Coma ÜB5

1) a) $f = g + h$

$K_{abs} f \le K_{abs}\, g + K_{abs}\, h$

$|f(x) - f(x_0)| = |g(x) + h(x) - g(x_0) - h(x_0)|$

$\le |g(x) - g(x_0)| + |h(x) - h(x_0)|$

$\le K_{abs}\, g\, |x - x_0| + K_{abs}\, h\, |x - x_0| + \dots$

$\le (K_{abs}\, g + K_{abs}\, h)\, |x - x_0| + \dots$

b) $f(x) = x^5 + |x|^3$

$K_{abs} f \le |g'| + |h'| = 5x^4 + 3x^2|$

$$K_{rel} f \le \frac{|x|\,(5x^4 + 3x^2)}{|x^5 + |x|^3|} = \frac{5x^2 + 3}{x^2 + 1}$$

c) $f(x) = \sin^2 x + \cos^2 x = 1$

$K_{abs} f = |f'| = 0$

$$K_{rel} f = \frac{|x|}{|f(x)|} K_{abs} = 0$$

$K_{abs} f \le |2 \cos x \sin x| + |-2 \sin x \cos x| = 4|\sin x \cos x|$

$x = \frac{\pi}{4} \rightsquigarrow K_{abs} f \le 2$

2) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ $\qquad K_{abs} = |f'(x)| = e^x \qquad K_{rel} = \frac{|x|}{|f(x)|} \cdot K_{abs} = |x|$

a) $x_1 = 0{,}5 \quad K_{abs} \approx 1{,}65 \quad K_{rel} = 0{,}5$

b) $x_2 = -2 \quad K_{abs} \approx 0{,}14 \quad K_{rel} = 2$

c) $x_3 = 2 \quad K_{abs} \approx 7{,}39 \quad K_{rel} = 2$

d) $x_4 = -0{,}5 \quad K_{abs} \approx 0{,}61 \quad K_{rel} = 0{,}5$

A3

a) $x_{k+1} + a x_k + b x_{k-1} = 0$

$$f_k(x_0) = \alpha(x_0)\lambda_1^{k+1} + \beta(x_0)\lambda_2^{k+1}$$

$$\alpha(x_0) = \frac{\lambda_2 x_{-1} - x_0}{\lambda_2 - \lambda_1} \qquad \beta(x_0) = \frac{x_0 - \lambda_1 x_{-1}}{\lambda_2 - \lambda_1}$$

λ_1, λ_2 Nullstellen von $p(\lambda) = \lambda^2 + a\lambda + b \quad |\lambda_2| > |\lambda_1|$

$$\kappa_{abs} f_k(x_0) = |f_k'(x_0)| = |\alpha'(x_0)\lambda_1^{k+1} + \beta'(x_0)\lambda_2^{k+1}|$$

$$\boxed{\alpha'(x_0) = \frac{-1}{\lambda_2 - \lambda_1} \quad \beta'(x_0) = \frac{1}{\lambda_2 - \lambda_1}}$$

$$= \left|\frac{\lambda_2^{k+1} - \lambda_1^{k+1}}{\lambda_2 - \lambda_1}\right| \leq C$$

$$|\lambda_2^{k+1} - \lambda_1^{k+1}| \leq C|\lambda_2 - \lambda_1| \iff |\lambda_2| \leq 1 \quad (|\lambda_1| < 1)$$

$$\lambda_{1/2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$\leadsto \quad a^2 \geq 4b$

$$|-a \pm \sqrt{a^2-4b}| \leq |a| + |\sqrt{a^2-4b}|$$

$$|a| + \sqrt{a^2-4b} \leq 2$$

$$a^2 - 4b \leq (2-|a|)^2$$

$$\leadsto \quad b \geq 4|a| - 4$$

Stabilität ⇝ Algorithmus

Algorithmus: Zerlegung der Funktion f in Elementarfunktion und Grundrechenarten $\bigcirc$

$$f(x_0) = g^1_{n_1} \circ g^1_{n_1-1} \cdots g^1_1(x_0) \bigcirc g^2_{n_2} \circ g^2_{n_2-1} \cdots g^2_1(x_0) \underset{\text{Grundrechenarten}}{\bigcirc} \cdots \bigcirc g^m_{n_m} \circ \cdots g^m_{n_m-1} g^m_1(x_0)$$

