Martingale Optimal Transport and Robust Finance

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(with Mathias Beiglböck and Nizar Touzi)

April 2015

Outline

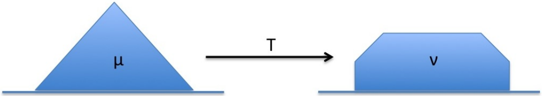
1. Monge–Kantorovich Optimal Transport

2. Martingale Optimal Transport

Monge Optimal Transport

Given:

- Probabilities μ, ν on $\mathbb{R}$.
- Reward (cost) function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.



Objective:

- Find a map $T : \mathbb{R} \to \mathbb{R}$ satisfying $\nu = T^{-1} \circ \mu$ such as to maximize the total reward,

$$\max_T \int f(x, T(x))\, \mu(dx).$$

Monge–Kantorovich Optimal Transport

Relaxation:

- Find a probability P on $\mathbb{R} \times \mathbb{R}$ with marginals μ, ν such as to maximize the reward:

$$\max_{P \in \Pi(\mu,\nu)} E^P[f(X,Y)], \quad \text{where} \quad \Pi(\mu,\nu) := \{P : \; P_1 = \mu, \; P_2 = \nu\}$$

  and $(X, Y) = \mathrm{Id}_{\mathbb{R}\times\mathbb{R}}$.

- $P \in \Pi(\mu,\nu)$ is a Monge transport if of the form $P = \mu \otimes \delta_{T(x)}$.

Example: Hoeffding–Frechet Coupling

Theorem: Let $f(x,y) = g(y-x)$ where g is strictly concave and sufficiently integrable. Then the optimal P is given by the Hoeffding–Frechet Coupling:

- P is the law of $((F_\mu)^{-1}, (F_\nu)^{-1})$ under the uniform measure on $[0,1]$.
- If μ is diffuse, P is of Monge type with $T = (F_\nu)^{-1} \circ F_\mu$.
- P is characterized by monotonicity:

$$\text{if } (x,y),(x',y') \in \operatorname{supp}(P) \text{ and if } x < x', \text{ then } y \leq y'.$$

Kantorovich Duality

- Buy $\varphi(X)$ at price $\mu(\varphi) := E^{\mu}[\varphi]$ and $\psi(Y)$ at $\nu(\psi)$ to superhedge,

$$f(X,Y) \leq \varphi(X) + \psi(Y).$$

- Then for all $P \in \Pi(\mu,\nu)$,

$$E^P[f(X,Y)] \leq E^P[\varphi(X) + \psi(Y)] = \mu(\varphi) + \nu(\psi).$$

- **Theorem** (Kantorovich, Kellerer): For any measurable $f \geq 0$,

$$\sup_{P \in \Pi(\mu,\nu)} E^P[f(X,Y)] = \inf_{\varphi,\psi} \mu(\varphi) + \nu(\psi)$$

and dual optimizers $\hat{\varphi}$, $\hat{\psi}$ exist.

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Application: Fundamental Theorem of Optimal Transport

Let $\Gamma = \{(x,y) : \hat{\varphi}(x) + \hat{\psi}(y) = f(x,y)\}$ and $P \in \Pi(\mu,\nu)$. TFAE:

(1) P is optimal.

(2) $P(\Gamma) = 1$.

(3) $\mathrm{supp}(P)$ is f-cyclically monotone P-a.s.; i.e.,

$$\sum_{i=1}^{n} f(x_i, y_i) \geq \sum_{i=1}^{n} f(x_i, y_{\sigma(i)}) \quad \forall\, (x_i, y_i) \in \mathrm{supp}(P), \quad \sigma \in \mathrm{Perm}(n).$$

(1)(2) If $P(\Gamma) < 1$, then P charges $\{(x,y) : \hat{\varphi}(x) + \hat{\psi}(y) > f(x,y)\}$ and thus $\mu(\hat{\varphi}) + \nu(\hat{\psi}) > E^P[f(X,Y)]$.

(2)(1) If $P(\Gamma) = 1$, then $\mu(\hat{\varphi}) + \nu(\hat{\psi}) = E^P[f(X,Y)]$, hence $P, \hat{\varphi}, \hat{\psi}$ are optimal.

