Generalized Vanna-Volga Method and Its Applications

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Abstract

We give a general treatment of the Vanna-Volga mark-to-market volatility smile correction in application to pricing of contracts with European exercise on a single underlying. The method remains applicable in cases of delayed or misaligned expiries and absolute dividends. It is also applied to cases of time-dependent instantaneous volatility, multiple underlying assets and random interest rates. We also offer computation of the underlying volatility from market data and most valuable correction using more than three traded options.

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1 Introduction

The Vanna-Volga (VV) approach is a group of methods that consider an option price as a Black-Scholes (BS) price corrected by hedging costs caused by stochasticity in price-forming factors (volatility, interest rate, etc.) observed from real markets. While accounting for stochasticity in volatility, it differs from stochastic volatility models (Heston, SABR) in the following way. Rather than considering a separate process for the instantaneous volatility and defining the price as the risk-neutral expectation, the VV puts much more weight on hedging. An option price is treated as a value of the replicating BS portfolio plus corrections offsetting randomness in volatility. At each moment of time the underlying is approximated by a BS process with estimated parameters (volatility). Viewed as diffusion processes themselves, these estimates add extra terms into the Itô diffferential of an option price. The goal of hedging is offsetting these extra Itô terms. The VV achieves this goal by including into the portfolio listed options traded in real markets. Positions of these options are found from equations zeroing out the difference between the BS price of the target option and listed options within each additional Itô term. As will be shown below, this formulation of VV results in a widely applicable option pricing framework, accounting for implied volatility strike-dependence (smile, skew).

The systematic formulation of Vanna-Volga, assuming volatility a random factor, had been given in the Castagna-Mercurio's paper [3]. In earlier publications ([7] by Lipton, McGee or [10]by Wystup) paper used VV as a common sense adjustment for FX barriers.

Following ideas of [3], we introduce a random variable of fair value implied volatility as an estimate of the underlying volatility obtained under an assumption that the underlying is identified wth a standard BS process with a constant volatility. The fair value volatility makes the Itô expansion with an additional random factor in volatility, proposed in ref. to [3], much more natural. The usage of the VV is generalized from the standard case of OTC BF/RR/ATM pivot option quotes to a general situation of given three or more listed pivot options in FX, equity or commodity. The method is also considered in other contexts. It is extended to the case of time-dependent instantaneous volatility with a random factor. It is considered together with stochasticity in interest rate (two rates, foreign and domestic, for FX). Another extension is applied to multiple underlyings.

The content is organized as follows. Section 2 gives a definition of the fair value implied volatility and derives the three-pivot VV correction from the Itô expansion identically to [3] in a way applicable to an option of any type. Section 3 gives fast explicit formulas for coefficients x_i in case of any target ([3] gives formulas for a vanilla only) and greeks. It also lists major limitations. Section 4 considers an instantaneous fair value volatility time-curve (required to price path-dependent exotics) as a function of a deterministic time-curve and random factor. Only additive and multiplicative cases, agreeing with most common parallel and proportional shift of the deterministic time-curve, required to compute volatility greeks numerically, are considered here. It is definitely possible to consider frameworks with more elaborate shifting algorithms. Section 5 gives a reasonable (but not the only possible) way of computing a fair

value volatility from prices of listed European options. It additionally incorporates user-definite weights or computes these weights from available from markets traded volumes at each strike. Section 6 presents a multi-asset application of the VV meant for equity baskets, spreads, indices, options foreign currencies. Section 7 applies the VV to options with random factors in both volatility and interest rate (both foreign and domestic rates for FX). Section 8 considers the correction using 3 linear combinations of more than 3 listed options.

2 Generalized one-dimensional approach

This section shows arguments from [3] may be modified extended to contracts of other types. Here and below, any considered underlying S_t will be an equity, FX, or commodity symbol, on which an option contract, maturing at time $t = T > 0$ and exercised in the European way, should be priced at time $t = 0$. The underlying is assumed observable at any time t together with European options on this underlying expiring at time $T > t$ with no other information about S available at the same time. For each t we will assume S to be a standard BS log-normal diffusion process at $t \leq \tau < T$ with volatility σ_t at each $\tau \geq t$. Variable σ_t is therefore a real time *estimate* of the underlying (not an option contract) implied volatility (if it makes any sense at al) at on $[t, T]$ under a BS log-normal assumption to the best of our knowledge (since no other information is available). Now it becomes natural to treat σ_t as a random variable, obtained from market information available at t. σ_t will be called a **fair value implied volatility**. It may be obtained from European options statistically, see 5, or believed to be an ATM volatility or even something else. If options at more than one expiry times T_i are available, more than one fair value implied volatilities σ_i will be available at t. In this case a raw discrete time-dependent instantaneous fair value volatility curve may be computed, see 4

Additionally σ_t will also be assumed a log-normal diffusion process. We will also fix three listed "pivot" European vanilla calls (puts) C_i at strikes K_i, maturing at T or later, and traded openly at public markets with quotes available at any time. The main goal will be to find the market price O_{Mkt} of an arbitrary option contract O, with European exercise at T as its Black-Scholes price O_{BS} computed from the current "fair value" implied volatility, plus a correction consistently accounting for a possible dependence of implied volatility of pivot market prices C_i^{Mkt} from strikes K_i (smile, skew).

Following [3], we will assume that the market follows the standard BS assumption with a random but strike-independent implied volatility, which leads to the assumption from [3] that option prices follow the BS PDE (with a random volatility). We will try to replicate O_{BS} by forming a risk-neutral portfolio Π^{BS} of 1 unit of O, Δ_t units of S in counter-position, and x_i units of each pivot in counter-position. The change $d\Pi^{BS}$ of the value of this portfolio in a small time dt may be expressed using the Itô expansion:

$$dΠ^{BS} = dO^{BS}(t) - Δ_t dS_t - \sum_{i=1}^{3} x_i dC_i^{BS} = \left[\frac{\partial O^{BS}}{\partial S_t} - Δ_t - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial S_t} \right] dS_t$$

$$+ \left[\left(\frac{\partial O^{BS}}{\partial t} - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial t} \right) + \frac{1}{2} σ^2 S^2 \left(\frac{\partial^2 O^{BS}}{\partial S^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial S^2} \right) \right] dt$$

$$+ \left[\frac{\partial O^{BS}}{\partial σ_t} - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial σ_t} \right] dσ_t + \frac{1}{2} \left[\frac{\partial^2 O^{BS}}{\partial σ_t^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial σ_t^2} \right] (dσ_t)^2$$

$$+ \left[\frac{\partial^2 O^{BS}}{\partial S_t \partial σ_t} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial S_t \partial σ_t} \right] dS_t dσ_t. \tag{1}$$

