Sets, Logic and Boolean Algebra

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This is just a brief summary of the contents of the lecture. Please note: most of the calculations and demonstrations are neglected.

Set Theory

Definition 0.1 • *set, elements, universal set*

- *subset, proper subset, set equality*
- *cardinality, infinite set*
- *complement, intersection, union, difference*
- *covering, partition, Cartesian product*[1], *power set*

Fundamental properties of sets[2]:

Idempotence
$A \cup A = A$
$A \cap A = A$

Associativity
$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity
$A \cup B = B \cup A$
$A \cap B = B \cap A$

Distributivity ($\cap$ over $\cup$)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Complement
$A \cup \overline{A} = U$
$A \cap \overline{A} = \emptyset$

Involution
$\overline{\overline{A}} = A$

Domination
$A \cup U = U$
$A \cap \emptyset = \emptyset$

Identity
$A \cup \emptyset = A$
$A \cap U = A$

De Morgan's Laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Distributivity ($\cup$ over $\cap$)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Complement (continued)
$\overline{\emptyset} = U$
$\overline{U} = \emptyset$

Exercise 1 *Prove all the above statements.*

Exercise 2 *Work on the compatibility between Cartisian product and $\cap$, $\cup$ and $\overline{()}$.*

[1] Note that strictly speaking Cartesian product is not associative, due to the definition using ordered pairs; but in general you could assume the associativity using the canonical isomorphism.

[2] The notation for complement, $\overline{A}$ will be a bit confusing; alternatively we could use $\sim A$, or $\neg A$.

Propositional Logic

Definition 0.2 *A statement is an assertion that could be labeled true or false.*

Consider the set of statements S, as well as the set $\{\text{true} \equiv 1, \text{false} \equiv 0\}$, the valuation of the statement can be viewed as a map $S \to \{0, 1\}$.

The value is an incomplete description of the statements.

On the set of statements, we can have many operations. Operations are maps between (Cartesian product of) sets of statements. Examples are as follows.

Example 1 • *NOT* $\neg$*: "Roses are red."*$\mapsto$*"Roses are not red."*

- *AND* $\wedge$*:* $($*"Roses are red.", "Violets are blue."*$) \mapsto$ *"Roses are red, and violets are blue."*
- *OR* $\vee$*:* $($*"Roses are red.", "Violets are blue."*$) \mapsto$ *"Roses are red, or violets are blue."*

To understand the nature of those operations, we look into the truth table of those operations. Note that the tables are useful to distinguish different operations. I.e., if the trube tables are different, the corresponding operations are not equivalent.

Example 2 *The non-associativity* $A \wedge (B \vee C)$ *versus* $(A \wedge B) \vee C$*.*

Operations can act on operations to produce new ones. [3]

[3] You should probably consider functions on the reals, which compose to make new functions.

Example 3

Implication $\to$*:* $($*"Roses are red.", "Violets are blue."*$) \mapsto$ *"If Roses are red, violets are blue."*

Biconditonal $\leftrightarrow$*:* $($*"Roses are red.", "Violets are blue."*$) \mapsto$ *"Roses are red if and only violets are blue.".*

The truth table for operations imposes logical equivalence between two operations. Note: equivalence relation does not mean being identical — it is entirely possible to have two operations that give rise to different compound statements for the same input. Consider for example $(P \to Q) \wedge (Q \to P)$ vs $P \leftrightarrow Q$. We denote the logical equivalence by $\Leftrightarrow$.

There are operations whose every truth value is true, which we call tautologies. Likewise, there are operations whose every truth value is false, which we call contradictions.

Example 4 • $P \vee \neg P$

- $P \wedge \neg P$

Definition 0.3 *If for operations* A, B*, we have that* $(A \to B) \Leftrightarrow \mathbf{T}$*, we say that* A *infers* B*, and denote by* $A \Rightarrow B$*.*

Theorem 0.1 (The substitution principles) • *Substituting an Equivalent Statement: If $A \Leftrightarrow B$, and A is a component of an operation, C, then B may be substituted for A without changing the T/F value of C.*

- *Replacing a Logic Variable in a Tautology: If B is a logic variable in a tautology, C, and A is any operation, then A may be substituted for every occurrence of B in C and C will still be a tautology.*
- *Using a Rule of Inference: If $A \Rightarrow B$, A evaluates to* **T**, *and A is a component of a statement, C, then B may be substituted for A without changing the T/F value of C C.*

Exercise 3 • *Show that $[(R \vee P) \rightarrow (Q \vee Q)] \leftrightarrow [R \vee (P \rightarrow Q)]$ is tautological.*

- *Show that $(P \rightarrow Q) \Leftrightarrow [(P \wedge \neg Q) \rightarrow \neg P]$.*

Boolean algebra

Definition 0.4 *A Boolean algebra is a set B together with binary operations $+$ and $\cdot$ together with an unary operation $\neg$, such that the following axioms hold:*

- *identity: there exists two distinguished elements 0 and 1, such that $x + 0 = x$ and $x \cdot 1 = x$, for all $x \in B$.*
- *complement: $x + \neg x = 1$ and $x \cdot \neg x = 0$.*
- *commutativity*
- *distributivity*

Definition 0.5 *A boolean expression on a Boolean algebra $(B, +, \cdot, \neg)$ is an algebraic experession that is composed using elements from B, those operations, and variables whose possible values are in B.*[4]

[4] Compare the notion in p261 of the textbook.

Example 5 • $(\mathcal{S}, \cup, \cap, \sim)$.

- $(Propositions, \vee, \wedge, \neg)$.

Theorem 0.2 (Stone's representation theorem, finite case) *Every finite Boolean algebra A is isomorphic to the Boolean algebra $(\mathcal{P}(S), \cup, \cap, \overline{()})$ for some finite set S.*

The Boolean algebra in the propositional calculus is the minimum, nontrivial one.

Example 6 *Show that the Boolean algebra $(\{0, 1\}, +, \cdot, \neg)$ is isomorphic to $(\mathcal{P}(S), \cup, \cap, \sim)$ for some finite set S.*

The duality principle for Boolean algebras holds: Let T be a theorem that is valid over a Boolean algebra. Then if all Os and 1s are exchanged (with a suitable change in parentheses), and if all $+$ and $\cdot$ are exchanged, the result is also a theorem that is valid over the Boolean algebra.

Syllogism Logic