A market model for inflation

N. BELGRADE*, E. BENHAMOU† & E. KOEHLER ‡§

January 2004.

Abstract

The various macro econometrics model for inflation are helpless when it comes to the pricing of inflation derivatives. The only article targeting inflation option pricing, the Jarrow Yildirim model [8], relies on non observable data. This makes the estimation of the model parameters a non trivial problem. In addition, their framework do not examine any relationship between the most liquid inflation derivatives instruments: the year to year and zero coupon swap. To fill this gap, we see how to derive a model on inflation, based on traded and liquid market instrument. Applying the same strategy as the one for a market model on interest rates, we derive no-arbitrage relationship between zero coupon and year to year swaps. We explain how to compute the convexity adjustment and what relationship the volatility surface should satisfy. Within this framework, it becomes much easier to estimate model parameters and to price inflation derivatives in a consistent way.

- **Keywords:** *inflation index, forward, zero-coupon, year-on-year, volatility cube, convexity adjustment.*

AMS Classification 60G35

*CDC IXIS-CM R&D and Maison des Sciences Economiques. Adress: CDC IXIS 47 Quai d'Austerlitz 75013 Paris, France. Maison des Sciences Economiques, University of Paris 1 Panthéon-Sorbonne, 116-118 Boulevard de l'Hôpital 75013 Paris, France. E-mail: nbelgrade@cdcixis-cm.com.

†CDC IXIS-CM R&D, 47 Quai d'Austerlitz 75013 Paris, France. E-mail: ebenhamou@cdcixis-cm.com

‡Risk Department CDC IXIS (26-28 Rue Neuve Tolbiac, 75658 Paris Cedex 13, France, (+33 1 58 55 59 68) and Associated Professor at Paris I, Sorbonne University (Maison des Sciences Economiques, University of Paris 1 Panthéon-Sorbonne, 116-118 Boulevard de l'Hôpital 75013 Paris). Email e.koehler@cdcixis.com

§The ideas expressed herein are the authors' ones and do not necessarily represent the ones of CDC IXIS CM.

1 Introduction

The standard approach for modelling inflation is based on econometrics models. Their aim is to forecast inflation rate, provided a time series of data. Using sophisticated version of the so-called "Taylor" rule, economists show how to relate inflation rate to various macro economic indexes such as short term interest rates and monetary policy target.

When it comes to pricing inflation derivatives, this framework is helpless for many reasons:

- First, it does not provide any information concerning the hedging strategy making this approach of poor usage for derivatives trading desk.
- Second, it does not offer any relationship between the various traded instruments. These relationships are crucial to provide a model consistent with traded securities.
- Third, it uses discrete time modelling. This makes it not easy to tackle complex options where the set up is based on continuous time modelling.

Surprisingly, there is not much in the literature on option pricing on inflation derivatives. The only paper by Jarrow and Yildirim [8] uses the interest rate curve as a starting point. The authors model the inflation rate as an exchange rate between the nominal and real zero-coupon bonds. Their key assumptions are deterministic volatilities and non zero correlation between the different factors. Using no-arbitrage relationship, they derive a model similar to a 3 factor HJM model. However, this model has the major drawback to use non observable parameters. In order to infer the inflation rate, one needs to fit a model on the real interest rate curve, which is even harder to estimate than the inflation rate itself. A second drawback is to provide no link between zero-coupon and year-on-year products.

The two disadvantages of the Jarrow and Yildirim model are precisely the motivation of our model. Adapting the concept of market model, we explain how to use consistent information between the zero-coupon and the year-on-year swap market. The first result concerns volatility information. The consistent relationship for the volatility market is first examined in the general framework of a market model. We then compute explicitly this relationship in the case of various model assumptions like Black-Scholes, homogeneous and Hull and White volatilities. We then see how to compute convexity adjustment using the market model.

2 Primer on inflation modelling

2.1 Product overview

Over the last 3 years, the inflation swap market had been exploding with monthly transaction volume around 100 millions Euro in 2001, 500 millions in 2002 and 1500 in 2003 (source ICAP). In 2003, 56% of the transactions concerned maturity below 7 years, with 28% between 1 and 4 years. The potential explanation of this tremendous growth are numerous, ranging from

- interest for competitive inflation product (due to anticipated deflation)
- bigger liquidity provided by raising government inflation linked bond issue
- interest from corporate to issue inflation linked debt (DEXMA,RFF,...)
- ability to offer capital guaranteed structure guaranteed not in notional but real term.

The two liquid instruments are the zero-coupon swap and the year-on-year swap. More precisely:

- a zero-coupon swap (in its payer version) pays the inflation return

$$CPI(T)/CPI(0) - 1.$$

  versus receiving a pre-agreed zero-coupon rate $(1 + Zc)^T - 1$. By far, zero-coupon swaps are the most liquid instruments.

- the year-on-year swap (also referred to as year-on-year, or annual swap) pays in its payer annual form version the annualized CPI return. At time $T_i + 1$, the inflation leg pays $CPI\left(T_{i+1}\right)/CPI\left(T_i\right) - 1$ versus receiving a fix leg paying S.
- Last but not least, inflation bonds pays the compounded inflation return over time $CPI(T)/CPI(0)$

2.2 Modelling issue

To model inflation, one may think to use the numerous recent models in time series analysis targeting inflation using discrete time modelling. One can find equivalents in continuous time but these models remain inefficient for the evaluation of the inflation linked products. The major challenge comes from the difference of probability measures between the historical and risk neutral ones. Econometric models are derived under historical probability while option pricing requires to use the risk neutral probability [1]. This therefore prevents us from using econometric models.

A first sight solution may be to use an adaptation of interest modeling. However, the inflation market offers some additional challenging features:

- multi-curve environment: because of the inability to lock in a inflation zero-coupon with its notional compounded by the inflation return, static replication of the year-on-year curve from the zero-coupon one is impossible. Hence the year-on-year curve has to account for the additional convexity adjustment.
- correlation modelling: inflation should be rigorously connected to interest rate as the correlation structure between forward CPI and interest rate has to be used for the convexity adjustment.
- multi-asset pricing dimension: because of the correlation between interest rates and inflation rates.

2.3 Modelling assumptions

Ignoring for now the multi-asset pricing dimension between interest rates and inflation, we examine how to provide consistent information between the various inflation markets. Our modelling target is to provide a model that can be

- simple enough to have only a few parameters.
- robust enough to replicate market prices.

At first sight, the CPI can be modelled as:
- either n sampling of one single process observed at different times
- or a single sampling of n different processes observed each at one time.

This fundamental difference can be related to the interest rate markets. We could think of the Libor or swap rate as one single instrument observed at different time (assumption made when doing a pricing of swaption and or cap in Black-Scholes) or we could see forward Libor rates as independent rates. The latter is the approach taken in the Libor market models (also referred to as forward rate models or BGM or Jamshidian models). Obviously the latter could also be seen as an extension of the first methodology but with a much richer information on the correlation structure. From an econometric point of view, the first assumption (respectively the second one) corresponds to a heteroscedastic process without (respectively with) autocorrelation on errors.

