Homework 5

July 29, 2020

1 Homework #5

Due July 29 @ 11:59pm

1.1 Submission requirements

Upload a **single PDF file** of your IJulia notebook for this entire assigment. Clearly denote which question each section of your PDF corresponds to.

1.2 Problem 1 – Nonconvex Quadratics

Suppose you have the constraint:

$2x^2 + y^2 + 2z^2 - 3xy + xz - 4yz \leq 0 \qquad (1)$

(a) Write constraint (1) in the standard form $v^T Q v \leq 0$ where Q is a symmetric matrix. What is Q and what is v?

Ans: vector $v = [x, y, z]$, symmetric matrix Q is $\begin{bmatrix} 2 & -1.5 & 0.5 \\ -1.5 & 1 & -2 \\ 0.5 & -2 & 2 \end{bmatrix}$ contstraint is $v^T Q v \leq 0$.

(b) This constraint is not convex (i.e., the set of points satisfying the constraint is not an ellipsoid). Explain why this is the case. *Hint*: You can perform an orthogonal decomposition of a symmetric matrix Q in Julia like this:

[36]:
```
Q = [2 -1.5 0.5; -1.5 1 -2; 0.5 -2 2]

using LinearAlgebra
(L,U) = (eigvals(Q),eigvecs(Q)) # L is the vector of eigenvalues and U is␣
 ↪orthogonal
U * Diagonal(L) * U' # this is equal to Q (as long as Q was symmetric to begin␣
 ↪with)

#take the square root of Diagonal(L) if all of its eigenvalues are non-negative
P = U * sqrt(Diagonal(L))
println(P)
```

```
DomainError with -0.8701601197116661:
```

```
   sqrt will only return a complex result if called with a complex argument.␣
↪Try sqrt(Complex(x)).



       Stacktrace:

        [1] throw_complex_domainerror(::Symbol, ::Float64) at ./math.jl:32

        [2] sqrt at ./math.jl:492 [inlined]

        [3] _broadcast_getindex_evalf at ./broadcast.jl:630 [inlined]

        [4] _broadcast_getindex at ./broadcast.jl:603 [inlined]

        [5] getindex at ./broadcast.jl:563 [inlined]

        [6] macro expansion at ./broadcast.jl:909 [inlined]

        [7] macro expansion at ./simdloop.jl:77 [inlined]

        [8] copyto! at ./broadcast.jl:908 [inlined]

        [9] copyto! at ./broadcast.jl:863 [inlined]

        [10] copy at ./broadcast.jl:839 [inlined]

        [11] materialize at ./broadcast.jl:819 [inlined]

        [12] sqrt(::Diagonal{Float64,Array{Float64,1}}) at /Users/julia/
↪buildbot/worker/package_macos64/build/usr/share/julia/stdlib/v1.3/
↪LinearAlgebra/src/diagonal.jl:555

        [13] top-level scope at In[36]:6
```

Ans: The constraint is convex if and only if the matrix Q is symmetric and positive semidefinite. Since Q is not positive semidefinite for that we cannot take the sqaure root of it, the constraint is not convex.

(c) We can write constraint (1) in norm format as follows:

$\|Av\|_2^2 - \|Bv\|_2^2 \leq 0 \qquad (2)$

Find matrices A and B that make this constraint equivalent to (1).

Ans: $\|Av\|_2^2 - \|Bv\|_2^2 \leq 0$ is equivalent to $(v^T A^T Av) - (v^T B^T Bv) \leq 0$, hence equivalent to $v^T(A^T A - B^T B)v \leq 0$. Therefore, $Q = A^T A - B^T B$. Solve this using julia.

