Visualizing Harmonic Analysis Transforms

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Abstract

The Laplace transform is the more general harmonic transform. The z- and the Fourier-transform are specializations of the former. A visualization of the connections between the different continuous and discrete transforms is presented in this short note. The visualization can be of educational value.

Enter Laplace

The Laplace transform of a real function f is defined as

$$\mathcal{L}(s) = \int_0^\infty e^{-st} f(t) dt \qquad\qquad s \in \mathbb{C}$$

The two-sided Laplace transform is defined as

$$\mathcal{L}(s) = \int_{-\infty}^\infty e^{-st} f(t) dt \qquad\qquad s \in \mathbb{C}$$

In each case we have a product of the function e^{-st} with the real function f.

The Fourier transform is just a specialization of the Laplace transform, when the complex number s is of the form $s = i\omega$, that is purely imaginary. Thus the Fourier transform F of f is given by

$$F(\omega/2\pi) = \int_{-\infty}^\infty e^{-i\omega t} f(t) dt$$

The function $e^{-i\omega t}$, for $-\infty < \omega < \infty$ and $-\infty < t < \infty$, can be represented by a matrix E with continuous indices of the form:

$$E = \begin{array}{cc} \begin{pmatrix} & \vdots & \\ \cdots & e^{-i\omega t} & \cdots \\ & \vdots & \end{pmatrix} & \updownarrow \omega \\ -\infty \leftarrow \; t \; \rightarrow \infty & \end{array}$$

where the entries in the matrix are all values of $e^{-i\omega t}$ arranged like points in a plane. The infinite dimensional vector f can contain all values of $f(t)$ for $-\infty < t < \infty$ and we then write

$$\int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \begin{pmatrix} & \vdots & \\ \cdots & e^{-i\omega t} & \cdots \\ & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ f(t) \\ \vdots \end{pmatrix} = Ef$$

The continuous multiplication operation is an integration. This notation is just a way of visualizing the Fourier transform as a linear operator acting on a function (as is done in quantum mechanics with the Dirac notation).

Visualizing $\mathcal{L}$ and $\mathcal{F}$

We can generalize the visualization shown above to three variables, and represent the Laplace operator by a three-dimensional arrangement of the values $e^{-(a+i\omega)t}$, as shown in Fig.1.

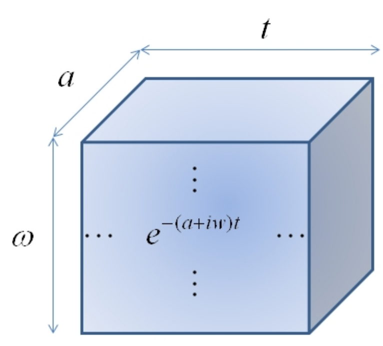

Figure 1: The volume of values $e^{-(a+i\omega)t}$

The cube in Fig.1 contains at each position the value of $e^{-(a+i\omega)t}$, for a, ω, and t running from $-\infty$ to ∞.

The Laplace transform of f is the product of the cube with f. We can think of this product as the weighted sum of all vertical cuts through the cube, as shown in Fig.2. The product with the function f "collapses" the cube onto a square. We can think of a product of an operator with a function as dimensionality reduction in the direction of the product.

