Project2 ---Quantum Fourier Transform

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Quantum Fourier Transform

Quantum Fourier Transform is a quantum implementation of the discrete Fourier transform. As we all know that Fourier Analysis is a tool to describe the internal frequencies of a function. Quantum Fourier Transform is the quantum analogue of the discrete Fourier transfrom which performs a linear transformation on quantum bits.

Quantum Fourier Transform (QFT) is exponentially faster than the famous Fast Fourier Transform (FFT) of classical computers. In a quantum computer, the duscrete fourier transform (DFT) can be implemented as a quantum circuit consisting of only $O(n^2)$ Hadamard gates and controlled phase gates. However, the classical DFT takes $O(n2^n)$ gates.

QFT Operation

Similar to DFT, the QFT acts on a quantum state $|X >= \sum_{i=0}^{N-1} x_i |i>$ to its Fourier transform $|Y >= \sum_{k=0}^{N-1} y_k |k>$. And y_k can be expressed as:

$$y_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n w_N^{nk}, \quad k = 0, 1, 2, \dots N-1$$

where $w_N^{nk} = e^{i2\pi \frac{nk}{N}}$

The inverse QFT acts similarly but with:

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k w_N^{-nk}, \quad n = 0, 1, 2, \dots N-1$$

The process of QFT can also be expressed as a mapping process:

$$QFT : |X >= \sum_{i=0}^{N-1} x_i |i> --> |Y >= \sum_{k=0}^{N-1} y_k |k>$$
$$QFT : |X >= \sum_{i=0}^{N-1} x_i |i> --> |Y >= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{i=0}^{N-1} x_i w_N^{ik} |k>$$
$$QFT : |i> --> \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} w_N^{ik} |k>$$

The mapping process of QFT can also be intuitively viewed as a transformation between two bases. As we know H-gate can transform between the basis state $|0>$ and $|1>$ to the basis $|+>$ and $|->$. Similarly, the basis states in the QFT should be able to transformed by using those qubit gate operation. These process can be symbolically expressed as:

$$QFT|x>=|\hat{x}>$$

According to the definition of $y_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n w_N^{nk}, \quad k = 0, 1, 2, \ldots N-1$, the QFT operation can be written in a matrix format :

$$QFT_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & w^3 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & w^6 & \ldots & 2^{2(N-1)} \\ 1 & w^3 & w^6 & w^9 & \ldots & 2^{3(N-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & w^{N-1} & w^{2(N-1)} & w^{3(N-1)} & \ldots & 2^{(N-1)(N-1)} \end{pmatrix}$$

If we re-write do some simplication with the martrix form of QFT_N and re-write it to a form more close to qubits (a tensor product of qubit), QFT_N can be re-written as:

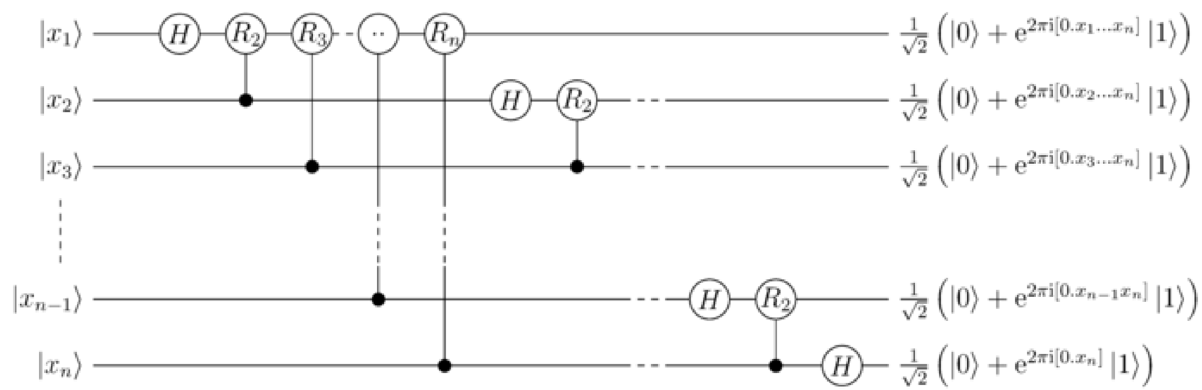
$$QFT|x_1x_2x_3\ldots x_n>= \frac{1}{\sqrt{N}}(|0> +e^{2\pi i[0,x_n]}|1>) \otimes (|0> +e^{2\pi i[0,x_{n-1}x_n]}|1>) \otimes \ldots \otimes (|0> +e^{2\pi i[0,x_1,x_2,\ldots x_n]}|1>)$$

where $[0, x_1, \ldots, x_m] = \sum_{k=1}^{m} x_k 2^{-k}$

The expression above implies the QFT can be implemented by the Hadamard gate (H) and phase gate R_n, where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and

