A Hybrid Markov-Functional Model with Simultaneous Calibration to Interest Rate and FX Smile

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Abstract

In this paper we present a Markov functional hybrid interest rate / fx model which allows the calibration of a given market volatility surface in both dimension simultaneously. We extend the approach introduced in [9] by introducing a functional for the FX which allows a fast, yet accurate calibration to a given market fx volatility surface. This calibration procedure comes as an additional step to the known calibration of the LIBOR functional.

Contents

1 Introduction

In this paper we present a Markov-functional hybrid interest rate / fx model which allows the calibration of a given market volatility surface in both dimension simultaneously.

Markov-functional models were first considered by Hunt, Kennedy and Pelsser [14] (2000), see also [12], [18]. The LIBOR Markov-functional model, models a market rate, the forward LIBOR as a functional of a low dimensional Markovian process. Thus, like for low dimensional short rate models, a Markov-functional model may be implemented on a low dimensional lattice, resulting in a fast and robust pricing algorithm. Since we are free to choose the underlying Markov process, it may be chosen such that large time-step conditional expectations are given by a convolution with a transition probability that is known analytically. This enables us to do large time-steps.

Since the LIBOR Markov-functional model models a market rate the market prices of corresponding options (given as implied Black volatilities) are the natural calibration products, similar to a LIBOR market model [4]. Due to the setup of Markov-functional models it is possible to imply the shape of the functional directly from such Market option prices. Thus, the calibration to implied volatility is fast, robust and exact.

The hybrid Markov-functional model extends the single-currency Markov-functional model to a consistent model of the evolution of interest rates and another stochastic underlying. We consider the FX rate as the second underlying but this could also be an equity process. The hybrid Markov-functional model was first introduced independently by Fries & Rott [9] and Antonov & Lee[2] in 2004. These approaches focus on the general theory and exemplary show how to model a flat volatility surface for the second underlying. In [9], the functionals considered for the second underlying were given by a one-parameter family. In this paper we replace this functional by a more general family of functionals which allows a robust calibration to a given FX volatility surface exhibiting smile/skew effects.

1.1 Outline of the Paper

In Section 3 we introduce the single currency Markov-Functional Model as published by Hunt, Kennedy and Pelsser in [14]. We suggest some further enhancement to the LIBOR Markov-Functional Model to improve the accuracy of the Swaption smile as replicated in the model.

In Section 4 we reconsider the approach by Fries and Rott[9]. They already point out that a flat volatility surface can be captured by an exponential functional form modeled over a normal distributed driving process. We introduce a functional form which will fit to a general volatility smile in the second underlying (FX). The calibration procedure is an additional step and retains the calibration to the volatility smile in the first underlying (LIBOR).

We will conclude by presenting some numerical results obtained from an implementation on a two dimensional Lattice.

2 Preliminaries and some Notation

The ideas presented in this paper rely on a knowledge of the general $\mathfrak{L}^1$ theory of option pricing. A full and rigorous treatment of the background theory can be found in for example [12].

$(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ denotes a filtered probability space where $\mathcal{F}_t$ is the augmented natural filtration generated by a Brownian motion W.

A Pure Discount Bond or Zero Coupon Bond paying unit amount at time T is an $\{\mathcal{F}_t\}$-adapted stochastic process defined on the filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ for $0 \leq t \leq T$ denoted by $P(T)$. T is called the maturity of the Bond. $P(T;t)$ denotes the time t value of the Bond and defines a $\{\mathcal{F}_t\}$-measurable r.v., while $P(T;t,\omega)$ denotes its value for some path $\omega \in \Omega$.

Respectively the forward LIBOR for the time period $[S,T]$ will be denoted by $L(S,T)$, its value at time t by $L(S,T;t)$ and $L(S,T;t,\omega)$ denotes its value for some path $\omega \in \Omega$. $L(S,T)$ is determined by

$$1 + L(S,T;\cdot)(T-S) := \frac{P(S;\cdot)}{P(T;\cdot)}$$

3 The Markov-Functional Model

The Markov-Functional framework was first introduced by Hunt, Kennedy and Pelsser [14] in 2000. In this chapter we summarize the ideas given in [14], [12], [18] and [9]. Moreover we describe some enhancements and give some suggestions for further optimizations.

The defining characteristics are:

- Pure discount bond prices are a function of some low dimensional process which is Markovian in some martingale measure $\Rightarrow$ Implementation is efficient.

- The freedom to choose the functional form connecting the driving process to the bond prices $\Rightarrow$ Possible to fit the marginal distributions of market interest rates, smiles & skews.

- The freedom to choose the law of the driving process $\Rightarrow$ Allows to make the model realistic, to capture the joint distribution of the considered interest rates.

