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Theory of automatic control 1

Modelling and Input-Output Properties of Dynamic Systems

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1 Assignment – Two liquids in series

We study mass balance of two tanks with straight vertical walls that hold a liquid and are connected in series. The sketch of the system is given in Fig. 1. The liquid enters a tank on its top. The flow-rate of this stream $q_0(t)$ is freely adjustable. An out-flowing streams leave the ith tank at its bottom with the flow-rate that is driven by the gravity, i.e., Torricelli's law $q_i(t) = k_{ii}\sqrt{h_i(t)}$, where k_{ii} is valve constant and $h_i(t)$ is the height of liquid in the tank. Two streams enter the second tank, $q_0(t)$ and $q_1(t)$. The cross-sectional area of the ith tank is denoted as F_i . We can assume that the density of the liquid (ρ) is constant everywhere. We can only measure the level in the second tank.

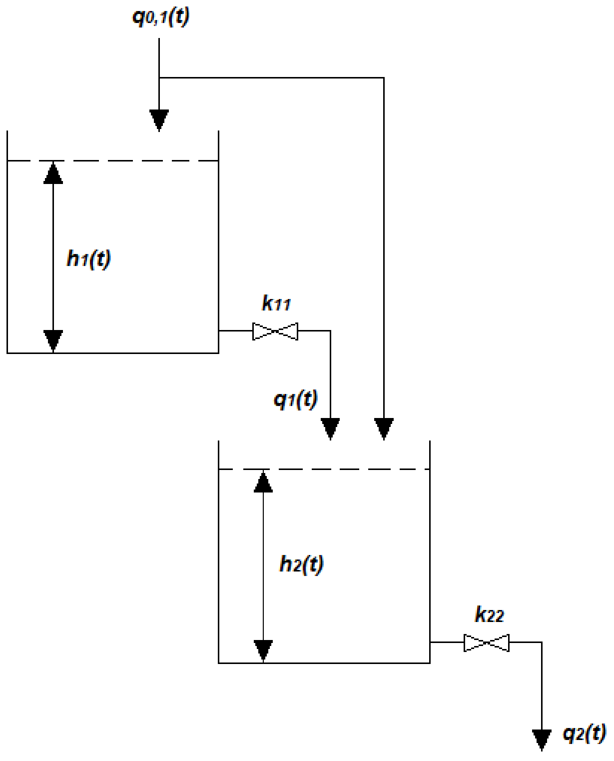

Figure 1: Schematic diagram of the two-tank system

1.1 The parameters of the system

$q_{0,1}^{s} = 1\ \mathrm{m^3/s}$

$F_1 = 0{,}1\ \mathrm{m^2}$

$F_2 = 0{,}8\ \mathrm{m^2}$

$k_{11} = 1{,}3\ \mathrm{m^{2,5}/s}$

$k_{22} = 1{,}0\ \mathrm{m^{2,5}/s}$

2 Tasks

2.1 Variables of the system

- Input variables: entering flow-rate $q_{0,1}(t)$
- Output variables: height of liquid in the second tank $h_2(t)$
- State variables: height of liquid in tanks $h_1(t), h_2(t)$

2.2 Mass balance of liquid in the tanks and dynamic model

$$\dot{m}_{0,1}(t) = \dot{m}_1(t) + \frac{dm_1(t)}{dt} \qquad [1]$$

$$\dot{m}_{0,1}(t) + \dot{m}_1(t) = \dot{m}_2(t) + \frac{dm_2(t)}{dt} \qquad [2]$$

where $\dot{m}_i(t)$ is mass flow and $\frac{dm_i(t)}{dt}$ is accumulation in the system.

Mass flow can be rewritten as:

$$\dot{m}_i = q_i \rho \qquad [3]$$

$$m_i = V_i \rho \qquad [4]$$

where q is volumetric flow, V is volume and ρ is the density of substance.

$$q_{0,1}(t)\rho = q_1(t)\rho + \frac{d[V_1(t)\rho]}{dt} \qquad [5]$$

$$q_{0,1}(t)\rho \ + \ q_1(t)\rho = q_2(t)\rho + \frac{d[V_2(t)\rho]}{dt} \qquad [6]$$

