Freie Universität Berlin

Introduction to Robotics

Coordinates, Transformations, Representations

WS 2015 / 16

Prof. Dr. Daniel Göhring
Intelligent Systems and Robotics
Department of Computer Science
Freie Universität Berlin

Freie Universität Berlin

Agenda

Spatial Descriptions

- Joint Types
- Transformations
- Representations

Preliminaries

- Kinematics
  - the study of classical mechanics which describes the motion of points, bodies and systems of bodies without consideration of the causes of motion
- Rigid Body
  - a body is rigid, when the mutual distance of every pair of specified points in it is invariable *(Whittaker, 1904)*
- Dynamics
  - the study of the forces which produced that motion *(J.S. Beggs)*

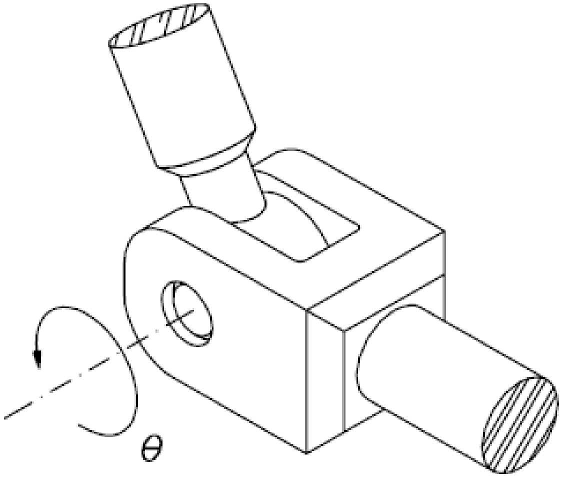
Kinematic Chain

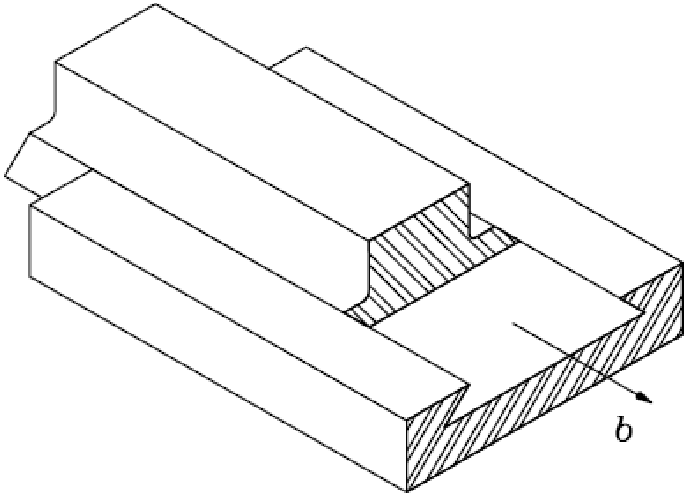
- Assembly of rigid bodies connected by joints
- Rigid bodies within a kinematic chain are also referred to as *links*
- Joints between two links are also referred to as *kinematic pairs*

Possible Joint Types

Mainly:

Revolute joint, Prismatic joint





(Angeles)

Other types include spherical, cylindrical joints.

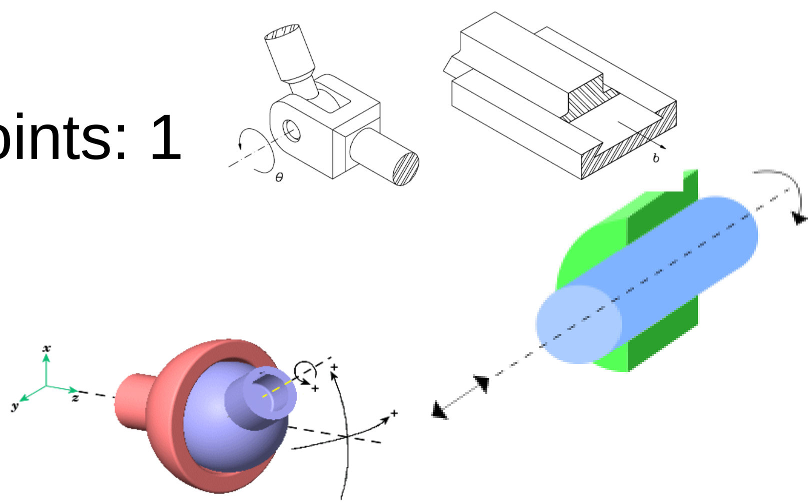
Freie Universität Berlin

Joints and Degrees of Freedom

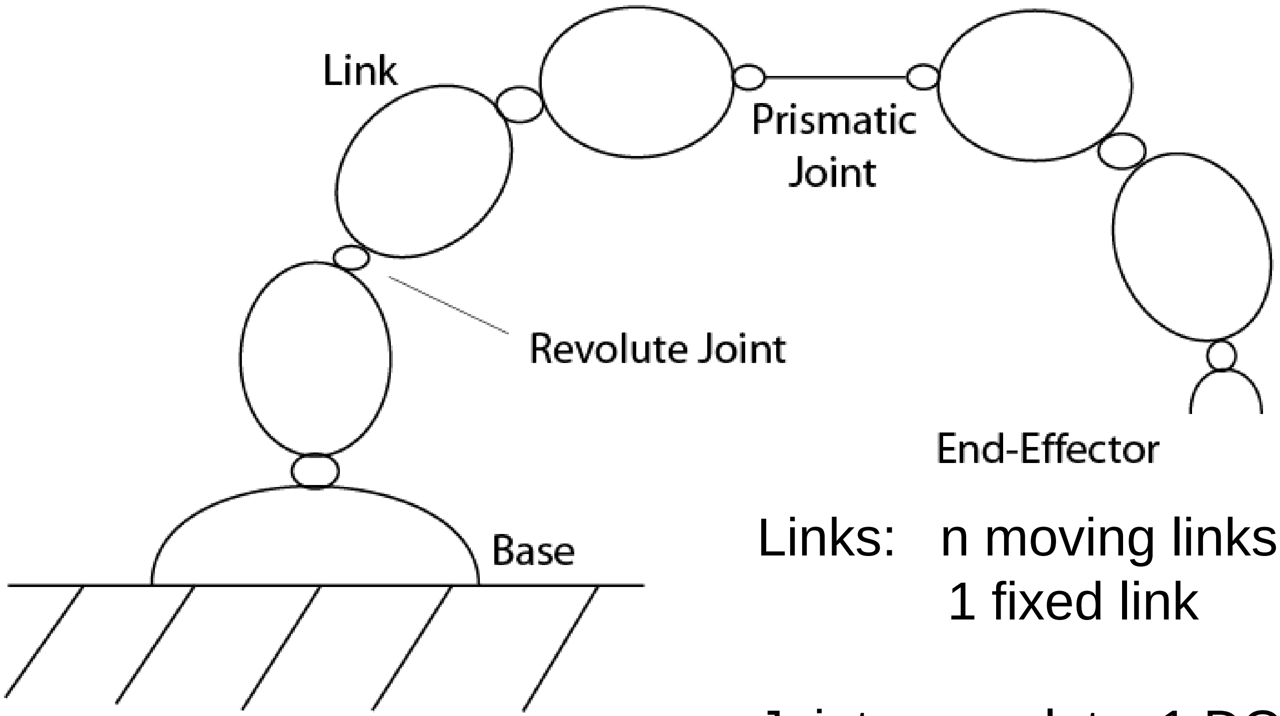
- Every joint allows a certain type of motion and forbids others

- The number of independent parameters that define the configuration of a mechanical system

- For revolute and prismatic joints: 1
- cylindrical joints: 2
- spherical joints: 3



Kinematic Chain, Example



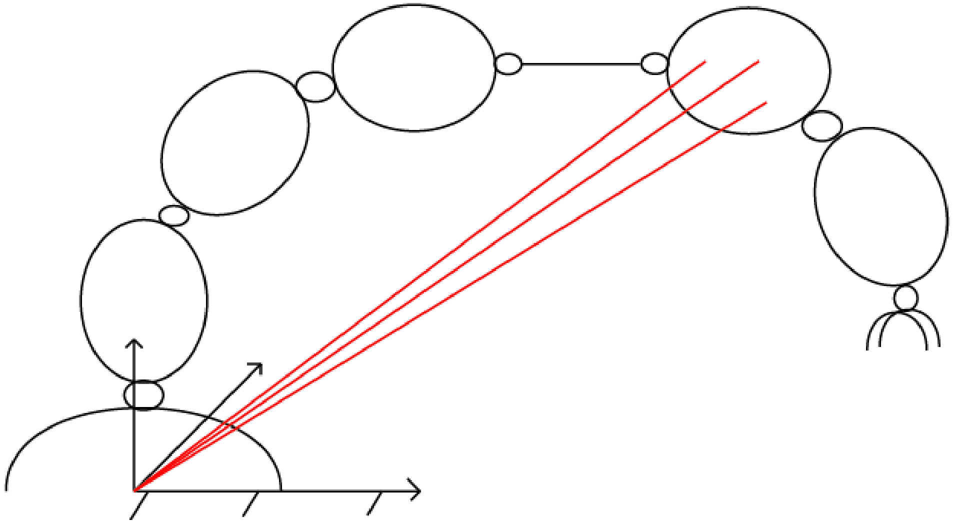
Links: n moving links
1 fixed link

Joints: revolute, 1 DOF
prismatic, 1 DOF

Configuration Parameters

- a set of parameters that describes the full configuration of a (mechanical) system

- the position of a link in 3d space can be defined by three 3-dim. vectors



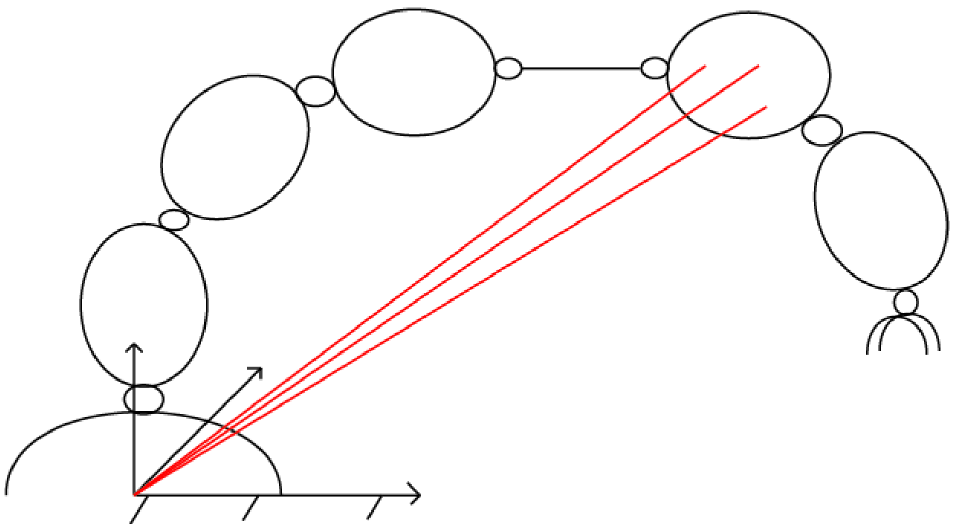
Number of Degrees of Freedom

- Body in 3 dimensional space: 3 for position, 3 for rotation

- Body in 2 dimensional space: 2 for position, 1 for rotation

Configuration Parameters

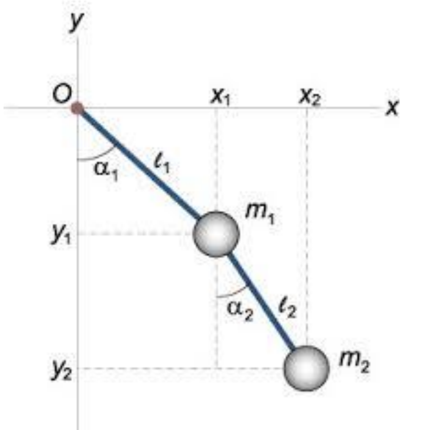
- a set of parameters that describes the full configuration of a system
- the position of a link can be defined by three vectors



