta}(0)}$$

hence

$$\sum_{k=0}^{k_1} w(k) s(k) \geq -\frac{1}{2} \tilde{\theta}^T (0) F^{-1}(0) \tilde{\theta}(0)$$

Summary

1. Big picture

   Series-parallel adjustable system (equation-error method)

   Parallel adjustable system (output-error method)

2. RLS-based parallel PAA

   Formulas

   Stability requirement for PAAs with fixed adaptation gain

   Stability requirement for PAAs with time-varying adaptation gain

3. Parallel PAAs with relaxed SPR requirements

4. PAAs with time-varying adaptation gains (revisit)