Choosing $\mathbf{x} = (x_1, x_2, x_3)$ to zero out the last three summands and, with $\mathbf{x}$ known and matching $Δ = \frac{\partial U}{\partial S}(t, S_t)$, $U = O - \sum_{i=1}^{3} x_i C_i$, to zero out the first summand, leaves only the dt summand. The assumption that prices of both O^{BS} and C_i^{BS} follow the BS equation immediately results in the self-financing (perfect hedge) property $dΠ^{BS} = rΠ^{BS} dt$. Since the BS arguments hold for any instrument O of S (see [9]), the procedure above is applicable to any instrument O, not just European vanillas (the only option type considered in [3]). The mixed and repeated zeroed out second derivatives are called $Vanna$ and $Volga$, respectively. The coefficient vector $\mathbf{x}$ may be found from

$$\mathbf{w}_O = \mathbf{V}\mathbf{x} \tag{2}$$

where

$$\mathbf{w}_O = \begin{pmatrix} v_O \\ Vanna_O \\ Volga_O \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_1 & v_2 & v_3 \\ Vanna_1 & Vanna_2 & Vanna_3 \\ Volga_1 & Volga_2 & Volga_3 \end{pmatrix}.$$

As stated in [3], for European vanillas with vega zeroed out, the second order term in the dt summand of (1) also vanishes. While this does not hold for a general contract O, it should be fortunately mentioned that for BS standard hedging arguments (always accounting for both terms in dt) it is irrelevant. At the same time, it should be specially stressed that, differently from [3], x should be seen as a function of time providing a local, dynamic (rather than in [3] global, static) hedge for portfolio $Π$. Everything may be briefly summarized within

Proposition 1 *Under the assumption for the underlying S to follow a BS process with a stochastic but strike-independent implied volatility, any option contract O expiring at T may be locally (dynamically) perfectly hedged by a portfolio of one unit of O, $Δ$ units of the underlying, and x_i units of pivot $C_i, i = 1, \ldots, 3$ expiring at T or later, where $\mathbf{x}$ may be found from (2). Due to the put-call parity calls any replacement of call C_i with put P_i at pivot strike K_i changes $Δ$ but does not change x_i.*

The vector $\mathbf{x}$ just found, and the dynamic hedging argument above, will now be applied to give a consistent formulation of a corrected market price. Differently from the original argument (see p. 108 of [3]), inapplicable to a general contract O, our alternative arguments

will be based on the concept of self-financing, see for example [5]. It is well known that, with the underlying S following standard BS assumptions with constant (or dependent on time only) implied volatility, the risk-neutral price of any contract O with European exercise may be replicated by a self-financing portfolio. Not having enough information from real markets to construct a risk-neutral measure and define a truly risk-neutral price, we will still attempt to form a self-financing portfolio of one unit of O, given pivots and the underlying based instead on the following consistency requirement: **In the case of the underlying following the BS assumption with stochastic-in-time but strike-independent implied volatility, the portfolio will degenerate to the Vanna-Volga portfolio from Proposition 1**.

We will choose an ample partition of $[0, T]$ into $0 = t_0 < \ldots < t_n = T$ with sufficiently small δt and follow the backward induction. The consistency requirement immediately implies that, at each time t_j, we choose vector $\mathbf{x_j}$ of pivot positions within the portfolio to have

$$\Pi^{Mkt} = O^{Mkt} - \Delta^{Mkt} dS - \sum_{i=1}^{3} x_i C_i^{Mkt}.$$

At time $t_n = T$, clearly $O^{Mkt} = O^{BS} = P_O$ (where P_O is the payoff for O of any complexity—exotic, time-dependent, for example) and $O^{BS} = 0$ at any time $t > T$, immediately meaning that $\mathbf{x_n} = \mathbf{0}$. With Δ chosen as above for both portfolios, we immediately get $\Pi^{Mkt} = \Pi^{BS}$ at $t = T$, even with $C_i^{Mkt} \neq C_i^{BS}$ (each C_i may last far beyond T and expiries of C_i may even be misaligned). Now, with established $\Pi^{Mkt} = \Pi^{BS}$ at each $t \geq t_{j+1}$, we consider the self-financing Π^{Mkt} we are looking for and Π^{BS}, self-financing under the stochastic BS settings as was found above. Self-financing exactly means

$$\delta \Pi^{Mkt}(t_j) = r \delta t \Pi^{Mkt}(t_j) + o(\delta t),$$

$$\delta \Pi^{BS}(t_j) = r \delta t \Pi^{BS}(t_j) + o(\delta t).$$

Since $\Pi^{Mkt}(t_{j+1}) = \Pi^{BS}(t_{j+1})$, we get

$$(\Pi^{Mkt} - \Pi^{BS})(t_j) = r \delta t (\Pi^{Mkt} - \Pi^{BS})(t_j) + o(\delta t)$$

or better to write

$$(1 - rdt)(\Pi^{Mkt} - \Pi^{BS})(t_j) = 0$$

or

$$(\Pi^{Mkt} - \Pi^{BS})(t_j) = 0.$$

With U and Δ defined above, and $U^{diff} = U^{Mkt} - U^{BS}$, it is just

$$(U^{diff} - \frac{\partial U^{diff}}{\partial S} S)(t_j) = 0,$$

meaning that $U^{diff} = C(t_j)S_{t_j}$. Since always $S > 0$ and $U^{diff}(t_{j+1}) = C(t_j)S_{t_{j+1}} = 0$, we get $C(t_j) = 0$, consequently

$$(U^{Mkt} - U^{BS})(t_j) = U^{diff}(t_j) = 0$$

and (again recalling the definition of Δ) finally,

$$\Pi^{Mkt}(t_j) = \Pi^{BS}(t_j).$$

We proved

Proposition 2 *Under the consistency requirement above, there is a unique self-financing portfolio* $\Pi^{Mkt} = O^{Mkt} - \Delta^{Mkt}dS - \sum\limits_{i=1}^{3} x_i C_i^{Mkt}$, *satisfying the property* $\Pi^{Mkt} = \Pi^{BS}$ *at any* $0 \le t \le T$, *immediately resulting in*

$$\mathbf{O^{Mkt} = O^{BS} + \sum_{i=1}^{3} x_i(C^{Mkt} - C^{BS})} \tag{3}$$

for any contract O. *Coefficients* $\mathbf{x}$ *are determined from (2) and depend on t. Under a non-stochastic interest rate at each strike* K_i *the equality*

$$C^{Mkt} - C^{BS} = P^{Mkt} - P^{BS}, \tag{4}$$

immediately following from the put call parity (even in case of absolute dividends), indicates that calls and put pivots may be used interchangeably.