9 parameters per link

Configuration Parameters

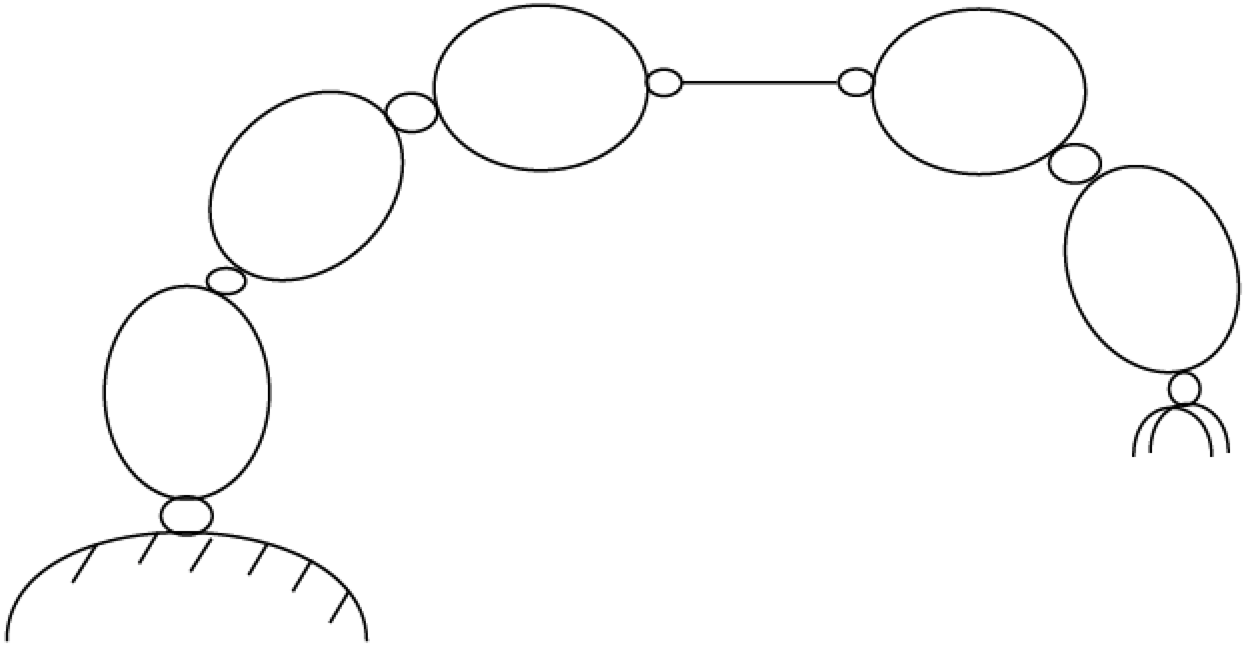
- The parameters that define the configuration of a mechanical system are called *Generalized Coordinates*
  - not necessarily cartesian
  - independent (advantageous, but not always poss.)
  - complete



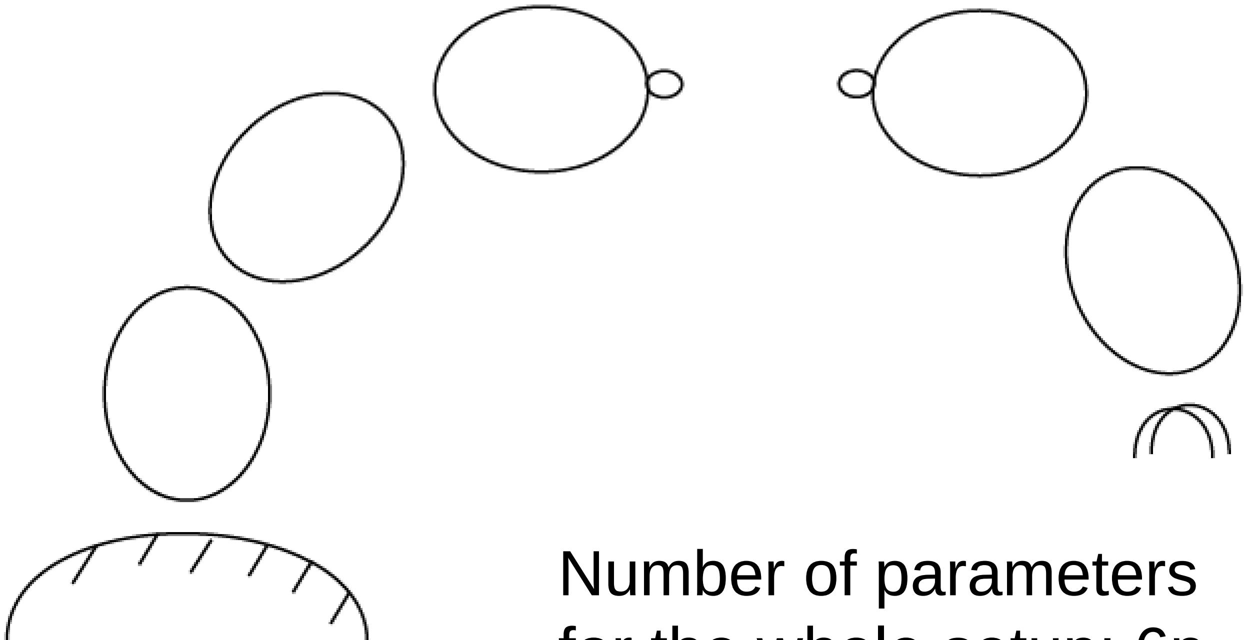
- The vector space defined by these coordinates is called *configuration space*

- Degrees of Freedom of a system
  - number of generalized coordinates

Generalized Coordinates



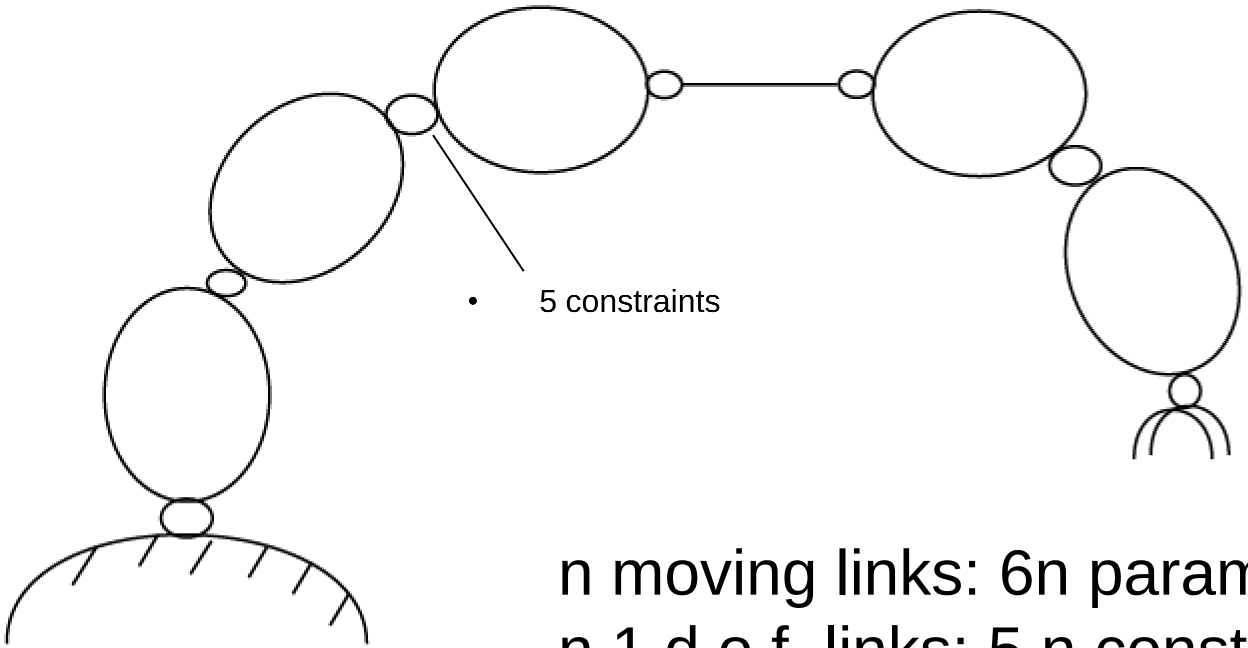
Generalized Coordinates



6 parameters:
3 for position
3 for orientation

Number of parameters
for the whole setup: 6n

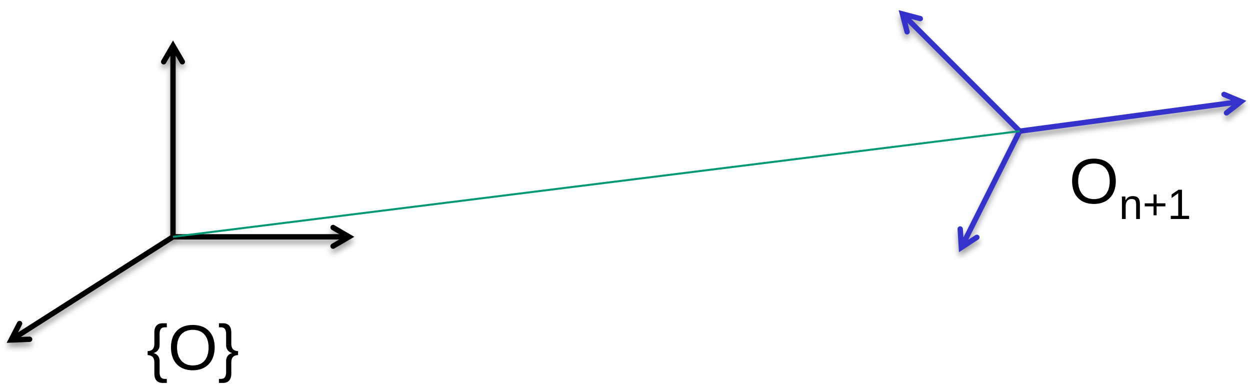
Generalized Coordinates



n moving links: 6n parameters
n 1 d.o.f. links: 5 n constraints
d.o.f. (system): 6n – 5n = n