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Definition 1 (Markov-functional):
An interest rate model is said to be Markov-Functional if there exists some numeraire pair $(N, \mathbb{N})$ *and some real-valued stochastic process X such that:*

- *The process* X *is a (time-inhomogeneous) Markov process under the measure* $\mathbb{N}$.
- *The pure discount bond prices are stochastic processes of the form*
$$P(S; t, X_t(\omega)), \quad 0 \leq t \leq S \text{ for all } S \leq \hat{T}.$$
which we will use as a convenient shorthand notation for the composition of some functional $P(S; t, \cdot)$ *and the real-valued stochastic process* $X_t(\omega)$*:*
$$P(S; t, X_t(\omega)) := P(S; t, \cdot) \circ X_t(\omega).$$
- *The numeraire* N *is a price process, a stochastic process of the form*
$$N(t, X_t(\omega)) \quad 0 \leq t \leq \hat{T}.$$
which is a shorthand notation for the composition of some functional $N(t, \cdot)$ *and the real-valued stochastic process* $X_t(\omega)$*:*
$$N(t, X_t(\omega)) := N(t, \cdot) \circ X_t(\omega).$$

where $\hat{T}$ *is the model horizon.*

Remark 2 (Numeraire): The numeraire and the bonds might be relaxed to path dependent processes, depending on $\{X_s\}_{s<t}$. This is relevant for an implementation in the spot rate measure[1], see [9].

To completely specify a Markov-Functional Model it is sufficient to know

(i) The law of the process X under $\mathbb{N}$.

(ii) The functional form of the discount factors on the boundary, i.e. $\xi \rightarrow P(S; S, \xi)$ for $S \in [0, \hat{T}]$.

(iii) The functional form of the numeraire $\xi \rightarrow N(t, \xi)$ for $0 \leq t \leq \hat{T}$.

From this we can recover the discount factors on the interior of the region via the martingale property for numeraire rebased assets under $\mathbb{N}$ by the fundamental valuation formula[2] :

$$P(S; t, X_t) = N(t, X_t) E_{\mathbb{N}}[\frac{P(S; S, X_S)}{N(S, X_S)} | \mathcal{F}_t]$$

[1] We actually think that an implementation in the spot rate measure has considerable numerical advantages over the terminal measure. Further research is in preparation.

[2] For a derivation see [12]

3.1 The LIBOR Markov-Functional Model(LMFM) under the Terminal Measure

We consider a time discretization of the interval $[0, \hat{T}]$ into n subintervals given by $0 =: T_0 < T_1 < \ldots < T_n := \hat{T}$. Define

$$L^i = L(T_i, T_{i+1}) \quad 0 \leq i \leq n-1$$

The specification of the model:

(i) $dX_t = \sigma(t) dW_t \quad$ under $\mathbb{N}$, $X_0 = x_0 = 0$

(ii) $P(S, S) \equiv 1$

(iii) $N(t, X_t) = P(T_n; t, X_t)$

where W is a $\mathbb{N}$-Brownian motion, adapted to the filtration $\{\mathcal{F}_t\}$.

Remark 3 (Free Parameters): The free parameters of the model are

(i) the specification of the driving process X, i.e. the deterministic function $\sigma(t)$

(ii) the specification of the numeraire functional $\xi \to N(T_i, \xi)$ for all $0 \leq i \leq n$

For the time being we want to leave $\sigma(t)$ to be specified and analyzed later. We will derive the numeraire functional form from LIBOR's and infer, i.e. calibrate the functional forms of the LIBOR's from market option prices. This gives us the LIBOR Markov-Functional Model.

By definition the LIBOR L^i, seen on its fixing date T_i is given by

$$1 + L^i(T_i, X_{T_i}(\omega))(T_{i+1} - T_i) = \frac{P(T_i; T_i, X_{T_i}(\omega))}{P(T_{i+1}; T_i, X_{T_i}(\omega))} = \frac{1}{P(T_{i+1}; T_i, X_{T_i}(\omega))}$$

Applying the valuation formula for $X_{T_i}(\omega) = \xi$ we get[3]:

$$1 + L^i(T_i, \xi)(T_{i+1} - T_i) = \frac{1}{N(T_i, \xi) E_{\mathbb{N}}[\frac{1}{N(T_{i+1}, X(T_{i+1}))} | (T_i, \xi)]}$$

$$N(T_i, \xi) = \frac{1}{E_{\mathbb{N}}[\frac{1}{N(T_{i+1}, X(T_{i+1}))} | (T_i, \xi)](1 + L^i(T_i, \xi)(T_{i+1} - T_i))}$$

[3] $E_{\mathbb{N}}[\ldots | (T_i, \xi)] := E_{\mathbb{N}}[\ldots | \{\omega | X(T_i)(\omega) = \xi\}]$

Thus, given $\xi \rightarrow L^i(T_i, \xi)$, this gives a backward induction step $T_{i+1} \rightarrow T_i$ to calculate $N(T_i)$ from $N(T_{i+1})$. The induction start is trivially given by $N(T_n, \xi) = 1 \quad \forall \xi$.