(2)(3) This argument even shows: if $\tilde{P}(\Gamma) = 1$, then $\tilde{P}$ is an optimal transport between its own marginals. Apply this with discrete $\tilde{P}$ $\Rightarrow \Gamma$ is cyclically monotone.

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Dynamic Hedging

- Dynamically tradable underlying $S = (S_0, S_1, S_2)$.
- Semi-static superhedge:

$$f((S_t)_t) \leq \varphi(S_1) + \psi(S_2) + H_0(S_1 - S_0) + H_1(S_2 - S_1).$$

- With $S_0 = 0$, $S_1 = X \sim \mu$, $S_2 = Y \sim \nu$ and normalization $H_0 = 0$:

$$f(X, Y) \leq \varphi(X) + \psi(Y) + h(X)(Y - X).$$

- Formally, duality with $P \in \Pi(\mu, \nu)$ satisfying the constraint that

$$E^P[h(X)(Y - X)] = 0 \quad \forall h; \quad \text{i.e.} \quad E^P[Y|X] = X.$$

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Martingale Transport

- Set of martingale transports:

$$\mathcal{M}(\mu,\nu) = \{P \in \Pi(\mu,\nu) : E^P[Y|X] = X\}.$$

- **Theorem** (Strassen): $\mathcal{M}(\mu,\nu)$ is nonempty iff $\mu \leq_c \nu$; i.e.,

$$\mu(\phi) \leq \nu(\phi) \quad \forall\, \phi \text{ convex.}$$

- Martingale Optimal Transport problem: Given $\mu \leq_c \nu$,

$$\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)].$$

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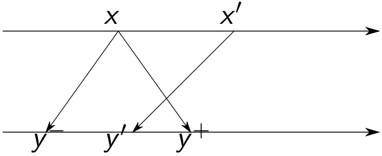

Figure : Forbidden Configuration.

Beiglböck–Juillet Coupling for Uniform Marginals

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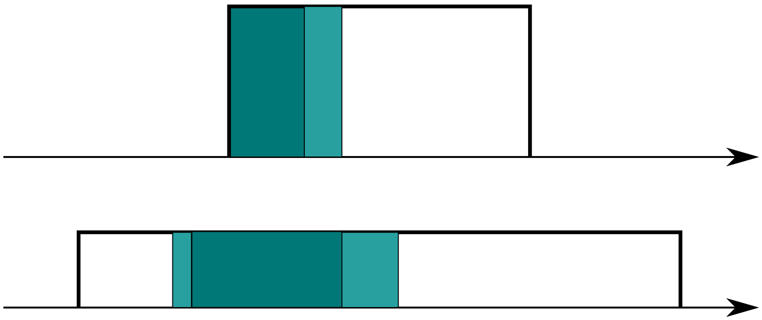

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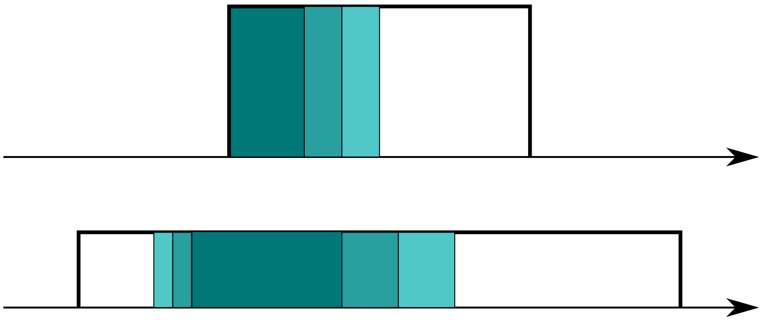

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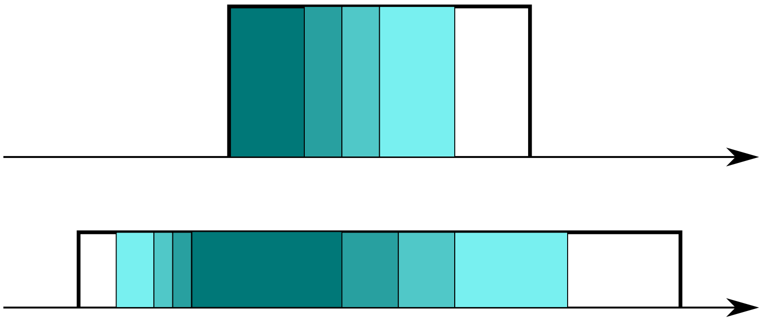