Typically, inflation structures are based on CPI data fixing at various dates denoted by $(T_i)_{1\leq i\leq n}$. Let us denote by $CPI\,(s, T_i, T_i+\delta_i)$ the CPI rate observed at time s that fixes at time T_i and applying for a period δ_i. Typically, the tenors δ_i are the same and are all equal to 1 month. In the following, we will drop the index i and denote by δ. From a concrete point of view, δ of 1 month means monthly CPI data. Note that compared to Libor modelling, the tenor δ is slightly different in the sense that it is not an interest period. However, writing the CPI rate like this shows the similarities between the two instruments. In order to simplify even further notations, we will write $CPI\,(s, T_i)$ dropping the third term in the parenthesis.

A simple but rich enough framework is to assume a market model for inflation where the forward CPI return is modelled as a diffusion with a deterministic volatility structure. For this, we consider the filtered probability space $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ where $\mathbb{P}$ is the historical probability, and $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration[2] generated by the standard multi $n+1$ dimensional Brownian motion $\left\{\left(W^i, B\right)(t)_{1\leq i\leq n}, t\geq 0\right\}$, with correlation matrixes given by Ξ with the various terms of this matrix are given by

$$\left\{\begin{array}{ll} d\left\langle W^i_{\cdot}, W^j_{\cdot}\right\rangle(t) & = \rho^{Inf}_{i,j}\,dt, \\ d\left\langle W^i_{\cdot}, B_{\cdot}\right\rangle(t) & = \rho^{B,I}_{i}\,dt. \end{array}\right. \tag{1}$$

[1] The passage between the two is made by the determination the market risk premium which is a parameter which still complicates the estimate of the models.

[2] $\forall t\geq 0,\ \mathcal{F}_t = \sigma\left\{\left(W^i(s)\right)_{1\leq i\leq n}, 0\leq s\leq t\right\}$.

Where $\{B(t), t \geq 0\}$ is the Brownian motion driving the zero-coupon bond

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + \Gamma(t,T)\,dB(t)\,,\ \ t \leq T.$$

In this framework, the evolution of the forward CPI under the risk neutral probability measure $\mathbb{Q}$ corresponds to a geometric Brownian motion:

$$\frac{dCPI(t,T_i)}{CPI(t,T_i)} = \mu(t,T_i)\,dt + \sigma(t,T_i)\,dW^i(t)\,, \tag{2}$$

where the volatility structure $\sigma(t,T_i)$ and the drift $\mu(t,T_i)$ are deterministic. Specific forms of volatility are

- Black-Scholes [2] (independent of date of observation)

$$\sigma(t,T_i) = \sigma_i,$$

- Homogeneous case ($\sigma(t,T_i) = f(T_i - t)$):
  - Hull and White [6] type volatility (potentially time dependent)

$$\sigma(t,T_i) = \sigma_i(t)\,e^{-\lambda_i(t)(T_i-t)},$$

  - Integrated Hull and White volatility (potentially time dependent)

$$\sigma(t,T_i) = \sigma_i(t)\,\frac{1-e^{-\lambda_i(t)(T_i-t)}}{\lambda_i(t)}.$$

- Many other type similar to the interest rate models like Mercurio Moraleda types and so on....

3 Consistent modelling of inflation volatility

When examining CPI products, we can notice that they are based on a ratio of two CPIs $\frac{CPI(.,T_i)}{CPI(.,T_j)}$. Including a strike, a vanilla inflation option (let) can be parameterized by the following three parameters:

- Date of fixing of the CPI used in the denominator of the ratio
- Difference between the fixing date of the CPI in the numerator
- strike of the option.

An option on year-on-year inflation returns will therefore be a strip of call or put on $\frac{CPI(.,T_i)}{CPI(.,T_{i-1})}$.

An option on zero-coupon will therefore be a call or put on $\frac{CPI(.,T_i)}{CPI(.,T_0)}$.

Although the volatility markets between the year-on-year and zero-coupon option may at first sight look different, there are some connections. A year-on-year option may 1 day become a zero-coupon one as soon as the CPI of the denominator has fixed. The modelling of the inflation volatility need to account for this.

3.1 A general *vol cube*

A simple idea is to parameterize the vol structure in term of

- the fixing date of the CPI in the denominator date denoted by T,
- either the tenor maturity denoted by δ representing the maturity between the numerator and the denominator or the numerator date denoted by $T+\delta$,
- the strikes denoted by K.

Definition 1 *We call* ***vol cube*** *and we denote by* ψ *the* 3 *dimensional deterministic function of a fixing date* T*, a tenor* δ *and a strike* K *defined as:*

$$\begin{array}{llll}\psi: & \mathbb{R}^+ \times \mathbb{R}^+_* \times \mathbb{R} & \rightarrow & \mathbb{R}^+ \\ & (T, \delta, K) & \mapsto & \psi(T, \delta, K) = Vol\left(\max/\min\left(\frac{CPI(T+\delta,T+\delta)}{CPI(T,T)} - K, 0\right)\right).\end{array}$$

Each of the three plans of the cube will be a real matrix obtained by fixing one variable among the three.

The market provides indication/information about only year-on-year options (the most liquid ones) and zero coupon. Hence, looking at the *vol cube*, we can only get from the market two volatility surfaces (somehow orthogonal) of the volatility cubes which are $\psi(T, T+\delta, K)_{T \geq 0}$ and $\psi(0, T, K)_{T>0}$. For different values of the strike K, a zero-coupon option of maturity one year being the first option year-on-year of tenor one year, the two surfaces are dependant ($\forall K \in \mathbb{R}$, $\psi(0, T, K)|_{T=\delta} = \psi(T, T+\delta, K)|_{T=0}$). A first objective would be thus, to bind these two surfaces in order to find the consistency relationships between them. The modelling issue is to provide a way to interpolate/extrapolate volatility information for forward starting structure different from year-on-year information.

3.2 Options on inflation description

As mentioned above, options on inflation available in the market are vanilla options, zero-coupon and year-on-year . For some $(\alpha, \beta) \in \mathbb{R}^+_* \times \mathbb{R}^+$, the general payoff of:

- an option on zero-coupon inflation is written as:

$$\max/\min\left(\alpha\frac{CPI(T_i,T_i)}{CPI(0,0)} - \beta, (1+k_0)^{T_i}\right). \tag{3}$$

  Here the nominator is fixed and known at $t = 0$. Let's remark the particular expression of the strike $(1+k_0)^{T_i}$, formulate as an actuarial rate. In the special case $(\alpha, \beta) = (1, 0)$, we simply compare the inflation to an zero-coupon swap level.

