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HW 1.

Chapter 4

4.2. (a) The results are generated using code provided on Prof. A. Greenbaum's web page.

```
bisect(4,5,5*10^-7)
  initial interval:  a=4, b=5, fa=4.959815e+01, fb=-5
  a=4.500000e+00, b=5, f(a)=4.000857e+01, f(b)=-5, interval length = 5.000000e-01
  a=4.750000e+00, b=5, f(a)=2.389607e+01, f(b)=-5, interval length = 2.500000e-01
  a=4.875000e+00, b=5, f(a)=1.137177e+01, f(b)=-5, interval length = 1.250000e-01
  a=4.937500e+00, b=5, f(a)=3.713829e+00, f(b)=-5, interval length = 6.250000e-02
  a=4.937500e+00, b=4.968750e+00, f(a)=3.713829e+00, f(b)=-5.047823e-01, interval length = 3.125000e-02
  a=4.953125e+00, b=4.968750e+00, f(a)=1.638289e+00, f(b)=-5.047823e-01, interval length = 1.562500e-02
  a=4.960938e+00, b=4.968750e+00, f(a)=5.752945e-01, f(b)=-5.047823e-01, interval length = 7.812500e-03
  a=4.964844e+00, b=4.968750e+00, f(a)=3.740406e-02, f(b)=-5.047823e-01, interval length = 3.906250e-03
  a=4.964844e+00, b=4.966797e+00, f(a)=3.740406e-02, f(b)=-2.331505e-01, interval length = 1.953125e-03
  a=4.964844e+00, b=4.965820e+00, f(a)=3.740406e-02, f(b)=-9.773880e-02, interval length = 9.765625e-04
  a=4.964844e+00, b=4.965332e+00, f(a)=3.740406e-02, f(b)=-3.013378e-02, interval length = 4.882812e-04
  a=4.965088e+00, b=4.965332e+00, f(a)=3.643536e-03, f(b)=-3.013378e-02, interval length = 2.441406e-04
  a=4.965088e+00, b=4.965210e+00, f(a)=3.643536e-03, f(b)=-1.324302e-02, interval length = 1.220703e-04
  a=4.965088e+00, b=4.965149e+00, f(a)=3.643536e-03, f(b)=-4.799219e-03, interval length = 6.103516e-05
  a=4.965088e+00, b=4.965118e+00, f(a)=3.643536e-03, f(b)=-5.777108e-04, interval length = 3.051758e-05
  a=4.965103e+00, b=4.965118e+00, f(a)=1.532945e-03, f(b)=-5.777108e-04, interval length = 1.525879e-05
  a=4.965111e+00, b=4.965118e+00, f(a)=4.776254e-04, f(b)=-5.777108e-04, interval length = 7.629395e-06
  a=4.965111e+00, b=4.965115e+00, f(a)=4.776254e-04, f(b)=-5.004066e-05, interval length = 3.814697e-06
  a=4.965113e+00, b=4.965115e+00, f(a)=2.137929e-04, f(b)=-5.004066e-05, interval length = 1.907349e-06
  a=4.965114e+00, b=4.965115e+00, f(a)=8.187623e-05, f(b)=-5.004066e-05, interval length = 9.536743e-07
  approximate zero c = 4.96511412, f(c) = 1.591782e-05, number of iterations = 20
```

Bisection Method converges linearly with a factor of 2. Thus from accuracy 10^{-6} to accuracy 10^{-12}, the difference of accuracy is 10^6. In order to reach that, it requires $\log_2 10^6 \approx 19.9 = 20$ more steps.

(b) Modifying Prof. A. Greenbaum, generated the following result.

```
newton(5,10^-8)
  initial guess:  c=5, fc=-5
  c=4.96631026500457, fc=-1.656428e-01, err = 3.368973e-02
  c=4.96511568630146, fc=-2.012018e-04, err = 1.194579e-03
  c=4.96511423174643, fc=-2.978897e-10, err = 1.454555e-06
  approximate zero c = 4.9651142317, f(c) = -2.978897e-10, number of iterations = 3
```

According to the property of newton's method, $\lim_{k\to\infty} e_{k+1}/e_k^2 = \left|f''(x_*)/2f'(x_*)\right| = \left|\frac{e^{x_*}\cdot(x_*-3)}{e^{x_*}\cdot(x_*-4)}\right|\cdot\frac{1}{2} = \frac{1}{2}\left|\frac{x_*-3}{x_*-4}\right|$

where $x_* \approx 4.965$, thus $C_* = \frac{1}{2}\left|(x_*-3)/(x_*-4)\right| \approx 1$

Thus, we know $e_{k+1}/e_k^2 \approx 1$, which is consistent with the data above.

