Problem Set 7

Due Date: July 6, 2018 in class.
Name: ________________________
Date: ________________________

Instructions

Answer all the questions in the Homework part and hand in your solutions by the due date. You are strongly encouraged to try the exercise part as well.

Homework

1. Determine whether each of these statements is true or false.
   a) $x \in \{x\}$; b) $\{x\} \subseteq \{x\}$; c) $\{x\} \in \{x\}$; d) $\{x\} \subseteq \{\{x\}\}$;
   e) $\emptyset \subseteq \{x\}$; f) $\emptyset \in \{x\}$.

2. Suppose that we want to prove that
   $$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$
   for all positive integers n. a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails. b) Show that mathematical induction can be used to prove the stronger inequality
   $$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$
   for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

3. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?
   a) a_0^2, $2a_0a_1$, $a_1^2+2a_0a_2$, $2a_0a_3+2a_1a_2$, $2a_0a_4+2a_1a_3+a_2^2, \cdots$; b) a_0, a_0+a_1, $a_0+a_1+a_2$, $a_0+a_1+a_2+a_3, \cdots$.

4. Given two finite sets S and T. Let $\mathrm{Hom}(S,T)$ be the set of all morphisms/maps from S to T. What is the cardinality (the number of elements) of $\mathrm{Hom}(S,T)$?

Exercise

1. Let $A_i = \{1, 2, 3, \cdots, i\}$ for $i = 1, 2, 3, \cdots$ Find a) $\cap_{i=1}^{n} A_i$ b) $\cup_{i=1}^{n} A_i$.

2. Count (exactly) the execution number of line 5 in each of the following algorithms.

(a)
```
i = 1
sum = 0
while i < n
    for j = 1 to i
        sum = sum + a[i,j]
    i = i + 1
```

(b)
```
sum = 0
for i = 1 to n
    j = 1
    while j < i
        sum = sum + a[i,j]
        j = j + 1
```

(c)
```
sum = 0
i = 1
j = 1
    while i + j < n
        sum = sum + a[i,j]
        j = j + 1
        i = i + 1
```

3. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \cdots$ if a) $a_n = 6n$; b) $a_n = 2^n + 1$; c) $a_n = n^2$; d) $a_n = 5$.

4. Solve the recursive relation $(n+1)a_{n+1} = a_n + \frac{1}{n!}$ for $n > 0$ and $a_0 = 1$ using the generating function method.