[74]:
```
println(Diagonal(L))
#diagonize L and turn it into L_pos - L_neg
L_pos = [0.0 0.0 0.0; 0.0 1.518494119904299 0.0; 0.0 0.0 4.351665999807366]
L_neg = [-0.8701601197116661 0.0 0.0; 0.0 0 0.0; 0.0 0.0 0]

A = U * sqrt(L_pos)
B = U * sqrt(L_neg)
println("A is " ,A)
println("B is " ,B)
println(A*A'-B*B') # check: it should equals Q
```

```
[-0.8701601197116661 0.0 0.0; 0.0 1.518494119904299 0.0; 0.0 0.0
4.351665999807366]
A is [0.0 -0.9764959351583015 1.0685819183944123; -0.0 -0.06381999098002426
-1.2462660599533772; 0.0 0.7489170963964722 1.2870973511340034]
B is Complex{Float64}[0.0 - 0.3088877266907329im 0.0 + 0.0im 0.0 + 0.0im; -0.0 -
0.7464931904849552im -0.0 - 0.0im -0.0 - 0.0im; 0.0 - 0.46636510222259325im 0.0
+ 0.0im 0.0 + 0.0im]
Complex{Float64}[1.9999999999999993 - 0.0im -1.5000000000000022 - 0.0im
0.499999999999999 - 0.0im; -1.5000000000000022 - 0.0im 0.9999999999999974 -
0.0im -1.9999999999999976 - 0.0im; 0.499999999999999 - 0.0im -1.9999999999999976
- 0.0im 2.0000000000000004 - 0.0im]
```

(d) Explain how to find (x, y, z) that satisfy the above constraint but make $2x^2 + y^2 + 2z^2$ arbitrarily large.

Ans: Define $w = U^T x$, $x^T Q x = \lambda_1 w_1^2 + \lambda_2 w_2^2 + \lambda_3 w_3^2$, since $x^T Q x \leq 0$ and $\lambda_1 < 0$, $w_2 = w_3 = 0$. $w_1 = a_{11}x + a_{12}y + a_{13}z$, then $2x^2 + y^2 + 2z^2$ can be arbitrarily large.

1.3 Problem 2 – Circles within Circles

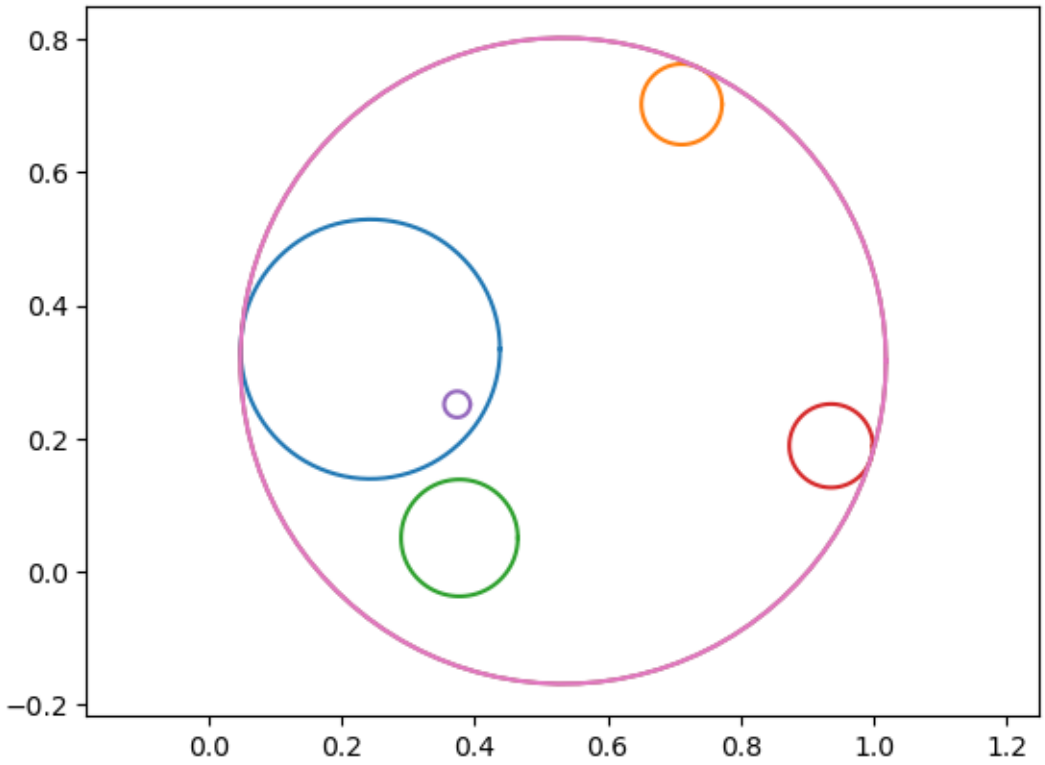
Formulate a convex program to solve the minimum enclosing ball (MEB) problem. You need to determine the center z of a ball, and its radius d so that each of the circles centered at c_i with radius r_i are enclosed in your generated circle. Your task is to find the cirle of smallest radius that encircles all the other circles.