Taylor expansion of $f(x)$ about x_*: $f(x_k) = f(x_*) + (x_k - x_*)\cdot f'(x_*) + \frac{1}{2}(x_k - x_*)\cdot f''(\xi) = 0 + e_k \cdot f'(x_*)$

$\Leftrightarrow f(x_k) - f(x_*) = e_k \cdot f'(x_*)$ where $f'(x_*) = e^{x_*}\cdot(x_*-4) \approx -138$

thus $|err(f(x_k))| \approx 140\, e_k \approx 10^2 e_k$

Therefore, in order than $|f(x_k)|$ is controled within 10^{-16}, e_k should be $\le 10^{-18}$.

The third iteration reached $e_3 \approx 10^{-6}$, thus $e_4 \approx 10^{-12}$, $e_5 \approx 10^{-24}$.

$\therefore$ At the 5^{th} iteration, $|f(x_k)|$ will definitely be under 10^{-16}.

Therefore, need 2 more steps. In total 5 iterations are needed.

(c)

```
initial guesses: x0=4, x1=5, fx0=4.959815e+01, fx1=-5
  x0=5, x1=4.908422e+00, fx0=-5, fx1=7.402024e+00, fpc = -5.459815e+01, err = 3.488577e-02
  x0=4.908422e+00, x1=4.963079e+00, fx0=7.402024e+00, fx1=2.808942e-01, fpc = -1.354255e+02, err = -2.034895e-03
  x0=4.963079e+00, x1=4.965235e+00, fx0=2.808942e-01, fx1=-1.675050e-02, fpc = -1.302864e+02, err = 1.210804e-04
  x0=4.965235e+00, x1=4.965114e+00, fx0=-1.675050e-02, fx1=3.473149e-05, fpc = -1.380557e+02, err = -2.510864e-07
  x0=4.965114e+00, x1=4.965114e+00, fx0=3.473149e-05, fx1=4.280999e-09, fpc = -1.383419e+02, err = -3.094858e-11
  approximate zero c = 4.9651142317, f(c) = 4.280999e-09, number of iterations = 5
```

Proved in textbook, the order of convergent is $(1+\sqrt{5})/2$ while C_* is the same.

$\therefore e_{k+1} \approx e_k^{1.62}$ According to the data, $e_5 \approx 3\times10^{-11}$.

Thus $e_6 \approx 5\times10^{-17}$ and e_7 must be less than 10^{-18}

Therefore, two more steps are needed. In total 7 iterations are needed.

4.3. Compute $\frac{1}{R}$ by finding zero of $f(x) = x^{-1} - R$.
then $f'(x) = -x^{-2}$ then $f(x_k)/f'(x_k) = \frac{x^{-1}-R}{-x^{-2}} = -x + Rx^2$
Suppose initial point is close enough to x_*
Then $x_1 = x_0 - (-x_0 + R x_0^2) = 2x_0 - R x_0^2$
...
$$x_{k+1} = x_k - (-x_k + R x_k^2) = 2x_k - R x_k^2$$
① approximate $1/3 \Leftrightarrow x_* = \frac{1}{3} \therefore R = 3$
given $x_0 = 0.5$
$x_1 = 2\cdot 0.5 - 3\cdot 0.5^2 = 1 - 0.75 = 0.25$ err $= 8\times 10^{-2}$
$x_2 = 2\cdot 0.25 - 3\cdot 0.0625 = 0.3125$ err $= 2\times 10^{-2}$
$x_3 = 2\cdot 0.3125 - 3\cdot 0.09765625 = 0.3203125$ err $= 1.3\times 10^{-3}$
Therefore, x_k is converging to $1/3$ quadratically.
if given $x_0 = 1$.
$x_1 = 2\cdot 1 - 3\cdot 1^2 = -1$
$x_2 = 2\cdot(-1) - 3\cdot 1^2 = -5$
$x_3 = 2\cdot(-5) - 3\cdot 5^2 = -10 - 75 = -85$
In this case, x_k will diverge.

