Local Volatility in Multi Dimensions

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Outline

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- Minimal multi asset models.
- Minimal models and arbitrage.
- Discrete case and calibration by Monte-Carlo.
- Foreign exchange, interest rates, equities.
- Conclusion.

References

- Andreasen, J (2020): “Multi-Factor Cheyette Monte-Carlo Calibration.” *Saxo WP*.
- Austing, P (2011): “Repricing the Cross Smile: An Analytic Joint Density.” *Risk* July.
- Dupire, B (1994): “Pricing with a Smile.” *Risk* July.
- Guyon, J (2014): “Local Correlation Familes.” *Risk* February.
- Gyöngy, I (1986): “Mimicking the One-Dimensional Marginal Distributions of Processes having and Ito Differential.” *Probability Theory and Related Fields* 71.
- McCloud, P (2011): “The CMS Triangle Arbitrage.” *Risk* January.

- Piterbarg, V (2011): “Farka’s Lemma, Spread Options, and Linear Programming“. *Risk* August.

- Shelton, D (2015): “Interpolating the Smile with Path-Dependent Local Volatility.” *ICBI Global Derivatives*.

Prelude

- To simplify the exposition and save time I will work with the model in its simplest form.

- It is relatively straightforward to generalise the model presented here to FX and equities.

- … but interest rates are more complicated and you will have to consult future material about that.

Multi Asset Arbitrage

- Consider a market with stocks $s_1,\ldots,s_I$ and bank account s_0.
- Assume interest rates and dividends are zero, and set the start prices to be $s_i(0)=0$.
- Assume you can trade variance contracts on all spreads

$$v_{ij}(t)=PV[(s_i(t)-s_j(t))^2] \tag{1}$$

- We note that covariance matrix is given by

$$g_{ij}=PV[s_i s_j]=\frac{1}{2}PV[s_i^2+s_j^2-(s_i-s_j)^2]=\frac{1}{2}(v_{i0}+v_{j0}-v_{ij}) \tag{2}$$

- Absence of arbitrage implies that the covariance matrix

$$G(t) = \{g_{ij}(t)\} \qquad (3)$$

- … must be ***positive semi definite*** for all t.
- If not, there exist non-zero weights $\{w_i\}$ so that

$$PV[(\sum_i w_i s_i(t))^2] = \sum_i \sum_j w_i w_j PV[s_i(t) s_j(t)] = w' G(t) w < 0 \qquad (4)$$

- This is contradicting absence of arbitrage since:

$$(\sum_i w_i s_i(t))^2 \geq 0 \qquad (5)$$

- The arbitrage portfolio is in this case given by

$$\{\underbrace{(w_i w_j)}_{\substack{\textit{portfolio}\\ \textit{weight}\\ ij}} \cdot \underbrace{g_{ij}}_{\substack{\text{cov}\,ij\\ \textit{contract}}}\} \tag{6}$$

Multi Asset Arbitrage -- Notes

- We can sharpen a bit: Positive definiteness has to hold for

$$\{g_{ij}(t_2)-g_{ij}(t_1)\} \tag{7}$$

- … for ***all*** pairs $t_1<t_2$.

- Identification of arbitrage: any symmetric matrix G can be written as

$$G=O\Lambda O' \tag{8}$$

- …. where $\Lambda=Diag(\lambda_1,\ldots,\lambda_N)$ is a diagonal matrix of eigenvalues and O is an orthogonal matrix of eigenvectors, i.e. $OO'=I$.

- If $\lambda_j < 0$ then $w_i = O_{ij}$ is a set of arbitrage weights.

- … with the arbitrage portfolio given by

$$\{(w_i w_j) \cdot g_{ij}\} \tag{9}$$

Minimal Multi Asset Models

- … is a multi asset local volatility model

$$
\begin{aligned}
& ds_i = \sigma_i(t, s_i)\, dW_i \\
& dW_i \cdot dW_j = \rho_{ij}(t, s_i, s_j)\, dt
\end{aligned}
\qquad (10)
$$

- … where the local correlation is given from the volatility of the spread

$$
\begin{aligned}
& (d(s_i - s_j))^2 / dt = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j = \sigma_{ij}^2 \\
& \Downarrow \\
& \rho_{ij}(t, s_i, s_j) = \frac{\sigma_i(t, s_i)^2 + \sigma_j(t, s_j)^2 - \sigma_{ij}(t, s_i - s_j)^2}{2\sigma_i(t, s_i)\sigma_j(t, s_j)}
\end{aligned}
\qquad (11)
$$

- So the model is parameterised from the local spread volatilities $\{\sigma_{ij}(s_i - s_j)\}$ which are given as function of the spread levels.