- an option on year-on-year inflation as:

$$\max/\min\left(\alpha\frac{CPI(T_i,T_i)}{CPI(T_{i-1},T_{i-1})} - \beta, K\right). \tag{4}$$

  Here the nominator $CPI(T_{i-1}, T_{i-1})$ can not be known before $t = T_{i-1}$ and the strike is crude.

Let's note for a given strike K, $\sigma_{BS}(0, T_i)$ (respectively $\sigma_{BS}(T_{i-1}, T_i)$) the Black-Scholes volatility of the option zero-coupon inflation (respectively year-on-year inflation) with exercise date T_i. Keeping the same notation as the last paragraph, we can write for a fixed strike K^* :

$$\left\{\begin{array}{ll}\psi(0, T_i, K^*) & = \sigma_{BS}(0, T_i) \\ \psi(T_{i-1}, T_i - T_{i-1}, K^*) & = \sigma_{BS}(T_{i-1}, T_i)\end{array}\right. . \tag{5}$$

In the following section we'll determine the relation between the Black-Scholes volatilities of the zero-coupon options and the year-on-year ones for a fixed strike K^*, so we will use σ_{BS} instead ψ.

3.3 Model inter/extrapolation

Denoting by $\sigma(t, T_i)$ the volatility function:

$$\begin{array}{llll}\sigma: & \mathbb{R}^+ \times \mathbb{R}^+_* & \rightarrow & \mathbb{R}^+ \\ & (t, T_i) & \mapsto & \sigma(t, T_i)\end{array}$$

Equations (2) and (3) gives[3]:

$$\mathbb{V}\left[\ln\left(CPI(T_i, T_i)\right)\right] = T_i\sigma^2_{BS}(0, T_i) = \int_0^{T_i} \sigma^2(s, T_i)\, ds.$$

[3] $\mathbb{V}$ means variance. It does not depend on the measure.

So from (4), we obtain:

$$\begin{aligned}
& \mathbb{V}\left[\ln\left(\frac{CPI(T_i)}{CPI(T_{i-1})}\right)\right] \\
= \; & T_i.\sigma^2_{BS}\left(T_{i-1},T_i\right) \\
= \; & \int_0^{T_i}\sigma^2\left(s,T_i\right)ds+\int_0^{T_{i-1}}\sigma^2\left(s,T_{i-1}\right)ds-2\int_0^{T_i}\sigma\left(s,T_{i-1}\right)\sigma\left(s,T_i\right)d\left\langle W^{i-1}_{\cdot},W^{i}_{\cdot}\right\rangle(s) \\
= \; & \sigma^2_{BS}\left(0,T_i\right).T_i+\sigma^2_{BS}\left(0,T_{i-1}\right).T_{i-1}-2\rho^{Inf}_{i-1,i}\int_0^{T_{i-1}}\sigma\left(s,T_{i-1}\right)\sigma\left(s,T_i\right)ds.
\end{aligned}$$

where the instantaneous correlation $\rho^{Inf}_{i-1,i}$ is defined as in (1). The year-on-year volatility includes:

- the two corresponding zero-coupon volatilities
- the covariance between corresponding zero coupons.

Denoting by $\gamma\left(T_{i-1},T_i\right)\equiv\int_0^{T_{i-1}}\sigma\left(s,T_{i-1}\right)\sigma\left(s,T_i\right)ds$ the full correlation integrated covariance[4] between the two CPIs, we find the explicit relation between the year-on-year volatilities and the zero-coupon ones:

$$\boxed{\sigma_{BS}\left(T_{i-1},T_i\right)=\sqrt{\frac{T_i\sigma^2_{BS}\left(0,T_i\right)+T_{i-1}\sigma^2_{BS}\left(0,T_{i-1}\right)-2\rho^{Inf}_{i-1,i}\gamma\left(T_{i-1},T_i\right)}{T_i-T_{i-1}}}.} \tag{6}$$

This shows that the relationship between year-on-year and zero-coupon is model dependent through not only the instantaneous correlation $\rho^{Inf}_{i-1,i}$ but also the full correlation integrated covariance $\gamma\left(T_{i-1},T_i\right)$ which in terms depends on the volatility assumptions. In the next paragraph, we detail the result for various form of volatilities.

3.4 Specific form of the volatilities

3.4.1 Case of Black and Scholes

A Black and Scholes volatility is deterministic and homogeneous i.e. an one-dimensional positive function of time:

$$\sigma\left(t,T_i\right)=\sigma_i,\ \forall 0\leq t<T_i,$$

leading to:

$$\gamma\left(T_{i-1},T_i\right)=T_{i-1}\sigma_{i-1}\sigma_i,\ \forall 1\leq i\leq n. \tag{7}$$

And using the fact that

$$\sigma_i=\sigma_{BS}\left(0,T_i\right).$$

Hence, the the year-on-year volatilities become a function of the zero-coupon ones only:

$$\boxed{\sigma_{BS}\left(T_{i-1},T_i\right)=\sqrt{\frac{T_i\sigma^2_{BS}\left(0,T_i\right)+T_{i-1}\sigma^2_{BS}\left(0,T_{i-1}\right)-2\rho^{Inf}_{i-1,i}\sigma_{BS}\left(0,T_i\right)\sigma_{BS}\left(0,T_{i-1}\right)}{T_i-T_{i-1}}}.} \tag{8}$$

3.4.2 Case of Hull and White

In the Hull and White (respectively the integrated) framework, we have an explicit exponential form of volatility function increasing (respectively decreasing) by time. We can distinguish these special cases $\forall 0\leq t<T_i$:

- Hull and White potentially time dependent[5] $\sigma\left(t,T_i\right)=\sigma_i\left(t\right)e^{-\lambda_i\left(T_i-t\right)}$:

$$T_i.\sigma^2_{B\&S}\left(0,T_i\right) = e^{-2\lambda_iT_i}\sum_{j=1}^{n_i}\sigma^2\left(j\right)\frac{e^{2\lambda_iT_j}-e^{2\lambda_iT_{j-1}}}{2\lambda_i}, \tag{9}$$

$$\gamma\left(T_{i-1},T_i\right) = e^{-\left(\lambda_{i-1}T_{i-1}+\lambda_iT_i\right)}\sum_{j=1}^{n_{i-1}}\sigma^2\left(j\right)\frac{e^{\left(\lambda_{i-1}+\lambda_i\right)T_j}-e^{\left(\lambda_{i-1}+\lambda_i\right)T_{j-1}}}{\lambda_{i-1}+\lambda_i}. \tag{10}$$

[4] We call this full correlation integrated covariance to mean that this would be the covariance if the instantaneous correlation $\rho^{Inf}_{i-1,i}$ were equal to 1.

