Problem Set 1: 2.1-2.12

Due Date: September 28 Monday before class, 2020.
Name: ______________________
Date: ______________________

Instructions

Write down your solutions on A4 papers and hand them in by the due date. You may discuss with your classmates about the strategy to solve the problems, but each of you should come up with a full solution on your own. Your solutions will be graded based on efforts, as opposed to the correctness.

Problems

1. Prove that $GL_n(\mathbb{R})$ is a group.

2. Prove that a nonempty subset H of a group G is a subgroup if for all $x, y \in H$ the element xy^{-1} is also in H.

3. Prove that in any group G the orders of ab and of ba are equal, $\forall a, b \in G$.

4. Find all possible homomorphisms $C_{12} \to C_{30}$.

5. Show that the functions $f = \frac{1}{x}$, $g = \frac{x-1}{x}$ generate a group of functions, the law of composition being composition of functions, which is isomorphic to the symmetric group S_3.

6. Let H be a subgroup of G, and let $g \in G$. The conjugate subgroup gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, where $h \in H$. Prove that gHg^{-1} is a subgroup of G.

7. Prove that G is an Abelian group iff $G \to G : g \mapsto g^{-1}$ is an isomorphism.

8. Consider the set U of real 3×3 matrices of the form $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

   1) Prove that U is a subgroup of $SL_3(\mathbb{R})$.
   2) **Prove or disprove** U is a normal subgroup of $SL_3(\mathbb{R})$.

9. Let H, K be subgroups of a group G of orders 3, 5 respectively. Prove that $H \cap K = \{1\}$.

10. Prove that a group of order 30 can have at most 7 subgroups of order 5.

11. Let G be a finite group whose order is a product of two integers: $n = ab$. Let H, K be subgroups of G of orders a and b respectively. Assume that $H \cap K = \{1\}$.
    1) Prove that $HK = G$.
    2) Is G isomorphic to the product group $H \times K$?

12. Prove that the subset H of $GL_n(\mathbb{R})$ of matrices whose determinant is positive forms a normal subgroup, and describe the quotient group $GL_n(\mathbb{R})/H$.

13. Determine the automorphism groups of C_4 and $C_2 \times C_2$.

14. Determine the automorphism groups of $\mathbb{Z}^+$ and $\mathbb{Q}^+$.

15. Compute the partition on S_3 by the conjugation relation.