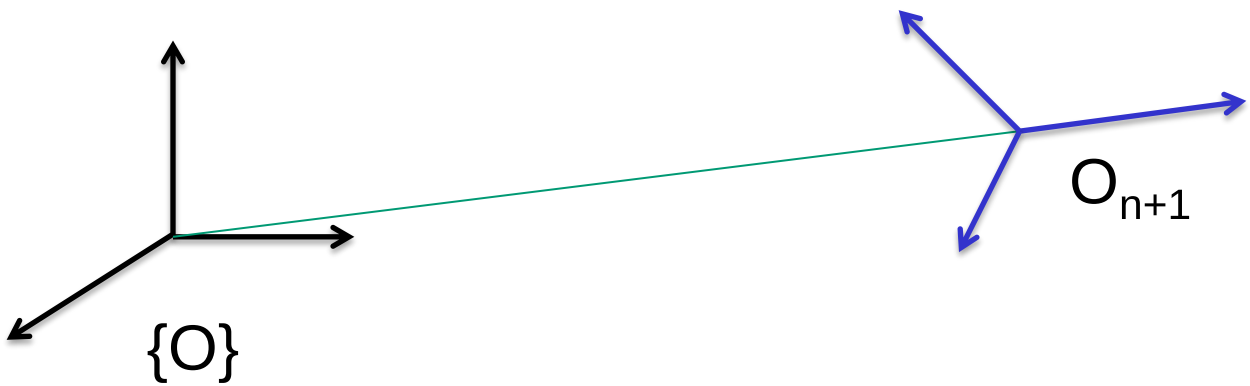
X

Spatial Descriptions for Endeffector



A set of parameters $(x_1, x_2, \ldots, x_m)$ that completely specifies the position and orientation of the endeffector with respect to {O}

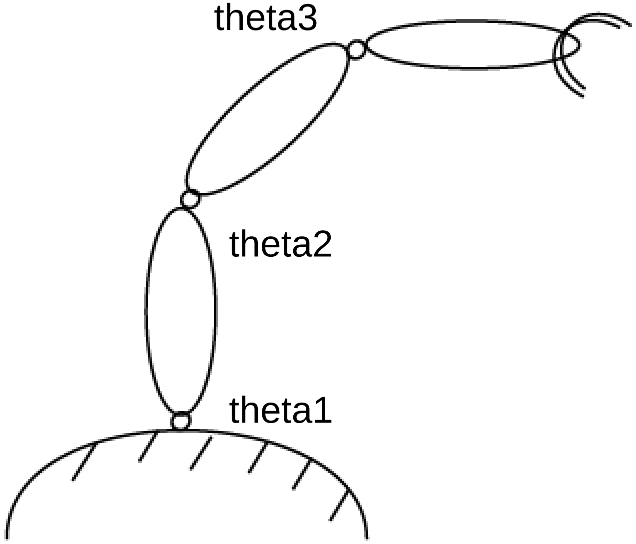
Endeffector Configuration Params

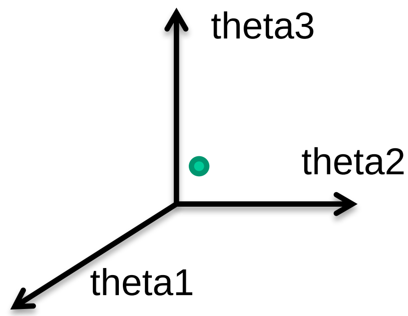


O_{n+1}: Operational point

- a set of independent m_0 $(x_1, x_2, \dots, x_{m0})$ configuration parameters
- m_0: number of d.o.f. of the end-effector

Joint Coordinates and Joint Space

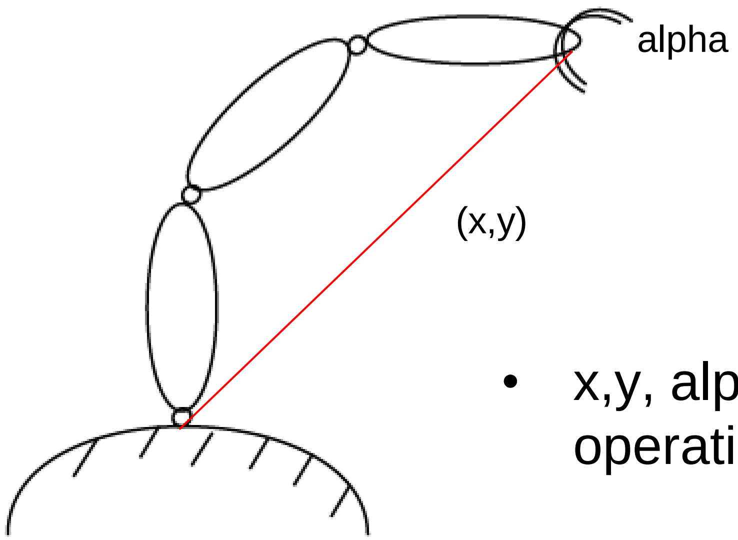


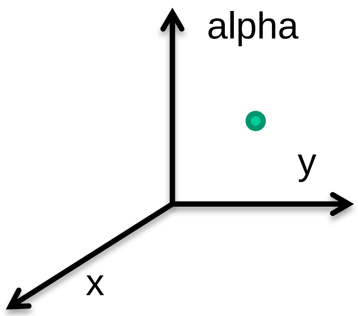


- important for motion planning
- obstacles can be mapped to this joint space

Freie Universität Berlin

Operational Coordinates



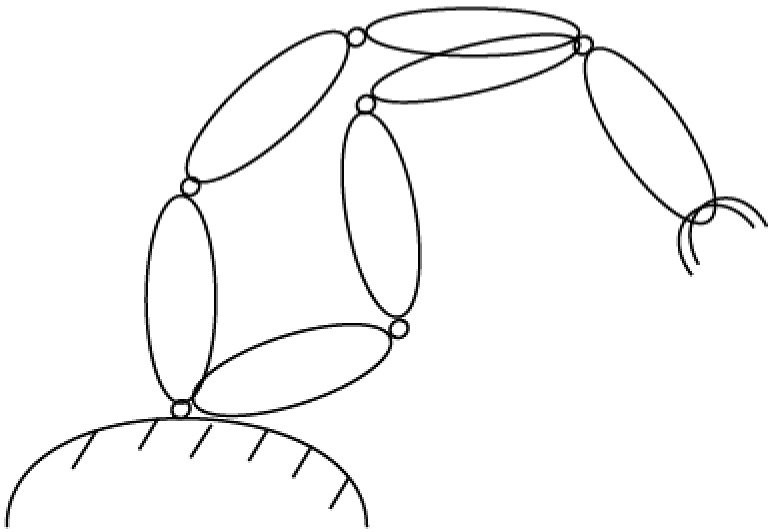


- x,y, alpha are operational coordinates in operational space

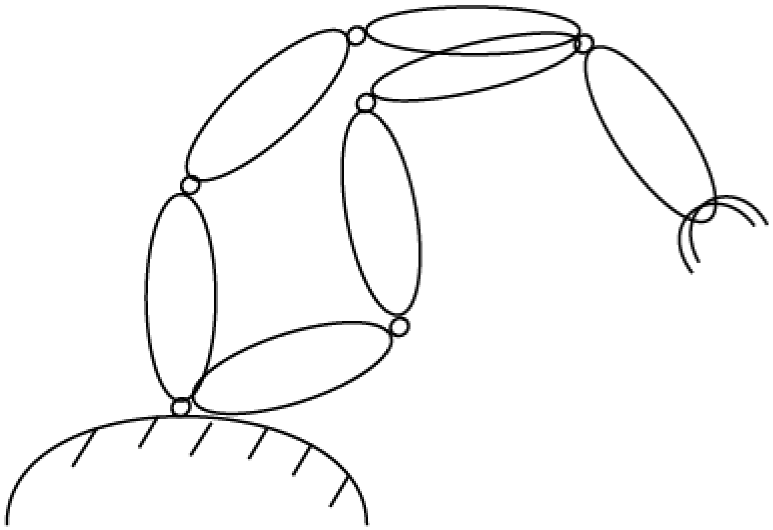
- robot is reduced to a point theta
- end effector is reduced to point (x,y,alpha)

Redundancy

- imagine introducing one more joint
- in **2d**: 4 joints, 3 degrees of freedom



Redundancy

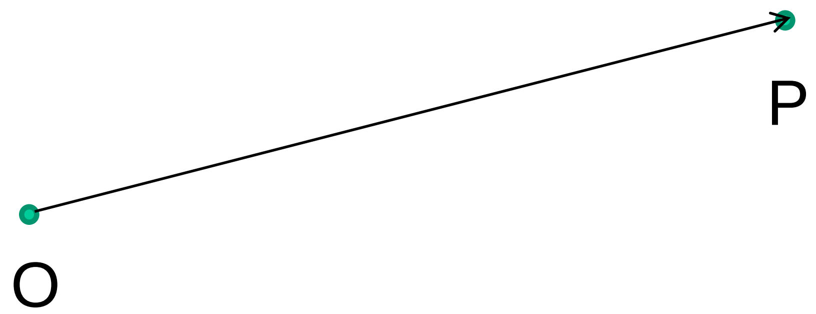


- a robot is said to be redundant if the number of d.o.f. of the robot *n* is greater than the number of d.o.f. of the end-effector m_0 : $n > m_0$. Here $\mathrm{m}_0 = 3$, $\mathrm{n} = 4$
- Degree of redundancy is measured by: $n - m_0$

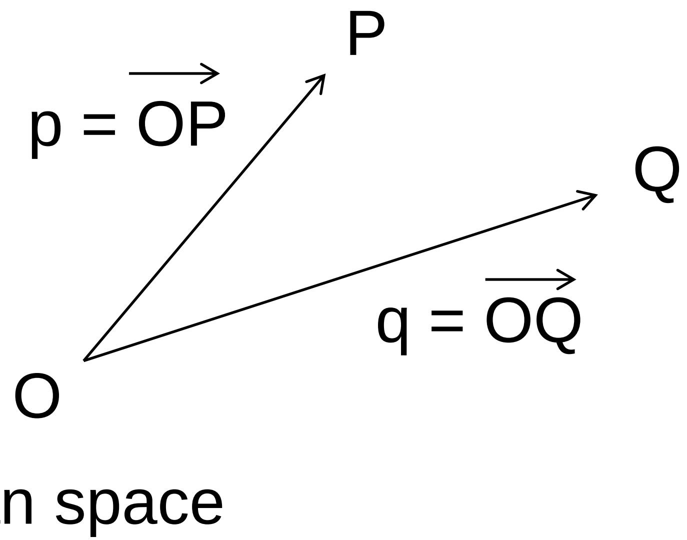
Position of a point

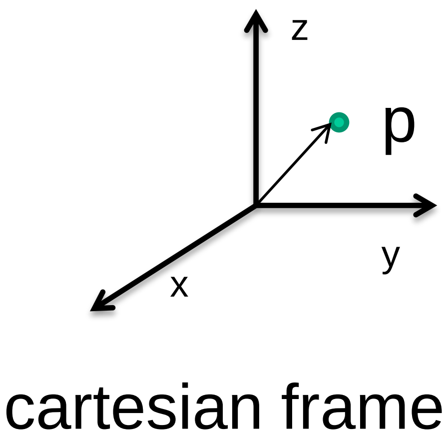
A point P is usually defined in relation to a fixed origin O.