The expression

$$O^{VV} = \sum_{i=1}^{3} x_i(C^{Mkt} - C^{BS})$$

will be called the Vanna-Volga (VV) correction. It is equal to a linear combination of mark-to-markets (MTM) of three given pivots (where mark-to-market of an option contract is defined as the difference between the real market price and modelled price). The major result proved in Proposition 2 coincides with (7) from [3], but was obtained using different arguments, extensible to contracts O of any complexity.

The explicit expression from [3] of O^{Mkt} through υ, $Vanna$ and $Volga$ (indicating the non-linearity of the correction) also holds for any contract O:

$$\begin{aligned} O^{Mkt} =\quad & O^{BS} + y_\upsilon \upsilon + y_{Vanna}Vanna + y_{Volga}Volga, \\ \mathbf{y} =\quad & \mathbf{m^t V^{-1}}, \quad \mathbf{m} = \mathbf{C^{Mkt} - C^{BS}}. \end{aligned} \tag{5}$$

3 Practical aspects

This section gives explicit formulas for x_i in case of any target (not just European vanilla), simplified formulas for greeks and considers other practical issues. In the default case of relative (proportional) dividends, v, $Vanna$, and $Volga$ for European pivots at $K_p = K_1, K_2, K_3$, expiring at $T_p = T_1, T_2, T_3$, may be expressed in a standard way,

$$
\begin{aligned}
v &= Se^{-(r-d)T_p}\sqrt{T_p}\, n_{(0,1)}(d_1(K_p)), \\
Volga &= \frac{v}{\sigma}d_1 d_2, \\
Vanna &= -\frac{v}{S\sigma\sqrt{T_p}}d_2,
\end{aligned}
\tag{6}
$$

which, in the case of aligned pivot expiries $T_p = T_1 = T_2 = T_3$, results in an efficient expression for $\mathbf{x}$:

$$
\begin{aligned}
\hat{x}_3 &= \frac{\sigma^2 T_p}{\ln\frac{K_3}{K_2}\ln\frac{K_3}{K_1}}\left[\sigma Volga_O + d_2(K_1)\,d_2(K_2)\,v_O + S\sigma\sqrt{T_p}Vanna_O\,(d_1(K_1)+d_2(K_2))\right]; \\
\hat{x}_2 &= -\frac{\sigma^2 T_p}{\ln\frac{K_3}{K_2}\ln\frac{K_2}{K_1}}\left[\sigma Volga_O + d_2(K_1)\,d_2(K_3)\,v_O + S\sigma\sqrt{T_p}Vanna_O\,(d_1(K_1)+d_2(K_3))\right]; \\
\hat{x}_1 &= v_O - (\hat{x}_2 + \hat{x}_3); \\
x_i &= \hat{x}_i/v_i, \ \ i = 1,2,3,
\end{aligned}
\tag{7}
$$

which remains valid even if O expires at $T < T_p$. If O is a European vanilla with strike $K \neq K_i$ expiring at $T = T_p$, the expression may be further simplified into the expression identical to (6) from [3]:

$$
\hat{x}_1 = \frac{v_O \ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}; \quad \hat{x}_2 = \frac{v_O \ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}; \quad \hat{x}_3 = \frac{v_O \ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}; \quad x_i = \hat{x}_i/v_i.
\tag{8}
$$

For a portfolio of options on the same underlying, it may be easily seen that while $\mathbf{x}$ should be computed separately for each option, $\mathbf{V}$ does not have to be recomputed for each option.

It is a common practice in FX to consider OTC Butterfly/Risk Reversal/ATM quotes instead of market pivots with the fair value implied volatility chosen to be σ_{ATM}. Details of recomputation standard for ATM, $25\%\Delta$ and $-25\%\Delta$ call or put pivots from OTC BF/RR/ATM quotes are available from [4].

If pivot expiries are misaligned, the "fair value" implied volatility σ is considered on the interval $[0, T_p]$, where $T_p = \max(T_i, T)$. While standard expressions (6) still hold separately, formulas of type (7) cannot be obtained and $\mathbf{x}$ should be computed by solving (2).

In the case of absolute dividends, there are no formulas of type (6) for v, $Vanna$, and $Volga$ even for European pivots, and $\mathbf{x}$ should also be computed by solving (2). In this case, since

there is no analytical price even for a European vanilla, precomputing $\mathbf{V}$ for a portfolio saves time drastically.

Greeks may be computed by differentiating (3):

$$
\begin{aligned}
O^{Mkt'} &= O^{BS'} + \sum_{i=1}^{3} x_i(C_i^{Mkt'} - C_i^{BS'}) + \sum_{i=1}^{3} x_i'(C_i^{Mkt} - C_i^{BS}); \\
O^{Mkt''} &= O^{BS''} + \sum_{i=1}^{3} x_i(C_i^{Mkt''} - C_i^{BS''}) \\
&\quad + \sum_{i=1}^{3} x_i'(C_i^{Mkt'} - C_i^{BS'}) + \sum_{i=1}^{3} x_i''(C_i^{Mkt} - C_i^{BS}),
\end{aligned}
\tag{9}
$$

where the first derivative means any first order greek $(\theta, \upsilon, \delta, \rho, dye^1, kite^2)$, and the second derivative means any second order greek $(\gamma, kiteGamma^3)$. All derivatives (greeks) of pivots C^{BS} may be computed from the standard BS formula. Derivatives of O^{BS} should be served by the mathematical routine computing O^{BS} and are not always available in an analytical form.

Because of an insufficiency of market information about C_i^{Mkt} for computing their derivatives, the following approximation of pivot derivatives is used instead:

- Implied volatility is computed at each strike K_i.

- The volatility just computed is re-applied to compute greeks using the standard BS formula.

For a European vanilla, derivatives of $\mathbf{x}$ may be easily computed from (8). For a general contract O, they have to be computed by solving a linear system obtained by differentiation of (2):

$$
\mathbf{V}\mathbf{x}' = \mathbf{w}'_o - \mathbf{V}'\mathbf{x}; \quad \mathbf{V}\mathbf{x}'' = \mathbf{w}''_o - \mathbf{V}''\mathbf{x} - \mathbf{V}'\mathbf{x}'
\tag{10}
$$

where both derivatives of pivot-determined $\mathbf{V}$ are easily computible from the BS formula. Derivatives of $\mathbf{w}$ should be served by the mathematical routine computing O^{BS}. It should be mentioned that, even in the case of $\mathbf{x}$ being available analytically from (7), further differentiation for computing derivatives of $\mathbf{x}$ is tedious and (10) may be a better choice. In the case of discrete dividends, this is the only choice.