Approximation von f: $\tilde{f}(\varepsilon, x_0)$

z.B. mit $\tilde{g}_i(x) = rd(g_i(x)) = g_i(x)(1+\varepsilon)$

Auswertungsfehler: $f(x_0) - \tilde{f}(\varepsilon, x_0)$

relative Stabilität

$$\frac{|f(x_0) - \tilde{f}(\varepsilon, x_0)|}{|f(x_0)|} \le \sigma_{rel} \|\varepsilon\| + \mathcal{O}(\|\varepsilon\|)$$

Relativen Gesamtfehler: Eingabefehler $\cdot \kappa_{rel}$ + Auswertungsfehler $\cdot \sigma_{rel}$

$$\frac{|f(x_0) - \tilde{f}(\varepsilon, \tilde{x}_0)|}{|f(x_0)|} \le \frac{|f(x_0) - f(\tilde{x}_0)|}{|f(x_0)|} + \frac{|f(\tilde{x}_0) - f(\varepsilon, \tilde{x}_0)|}{|f(\tilde{x}_0)|} \frac{|f(\tilde{x}_0)|}{|f(x_0)|}$$

$$\le \kappa_{rel} \frac{|x_0 - \tilde{x}_0|}{|x_0|} + \sigma_{rel}(\tilde{x}_0) \|\varepsilon\|$$

$$\boxed{\frac{|f(\tilde{x}_0) - f(x_0)|}{|f(x_0)|} + 1}$$

$f(x) = g(x) \bigcirc h(x) \leadsto \sigma_{rel}\, f \le 1 + \kappa_0 \max\{\sigma_g, \sigma_h\}$

$f(x) = g(h(x)) \leadsto \sigma_{rel}\, f \le 1 + \kappa_g \sigma_h$

$\sigma_{rel\, f} = 1 + \kappa_n (1 + \kappa_{n-1} (1 + \dots \kappa_3 (1 + \kappa_2) \dots))$

⇝ Schlecht konditionierte Elementarfunktionen vermeiden

Rekursive Stabilitätsberechnung

Bsp: $f(x_1, x_2) = (x_1 - x_2)^2 = x_1^2 - 2x_1 x_2 + x_2^2$

1.) $f(x_1, x_2) = f_2(f_1(x_1, x_2))$ $\qquad f_1 = x - y,\ f_2 = x^2$

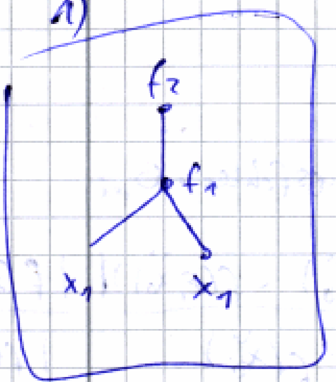
2) $f(x_1, x_2) = f_5(f_1(f_2(x_1), f_4(f_3(x_1, x_2))), f_2(x_2))$

$f_3(x, y) = x \cdot y$

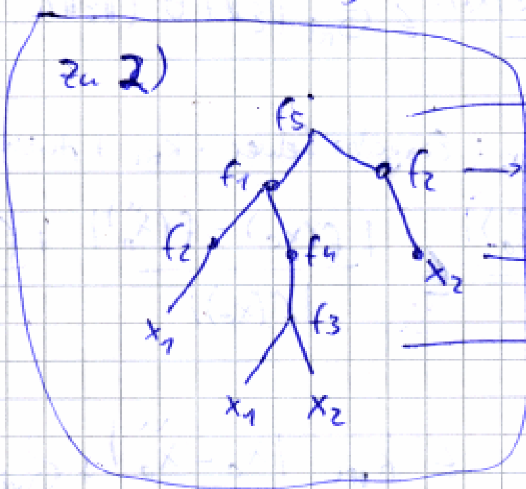
$f_4(x) = 2x$

$f_5(x, y) = x + y$

1)



$\sigma_{rel} \leq 1 + \kappa_{f_2} = 3$



$1 + 2\kappa_{sub}$

$\sigma_{rel} \leq 1 + \kappa_{f_5} \cdot \max\{f_1, f_2\}$ $\qquad (1 + 2\kappa_{sub})$

$\sigma^{\beta_{3,1}} \leq 1 + \kappa_{f_1} \max\{\sigma^{\beta_{2,1}}, \sigma^{\beta_{2,2}}\} = 1 + 2\kappa_{sub}$

$\sigma^{\beta_{2,1}} = 1$ $\qquad \sigma^{\beta_{2,2}} \leq 1 + \kappa_{f_4} = 1 + 1 = 2$

$\sigma_{rel}^{\beta_1} = 1$

$\sigma_{rel} \leq 1$