[5] $\sigma\left(t\right)$ is a step wise function: $\sigma\left(t\right)=\sum_{i=1}^{n}\sigma\left(i\right)\mathbb{I}_{\left\{T_{i-1}\leq i<T_i\right\}},\ \forall 0\leq t\leq T_n$

- Integrated Hull and White, potentially time dependent $\sigma\left(t,T_i\right)=\sigma_i\left(t\right)\frac{1-e^{-\lambda_i\left(T_i-t\right)}}{\lambda_i}$:

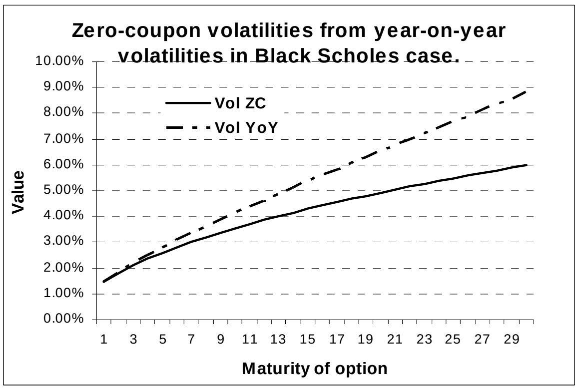
$$T_i.\sigma_{B\&S}^2\left(0,T_i\right) = \sum_{j=1}^{n_i}\sigma^2\left(j\right)\frac{2\lambda_i(T_j-T_{j-1})-4e^{-\lambda_iT_i}\left(e^{\lambda_iT_j}-e^{\lambda_iT_{j-1}}\right)+e^{-2\lambda_iT_i}\left(e^{2\lambda_iT_j}-e^{2\lambda_iT_{j-1}}\right)}{2\lambda_i^3}, \tag{11}$$

$$\gamma\left(T_{i-1},T_i\right) = \sum_{j=1}^{n_{i-1}}\frac{\sigma^2(j)}{\lambda_{i-1}\lambda_i}\left\{\begin{array}{c} T_j-T_{j-1}-e^{-\lambda_{i-1}T_{i-1}}\frac{e^{\lambda_{i-1}T_j}-e^{\lambda_{i-1}T_{j-1}}}{\lambda_{i-1}} \\ +e^{-(\lambda_{i-1}T_{i-1}+\lambda_iT_i)}\frac{e^{(\lambda_{i-1}+\lambda_i)T_j}-e^{(\lambda_{i-1}+\lambda_i)T_{j-1}}}{\lambda_{i-1}+\lambda_i} \\ -e^{-\lambda_iT_i}\frac{e^{\lambda_iT_j}-e^{\lambda_iT_{j-1}}}{\lambda_i}\end{array}\right\}. \tag{12}$$

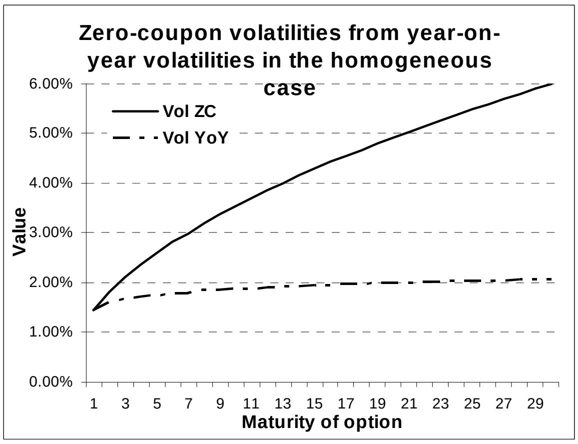
3.5 Numerical results

We present below a set of zero-coupon and year-on-year volatilities in each case of volatility function with fixed values of parameters. Except in the Black-Scholes case, the zero-coupon volatilities do not correspond to the real market data. This inconsistency results from the choice of volatility function parameters.

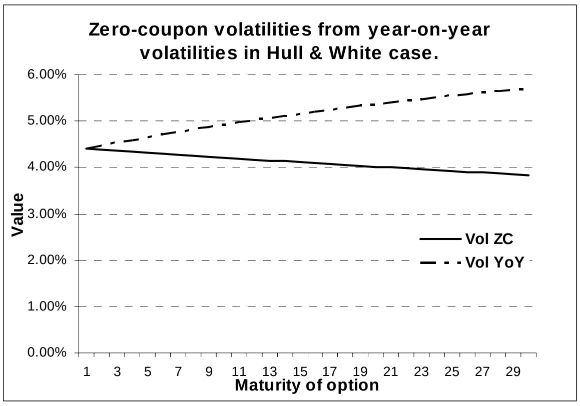
Example 2 (Black-Scholes case with $\rho_{j,i}^{Inf}=0.98,\forall j<i$) *In this case, the year-on-year volatilities are higher than the zero-coupon ones. We explain this in the subsection 5.1.*



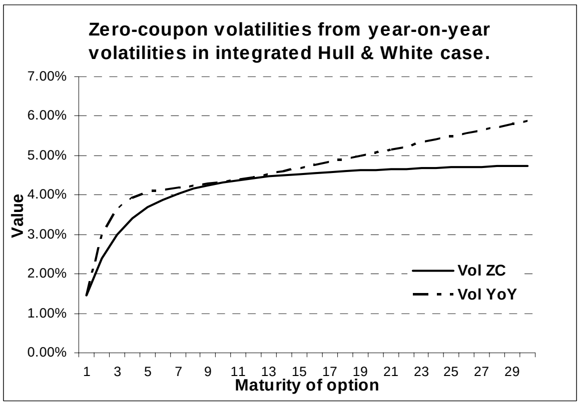
Example 3 (Homogeneous case with $\rho_{j,i}^{Inf}=1,\forall j<i$) *In this case, the year-on-year volatilities are lower than the zero-coupon ones. But they can't attain a minimum level. We explain this in the subsection 5.1.*



Example 4 (Hull & White case with $\rho_{j,i}^{Inf} = 0.98, \forall j < i$)



Example 5 (Integrated Hull & White case with $\rho_{j,i}^{Inf} = 0.98, \forall j < i$)



4 Convexity adjustment

As for the CMS (see [1] for a review on CMS pricing), the convexity adjustment of the inflation swaps results from the difference of martingale measures between the numerator and the denominator.

4.1 Intuition

The forward $CPI(t, T_i)$ fixing at time T_i is obviously a martingale under its payment probability measure $\mathbb{Q}_{T_i}$. Similarly, for the forward $CPI(t, T_j)$ fixing at time T_j under the probability measure $\mathbb{Q}_{T_j}$, but not $\mathbb{Q}_{T_i}$. Consequently, the expected value under the probability measure $\mathbb{Q}_{T_i}$ of the ratio of the two CPIs (with time $T_i > T_j$) has to take into account various convexity adjustments[6]:

- $CPI(t, T_j)$ is not a martingale under the $\mathbb{Q}_{T_i}$ measure. Hence it has to be adjusted to account for the change of measure between $\mathbb{Q}_{T_j}$ and $\mathbb{Q}_{T_i}$. This adjustment should intuitively depend on the covariance between the forward interest bond volatility (between T_j and T_i) and the forward inflation rate in the denominator $CPI(t, T_j)$. This change of measure is similar to the CMS adjustment.
- In addition, we pay $CPI(t, T_i) / CPI(t, T_j)$. Because of the correlation between these two inflation forward rates, we need to account for the their joint move. The adjustment should intuitively be depending on the covariance between these two CPI rates. This is similar in a sense to a quanto adjustment.