The induction step from $T_{i+1} \rightarrow T_i$:
We calibrate the model to digital caplets which is the same as calibrating the model to caplets, see [9].

Assume that $\xi \rightarrow L^k(T_k, \xi)$ for $k \geq i+1$ has already been determined. Thus the numeraire functionals $N(T_k)$ are known for $k \geq i+1$. Let $V^{market}_{K,Ti}(T_0)$ denote the market price of the digital caplet with fixing date T_i, paying

$$V^{market}_{K,T_i}(T_{i+1}) = \begin{cases} 1, & \text{if } L(T_i) \geq K \\ 0, & \text{if } L(T_i) < K \end{cases}$$

Furthermore assume that these prices are monotone in K[4] and known for arbitrary strikes K.

We model the functional $\xi \rightarrow L(T_i, \xi)$ as a monotone function in ξ. For a fixed $x^* \in \mathbb{R}$ the payoff of a digital caplet with strike $L(T_i, x^*)$ and payment date T_{i+1} is then given by

$$\xi \rightarrow V^{model}_{L^i(T_i,x^*),T_i}(T_{i+1}, \xi) = \begin{cases} 1, & \text{if } \xi \geq x^* \\ 0, & \text{if } \xi < x^* \end{cases}$$

Hence the model price is

$$V^{model}_{L^i(T_i,x^*),T_i}(T_0) = N(0)E[\frac{1_{[x^*,\infty)}}{N(T_{i+1})}|(T_0, x_0)]$$

Note that the right hand side can be calculated for any given x^* from the information available from the previous induction step, namely $N(T_{i+1})$, and does not depend on $L^i(T_i, x^*)$. The model price can be calculated for any given x^* without knowing $L(T_i, x^*)$ in the current induction step. We will thus write $V^{model}_{x^*,Ti} := V^{model}_{L^i(T_i,x^*),T_i}$. The requirement to have the model calibrated to market prices for each strike $K = L^i(T_i, x^*)$ implies:

$$V^{model}_{x^*,T_i}(T_0) = V^{market}_{K,T_i}(T_0)$$

Now we find $L^i(T_i, x^*) = K$ by inverting the market-price formula.

As a result this method gives functionals $\xi \rightarrow L^i(T_i, \xi)$, monotone in ξ, calibrated to caplets of all strikes.

3.1.1 The additional calibration of Co-terminal Swaptions

The parameter σ. The not yet specified parameter σ controls the autocorrelation between the different time steps and therefore the mean reversion of

[4] The prices are monotone in K if the market prices do not allow for an arbitrage.

the model. If we choose $\sigma_t = \exp(at)$ the parameter a will be referred to as the mean-reversion parameter of the model. The parameter a induces a mean reversion on $T_i \rightarrow L^i(T_i, X_{T_i})$. For a detailed discussion see [18], [14].

Remark 4 (Mean-Reversion): The mean-reversion parameter has no effect on the caplet prices but is essential to calibrate Swaption prices [18].

Further Optimization

The Joint Distribution So far we just allowed one single deterministic parameter "a", the mean reversion factor, to calibrate a whole set of swaptions. In the single currency setup we do have the not yet introduced freedom to choose a drift term for the process X_t :

$$dX_t = \mu(t, X_t)dt + \sigma(t)dW_t \quad \text{under } \mathbb{N},\, X_0 = x_0 = 0$$

It indirectly determines where the mean reversion leads to, the so called mean-reversion level. This leaves the freedom to calibrate for more than one co-terminal swaption at each time T_i and will make the model more realistic[5]. The introduction of the drift term is much more straightforward in the spot measure than in the terminal measure.

For our discussion the final extension of the single currency model is to allow a in $\sigma(t) = \exp(at)$ to be time dependent. This enables one to calibrate swaptions to each option maturity T_i.

Further Reading The interested reader will find several papers about a multidimensional extension of the single currency model, for example [13], [10], [20], here a higher dimensional driving process for the numeraire process and therefore the bonds is considered. This leads to even a greater flexibility in the joint distribution.

[5] Further research on this topic is in preparation.