② Convergent interval.

还没做.

4.4. $\sqrt{2}$ is the root of $f(x) = x^2 - 2$.
thus $f'(x) = 2x$.
By Newton's Method formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$
$\therefore x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} = x_k - \frac{1}{2}x_k + \frac{1}{x_k}$

$\therefore x_{k+1} = \frac{1}{2}x_k + 1/x_k$ (*)
we know that $1.4^2 = 1.96$, approximately 2.
thus take $x_0 = 1.4$, apply (*) iteratively
$x_1 = 0.7 + 1/1.4 = 0.7 + 0.7142857 = 1.41428571$
$x_2 = \frac{1}{2}x_1 + 1/x_1 = 1.414213564$ err $\approx 10^{-7}$
$x_3 = \frac{1}{2}x_2 + 1/x_2 = 1.414213562$
Therefore, $x_3 = 1.41421356$ approximate $\sqrt{2}$ with accuracy greater than 6 decimal pts.

4.6. $h(x) = x^4/4 - 3x$ using newton's method to find minimum.

a). $h(x)$ reaches minimum or maximum at roots of $h'(x)=0$

$\therefore$ $h'(x) = x^3 - 3 = 0$ $\Rightarrow$ $x = \sqrt[3]{3}$ is a critical pt.

verify: $h''(x) = 3x^2 \geq 0$ thus concave $\therefore x = \sqrt[3]{3}$ is the minimum pt.

- for $f(x) = h'(x) = x^3 - 3$, $f'(x) = 3x^2$

$\therefore$ by newton's formula, $x_{k+1} = x_k - f(x_k)/f'(x_k) = x_k - \frac{x_k^3 - 3}{3x_k^2}$

$= x_k - \frac{1}{3}x_k + 1/x_k^2 = \frac{2}{3}x_k + \frac{1}{x_k^2}$

Thus newton's method gives recursive relation: $x_{k+1} = \frac{2}{3}x_k + 1/x_k^2$

b) let $x_0 = 1$, then $x_1 = \frac{2}{3} + 1 = 1.6\dot{6}$

by calculator, $\sqrt[3]{3} = 1.4422$, so the accuracy is 10^0.

c) step 0: $f(0) = -3$, $f(2) = 5$, $f(4) = 61$ pick $[0,2]$.

Step 1: $f(0) = -3$, $f(1) = -2$, $f(2) = 5$ pick $[1,2]$.

Step 2: $f(1) = -2$, $f(1.5) = 0.375$, $f(2) = 5$ pick $[1, 1.5]$

Thus the approximate root is $1+1.5/2 = 1.25$

4.9. for $f(x) = \sin x + 1$, it's function value is alway non-negative.

Thus cannot apply Rolle's Theorem to find the root.

Bisection Method requires $f(a) \cdot f(b) < 0$.

Exercise

2. Consider the initial interval to be $2\Delta_0$, denote root as C_*

mid-point of that interval as x_0.

Then initially, $|C_0 - C_*| < \Delta_0$

Since the bisection method restrict the previous interval to half of it,

So the new interval will be half the length of the previous interval.

i.e. $\Delta_{k+1} = \frac{1}{2}\Delta_k$ Denote $e_k = |C_k - C_*|$

$\therefore$ for that interval, $e_k \leq \Delta_k = \frac{1}{2}\Delta_{k-1} = \frac{1}{2^k}\Delta_0 = \frac{1}{2^k}e_0$

$\therefore$ $e_k = C_k - C_* = \frac{1}{2^k} \cdot \delta_0$ for some $\delta_0 \in [-\Delta, \Delta]$

$\therefore$ $C_k = C_* + \frac{1}{2^k}\delta_0 = C_* + O(\frac{1}{2^k})$.

Therefore, $\{C_n\}$ conv. to C_* with the rate of $O(\frac{1}{2^n})$.