The adjustment is therefore computed in two steps:

- change of measure between $\mathbb{Q}_{T_j}$ and $\mathbb{Q}_{T_i}$.

[6] The forward of the ratio of CPI is not equal to the ratio of the forward CPIs. One calls more or less improperly this adjustment a convexity adjustment by extension from the one used in interest rates for various change of measures like CMS and in-arrears.

- computation of the expected ratio.

Definition 6 *We call the* ***Inflation Convexity Adjustment*** *at time t between T_j and T_i, and we denote $\lambda_{Cvx}(t,T_j,T_i)$, the difference between the forward ratio*[7] $\mathbb{E}^{\mathbb{Q}_{T_i}}\left[\left.\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}\right|\mathcal{F}_t\right]$ *and both zero-coupon swaps corresponding to the two dates which frame it* $\frac{\mathbb{E}^{\mathbb{Q}_{T_i}}[CPI(T_i,T_i)|\mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}_{T_j}}[CPI(T_j,T_j)|\mathcal{F}_t]}$[8]*, where $\mathbb{Q}_{T_i}$ is the T_i-forward neutral probability, so:*

$$\lambda_{Cvx}(t,T_j,T_i)=\mathbb{E}^{\mathbb{Q}_{T_i}}\left[\left.\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}\right|\mathcal{F}_t\right]-\frac{CPI(t,T_i)}{CPI(t,T_j)}.$$

4.2 General framework

The forward $CPI(t,T_i)$ fixing at time T_i being a martingale under its payment probability measure $\mathbb{Q}_{T_i}$, it's dynamic is as follows:

$$\frac{dCPI(t,T_i)}{CPI(t,T_i)}=\sigma(t,T_i)\,dW_{T_i}(t), \tag{13}$$

where $(W_{T_i}(t))_{1\leq i\leq n}$ is a n dimensional Brownian motion under $\mathbb{Q}_{T_i}$. It is well known that the relationship between the risk neutral measure and forward measure is given by

$$dW_{T_i}(t)\equiv dW^i(t)-\Gamma(t,T_i)\,dt,\ \forall t\geq 0,\ \forall 1\leq i\leq n,$$

where $\Gamma(t,T_i)$ is the lognormal volatility of the zero-coupon bond $B(t,T_i)$.

This change of measure forces us to specify an implied correlation between zero-coupon bonds and CPI forwards[9]. We suppose that zero-coupon bonds and CPI forwards are driven by different Brownian motions $W^i(t)$ and $B(t)$ with the correlation from (1) :

$$\left\langle dW^i(t),dB(t)\right\rangle=\rho_i^{B,I}dt. \tag{14}$$

Solving the SDE (13) for T_j and T_i under the same probability $\mathbb{Q}_{T_i}$ leads to:

$$\begin{aligned}
&\ln\left(\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}/\frac{CPI(0,T_j)}{CPI(0,T_i)}\right)\\
=\ &\int_0^{T_i}\sigma(s,T_i)\,dW_{T_i}(s)-\int_0^{T_j}\rho_{j,i}^{Inf}\sigma(s,T_j)\,dW_{T_i}(s)\\
&-\sqrt{1-\left(\rho_{j,i}^{Inf}\right)^2}\sigma(s,T_j)\,dW_{T_i}^{\perp}(s)-\frac{1}{2}\left(\sigma^2(s,T_i)-\sigma^2(s,T_j)\right)ds\\
&+\rho_j^{B,I}\int_0^{T_j}\sigma(s,T_j)\left\{\Gamma(s,T_i)-\Gamma(s,T_j)\right\}ds.
\end{aligned}$$

The computation of the expectation of the forward CPI is then simply given by:

$$\mathbb{E}^{\mathbb{Q}_{T_i}}\left[\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}\right]/\frac{CPI(0,T_j)}{CPI(0,T_i)}=e^{\int_0^{T_j}\sigma(s,T_j)\left\{\sigma(s,T_j)-\rho_{j,i}^{Inf}\sigma(s,T_i)\right\}+\rho_j^{B,I}\sigma(s,T_j)\left\{\Gamma(s,T_i)-\Gamma(s,T_j)\right\}ds}.$$

The convexity adjustment at $t=0$, defined by $\lambda_{Cvx}(0,T_j,T_i)=\mathbb{E}^{\mathbb{Q}_{T_i}}\left[\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}\right]-\frac{CPI(0,T_j)}{CPI(0,T_i)}$ is equal to:

$$\begin{aligned}
&\lambda_{Cvx}(0,T_j,T_i)/\frac{CPI(0,T_j)}{CPI(0,T_i)}\\
=\ &e^{\int_0^{T_j}\sigma(s,T_j)\left(\sigma(s,T_j)-\rho_{j,i}^{Inf}\sigma(s,T_i)+\rho_j^{B,I}\left\{\Gamma(s,T_i)-\Gamma(s,T_j)\right\}\right)ds}-1.
\end{aligned}$$

This shows that this convexity adjustment depends on

- the covariance between the two CPI forwards$\frac{CPI(T_j,T_i)}{CPI(T_j,T_j)}$,

[7] Which corresponds to a zero coupon forward swap $\mathbb{E}^{\mathbb{Q}}\left[\left.e^{-\int_t^{T_i}r(s)ds}\frac{CPI(T_i,T_i)}{CPI(T_j,T_j)}\right|\mathcal{F}_t\right]$ with a discount factor $B(0,T_i)$ meadows.

[8] This ratio is called *naïve CPI forward.*

[9] This implied correlation should not be the same as the one between inflation and interest rate used in Economic's models. This latter is negative whereas the implied one could be positive.

- the covariance between the zero-coupon forward bond $B(t,T_j,T_i)=\frac{B(t,T_i)}{B(t,T_j)}$ observed at time T_j and the forward CPI $CPI(T_j,T_j)$as nominal and inflation securities covariances.

If we note the correlation between CPI forward $CPI(T_j,T_j)$ and of the zero-coupon forward bond $B(T_j,T_j,T_i)$ as

$$\zeta_{j,i}^{I,B}\equiv\frac{\rho_j^{B,I}\int_0^{T_j}\sigma(s,T_j)(\Gamma(s,T_i)-\Gamma(s,T_j))ds}{T_j\ \sigma_{BS}(0,T_j)\ \Gamma_{BS}(0,T_j,T_i)},$$

where

$$T_j\Gamma_{BS}^2(0,T_j,T_i)\equiv\int_0^{T_j}\left(\Gamma(s,T_i)-\Gamma(s,T_j)\right)^2ds$$

is the integrated volatility of $B(T_j,T_j,T_i)$ and replacing by the equation (6), we can write $\lambda_{Cvx}(0,T_j,T_i)$ as only a sum of volatilities:

$$\boxed{\begin{array}{rl}\frac{\lambda_{Cvx}(0,T_j,T_i)}{\frac{CPI(0,T_j)}{CPI(0,T_i)}} = & e^{T_j\sigma_{BS}(0,T_j)\zeta_{j,i}^{I,B}\Gamma_{BS}(0,T_j,T_i)} \\ & .e^{-\frac{1}{2}\left\{T_i\sigma_{BS}^2(0,T_i)-T_j\sigma_{BS}^2(0,T_j)+(T_i-T_j)\sigma_{BS}^2(T_j,T_i)\right\}}-1.\end{array}} \tag{15}$$

4.3 Assumption about the forward bond volatility

Because of the uncertainty on the estimation of the instantaneous correlation between the forward bond and the CPI, we take as an input the new integrated correlation $\zeta_{j,i}^{I,B}$. In the case of constant CPI volatility $\sigma(s,T_j)$ and forward bond volatility $(\Gamma(s,T_i)-\Gamma(s,T_j))$, notice that the two correlations: the instantaneous $\rho_i^{B,I}$ and integrated one $\zeta_{j,i}^{I,B}$ are the same.

Using the BGM model (see [4]), the volatility of the forward bond can be read directly from the volatility structure of the caplets. This comes from the fact that

$$B(t,T_j,T_i)=\prod_{k=j..i-1}\frac{1}{1+\delta_kF(t,T_k,T_k+\delta_k)},$$

where $F(t,T_k,T_k+\delta_k)$ is the forward Libor of period δ_k fixing at time T_k and paid at time $T_k+\delta_k$ and observed at time t.

Applying Itô (and looking only at the stochastic part) leads immediately to

$$dB(t,T_j,T_i)=\sum_{k=j..i-1}\frac{B(t,T_j,T_i)}{1+\delta_kF(t,T_k,T_k+\delta_k)}\delta_k\sigma^F(t,T_k)F(t,T_k,T_k+\delta_k)\,dB_{\mathbb{Q}_{T_i}}^k(t),$$

where $\sigma^F(t,T_k)$ is the lognormal volatility of the forward Libor $F(t,T_k,T_k+\delta_k)$ and where the diffusion is taken under the $\mathbb{Q}_{T_i}$ probability measure. This means that the forward bond volatility is approximately given by

$$\Gamma_{BS}(0,T_j,T_i)=\sum_{k=j..i-1}\frac{\delta_kF(0,T_k,T_k+\delta_k)\,\sigma^F(0,T_k)}{1+\delta_kF(0,T_k,T_k+\delta_k)},$$

where we have approximated the forward by its current value.

4.4 Specific form of volatilities

4.4.1 Case of Black and Scholes

In this case from (7), we get:

$$\gamma(T_j,T_i)=T_j\sigma_{BS}(0,T_i)\,\sigma_{BS}(0,T_j),\ \forall 1\leq i\leq n.$$

Hence, the inflation convexity adjustment become a function of the inflation zero-coupon volatilities and the zero-coupon forward bond ones only:

$$\boxed{\frac{\lambda_{Cvx}(0,T_j,T_i)}{\frac{CPI(0,T_j)}{CPI(0,T_i)}}=e^{T_j\sigma_{BS}(0,T_j)\left(\zeta_{j,i}^{I,B}\Gamma_{BS}(0,T_j,T_i)+\sigma_{BS}(0,T_j)-\rho_{j,i}^{Inf}\sigma_{BS}(0,T_i)\right)}-1.} \tag{16}$$

4.4.2 Case of Hull and White

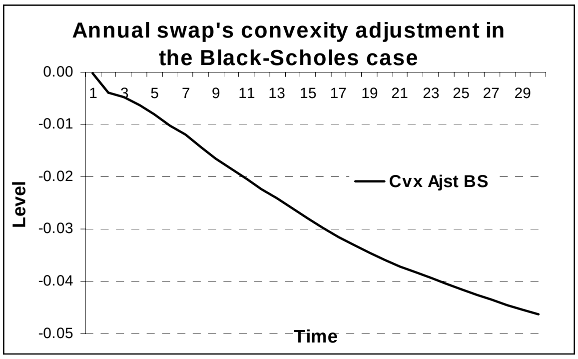
We have only to replace the explicit expressions of covariances $\gamma(T_j, T_i)$ and the variances $T_j \sigma_{BS}^2(0, T_j)$, from (9) or (11) in (15):

$$\boxed{\frac{\lambda_{Cvx}(0,T_j,T_i)}{\frac{CPI(0,T_j)}{CPI(0,T_i)}} = \begin{array}{l} e^{\rho_j^{B,I} T_j \sigma_{BS}(0,T_j) \zeta_{j,i}^{I,B} \Gamma_{BS}(0,T_j,T_i)} \\ .e^{-\frac{1}{2}\left\{T_i \sigma_{BS}^2(0,T_i) - T_j \sigma_{BS}^2(0,T_j) + (T_i - T_j)\sigma_{BS}^2(T_j,T_i)\right\}} - 1. \end{array}} \tag{17}$$

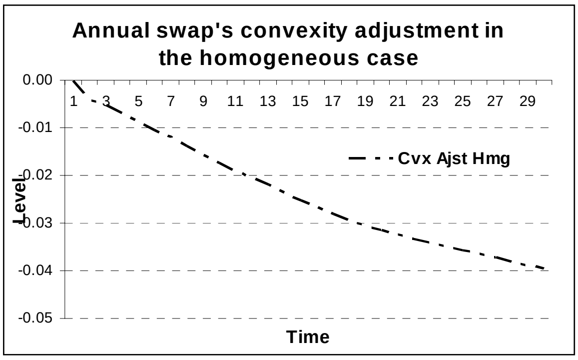
4.5 Numerical Results

Keeping the data in the section (3.5) we present the following results about convexity adjustment of year-on-year swaps:

Example 7 (Black-Scholes case with $\zeta_{j,i}^{I,B} = 0.3$) *In this example the year-on-year swap's convexity adjustments is negative and decreasing with time. This means that the year-on-year swap rate is at least lower than a year-on-year swap rate whose legs are priced with the raw CPI forwards.*



Example 8 (Homogenous case with $\zeta_{j,i}^{I,B} = 0.3, \forall j < i$) *In this case the year-on-year swap's convexity adjustment is greater than in the Black-Scholes case, because for the same other data, the homogenous year-on-year volatilities are lower then the Black-Scholes ones.*



5 Coherence tests

From the preceding sections, one can notice that the market's unknowns are threesome: **the Black-Scholes volatilities of zero-coupon options, the year-on-year ones and the CPIs' implicit correlations**. However, these three data are not observable at the same time. In fact, one can find some consistency relationship between these three data according to the assumed volatility function. We present conditions of level for volatilities and confidence intervals for each component of this trio.

5.1 Level volatility condition

The aim of this subsection, is to provide a minimal level for year-on-year volatilities from a zero-coupon volatility curve.

In the log-normal *deterministic-volatility* model, we have the inequality of the Black-Scholes year-on-year volatility of maturity T and tenor δ:

$$\delta\sigma_{BS}^2\left(T-\delta,T\right) \geq \int_{T-\delta}^{T} \sigma^2\left(s,T\right)ds. \tag{18}$$

If we consider the two chief forms of volatility function of the last sections, we find respectively:

- Black-Scholes type $\sigma\left(s,T\right)=\sigma\left(T\right)$:

$$\int_{T-\delta}^{T} \sigma^2\left(s,T\right)ds = \delta\sigma^2\left(T\right) = \delta\sigma_{BS}^2\left(0,T\right),$$

  so

$$\boxed{\sigma_{BS}\left(T-\delta,T\right) \geq \sigma_{BS}\left(0,T\right).} \tag{19}$$

  This implies that the year-on-year volatility curve must to be above the zero-coupon one. This does not correspond to the reality of the market.

- Homogeneous case $\sigma\left(s,T\right)=f\left(T-s\right)$:

$$\begin{aligned}\int_{T-\delta}^{T} \sigma^2\left(s,T\right)ds &= \int_{T-\delta}^{T} f^2\left(T-s\right)ds \overset{v=T-s}{=} \int_0^{\delta} f^2\left(v\right)dv \\ &\overset{u=\delta-v}{=} \int_0^{\delta} f^2\left(\delta-u\right)du \\ &= \int_0^{\delta} \sigma^2\left(u,\delta\right)du = \delta\sigma_{BS}^2\left(0,\delta\right).\end{aligned}$$

  Then

$$\boxed{\sigma_{BS}\left(T-\delta,T\right) \geq \sigma_{BS}\left(0,\delta\right).} \tag{20}$$

  This means that the value of the volatility of a year-on-year option of a fixed maturity and a fixed tenor, is at least greater than the volatility of a zero-coupon option of a maturity which is worth the tenor (the first zero-coupon volatility and then the year-on-year one when $\delta = 1$ *year*).

5.2 Triangle bounds

We allow ourselves to name the trio Black-Scholes zero-coupon, year-on-year volatilities and CPIs' correlation implicit the "*market data triangle*". Independently from any model or an hypothesis on volatility function, we want to have an order of size of one of the heads of the *market data triangle*, by fixing the two others. To have a intuitive size of each component, we use the main relation (6) rewritten as follows:

$$\delta\sigma_{BS}^2\left(T-\delta,T\right) = T\sigma_{BS}^2\left(0,T\right) + \left(T-\delta\right)\sigma_{BS}^2\left(0,T-\delta\right) - 2\rho_{T-\delta,T}\gamma\left(T-\delta,T\right). \tag{21}$$

The different intervals are obtained according to the choice of the volatility function. As the Black-Scholes case doesn't match to the market, we will restrict to the homogeneous case, in which we can distinguish two cases $\frac{\partial\sigma(s,T)}{\partial T} \geq 0$[10] and $\frac{\partial\sigma(s,T)}{\partial T} \leq 0$[11]. These inequalities involve the relation between zero-coupon, year-on-year volatilities and covariance below:

$$\left\{\begin{array}{l}\sigma\left(s,T-\delta\right) \gtreqless \sigma\left(s,T\right) \\ 0 \leq \sigma\left(s,T\right) \leq 1\end{array}\right. \Rightarrow T\sigma_{BS}^2\left(0,T\right) - \delta\sigma_{BS}^2\left(0,\delta\right) \gtreqless \gamma\left(T-\delta,T\right) \gtreqless \left(T-\delta\right)\sigma_{BS}^2\left(0,T-\delta\right). \tag{22}$$

It leads to these interval bounds of each component of the *market data triangle*, resumed on the tables below:

[10] For example, the integrated Hull and White volatility $\sigma\left(t,T\right)=\sigma_T\left(T-t\right)\frac{1-e^{-\lambda_T(t)(T-t)}}{\lambda_T(t)}$.

[11] For example, the Hull and White type volatility $\sigma\left(t,T_i\right)=\sigma_T\left(T-t\right)e^{-\lambda_T(t)(T-t)}$.

(∗) For $\kappa \in \left\{\rho_{T-\delta,T}, \sigma_{BS}\left(T-\delta,T\right)\right\}$,

$$\boxed{\kappa \in \left[\min\left(\kappa_1,\kappa_2\right), \max\left(\kappa_1,\kappa_2\right)\right],} \tag{23}$$

where

	κ_2	κ_2
$\rho_{T-\delta,T}$	$\frac{1}{2}\frac{T\sigma^2_{BS}(0,T)+(T-\delta)\sigma^2_{BS}(0,T-\delta)-\delta\sigma^2_{BS}(T-\delta,T)}{\left(T\sigma^2_{BS}(0,T)-\delta\sigma^2_{BS}(0,\delta)\right)}$	$\frac{1}{2}\left(1+\frac{T\sigma^2_{BS}(0,T)-\delta\sigma^2_{BS}(T-\delta,T)}{(T-\delta)\sigma^2_{BS}(0,T-\delta)}\right)$
$\sigma_{BS}(T-\delta,T)$	$\sqrt{\frac{\left(1-2\rho_{T-\delta,T}\right)T\sigma^2_{BS}(0,T)+(T-\delta)\sigma^2(0,T-\delta)+2\rho_{T-\delta,T}\delta\sigma^2(0,\delta)}{\delta}}$	$\sqrt{\frac{T\sigma^2_{BS}(0,T)+\left(1-2\rho_{T-\delta,T}\right)(T-\delta)\sigma^2(0,T-\delta)}{\delta}}$

(∗∗) For $\sigma_{BS}\left(0,T\right)$, we have

$$\boxed{\begin{array}{l}\sigma_{BS}\left(0,T\right) \in \left[\min\left(\sigma_1,\sigma_2\right), \max\left(\sigma_1,\sigma_2\right)\right] \|_{\left\{0\leq\rho\leq\frac{1}{2}\right\}} \\ +\left[0,\min\left(\sigma_1,\sigma_2\right)\right] \cup \left[\max\left(\sigma_1,\sigma_2\right), +\infty\right[\|_{\left\{\frac{1}{2}\leq\rho\leq1\right\}}.\end{array}} \tag{24}$$