4 The Hybrid Markov-Functional Model

After this review of the single Currency model we are now ready to introduce a two-dimensional Markov-Functional Model in the sense that we consistently model two underlyings at the same time. Here we will consider the second underlying to be the stochastic FX rate but we will see that this is arbitrary and can also model an equity process.

4.1 Introduction

The Hybrid Markov-Functional model was first introduced independently by C. Fries and M. Rott [9] and A. Antonov and H. Lee [2] in 2004. Antonov and Lee work with an exponential functional form for the second process. Fries and Rott also consider this approach as one example but do mention that this will just model a flat volatility surface and suggest further research on this topic. In this chapter we extend the approach by Fries and Rott by introducing a functional form to fit smile effects and consider the degrees of freedom given.

4.2 The Model

In addition to the single currency Markov-Functional interest rate model we introduce[6]

(i) a process Y which is a (time-inhomogeneous) Markov process under the measure $\mathbb{N}$

(ii) a second Underlying U as a price process, a stochastic process of the form

$$U(t, Y_t(\omega)), \quad 0 \leq t \leq \hat{T}$$

defined by $U(t, Y_t(\omega)) := U(t, \cdot) \circ Y_t(\omega)$.

Specification: To completely specify the two-dimensional Markov-Functional Model it is sufficient to specify

(i) a single currency Markov-Functional Model as introduced in the previous chapter

(ii) the law of the process Y under $\mathbb{N}$

(iii) the functional form of the second underlying $\eta \to U(t, \eta)$ for $0 \leq t \leq \hat{T}$

From this we can price any $\{\mathcal{F}_t\}$- measurable non path-dependent derivative which depend on no more than the term structure of the interest rate and

[6] This generalizes the ideas of the Cross-Currency model which can be found in [2] and [9].

the second underlying. For the valuation we use the fundamental valuation formula [12]. This is postulated by the martingale property for numeraire rebased tradeable assets of the economy under $\mathbb{N}$.

Free Parameters: The desirable free parameters of the model are

(i) The free parameters of the one-dimensional model.

(ii) The variance of the driving process Y.

(iii) The covariance between the processes X and Y.

(iv) The specification of the functional U.

Hence, to guarantee that the model is arbitrage free we have to make sure that $\frac{U(t)}{N(t)}$ is a martingale under $\mathbb{N}$, i.e.

$$\frac{U(t)}{N(t)} = E_{\mathbb{N}}[\frac{U(s)}{N(s)}|\mathcal{F}_t] \text{ for all } s > t \qquad \text{(no-arbitrage condition)}$$

This can be satisfied by choosing the right drift of the driving process Y.

A note on the drift of the Y-process, see [9]: In the single currency Markov-Functional Model we could arbitrary choose a driving process without imposing any restrictions on the functional form of the LIBOR. The model would always remain arbitrage free and we first naturally opted for a zero-drift process. In the two-dimensional model the freedom to choose a y-drift is gone, as for a given functional U we now need a corresponding drift for the model to remain arbitrage free. If we would a priori insist on a specific drift, e.g. a zero drift, specifying the functional form of $U(\hat{T}, \cdot)$ would automatically determine all other functional forms $U(t, \cdot), 0 \leq t \leq \hat{T}$ by the martingale property from above. The only free parameter remaining for U would be the variance of the process Y.

The reason for the 'freedom' to choose the drift for the process X is due to the ability to choose the numeraire functional. Because of

$$P(T_{i+1}; T_i, X_{T_i}) = N(T_i, X_{T_i}) E_{\mathbb{N}}[\frac{1}{N(T_{i+1}, X_{T_{i+1}})}|\mathcal{F}_t]$$

$T_i \rightarrow P(T_{i+1}; T_i, X_{T_i})$ is a $\mathbb{N}$-martingale for any functional form $\xi \rightarrow N(T_i, \xi)$. Conversely, for any functional form $\xi \rightarrow P(T_{i+1}; T_i, \xi)$ we may choose the numeraire $\xi \rightarrow N(T_i, \xi)$ such that $T_i \rightarrow P(T_{i+1}; T_i, X_{T_i})$ is a $\mathbb{N}$-martingale. This freedom is gone once the numeraire has been chosen.

Remark 5 (Drift / no-arbitrage condition): One can also derive a cross-currency model focusing on IR/IR-Hybrids with a zero drift in all dimensions as currently introduced by Simon Johnson and Sergio Dutra [15]. In their paper they introduce a model with three driving processes modeling the three stochastic processes: domestic interest rates, foreign interest rates and the FX process. The focus is on a calibration of a general volatility surface in the domestic and foreign interest rate. Therefore Johnson and Dutra sacrifice the calibration of the FX functional to satisfy the no-arbitrage condition. One can either choose for a zero drift and to sacrifice the calibration of FX products to solve the no-arbitrage condition by the FX functional or if one wants to calibrate FX options in addition to an existing IR Markov-functional one has to introduce a drift. We will consider the latter approach. The approach by Johnson and Dutra is especially applicable to value quantos.

