习题(35)

35.1 设X_1, X_2, X_3为总体X的样本,证明下列统计量都是总体均值$E(X)$的无偏估计量:

$$\gamma_1=\frac{2}{5}\cdot X_1+\frac{1}{5}\cdot X_2+\frac{2}{5}\cdot X_3,$$

$$\gamma_2=\frac{1}{6}\cdot X_1+\frac{1}{3}\cdot X_2+\frac{1}{2}\cdot X_3,$$

$$\gamma_3=\frac{1}{7}\cdot X_1+\frac{3}{14}\cdot X_2+\frac{9}{14}\cdot X_3.$$

并问哪一个无偏估计量最有效?

35.2 设总体X的密度函数

$$f(x)=\begin{cases}\frac{x}{\theta}\cdot\exp\{-\frac{x^2}{2\theta}\} & , \quad x>0 \\ 0 & , \quad x\le 0\end{cases},$$

其中$\theta>0$为未知参数. $X_1, X_2, \cdots, X_n$为X的样本,试求θ的矩估计量$\hat{\theta}$;并证明$\hat{\theta}$是θ的相合估计量.

35.3 设分别自总体$N(\mu_1,\sigma^2)$和$N(\mu_2,\sigma^2)$中抽取容量为n_1, n_2的两个独立样本.其样本方差分别为S_1^2, S_2^2.试证:对于任何常数a, b $(a+b=1)$,统计量$aS_1^2+bS_2^2$都是σ^2的无偏估计;并确定常数a, b,使$D(aS_1^2+bS_2^2)$达到最小.

35.4 设总体X的密度函数为

$$f(x)=\begin{cases}\frac{1}{\theta} & , \quad 0\le x\le\theta \\ 0 & , \quad \text{其他}\end{cases},$$

X_1, X_2, X_3为X的样本,试证:$4X_{(1)}$及$\frac{4}{3}X_{(3)}$都是未知参数θ的无偏估计;并问哪个较有效?

习题(35)参考解答

35.1 证 由X_1, X_2, X_3为总体X的样本,则

$$E(X_i)=E(X),\ D(X_i)=D(X),\ i=1,2,3$$

$$E(\gamma_1)=\frac{2}{5}\cdot E(X_1)+\frac{1}{5}\cdot E(X_2)+\frac{2}{5}\cdot E(X_3)=E(X).$$

同理,得

$$E(\gamma_2)=E(X),E(\gamma_3)=E(X).$$

故$\gamma_1,\gamma_2,\gamma_3$均为$E(X)$的无偏估计量.又由

$$D(\gamma_1)=(\frac{2}{5})^2\cdot D(X_1)+(\frac{1}{5})^2\cdot D(X_2)+(\frac{2}{5})^2\cdot D(X_3)=\frac{9}{25}\cdot D(X),$$

同理,得

$$D(\gamma_2)=\frac{7}{18}\cdot D(X),\ D(\gamma_3)=\frac{47}{98}\cdot D(X).$$

且$\frac{9}{25}<\frac{7}{18}<\frac{47}{98}$,所以,估计量$\gamma_1$最有效. ♣

35.2 解: 由

$$E(X)=\int_0^{+\infty}x\cdot\frac{x}{\theta}\cdot\exp\{-\frac{x^2}{2\theta}\}dx=\sqrt{\theta}\cdot\int_0^{+\infty}t^2\cdot\exp\{-\frac{t^2}{2}\}dt$$

$$=\sqrt{\theta}\cdot\frac{\sqrt{2\pi}}{2}\cdot\int_{-\infty}^{+\infty}t^2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt=\sqrt{\theta}\cdot\sqrt{\frac{\pi}{2}},$$

根据矩估计法,令

$$\overline{X}=E(X)\qquad \overline{X}=\sqrt{\theta}\cdot\sqrt{\frac{\pi}{2}}.$$

解得θ的矩估计量为$\hat{\theta}=\frac{2}{\pi}\cdot\overline{X}^2$.又当$n\to\infty$时,由

$$\overline{X}\xrightarrow{P}E(X)=\sqrt{\theta}\cdot\sqrt{\frac{\pi}{2}},$$

则

$$\hat{\theta}=\frac{2}{\pi}\cdot\overline{X}^2\xrightarrow{P}\frac{2}{\pi}\cdot[E(X)]^2=\theta,$$

即得$\hat{\theta}$是θ的相合估计. ♣

35.3 解: 由于$E(S_1^2)=\sigma^2$,$E(S_2^2)=\sigma^2$,且在$a+b=1$时,有

$$E(aS_1^2+bS_2^2)=a\cdot E(S_1^2)+b\cdot E(S_2^2)=(a+b)\sigma^2=\sigma^2,$$

所以,$aS_1^2+bS_2^2$都是σ^2的无偏估计.

