Stochastic interest rates for local volatility hybrids models

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Abstract: This paper studies the impact of stochastic interest rates for local volatility hybrids. Our research shows that it is possible to explicitly determine the bias between the local volatility of a model with stochastic interest rates and the local volatility of the same model, but with deterministic interest rates as a function between the correlation of the stochastic interest rates and the digital at the local strike. The paper will show that this bias can be expressed in a simpler form under the assumption of a diffusion of the stochastic interest rates, enabling us to compute a fast calibration for a hybrid model with stochastic interest rates. This bias leads to a decrease in the value of the local volatility as a result of the induced volatility caused by the stochastic drift. Numerical results illustrate the importance of the bias and confirm that some stochastic noise arises from the stochastic drift.

1. Motivation

It is well known that hybrid modeling is one of the most challenging tasks for quants. The difficulty arises from the multi-dimensional nature of the problem and stochastic drift arising from stochastic interest rates. As the drift term is under stochastic interest rates not any more a simple deterministic function, it becomes much harder to calibrate and simulate equity and fx models. Intuitively, a stochastic interest rate induces some stochasticity in the drift of the equity or fx asset and therefore creates some additional volatility. Hence, if one calibrates the volatility of an equity or fx model, assuming deterministic drift, and then uses this volatility in the same model but with stochastic interet rates, one finds that the new model has an overall volatility that is bigger than that of the initial model. Hence, if one has calibrated the volatility to match the price of liquid options under a deterministic drift assumption (namely assuming deterministic interest rates) and uses the calibrated results in the same model but now with stochastic interest rates, one finds that the new model does not match option prices any more and overprices these options.

One has to therefore account for stochastic drift when calibrating the volatility. The bad news is that standard closed formulas cannot apply directly as they assume deterministic interest rates. Surprisingly, there is not much in the literature, as standard practice is to use simple diffusion for the equity or FX models like a Black Scholes and a Hull and White model for the interest rates. In this particular case, one can find some closed forms. A recent work from Pitterbarg [Pit] goes a step further and shows how to construct a stochastic volatility FX model with stochastic interest rates. But there is no work on the combination of stochastic interest rates and local volatility models. This is precisely the motivation of our work.

2. Deriving a How to integrate stochastic drift

a. Impact of a stochastic interest rates on an equity model

Before looking at the impact of stochastic interest rates in a local volatility, let us mention that the issue of stochastic drift can be elegantly handled by the change of forward measure in the case of a derivative that does not depend explicitly on an interest rate. As shown in Geman et al [Gem], one can show that the payoff can be computed under the forward measure as follows:

$$E\left(e^{-\int_0^T r_t dt} X_T\right) = B(0,T) E^T(X_T) \tag{2.1}$$

where r_t is the risk free rate available at time t, $B(0,T)$ is the price of a zero coupon bond maturing at time T and E^T is the expectation under the forward measure

In this case, there is no need to model the stochasticity arising from interest rates as the payoff is only sensitive to the discount factor value. The real question arises when we cannot use the change of numeraire as the financial product itself does really depend on the interest rates.

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Consider a jump diffusion extension of the Dupire [Dup] model as in Andersen, Andreasen [And], with stochastic interest rates r_t given by:

$$\frac{dS_t}{S_t^-} = \left(r_t - q_t\right)dt - \lambda_t m_t + \sigma\left(t, S_t^-\right)dW_t^S + \left(J_t - 1\right)dN_t \tag{2.2}$$

where q_t is the traditional dividend rate, W_t^S is a Brownian motion, J_t is a positive jump magnitude lognormally distributed with mean m_t and standard deviation Σ, N_t is a Poisson process with density λ_t counting the number of jumps. J_t and N_t are assumed to be mutually independent and also independent from the Brownian motion W_t^S and the stochastic interest rates r_t.

Let us denote by $\sigma^{hyb}(T,K)$ the local volatility of the hybrid model with stochastic interest rates for the maturity T and strike K, and by $\sigma^{\det}(T,K)$ the local volatility of the same model but with deterministic interest rates. We know that the local volatility of the jump diffusion model with deterministic interest rates is given by the Andersen Andreasen formula (See [And]):

$$\left[\sigma^{\det}(T,K)\right]^2 = 2\frac{\partial_T C_{T,K} - \left(q_T + \lambda_T m_T - r_T\right)K\partial_K C_{T,K} - E\left[J_T C_{T,K/J_T}\right]\lambda_T + \left(q_T + \lambda_T\left(m_T + 1\right)\right)C_{T,K}}{\partial_{KK}C_{T,K}K^2} \tag{2.3}$$

The following proposition shows that the two local volatilities are related with a bias related to the correlation between the digital at the local strike paying the difference between the stochastic and deterministic interest rates.

