Problem Set 1: Group and Group Action

Due Date: September 30, 2019 in class.
Name: ________________________
Date: ________________________

Instructions

Write down your solutions on A4 papers and hand them in by the due date. You may discuss with your classmates about the strategy to solve the problems, but each of you should come up with a full solution on your own. Your solutions will be graded based on efforts, as opposed to the correctness.

Problems

1. Prove that $GL_n(\mathbb{R})$ is a group.

2. Prove that a nonempty subset H of a group G is a subgroup if for all $x, y \in H$ the element xy^{-1} is also in H.

3. Prove that in any group the orders of ab and of ba are equal.

4. Show that the functions $f = \frac{1}{x}$, $g = \frac{x-1}{x}$ generate a group of functions, the law of composition being composition of functions, which is isomorphic to the symmetric group S_3.

5. Let H be a subgroup of G, and let $g \in G$. The conjugate subgroup gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, where $h \in H$. Prove that gHg^{-1} is a subgroup of G.

6. Consider the set U of real 3×3 matrices of the form $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$
   1) Prove that U is a subgroup of $SL_3(\mathbb{R})$.
   2) **Prove or disprove** U is a normal subgroup of $SL_3(\mathbb{R})$.

7. Let H, K be subgroups of a group G of orders 3, 5 respectively. Prove that $H \cap K = \{1\}$.

8. Prove that a group of order 30 can have at most 7 subgroups of order 5.

9. Let G be a finite group whose order is a product of two integers: $n = ab$. Let H, K be subgroups of G of orders a and b respectively. Assume that $H \cap K = \{1\}$.
   1) Prove that $HK = G$.
   2) Is G isomorphic to the product group $H \times K$?

10. Prove the associative and commutative laws for multiplication in $\mathbb{Z}/n\mathbb{Z}$.

11. Prove that the subset H of $GL_n(\mathbb{R})$ of matrices whose determinant is positive forms a normal subgroup, and describe the quotient group $GL_n(\mathbb{R})/H$.

12. Determine the automorphism group of C_4 and $C_2 \times C_2$.

13. Let $GL_n(\mathbb{R})$ operate on the set $\mathbb{R}^n$ by left multiplication. 1) Describe the decomposition of $\mathbb{R}^n$ into orbits for this operation. 2) What is the stabilizer of $e_1 = (1, 0, 0, \cdots, 0)$?

14. What is the stabilizer of the coset aH for the operation of G on G/H?

15. Use the counting formula to determine the orders of the group of rotational symmetries of a cube and of the group of rotational symmetries of a tetrahedron.

16. 1) Prove that if H and K are subgroups of finite index of a group G, then the intersection $H \cap K$ is also of finite index.
    2) Show by example that the index $[H : H \cap K]$ need not divide $[G : K]$.

17. A group G operates faithfully on a set S of five elements, and there are two orbits, one of order 3 and one of order 2. What are the possibilities for G?

18. Prove that the map $\phi : G \to \mathrm{Aut}(G)$ defined by $g \mapsto$ conjugation by g is a homomorphism, and determine its kernel.

19. Let G be a group of order n which operates nontrivially on a set of order r. Prove that if $n > r!$, then G has a proper normal subgroup.

20. Let G be a finite group operating on a finite set S. For each element $g \in G$, let S^g denotes the subset of elements of S fixed by g: $S^g = \{s \in S | g.s = s\}$. Prove Burnside's Formula

$$|G| \cdot (\text{number of orbits}) = \sum_{g \in G} |S^g|.$$