Volume V can be rewritten as:

$$V_i = F_i h_i(t) \qquad [7]$$

After these modifications the mass balance of system can be written as:

$$q_{0,1}(t) = q_1(t) + F_1 \frac{dh_1(t)}{dt} \qquad [8]$$

$$q_{0,1}(t) \ + \ q_1(t) = q_2(t) + F_2 \frac{dh_2(t)}{dt} \qquad [9]$$

Where flow rates are defined as follow (Torricelli's law):

$$q_1(t) = k_{11}\sqrt{h_1(t)} \qquad [10]$$

$$q_2(t) = k_{22}\sqrt{h_2(t)} \qquad [11]$$

where k_{11} and k_{22} are valve constants and $h_1(t)$ and $h_2(t)$ are height of liquid in tanks.

$$q_{0,1}(t) = k_{11}\sqrt{h_1(t)} + F_1 \frac{dh_1(t)}{dt} \qquad [12]$$

$$q_{0,1}(t) \ + \ k_{11}\sqrt{h_1(t)} = k_{22}\sqrt{h_2(t)} + F_2 \frac{dh_2(t)}{dt} \qquad [13]$$

Dynamic mathematical model:

$$F_1 \frac{dh_1(t)}{dt} = q_{0,1}(t) - k_{11}\sqrt{h_1(t)} \qquad [14]$$

$$F_2 \frac{dh_2(t)}{dt} = q_{0,1}(t) \ + \ k_{11}\sqrt{h_1(t)} - k_{22}\sqrt{h_2(t)} \qquad [15]$$

2.3 The steady-state values of the liquid levels in the tanks for the given input q_0^s

Mathematical model in steady-state:

$$F_1 \frac{dh_1^s}{dt} = q_{0,1}^s - k_{11}\sqrt{h_1^s} \qquad [16]$$

$$F_2 \frac{dh_2^s}{dt} = q_{0,1}^s + \ k_{11}\sqrt{h_1^s} - k_{22}\sqrt{h_2^s} \qquad [17]$$

Steady-state values are define when $\frac{dh_i^s}{dt} = 0$ and all variables are time-invariable. After that steady-state values are expressed as:

$$h_1^s = \left(\frac{q_{0,1}^s}{k_{11}}\right)^2 = 0{,}5917 \text{ m} \qquad [18]$$

$$h_2^s = \left(\frac{q_{0,1}^s + k_{11}\sqrt{h_1^s}}{k_{22}}\right)^2 = 4{,}0 \text{ m} \qquad [19]$$

2.4 Linearisation of the model

The first step of linearisation is make the difference between dynamic mathematical model and mathematical model in steady-state:

$$\frac{d(h_1(t) - h_1^s)}{dt} = \frac{1}{F_1}(q_{0,1}(t) - q_{0,1}^s) - \frac{1}{F_1}(k_{11}\sqrt{h_1(t)} - k_{11}\sqrt{h_1^s}\,) \qquad [20]$$

$$\frac{d(h_2(t)-h_2^s)}{dt}=\frac{1}{F_2}(q_{0,1}(t)-q_{0,1}^s)+\frac{1}{F_2}(k_{11}\sqrt{h_1(t)}-k_{11}\sqrt{h_1^s})-\frac{1}{F_2}(k_{22}\sqrt{h_2(t)}-k_{22}\sqrt{h_2^s}) \quad [21]$$

Deviation variables are defined as follows:

- Input deviation variables: $u(t) = q_{0,1}(t) - q_{0,1}^s$
- State deviation variables: $x_1(t) = h_1(t) - h_1^s$
  $x_2(t) = h_2(t) - h_2^s$
- Output deviation variables: $y(t) = x_2(t)$

The state space of the linearised model is defined as:

$$\frac{dx_1(t)}{dt}=\frac{1}{F_1}u(t)-\frac{k_1}{F_1}x_1(t) \quad ; \quad x_1(0)=0 \quad [22]$$

$$\frac{dx_2(t)}{dt}=\frac{1}{F_2}u(t)+\frac{k_1}{F_2}x_1(t)-\frac{k_2}{F_2}x_2(t) \quad ; \quad x_2(0)=0 \quad [23]$$

$$y(t) = x_2(t) \quad [24]$$

Constants k_1 and k_2 are defined as:

$$k_1=\frac{k_{11}}{2\sqrt{h_1^s}} \quad [25]$$

$$k_2=\frac{k_{22}}{2\sqrt{h_2^s}} \quad [26]$$

The matrices of state-space model are defined as:

