Problem Set 4: Field and Galois Theory

Due Date: Dec. 30 in class.
Name: ______________________
Date: ______________________

Instructions

Write down your solutions on A4 papers and hand them in by the due date. You may discuss with your classmates about the strategy to solve the problems, but each of you should come up with a full solution on your own. Your solutions will be graded based on efforts, as opposed to the correctness. Star-ed problems are optional by the time of due date. However, to fulfill the goal of this class, you should try to work on them when there is time.

Problems

1. Let α be a complex root of the polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a + b\alpha + c\alpha^2$, with $a, b, c \in \mathbb{Q}$.

2. Let $\beta = \omega \cdot 2^{\frac{1}{3}}$, where $\omega = e^{2\pi i/3}$, and let $K = Q(\beta)$. Prove that the equation $x_1^2 + \cdots + x_k^2 = -1$ has no solution with $x_i \in K$.

3. Let $\xi_n = e^{2\pi i/n}$. Prove that $\xi_5 \notin \mathbb{Q}(\xi_7)$.

4. Let α and β be complex numbers. Prove that if $\alpha+\beta$ and $\alpha\beta$ are algebraic numbers, then α and β are also algebraic numbers.

5. A field extension K/F is an algebraic extension if every element of K is algebraic over F. Let K/F and L/K be algebraic field extensions. Prove that L/F is an algebraic extension.

6. Determine the irreducible polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over the following fields.
   1) $\mathbb{Q}$,
   2) $\mathbb{Q}(\sqrt{10})$,
   3) $\mathbb{Q}(\sqrt{15})$.

7. Classify quadratic extensions of $\mathbb{Q}$.

8. Determine the number of irreducible polynomials of degree 3 over $\mathbb{F}_3$ and over $\mathbb{F}_5$.

9. Factor $x^9 - x$ and $x^{27} - x$ in $\mathbb{F}_3$.

10. Let F be a finite field, and let $f(x)$ be a non-constant polynomial whose derivative is the zero polynomial. Prove that f cannot be irreducible over F.

11. Let p be a prime integer, and let $q = p^r$ and $q' = p^k$. For which values of rand k does $x^{q'} - x$ divide $x^q - x$ in $\mathbb{Z}[x]$?

12. Prove that every finite extension of a finite field has a primitive element.

13. Determine all primitive elements for the extension $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ of $\mathbb{Q}$.

14. * Let f be a polynomial of degree n with coefficients in F and let K be a splitting field for f over F. Prove that $[K : F]$ divides $n!$.

15. * Let K be the splitting field over $\mathbb{Q}$ of $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$. Determine all automorphisms of K.

16. * Let $K = Q(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine $[K : \mathbb{Q}]$, prove that K is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

17. * Let K/F be a Galois extension with Galois group G, and let H be a subgroup of G. Prove that there exists an element $\beta \in K$ whose stabilizer is equal to H.