P can be denoted as vector OP or just as p

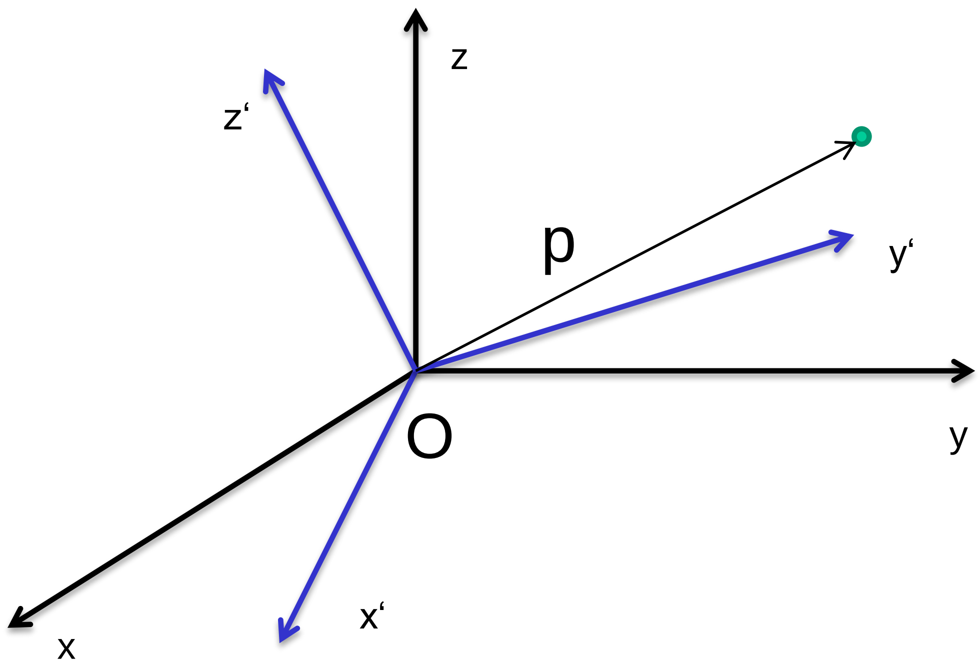


Rigid Body Configuration





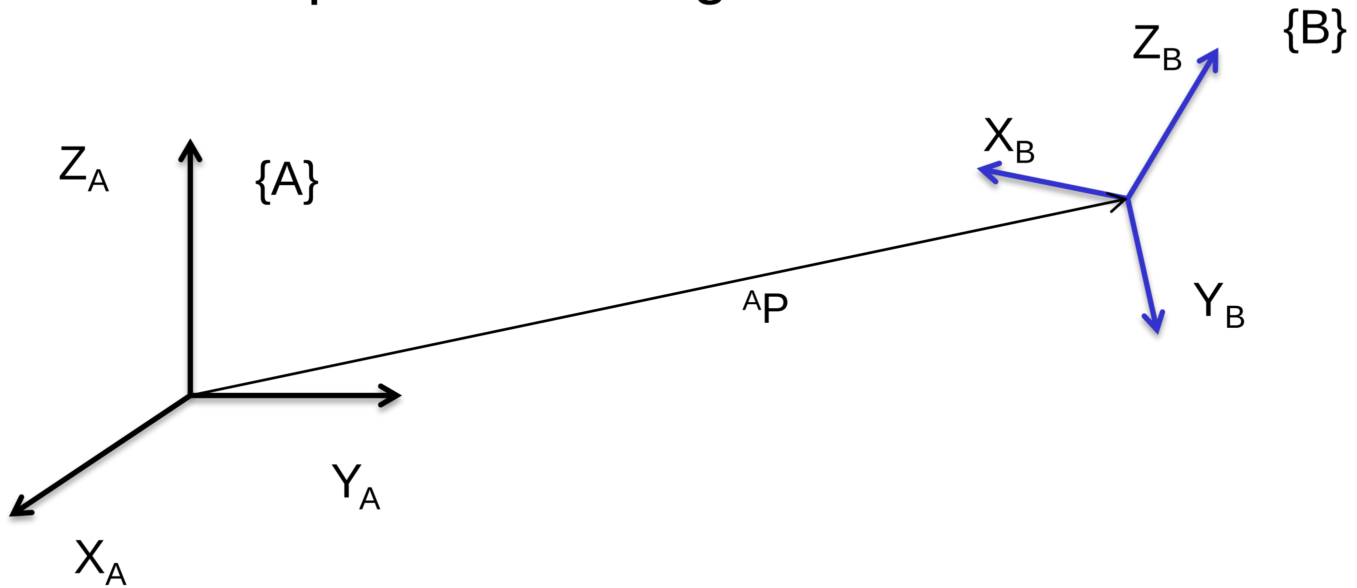
Coordinate Frames (same Origin)



Transformations describe relations of different coordinate frames to each other

Rigid Body Configuration

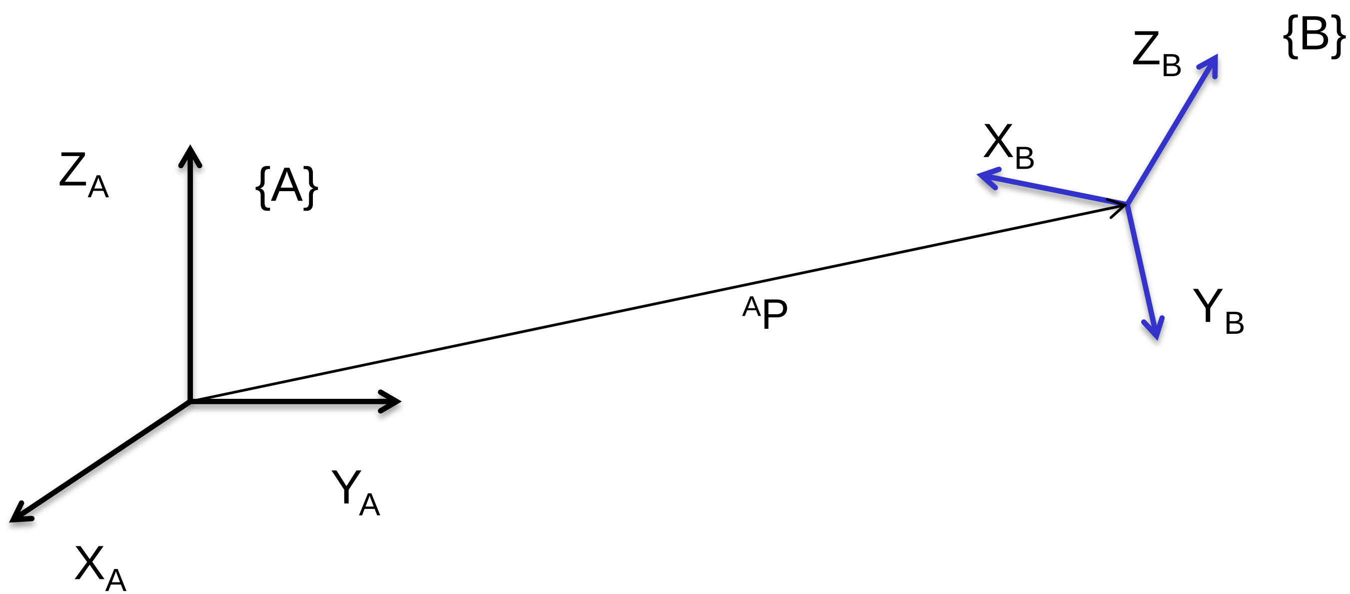
- Relationship between origins



- How to describe frame {B} with respect to fixed frame {A}, ${}^{A}P$ defines the origin of frame {B}.

Rigid Body Configuration

(source: Khatib)



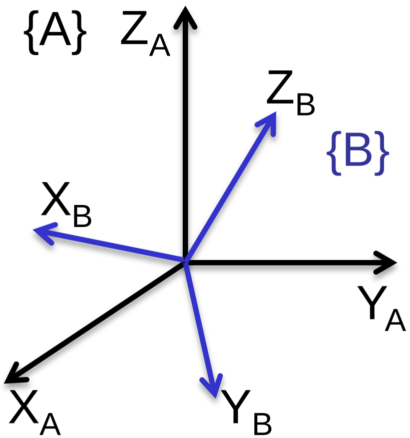
Position: ^{A}P

Orientation: $\{^{A}X_B, {}^{A}Y_B, {}^{A}Z_B\}$

rotations of base vectors of frame {B} with respect to frame {A} → **Rotation Matrix**

Rotation Matrix

(source: Khatib)



$$ {}^A_B R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ {}^A X_B = {}^A_B R ({}^B X_B) \qquad {}^B X_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

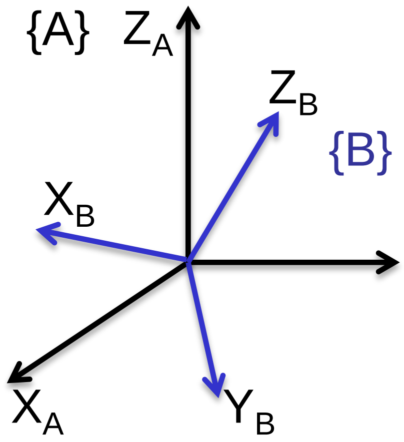
$$ {}^A X_B = {}^A_B R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad {}^A Y_B = {}^A_B R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad {}^A Z_B = {}^A_B R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ \boxed{ {}^A_B R = \begin{bmatrix} {}^A X_B & {}^A Y_B & {}^A Z_B \end{bmatrix} } $$

columns of R are the components of axis X,Y,Z of frame {B} in reference frame {A}

Rotation Matrix

(source: Khatib)



calculate $^{A}X_B$

using the Dot-product:

$$^{A}X_B = \begin{bmatrix} X_B \cdot X_A \\ X_B \cdot Y_A \\ X_B \cdot Z_A \end{bmatrix}$$

$$^{A}_{B}R = \begin{bmatrix} X_B \cdot X_A & Y_B \cdot X_A & Z_B \cdot X_A \\ X_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ X_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{bmatrix} = {}^{B}X_A^T$$

Freie Universität Berlin

Properties of the Rotation Matrix *(source: Khatib)*

$$ {}^A_B R = \begin{bmatrix} {}^A X_B & {}^A Y_B & {}^A Z_B \end{bmatrix} = \begin{bmatrix} {}^B X_A^T \\ {}^B Y_A^T \\ {}^B Z_A^T \end{bmatrix} = \begin{bmatrix} {}^B X_A & {}^B Y_A & {}^B Z_A \end{bmatrix}^T = {}^B_A R^T $$

$$ {}^A_B R = {}^B_A R^T $$

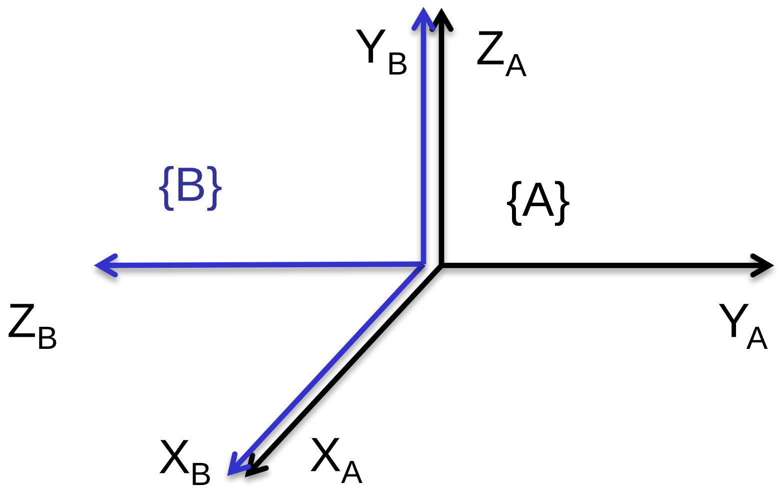
Inverse of the Rotation Matrix:

$$ {}^A_B R^{-1} = {}^B_A R = {}^A_B R^T $$

Orthonormal matrix (orthogonal unit vectors).

All orthonormal matrices (length of column vectors 1 and dot product of column vectors 0) have this property.