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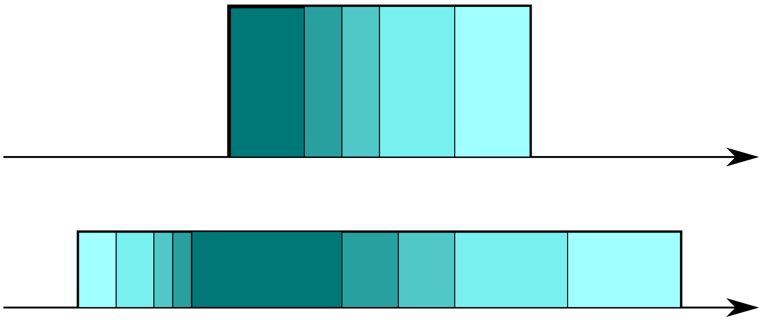

Duality for Martingale Optimal Transport

In analogy to Monge–Kantorovich duality we want:

(1) No duality gap:

$$\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)] = \inf_{\varphi,\psi,h} \mu(\varphi) + \nu(\psi).$$

(2) Dual existence: $\hat{\varphi}$, $\hat{\psi}$, $\hat{h}$.

Theorem (Beiglböck, Henry-Labordère, Penkner):

- For upper semicontinuous $f \leq 0$, there is no duality gap.
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An Example with Duality Gap

- Let f be the bounded, lower semicontinuous function

$$f(x,y) = \mathbf{1}_{x \neq y} = \begin{cases} 0 & \text{on the diagonal,} \\ 1 & \text{off the diagonal.} \end{cases}$$

- Let $\mu = \nu =$ Lebesgue measure on $[0,1]$.
- There exists a unique martingale transport P, concentrated on the diagonal ($T(x) = x$).
- Primal value: $\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)] = 0$.
- Dual optimizers exist, $\hat{\varphi} = 1$, $\hat{\psi} = 0$, $\hat{h} = 0$ but
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Financial Intuition

Earlier work considered model uncertainty over a set $\mathcal{P}$ of real-world models (with finitely many traded options). $\mathcal{P}$-q.s. version of the FTAP:

Theorem (Bouchard, N.):
Let $\mathcal{M} = \{\text{calibrated martingale measures } Q \ll \mathcal{P}\}$. Then

$$\text{No arbitrage NA}(\mathcal{P}) \iff \mathcal{P} \text{ and } \mathcal{M} \text{ have same polar sets}$$

and under this condition, quasi-sure duality holds with existence.

Reverse-engineered:

To avoid arbitrage, neglect events not seen by $\mathcal{M}$.

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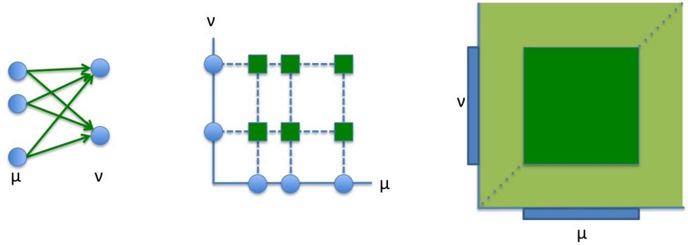
Ordinary and Martingale OT: What is the Difference?

- In ordinary OT, all roads $x \to y$ can be used.
- E.g., in the discrete case, $\mu \times \nu$ already has full support.



- **Theorem** (Kellerer): $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\Pi(\mu,\nu)$-polar if and only if

$$A \subseteq (N_1 \times \mathbb{R}) \cup (\mathbb{R} \times N_2), \quad \text{where} \quad \mu(N_1) = \nu(N_2) = 0.$$

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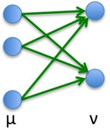