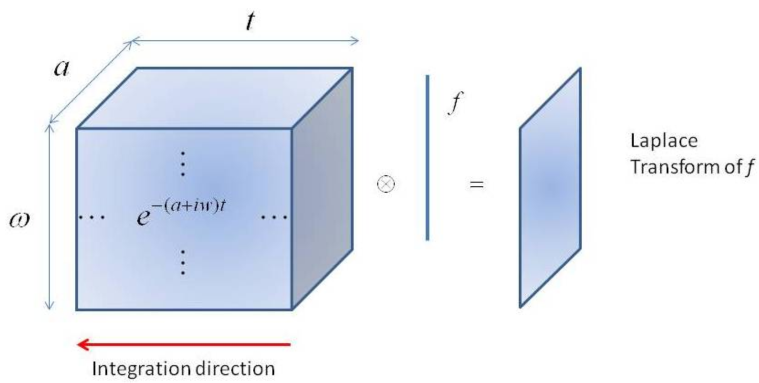

Figure 2: The Laplace transform as dimensionality reduction

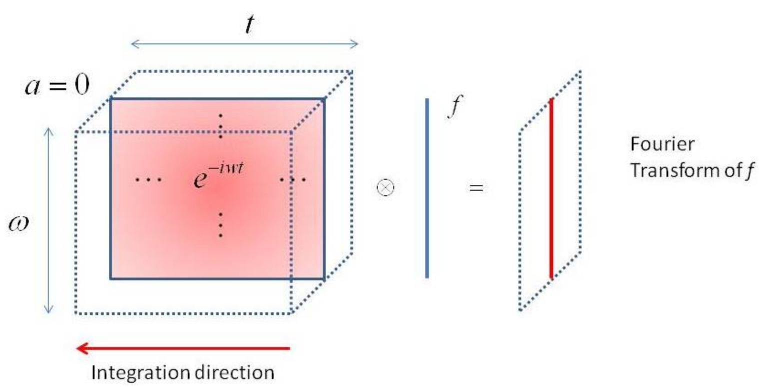

Figure 3: Specialization of the Laplace transform for $a = 0$

Fig.3 shows the specialization of the Laplace transform for the case $a = 0$, that is, for the Fourier transform. This represents the product of the vector with a cut of the cube at $a = 0$. In the case of the Fourier transform we have to scale the argument ω by 2π, but that is just a detail. The Laplace transform $\mathcal{L}(s)$ is thus a function of a complex argument, while the Fourier transform is a function of a real argument ω (Fig.4).

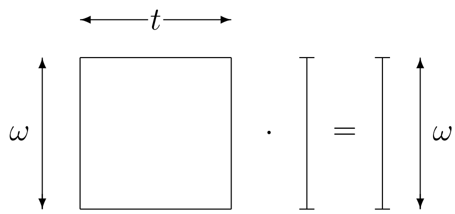

Figure 4: Fourier transform of a vector

The z-Transform

The bilateral z-transform is a specialization of the Laplace transform for discrete values of t, i.e. $t =, \ldots, -2, -1, -0, 1, 2, \ldots$. Therefore

$$F(z) = \sum_{k=\infty}^{\infty} z^{-k} f(k)$$

In this case, the function being transformed is discrete. Since any complex number z can be written as $z = e^{a+i\omega}$, we have the specialization shown in Fig. **??** where the vector of boxes represents a discrete vector.

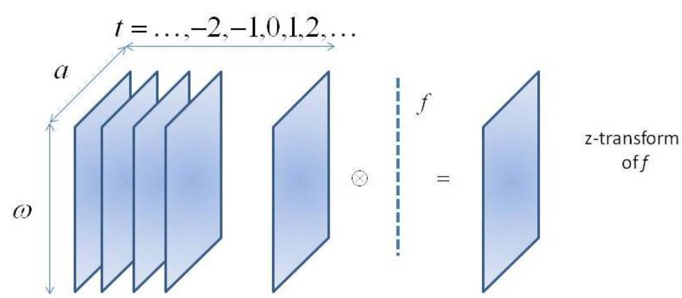

Figure 5: The z-transform of a vector

Fourier series

In the case of continuous F, but a set of discrete frequencies $\omega_1, \omega_2, \ldots \omega_N$, we can compute a Fourier series as

$$F(\omega_k/2\pi) = \int e^{-i\omega_k t} f(t)dt, \qquad \text{for } k = 1, \ldots, N$$

This is the specialization shown in Fig.6.

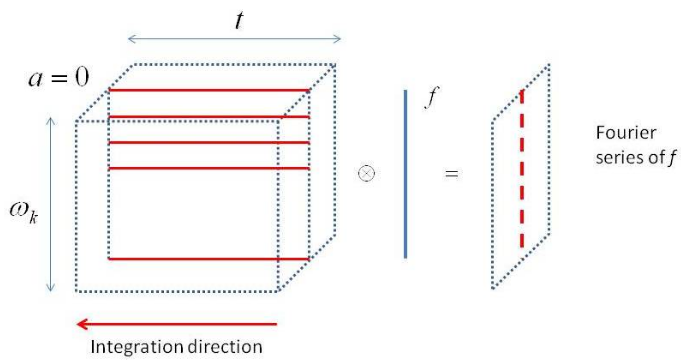

Figure 6: The Fourier series of a vector

We can also handle the case of a discrete function f, by specializing the t's, as shown in Fig.7. This is called the discrete frequency Fourier transform, the dual of the Fourier series.This is just a special case of the z-transform.

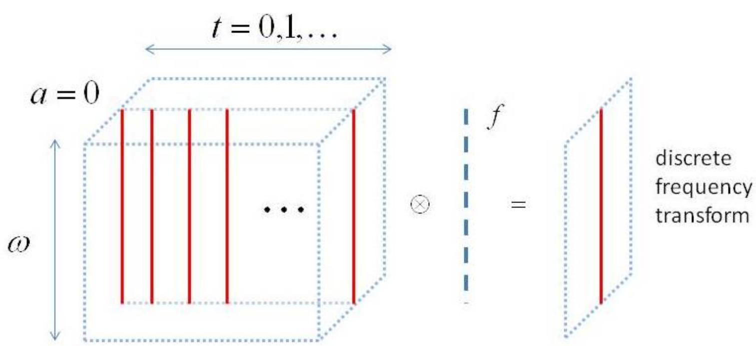

Figure 7: Discrete frequency Fourier transform of a discrete function

Specializing the Fourier transform for discrete frequencies and discrete values of t, leaves us with a conventional matrix-vector multiplication for the computation of the discrete Fourier transform.