Specification of the driving processes

$$\begin{aligned} dX_t &= \sigma_x(t)dW_t^{(1)} & X_0 &= x_0 \\ dY_t &= \mu(t, X_t, Y_t)dt + \sigma_y(t)dW_t^{(2)} & Y_0 &= y_0 \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ have covariance ρ.

A note on the importance of early discretization, see [9]: For a discrete model it is important to derive the drift for the discrete version of the driving process from no-arbitrage considerations. If we instead derived the drift of the time-continuous process from no-arbitrage considerations and then use a discretization scheme with an approximation of the time-continuous drift, it would only guarantee that the discretised model satisfies the no-arbitrage condition up to a discretization error.

The discrete approximation of the driving processes[7]

Since we are only interested in functionals of (x, y) at the T_i's, we consider a discretization of the driving processes. Using an Euler discretization yields

$$\begin{aligned} \Delta X(T_i - 1) &= \sigma_{x,i-1}\sqrt{\Delta T_{i-1}}\Delta W^{(1)} \\ \Delta Y(T_i - 1) &= \mu(T_{i-1}, X_{T_i-1}, Y_{T_i-1}) + \sigma_{y,i-1}\sqrt{\Delta T_{i-1}}\Delta W^{(2)} \end{aligned}$$

with

$$\begin{aligned} \Delta T_{i-1} &= (T_i - T_{i-1}) \\ \begin{pmatrix} \Delta W_1 \\ \Delta W_2 \end{pmatrix} &\sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \\ \sigma_{x,i-1} &:= \sqrt{\frac{1}{\Delta T_{i-1}}\int_{T_{i-1}}^{T_i} \sigma_x^2(t)dt} \\ \sigma_{y,i-1} &:= \sqrt{\frac{1}{\Delta T_{i-1}}\int_{T_{i-1}}^{T_i} \sigma_y^2(t)dt} \end{aligned}$$

[7] see [9].

From now on X_{T_i},Y_{T_i} denote the Euler scheme approximation of

$$x_0 + \int_0^{T_i} dX_t$$
$$y_0 + \int_0^{T_i} dY_t$$

given by

$$\begin{aligned}
X_{T_0} &= x_0 \\
Y_{T_0} &= y_0 \\
X_{T_i} &= X_{T_{i-1}} + \Delta X_{T_{i-1}} \\
Y_{T_i} &= Y_{T_{i-1}} + \Delta Y_{T_{i-1}}
\end{aligned}$$

and we will consider functionals of the time-discrete process and not functionals of the time-continuous process.

4.3 The Two-Factor Cross-Currency Model (stochastic FX rates)

In the two-factor cross currency model[8] we will model stochastic domestic interest rates and stochastic FX rates. The required foreign interest rates will be assumed to be deterministic - for the time being.

We will model $U(T_i, Y_{T_i})$ as the two processes $U(T_i, Y_{T_i}) = FX(T_i, Y_{T_i})\tilde{P}(T_n; T_i)$ where $\tilde{P}$ denotes the deterministic price process of the foreign discount bond. That means we directly calibrate the functional form $FX(T_i, \cdot)$.

Therefore the drift condition is given by

$$\begin{aligned}
& \frac{U(T_i)}{N(T_i)} = E_{\mathbb{N}}[\frac{U(T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}] \\
\Longleftrightarrow \quad & \frac{FX(T_i)\tilde{P}(T_n; T_i)}{N(T_i)} = E_{\mathbb{N}}[\frac{FX(T_{i+1})\tilde{P}(T_n; T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}] \\
\Longleftrightarrow \quad & \frac{FX(T_i)\tilde{P}(T_{i+1}; T_i)}{N(T_i)} = E_{\mathbb{N}}[\frac{FX(T_{i+1})\tilde{P}(T_{i+1}; T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}] \\
\Longleftrightarrow \quad & \frac{FX(T_i)\tilde{P}(T_{i+1}; T_i)}{N(T_i)} = E_{\mathbb{N}}[\frac{FX(T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}] \quad \text{(drift condition)}
\end{aligned}$$

4.3.1 The General Calibration Procedure

In the two-dimensional model we have a dependency between the functional form at time T_i and the drift of the driving process up to time T_i. This requires a

[8] The two-factor cross currency model was first introduced in [9]

forward calibration and that we have to determine the whole functional before calculating any model prices. We can no longer calibrate the functional point by point as in the first dimension.