$$A=\begin{bmatrix}-\frac{k_1}{F_1} & 0\\ \frac{k_1}{F_2} & -\frac{k_2}{F_2}\end{bmatrix}=\begin{bmatrix}-8{,}4500 & 0\\ 1.0563 & -0.3125\end{bmatrix} \quad [27]$$

$$B=\begin{bmatrix}\frac{1}{F_1}\\ \frac{1}{F_2}\end{bmatrix}=\begin{bmatrix}10{,}00\\ 1{,}25\end{bmatrix} \quad [28]$$

$$C=[0 \quad 1] \quad [29]$$

$$D=0 \quad [30]$$

2.5 Simulink scheme of non-linear and linearised model

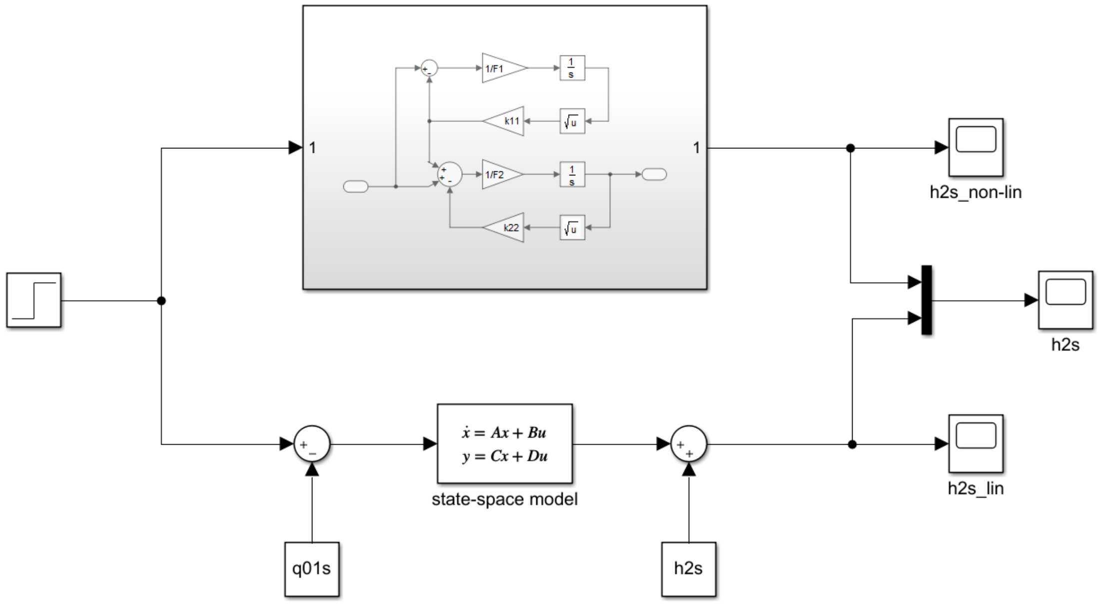

Figure 2: Simulink scheme of non-linear and linearised model

2.6 Non-linear and linearised model

In the following figure is simulated response of non-linear and linearised model in case that we perform +10% step in the input variable. We can judge that the non-linear model is more accurate but the deviation is minimal.