In the case that derivatives of $\mathbf{w}$ (as derivatives of even harder order of O^{BS}) are hard to compute within a routine computing O^{BS}, simplified greeks, obtained by ignoring both derivatives of $\mathbf{x}$, may be used instead:

$$
\begin{aligned}
O^{Mkt'} &= O^{BS'} + \sum_{i=1}^{3} x_i(C_i^{Mkt'} - C_i^{BS'}); \\
O^{Mkt''} &= O^{BS''} + \sum_{i=1}^{3} x_i(C_i^{Mkt''} - C_i^{BS''}).
\end{aligned}
\tag{11}
$$

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[1] sensitivity of the price to proportional dividend yield

[2] sensitivity of the price with respect to changes in strike

[3] convexity of the price with respect to changes in strike

They provide a reasonably good approximation of greeks under the following (usually acceptable) assumptions:

- The hedging time interval is very short $\delta t << T$.

- The volatility process compared to the underlying is noticeably slower.

It should be mentioned however that VV is not free of limitations. (3) does not guarantee the corrected price to be within logical bounds. Two following issues may be considered as major reasons:

- High deviation of implied volatilities from FVV. The market violates the previously made assumption that option prices follow the BS PDE, which would require the implied volatility curve to be near constant.

- A discontinuity in Vega, Vanna or Volga for some options. Typical examples are barriers, whose delta and consequently vanna become discontinuous at the barrier. It immediately follows from (3) that O^{Mkt} in this case also becomes discontinuous.

There is no universal solution following from the description of the method. Practitioners introduce empiric adjustments, specific to every option type. The choice of these adjustments is beyond the scope of this document. Recommendations for barriers can be found in [11], [1].

4 Applications to time-dependent volatility

This section applies Vanna-Volga to cases of time-dependent volatility represented as a sum or product of a deterministic function of time and random factor. In order to correct an option whose life span covers more than one pivot expiry, we have to drop the assumption of the "fair value" implied volatility to be constant. Even computing O^{BS} under standard BS conditions requires an instantaneous volatility time-curve, which is computed from the given implied time-curve. The VV may be still seemlessly applied using pivots with the closest later expiry T_p if the following assumption is made. Dealing with a raw discrete fair value time-curve implied from a real market, we will assume that at each given t and increasing expiries T_j, the fair value implied volatilities $\sigma(t, T_j)$ admit a commonly accepted representation $\sigma(t, T_j) = \sigma_{deterministic}(t, T_j) + \sigma_{stochastic}(t)$ where $\sigma_{stochastic}(t)$ is same for all T_j and depends on t only. In this case, for computing volatility greeks (if analytic computation is impossible or difficult), we give a parallel shift to the whole implied fair value volatility curve, recompute the perturbed instantaneous fair value volatility, reprice O and get a perturbed price value. The approximate values of volatility greeks may be computed by finite differencing. An alternative, multiplicative representation, $\sigma(t, T_j) = \sigma_{deterministic}(t, T_j)\sigma_{stochastic}(t)$, looking less common but more natural, is also possible. In this case, the implied fair value time-curve is perturbed proportionally. On the pivot side, v and $Vanna$ will be multiplied by $\sigma(t, T_j)$ and $Volga$ by $\sigma^2(t, T_j)$.

In this section, we will also try a more efficient alternative way of computing the instantaneous fair value curve once, and computing volatility greeks internally within the target by considering a stochastic factor within the time-dependent instantaneous (not implied) volatility curve. This way is also much more appropriate for Monte Carlo based pricers of O^{BS} because of its immediate integrability into on-path algorithms of greek computation.

Given fair value implied volatilities for multiple expiries, the raw "fair value" instantaneous volatility curve may be obtained by square differencing. If required, the instantaneous raw curve may later be approximated by a smooth function at each fixed $t < T$. For any $\tau \in (t, T]$, we will assume the instantaneous volatility to be a deterministic function of $\tau > t$ and random factor w following a log-normal BS diffusion process: $\sigma_t(\tau) = F(\tau, \omega_t)$, where ω_t is random in t but constant in $\tau \in (t, T]$. In the way it was done for σ_t in section (2), we will instead apply Vanna-Volga to ω_t and get

$$\widehat{\mathbf{w}_o} = \widehat{\mathbf{V}}\mathbf{x};$$

$$\widehat{\mathbf{w}_O} = \begin{pmatrix} \widehat{\upsilon_O} \\ \widehat{Vanna_O} \\ \widehat{Volga_O} \end{pmatrix};$$

$$\mathbf{V} = \begin{pmatrix} \widehat{\upsilon_1} & \widehat{\upsilon_2} & \widehat{\upsilon_3} \\ \widehat{Vanna_1} & \widehat{Vanna_2} & \widehat{Vanna_3} \\ \widehat{Volga_1} & \widehat{Volga_2} & \widehat{Volga_3} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} \widehat{\upsilon} &= \frac{\partial O^{BS}}{\partial \omega} &= \upsilon \frac{\partial F}{\partial \omega}; \\ \widehat{Vanna} &= \frac{\partial^2 O^{BS}}{\partial \omega \partial S} &= Vanna \frac{\partial F}{\partial \omega}; \\ \widehat{Volga} &= \frac{\partial^2 O^{BS}}{\partial \omega^2} &= \frac{\partial \widehat{\upsilon}}{\partial \omega} = Volga \left(\frac{\partial F}{\partial \omega}\right)^2 + \upsilon \frac{\partial^2 F}{\partial \omega^2}. \end{aligned} \tag{13}$$

For European vanilla pivots, the price for each of them at each t may be re-expressed as the BS price of a standard European vanilla with the at-expiry implied (quadratic average) volatility $\sigma_T = \sqrt{\frac{1}{T-t} \int_t^T (F(\tau, \omega))^2 d\tau}$. While derivatives for pivots are always obtainable analytically by differentiation within the integral, the derivatives of O^{BS} may be hard to compute. The following additive and multiplicative models are natural and easy, while other easy settings may also be possible.

The **additive** model has a representation

$$\sigma(\tau, \omega_t) = \sigma_0(\tau) + \omega_t, \quad \omega_0 = 0. \tag{14}$$

As may be easily seen, finite difference computation of volatility-related greeks following from this model immediately implies parallel shifting of the volatility curve, very common in practice.

With the at-expiry quadratic average σ_T being the "fair value" implied volatility, and with flat v, $Vanna$, and $Volga$ computed for European pivots as if the volatility were constant and equal to σ_T in the usual case of proportional dividends, we get

$$
\begin{array}{rcl}
\widehat{v} & = & \dfrac{\partial O_{BS}(...,\sigma_T,...)}{\partial \sigma_T} \dfrac{\partial\left(\sqrt{\frac{1}{T}\int\limits_0^T (\sigma(\tau)+\omega)^2 d\tau}\right)}{\partial \omega}(\omega_0 = 0) = \lambda v, \\[2ex]
\widehat{Vanna} & = & \lambda Vanna, \\
\widehat{Volga} & = & \lambda^2 Volga + \left(1 - \lambda^2\right) v\big/\sigma_T, \\[1ex]
\lambda & = & \dfrac{\frac{1}{T}\int\limits_0^T \sigma(\tau)d\tau}{\sigma_T} = \dfrac{\sigma_{Avg}}{\sigma_T}.
\end{array}
\tag{15}
$$

For O^{BS}, if greeks are not computible analytically, they may be obtained using the parallel shift. In the case of absolute dividends, parallel shifting will be required for pivots, too.

