Problem Set 2: p-Group, Field and Vector Space

Due Date: October 23 in class.
Name: ________________________
Date: ________________________

Instructions

Write down your solutions on A4 papers and hand them in by the due date. You may discuss with your classmates about the strategy to solve the problems, but each of you should come up with a full solution on your own. Your solutions will be graded based on efforts, as opposed to the correctness.

Problems

1. Rule out as many of the following as possible as Class Equations for a group of order 10: $1+1+1+2+5$, $1+2+2+5$, $1+2+3+4$, $1+1+2+2+2+2$.

2. Determine the class equation for the dihedral groups D_5 and D_6.

3. List all subgroups of the dihedral group D_4, and divide them into conjugacy classes.

4. Prove or disprove: A_5 is the only proper normal subgroup of S_5.

5. Let H be a subgroup of a group G. Prove or disprove: The normalizer $N(H)$ is a normal subgroup of the group G.

6. Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime dividing $|G|$. Prove that H is in the center $Z(G)$.

7. Let K be a normal subgroup of order 2 of a group G, and let $\bar{G} = G/K$. Let $\bar{C}$ be a conjugacy class in $\bar{G}$. Let S be the inverse image of $\bar{C}$ in G. Prove that one of the following two cases occurs.
(a) $S = C$ is a single conjugacy class and $|C| = 2|\bar{C}|$.
(b) $S = C_1 \cup C_2$ is made up of two conjugacy classes and $|C_1| = |C_2| = |C|$.

8. How many elements of order 5 are contained in a group of order 20?

9. Prove that no group of order pq, where p and q are prime, is simple.

10. Let G be a group of order $p^e m$. Prove that G contains a subgroup of order p^r for every integer $r \leq e$.

11. Prove that if G has order $n = p^e a$ where $1 \leq a < p$ and $e \geq 1$, then G has a nontrivial proper normal subgroup unless $n = p$.

12. Classify groups of order 33.

13. Determine the Class Equation and list the conjugacy classes for A_4.

14. Prove that the symmetric group S_n is generated by the cycles $(1\,2\,\cdots\,n)$ and $(1\,2)$.

15. Prove that the set of numbers of the form $a+b\sqrt{2}$, where a, b are rational numbers, is a field.

16. Solve completely the systems of linear equations $AX = B$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ in the following two cases:

    1) $B = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and the equation holds in $\mathbb{F}_7$.

    2) $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and the equation holds in $\mathbb{F}_3$.

17. Find a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$. (Hint: You might want to solve the next problem first.)

18. Compute the order of $SL_2(\mathbb{F}_p)$.

19. Let F be a field which is not of characteristic 2, and let $x^2+bx+c=0$ be a quadratic equation with coefficients in F. Assume that the discriminant b^2-4c is a square in F, that is, that there is an element $\delta \in F$ such that $\delta^2 = b^2-4c$. Prove that the quadratic formula $x = (-b+\delta)/2$ solves the quadratic equation in F, and that if the discriminant is not a square the polynomial has no root in F.