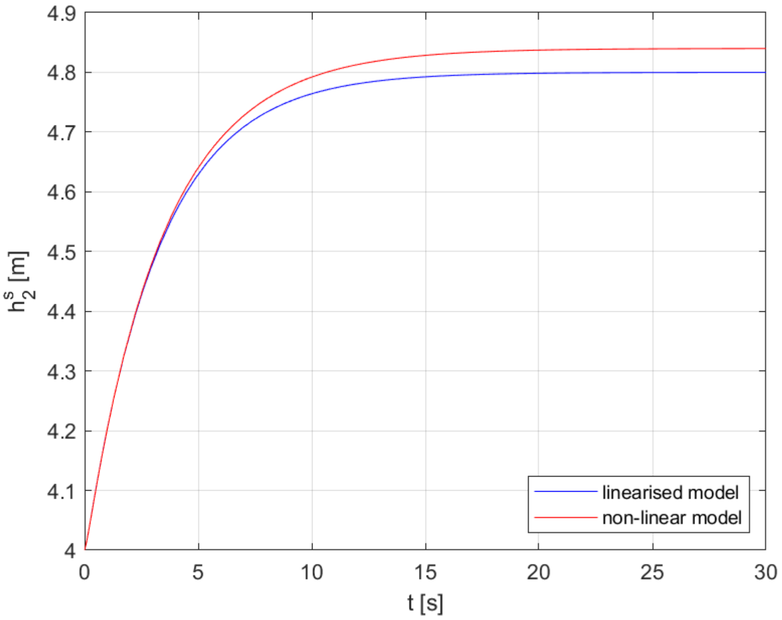

Figure 3: Response of non-linear and linearised model

2.7 Transfer function of the system

$$G(s) = C(sI - A)^{-1}B + D \quad [31]$$

$$G(s) = [\,0 \quad 1\,]\begin{bmatrix} s + \frac{k_1}{F_1} & 0 \\ -\frac{k_1}{F_1} & s + \frac{k_2}{F_2} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{F_1} \\ \frac{1}{F_2} \end{bmatrix} + 0 \quad [32]$$

$$\begin{bmatrix} s + \frac{k_1}{F_1} & 0 \\ -\frac{k_1}{F_1} & s + \frac{k_2}{F_2} \end{bmatrix}^{-1} = \frac{1}{s^2 + s\frac{k_2}{F_2} + s\frac{k_1}{F_1} + \frac{k_1 k_2}{F_1\,F_2}}\begin{bmatrix} s + \frac{k_2}{F_2} & 0 \\ \frac{k_1}{F_2} & s + \frac{k_1}{F_1} \end{bmatrix} \quad [33]$$

$$G(s) = [\,0 \quad 1\,]\,\frac{1}{(s + \frac{k_1}{F_1})(s + \frac{k_2}{F_2})}\begin{bmatrix} s + \frac{k_2}{F_2} & 0 \\ \frac{k_1}{F_2} & s + \frac{k_1}{F_1} \end{bmatrix}\begin{bmatrix} \frac{1}{F_1} \\ \frac{1}{F_2} \end{bmatrix} \quad [34]$$

$$G(s) = \frac{\left(\frac{k_1}{F_1 F_2} + \frac{s}{F_2} + \frac{k_1}{F_1 F_2}\right)}{s^2 + s\left(\frac{k_2}{F_2} + \frac{k_1}{F_1}\right) + \frac{k_1 k_2}{F_1\,F_2}} \quad [35]$$

Derived transfer function with numeric values:

$$G(s) = \frac{1{,}25s + 21{,}13}{s^2 + 8{,}762s + 2{,}641} \quad [36]$$

2.8 State-space representation and transfer function

In the following figure is simulated comparison of the I/O state-space model and derived transfer function. We can judge that the transfer function was derived correctly because the responses are equivalent.

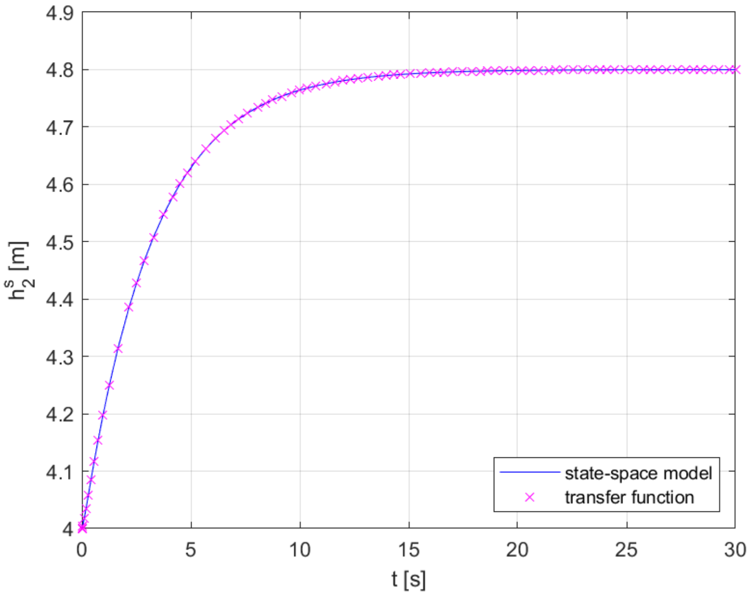

Figure 4: Comparison of the I/O state-space model and transfer function

3 Conclusion

The aim of this assignment was derive a dynamic mathematical model of system two tanks connected in series. At the first, we derived dynamic model from the mass balance of the system. This model was non-linear, therefore it was necessary to perform linearisation and get linearised model using steady-state model. After that we compared non-linear model with linearised model by simulation in Simulink/Matlab and convert a state-space representation of system to transfer function. Correctness of derived transfer function we verified using a Simulink scheme.