In the case of standard vanilla pivots with proportional dividends, computation of $\mathbf{x}$ can be made faster if we always use the flat σ_T for pivots and apply the inverse correction to O^{BS}:

$$
\begin{array}{rcl}
\widetilde{v_O} & = & \widehat{v}_O/\lambda, \\
\widetilde{Vanna_O} & = & \widehat{Vanna_O}\big/\lambda, \\
\widetilde{Volga_O} & = & \left(\widehat{Volga_O} - \left(1 - \lambda^2\right)\widetilde{v}_O/\sigma_T\right)\big/\lambda^2,
\end{array}
\tag{16}
$$

after which we end up with a modified $\mathbf{w_O}$ but original $\mathbf{V}$ from (2), and (2) again may be replaced by faster (7).

The **multiplicative** model is immediately adapted to the log-normal nature of ω_t, has a much more natural look,

$$
\sigma(\tau, \omega_t) = \sigma_0(\tau)\omega_t, \quad \omega_0 = 1,
\tag{17}
$$

and much simpler (immediately computible in the case of proportional dividends) greeks for pivots:

$$
\begin{array}{rcl}
\widehat{v} & = & \sigma_T v; \\
\widehat{Vanna} & = & \sigma_T Vanna; \\
\widehat{Volga} & = & \sigma_T^2 Volga.
\end{array}
$$

The only disadvantage is that for O^{BS} (and in the case of absolute dividends, for pivots, too) a very "unusual" proportional (rather than parallel) shift of the volatility curve needs to be applied for differentiation:

$$
\tilde{v} = \frac{\partial O_{BS}\left(...,\sigma(t)\omega,...\right)}{\partial \omega}(\omega = 1) \approx \frac{O_{BS}\left(...,\sigma(t)(1+h),...\right) - O_{BS}\left(...,\sigma(t),...\right)}{h}.
\tag{18}
$$

The inverse correction on the target side in order to solve the flat (2) is immediate. It is possible to make one more step forward and consider proportional shifts of the implied volatility curve as well.

5 Automation of determining the fair value volatility

This section computes the fair value volatility, not specified at input, from listed prices of traded pivots, using a simple algorithm based on least squares. It may be viewed as an "objective" alternative to a "pre-specified" fair value based on subjective "market-related" or "business-related" reasons. Given at time t European vanilla calls C_i^{Mkt} at strikes $K_i, i = 1, \ldots, n$, expiring at time T, we will define the fair value volatility as

$$\sigma_{fair} = \arg \min_{\sigma \in \left[\sigma_{left}, \sigma_{right}\right]} \sum_{i=1}^{n} \left[W_i \left(C_i^{Mkt} - C_i^{BS}\left(\sigma\right) \right) \right]^2$$

where $\sigma_{left} = \min \sigma_i, \sigma_{right} = \max \sigma_i$, each σ_i is the implied BS volatility for each C_i, given S_t, r_t, d_t. W_i are given weights. A definition of weights based on traded volumes may be found below. Under an assumption that the process for the underlying is still close to BS in the first approximation, the minimization problem for σ_{fair} above may be replaced with a first order Taylor expansion

$$\sigma_{fair} = \arg \min_{\sigma \in \left[\sigma_{left}, \sigma_{right}\right]} \sum_{i=1}^{n} \left[W_i \upsilon_i \left(\sigma_i - \sigma \right) \right]^2$$

from which we compute

$$\sigma_{fair} = \frac{\sum_{i=1}^{n} \left(W_i \upsilon_i \right)^2 \sigma_i}{\sum_{i=1}^{n} \left(W_i \upsilon_i \right)^2}.$$

Due to the put-call parity, call C_i and put P_i at strike K_i may be used interchangeably. The volume-based weights may be computed from traded volumes V_i^{Call}, V_i^{Put} at strike K_i as $W_i = \frac{V_i}{V}$, where $V_i = V_i^{Call} + V_i^{Put}$ and $V = \sum_{i=1}^{n} V_i$. This definition of volume weighting accounts for liquidity at strike K_i on both call and put sides. With volumes ignored or unspecified, all V_i are set to $\frac{1}{n}$. All W_i degenerate to the same value of $\frac{1}{n}$ and may be dropped.

If pivots at multiple expiries T_j are given at time t, it becomes possible to find the implied fair value $\sigma_{fair}(t, T_j)$ for each T_j. The raw instantaneous fair value volatility curve is obtainable by square differencing.

6 Multi-dimensional cases

This section offers a simplistic extension of VV to an option O on a vector $\mathbf{S}$ of log-normal underlyings of dimension m with log-normal fair value implied volatilities σ. For each S_i, European vanilla calls C_{ij} at strikes $K_{ij}, j = 1, \ldots, 3$, are listed and traded openly at public markets. While more general settings may also be possible, we will make a natural simplifying

---

assumption of $\rho(\sigma_i, S_j) = 0, i \neq j$, where ρ is the coefficient of correlation between Brownian components. Under this assumption,

$$dw_i^S dw_j^S = \rho_{ij}^S dt, dw_i^\sigma dw_j^\sigma = \rho_{ij}^\sigma dt \tag{19}$$

$$dS_i dS_j = \rho_{ij}^S \sigma_i \sigma_j S_i S_j dt, d\sigma_i d\sigma_j = \rho_{ij}^\sigma \omega_i \omega_j \sigma_i \sigma_j dt \tag{20}$$

$$dS_i d\sigma_j = 0, i \neq j \tag{21}$$

where $\boldsymbol{\omega}$ is the volatility of volatility $\boldsymbol{\sigma}$. Similar to section 2, we will form a BS portfolio Π^{BS} of one unit of O, Δ_i units of S_i in counterposition, and $\mathbf{x}_i = \{x_{ij}\}$ units of $\mathbf{C}_i = \{C_{ij}\}, j = 1, \ldots, 3$, also in counterposition, and apply to it the Itô expansion with (19-21) taken into account:

