In-Class Quiz 1 (20 min, 10 points)

Name: ____________________

Stud ID No.: ____________________

1. Right or wrong: A group of even order must contain an element of order 2?(3pt)

   Ans: Right. Otherwise, suppose a group G of order $2n$ has no element of order 2, ie, $\forall g \neq 1 \in G$, $g \neq g^{-1}$. Consider the pairs $\{g, g^{-1}\} \subseteq G$: if $g = g^{-1}$, then the set $\{g, g^{-1}\}$ has cardinality 1, otherwise $|\{g, g^{-1}\}| = 2$. Since there are $2n-1$ elements in G which are not unit, there has to be at least one set $\{g, g^{-1}\}$ which consists of only one element. But then $g = g^{-1}$ and $g \neq 1$, which is an order 2 element. That leads to a contraduction.

2. Let H, K be subgroups of a group G, and let $g \in G$. The set $HgK = \{x \in G | x = hgk \text{ for some } h \in H, k \in K\}$ is called a double coset.
   (a) Prove that the double cosets give a partition of G. (4pt)

   Define a relation R on G: $(g, g') \in R \subseteq G \times G$ iff $\exists h \in H$, $k \in K$ such that $g' = hgk'$. The students should check that this is an equivalence relation.

   Claim: $\forall x, y \in G$ are in the same double coset iff $(x, y) \in R$. Proof skipped.

   Now from the correspondence between equivalence relation and the partition, double cosets gives a partition for G.

   Alternatively, one can prove that 1) $\cup_{g \in G} HgK = G$ and 2) if $HgK \cap Hg'K \neq \emptyset$, then $HgK = Hg'K$.

   For 1), obviously $\cup_{g \in G} HgK \subseteq G$. Since $1 \in H$ and $1 \in K$, $G \subseteq \cup_{g \in G} HgK$. For 2), suppose $\exists h, h' \in H$ and $k, k' \in K$ such that $hgk = h'g'k'$, then $g = h^{-1}h'g'k'k^{-1}$, so $HgK \subseteq Hg'K$. Similar $g' = h'^{-1}hgkk'^{-1}$, so $Hg'K \subseteq HgK$. This completes the proof.

   (b) Do all double cosets have the same order? Verify your answer. (3pt)

   Not necessarily. Consider the case $G = S_3 =< x, y | y^2 = x^3 = 1, x^2y = yx >$, $H = K = \{xy, 1\}$, so we have that $H1K = \{1, xy\}$, and $HyK = \{y, xyy1 = x, 1yxy = x^2, xyyxy = x^2y\}$.