Proposition 2.1

The bias between the local volatility $\sigma^{hyb}(T,K)$ of the model with stochastic interest rates and the local volatility $\sigma^{\det}(T,K)$ of the same model but with deterministic interest rates can be expressed in terms of the correlation between the local digital and the spread between the stochastic interest rates and its deterministic value as follows:

$$\left[\sigma^{hyb}(T,K)\right]^2 = \left[\sigma^{\det}(T,K)\right]^2 - \frac{2B(0,T)}{\partial_{KK}C_{T,K}K}E^T\left[\left(r_T^{hyb} - r_T^{\det}\right)1_{\{S_T^- - K > 0\}}\middle| S_T^- = K\right] \tag{2.4),}$$

where $r_T^{hyb} - r_T^{\det}$ are respectively the interest rates in the model with stochastic and deterministic interest rates and $\sigma^{hyb}(T,K) = E\left[\sigma^2\left(T, S_T^-\right)\middle| S_T^- = K\right]$ is the expected local volatility at the strike K.

Proof: see Appendix 6, Proof 5.1. □

It is remarkable to see that the presence of jumps does not play any additional role between the model with stochastic interest rates and the one with deterministic interest rates. This is a straightforward consequence of the independence between Brownian motion, interest rates and the jump processes.

b. Relating the bias with the volatility of forward rates

Following a work of Balland [Bal] on stochastic interest rates in FX model, we show how to simplify the bias between the local volatility under stochastic interest rates and the one of the same model but with deterministic interest rates. To make the expression (2.4) useful, it is appropriate to give some additional information. Without loss of generality, and by means of the martingale representation theorem, we can assume that the stochastic interest rates can be represented by a pure diffusion process as follows:

$$dr_t^{hyb} = \gamma\left(t, r_t^{hyb}\right)dW_t^r \tag{2.5).}$$

We will assume that the Brownian motion driving stochastic interest rates and the one driving the equity are correlated with constant correlation:

$$\left\langle dW_t^S, dW_t^r \right\rangle = \rho_{S,r}dt \tag{2.6}$$

Under these assumptions, we can express the bias between the local volatility of a model with stochastic interest rates and the local volatility of the same model but with deterministic interest rates in a simple form as follows:

Proposition 2.2

$$\left[\sigma_{loc}^{hyb}(T,K)\right]^2 = \left[\sigma_{loc}^{\det}(T,K)\right]^2 - 2Cov\left(X_T, r_T \middle| X_T = \ln K\right) \quad (2.7),$$

where $X_T = \ln S_T$ and $r_T = r_0 + \int_0^T \gamma\left(t, r_t^{hyb}\right) dW_t^r$.

Proof: see Appendix 6, Proof 6.2.□.

Assuming a positive correlation between $X_T = \ln S_T$ and r_T, formula (2.7) proves that the local volatility for the model with a stochastic interest rates drift should be lower than the one corresponding to the same model but with deterministic interest rates. Intuitively, the stochastic interest rates drift creates some stochasticity that must be deduced from the local volatility to have the same level of global volatility. Numerical results in the next section show the importance of this bias.

c. Computing the bias iteratively

The expression 2.7 is not straightforward to calculate as the covariance term implies the knowledge of the local volatility for the hybrid process $\sigma_{loc}^{hyb}(T,K)$, which is the result of formula (2.7). However, formula (2.7) can be computed iteratively by a fixed point iteration procedure. For a given maturity T and a strike K, the first step is to approximate the underlying process $X_T = \ln S_T$ with stochastic interest rates by its equivalent without stochastic interest rates at the same maturity T and strike K. In this first step, the covariance term in expression (2.7) is approximated as follows:

$$Cov(X_T, r_T) = Cov\left(\int_0^T \sigma^{hyb}(u, S_u) dW_u^S, \int_0^T \gamma(u,r) dW_u^r\right) \approx Cov\left(\int_0^T \sigma^{\det}(u,K) dW_u^S, \int_0^T \gamma(u,r) dW_u^r\right) \quad (2.8).$$