$$
\begin{aligned}
d\Pi^{BS} &= dO^{BS} - \boldsymbol{\Delta}^{\mathbf{T}} d\mathbf{S} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} d\mathbf{C}_i^{BS} = \left[\frac{\partial O^{BS}}{\partial \mathbf{S}} - \boldsymbol{\Delta} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \frac{\partial \mathbf{C}_i^{BS}}{\partial \mathbf{S}} \right]^{\mathbf{T}} d\mathbf{S} \\
&+ \left[\frac{\partial}{\partial t} \left(O^{BS} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \mathbf{C}_i^{BS} \right) + \frac{1}{2} (\boldsymbol{\sigma} * \mathbf{S})^{\mathbf{T}} \left[\rho_{ij}^S \frac{\partial^2}{\partial S_i \partial S_j} \right] \left(O^{BS} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \mathbf{C}_i^{BS} \right) (\boldsymbol{\sigma} * \mathbf{S}) \right] dt \\
&+ \left[\frac{\partial}{\partial \boldsymbol{\sigma}} \left(O^{BS} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \mathbf{C}_i^{BS} \right) \right]^{\mathbf{T}} d\boldsymbol{\sigma} \\
&+ \frac{1}{2} (\boldsymbol{\omega} * \boldsymbol{\sigma})^{\mathbf{T}} \left[\rho_{ij}^\sigma \frac{\partial^2}{\partial \sigma_i \partial \sigma_j} \right] \left(O^{BS} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \mathbf{C}_i^{BS} \right) (\boldsymbol{\omega} * \boldsymbol{\sigma}) dt \\
&+ \sum_{i=1}^{m} \frac{\partial^2}{\partial S_i \partial \sigma_i} \left(O^{BS} - \sum_{i=1}^{m} \mathbf{x}_i^{\mathbf{T}} \mathbf{C}_i^{BS} \right) dS_i d\sigma_i \, .
\end{aligned}
\tag{22}
$$

Additionally, $\frac{\partial \mathbf{C}_i}{\partial S_j} = \frac{\partial \mathbf{C}_i}{\partial \sigma_j} = 0, i \neq j$. With all this, the 0-condition for the last 3 terms, making Π^{BS} self-financing, will look like the following:

$$
\mathbf{w}_{Oi} = \mathbf{V}_i \mathbf{x}_i, \quad \mathbf{w}_{Oi} = \begin{pmatrix} v_{Oi} \\ Vanna_{Oi} \\ Volga_{Oi} \end{pmatrix}, \quad \mathbf{V}_i = \begin{pmatrix} v_{i1} & v_{i2} & v_{i3} \\ Vanna_{i1} & Vanna_{i2} & Vanna_{i3} \\ Volga_{i1} & Volga_{i2} & Volga_{i3} \end{pmatrix}; \quad \tag{23}
$$

$$
v_{Oi} = \frac{\partial O^{BS}}{\partial S_i}, \quad Vanna_{Oi} = \frac{\partial^2 O^{BS}}{\partial S_i \partial \sigma_i}, \quad Volga_{Oi} = \frac{1}{\omega_i \sigma_i} \sum_{j=1}^{m} \rho_{ij}^\sigma \omega_j \sigma_j \frac{\partial^2 O^{BS}}{\partial \sigma_i \partial \sigma_i}, i = 1, \ldots, m. \tag{24}
$$

As immediately seen from (23-24), $\mathbf{x}_i$ may be computed block-wise, independently for each i. To each underlying S_i, having proportional dividends, formulas (7) immediately apply. Each underlying also independently admits a time-dependent instantaneous volatility curve with independent modification of greeks from Section 4. With known $\mathbf{x}_i, i = 1, \ldots, m$, the price with the MTM correction immediately equals

$$
O^{Mkt} = O^{BS} + \sum_{i=1}^{m} \sum_{k=1}^{3} x_{ik} \left(C_{ik}^{Mkt} - C_{ik}^{BS} \right). \tag{25}
$$

Obtaining ω_i and ρ_{ij}^σ statistically may, in general, require additional efforts. Computing mixed second derivatives for $Volga$ may also be time-consuming in the case of a high number of underlyings. The following (and, by no means, the only) examples below may be considered as easy shortcuts.

Options in foreign currency may be considered as contracts on two underlyings with S_1 being an equity or commodity, priced in domestic currency, and S_2 an FX rate with $\rho_{12}^\sigma = 0$. The same shortcut immediately applies to **uncorrelated baskets** with $\rho^\sigma = 0$.

In many cases, **spreads** may be considered as options on **pairs** of underlyings with $\omega_1 = \omega_2$ and $\rho_{12}^\sigma = \pm 1$.

Industry indices may be considered as weighted baskets with $\omega_i = const$ and $\rho_{ij}^\sigma = 1$.

Cross-industry indices may be considered as weighted baskets of smaller components (each component being a weighted basket itself) with $\omega = const$ and $\rho^\sigma = 1$ within each component and $\rho^\sigma = 0$ between components.

7 Random factor in interest rate

In this section, VV will be applied to randomness in interest rate. The underlying will be assumed a BS process with a fair value implied interest rate r_t constant for each t at each $\tau \in (t, T_p]$ and, as a process in t, to follow a generalized additive one-factor diffusion

$$r_t = C(t) + Z_t; dZ_t = F(t, Z_t)dt + G(t, Z_t)dw, \tag{26}$$

where $C(t)$ is a deterministic function and Z_t is a Brownian stochastic factor. While commonly accepted models used for equity/commodity/FX option pricing purposes view the interest rate as a substantially slower process updated within models once a while (once per day or so) and assumed deterministic between updates, an impact of randomness in interest rate on option prices may be in reality more noticeable. Moreover, some random interest rate changes are observable in real time, not once a while. Indeed, we will make a very liberal assumption that, within the existing market, there is at least one underlying S (equity or instantly replicated ETF like common indices) for which European call and put are presently traded in real time. For the considered option on S, with strike K_i expiring at time $T \geq t$, we may consider the Standard European put-call parity $Q(t, T, K_i) = C(t, T, K_i) - P(t, T, K_i) = Se^{-d_t(T-t)} - Ke^{-r_t(T-t)}$ (if spot interest rate and dividend yield are used instead of implied ones, time products may be consistently replaced by integrals; the absolute dividend term $S - D_t$ may also be used instead of the proportional dividend term $Se^{-d_t(T-t)}$). If more than one strike is traded, elimination of randomness due to S immediately results in the value of the interest rate discount (standard 0-coupon bond) $B(t, T) = e^{-r_t(T-t)} = \frac{Q_1 - Q_0}{K_0 - K_1}$, expiring at T and the same for any pair of strikes K_0, K_1. The value of $B^{BS}(t, T)$ is used (computed) from the interest rate curve (usually deterministic) passed into the BS model, while $B^{Mkt}(t, T)$ is obtained from the market. It should be stressed that, in general, at strike K_i values of Q^{Mkt} and Q^{BS} do not coincide and

(4) as stated in Proposition 2 does not hold either. It is still easy, however, to modify (4) to make it hold in forms

$$C^{Mkt} - C^{BS} = (P^{Mkt} - P^{BS}) - K_i(B^{Mkt}(t,T) - B^{BS}(t,T)) \tag{27}$$

$$C^{Mkt} - C^{BS} = (P^{Mkt} - P^{BS}) + S(B_f^{Mkt}(t,T) - B_f^{BS}(t,T)) - K_i(B_d^{Mkt}(t,T) - B_d^{BS}(t,T)) \tag{28}$$

respectively for equity options with stochastic domestic bond or FX options with foreign and domestic bonds both stochastic.