You can use the following code to start building your model. Start with $n = 5$ circles to help with debugging, but display the output of your model for both $n = 5$ and a very large value of n, such as $n = 200$.

[52]:
```
n = 5
using Random, PyPlot
r = [rand()*0.2 for i in 1:n] # randomly generate n radii with length between 0␣
 ↪and 0.2
# randomly generate n points with (x,y) coords between 0 and 1
c = [rand() for i in 1:n, j in 1:2]

t = range(0,stop=2*pi,length=100) # parameter that traverses the circle
```

```
# for each of the n points
for i in 1:n
    # plot circle radius r[i] with center (x[i],y[i])
    plot( c[i,1] .+ r[i]*cos.(t), c[i,2] .+ r[i]*sin.(t))
end

axis("equal"); # make x and y scales equal

#build model
using Pkg
Pkg.add("Ipopt")
using JuMP,Ipopt

m = Model(Ipopt.Optimizer)

@variable(m, x[1:2])# variable for coordinates of circle center
@variable(m, d>=0)# variable representing radius of the circle

@NLobjective(m, Min, pi*(d^2))# we want to minimize the area

@NLconstraint(m, constr[i in 1:n], (x[1]-c[i,1])^2 + (x[2]-c[i,2])^2 <=␣
 ↪(d-r[i])^2 )

optimize!(m)
t=range(0,stop=2pi,length=100)# parameter that traverses the circle
# plot circle radius d with center (x1,x2); (orange circe)
plot( value(x[1]).+ value(d)*cos.(t), value(x[2]).+ value(d)*sin.(t));
```

```
Resolving package versions…
```




```
  Updating `~/.julia/environments/v1.3/Project.toml`
 [no changes]
  Updating `~/.julia/environments/v1.3/Manifest.toml`
 [no changes]
This is Ipopt version 3.13.2, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian…:        0
Number of nonzeros in inequality constraint Jacobian.:       15
Number of nonzeros in Lagrangian Hessian…:       16

Total number of variables…:        3
                     variables with only lower bounds:        1
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints…:        0
Total number of inequality constraints…:        5
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        5

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
```

