Homework 1

Due: Friday, Sept 5, 11:59 PM PT

1. Consider a *stable roommate problem* involving four students (A, B, C, D): Each student ranks the other three in a strict order of preference. A matching involves forming two pairs of the students, and it is considered stable if no two *separated* (not matched) students would prefer each other over their current roommates.
   Prove or disprove: A stable matching always exists in this scenario.

   Solution:
   False. A stable matching need not exist.
   Consider the following list of preferences. Note A, B, and C all prefer D the least.
   • A : B > C > D
   • B : C > A > D
   • C : A > B > D
   • D : A > B > C
   Now, there can only be 3 sets of disjoint roommate pairs.
   • If the students are divided as (A, B) and (C, D), then (B, C) causes instability, since C prefers B over D and B prefers C over A.
   • If the students are divided as (A, C) and (B, D), then (A, B) causes instability, since B prefers A over D and A prefers B over C.
   • If the students are divided as (A, D) and (B, C), then (A, C) cause an instability, since A prefers C over D and C prefers A over B.
   Thus, every matching is unstable, and no stable matching exists with this list of preferences.

   Rubrics:
   - 3 points for mentioning that a stable matching need not exist
   - 7 points for the correct explanation.

2. Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.
   Prove or disprove: In every stable matching for this instance, m is matched with w.

   Solution:
   True. Suppose S is a stable matching that contains the pairs (m, w0) and (m0, w), for some m0 ≠ m, w0 ≠ w. Clearly, the pairing (m, w) is preferred by both m and w over their current pairings, contradicting the stability of S.

   Rubric:
   - 3 points for correctly identifying the statement as True.

- 7 points for providing a correct explanation.

3. Consider the stable matching problem with n women and n men (where $n \geq 2$). Prove or disprove: It is not possible for any preference lists, that the Gale-Shapley algorithm with men proposing can match every woman to her least preferred man.

   Solution:
   False. If every man's first preference in women is unique and for that particular woman, the corresponding man is their least preferred man, it would serve as the required counter-example.

   For example:
   M1: W1>W2
   M2: W2>W1

   W1: M2>M1
   W2: M1>M2

   M1 proposes to W1, and she accepts.
   M2 proposes to W2, and she accepts.

   Rubric:
   - 3 points for stating it is True
   - 7 points for correct explanation:

4. There are six students in a class: Harry, Ron, Hermione, Luna, Neville, and Ginny. This class requires them to pair up and work on pair programming. Each has preferences over who they want to be paired with. The preferences are:

   Harry: Hermione > Neville > Ron > Ginny > Luna
   Ron: Ginny > Neville > Hermione > Harry > Luna
   Hermione: Ron > Neville > Ginny > Harry > Luna
   Luna : Harry > Ron > Ginny > Hermione > Neville
   Neville: Harry > Ron > Hermione > Ginny > Luna
   Ginny: Neville > Harry > Hermione > Ron > Luna

   A given matching (of students into pairs) is not stable if there are two potential partners who are not currently paired but prefer each other over their respective current partners. Prove or Disprove: No matching is stable for the preferences given above.

   Solution:
   Notice that Luna is the least preferred by everyone else. Now, someone must get matched with Luna, say A.

Among the remaining students, there must be a student, say B, who prefers A the most - this is because everyone other than Luna (i.e. Harry, Ron, Hermione, Neville, and Ginny) appear as the first preference for someone other than Luna. Then, suppose this B is paired with C (someone other than A, as A is paired with Luna). Now, A prefers B over their current partner, Luna, and B prefers A over C; thus, there will always be an instability.

Rubric:

- 5 points for mentioning that Harry is last on everybody's list.
- 5 points for mentioning that every other student has a unique, most preferred partner.

Ungraded Problems:

1. Prove or Disprove: It is possible to have an instance of the Stable Matching problem in which two women have the same best valid partner.