In order to account for the stochastic factor in the interest rate when hedging the target price O^{BS}, we will add to the portfolio Π^{BS} from Section 2 an additional term, yB^{BS}. Since the stochastic factor in r_t is assumed additive instead of dZ_t, we may consistently use dr_t everywhere. We will assume pivots in all strikes to expire at the same time T_p. Additionally, we will make a formal assumption that the interest rate is substantially slower than both underlying and volatility $((dr)^2 \ll (dS)^2, (dr)^2 \ll (d\sigma)^2)$ and uncorrelated with them $(|drdS| \ll (dS)^2, |drdS| \ll (d\sigma)^2)$ which means that all second order terms containing dr will be ignored. The Itô expansion of $d\Pi^{BS}$ will now contain the interest rate term:

$$
\begin{aligned}
d\Pi^{BS} = \quad & dO^{BS}(t) - \Delta_t dS_t - \sum_{i=1}^{3} x_i dC_i^{BS} = \left[\frac{\partial O^{BS}}{\partial S_t} - \Delta_t - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial S_t} \right] dS_t \\
& + \left[\left(\frac{\partial O^{BS}}{\partial t} - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial t} \right) + \tfrac{1}{2}\sigma^2 S^2 \left(\frac{\partial^2 O^{BS}}{\partial S^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial S^2} \right) \right] dt \\
& + \left[\frac{\partial O^{BS}}{\partial \sigma_t} - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial \sigma_t} \right] d\sigma_t + \frac{1}{2} \left[\frac{\partial^2 O^{BS}}{\partial \sigma_t^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial \sigma_t^2} \right] (d\sigma_t)^2 \\
& + \left[\frac{\partial^2 O^{BS}}{\partial S_t \partial \sigma_t} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}}{\partial S_t \partial \sigma_t} \right] dS_t d\sigma_t + \left[\frac{\partial O^{BS}}{\partial r_t} - \left(\sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}}{\partial r_t} + y \frac{\partial B_t^{BS}}{\partial r_t} \right) \right] dr_t.
\end{aligned}
\tag{29}
$$

To fulfill the self-financing condition for Π^{BS}, we solve for x_i as before and, with x_i just computed and ready to use, zero-out the new interest rate-related addition. With $\mathbf{x}$ found from $\mathbf{w}_O = \mathbf{V}\mathbf{x}$ as before and $\frac{\partial B_t^{BS}}{\partial r_t} = -(T_p - t)B_t^{BS}$, we arrive at

$$ y = \frac{\displaystyle\sum_{i=1}^{m} x_i \rho_i - \rho_O}{(T_p - t)\, B_t}, \tag{30} $$

with $m = 3$. In order to cover the later Section 8, we will assume any $m \geq 3$, which gives the correction formula at time t the final look

$$ O^{Mkt} = O^{BS} + \sum_{i=1}^{m} x_i(C^{Mkt} - C^{BS}) + y(B^{Mkt}(T_p) - B^{BS}(T_p)). \tag{31} $$

In cases of no randomness in r_t or unspecified $B^{Mkt}(T_p)$, it immediately degenerates to (3).

If O is an FX option with both domestic and foreign discount factors $B_d^{Mkt}(T_p)$ and $B_f^{Mkt}(T_p)$ available at time t, correction (31) may be further generalized into

$$O^{Mkt} = O^{BS} + \sum_{i=1}^{m} x_i(C^{Mkt} - C^{BS}) + y(B_d^{Mkt}(T_p) - B_d^{BS}(T_p)) + z(B_f^{Mkt}(T_p) - B_f^{BS}(T_p)), \ (32)$$

where y is computed using (30) with B_d plugged in and, similarly,

$$z = \frac{\sum_{i=1}^{m} x_i Dye_i - Dye_O}{(T_p - t) B_{f_t}}. \tag{33}$$

Writing m instead of 3 was chosen intentionally to stress that all above will remain unchanged in cases of more than 3 pivots used, see Section 8 below. In cases of the option expiry T_O covering one or more expiries before T_p, the instantaneous interest rate r_t is treated as a time-dependent curve (as it was done for volatility) with the stochastic factor applied additively to the whole curve. The target rho is computed by parallel shifting of the interest rate curve with no additional correction required.

Despite y and z changes, if, at one or more strikes K_i, call pivots C_i are replaced by put pivots P_i, the value of correction remains unchanged.

Proposition 3 *Values of corrections* (31) *and* (32) *will not change if one or more pivot calls are altered to pivot puts.*

We will prove the proposition for equity options with (31) where the pivot call C_m at strike K_m replaced with pivot put P_m. From (30), with all x_i unchanged and ρ_i^{Put} obtained from ρ_i^{Call} by differentiation of the standard European put-call parity, we have

$$(T_p - t) \, y^{AllCalls} B_t = -\rho_O + \sum_{i=1}^{m} x_i^{Call} \rho_i^{Call} = -\rho_O + \sum_{i=1}^{m-1} x_i \rho_i^{Call} + x_m \left(\rho_m^{Put} + (T_p - t) K_m B_t \right),$$

from which we get

$$y^{AllButLastCalls} = y^{AllCalls} - x_m K_m.$$

Value $y^{AllButLastCalls}$ is used within correction (31) with all but last pivots being calls

$$O^{VV\,AllButLastCalls} = \sum_{i=1}^{m-1} x_i \left(C_i^{Mkt} - C_i^{BS} \right) + x_m \left(P_m^{Mkt} - P_m^{BS} \right) + y^{AllButLastCalls} \left(B_t^{Mkt} (T_p) - B_t^{BS} (T_p) \right).$$

The put-call parity (28) gives the final touch,

$$\begin{aligned} O^{VV\,AllButLastCalls} &= \sum_{i=1}^{m} x_i \left(C_i^{Mkt} - C_i^{BS} \right) + x_m K_m \left(B_t^{Mkt} - B_t^{BS} \right) \\ &\quad + y^{AllCalls} \left(B_t^{Mkt} - B_t^{BS} \right) - x_m K_m \left(B_t^{Mkt} - B_t^{BS} \right) \\ &= O^{VV\,AllCalls}, \end{aligned}$$

proving the proposition. Any case of more than one pivot follows by repetition immediately. Extension to FX options with (32) is also obvious. Without any changes, it will work with pivots recomputed from BF/RR/ATM quotes.