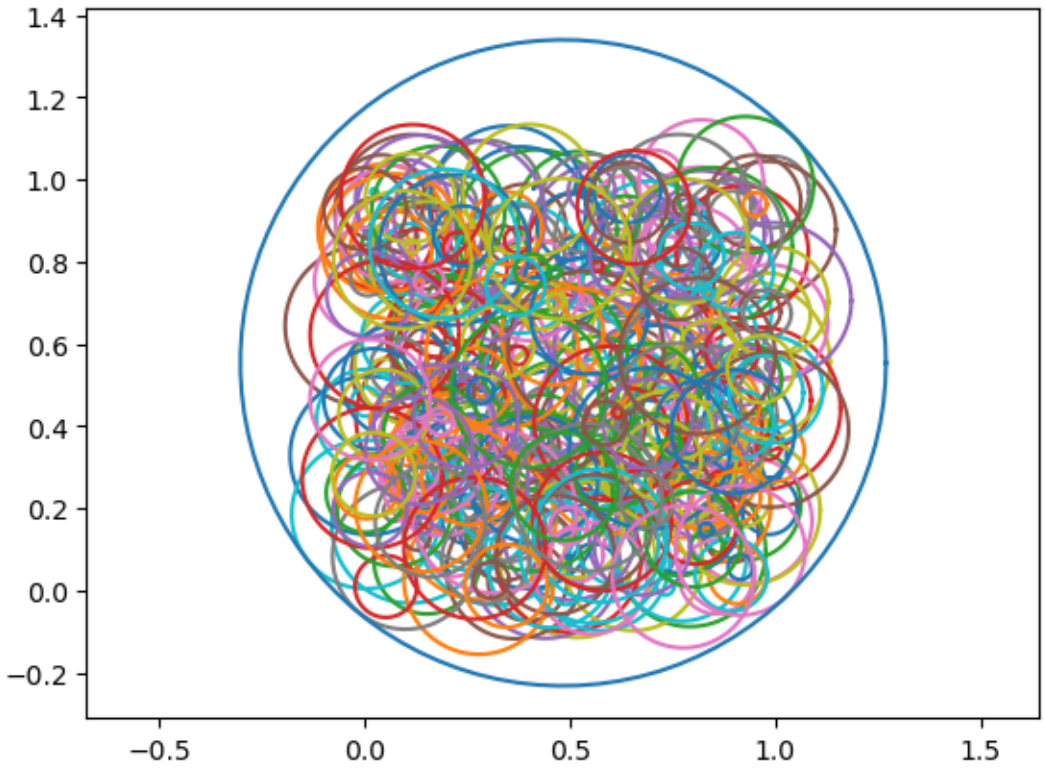
```
   0  3.1415864e-04 9.97e-01 1.24e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  6.5624396e-02 2.54e-01 2.03e+01  -1.0 4.67e-01    -  5.80e-02 1.00e+00f  1
   2  8.3360364e-02 2.15e-01 1.74e+01  -1.0 3.37e-01   2.0 1.00e+00 1.89e-01h  1
   3  3.5227950e-01 8.64e-02 1.43e+01  -1.0 3.49e-01   1.5 1.00e+00 4.92e-01h  1
   4  1.1661677e+00 0.00e+00 3.40e+01  -1.0 2.74e-01    -  3.04e-01 1.00e+00f  1
   5  1.2527991e+00 0.00e+00 6.29e+00  -1.0 1.35e-01   1.0 1.00e+00 1.00e+00h  1
   6  1.0239001e+00 0.00e+00 2.31e-01  -1.0 1.02e-01    -  1.00e+00 1.00e+00f  1
   7  7.9730222e-01 0.00e+00 1.25e-02  -1.7 9.75e-02    -  1.00e+00 1.00e+00h  1
   8  7.4819585e-01 0.00e+00 6.33e-02  -2.5 1.80e-02    -  1.00e+00 8.76e-01h  1
   9  7.4947098e-01 0.00e+00 9.64e-04  -2.5 6.15e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  7.4103690e-01 0.00e+00 3.28e-04  -3.8 5.72e-03    -  1.00e+00 1.00e+00h  1
  11  7.4055185e-01 6.64e-07 1.04e-05  -5.7 1.16e-03    -  1.00e+00 1.00e+00h  1
  12  7.4053863e-01 7.29e-10 8.25e-09  -8.6 4.06e-05    -  1.00e+00 1.00e+00h  1
  13  7.4053862e-01 0.00e+00 1.00e-14  -9.0 4.52e-08    -  1.00e+00 1.00e+00h  1

Number of Iterations…: 13

                                   (scaled)                 (unscaled)
Objective…:   7.4053861765729734e-01    7.4053861765729734e-01
Dual infeasibility…:   1.0002480044416343e-14    1.0002480044416343e-14
Constraint violation…:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity…:   9.0911049503938839e-10    9.0911049503938839e-10
Overall NLP error…:   9.0911049503938839e-10    9.0911049503938839e-10


Number of objective function evaluations             = 14
Number of objective gradient evaluations             = 14
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 14
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 14
Number of Lagrangian Hessian evaluations             = 13
Total CPU secs in IPOPT (w/o function evaluations)   =      0.011
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
This is Ipopt version 3.13.2, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian…:        0
Number of nonzeros in inequality constraint Jacobian.:       15
Number of nonzeros in Lagrangian Hessian…:       16

Total number of variables…:        3
                     variables with only lower bounds:        1
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
```

```
Total number of equality constraints…:        0
Total number of inequality constraints…:        5
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        5

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.1415864e-04 9.97e-01 1.24e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  6.5624396e-02 2.54e-01 2.03e+01  -1.0 4.67e-01    -  5.80e-02 1.00e+00f  1
   2  8.3360364e-02 2.15e-01 1.74e+01  -1.0 3.37e-01   2.0 1.00e+00 1.89e-01h  1
   3  3.5227950e-01 8.64e-02 1.43e+01  -1.0 3.49e-01   1.5 1.00e+00 4.92e-01h  1
   4  1.1661677e+00 0.00e+00 3.40e+01  -1.0 2.74e-01    -  3.04e-01 1.00e+00f  1
   5  1.2527991e+00 0.00e+00 6.29e+00  -1.0 1.35e-01   1.0 1.00e+00 1.00e+00h  1
   6  1.0239001e+00 0.00e+00 2.31e-01  -1.0 1.02e-01    -  1.00e+00 1.00e+00f  1
   7  7.9730222e-01 0.00e+00 1.25e-02  -1.7 9.75e-02    -  1.00e+00 1.00e+00h  1
   8  7.4819585e-01 0.00e+00 6.33e-02  -2.5 1.80e-02    -  1.00e+00 8.76e-01h  1
   9  7.4947098e-01 0.00e+00 9.64e-04  -2.5 6.15e-03    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  7.4103690e-01 0.00e+00 3.28e-04  -3.8 5.72e-03    -  1.00e+00 1.00e+00h  1
  11  7.4055185e-01 6.64e-07 1.04e-05  -5.7 1.16e-03    -  1.00e+00 1.00e+00h  1
  12  7.4053863e-01 7.29e-10 8.25e-09  -8.6 4.06e-05    -  1.00e+00 1.00e+00h  1
  13  7.4053862e-01 0.00e+00 1.00e-14  -9.0 4.52e-08    -  1.00e+00 1.00e+00h  1

Number of Iterations…: 13

                                   (scaled)                 (unscaled)
Objective…:   7.4053861765729734e-01    7.4053861765729734e-01
Dual infeasibility…:   1.0002480044416343e-14    1.0002480044416343e-14
Constraint violation…:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity…:   9.0911049503938839e-10    9.0911049503938839e-10
Overall NLP error…:   9.0911049503938839e-10    9.0911049503938839e-10


Number of objective function evaluations             = 14
Number of objective gradient evaluations             = 14
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 14
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 14
Number of Lagrangian Hessian evaluations             = 13
Total CPU secs in IPOPT (w/o function evaluations)   =      0.011
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

[58]:
```
n = 400
using Random, PyPlot
```