Before we can start the procedure we have to choose a family of reasonable functionals with enough free parameters to allow a good fit to market prices. Obviously we set FX(0) to be the current FX rate observed in the market. Given $FX(T_i)$ a multi-dimensional root finder[9] is used to take a first guess of the parameters of $FX(T_{i+1},\cdot)$. Then the drift $\mu(T_i,\cdot,\cdot)$ has to be adjusted. Now the model prices of FX-Options with different strikes can be calculated and compared to the corresponding market prices. The error in the model prices is now used by the root finder to set the next guess until the prices are reasonable replicated in the model.

- for $T_i,\ldots,T_n$
  - optimize the FX functional with a multi-dimensional root finder
    - set FX parameters
    - solve and set corresponding drift
    - return the error between the market and model prices of specified FX options to the solver
    - continue the optimization till a certain threshold
  - set FX to optimum
  - go to the next time step

4.4 Calibration of a Flat FX Volatility Surface

4.4.1 The Functional Form

As a functional form[10] in the case of a flat volatility surface we choose:

$$\eta \rightarrow FX(T_i,\eta)$$

$$FX(T_i,\eta) = a(T_i)\cdot\exp(b(T_i)\cdot\eta)$$

and the driving process is given as stated above by

$$dY(T_i) = \mu(t, X_{T_i}, Y_{T_i})dt + \sigma_{y,i-1}\sqrt{\Delta T_{i-1}}dW^{(2)}_{T_i} \quad \text{with} \quad Y_0 = y_0$$

The idea behind this approach: If we consider that

$$d(FX(t)) = FX(t)\cdot\mu(t)dt + FX(t)\cdot\sigma(t)dW_t$$

[9] One can for example use the Levenberg-Marquardt algorithm

[10] [2] and [9] also use an exponential functional form but with less degrees of freedom.

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we get by Ito's formula

$$\begin{aligned} d(\log FX(t)) &= \frac{1}{FX(t)} \cdot d(FX(t)) - 1/2 \cdot \frac{1}{(FX(t))^2} \cdot (d(FX(t))^2 \\ &= (\mu(t) - 1/2\sigma^2(t))dt + \sigma(t)dW_t \end{aligned}$$

$$\Rightarrow \log FX(t) \sim N(log(FX(0)) + \int_0^t \mu(s) - 1/2\sigma^2(s)ds, \int_0^t \sigma^2(s)ds)$$
$$\Rightarrow f(0) = FX(0) \qquad \text{and} \qquad d\hat{y}(t) = (\mu(t) - 1/2\sigma^2(t))dt + \sigma(t)dW_t$$

For a fixed σ_y and $a(T_i) = FX(0)\ \forall i$ the σ of the log-normal distribution is determined by b such that:

$$\begin{aligned} \hat{Y}_{T_i} &= b(T_i) \cdot Y_{T_i} \\ \Rightarrow b(T_i) &= \frac{\int_0^{T_i} \sigma(s)ds}{\int_0^{T_i} \sigma_y(s)ds} \end{aligned}$$

4.4.2 The Drift Equation for $\rho = 0$

$$\begin{aligned} & \frac{FX(T_i)\tilde{P}(T_{i+1};T_i)}{N(T_i)} = E_{\mathbb{N}}[\frac{FX(T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}] \\ \Longleftrightarrow\quad & FX(T_i) = \frac{1}{\tilde{P}(T_{i+1};T_i)} \cdot E_{\mathbb{N}}[FX(T_{i+1})|\mathcal{F}_{T_i}] \cdot N(T_i) \cdot E_{\mathbb{N}}[\frac{1}{N(T_{i+1})}|\mathcal{F}_{T_i}] \\ \Longleftrightarrow\quad & FX(T_i, Y_{T_i}) = \frac{P(T_{i+1};T_i,X_{T_i})}{\tilde{P}(T_{i+1};T_i)} \cdot E_{\mathbb{N}}[FX(T_{i+1}, Y_{T_{i+1}})|\mathcal{F}_{T_i}] \end{aligned}$$