   Solution:
   False. When women propose in GS, they end up with their best valid partner. Thus, there needs to be a unique best valid partner for each woman, since, if two women had the same best valid partner, one wouldn't end up with that partner in the matching obtained by GS with women proposing, which is a contradiction.

   Rubrics:
   - 2 points for mentioning it is False.
   - 3 points for correct explanation.

2. Prove or Disprove: In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

   Solution:
   This is false. Consider the case with two men and two women where m prefers w to w'. m' prefers w' to w. w prefers m' to m. w' prefers m to m'. Note that there are no pairs at all where each ranks the other first, so clearly no such pair can show up in a stable matching.

   Rubrics:
   - 3 points for mentioning it is False.
   - 7 points for explanation.

3. Consider the stable matching problem with n women and n men (where $n \geq 2$). Suppose the preference lists are such that for any woman w, her most preferred man m, does not have w as his first preference
   Prove or disprove: It is possible that the Gale-Shapley algorithm with men proposing can match every woman to her most preferred man.

   Solution:
   True. Consider this example:
   W1: M1>M3>M2
   W2: M2>M1>M3
   W3: M3>M1>M2

   M1: W2>W1>W3

M2: W1>W2>W3
M3: W1>W3>W2

M1 proposes to W2, she accepts
M2 proposes to W1, she accepts
M3 proposes to W1, she accepts, now M2 is free
M2 proposes to W2, she accepts, now M1 is free
M1 proposes to W1, she accepts, now M3 is free
M3 proposes to W3, she accepts.
As we can see, all women end up with their most preferred partner, and men don't.

Rubrics:

- 3 points for stating it is True.
- 7 points for showing the example.

4. Suppose the Gale-Shapley algorithm with men proposing is run once. Then, a particular woman w falsely reports her preferences by swapping two men's positions in her actual preference list. With this alteration, the Gale-Shapley algorithm with men proposing is run once again.
Prove or disprove: It is possible for the woman w to get a more preferred partner (as per her actual preference list), by doing the swapping alteration in her preferences.

Solution:
True. Assume we have three men, m1, m2, and m3, and three women, w1, w2, and w3 with preferences as given in the table below.
Column w3 shows the true preferences of woman w3, while in column w′ she pretends she prefers man m3 to m1.

m_1	m_2	m_3	w_1	w_2	w_3	(w_3')
w_3	w_1	w_3	m_1	m_1	m_2	m_2
w_1	w_3	w_1	m_2	m_2	m_1	m_3
w_2	w_2	w_2	m_3	m_3	m_3	m_1

First let us consider one possible execution of the G-S algorithm with the true preference list of w3.

m_1	w_3			w_3
m_2		w_1		w_1
m_3			$[w_3][w_1]w_2$	w_2

First, m1 proposes to w3, then m2 proposes to w1. Then m3 proposes to w2 and w1 and gets rejected, finally proposes to w2 and is accepted. This forms pairs (m1, w3), (m2, w1) and (m3, w2), thus pairing w3 with m1, who is her second choice.

Now consider the execution of the G-S algorithm when w3 pretends she prefers m3 to m1 (see column w′).
Then the execution might look as follows:

m_1	w_3		$-$	w_1			w_1
m_2		w_1		$-$	w_3		w_3
m_3			w_3		$-$	$[w_1]w_2$	w_2

Man m1 proposes to w3, m2 to w1. Then m3 to w3, she accepts the proposal, leaving m1 alone. Then m1 proposes to w1, which causes w1 to leave her current partner m2, who consequently proposes to w3 (and that is exactly what w3 prefers). Finally, the algorithm pairs up m3 (recently left by w3) and w2. As we see, w3 ends up with the man m2, who is her true favorite. Thus, we conclude that by falsely switching the order of her preferences, a woman may be able to get a more desirable partner in the G-S algorithm.

Rubrics:

- 3 points for stating it is True.
- 7 points for showing the correct example.