The generalization of corrections (30), (32) to multi-asset options from Section 6 is also easily obtained:

$$O^{Mkt} = O^{BS} + \sum_{i=1}^{m}\sum_{k=1}^{3} x_{ik}\left(C_{ik}^{Mkt} - C_{ik}^{BS}\right) + y(B_d^{Mkt}(T_p) - B_d^{BS}(T)) + z(B_f^{Mkt}(T_p) - B_f^{BS}(T_p)).$$
(34)

The following consideration will justify the corrections above in the case where r_t is a spot (instantaneous rather than implied) interest rate curve. In the standard SDE for a BS underlying, $\frac{dS}{S} = rdt + \sigma dw$, the stochastic spot interest rate term rdt may be replaced by $-dR$, where $R(t,T) = \int_t^T r(\tau)d\tau$ is the accumulated interest, immediately obtainable from the yield curve. $R(t,T)$ can be used instead of the rt term in the BS price formula for European vanillas. The forward rate $f(t,T) = \frac{dR(t,T)}{dT}$ is also immediately available from the yield curve and may be consistently used for pivot pricing, see [8]. We will also assume it follows the HJM model (see [2]),

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW(u),$$
(35)

where $f(0,T)$ is the time-dependent initial forward rate curve. It is known that commonly used short rate models with term structure of type (26) (HW, CIR, BK), when converted into forwards, may be considered as special cases of HJM, see [8]. As a result, the time-dependent spot interest rate $r(t) = f(t,t) = -\frac{d}{dt}R(t,T)\big|_{T=t} = f(0,t) + Z_t$, where Z_t is a generalized Brownian factor of type (26), may be, for VV correction purposes, viewed as an additive model with numeric derivatives obtained by parallel shifting. Because of the linearity of the interest rate integral average in the additive model (unlike the quadratic average for volatility), no changes in corrections above are required.

Coming back to the assumption of negligibility of all second order terms containing dr_t, it should be mentioned that this assumption was really necessary. As may be seen from the expressions below (courtesy of Anatoly Gormin), for pivots, all second order greeks involving differentiation by r are expressible as linear combinations of $v, Vanna, Volga, \rho, Dye$:

$$
\begin{array}{rcl}
\rho 2 &=& \frac{\partial \rho}{\partial r} = \left(-\rho + \frac{v}{\sigma}\right)\tau, \\
\rho \delta &=& \frac{\partial \rho}{\partial S} = \frac{v}{\sigma S}, \\
\rho v &=& \frac{\partial \rho}{\partial \sigma} = -\frac{v}{\sigma}d_1\sqrt{\tau} = (SVanna - v)\tau, \\
\rho Dye &=& -\frac{v}{\sigma}\tau \\
DyeDye &=& \frac{\partial Dye}{\partial d} = \left(-Dye + \frac{v}{\sigma}\right)\tau, \\
Dye\delta &=& \frac{\partial Dye}{\partial S} = -\tau\Delta - \rho\delta, \\
Dyev &=& \frac{\partial Dye}{\partial \sigma} = \frac{v}{\sigma}d_2\sqrt{\tau} = -S\tau Vanna,
\end{array}
$$

As a result the rows of these greeks for pivots will be automatically zeroed out just because $v, Vanna, Volga, \rho, Dye$ were zeroed out. At the same time, on the target side, they will remain non-zero, which will immediately mean no hedging solution $\mathbf{x}, y, z$. Fortunately, the assumption of negligibility can always be made in the BS context. The case of non-negligible second order interest rate dynamics would make the applicability of all BS-based models to equity/commodity/FX option pricing very questionable. Under such circumstances these options would rather have to be viewed as interest rate products.

8 Arbitrary number of pivots

This section offers the "most valuable" correction in case more than 3 available traded pivots are available. Consider a general VV scheme $\mathbf{w}_O = \mathbf{V}\mathbf{x}$ with $m = 3$ independent equations and $n > m$ available pivots. The excessive number of pivots may offer multiple choices of m pivots required for computing $\mathbf{x}$ and, as a result, multiple choices of computed $\mathbf{x}$. A Principal Component algorithm (optionally with weights), choosing m most contributing linear combinations of pivots, will be just one of these choices.

Given tolerance *tol* and matrix $\mathbf{V}$ of m-dimensional pivot columns $\mathbf{p}_1, \ldots, \mathbf{p}_n$ with arbitrary m, n and diagonal matrix $\mathbf{W}$ of weights (identity matrix $\mathbf{I}$ if not specified or volume weights W_i), for non-negative symmetric matrix $\mathbf{A} = \mathbf{W}^t \mathbf{V}^t \mathbf{V} \mathbf{W}$ of size $n \times n$:

1. Consider the eigenvalue decomposition of $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ sorted descendingly and conditioning number $c(\mathbf{A}) = \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\|$ in the maximal eigenvalue norm.

2. If $\frac{1}{c(\mathbf{A})} = \frac{\lambda_n}{\lambda_1} < tol$ on a real computer (ill-conditioning), choose the maximal $k \leq \min(m, n)$ for which $\frac{1}{c_k(\mathbf{A})} = \frac{\lambda_k}{\lambda_1} \geq tol$.

3. For eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\mathbf{A}$, compute normalized eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$; group columns $\mathbf{e}_i$ into matrix $\mathbf{E}$.

4. Consider a $k \times m$ matrix $\hat{\mathbf{V}} = \mathbf{V}\mathbf{W}\mathbf{E}$ of principal components $\hat{\mathbf{p}}_i = \mathbf{V}\mathbf{W}\mathbf{e}_i$.

5. For each O within the given portfolio,

   - solve $\mathbf{w}_O = \hat{\mathbf{V}}\mathbf{z}$ for $\mathbf{z}$ precisely if $k = m$ or, in least-square approximation from $\hat{\mathbf{V}}^t \mathbf{w}_O = \hat{\mathbf{V}}^t \hat{\mathbf{V}} \mathbf{z} = \mathbf{E}^t \mathbf{A} \mathbf{E} \mathbf{z}$ (matrix $\mathbf{E}^t \mathbf{A} \mathbf{E}$ is diagonal with elements $\lambda_i > 0$), if $k < m$;

   - make correction $O^{Mkt} = O^{BS} + \sum_{i=1}^{n} x_i (C_i^{Mkt} - C_i^{BS})$, where $\mathbf{x} = \mathbf{W}\mathbf{E}\mathbf{z}$.

In cases of FX, instead of additional pivots, additional quotes of OTC BF/RR pairs for less standard values of $\Delta \neq 25\%$ (but $\Delta < 50\%$) may be used. Pivots may be recomputed in the way it is done for $\Delta = 25\%$, see [4].

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9 Conclusion

Robustness, simplicity and extensibility make the VV method a reasonable choice for fast price correction algorithms, accounting for volatility smile. Because of limitation the method however should be used with care.

Acknowledgments

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