- The model is constructed as to be able to fit the initial option prices

$$c_{ij}(t,k) = E[(s_i(t) - s_j(t) - k)^+] \tag{12}$$

- … through the Dupire equation

$$0 = -\frac{\partial c_{ij}}{\partial t} + \frac{1}{2}\sigma_{ij}(t,k)^2 \frac{\partial^2 c_{ij}}{\partial k^2} \tag{13}$$

- Absence of arbitrage is dictated through the usual conditions

$$\frac{\partial c_{ij}}{\partial t} > 0 \quad , \frac{\partial^2 c_{ij}}{\partial k^2} > 0 \tag{14}$$

- … ***plus*** the correlation matrix given by (11):

$$\{\rho(t, s_i, s_j)\} \tag{15}$$

- … needs to be bounded in $[-1,1]$ and ***positive definite***.

- The construction through spread volatility rather than correlation is similar to Austing (2011).

Minimal Model and Arbitrage

- If a minimal model exists then there is absence of arbitrage.
- Does absence of arbitrage imply the existence of a minimal model?
- Unfortunately not. Counter example:

$$
\begin{aligned}
&ds_1 = \sigma(s_1, s_2)\, dW_1 \\
&ds_2 = \sigma(s_1, s_2)\, dW_2 \\
&\sigma(s_1, s_2) = \underline{\sigma} + (\bar{\sigma} - \underline{\sigma}) 1_{s_1 - s_2 = k} \quad , dW_1 \cdot dW_2 = 0
\end{aligned}
\tag{16}
$$

- … for some constants $\underline{\sigma} < \bar{\sigma}$.

- Minimal model correlation:

$$
\begin{aligned}
\rho(s_1,s_2) &= \frac{1}{2}\frac{E[(ds_1)^2|s_1]+E[(ds_2)^2|s_2]-E[(ds_1-ds_2)^2|s_1-s_2]}{(E[(ds_1)^2|s_1]E[(ds_2)^2|s_2])^{1/2}} \\
&= \frac{1}{2}\frac{\underline{\sigma}^2+\underline{\sigma}^2-(2\underline{\sigma}^2+2(\bar{\sigma}^2-\underline{\sigma}^2)1_{s_1-s_2=k})}{\underline{\sigma}^2} \qquad (17) \\
&= -\frac{\bar{\sigma}^2}{\underline{\sigma}^2}1_{s_1-s_2=k}
\end{aligned}
$$

- … so $\rho(s_1,s_2)<-1$ on $\{s_1-s_2=k\}$.

- Obviously, a quite specific case but it suggests that if the volatility smiles are more pronounced in the spread directions than in the primal directions, then it may be that the spread is the overall volatility driver …

Discrete Time

- For several reasons it is beneficial to consider the discrete time case.
- First, models live in computers and computers live in discrete time.
- Secondly, in real applications the model setup will have to be somewhat modified relative to what we have outlined so far.
- Thirdly, it would be nice to be able to handle various model extensions such as stochastic volatility and stochastic interest rates.
- It turns out that these modifications and extensions are relatively straightforward to handle in discrete time.

Discrete Time Minimal Model

- An Euler discretisation of the model on the time grid $\{t_h\}$ is

$$\Delta s_i(t_h) = \sigma_i(t_h, s_i(t_h))\Delta W_i(t_h)$$

$$\{\Delta W_i(t_h)\} \sim N(0, \{\rho_{ij}(t_h)\}) \qquad (18)$$

$$\rho_{ij}(t_h, s_i, s_j) = \frac{\sigma_i(t_h, s_i)^2 + \sigma_j(t_h, s_j)^2 - \sigma_{ij}(t_h, s_i - s_j)^2}{2\sigma_i(t_h, s_i)\sigma_j(t_h, s_j)}$$

- … where we have used the notation $\Delta x(t_h) = x(t_{h+1}) - x(t_h)$.

- As in the continuous time case, the model is specified through spread volatility rather than correlation.

