Problem Set 4

Due Date: May 4, 2018 in class.
Name: ______________________
Date: ______________________

Instructions

Answer all the questions in the Homework part and hand in your solutions by the due date. You are strongly encouraged to try the exercise part as well.

Homework

1. Prove that $p(n) \leq \frac{p(n-1)+p(n+1)}{2}$. (Hints: Write the inequality as $p(n+1) - p(n) > p(n) - p(n-1)$. Interpret $p(n+1) - p(n)$ as the number of partitions of $n+1$ that do not contain a "1" as a summand. (You need to give adequate justification for this assertion.) Create a similar interpretation for $p(n) - p(n-1)$.)

2. Express the polynomial $2x^5 - 4x^4 + x^3 + 7x^2 - 9x - 9$ as a linear combination of falling factorials.

3. Determine if the following pairs of Latin squares are orthogonal.

   (1) $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

   (2) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$.

4. How many points and lines are in a finite projective plane of order 3? How many points are on each line? How many lines contain each point? Draw a diagram of a finite projective plane of order 3.

Exercise

1. Show that $p(n,k)$ satisfies the recursive relation $p(n,k) = \sum_{i=0}^{k} p(n-k,i)$ for $0 < k \leq n$, with base conditions $p(n,1) = 1$ and $p(n,k) = 0$ if $k > n$.

2. Define $q(n,k)$ to be the number of partitions of n for which the largest summand is exactly k. Show that $q(n,k) = \sum_{i=0}^{k} q(n-k,i)$ for $0 < k \leq n$, with base conditions $q(n,1) = 1$ and $q(n,k) = 0$ if $k > n$. Now combining the result of the previous problem, conclude that $p(n,k)$ also represents the number of partitions of n for which the largest summand is exactly k.

3. Let $T(n,k)$ be the number of ways to partition a set of n elements into exactly k nonempty, ordered subsets. Prove that $T(n,k) = \sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) (k-i)^n$ for $0 < k \leq n$. (Hints: There are k^n ways to create the partitions if empty subsets are allowed. This count contains illegal partitions, so subtract the number of partitions with at least one empty subset. Now adjust for oversubtracting in the previous step, etc.)

4. Prove that $T(n,k) = k!S(n,k)$. Then use the result of the previous problem to show that $S(n,k) = \sum_{i=0}^{k} \frac{(-1)^i}{k!} \left(\begin{array}{c} k \\ i \end{array} \right) (k-i)^n$ for $0 < k \leq n$.

5. What is the numeric coefficient of $x(x-1)(x-2)(x-3)(x-4)$ when x^6 is written as a linear combination of falling factorials?

6. What is the numeric coefficient of x^5 when $x(x-1)(x-2)(x-3)(x-4)$ is written as a linear combination of powers of x?

7. A Latin square is called self-orthogonal if it is orthogonal to its transpose. Prove that there are no self-orthogonal Latin squares of order 3.

8. Prove that $FPP1$, $FPP2$, and $FPP3'$ imply $FPP3$.