Example

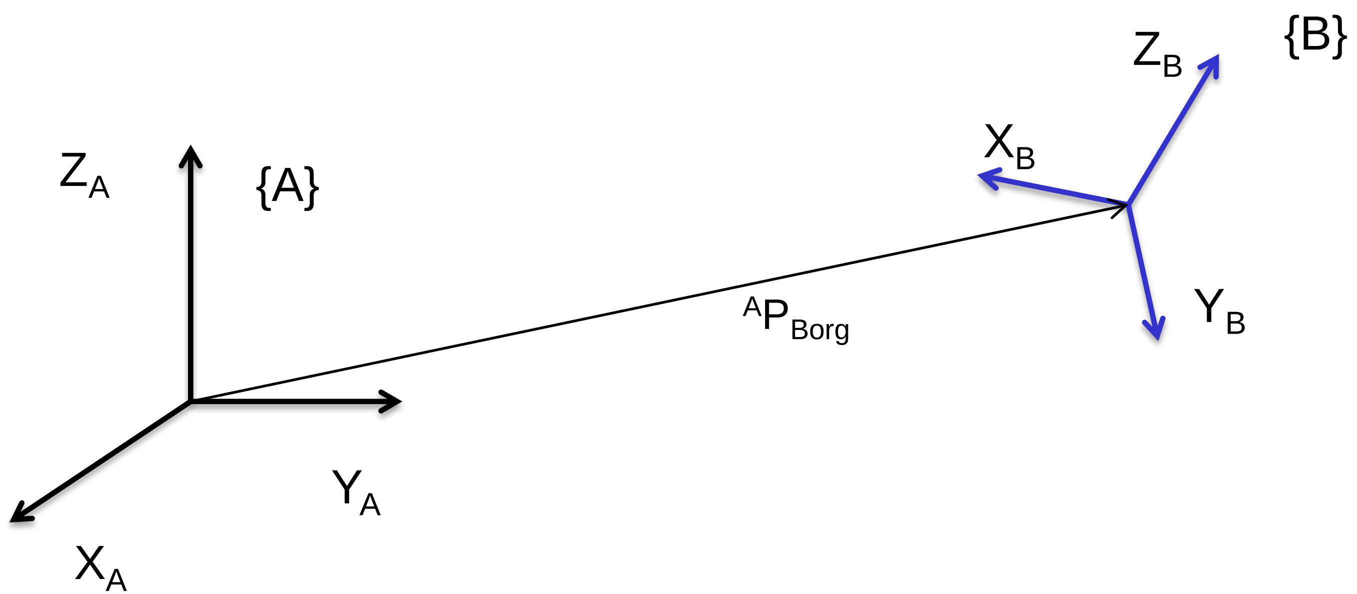


$$ {}_B^A R = \begin{bmatrix} X_B \cdot X_A & Y_B \cdot X_A & Z_B \cdot X_A \\ X_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ X_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{bmatrix} $$

$$ {}_B^A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \boxed{\begin{bmatrix} {}^B X_A^T \\ {}^B Y_A^T \\ {}^B Z_A^T \end{bmatrix}} $$

$$ \boxed{\begin{bmatrix} {}^A X_B & {}^A Y_B & {}^A Z_B \end{bmatrix}} $$

Full Description of a Frame

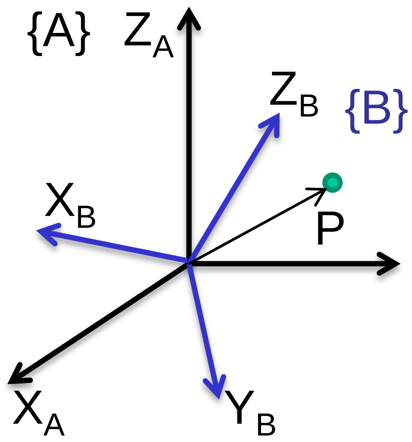


$$\{B\} = [{}^A_BR \quad {}^AP_{Borg}]$$

Freie Universität Berlin

Interpretations of Rotation Matrices

- Mapping
  - changes the description of the same vector from frame to frame
  - for rotated frames vector stays the same (same length)

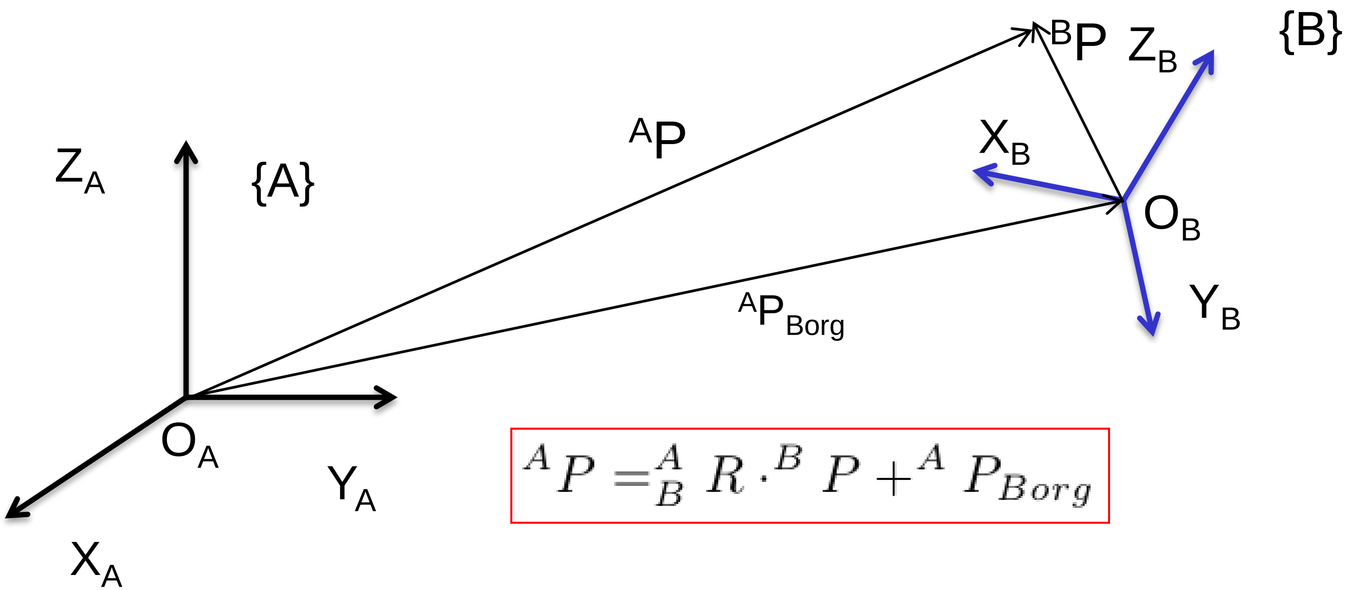


$$^{A}P = \begin{bmatrix} {^{B}X_A} \cdot {^{B}P} \\ {^{B}Y_A} \cdot {^{B}P} \\ {^{B}Z_A} \cdot {^{B}P} \end{bmatrix} = \begin{bmatrix} {^{B}X_A^T} \\ {^{B}Y_A^T} \\ {^{B}Z_A^T} \end{bmatrix} \cdot {^{B}P}$$

If P is given in {B}

$$^{A}P = {^{A}_{B}R} \cdot {^{B}P}$$

Homogenious Form



$$\boxed{{}^A P = {}^A_B R \cdot {}^B P + {}^A P_{Borg}}$$

$$\boxed{\left[\begin{array}{c} {}^A P \\ \hline 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^A_B R & & {}^A P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} {}^B P \\ \hline 1 \end{array}\right]}$$

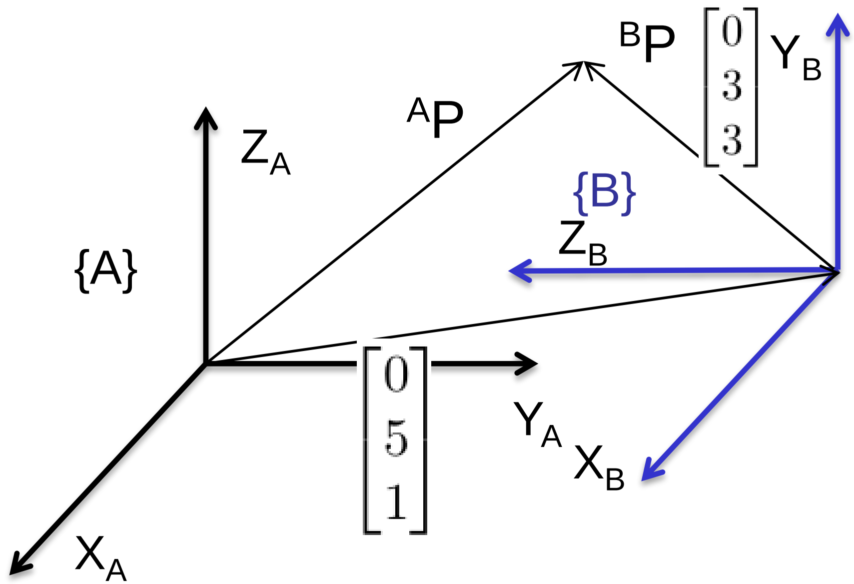
$${}^A P = {}^A_B T \cdot {}^B P$$

(4x1) (4x4) (4x1)

Freie Universität Berlin

Example, Homog. Transform

(Khatib)



$$^A_BR = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$^AP = ^A_BT \cdot {}^BP$$

$$^A_BT = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & -1 & 5\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$^AP = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & -1 & 5\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\3\\3\\1\end{bmatrix} = \begin{bmatrix}0\\2\\4\\1\end{bmatrix}$$

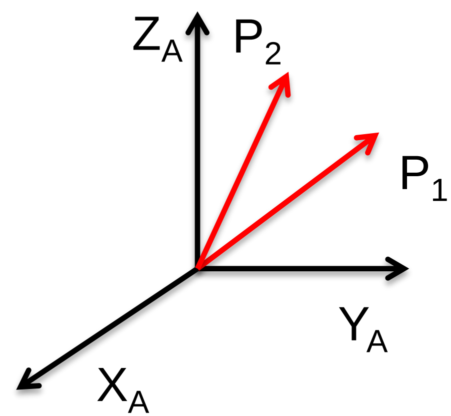
Operators

- in contrast to mapping (description of a vector changes), operators change the vector in a given frame

- Rotational Operator:

$$R_k(\theta) : P_1 \rightarrow P_2$$

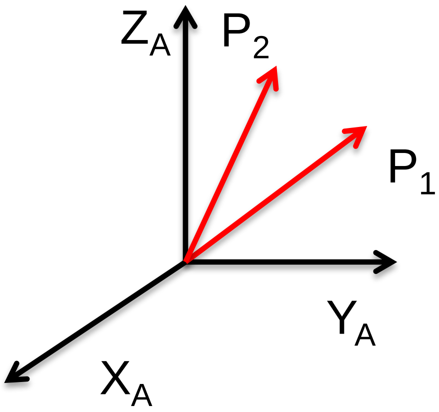
$$P_2 = R_k(\theta) P_1$$



- Example:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Example Rotational Operation