4.4.3 The Analytic Solution for $\rho = 0$

For the functional form $FX(t,\cdot)$ we can find an analytic solution :

$$
\begin{aligned}
&E(FX(T_{i+1},Y_{T_{i+1}})|(T_i,X_{T_i}=\xi,Y_{T_i}=\eta))\\
&=a_{T_{i+1}}\cdot\exp(b_{T_{i+1}}\cdot(\eta+\mu_{T_i}(\xi,\eta)\cdot\Delta T_i)+\frac{b^2_{T_{i+1}}}{2}\cdot\sigma^2_{T_i}\Delta T_i)\\
&\Longleftrightarrow\\
&FX((T_i,\eta)\cdot\frac{\tilde{P}(T_{i+1};T_i)}{P(T_{i+1};T_i,\xi)}\cdot\frac{1}{a_{T_{i+1}}}\\
&=\exp(b_{T_{i+1}}\cdot(\eta+\mu_{Ti}(\xi,\eta)\cdot\Delta T_i)+\frac{b^2_{T_{i+1}}}{2}\cdot\sigma^2_{T_i}\Delta T_i)\\
&\Longleftrightarrow\\
&\log(FX((T_i,\eta)\cdot\frac{\tilde{P}(T_{i+1};T_i)}{P(T_{i+1};T_i,\xi)}\cdot\frac{1}{a_{T_{i+1}}})\cdot\frac{1}{b_{T_{i+1}}}\\
&=\eta+\mu_{Ti}(\xi,\eta)\cdot\Delta T_i+\frac{b_{T_{i+1}}}{2}\cdot\sigma^2_{T_i}\Delta T_i\\
&\Longleftrightarrow\\
&\mu_{T_i}(\xi,\eta)\cdot\Delta T_i\\
&=\log(FX((T_i,\eta)\cdot\frac{\tilde{P}(T_{i+1};T_i)}{P(T_{i+1};T_i,\xi)}\cdot\frac{1}{a_{T_{i+1}}})\cdot\frac{1}{b_{T_{i+1}}}-\eta-\frac{b_{T_{i+1}}}{2}\cdot\sigma^2_{T_i}\Delta T_i
\end{aligned}
$$

Within a numerical implementation of the model the above analytical solution will not provide an arbitrage free model due to the discretization error. We thus propose a numerical calculation of the solution of the discretised drift equation.

4.4.4 The Numerical Solution of the Drift Equation

The analytic drift is a good guess in the center of the grid. Here the approximation error of the numerical implementation is small. At the sides the extrapolation error implies that the drift equation in the model does not hold any more. A first evidence for a drift which is not consistent with the drift equation with respect to the used integration and extrapolation method is an error in the FX-forward. Furthermore the above calculation just holds for $\rho = 0$.

To guarantee that the numerical model is arbitrage free we need to solve the drift equation numerically for every conditional state (x, y) at every time step $T_i \rightarrow T_{i+1}$ with a one dimensional root finder.

4.5 Calibration of a general FX Volatility Surface

4.5.1 The Functional Form

To fit smile effects we suggest the same form as the one for a flat calibration with two local correction terms [11]:

$$\eta \to FX(T_i, \eta)$$

$$\begin{aligned} FX(T_i, \eta) &= a(T_i) \cdot \exp(b(T_i) \cdot \eta) \\ &\quad + d_1(T_i) \cdot \exp(-c_1(T_i) \cdot (\eta - m_1(T_i))^2) \\ &\quad + d_2(T_i) \cdot \exp(-c_2(T_i) \cdot (\eta - m_2(T_i))^2) \end{aligned}$$

where

$$m = \text{state of the correction}$$

$$d = \text{impact of the correction}$$

$$c = \text{radius of the correction } (c > 0)$$

The most significant impact on the functional is given between the inflection points of the correction terms : $m \pm \sqrt{\frac{1}{2c}}$.

Keep in mind that these additional terms also have an impact on the drifts. Here the radius is much larger. A change of one of these correction terms will change all FX option prices even if c is large.

4.5.2 The Drift Equation

With the correction terms we are not able to find an analytic solution for the drift equation, and as we have seen in the flat case the analytic solution is not the best choice anyway. So we are using the numerical drift calculation from above.

4.5.3 How this functional generates a smile effect and how to choose the parameters

The local correction terms has the effect that if d is positive the probability density function of the r.v. FX shifts mass from the left of m to the right and vice versa if d is negative.

With the two correction terms placed to both sides of ATM we can create the so called fat tails which generate the smile effect of the volatility surface.

[11] Pelsser discusses a functional with a similar correction term in the context of the one dimensional model in [18].