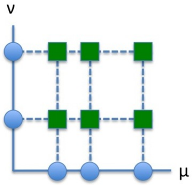

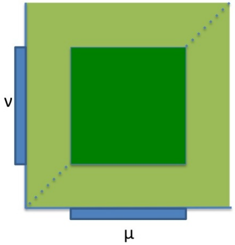

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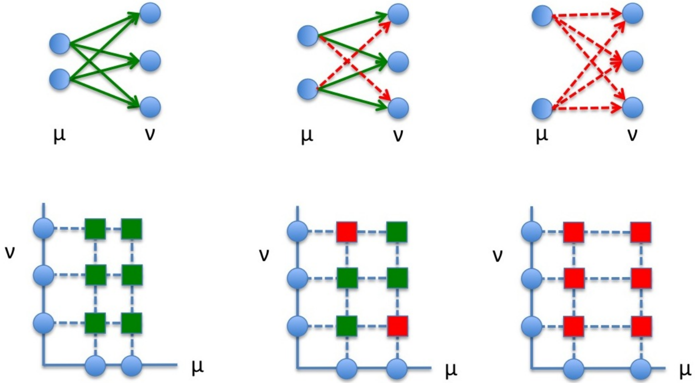
Obstructions for Martingale Transport

- In martingale OT, some roads $x \to y$ can be blocked.



Potential Functions

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- $\mu \leq_c \nu \iff u_\mu \leq u_\nu$.
- If

  $$u_\mu(x) = u_\nu(x); \quad \text{i.e.} \quad E[|X-x|] = E[|Y-x|] \qquad (*),$$

  then x is a barrier for any martingale transport:

1. Jensen: $|X-x| = |E[Y|X]-x| = |E[Y-x|X]| \leq E[|Y-x|\,|X]$
2. Under $(*)$, it follows that $|X-x| = E[|Y-x|\,|X]$ a.s. Hence,

   $$E[|Y-x|\mathbf{1}_{X\geq x}] = E[|X-x|\mathbf{1}_{X\geq x}] = E[(X-x)\mathbf{1}_{X\geq x}] = E[(Y-x)\mathbf{1}_{X\geq x}]$$

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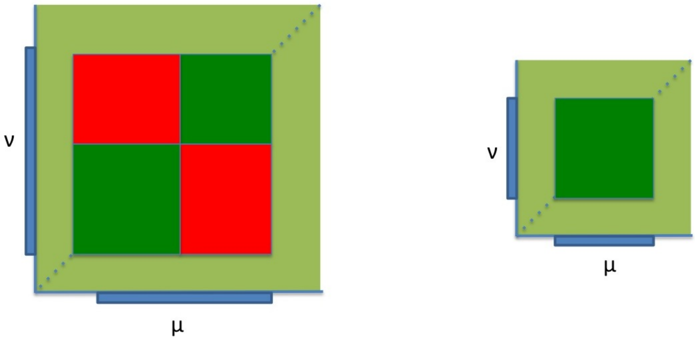
Structure of $\mathcal{M}(\mu,\nu)$-polar Sets



Theorem: "These are precisely the $\mathcal{M}(\mu,\nu)$-polar sets."

Duality Result

Theorem

Let $f \geq 0$ be measurable and consider the quasi-sure relaxation of the dual problem:

$$f(X,Y) \leq \varphi(X) + \psi(Y) + h(X)(Y - X) \quad \mathcal{M}(\mu,\nu)\text{-q.s.}$$

Then,

(1) *there is no duality gap,*

(2) *dual optimizers $\hat{\varphi}$, $\hat{\psi}$, $\hat{h}$ exist.*

- The superhedge is pointwise on each component (e.g., $\mu = \delta_{x_0}$).
- Dual existence in the pointwise formulation typically fails as soon as there is more than one component.
- Application as in the FTOT.

Key Idea for the Proof

Core step: make almost-optimal φ_n, ψ_n converge.

- Control φ_n, ψ_n by a single, convex function χ_n.
- Suppose there is only one component; thus $\nu - \mu >_c 0$.
- After a normalization, $\chi_n(0) = \chi_n'(0) = 0$ and

$$0 \leq \int \chi_n \, d(\nu - \mu) \leq \text{const.}$$

- This bounds the convexity of χ_n.

$\Rightarrow$ Relative compactness of (χ_n).

$\Rightarrow$ Relative compactness of (φ_n, ψ_n); Komlos.

Conclusion

- The quasi-sure formulation ("model uncertainty") seems to be a natural setup for the Martingale Optimal Transport problem.

- One may expect this to be true for a larger class of problems; in particular, if discontinuous reward functions are involved or attainment is desired.

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