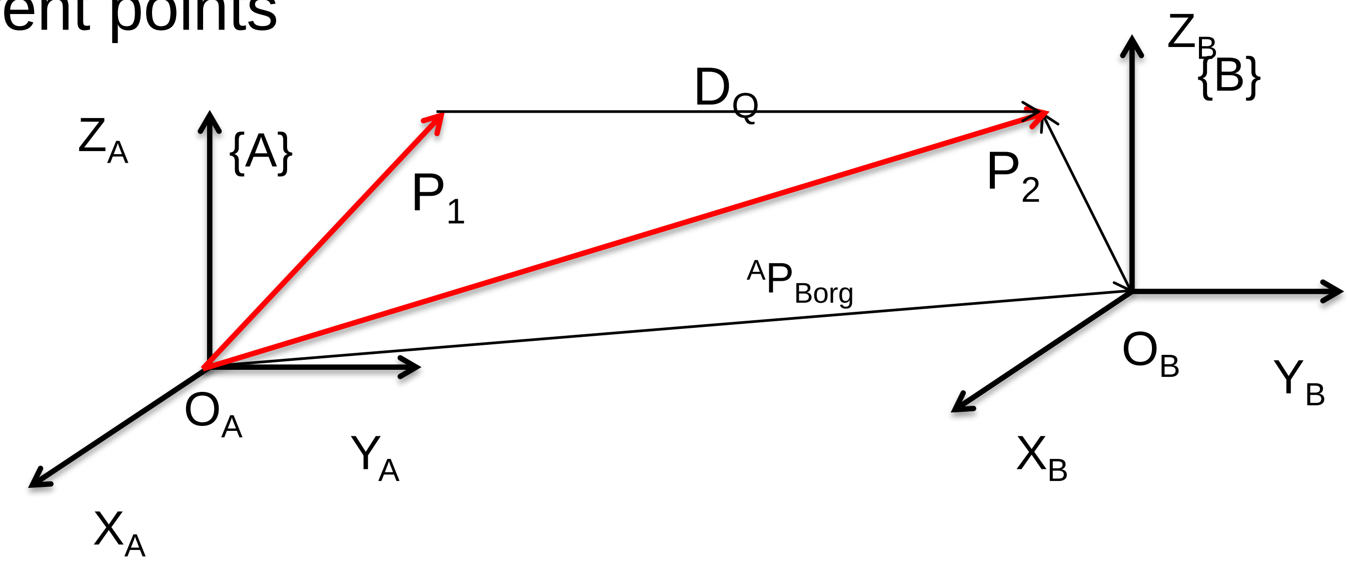


$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$P_2 = R_k(\approx 0.64)P_1 = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translational Operators

- Mapping: two different vectors for the same point
- Translational Operator: two different vectors for two different points



Homogenious Tranformation

- Transformational operator in homogenious coordinates

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^A P_2 = {}^A D_Q \cdot {}^A P_1$$

Inverse Transformation

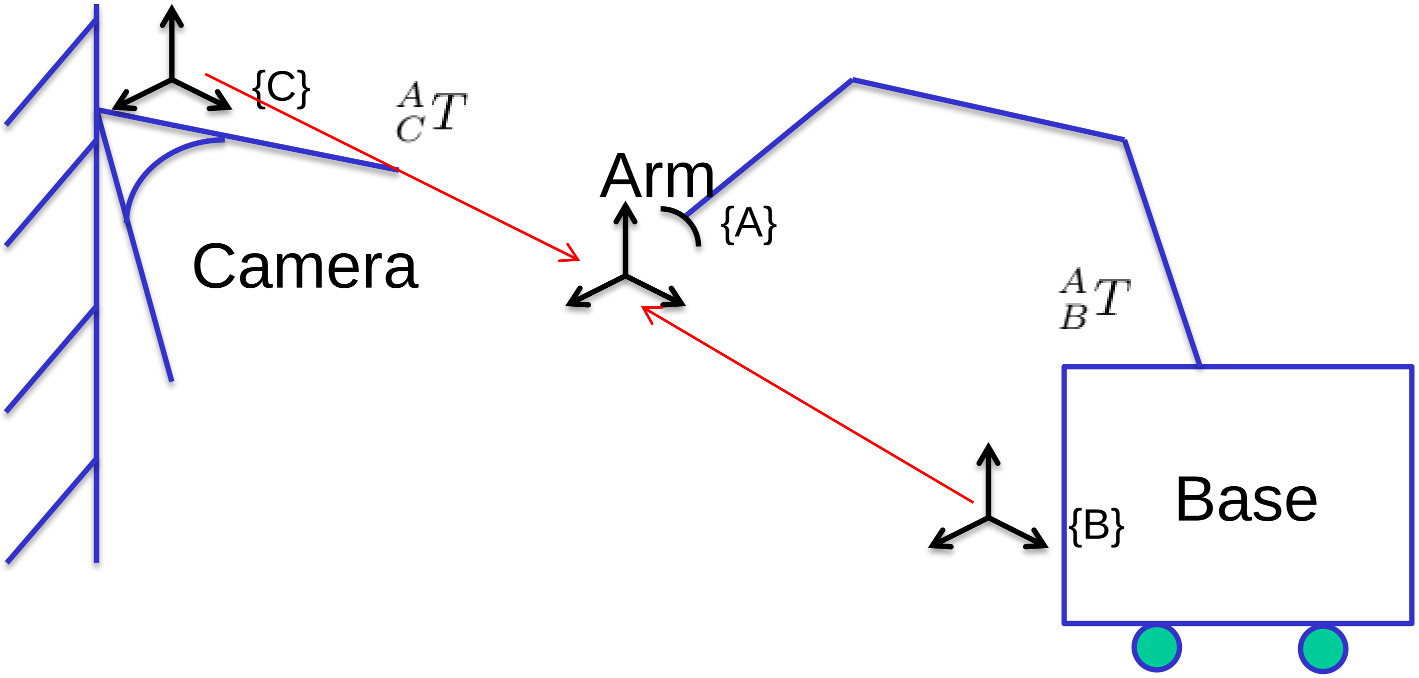
- Definition of transformation:

$$ {}^{A}_{B}T = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

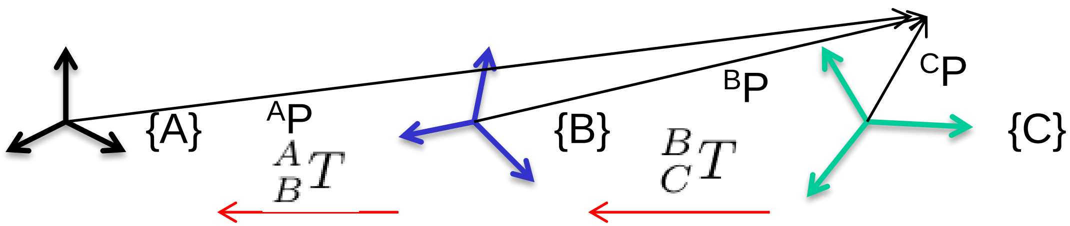
- Inverse:

$$ {}^{B}_{A}T = \begin{bmatrix} & {}^{B}_{A}R & & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Transformation between Frames



Compound Transformations



$$^BP = ^B_CT \cdot ^CP$$

$$^AP = ^A_BT \cdot ^BP$$

$$^AP = ^A_BT \cdot ^B_CT \cdot ^CP$$

$$\boxed{^A_CT = ^A_BT \cdot ^B_CT}$$

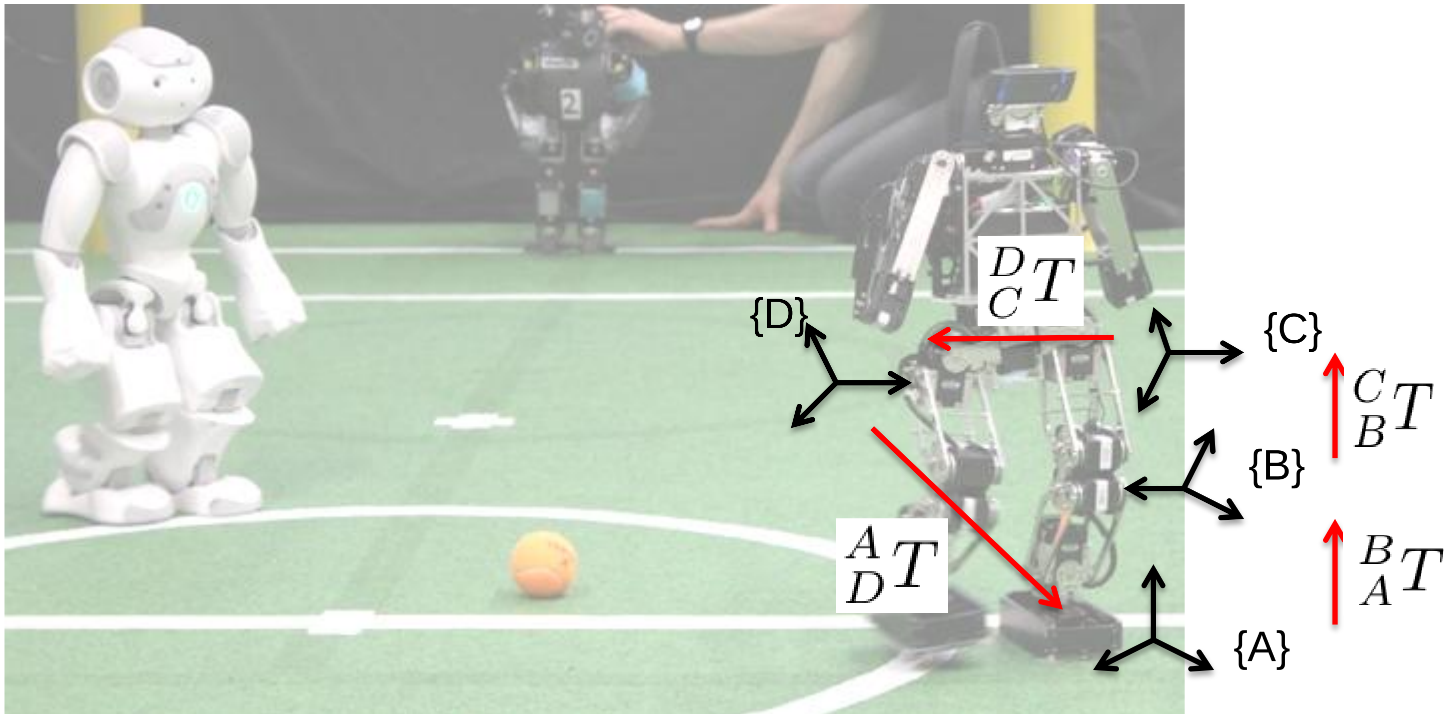
Compound Transformations

$$ {}^{A}_{C}T = {}^{A}_{B}T \cdot {}^{B}_{C}T $$

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R \cdot {}^{B}_{C}R & {}^{A}_{B}R \cdot {}^{B}P_{Corg} + {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Freie Universität Berlin

Transformation between Frames



$$\boxed{{}^{A}_{D}T \cdot {}^{D}_{C}T \cdot {}^{C}_{B}T \cdot {}^{B}_{A}T = I}$$

$${}^{C}_{C}T^{-1} = {}^{C}_{D}T^{-1} \cdot {}^{D}_{A}T^{-1} \cdot {}^{A}_{B}T^{-1} \cdot {}^{B}_{C}T^{-1} = I$$

Representations

- Transformation matrices change with joint angles

- How can the position and orientation of the endeffector be described (which part of T)?

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

Representations, contd.