$R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/N} \end{pmatrix}$.

The figure below shows a circuit that can implement an N point QFT. It is important to note that the order of the qubits is reversed in the output state.



Example: 1-qubit QFT $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

In this case $x_0 = \alpha$, $x_1 = \beta$, and then

$$y_0 = \frac{1}{\sqrt{2}}\left(\alpha\exp\left(2\pi i\frac{0\times 0}{2}\right) + \beta\exp\left(2\pi i\frac{1\times 0}{2}\right)\right) = \frac{1}{\sqrt{2}}(\alpha+\beta)$$
$$y_1 = \frac{1}{\sqrt{2}}\left(\alpha\exp\left(2\pi i\frac{0\times 1}{2}\right) + \beta\exp\left(2\pi i\frac{1\times 1}{2}\right)\right) = \frac{1}{\sqrt{2}}(\alpha-\beta)$$

This can also be written as a matrice form:

$$|y_1 y_2 >= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}|x_1 x_2 >= H|x_1 x_2 >$$

This matches with the expression derived earlier for N=2.

Example: 4-qubit QFT $|y_4 y_3 y_2 y_1\rangle = QFT_8|x_4 x_3 x_2 x_1\rangle$

According to the general expression derived earlier:

$$QFT|x_1x_2x_3x_4 >= \frac{1}{\sqrt{N}}(|0> +e^{2\pi i[0,x_4]}|1>)\otimes(|0> +e^{2\pi i[0,x_3x_4]}|1>)\otimes(|0> +e^{2\pi i[0,x_2,x_3,x_4]}|1>)\otimes(|0> +e^{2\pi i[0,x_1,x_2,x_3,x_4]}|1>)$$

And this can also be written as an expanded form:

$$QFT|x_1x_2x_3x_4 >= \frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2}x_4\right)|1\rangle\right]\otimes\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^2}x_4 + \frac{2\pi i}{2}x_3\right)|1\rangle\right]$$
$$\otimes\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^3}x_4 + \frac{2\pi i}{2^2}x_3 + \frac{2\pi i}{2}x_2\right)|1\rangle\right]\otimes\frac{1}{\sqrt{2}}\left[|0\rangle + \exp\left(\frac{2\pi i}{2^4}x_4 + \frac{2\pi i}{2^3}x_3 + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)|1\rangle\right]$$

Note that the order of the qubits needs to be reversed in the output state.

Quantum Circuit Implementation in Qiskit

In [18]:
```
import numpy as np
%matplotlib notebook
import matplotlib.pyplot as plt
from qiskit import QuantumCircuit, ClassicalRegister, QuantumRegister, Aer, transpile
from qiskit import BasicAer # for simulating circuits
from qiskit.tools.visualization import plot_histogram, plot_bloch_multivector
from qiskit import execute
#from qiskit_textbook.widgets import scalable_circuit
```

Define a general qft rotation:

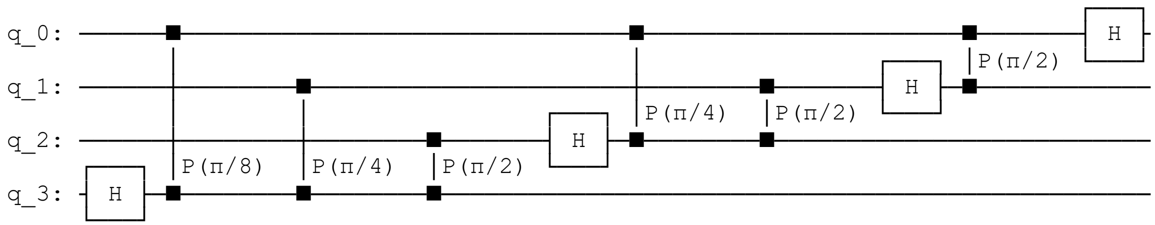
In [19]:
```
def qft_rotations(circuit, n):
    #Performs qft on the first n qubits in circuit (without swaps)
    if n == 0:
        return circuit
    n -= 1
    circuit.h(n)
    for qubit in range(n):
        circuit.cp(np.pi/2**(n-qubit), qubit, n)
    # the next qubits (we reduced n by one earlier in the function)
    qft_rotations(circuit, n)
```