又由于S_1^2,S_2^2相互独立,且

$$\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2(n_1-1), \quad \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2(n_2-1)$$

$$D(\frac{(n_1-1)S_1^2}{\sigma^2}) = 2(n_1-1), \quad D(\frac{(n_2-1)S_2^2}{\sigma^2}) = 2(n_2-1)$$

$$D(S_1^2) = \frac{2\sigma^4}{n_1-1}, \qquad D(S_2^2) = \frac{2\sigma^4}{n_2-1}$$

$$D(aS_1^2+bS_2^2) = a^2 \cdot D(S_1^2) + b^2 \cdot D(S_2^2) = a^2 \cdot \frac{2\sigma^4}{n_1-1} + b^2 \cdot \frac{2\sigma^4}{n_2-1}$$

$$= 2(\frac{a^2}{n_1-1} + \frac{b^2}{n_2-1})\sigma^4 .$$

于是问题转化为:要确定常数a,b,在条件$a+b=1$下,使得$\frac{a^2}{n_1-1}+\frac{b^2}{n_2-1}$达到最小.

由$a+b=1$ $\quad b=1-a$.记$L(a)=\frac{a^2}{n_1-1}+\frac{(1-a)^2}{n_2-1}$,令

$$\frac{d}{da}L(a)=0 \qquad \frac{2a}{n_1-1}-\frac{2(1-a)}{n_2-1}=0 .$$

则所求常数a,b分别为

$$a=\frac{n_1-1}{n_1+n_2-2}, b=\frac{n_2-1}{n_1+n_2-2}$$

时,$D(aS_1^2+bS_2^2)$达到最小. ♣

35.4 解: 考虑更一般情形,对于容量为n的样本$X_1, X_2, \cdots, X_n$,将比较$(n+1)X_{(1)}$与$\frac{n+1}{n}X_{(n)}$.

已知总体X的密度函数为$f(x)$,则X的分布函数为

$$F(x)=\begin{cases} 0 , & x<0 \\ \frac{x}{\theta} , & 0 \le x \le \theta \\ 1 , & x>\theta \end{cases}.$$

对于X的样本$X_1, X_2, \cdots, X_n$,则$X_{(1)}, X_{(n)}$的密度函数分别为

$$f_{(1)}(x)=n\cdot[1-F(x)]^{n-1}\cdot f(x)=\begin{cases} \frac{n}{\theta}\cdot(1-\frac{x}{\theta})^{n-1} , & 0 \le x \le \theta \\ 0 , & \text{其他} \end{cases},$$

$$f_{(n)}(x)=n\cdot[F(x)]^{n-1}\cdot f(x)=\begin{cases} \frac{n}{\theta^n}\cdot x^{n-1} , & 0 \le x \le \theta \\ 0 , & \text{其他} \end{cases}.$$

则

$$E(X_{(1)})=\int_{-\infty}^{+\infty}x\cdot f_{(1)}(x)dx=\int_{0}^{\theta}x\cdot\frac{n}{\theta}\cdot(1-\frac{x}{\theta})^{n-1}dx=\frac{1}{n+1}\cdot\theta,$$

$$E(X_{(n)})=\int_{-\infty}^{+\infty}x\cdot f_{(n)}(x)dx=\int_{0}^{\theta}x\cdot\frac{n}{\theta^n}\cdot x^{n-1}dx=\frac{n}{n+1}\cdot\theta$$

$$E[(n+1)X_{(1)}]=\theta,\ E(\frac{n+1}{n}X_{(n)})=\theta.$$

即知$(n+1)X_{(1)}$及$\frac{n+1}{n}X_{(n)}$是θ的无偏估计.本题中$n=3$,则得$4X_{(1)}$及$\frac{4}{3}X_{(3)}$都是θ的无偏估计.

又由

$$E(X_{(1)}^2)=\int_{0}^{\theta}x^2\cdot\frac{n}{\theta}\cdot(1-\frac{x}{\theta})^{n-1}dx=\frac{2}{(n+1)(n+2)}\cdot\theta^2,$$

$$E(X_{(n)}^2)=\int_{0}^{\theta}x^2\cdot\frac{n}{\theta^n}\cdot x^{n-1}dx=\frac{n}{n+2}\cdot\theta^2$$

$$\begin{aligned}D((n+1)X_{(1)})&=(n+1)^2\cdot D(X_{(1)})=(n+1)^2\cdot[E(X_{(1)}^2)-(E(X_{(1)}))^2]\\&=(n+1)^2\cdot[\frac{2}{(n+1)(n+2)}\cdot\theta^2-(\frac{1}{n+1}\cdot\theta)^2]\\&=\frac{n}{n+2}\cdot\theta^2,\end{aligned}$$

$$\begin{aligned}D(\frac{n+1}{n}X_{(n)})&=(\frac{n+1}{n})^2\cdot D(X_{(n)})=(\frac{n+1}{n})^2\cdot[E(X_{(n)}^2)-(E(X_{(n)}))^2]\\&=(\frac{n+1}{n})^2\cdot[\frac{n}{n+2}\cdot\theta^2-(\frac{n}{n+1}\cdot\theta)^2]\\&=\frac{1}{n(n+2)}\cdot\theta^2.\end{aligned}$$

则当$n>1$时,有

$$\frac{1}{n(n+2)}\cdot\theta^2<\frac{n}{n+2}\cdot\theta^2,$$

即有

$$D(\frac{n+1}{n}X_{(n)})<D((n+1)X_{(1)}).$$

本题中$n=3$,则$D(\frac{4}{3}X_{(3)})<D(4X_{(1)})$,故$\frac{4}{3}X_{(3)}$较$4X_{(1)}$有效. ♣