```
r = [rand()*0.2 for i in 1:n] # randomly generate n radii with length between 0
 ↪and 0.2
# randomly generate n points with (x,y) coords between 0 and 1
c = [rand() for i in 1:n, j in 1:2]

t = range(0,stop=2*pi,length=100) # parameter that traverses the circle

# for each of the n points
for i in 1:n
    # plot circle radius r[i] with center (x[i],y[i])
    plot( c[i,1] .+ r[i]*cos.(t), c[i,2] .+ r[i]*sin.(t))
end

axis("equal"); # make x and y scales equal

#build model
using Pkg
Pkg.add("Ipopt")
using JuMP,Ipopt

m = Model(Ipopt.Optimizer)

@variable(m, x[1:2])# variable for coordinates of circle center
@variable(m, d>=0)# variable representing radius of the circle

@NLobjective(m, Min, pi*(d^2))# we want to minimize the area

@NLconstraint(m, constr[i in 1:n], (x[1]-c[i,1])^2 + (x[2]-c[i,2])^2 <=
 ↪(d-r[i])^2 )

optimize!(m)
t=range(0,stop=2pi,length=100)# parameter that traverses the circle
# plot circle radius d with center (x1,x2);
plot( value(x[1]).+ value(d)*cos.(t), value(x[2]).+ value(d)*sin.(t));
```

```
Resolving package versions…
```




```
  Updating `~/.julia/environments/v1.3/Project.toml`
 [no changes]
  Updating `~/.julia/environments/v1.3/Manifest.toml`
 [no changes]
This is Ipopt version 3.13.2, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian…:        0
Number of nonzeros in inequality constraint Jacobian.:     1200
Number of nonzeros in Lagrangian Hessian…:     1201

Total number of variables…:        3
                     variables with only lower bounds:        1
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints…:        0
Total number of inequality constraints…:      400
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:      400