3. ① $f(P_n) = (1-1-\frac{1}{n})^{10} = (\frac{1}{n})^{10}$

we have $(\frac{1}{2})^{10} > (\frac{1}{3})^{10} > (\frac{1}{4})^{10} > \cdots > 0$ s.t $(\frac{1}{2})^{10} \geq (\frac{1}{n})^{10}$ $\forall n \geq 1$

Besides, $(\frac{1}{2})^{10} = \frac{1}{1024} < 10^{-3}$

$\therefore$ $10^{-3} > (\frac{1}{2})^{10} \geq (\frac{1}{n})^{10}$ for all $n>1$.

i.e. $f(P_n) < 10^{-3}$ for all $n>1$.

② $|P-P_n| = \frac{1}{n}$, want $\frac{1}{n} < 10^{-3}$ we need $n>1000$

$\therefore$ $|P-P_n| < 10^{-3}$ when $n>1000$.

This phenomenon shows although the function value of a point being close to zero does not necessarily imply the point is close to the root. Similarly, the function value of a point being far from zero doesn't necessarily the point is not close to the root.
So the second criteria alone is not appropriate as a criteria to stop an iterative method.
Using 2nd criteria on $f(x) = (x-1)^{10}$ to find root x would cause the algorithm to stop too earlier, making the point not accuracy enough.
example: $f(x) = (x-1)^{1/100}$ where the root should be $x_* = 1$.
If we use the first criteria, even if two points are really close to 1, their function value can still differ a lot from 1.
Consider $P_{10^{99}}$ and $P_{10^{100}}$, then $|P_{10^{100}} - P_{10^{99}}| \approx 10^{-99}$ which is extremely close to 0 $\Rightarrow$ they are extremely close to the root $x_* = 1$.
However, $f(P_{10^{99}}) \approx 10^{-\frac{99}{100}} \approx 0.1$, $f(P_{10^{100}}) = 0.1$ are still not close to 0.

Consider criteria ③ $\frac{P_{10^{100}} - P_{10^{99}}}{P_{10^{100}}} \approx \frac{9 \times 10^{-100}}{1+10^{-100}} \approx 0.$

Yet the function value is still not close to 0 enough.
which means criteria ③ is still not a satisfactory criteria.

Thus, none of three criterias is appropriate to approximate the root of the functions mentioned above.

4. $f(x) = x \cdot (1+x^2)^{-\frac{1}{2}}$ $\therefore$ $f'(x) = (1+x^2)^{-\frac{1}{2}} + x \cdot [2x \cdot (1+x^2)^{-\frac{3}{2}} \cdot (-\frac{1}{2})]$

$= (1+x^2)^{-\frac{1}{2}} - x^2 \cdot (1+x^2)^{-3/2}$

$= (1+x^2)^{-\frac{3}{2}} \cdot [(1+x^2) - x^2] = (1+x^2)^{-\frac{3}{2}}$

$\therefore$ $f(x)/f'(x) = x \cdot (1+x^2)^{-\frac{1}{2}} \cdot (1+x^2)^{\frac{3}{2}} = x(1+x^2) = x + x^3$

By newton's method: $x_{k+1} = x_k - f(x)/f'(x) = x - x - x^3 = -x^3$

- Case $x_0 < -1$: the sequence obviously diverge.
- case $x_0 > 1$, similar as $x_0 < -1$, sequence diverge.
- case $-1 < x_0 < 1$:

  the sequence $\{x_n\} = \{x_0, -x_0^3, x_0^9, \cdots\}$

  where $x_k = x_0^{3^k} \cdot (-1)^k$

  Let $N \in \mathbb{N}$ be given. suppose $m, n > N$.

  $\because |x_0| < 1 \quad \therefore |x_0^{3^n}| \to 0$, i.e. for every $\frac{\varepsilon}{2}$, $\exists N$ s.t $\forall n > N$, $|x_0|^{3^n} \le \varepsilon/2$.

  Obviously, $|x_{n+1}| < |x_n| \quad \therefore |x_{N+1} - x_N| \le |x_{N+1}| + |x_N| < \varepsilon$

  $\because m, n > N \quad \therefore |x_m| < |x_N|$, $|x_n| < |x_N| \quad \therefore |x_m - x_n| \le |x_m| + |x_n| < \varepsilon$

  Therefore for all $\varepsilon > 0$, $\exists N > 0$ s.t for all $m, n > N$, $|x_m - x_n| < \varepsilon$.