In [20]:
```
n=4
q = QuantumRegister(n, 'q') # specify the number of qubits in the register and a name
circ = QuantumCircuit(q)
circ.draw()
```

Out[20]:
```
q_0:

q_1:

q_2:

q_3:
```

In [21]:
```
qft_rotations(circ , 4)
circ.draw()
```

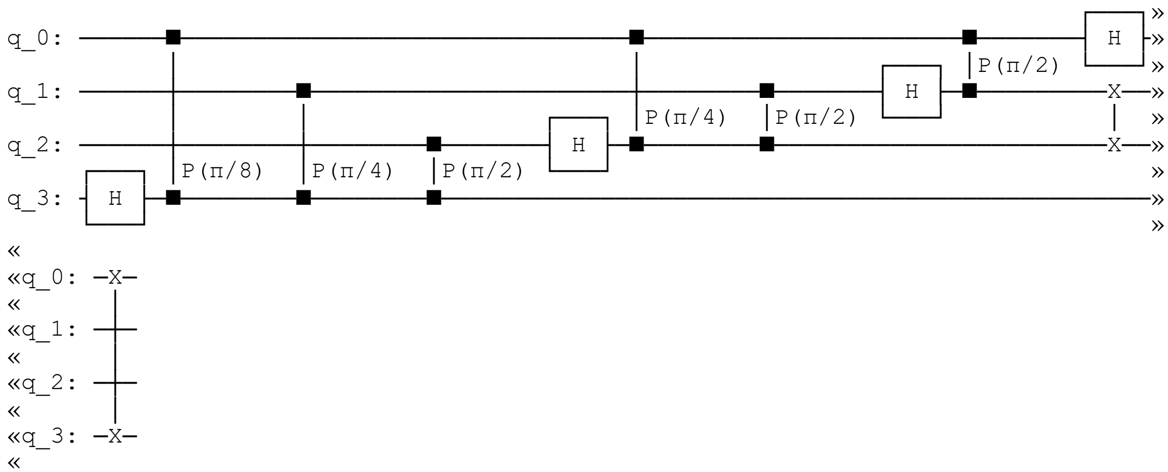
Out[21]:



In [22]:
```
def swap_registers(circuit, n):
    for qubit in range(n//2):
        circuit.swap(qubit, n-qubit-1)
    return circuit

def qft(circuit, n):
    qft_rotations(circuit, n)
    swap_registers(circuit, n)
    return circuit
```

In [23]:
```
# swap the output
for qubit in range(n//2):
    circ.swap(qubit, n-qubit-1)


#q = QuantumRegister(n, 'q') # specify the number of qubits in the register and a name
#circ = QuantumCircuit(q)
#qft(circ,n)
circ.draw()
```

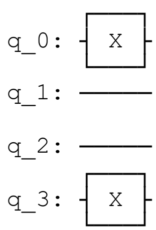
Out[23]:



Test with an input sequence 1001:

In [24]:
```
n=4
q = QuantumRegister(n, 'q') # specify the number of qubits in the register and a name
circ = QuantumCircuit(q)
circ.x(0)
circ.x(3)
circ.draw()
```

Out[24]:



