Problem Set 3: Ring and Factorization

Due Date: December 2nd, 2019 in class.
Name: ____________________
Date: ____________________

Instructions

Write down your solutions on A4 papers and hand them in by the due date. You may discuss with your classmates about the strategy to solve the problems, but each of you should come up with a full solution on your own. Your solutions will be graded based on efforts, as opposed to the correctness.

Problems

1. Prove that $7 + 2^{\frac{1}{3}}$ and $\sqrt{3} + \sqrt{5}$ are algebraic numbers.

2. For which positive integers n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 7x + 5$ in $[\mathbb{Z}/(n)][x]$? Verify your answer.

3. Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

4. Let $\phi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ be the homomorphism that sends $x \mapsto t + 1$ and $y \mapsto t^3 - 1$. Determine the kernel K of ϕ, and prove that every ideal I of $\mathbb{C}[x, y]$ that contains K can be generated by two elements.

5. An automorphism of a ring R is an isomorphism from R to itself. Let R be a ring, and let $f(y)$ be a polynomial in one variable with coefficients in R. Prove that the map $R[x, y] \to R[x, y]$ defined by $x \mapsto x + f(y)$, $y \mapsto y$ is an automorphism of $R[x, y]$.

6. Let I and J be ideals of a ring R.
   1) Prove that the set $I + J$ of elements of the form $x + y$, with $x \in I$ and $y \in J$, is an ideal. This ideal is called the sum of the ideals I and J.
   2) Prove that the intersection $I \cap J$ is an ideal.

7. Let I and J be ideals of a ring R.
   1) Show by example that the set of products $\{xy | x \in I, Y \in J\}$ need not be an ideal.
   2) Prove that the set of finite sums $\sum X_v Y_v$ of products of elements of I and J is an ideal. This ideal is called the product ideal, and is denoted by IJ.

8. Identify the following rings:
   1) $\mathbb{Z}[x]/(x^2 - 3, 2x + 4)$
   2) $\mathbb{Z}[i]/(2 + i)$.

9. Determine the structure of the ring R' obtained from $\mathbb{Z}$ by adjoining an element a satisfying each set of relations.
1) $2\alpha = 6$, $6\alpha = 15$,
2) $\alpha^3 + \alpha^2 + 1 = 0$, $\alpha^2 + \alpha = 0$.

10. Let $\phi : \mathbb{R}[x] \to \mathbb{C} \times \mathbb{C}$ be the homomorphism defined by $\phi(x) = (1, i)$ and $\phi(r) = (r, r)$ for $r \in \mathbb{R}$. Determine the kernel and the image of ϕ.

11. Let I and J be ideals of a ring R such that $I + J = R$.
1) Prove that $IJ = I \cap J$.
2) Prove the Chinese Remainder Theorem: For any pair a, b of elements of R, there is an element x such that $x \equiv a \text{modulo} I$ and $x \equiv b \text{modulo} J$. (The notation $x \equiv a \text{modulo} I$ means $x - a \in I$.)
3) Prove that if $IJ = 0$, then R is isomorphic to the product ring $(R/l) \times (R/J)$.

12. Let R be a domain. Prove that the polynomial ring $R[x]$ is a domain, and identify the units in $R[x]$.

13. Which principal ideals in $\mathbb{Z}[x]$ are maximal ideals?

14. Factor the following polynomials into irreducible factors in $\mathbb{F}_p[x]$.
1) $x^3 + x^2 + x + 1$, $p = 2$,
2) $x^2 - 3x - 3$, $p = 5$.

15. Prove that the following rings are Euclidean domains.
1) $\mathbb{Z}[w]$, $w = e^{2\pi/3}$,
2) $\mathbb{Z}[\sqrt{2}]$.

16. Prove that two integer polynomials are relatively prime elements of $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains an integer.

17. Factor $x^3 + x + 1$ in $\mathbb{F}_p[x]$, when $p = 2, 3$, and 5.

18. Let R be the ring $Z[\sqrt{-3}]$. Prove that an integer prime p is a prime element of R if and only if the polynomial $x^2 + 3$ is irreducible in $\mathbb{F}_p[x]$.

19. Prove that the integers in $\mathbb{Q}[\sqrt{d}]$ form a ring.

20. Let $R = \mathbb{Z}[\sqrt{-5}]$.
1) Decide whether or not 11 is an irreducible element of R and whether or not (11) is a prime ideal of R.
2) Factor the principal ideal (14) into prime ideals in $\mathbb{Z}[\sqrt{-5}]$.

21. Let $f = y^2 - x^3 - x$. Is the ring $\mathbb{C}[x, y]/(f)$ an integral domain?