The complete visualization

Finally, we are left with the following complete visualization of the transformations and their connections.

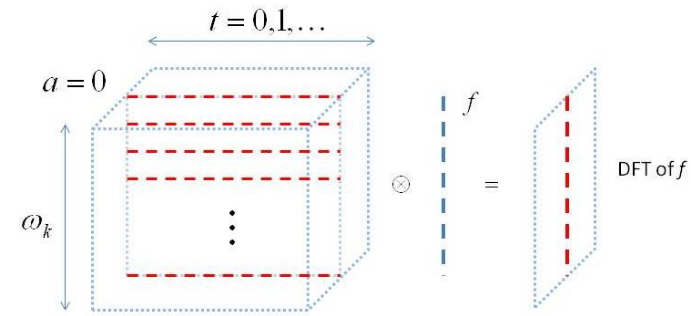

Figure 8: Visualization of the discrete Fourier transform

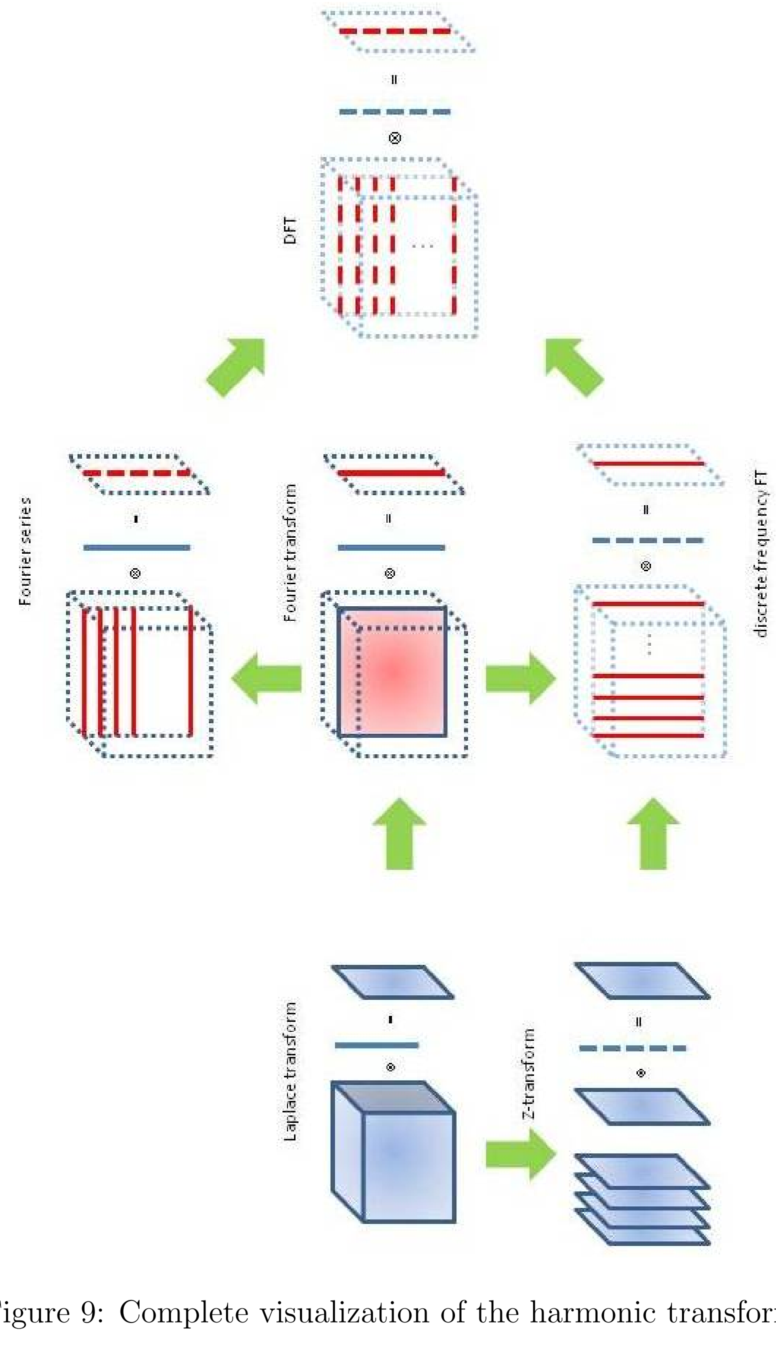

Figure 9: Complete visualization of the harmonic transforms