  Thus, it's a cauchy sequence, and $\{x_n\}$ converges to 0.

- case $x_0 = 1$ then $x_1 = -1$, $x_2 = 1$, $\cdots$

  the sequence oscillate but does not converge.

  $\{x_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$

- Case $x_0 = -1$, similar as $x_0 = 1$. $\{x_n\} = \{-1, 1, -1, \cdots\}$

5. $f(x) = e^x - x - 2$

$\therefore f'(x) = e^x - 1 \Rightarrow f(x)/f'(x) = \dfrac{e^x - 1 - x - 1}{e^x - 1} = 1 - \dfrac{x+1}{e^x - 1}$

$\therefore$ Newton's Method: $x_{k+1} = x_k - 1 + \dfrac{x_k + 1}{e^{x_k} - 1}$

- Suppose $x_0 > 0$, then consider $g(x) = x - 1 + \dfrac{x+1}{e^x - 1} = \dfrac{xe^x + 1}{e^x - 1} - 1$

  for $x > 0$, $xe^x + 1 > 0$, $e^x - 1 > 0$

  compare $\Delta = xe^x + 1 - (e^x - 1) = e^x(x-1) + 2$

  at $x = 0$, $\Delta = 1 \cdot (-1) + 2 = 1 > 0$, $\frac{d}{dx}\Delta = e^x + (x-1)e^x = x \cdot e^x > 0 \quad \forall x > 0$

  thus Δ is increasing for $x > 0$ $\therefore \Delta > 0$ for all $x > 0$.

  $\therefore xe^x + 1 > e^x - 1 > 0$ for all $x > 0 \Rightarrow \dfrac{xe^x + 1}{e^x - 1} > 1$ for all $x > 0$

  Therefore, $g(x) > 0$ for all $x > 0$. i.e., $x_{k+1} > 0$ if $x_k > 0$.

  As a result, $\{x_n\}$ will converge to a positive value if $x_0 > 0$.

- Similar proof works for $x_0 < 0$.

- $x_0 = 100$, $x_1 = \frac{100e^{100}+1}{e^{100}-1} - 1 \approx 100-1 = 99.$
for x_k big, $x_{k+1} = \frac{x_k \cdot e^{x_k}+1}{e^{x_k}-1} - 1 \approx x_k - 1.$

Recall $f(x) = e^x - x - 2$, $f(2)>0$, $f(1)<0$.
we know the root is within $(1,2)$.
thus, In order to reach $x_k \approx 2$, it takes about 98 iterations.

Now we know the root locates within $(1,2)$, suppose the error at that step is approximately $1/2$, i.e. $e_{98} = \frac{1}{2}$
then $e_{99} \approx C_* \cdot e_{98}^2$, $e_{100} = C_* \cdot (C_* \cdot e_{98}^2)^2$
we know that $C_* = |f''(x_*)/2f'(x_*)|$. $f'(x) = e^x - 1$ $\therefore f''(x) = e^x$
$\therefore$ $C_* = |e^{x_*}/2(e^{x_*}-1)|$.
Assume $x_* \approx 1.25$ (since the value of C_* doesn't differ a lot)
$C_* \approx 0.7$.
Therefore $e_{100} \approx 0.7^3 \cdot e_{98}^4 = 0.7^3 \cdot (\frac{1}{2})^4 \approx 0.02.$
$e_{101} \approx 0.7\, e_{100}^2 \approx 3\times10^{-4}$, $e_{102} \approx 0.7\, e_{101}^2 \approx 10^{-7}$
$\therefore$ At iteration 102 the accuracy reaches about 10^{-7}, which is more accurate than 6 decimal digits.

- if $x_0 = -100$ $\quad \because x_{k+1} = \frac{x_k e^{x_k}+1}{e^{x_k}-1} - 1$
$x_1 = \frac{-100e^{-100}+1}{e^{-100}-1} - 1 \approx \frac{1}{-1} - 1 = -2.$