- We require the matrix $P = \{\rho_{ij}\}$ to be positive definite.

Monte-Carlo Pricing

- In a Monte-Carlo simulation over samples $\{\omega\}$, the value of an option that expiries as time t_{h+1} can be written as a sum over Bachelier's formula

$$
\begin{aligned}
c_{ij}(t_{h+1},k) &= \frac{1}{N}\sum_{\omega} E_{t_h}[(\underbrace{s_i(t_{h+1})-s_j(t_{h+1})}_{Conditional\ Normal\ Distributed}-k)^{+}|\omega] \\
&= \frac{1}{N}\sum_{\omega} \underbrace{b(\Delta t_h,k;s_i-s_j,\sigma_{ij}(t_h,s_i-s_j))(t_h,\omega)}_{Bachelier's\ \ formula}
\end{aligned}
\tag{19}
$$

- … where $N=\#\{\omega\}$ is the number of samples and Bachelier's formula is

$$b(\tau,k;s,v)=(s-k)\Phi(x)+v\sqrt{\tau}\phi(x) \quad ,x=\frac{s-k}{v\sqrt{\tau}} \tag{20}$$

- This is so because over each time step, $s_i - s_j$ has a conditional normal distribution – due to the Euler discretisation.

- The pricing formula is ***exact*** within the discrete model.

Monte-Carlo Calibration

- If we wish to calibrate the model to the strikes $\{k_{ij}^1,\ldots,k_{ij}^L\}$ at expiry t_{h+1} then we parameterise the volatility function $\sigma_{ij}(t_h;s_i-s_j)$ with L parameters.

- … for example linear interpolation between the L strike points.

- We then solve the minimization problem

$$\inf_{\sigma_{ij}(t_h,\cdot)}\sum_l\Big(\underbrace{c(t_{h+1},k_{ij}^l)}_{mc\,\text{model}\,price}-\underbrace{\hat{c}(t_{h+1},k_{ij}^l)}_{market\,price}\Big)^2 \tag{21}$$

- Note that the calibration of $\{\sigma_{ij}(t_h,\cdot)\}$ is independent for different pairs (i,j).

- After calibration to the options for each spread pair (i, j) then can we construct the correlation matrix $P = \{\rho_{ij}\}$.

- The methodology can also be used for correlation structures that are not minimal.

- If we for example set

$$\sigma_{ij} = \sigma_{ij}(a_{ij} \cdot s) \qquad (22)$$

- … for constant vectors a_{ij}, then the calibration problem is still independent over the different pairs (i, j).

- We do, however, not yet have a methodology for optimal choice of directional vectors $\{a_{ij}\}$.

Positive Definiteness and Bootstrap

- The resulting correlation matrix P is not necessarily positive definite.

- To make it positive definite, decompose into the product $P = O\Lambda O'$, chop negative eigenvalues and rescale to obtain units along the diagonal.

- This procedure is not computationally costless.

- Once done with calibration of the time step $t_h \mapsto t_{h+1}$, we simulate forward to calibrate the model to the time step $t_{h+1} \mapsto t_{h+2}$.

Catch-Up and Discrete Quotes

- If fiddling with the correlation (or covariance) matrix is necessary then the model will not hit the option prices at the particular expiry.

- However, the bootstrap methodology will attempt to ***catch-up*** at the next expiry.

- This is so because the Monte-Carlo pricing/calibration (19) works a bit like updating local volatility according to

$$\sigma(t_h)^2 = 2 \underbrace{\frac{\hat{c}(t_{h+1}) - c(t_h)}{t_{h+1} - t_h}}_{\substack{maturity\ spread \\ computed\ as \\ market - \mathrm{model}}} [\delta_{kk} \hat{c}(t_{h+1})]^{-1} \tag{23}$$

- Equation (23) is a trick that has been used with success in finite difference implementation of local volatility models.

- Hence, the model fit will only be broken at the expiry with positive definiteness problems -- not necessarily at subsequent expiries.

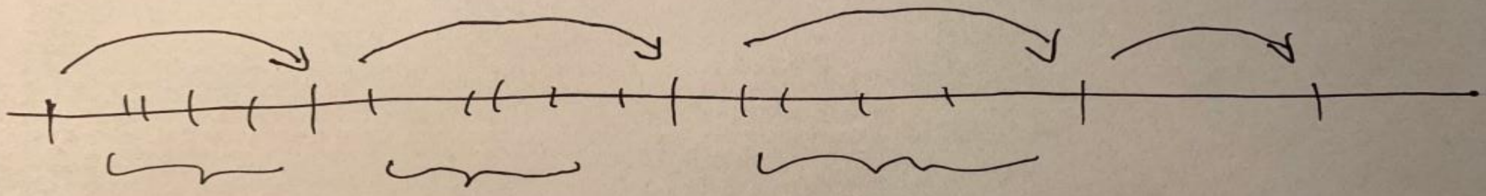
- Also, note we only calibrate to a discrete number of option strikes.

- Hence, we do not rely on perfectly smooth and arbitrage free volatility surfaces in all directions.

Timelines

- The calibration time line $\{t_h\}$ is fixed.
- But we can insert extra simulation time points as we wish inside each calibration time bucket $[t_h, t_{h+1}]$.
- As long as we keep the volatilities and correlations constant over these extra time points.
- In that sense, the model looks a bit like the model in Shelton (2015).
- Here, we use Monte-Carlo rather than numerical integration and this makes our model applicable to high dimensions.

- Bootstrap calibration by Monte-Carlo
- Using Normality of Euler Stepping.



- Any simulation time line after Calibration.
- Freezing Volatility & Correlation over each calibration time Bucket.

Applications and Extensions

- Foreign exchange: Note that log-normal form and currency translations are necessary.

- Equities: Calibrate to basket rather than spread options. Potentially, using notions of average correlation.

- Note that non-trivial dividend models can also be handled this way.

- Interest rates: Non-trivial but interesting. Both multifactor Cheyette and LMM type models can be constructed.

- The interest rate models can potentially calibrate simultaneously to cap/swaption smiles ***and*** smiles of spread and/or mid-curve options.

- Stochastic volatility and even rough volatility is straightforward.
- It is also possible to do models that simultaneously calibrate to SP500 and VIX smiles.
- … and more.

Numerical Implementation

- So far, we have implemented a multi factor Cheyette model for interest rates and a multi price model for FX and equities.

- Both with multi factor stochastic volatility.

- The intention is to combine the two model types to a Next Gen Beast.

- Both are implemented with extensive use of multi threading on CPUs.

- Adjoint differention (AAD) risk has been implemented for the interest rate model.

Numerical Performance

- Hardware is a standard 4 core CPU machine.

- 5 Ccy FX model calibration to 5 strikes in all crosses on expiries 1m, 2m, … 12m:

  - 8,192 paths: 0.46s
  - 65,536 paths: 3.32s

- 4 factor interest rate model. Calibration to 5 strikes in all crosses (spread options and mid curves) in tenors 3m, 2y, 10y, 30y on expiries 1y, 2y, … , 10y.

  - 8,192 paths: 0.45s
  - 65,536 paths: 3.44s

- 6 factor interest rate model. Calibration to 5 strikes in all crosses in tenors 3m, 2y, 5y, 10y, 20y, 30y on expiries 1y, 2y, … , 10y.

    - 8,192 paths: 1.00s
    - 65,536 paths: 7.13s

- Pure simulation times (without calibration) are roughly half.

- AD risk around a factor 10 slower than calibration/pricing.

- Numerical performance is definitely ok, but a tad slower than my audience is used to.

- The approach lends itself very well to TensorFlow/GPU acceleration.

Discrete Model Summary

- The method does not require full continuous surfaces of arbitrage free option prices. Discrete points are sufficient.

- If arbitrage or positive definiteness is broken at a particular time point, the methodology will attempt to catch-up at subsequent time-steps.

- It applies to cases without non-trivial forward equations such as interest rates and commodities.

- Calibration is discretely consistent with discrete Euler stepping. No approximation error.

- Non-minimal correlation can also be handled but we don’t know yet how to optimally choose the directions $\{a_{ij}\}$.

- Long time steps in the calibration, short time steps in pricing.

Conclusion

- We have presented an approach to multi factor local volatility with associated Monte-Carlo calibration methodology that is performing, flexible and general.

- Next steps:

    - Combining interest rate and price models in a 5G Beast.

    - Non-minimal correlation structures.

    - GPU/TensorFlow acceleration.

- The future is bright.