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
```

```
   0  3.1415864e-04 1.93e+00 1.66e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.8715516e-03 1.87e+00 1.90e+00  -1.0 4.46e-01    -  2.88e-02 3.32e-02f  1
   2  2.7477677e-03 1.86e+00 4.15e+01  -1.0 6.11e-01    -  4.82e-02 8.46e-03f  1
   3  1.1977085e-01 1.15e+00 2.10e+02  -1.0 5.15e-01    -  8.71e-02 4.07e-01f  1
   4  3.2543085e-01 1.02e+00 1.91e+02  -1.0 1.38e+00    -  1.40e-01 9.18e-02f  1
   5  3.5181592e-01 9.94e-01 1.87e+02  -1.0 6.20e-01    -  1.06e-01 2.06e-02h  1
   6  3.2763826e+00 0.00e+00 3.83e+02  -1.0 6.87e-01    -  2.58e-01 1.00e+00f  1
   7  3.0836917e+00 0.00e+00 3.14e+02  -1.0 3.50e-01    -  1.07e-01 1.84e-01h  1
   8  2.8735763e+00 2.58e-02 7.74e+01  -1.0 4.16e-01    -  8.92e-02 1.00e+00f  1
   9  4.1521286e+00 0.00e+00 1.83e+01  -1.0 4.25e-01    -  4.51e-01 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  6.3270018e+00 0.00e+00 7.76e+00  -1.0 7.19e-01    -  3.31e-01 1.00e+00f  1
  11  8.5481281e+00 0.00e+00 6.54e-01  -1.0 6.59e-01    -  4.68e-01 1.00e+00f  1
  12  6.8590134e+00 0.00e+00 2.17e-01  -1.7 4.15e-01    -  6.18e-01 1.00e+00h  1
  13  7.1030457e+00 0.00e+00 1.42e-02  -1.7 4.25e-01    -  1.00e+00 1.00e+00h  1
  14  3.2323501e+00 0.00e+00 8.83e-02  -2.5 1.68e+00    -  1.00e+00 9.16e-01h  1
  15  2.3408544e+00 0.00e+00 4.33e-02  -2.5 1.51e-01    -  1.00e+00 1.00e+00h  1
  16  2.2059698e+00 0.00e+00 2.16e-03  -2.5 4.56e-02    -  1.00e+00 1.00e+00h  1
  17  1.9865563e+00 0.00e+00 2.68e-01  -3.8 2.97e-01    -  1.00e+00 2.69e-01f  1
  18  1.9607147e+00 0.00e+00 1.16e+00  -3.8 8.96e-02    -  1.00e+00 1.57e-01h  1
  19  1.9423307e+00 4.22e-04 2.21e-01  -3.8 3.29e-02    -  7.20e-01 1.00e+00h  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  20  1.9418250e+00 1.16e-04 5.27e-03  -3.8 1.01e-02    -  1.00e+00 8.41e-01h  1
  21  1.9426394e+00 0.00e+00 2.05e-06  -3.8 1.80e-03    -  1.00e+00 1.00e+00h  1
  22  1.9419956e+00 1.75e-06 3.42e-02  -5.7 2.27e-03    -  9.93e-01 7.87e-01h  1
  23  1.9419559e+00 0.00e+00 1.26e-07  -5.7 3.44e-05    -  1.00e+00 1.00e+00h  1
  24  1.9419504e+00 3.39e-11 9.88e-06  -8.6 9.45e-06    -  1.00e+00 9.97e-01h  1
  25  1.9419504e+00 0.00e+00 2.86e-14  -8.6 1.38e-08    -  1.00e+00 1.00e+00f  1

Number of Iterations…: 25

                                   (scaled)                 (unscaled)
Objective…:   1.9419504140556154e+00    1.9419504140556154e+00
Dual infeasibility…:   2.8634332171460145e-14    2.8634332171460145e-14
Constraint violation…:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity…:   2.5059158220662008e-09    2.5059158220662008e-09
Overall NLP error…:   2.5059158220662008e-09    2.5059158220662008e-09


Number of objective function evaluations             = 26
Number of objective gradient evaluations             = 26
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 26
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 26
Number of Lagrangian Hessian evaluations             = 25
Total CPU secs in IPOPT (w/o function evaluations)   =      0.083
Total CPU secs in NLP function evaluations           =      0.012
```

```
EXIT: Optimal Solution Found.
```

[ ]: