Local Volatility in Multi Dimensions

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Outline

- Prelude.
- Multi asset arbitrage.
- Minimal multi asset models.
- Discrete case and calibration by Monte-Carlo.
- Foreign exchange, interest rates, equities.
- Conclusion.

References

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- Austing, P (2011): “Repricing the Cross Smile: An Analytic Joint Density.” *Risk* July.

- Dupire, B (1994): “Pricing with a Smile.” *Risk* July.

- Shelton, D (2015): “Interpolating the Smile with Path-Dependent Local Volatility.” *ICBI Global Derivatives*.

Prelude

- To simplify the exposition and save time I will work with the model in its simplest form.

- It is relatively straightforward to generalise the model presented here to FX and equities.

- … but interest rates are more complicated and you will have to consult future material about that.

Multi Asset Arbitrage

- Consider a market with stocks $s_1,\ldots,s_I$ and bank account s_0.

- Assume interest rates and dividends are zero, and set the start prices to be $s_i(0)=0$.

- Assume you can trade variance contracts on all spreads

$$v_{ij}(t)=PV[(s_i(t)-s_j(t))^2] \tag{1}$$

- We note that covariance matrix is given by

$$g_{ij}=PV[s_i s_j]=\frac{1}{2}PV[s_i^2+s_j^2-(s_i-s_j)^2]=\frac{1}{2}(v_{i0}+v_{j0}-v_{ij}) \tag{2}$$

- Absence of arbitrage implies that the covariance matrix

$$G(t) = \{g_{ij}(t)\} \qquad (3)$$

- … must be ***positive semi definite*** for all t.

- If not, there exist non-zero portfolio weights $\{w_i\}$ so that

$$PV[(\sum_i w_i s_i(t))^2] = \sum_i \sum_j w_i w_j PV[s_i(t) s_j(t)] = w' G(t) w < 0 \qquad (4)$$

- This is contradicting absence of arbitrage since:

$$(\sum_i w_i s_i(t))^2 \geq 0 \qquad (5)$$

Multi Asset Arbitrage -- Notes

- Positive definiteness has to hold for

$$\{g_{ij}(t_2) - g_{ij}(t_1)\} \tag{6}$$

- … for all pairs $t_1 < t_2$.

- Identification of arbitrage: any symmetric matrix G can be written as

$$G = O \Lambda O' \tag{7}$$

- …. where $\Lambda = Diag(\lambda_1, \ldots, \lambda_N)$ is a diagonal matrix of eigenvalues and O is an orthogonal matrix of eigenvectors, i.e. $OO' = I$.

- If $\lambda_j < 0$ then $w_i = O_{ij}$ is a set of arbitrage portfolio weights.

Minimal Multi Asset Models

- … is a multi asset local volatility model

$$
\begin{aligned}
& ds_i = \sigma_i(t,s_i)dW_i \quad ,i=1,\ldots,N \\
& dW_i \cdot dW_j = \rho_{ij}(t,s_i,s_j)dt
\end{aligned}
\tag{8}
$$

- … where the local correlation is given from the volatility of the spread

$$
\begin{aligned}
& (d(s_i - s_j))^2 / dt = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j = \sigma_{ij}^2 \\
& \Downarrow \\
& \rho_{ij}(t,s_i,s_j) = \frac{\sigma_i(t,s_i)^2 + \sigma_j(t,s_j)^2 - \sigma_{ij}(t,s_i - s_j)^2}{2\sigma_i(t,s_i)\sigma_j(t,s_j)}
\end{aligned}
\tag{9}
$$

- So the model is parameterised from the local spread volatilities $\{\sigma_{ij}(s_i - s_j)\}$ which are given as function of the spread levels.

- The model is constructed as to be able to fit the initial option prices

$$c_{ij}(t,k) = E[(s_i(t) - s_j(t) - k)^+] \tag{10}$$

- … through the Dupire equation

$$0 = -\frac{\partial c_{ij}}{\partial t} + \frac{1}{2}\sigma_{ij}(t,k)^2 \frac{\partial^2 c_{ij}}{\partial k^2} \tag{11}$$

- Absence of arbitrage is dictated through the usual conditions

$$\frac{\partial c_{ij}}{\partial t} > 0 \quad , \frac{\partial^2 c_{ij}}{\partial k^2} > 0 \tag{12}$$

- … ***plus*** the correlation matrix

$$\{\rho(t, s_i, s_j)\} \tag{13}$$

- … being bounded in $[-1,1]$ and ***positive definite***.

- The construction through spread volatility rather than correlation is similar to Austing (2011).

Discrete Time

- For several reasons it is beneficial to consider the discrete time case.
- First, models live in computers and computers live in discrete time.
- Secondly, in real applications the model setup will have to be somewhat modified relative to what we have outlined so far.
- Thirdly, it would be nice to be able to handle various model extensions such as stochastic volatility and stochastic interest rates.
- It turns out that these modifications and extensions are relatively straightforward to handle in discrete time.

Discrete Time Minimal Model

- An Euler discretisation of the model on the time grid $\{t_h\}$ is

$$\Delta s_i(t_h) = \sigma_i(t_h, s_i(t_h))\Delta W_i(t_h)$$

$$\{\Delta W_i(t_h)\} \sim N(0, \{\rho_{ij}(t_h)\}) \tag{14}$$

$$\rho_{ij}(t_h, s_i, s_j) = \frac{\sigma_i(t_h, s_i)^2 + \sigma_j(t_h, s_j)^2 - \sigma_{ij}(t_h, s_i - s_j)^2}{2\sigma_i(t_h, s_i)\sigma_j(t_h, s_j)}$$

- … where we have used the notation $\Delta x(t_h) = x(t_{h+1}) - x(t_h)$.

- As in the continuous time case, the model is specified through spread volatility rather than correlation.

- Absence of arbitrage requires the matrix $P = \{\rho_{ij}\}$ to be positive definite.

Monte-Carlo Pricing

- In a Monte-Carlo simulation, over samples $\{\omega\}$, the value of an option that expiries as time t_{h+1} can be written as a sum over Bachelier's formula

$$
\begin{aligned}
c_{ij}(t_{h+1},k) &= \frac{1}{N}\sum_{\omega} E_{t_h}[(\underbrace{s_i(t_{h+1})-s_j(t_{h+1})}_{\substack{Conditional\\ Normal\,Distributed}}-k)^+|\omega] \\
&= \frac{1}{N}\sum_{\omega} \underbrace{b(\Delta t_h,k;s_i-s_j,\sigma_{ij}(t_h,s_i-s_j))(t_h,\omega)}_{Bachelier's\ \ formula}
\end{aligned}
\tag{15}
$$

- … where $N=\#\{\omega\}$ is the number of samples and Bachelier's formula is

$$
b(\tau,k;s,v)=(s-k)\Phi(x)+v\sqrt{\tau}\phi(x) \quad ,x=\frac{s-k}{v\sqrt{\tau}} \tag{16}
$$

- This is so because over each time step $s_i - s_j$ has a conditional normal distribution – due to the Euler discretisation.

Monte-Carlo Calibration

- If we wish to calibrate the model to the strikes $\{k_{ij}^1,\ldots,k_{ij}^L\}$ at expiry t_{h+1} then we parameterise the volatility function $\sigma_{ij}(t_h;s_i - s_j)$ with L parameters.

- … for example linear interpolation between the L strike points.

- We then solve the minimization problem

$$\inf_{\sigma_{ij}(t_h,\cdot)} \sum_l (\underbrace{c(t_{h+1},k_{ij}^l)}_{mc\,\text{model}\,price} - \underbrace{\hat{c}(t_{h+1},k_{ij}^l)}_{market\,price})^2 \tag{17}$$

- Note that the calibration of $\{\sigma_{ij}(t_h,\cdot)\}$ is independent for different pairs (i, j).

- After calibration to the options for each spread pair (i, j) then we can construct the correlation matrix $P = \{\rho_{ij}\}$.

Positive Definiteness and Bootstrap

- The resulting correlation matrix P is not necessarily positive definite.

- To make it positive definite, decompose into the product $P = O\Lambda O'$, chop negative eigenvalues and rescale to obtain units along the diagonal.

- This procedure is not computationally costless.

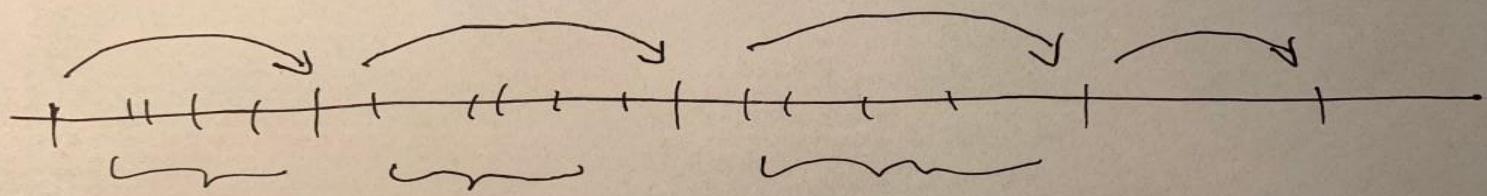
- Once done with calibration of the time step $t_h \mapsto t_{h+1}$, we simulate forward to calibrate the model to the time step $t_{h+1} \mapsto t_{h+2}$.

- We note that within MC error, calibration is ***exact***.

Timelines

- The calibration time line $\{t_h\}$ is fixed.
- However, we can insert extra simulation time points as we wish inside each calibration time bucket $[t_h, t_{h+1}]$.
- As long as we keep the volatilities and correlations constant over these extra time points.
- In that sense, the model looks a bit like the model in Shelton (2015).
- However, we use Monte-Carlo rather than numerical integration and this makes our model applicable to high dimensions.

- Bootstrap calibration by Monte-Carlo
- Using <u>Normality</u> of Euler Stepping.



- Any simulation time line after Calibration.
- Freezing Volatility & Correlation over each calibration time Bucket.

Applications and Extensions

- Foreign exchange: Obvious but note that log-normal form and quanto adjustments are necessary.

- Equities: In this case we would calibrate to basket rather than spread options. Potentially, using notions of average correlation.

- Note that non-trivial dividend modeling also can be handled this way.

- Interest rates: Non-trivial but interesting. Both multifactor Cheyette and LMM type models can constructed.

- The interest rate models can potentially calibrate simultaneously to cap and swaption smiles as well as smiles of spread and mid-curve options.

- Stochastic volatility and even rough volatility is straightforward.

- It is also possible to do models that simultaneously calibrate to SP500 and VIX smiles.

- … and more.

Numerical Implementation

- So far, we have implemented a multi factor Cheyette model for interest rates and a multi price model for FX and equities.
- Both with multi factor stochastic volatility.
- The intention is to combine the two model types to a Next Gen Beast.
- Both are implemented with extensive use of multi threading on CPUs.
- Adjoint differention (AAD) risk has been implemented for the interest rate model.

Numerical Performance

- Hardware is a standard 4 core CPU machine.

- 5 Ccy FX model calibration to 5 strikes in all crosses on expiries 1m, 2m, … 12m:

    - 8,192 paths: 0.46s
    - 65,536 paths: 3.32s

- 4 factor interest rate model. Calibration to 5 strikes in all crosses (spread options and mid curves) in tenors 3m, 2y, 10y, 30y on expiries 1y, 2y, … , 10y.

    - 8,192 paths: 0.45s
    - 65,536 paths: 3.44s

- 6 factor interest rate model. Calibration to 5 strikes in all crosses in tenors 3m, 2y, 5y, 10y, 20y, 30y on expiries 1y, 2y, … , 10y.

  - 8,192 paths: 1.00s
  - 65,536 paths: 7.13s

- Pure simulation times (without calibration) are roughly half.

- AD risk around a factor 10 slower than calibration/pricing.

- Numerical performance is definitely ok, but a tad slower than my audience is used to.

- The approach lends itself very well to TensorFlow/GPU acceleration.

Conclusion

- We have presented a new approach to multi factor local volatility with associated Monte-Carlo calibration methodology that is performing, flexible and general.

- Next steps:

    - Combining interest rate and price models in a 5G Beast.

    - Dynamic volatility surface modelling.

    - GPU/TensorFlow acceleration.

- The future is bright.