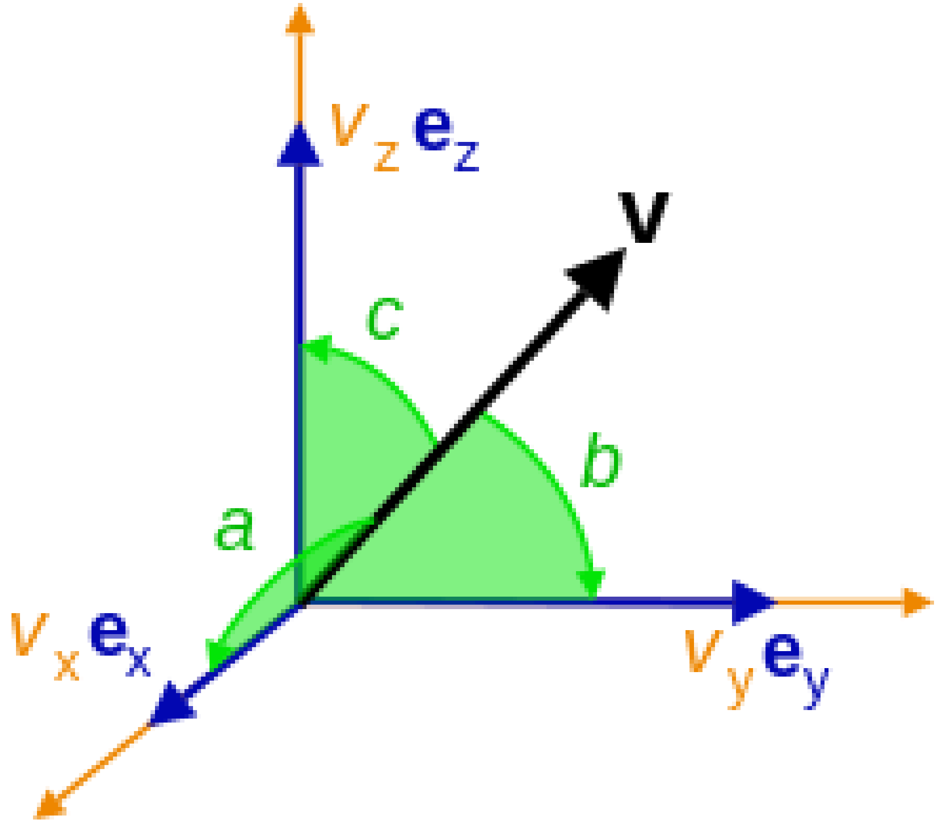
- Position Representations:
  - Cartesian (x,y,z)
  - Cylindrical (rho, theta, z)
  - Spherical (r, theta, phi)

- Rotation Representations:
  - Rotation Matrix (Dir. Cos.)
    - 6 Redundancies:
      - $|r_i| = 1$, $r_i \cdot r_j = 0$ (i != j)
    - Not applicable for motion interpolation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \rightarrow \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \\ r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

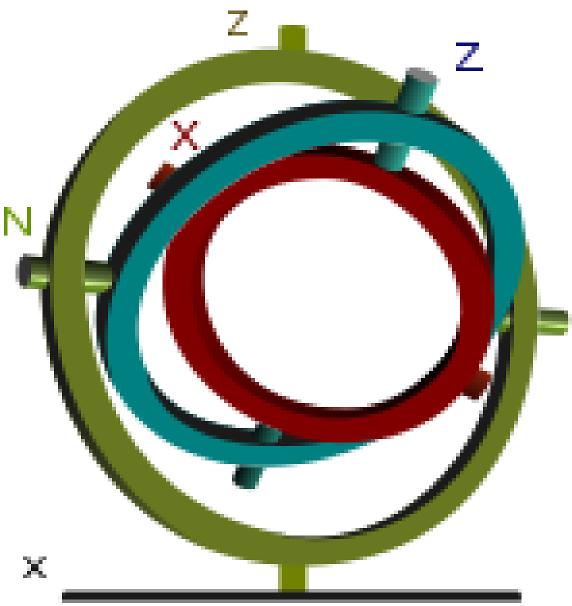
Rotational Representations

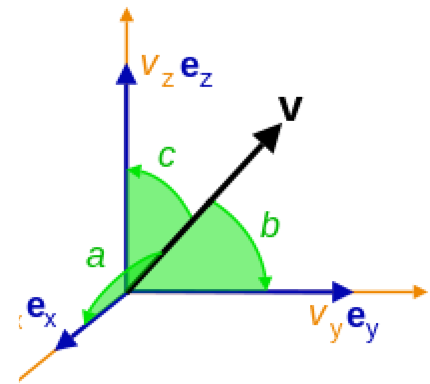
– Direction Cosines



Rotational Representations

- Direction Cosines
- Euler angles





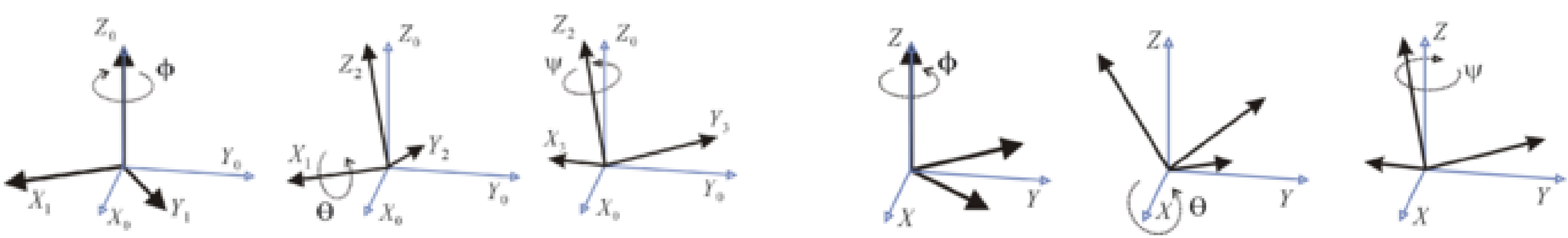
Source: Wikipedia

Three Angle Representations

- 24 different representations involving x,y,z
- Intrinsic (Relat. Axis, Euler) vs. Extrinsic (Fix Axis) (12 ea.)
- third axis is the same as first

  x-y-x, x-z-x, y-x-y, y-z-y, z-x-z, z-y-z
- rotation about all three axis:

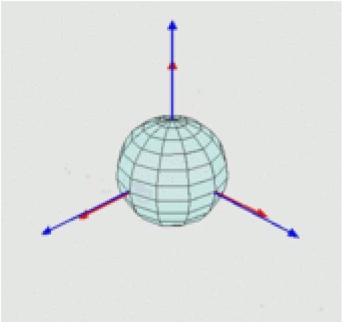
  x-y-z, x-z-y, y-x-z, y-z-x, z-y-x, z-x-y



Intrinsic vs. Extrinsic angles, Source: Wikipedia

Three Angle Representations

- Relative axis rotations sometimes written as: x-y‘-x“ ,
  Fixed axis as: x-y-x

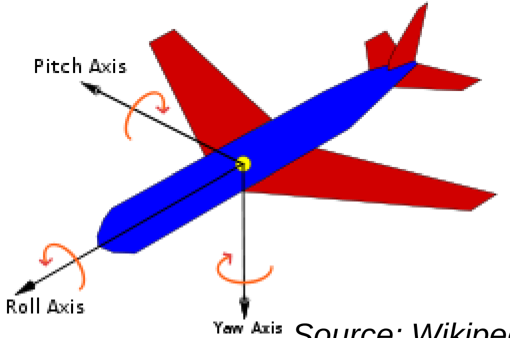


z-x‘-z“

Source: Wikipedia

- For each relative axis rotation there exists a fix axis one
  → 12 different representations

- In robotics and aviation: *roll, pitch, yaw* often used for three rotational axes

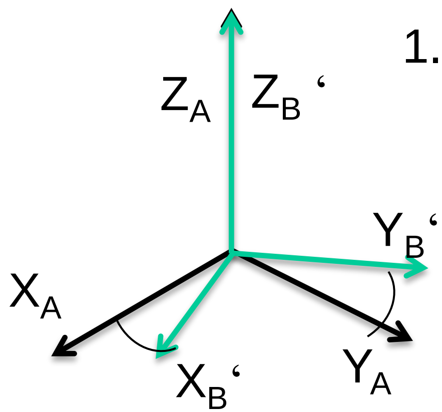


Source: Wikipedia

Freie Universität Berlin

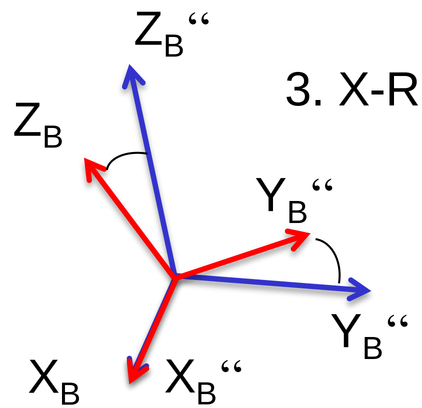
Euler Angles (Z,Y,X)

(repr. after Khatib)



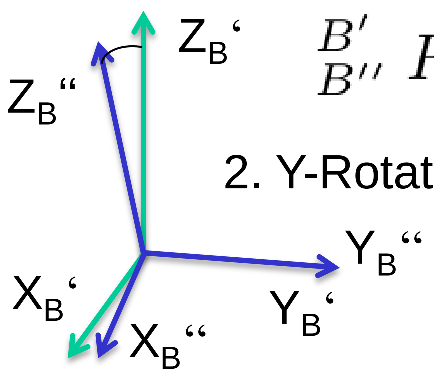
1. Z-Rotation

$${}^{A}_{B'}R$$



3. X-Rotation

$${}^{B''}_{B}R$$



$${}^{B'}_{B''}R$$

2. Y-Rotation

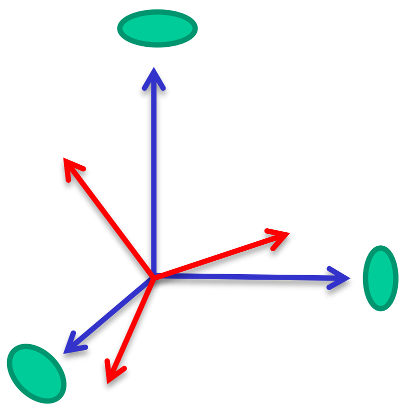
$${}^{A}_{B}R = {}^{A}_{B'}R \cdot {}^{B'}_{B''}R \cdot {}^{B''}_{B}R$$

$${}^{A}_{B}R = R_Z(\alpha) \cdot R_Y(\beta) \cdot R_X(\gamma)$$

Fixed Angles (Z,Y,X)

(repr. after Khatib)

- Almost similar, but rotational axes stay constant



Euler Angles (Z,Y,X)

(repr. Khatib)

$$ {}^{A}_{B}R = R_Z(\alpha) \cdot R_Y(\beta) \cdot R_X(\gamma) $$

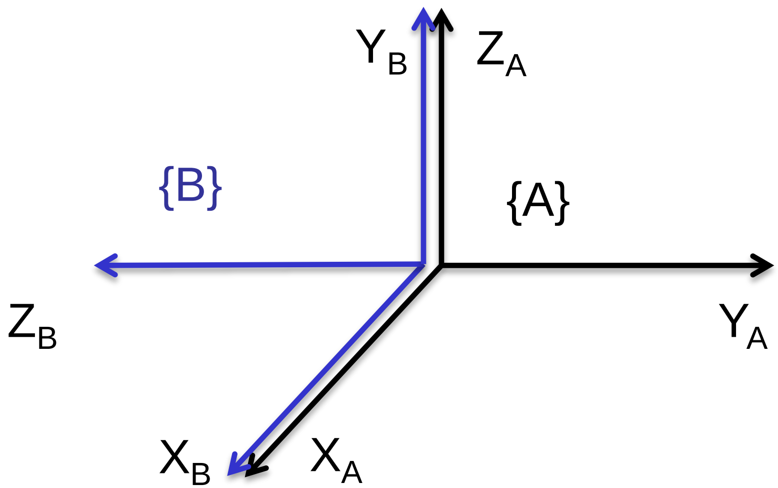
$$ \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\alpha \cdot \cos\beta & X & X \\ \sin\alpha \cdot \cos\beta & X & X \\ -\sin\beta & \cos\beta \cdot \sin\gamma & \cos\beta \cdot \cos\gamma \end{bmatrix} $$

Inverse Problem: How to find the angles for a given transformation?