Denoting by $\rho_{S,r}$ the correlation between the Brownian motion

$$\left\langle dW_u^S, dW_u^r \right\rangle = \rho_{S,r} dt \quad (2.9),$$

and assuming that the stochastic interest rates follows a Hull and White process, hence the interest rates volatility does not depend on the short rates and is gven by a deterministic function γ_s, we get that the first order local volatility $\sigma_{loc}^{hyb,1}(T,K)$ is given by:

$$\sigma_{loc}^{hyb,1}(T,K) = \sqrt{\left[\sigma_{loc}^{\det}(T,K)\right]^2 - 2\rho_{S,r}\int_0^T \sigma^{\det}(s,K)\gamma_s ds} \quad (2.10),$$

The second step is to re-use this approximation to compute the second order local volatility $\sigma_{loc}^{hyb,2}(t,K)$ in terms of first order local volatility as follows:

$$\sigma_{loc}^{hyb,2}(T,K) = \sqrt{\left[\sigma_{loc}^{\det}(T,K)\right]^2 - 2\rho_{S,r}\int_0^T \sigma_{loc}^{hyb,1}(s,K)\gamma_s ds} \quad (2.11),$$

We can iterate this computation until the result converges to the appropriate value with the local volatility at order n $\sigma_{loc}^{hyb,n}(t,K)$ computed follows:

$$\sigma_{loc}^{hyb,n}(T,K) = \sqrt{\left[\sigma_{loc}^{\det}(T,K)\right]^2 - 2\rho_{S,r}\int_0^T \sigma_{loc}^{hyb,n-1}(s,K)\gamma_s ds} \quad (2.12),$$

Three iterations in practice are enough to get a good convergence. This fixed point iteration procedure is the main result of the paper and is now illustrated in the next section.

3. Numerical results

The purpose of this numerical section is to illustrate the importance of the bias between the local volatility with stochastic interest rates and the local volatility without. As starting inputs we use Black Scholes volatility and calculate the local volatility under stochastic and deterministic interest rates. Table 1 provides the implied Black Scholes volatility for the Eurostoxx 50 index. Smiles are more pronounced for short term maturity and flatten for maturity like 5 years.

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	38.91%	30.34%	26.05%	23.91%	17.84%	17.84%	17.84%	17.84%	17.31%	16.79%
3M	36.56%	29.26%	25.61%	23.73%	18.31%	18.31%	18.31%	18.31%	17.76%	17.23%
6M	32.86%	27.21%	24.39%	23.05%	18.32%	18.32%	18.32%	18.32%	17.77%	17.23%
9M	31.40%	26.47%	24.03%	22.79%	18.45%	18.45%	18.45%	18.45%	17.90%	17.36%
1Y	30.30%	25.98%	23.75%	22.72%	18.72%	18.72%	18.72%	18.72%	18.16%	17.61%
2Y	27.75%	24.71%	23.18%	22.43%	19.01%	19.01%	19.01%	19.01%	18.44%	17.88%
3Y	24.56%	23.31%	22.69%	22.07%	18.94%	18.94%	18.94%	18.94%	18.37%	17.82%
4Y	26.57%	24.60%	23.54%	23.01%	19.97%	19.97%	19.97%	19.97%	19.38%	18.79%
5Y	26.15%	24.53%	23.67%	23.23%	20.34%	20.34%	20.34%	20.34%	19.73%	19.14%
10Y	25.73%	24.46%	23.80%	23.45%	20.71%	20.71%	20.71%	20.71%	20.08%	19.49%

Table 1: Implied BS volatility

Table 2 provides the resulting local volatility surface for the 1 factor model with jumps. To apply the Andersen Andreasen formula given by (2.3), we use a smoothened implied volatility surface. Following a methodology similar to the one explained in [Sep], we smooth the initial implied volatility surface by fitting a two steps parabolic functional form for the implied Black Scholes volatility as follows:

1) For a given strike, we compute the best fitting parabola given by $\sigma_{imp}(K,T) = a(K) + b(K)T + c(K)T^2$
2) Then, we fit a second parabola in strikes by determining the following values:
   a. $a(K) = \alpha_1 + \beta_1 \log\left(\frac{K}{S_0}\right) + \gamma_1 \log\left(\frac{K}{S_0}\right)^2$
   b. $b(K) = \alpha_2 + \beta_2 \log\left(\frac{K}{S_0}\right) + \gamma_2 \log\left(\frac{K}{S_0}\right)^2$
   c. $c(K) = \alpha_3 + \beta_3 \log\left(\frac{K}{S_0}\right) + \gamma_3 \log\left(\frac{K}{S_0}\right)^2$ s

We found the following values

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 22.51\% & -1.00\% & 0.20\% \\ -61.01\% & 24.14\% & -3.28\% \\ 168.32\% & -78.95\% & 10.70\% \end{pmatrix} \quad (3.1).$$

Once the parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ have been determined by least square minimization, we apply the Andersen Andreasen [And] formula given in (2.3) to compute the local volatility surface. For the jump parameters, we do an historical calibration as explained in Andersen Andreasen [And] and assume that the jump probability is equal to 1%, the jump size -10% and the jump volatility 5%. The resulting volatility surface is given in table 2 below**.**

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	37.94%	29.90%	25.84%	23.78%	17.94%	17.64%	14.49%	14.49%	14.49%	14.49%
3M	32.90%	27.42%	24.63%	23.18%	18.53%	18.46%	18.23%	17.60%	14.25%	14.25%
6M	27.29%	24.27%	22.69%	21.99%	18.30%	18.25%	18.14%	17.93%	16.87%	15.03%
9M	27.34%	24.54%	22.97%	22.17%	18.86%	18.83%	18.74%	18.61%	17.77%	16.72%
1Y	26.19%	24.03%	22.60%	22.17%	19.16%	19.12%	19.05%	18.94%	18.16%	17.30%
2Y	22.08%	22.25%	22.28%	21.83%	19.06%	19.03%	18.97%	18.89%	18.21%	17.47%
3Y	22.55%	22.95%	23.00%	22.70%	20.04%	20.00%	19.95%	19.89%	19.20%	18.49%
4Y	29.46%	26.74%	25.17%	24.98%	22.28%	22.25%	22.20%	22.15%	21.41%	20.66%
5Y	23.71%	23.90%	24.00%	24.01%	21.75%	21.70%	21.66%	21.61%	20.83%	20.16%
10Y	23.13%	23.43%	23.50%	23.35%	20.89%	20.85%	20.81%	20.75%	20.01%	19.33%

Table 2: 1F local volatility

To compute the formula (2.7) for a given maturity t and a strike K, we apply the iterative procedure using three iterations as described in section 2.c. To get the interest rates volatility, we use the ATM cap volatility and find the value for the deterministic interest rates volatilities γ_s given by the table 3.

Time (s)	γ_s
1M	1.04%
3M	1.20%
6M	1.36%
9M	1.26%
1Y	1.17%
2Y	0.99%
3Y	0.90%
4Y	0.85%
5Y	0.81%
10Y	0.81%

Table 3: interest rates volatility

For the equity interest rates correlation, we use a correlation of 40% obtained by an historical computation on two years of weekly data between the Eurostoxx 50 index and the 2Y swap rate. We provide in table 4, 5 and 6, the hybrid volatility for the order 1, 2 and 3. We see that three iterations are enough to obtain a good convergence.

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	37.90%	29.87%	25.80%	23.75%	17.91%	17.60%	14.45%	14.45%	14.45%	14.45%
3M	32.78%	27.31%	24.51%	23.07%	18.41%	18.35%	18.13%	17.49%	14.13%	14.13%
6M	27.01%	24.00%	22.43%	21.73%	18.05%	18.00%	17.90%	17.68%	16.64%	14.78%
9M	26.93%	24.14%	22.58%	21.79%	18.49%	18.46%	18.38%	18.24%	17.42%	16.36%
1Y	25.65%	23.50%	22.09%	21.67%	18.67%	18.64%	18.57%	18.47%	17.70%	16.84%
2Y	20.97%	21.25%	21.33%	20.89%	18.14%	18.11%	18.06%	17.99%	17.32%	16.59%
3Y	21.08%	21.60%	21.70%	21.42%	18.79%	18.75%	18.71%	18.65%	17.97%	17.28%
4Y	28.00%	25.24%	23.63%	23.46%	20.80%	20.77%	20.73%	20.69%	19.96%	19.22%
5Y	21.51%	21.85%	22.03%	22.08%	19.88%	19.83%	19.80%	19.76%	18.99%	18.33%
10Y	18.99%	19.47%	19.63%	19.51%	17.06%	17.02%	16.98%	16.93%	16.19%	15.51%

Table 4: local volatility for the hybrid model with stochastic interest rates computed at iteration 1

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	37.90%	29.87%	25.80%	23.75%	17.91%	17.60%	14.45%	14.45%	14.45%	14.45%
3M	32.78%	27.31%	24.51%	23.07%	18.41%	18.35%	18.13%	17.49%	14.13%	14.13%
6M	27.01%	24.00%	22.43%	21.74%	18.05%	18.01%	17.90%	17.68%	16.64%	14.79%
9M	26.93%	24.15%	22.58%	21.79%	18.50%	18.46%	18.38%	18.25%	17.43%	16.37%
1Y	25.65%	23.51%	22.09%	21.68%	18.68%	18.64%	18.58%	18.47%	17.71%	16.85%
2Y	21.01%	21.26%	21.34%	20.90%	18.16%	18.12%	18.07%	18.00%	17.33%	16.60%
3Y	21.11%	21.60%	21.70%	21.41%	18.79%	18.75%	18.71%	18.65%	17.97%	17.28%
4Y	28.01%	25.24%	23.63%	23.46%	20.81%	20.77%	20.74%	20.69%	19.96%	19.23%
5Y	21.55%	21.88%	22.04%	22.09%	19.90%	19.86%	19.82%	19.78%	19.01%	18.36%
10Y	19.40%	19.83%	19.97%	19.85%	17.45%	17.42%	17.38%	17.32%	16.61%	15.95%

Table 5: local volatility for the hybrid model with stochastic interest rates computed at iteration 2

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	37.90%	29.87%	25.80%	23.75%	17.91%	17.60%	14.45%	14.45%	14.45%	14.45%
3M	32.78%	27.31%	24.51%	23.07%	18.41%	18.35%	18.13%	17.49%	14.13%	14.13%
6M	27.01%	24.00%	22.43%	21.74%	18.05%	18.01%	17.90%	17.68%	16.64%	14.79%
9M	26.93%	24.15%	22.58%	21.79%	18.50%	18.46%	18.38%	18.25%	17.43%	16.37%
1Y	25.65%	23.51%	22.09%	21.68%	18.68%	18.64%	18.58%	18.47%	17.71%	16.85%
2Y	21.01%	21.26%	21.34%	20.90%	18.15%	18.12%	18.07%	18.00%	17.33%	16.60%
3Y	21.11%	21.60%	21.70%	21.41%	18.79%	18.75%	18.71%	18.65%	17.97%	17.28%
4Y	28.01%	25.24%	23.63%	23.46%	20.81%	20.77%	20.73%	20.69%	19.96%	19.22%
5Y	21.55%	21.87%	22.04%	22.08%	19.89%	19.85%	19.82%	19.77%	19.00%	18.36%
10Y	19.35%	19.79%	19.93%	19.81%	17.40%	17.36%	17.32%	17.27%	16.55%	15.88%

Table 6: local volatility for the hybrid model with stochastic interest rates computed at iteration 3

Defining the bias as the difference between the local volatility in the jump diffusion model with deterministic interest rates and the one with stochastic interest rates,

$$\text{Bias} = \left[\sigma^{hyb}(t,K)\right] - \sigma^{\det}(t,K) \quad (3.2).$$

we can measure the impact of a stochastic interest rates in a local volatility model as given in Table 7. This data tableholds the most interesting numerical result of the paper. This result can be obtained by the Misys Summit Pricing Partners integrated solution. It numerically confirms the intuition that some stochasticity comes from the stochastic interest rates drift. Logically, we see that this bias gets bigger for longer maturity as the difference between the stochastic and the deterministic interest rates becomes more and more striking. The impact can be substantial and be as large as 3.5 percents for a 10 year maturity.

Expiry\Strike	85.00%	90.00%	95.00%	100.00%	105.00%	110.00%	115.00%	120.00%	125.00%	130.00%
1M	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%
3M	0.12%	0.12%	0.12%	0.12%	0.11%	0.11%	0.11%	0.11%	0.11%	0.11%
6M	0.28%	0.27%	0.26%	0.26%	0.25%	0.25%	0.24%	0.24%	0.23%	0.24%
9M	0.41%	0.39%	0.38%	0.38%	0.37%	0.37%	0.36%	0.36%	0.35%	0.34%
1Y	0.54%	0.52%	0.51%	0.50%	0.48%	0.48%	0.47%	0.47%	0.46%	0.45%
2Y	1.07%	0.99%	0.94%	0.93%	0.91%	0.91%	0.90%	0.90%	0.88%	0.87%
3Y	1.44%	1.35%	1.30%	1.29%	1.25%	1.25%	1.25%	1.24%	1.23%	1.21%
4Y	1.45%	1.50%	1.54%	1.52%	1.48%	1.48%	1.47%	1.46%	1.45%	1.43%
5Y	2.16%	2.03%	1.96%	1.93%	1.85%	1.85%	1.85%	1.84%	1.83%	1.81%
10Y	3.78%	3.64%	3.57%	3.55%	3.49%	3.49%	3.49%	3.48%	3.46%	3.45%

Table 7: Impact of stochastic interest rates

4. Conclusion

This paper has presented a methodology for taking into account the stochastic interest rates drift in a local volatility jump diffusion model. It provides an analytical formula for the bias and confirms the hypothesis that stochastic interest rates drift creates some stochasticity. Numerical results show that this bias can be substantial for long maturity.

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6. Appendix

Proof 6.1 for the result (2.4)

Taking the differentiation of the discounted call payoff and interchanging the differentiation and the expectation operator leads to

$$d\left[E\left(e^{-\int_0^t r_s ds}\left(S_t-K\right)^+\right)\right]=E\left[-\left(e^{-\int_0^t r_s ds}\left(S_t-K\right)^+\right)r_t dt+e^{-\int_0^t r_s ds}d\left(S_t-K\right)^+\right] \tag{6.1}$$

Using the Tanaka formula for the call payoff convex function $(X-K)^+$ for the jump diffusion given by (2.2) helps us to express the second part of the above expression as:

$$\begin{aligned} d\left(S_t-K\right)^+ &= 1_{\{S_t^- - K>0\}}\left(r_t-q_t-\lambda_t m_t\right)S_t^- dt+\frac{1}{2}\delta\left(S_t^- - K\right)\left[\sigma\left(t,S_t^-\right)S_t^-\right]^2 dt \\ &+1_{\{S_t^- - K>0\}}\sigma\left(t,S_t^-\right)S_t^- dW_t^S+\left[\left(J_t S_t^- - K\right)^+-\left(S_t^- - K\right)^+\right]dN_t \end{aligned} \quad (6.2),$$

where $\delta()$ is the Dirac function and S_t^- is the left continuous value before the jump. Combining equations (6.1) and (6.2) gives the following expression for the differentiated discounted call payoff:

$$\begin{aligned} d\left[E\left(e^{-\int_0^t r_s ds}\left(S_t-K\right)^+\right)\right] &= E\left[-r_t\left(e^{-\int_0^t r_s ds}\left(S_t^- - K\right)^+\right)+e^{-\int_0^t r_s ds}\left(1_{\{S_t^- - K>0\}}\left(r_t-q_t-\lambda_t m_t\right)S_t^- +\frac{1}{2}\delta\left(S_t^- - K\right)\left(\sigma\left(t,S_t^-\right)S_t^-\right)^2\right)\right]dt \\ &+E\left[e^{-\int_0^t r_s ds}\left(\left(J_t S_t^- - K\right)^+-\left(S_t^- - K\right)^+\right)\right]\lambda_t dt \end{aligned} \quad (6.3).$$

The differentiation of the call option seen as a function of the maturity time T is trivially given by

$$dC_{T,K}=\partial_T C_{T,K}dt \quad (6.4).$$

Using the fact that second order derivatives of the call prices with respect to the strikes is equal to the density given by the Dirac function $\delta\left(S_T^- - K\right)$, we get that the two expressions for the dt terms should be equal leading to:

$$\begin{aligned} \partial_T C_{T,K} &= E\left[e^{-\int_0^T r_s ds}1_{\{S_T^- - K>0\}}\left(\left(-q_T-\lambda_T(m_T+1)\right)S_T^- + K\left(r_T+\lambda_T\right)\right)\right]+\frac{1}{2}\partial_{KK}C_{T,K}K^2 E\left[\sigma^2\left(T,S_T^-\right)\middle| S_T^- = K\right] \\ &+E\left[e^{-\int_0^T r_s ds}\left(\left(J_T S_T^- - K\right)^+\right)\right]\lambda_T \end{aligned} \tag{6.5}$$

The above equation can be rephrased into the well known Dupire formula but with stochastic interest rates:

$$E\left[\sigma^2\left(T,S_T^-\right)\middle| S_T^- = K\right]=2\frac{\partial_T C_{T,K}-E\left[e^{-\int_0^T r_s ds}1_{\{S_T^- - K>0\}}\left(\left(-q_T-\lambda_T\left(m_T+1\right)\right)S_T^- + K\left(r_T+\lambda_T\right)\right)\right]-E\left[e^{-\int_0^T r_s ds}\left(J_T S_T^- - K\right)^+\right]\lambda_T}{\partial_{KK}C_{T,K}K^2} \quad (6.6).$$

The above formula can be seen as an extension of the Andersen Andreasen [And] formula (2.3) for stochastic interest rates. A particular case of the above formula is given in the case of a model with deterministic interest rates and leads to the following formula:

$$\left[\sigma^{\det}(T,K)\right]^2 = 2\frac{\partial_T C_{T,K} - (q_T + \lambda_T m_T - r_T) K \partial_K C_{T,K} - E\left[J_T C_{T,K/J_T}\right]\lambda_T + (q_T + \lambda_T (m_T + 1)) C_{T,K}}{\partial_{KK} C_{T,K} K^2} \quad (6.7)$$

At this stage, it becomes easy to relate the local volatility $\sigma^{hyb}(T,K)$ of the model with stochastic interest rates and the local volatility $\sigma^{\det}(T,K)$ of the same model but with deterministic interest rates. Notice that the call put prices should be the same in the two models as they are calibrated on the same market data. Hence, in the two models, the terms:

$E\left[e^{-\int_0^T r_s ds}\left(J_T S_T^- - K\right)^+\right]$ and $E\left[e^{-\int_0^T r_s ds} 1_{\{S_T^- - K>0\}}\left\{\left(-q_T - \lambda_T (m_T + 1)\right)\left(S_T^- - K\right)^+\right\}\right]$ should be equal in the two models. Summarizing all of this with equations (6.6) and (6.7) provides the following bias;

$$\left[\sigma^{hyb}(T,K)\right]^2 = \left[\sigma^{\det}(T,K)\right]^2 - \frac{2}{\partial_{KK} C_{T,K} K} E\left[\left(e^{-\int_0^T r_s^{hyb} ds} r_T^{hyb} - e^{-\int_0^T r_s^{\det} ds} r_T^{\det}\right) 1_{\{S_T^- - K>0\}} \middle| S_T^- = K\right] \quad (6.8),$$

which leads to the result (2.4) under the forward measure.□

Proof 6.2 for the result (2.7)

With the assumption (2.5), equation (2.4) becomes

$$\left[\sigma^{hyb}(T,K)\right]^2 = \left[\sigma^{\det}(T,K)\right]^2 - \frac{2B(0,T)}{\partial_{KK} C_{T,K} K} E^T\left[1_{\{S_T^- - K>0\}} \int_0^T \gamma\left(t, r_t^{hyb}\right) dW_t^r\right] \quad (6.9).$$

Without loss of generality, we can assume that the normal volatility $\gamma\left(t, r_t^{hyb}\right)$ of the stochastic interest rates is hypo-elliptic[4], namely: $\left|\gamma\left(t, r_t^{hyb}\right) r_t^{hyb}\right| > \varepsilon r_t^{hyb}$

Under this condition, the Skorohod integral of the volatility function $\gamma\left(t, r_t^{hyb}\right)$ coincides with the Ito integral. The last integration can therefore be interpreted as the product of the Skorohod integral with the digital process $1_{\{S_T^- - K>0\}}$. Hence, we can reformulate it in terms of the Malliavin derivatives of the digital process, by means of the integration by parts (See Nualart [Nua]):

$$E^T\left[\int_0^T 1_{\{S_T^- > K\}} \gamma\left(t, r_t^{hyb}\right) dW_t^r\right] = E^T\left[\int_0^T \gamma\left(t, r_t^{hyb}\right) D_t^r 1_{\{S_T^- > K\}} dt\right] \quad (6.10),$$

where $D_t^r 1_{\{S_T^- > K\}}$ is the standard Malliavin derivative with respect to the Brownian motion W_t^r. But using the chain rule (see for instance Nualart [Nua]), one can simplify the Malliavin derivative as follows

$$D_t^r 1_{\{S_T^- > K\}} = \delta\left(X_T^- - \ln K\right) D_t^r X_T^- \quad (6.11),$$

where $X_t = \ln S_t$ is the log asset given by the following jump diffusion under the forward measure:

$$X_t = X_0 + \ln\left(\frac{B(t,T)}{B(0,T)}\right) + \int_0^t \sigma(u, X_u) dW_u^S - \frac{1}{2}\int_0^t \sigma^2(u, X_u) du + \int_0^t J(u, X_u) dN_u - \int_0^t E\left[J(u, X_u)\right] \lambda du$$

(6.12).Since $B(t,T)$ is an adapted process, its Malliavin derivative is zero.

Hence, the Malliavin derivative of X_T with respect to the Brownian motion W_t^r is given by the perturbed terms with respect to the Brownian motion W_t^r or any correlated terms, that is to say:

$$D_t^r 1_{\{S_T^- > K\}} = \delta\left(X_T^- - \ln K\right)\left[\rho_{S,r} D_t^S X_T^-\right] \quad (6.13).$$

The latter term is the Malliavin derivative with respect to the Brownian motion W_t^S. Recall that for a general jump diffusion process given by:

$$X_t = X_0 + \int_0^t \mu\left(s, X_s^-\right) ds + \int_0^t \sigma\left(s, X_s^-\right) dW_s + \int_0^t J\left(s, X_s^-\right) dN_s \quad (6.14),$$

[4] If our process is not hypo-elliptic, we can approximate our initial process by a series of hypo-elliptic processes that converge to our initial process.

the Malliavin derivatives can be expressed in terms of the first variation process as follows:

$$D_s X_t = \sigma(s, X_s)\frac{Y_t}{Y_u}1_{\{t>u\}} \tag{6.15),}$$

where the first variation process is:

$$\frac{dY_t}{Y_t^-} = \partial_X \mu(t, Y_t)dt + \partial_X \sigma(t, X_t)dW_t + \partial_X J(t, Y_t)dN_t$$
$$Y_{t=0} = 1 \tag{6.16),}$$

We can also notice that:

$$\frac{B(0,T)}{\partial_{KK} C_{T,K} K} E^T\left[\delta\left(X_T^- - \ln K\right)\right] = 1 \tag{6.17),}$$

Combining (6.10), (6.13), (6.15) and (6.17), one gets:

$$\left[\sigma^{hyb}(T,K)\right]^2 - \left[\sigma^{\det}(T,K)\right]^2 = -2\left(\int_0^T \gamma\left(t, r_t^{hyb}\right)\rho_{S,f} E^T\left[\sigma^{hyb}(t, X_t)\frac{Y_T}{Y_t} \middle| X_T = \ln K\right] dt\right) \tag{6.18).}$$

It is worth noticing at this stage that the term above is precisely the correlation term between the log asset $X_T = \ln S_T$ and the instantaneous rate $r_T = r_0 + \int_0^T \gamma\left(t, r_t^{hyb}\right) dW_t^r$ conditional to $X_T = \ln K$ since

$$\operatorname{cov}(X_T, r_T | X_T = \ln K) = \operatorname{cov}\left(X_T, \int_0^T \gamma\left(t, r_t^{hyb}\right) dW_t^r \middle| X_T = \ln K\right) = E^T\left[\left(\int_0^T \gamma\left(t, r_t^{hyb}\right)\rho_{S,f} E^T\left[\sigma^{hyb}(t, X_t)\frac{Y_T}{Y_t} \middle| X_T = \ln K\right] dt\right)\right] \tag{6.19).}$$

Combining (6.18) and (6.19) leads to the result. Note that this result was first found by Balland [Bal] in the case of no jumps and for a FX dynamics.□