In [25]:
```
qft(circ,n)
circ.draw()
```

Out[25]:
```
     ┌───┐                                                                       »
q_0: ┤ X ├──────■────────────────────────────────■──────────────────────■────────»
     └───┘      │                                │               ┌───┐  │P(π/2)  »
q_1: ───────────┼───────■────────────────────────┼───────■───────┤ H ├──■────────»
                │       │               ┌───┐    │P(π/4) │P(π/2) └───┘           »
q_2: ───────────┼───────┼───────■───────┤ H ├────■───────■───────────────────────»
     ┌───┐┌───┐ │P(π/8) │P(π/4) │P(π/2) └───┘                                    »
q_3: ┤ X ├┤ H ├─■───────■───────■────────────────────────────────────────────────»
     └───┘└───┘                                                                  »
«     ┌───┐   
«q_0: ┤ H ├─X─
«     └───┘ │ 
«q_1: ──X───┼─
«       │   │ 
«q_2: ──X───┼─
«           │ 
«q_3: ──────X─
«             
```

In [26]:
```
circ.h(q[0])
circ.draw()
```

Out[26]:
```
     ┌───┐                                                                       »
q_0: ┤ X ├──────■────────────────────────────────■──────────────────────■────────»
     └───┘      │                                │               ┌───┐  │P(π/2)  »
q_1: ───────────┼───────■────────────────────────┼───────■───────┤ H ├──■────────»
                │       │               ┌───┐    │P(π/4) │P(π/2) └───┘           »
q_2: ───────────┼───────┼───────■───────┤ H ├────■───────■───────────────────────»
     ┌───┐┌───┐ │P(π/8) │P(π/4) │P(π/2) └───┘                                    »
q_3: ┤ X ├┤ H ├─■───────■───────■────────────────────────────────────────────────»
     └───┘└───┘                                                                  »
«     ┌───┐   ┌───┐
«q_0: ┤ H ├─X─┤ H ├
«     └───┘ │ └───┘
«q_1: ──X───┼──────
«       │   │      
«q_2: ──X───┼──────
«           │      
«q_3: ──────X──────
«                  
```

In [27]:
```
backend = BasicAer.get_backend('statevector_simulator')
job = execute(circ, backend)
job.status()
```

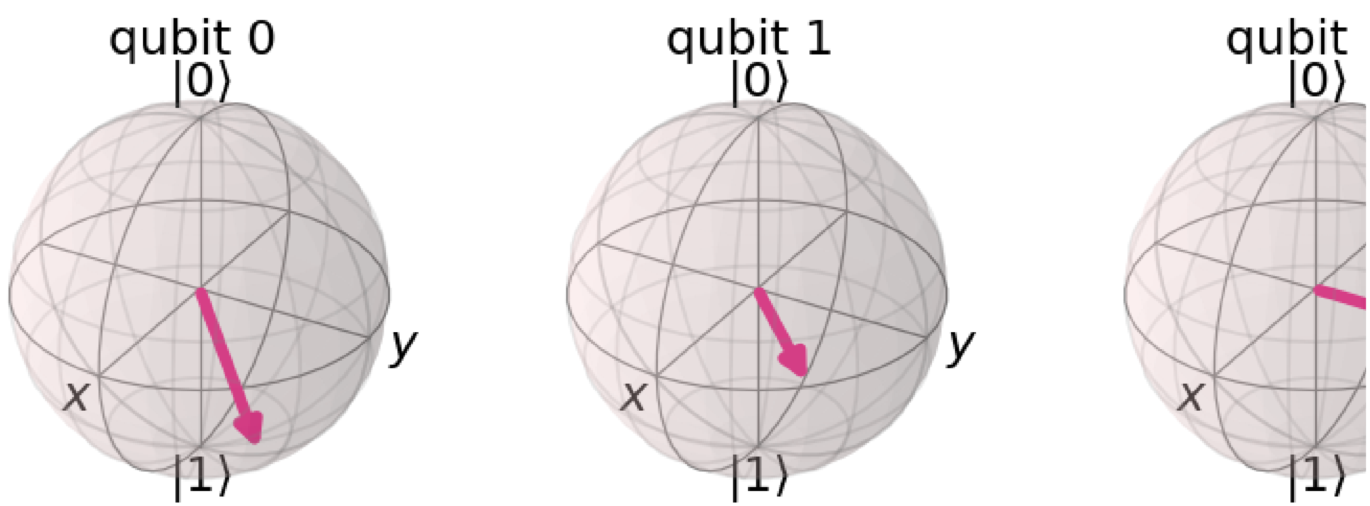
Out[27]: <JobStatus.DONE: 'job has successfully run'>