Example

(Khatib)



$${}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma)$$

$$\alpha = 0$$
$$\beta = 0$$
$$\gamma = 90^o$$

Euler Angles vs. Fixed Angles

(Khatib)

- X-Y-Z Fixed Angles:

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)$$

- Z-Y-X Euler Angles:

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Freie Universität Berlin

Inverse Problem

(Khatib)

- How to find the Euler Angles for a given matrix?

$$\text{Given } {}^A_BR, \text{ find } (\alpha, \beta, \gamma)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha \cdot c\beta & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma \\ s\alpha \cdot c\beta & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma \\ -s\beta & c\beta \cdot s\gamma & c\beta \cdot c\gamma \end{bmatrix}$$

$$\sin(\beta) = s\beta = \boxed{-r_{13}}$$

$$\cos(\beta) = c\beta = \sqrt{r_{11} + r_{21}}$$

$$\beta = \arctan \frac{\sin\beta}{\cos\beta}$$

- if $\beta = 90^o, \cos\beta = 0$ → Singularity of Representation

Inverse Problem

(Khatib)

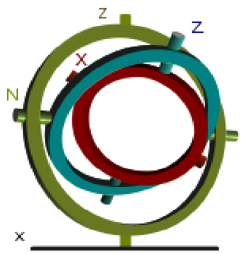
$$\beta = 90^o, \cos\beta = 0, \sin\beta = -1$$

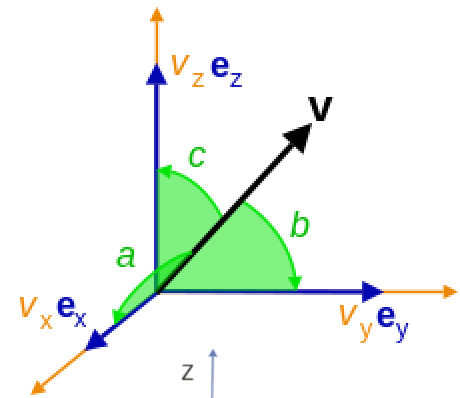
$$^A_BR = \begin{bmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{bmatrix}$$

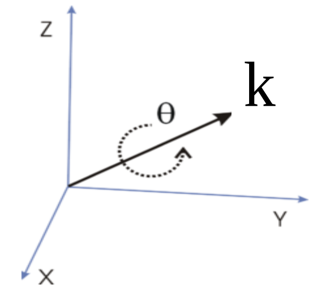
Every 3 parameter representation has a representational singularity (gimbal lock)

Rotational Representations

- Direction Cosines
- Euler angles
- Axis-Angle-Representation







Axis-Angle Representation

- *k* is unit vector (axis of rotation), with a_x, a_y, a_z as elements
- *θ* represents angle of rotation
- 3 parameters (product involving 4)

$$< \text{normalized axis, angle} > = < k, \theta > = \left(\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}, \theta \right)$$

$$\theta = \arccos\left(\frac{trace(R) - 1}{2}\right)$$

- Singularity for small angles

$$k = \frac{1}{2\sin(\theta)} \begin{bmatrix} R(3,2) - R(2,3) \\ R(1,3) - R(3,1) \\ R(2,1) - R(1,2) \end{bmatrix}$$

Freie Universität Berlin

Application on a Vector

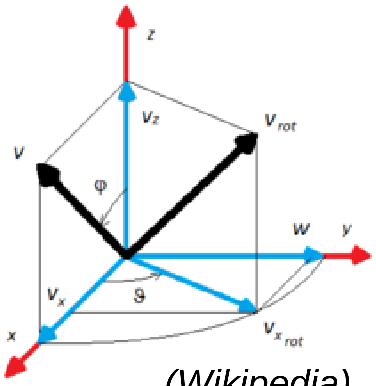
- Rodrigues‘ Formula to rotate a vector *v* around a unit vector *k* by angle *θ*:

$$v_{rot} = (\cos\theta)v + (\sin\theta)(k \times v) + k(k \cdot v)(1 - \cos\theta)$$

- Rotation matrix:
  - R = exp(*θ* K) , exponential map of (*θ* K)
  - K is cross product matrix (skew matrix): K· v = *k* x v

$$K = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$R = I + \sin(\theta)K + (1 - \cos(\theta))K^2$$



(Wikipedia)

Freie Universität Berlin

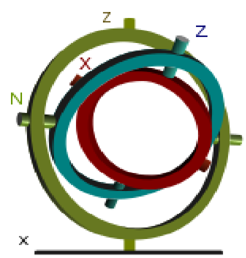
Rotational Representations

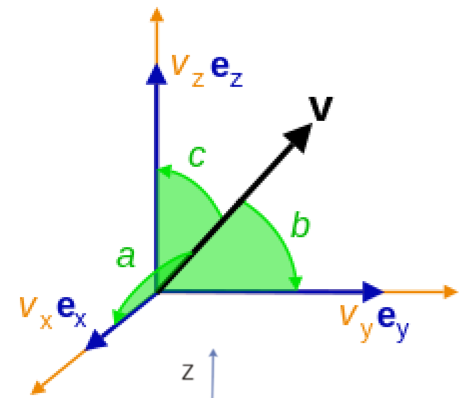
– Direction Cosines

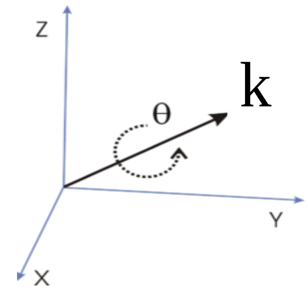
– Euler angles

– Axis-Angle-Representation

– Euler-Parameters (Quaternion-Rotation)





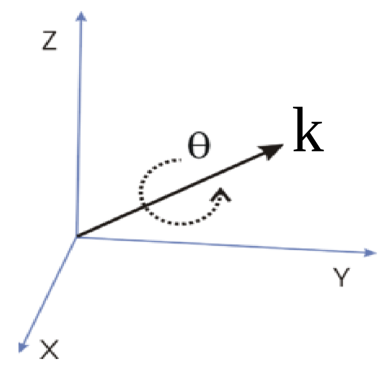


Euler Parameters

- All rotations with 3 parameters have singularities

- Let‘s try $\varepsilon \in \mathbf{R}^4$

$$\begin{aligned}
\varepsilon_0 &= \cos(\tfrac{\theta}{2}) \\
\varepsilon_1 &= k_X \sin(\tfrac{\theta}{2}) \\
\varepsilon_2 &= k_Y \sin(\tfrac{\theta}{2}) \\
\varepsilon_3 &= k_Z \sin(\tfrac{\theta}{2})
\end{aligned}$$



- k is unit vector in 4-dimensional space, (1 parameter is redundant) $\varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = 1$

Quaternion Space

- Euler parameters are a quaternion in scalar vector representation.

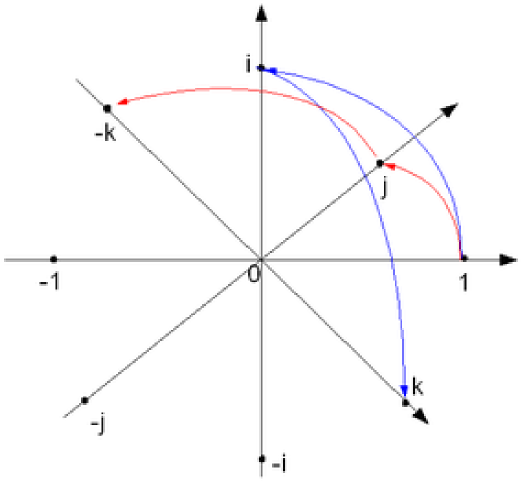
$$\varepsilon = \varepsilon_0 + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k$$

- Some interesting properties:

$$ij = k, jk = i, ki = j$$

$$ji = -k, kj = -i, ik = -j$$

$$i^2 = j^2 = k^2 = ijk = -1$$



Graphical representation of quaternion units product as 90°-rotation in 4D-space

ij = k
ji = -k
ij = -ji

(Wikipedia)

Inverse Problem

$$
{}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_0) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_0) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_0) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_0) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_0) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_0) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}
$$

$$
r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)
$$

$$
\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2 = 1-\varepsilon_0^2
$$

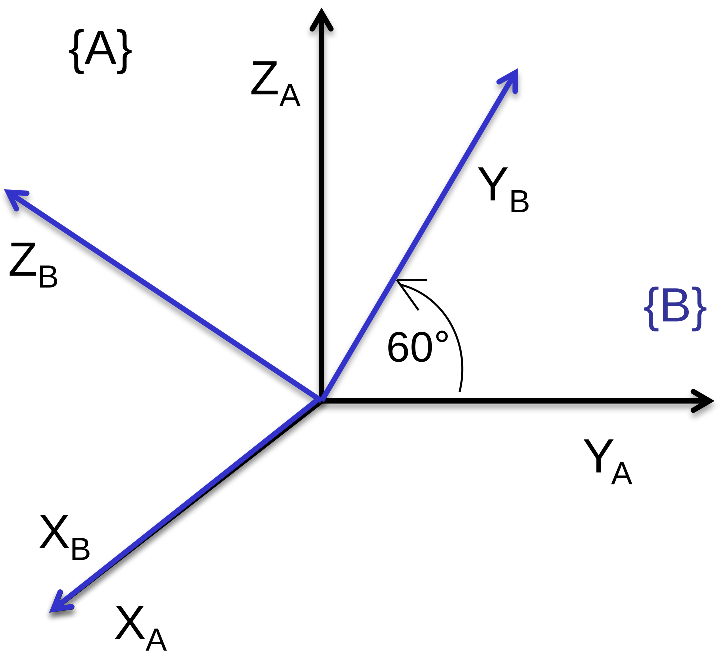
$$
\varepsilon_0 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}
$$

$$
\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_0}, \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_0}, \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_0}
$$

$$
\text{What if: } \varepsilon_0 = 0?
$$

- Lemma: Always at least one parameter is larger than ½, Apply this one for the computation, → No Singularity.

Example



Euler Params:

$$\varepsilon = (\sqrt{(}3/4), 1/2, 0, 0)$$

Rotation Matrix (DC):

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3/4} \\ 0 & \sqrt{3/4} & 1/2 \end{bmatrix}$$

Summary

- Joint space and operational space
- Homogenious Transformations for mapping between coordinate frames and for operations
- Rotations:
  - Rotations with three angles are minimal but have singularities, must be converted to matrix to rotate a vector
  - Quarternions: no singularities, must be converted to matrix to rotate a vector