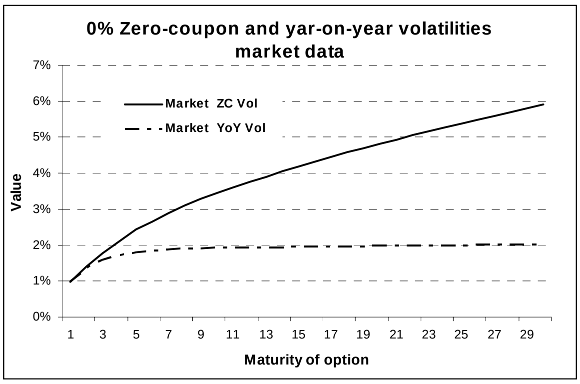
Where

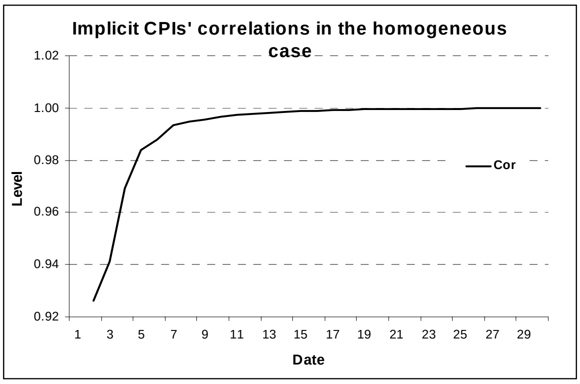
$$\begin{cases} \sigma_1 = & \sqrt{\frac{\delta\sigma^2(T-\delta,T)-(1-2\rho_\delta)(T-\delta)\sigma^2(0,T-\delta)}{T}} \\ \sigma_2 = & \sqrt{\frac{\delta\sigma^2(T-\delta,T)-(T-\delta)\sigma^2(0,T-\delta)-2\rho_\delta\delta\sigma^2(0,\delta)}{T(1-2\rho_\delta)}} \end{cases}.$$

5.3 Numerical Results

5.3.1 Triangle heads

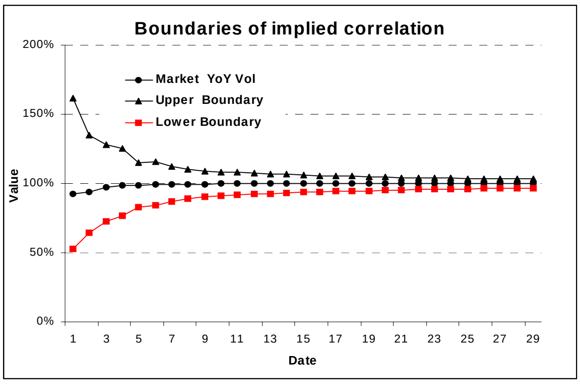
Example 9 (Implicit correlation from zero-coupon and year-on-year volatilities.) *We present below a set of market data of zero-coupon and year-on-year volatilities and the implicit CPIs' correlations deduced from a homogeneous volatility calibration function schema. Let's remark that the size are unrealistic at the medium and long term. This means that the market can be incoherent and that there is at least a freedom degree in excess.*



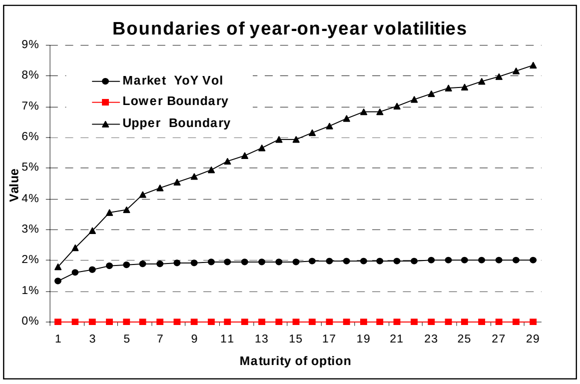


5.3.2 Triangle bounds

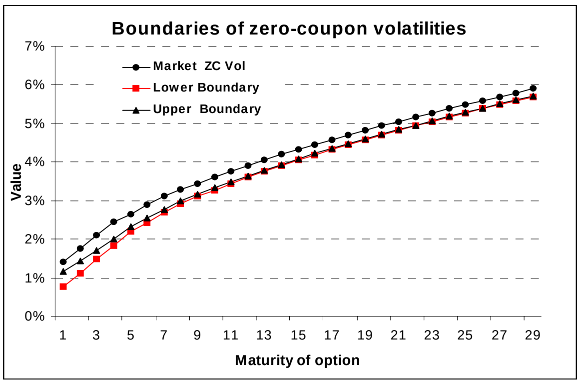
Example 10 (CPIs' implicit correlations bounds from volatilities) *We can notice that the implicit CPIs correlation implied by the market is always above 1. The confidence interval, the lower bound at least, aims at providing the trader a credible and intuitive level of the market correlations.*



Example 11 (Year-on-year volatilties' bounds from zero-coupon ones and correlations) *As the section* (5.1) *shows, the homogeneous year-on-year volatility curve is decreasing but doesn't go under the first point.*



Example 12 (Zero-coupon volatilties' bounds from year-on-year ones and correlations) *We can observe that in the beginning of the curve* $\sigma_{BS}(0,T) \in [\min(\sigma_1,\sigma_2), \max(\sigma_1,\sigma_2)]$ $\left(\Rightarrow 0 \le \rho \le \frac{1}{2}\right)$, *and after* $\sigma_{BS}(0,T) \in [\max(\sigma_1,\sigma_2), +\infty[\ \left(\Rightarrow \frac{1}{2} \le \rho \le 1\right)$. *This means that zero-coupon volatilities provide already an interval of the size of the CPIs' implicit correlations.*



6 Conclusion

In this paper, we derive a market model for the inflation derivatives. Under weak assumptions, we can set up a model driven only by the term structure of volatilities, describing CPIs forwards. This allows us in particular to relate *zero-coupon swaps (*swap market inputs) and *volatilities of year-on-year options (*inputs of the option market). This term structure of volatility as well as assumptions on the implicit correlations (between CPIs and CPI-zero-coupon nominal Bond) allows us to:

- to infer zero-coupon volatilities from the *vol cube information*,
- to price year-on-year swaps with consistent *convexity adjustments*.

Compared to previous models, the offered market models gives consistent assumptions between the zero-coupon and year-on-year inflation swap market.

Although it is not possible to estimate accurately implicit correlation, we show that these correlations should satisfy certain boundaries conditions. We notice that these boundaries conditions imply unrealistic level of correlation under certain model hypotheses. We also provide confidence interval for the three unknowns of the inflation market, leading to what we called the *"market data triangle"* inequalities. These relationship relates two of the unknowns to the remaining one.

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