In [28]:
```
result = job.result()
outputstate = result.get_statevector(circ, decimals=8)
print(outputstate)
probs = np.abs(outputstate)**2
print(probs)
```

```
[ 0.01345632-0.06764951j  0.34009707+0.06764951j  0.05735049-0.03832037j
  0.19264951+0.28832037j  0.06764951+0.01345632j -0.06764951+0.34009707j
  0.03832037+0.05735049j -0.28832037+0.19264951j -0.01345632+0.06764951j
 -0.34009707-0.06764951j -0.05735049+0.03832037j -0.19264951-0.28832037j
 -0.06764951-0.01345632j  0.06764951-0.34009707j -0.03832037-0.05735049j
  0.28832037-0.19264951j]
[0.00475753 0.12024247 0.00475753 0.12024247 0.00475753 0.12024247
 0.00475753 0.12024247 0.00475753 0.12024247 0.00475753 0.12024247
 0.00475753 0.12024247 0.00475753 0.12024247]
```

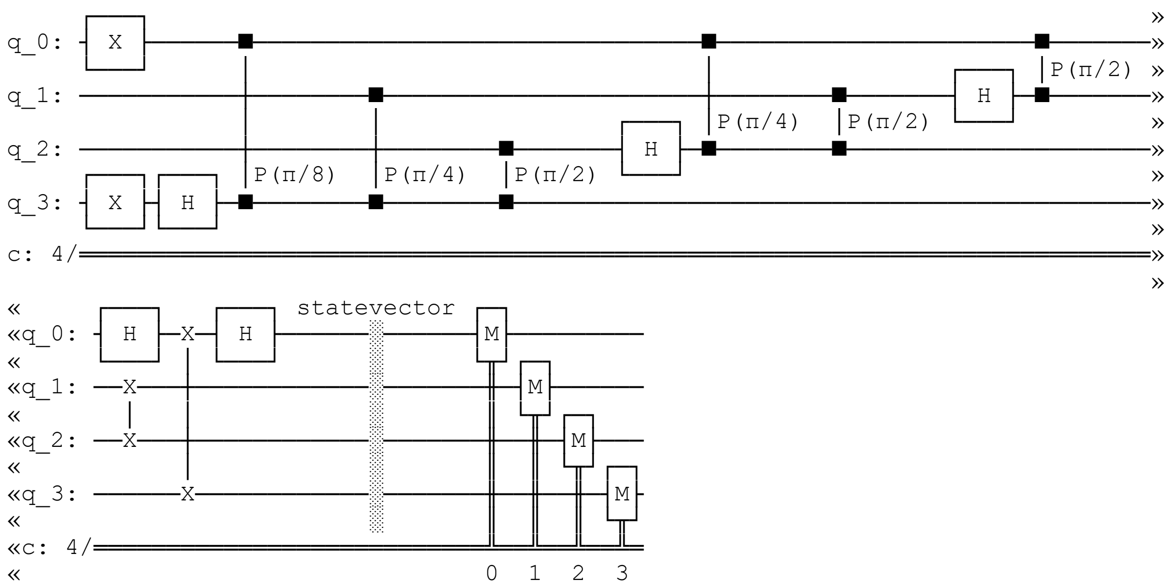
In [29]:
```
sim = Aer.get_backend("aer_simulator")
circ.save_statevector()
statevector = sim.run(circ).result().get_statevector()
plot_bloch_multivector(statevector)
```

Figure 1



In [30]:
```python
c = ClassicalRegister(n, 'c')
circ.add_register(c)
circ.measure(q[0], c[0])
circ.measure(q[1], c[1])
circ.measure(q[2], c[2])
circ.measure(q[3], c[3])

circ.draw()
```

Out[30]:



In [31]:
```python
#shots = 2048
#transpiled_circ = transpile(circ, backend, optimization_level=3)
#job = backend.run(transpiled_circ, shots=shots)

#result = job.result()

#counts = result.get_counts()
#print(counts)
#plot_histogram(counts)
```

Have some problems measuring the output state by using the `qasm_simulator` provided in the homework example. Need to fix this in the future.

In [ ]:
```python

```

In [ ]: