

DIGITAL FOREX OPTIONS

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ABSTRACT. Some pricing methods for forex digital options are described. The price in the Garhman-Kohlhagen model is first described, more for completeness than as a real proposal. The call-spread approximation approach is described.

1. INTRODUCTION

A digital options, also called a *binary option*, has the simplest pay-off, if not the most natural one. A digital call (resp. put) pays a fixed amount if an exchange rate is above (resp. below) a given level, called *strike*. The fixed amount can be paid in any of the two currencies involved in the exchange rate.

To fix the notations, the digital forex option will be represented by a payment amount N in a currency X , an exercise date θ , a strike rate K representing the number of currency unit D required to buy another currency unit F and a *payment* or *settlement* date t_p . The currency X can be D or F . The payment D is sometime called the *domestic* currency and the other the *foreign* currency. The exchange rate between the currencies at time t is denoted S_t .

The pay-off of a digital call, denoted $\text{Digital}(\text{Call}, \theta, K, NX)$, is NX (with $N > 0$) when the exchange rate is such that $S_\theta \geq K$ (and nothing otherwise). The pay-off of a put, denoted $\text{Digital}(\text{Put}, \theta, K, NX)$, is NX when the exchange rate is such that $S_\theta < K$ (and nothing otherwise).

Note that a portfolio of a call and a put with same strike, dates and payment amount will pay in all circumstances the fixed amount NX and is equivalent to a zero-coupon in currency X whatever the other currency is.

Note also that contrarily to vanilla options there is not a total symmetry. A call on the currency pair F/D paid in D is not a put on the reverse currency pair D/F paid in F . The paid amounts are not the same in both cases (one is in currency D and the other one in currency F).

The discount factor for the currency Y view from s for maturity t will be denoted $P_Y(s, t)$. The forward rate for (payment) date t and today rate S_0 is given by

$$F_0^t = \frac{P_F(0, t)}{P_D(0, t)} S_0.$$

2. PRICING IN GARMAN-KOHLHAGEN MODEL

The Garman-Kohlhagen model is the Black-Scholes model applied to the foreign exchange market, with one interest rate in each currency. The spot rate is suppose to follow a geometric Brownian motion with a deterministic volatility.

The digital option paying ND has a present value, when the volatility is σ ,

$$\text{PV}^{\text{D, Digital}}(S_0) = NP_D(0, t_p)N(\omega d_-)$$

where $\omega = 1$ for a call, $\omega = -1$ for a put and

$$d_{\pm} = \frac{\ln\left(\frac{F_0^{tp}}{K}\right) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}.$$

The delta of the present value with respect to the today's exchange rate is given by

$$\Delta_{\text{Spot}} = NP_D(0, t_p) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_-^2\right) \omega \frac{1}{\sigma\sqrt{t}} \frac{1}{S}.$$

This pricing formula is given more for historical reason and completeness than as a real pricing alternative. Even if a vanilla option smile (i.e. a curve of implied volatilities for each strike in the Garman-Kohlhagen model) is available, the pricing of the digital options with the above formula and the vanilla option implied volatility for the given strike will not give a price coherent with the vanilla options.

3. CALL SPREAD APPROXIMATION

It is possible to provide a model independent approximation of the digital price based on vanilla options prices. The approach, called *call-spread*, is presented below.

The approach is presented for a call, the symmetrical approach for puts is obvious. The idea is to approximate the digital call payment with the difference between two vanilla calls. Associated to the digital strike K , we construct two other strikes $K_- \leq K$ and $K_+ \geq K$. As a simple example, we propose $K_{\pm} = K(1 \pm \epsilon)$ with ϵ small with respect to 1 (like 0.001).

Let $\text{Vanilla}(\text{Call}, \theta, K, M)$ denote the vanilla option pay-off for an option to exchange MF with KMD . The digital call option paying in D currency is approximated, with $M = N/(K_+ - K_-)$, by

$$\text{Digital}(\text{Call}, \theta, K, ND) \simeq \text{Vanilla}(\text{Call}, \theta, K_-, M) - \text{Vanilla}(\text{Call}, \theta, K_+, M)$$

and the digital put option by

$$\text{Digital}(\text{Put}, \theta, K, ND) \simeq \text{Vanilla}(\text{Put}, \theta, K_+, M) - \text{Vanilla}(\text{Put}, \theta, K_-, M).$$

When $S_{\theta} < K_-$, both pay-off are 0; when $S_{\theta} > K_+$, both pay-off are ND . In between, the digital option pay-off jumps from 0 to ND at K while the portfolio of vanilla pay-off increases linearly from one to the other. When $K_+ - K_-$ is small the two pay-offs are the same in most cases and the expected value of the difference is small.

The digital call option paying in *foreign* currency is approximated, with $M = N/(1/K_- - 1/K_+)$ by

$$\text{Digital}(\text{Call}, \theta, K, ND) \simeq \text{Vanilla}(\text{Put}, \theta, 1/K_-, M) - \text{Vanilla}(\text{Put}, \theta, 1/K_+, M)$$

and the digital put option by

$$\text{Digital}(\text{Put}, \theta, K, ND) \simeq \text{Vanilla}(\text{Put}, \theta, 1/K_+, M) - \text{Vanilla}(\text{Put}, \theta, 1/K_-, M).$$

Note that in this case when the currency paid F/D is quoted in one order, the currency pair D/F underlying the vanilla one is in the opposite order. The choice to represent the digital on F/D paying in currency F by vanilla options on D/F is to ensure some coherency in the pricing. the digital pays in currency F and its is more natural to have the present value in the same currency. The vanilla options on a currency paid are traditionally priced in the second currency. By writing the digital paying in currency F as a portfolio of vanilla options on the pair D/F , the pricing of the portfolio is directly done in the expected currency. The drawback is that the currency pair is in the non-market order and when computing the sensitivity to the market quote (for delta or gamma), one has to take this inversion into account.

The advantages of this approach are clear. It is *model independent* (if one has the prices of the vanilla options of strikes K_{\pm}). The approximated pay-off is smoother than the exact pay-off of the digital, avoiding huge (potentially infinite) delta numbers. Another advantage is that it is independent of the way the smile is represented. Some other pricing techniques for digitals rely on

a specific representation, in particular with exactly three points, like the Vanna-Volga approach by [Castagna and Mercurio \(2007\)](#) and the quadratic approximation by [Malz \(1997\)](#).

Some drawbacks are also immediate. It is an approximation and close to the money the two pay-offs will not be the same. But if required the strike can be selected in such a way that the pay-off of the vanilla portfolios are always higher ($K_+ = K$; super-replication) or always lower ($K_- = K$) than the original digital.

Another drawback, less immediate but technically maybe more important, is that this approach require one extra level of smoothness¹ for vanilla options. The pay-off of the digital can be viewed as the differentiation of the vanilla with respect to the strike. The above approximation is then the differentiation ratio approximation. Any lack of smoothness of the smile with respect to the strike will be immediately apparent. The importance of smoothness is clear in the examples Section 5 where we have used linear and (double-)quadratic interpolations.

4. RISK FIGURES

The risk figures (present value, currency exposure, curve sensitivity, ...) are directly computed from the equivalent figures for the vanilla options. It is enough to be able to subtract the figures for the two vanilla options involved in the call-spread representation.

As mentioned earlier, the only point of attention is the derivatives with respect to the spot rate in case the digital pays in the first currency (F). In the call-spread representation, the currency order of the vanilla is opposite one.

Let $PV_{\text{Vanilla}}^{X/Y}(T)$ be the present value of the vanilla option on the ratio X/Y for a rate T (i.e. $1X = TY$). The present value relationship is

$$PV_{F,\text{Digital}} = PV_{D/F,\text{Vanilla}}(1/S) - PV_{D/F,\text{Vanilla}}(1/S).$$

The first order derivative (delta0 with respect to S is (using the notation of [OpenGamma Research \(2012\)](#))

$$\Delta_{\text{Relative},F,\text{Digital}} = \Delta_{\text{Relative},D/F,\text{Vanilla}}(1/S) - \Delta_{\text{Relative},D/F,\text{Vanilla}}(1/S).$$

5. INTERPOLATION - SIMPLE EXAMPLES

In this section we describe several price profiles and implied densities for digitals by call spread using different interpolation schemes. The interpolation schemes are relatively simple and not based on financial strong foundations.

The examples are for EUR/USD options paying an USD amount. The curves, ATM volatilities, risk reversal and strangles are realistic figures even if not actual market figures at any given date.

The first example uses linear interpolation, flat extrapolation and five market volatility figures (ATM, 10 and 25 risk reversal and strangles). The linear interpolation example is more for illustrative purposes than as a real proposal for reasons that will appear clearly in the graphs.

The spot exchange rate is 1.40 and the option expiry nine months in the future. As the data are given in delta, other expiries would give similar pictures with the moneyness scaled with a square-root of time rule. The price profile is given for the Garman-Kohlhagen (GK) model with vanilla option of same strike implied volatility and for the call spread approximation in Figure 1(a). By price profile we mean the price of a continuum of options with different strikes priced with the same market data. The price in the GK is given to show the incoherence introduced by that approach but also as a reference for the eyes in term of smoothness.

The implied density is plotted in Figure 1(b). The density is computed as the second order derivative of the price with respect to the strike. The second derivative is approximated by its finite difference approximation. The problem around the data points (where the smile is not smooth

¹To be exact, the problem is not the smoothness in a theoretical sense, but the existence of points with large second order derivatives. If the smile was smoothed to be C^∞ with a convolution of very small radius (lets say 10^{-10} , the results would be the same for any practical purpose.

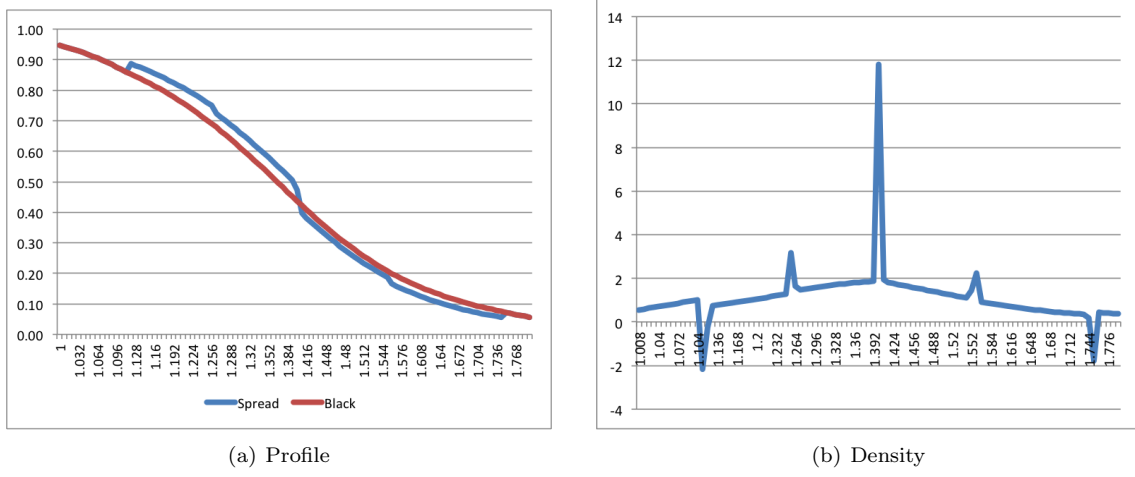


FIGURE 1. Example of price profile and density for linear interpolation and flat extrapolation.

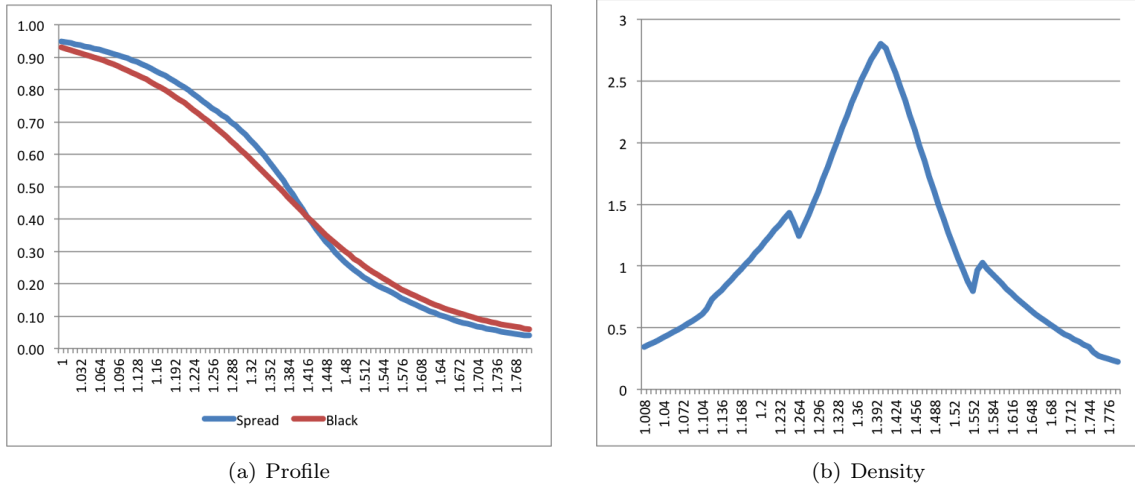


FIGURE 2. Example of price profile and density for double quadratic interpolation with 5 points and linear extrapolation.

is clear on the picture. Not only some points have some jumps in density (not nice in practice but theoretically not impossible), but in some places the density is even negative.

In the second example we use the same data but a double quadratic interpolation and a linear extrapolation. The similar profile and density graphs are given in Figure 2. The profile is a lot nicer even if there are still some kinks.

In the third example we use only three data points, a quadratic interpolation (degenerated double quadratic) and a linear extrapolation. The similar profile and density graphs are given in Figure 3.

6. INTERPOLATION – OTHER METHODS

The description of better interpolation schemes for smiles can be found in the OpenGamma note [White \(2012\)](#).

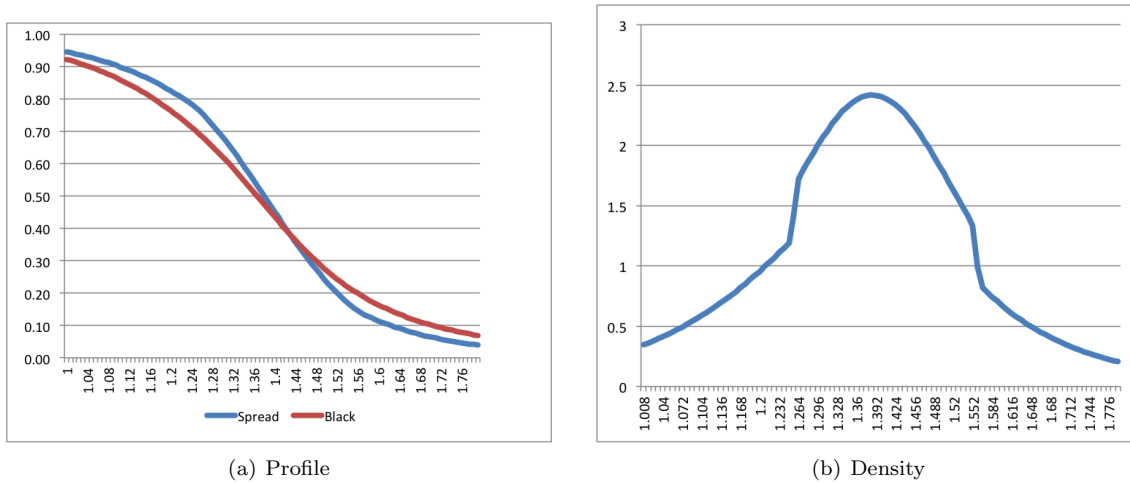


FIGURE 3. Example of price profile and density for quadratic interpolation with 3 points and linear extrapolation.

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