In-Class Quiz 2 (20 min, 10 points)

Name: ________________________
Stud ID No.: ________________________

1. Let R be an integral domain that contains a field F as subring and that is finite-dimensional when viewed as vector space over F. Prove that R is a field. (5pts)

   Suppse that $\dim_F(R) = n < \infty$. For any nonzero element $x \in R$, we have that $\{1, x^1, x^2, \cdots, x^n\}$ are linearly dependent, ie, there exists $f_0, \cdots, f_{n-1} \in F$, such that $\sum_{i=0}^{nn} f_i x^i = 0$. There is a minimal $j < n$ such that f_j is nonzero. If $j \neq 0$, then the above expression is divisible by x^j. Since R is an integral domain, division preserves the equality. So we have that $\sum_{i=j}^{n} f_i x^{i-j} = 0$, which is $1 + x(\sum_{i=j+1}^{n} f_j^{-1} f_i x^{n-j-1}) = 0$. So we have found that $x^{-1} = -\sum_{i=j+1}^{n} f_j^{-1} f_i x^{n-j-1} \in R$. So R is a field. □

2. Is $\mathbb{Z}/(8)$ isomorphic to the product ring $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$? (2pts) Briefly verify your answer. (3pts)

   Checking the additive group structure would lead to a negative answer. Suppose there is an isomorphism ϕ, it must send $1 \in \mathbb{Z}/(8)$ to $(1, 1) \in \mathbb{Z}/(2) \times \mathbb{Z}/(4)$. But then $\phi(1 + 1 + 1 + 1) = \phi(4) = 0$, so ϕ has nontrivial kernel. □