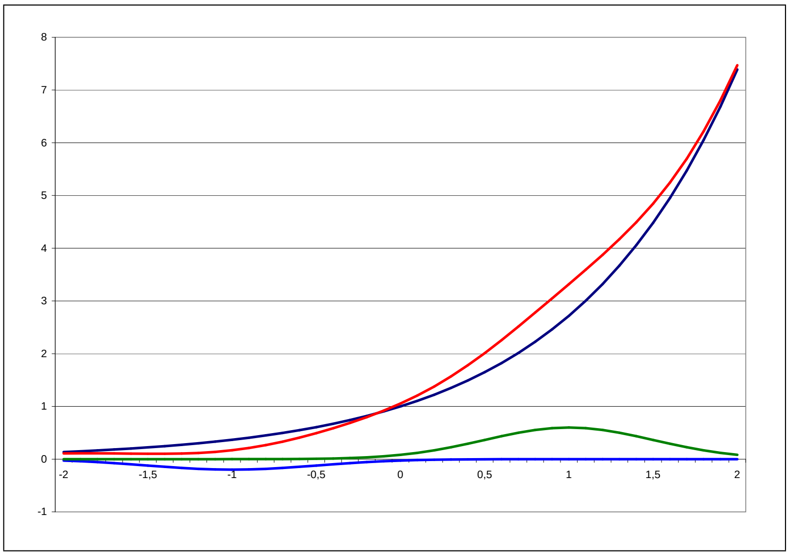


Figure 4.1: Plot of $\exp(x)$ (black), the two correction terms (green, blue) and the resulting FX functional (red).

To fit the volatility smile we use an ATM, one in the money and one out of the money FX-option. It seems to be best to chose m_1 and m_2 in the middle of these three in terms of the driving process Y. a_{T_i} is chosen such that the ATM strike corresponds to $Y_{T_i} = 0$. We chose the corresponding c to be a function of the distance between the point $m_{1/2}$ and ATM, $c_{1/2} = \frac{const}{m_{1/2}^2}$. To stabilize the calibration procedure the constant left to determine c has to be large enough to ensure that the correction terms does not effect the functional form at the sides of the grid, but small enough to have a smooth effect around $m_{1/2}$.

If c exceeds its lower limit the modeling range changes by a change in the correction term which has the consequence that the solver will become unstable.

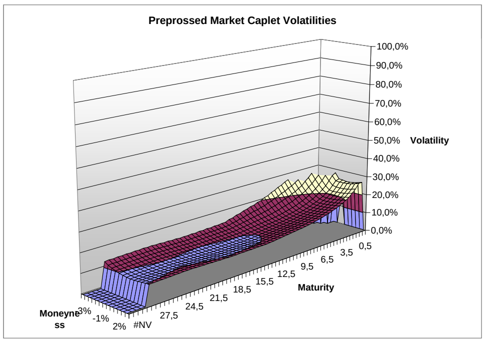
4.6 Free Parameters

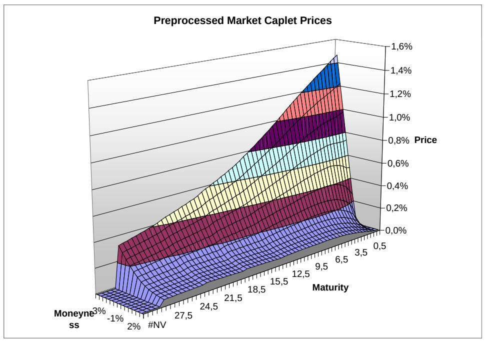
Remark 6 (σ, ρ)**:** We are still left with the free parameter σ which can be used to calibrate autocorrelation depending products and the parameter ρ to choose the correlation between the FX and the domestic interest rate processes. If there is no need to adjust σ one should use $\sigma(t) = \exp(at)$ with a such that the parameter b does not move too much away from its value at T_1 over time. This guarantees that the ratio of σ of the FX rate and the driving process stays the same over time and therefore a higher numerical quality of the model.

5 Empirical Results

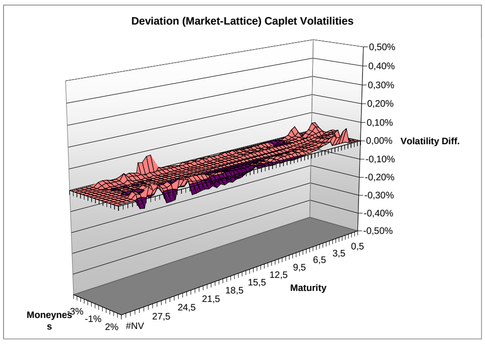
5.1 First Dimension: The Interest Rate Calibration

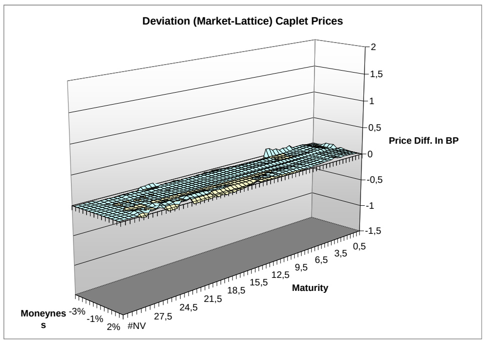
(i) Caplet sample market data:



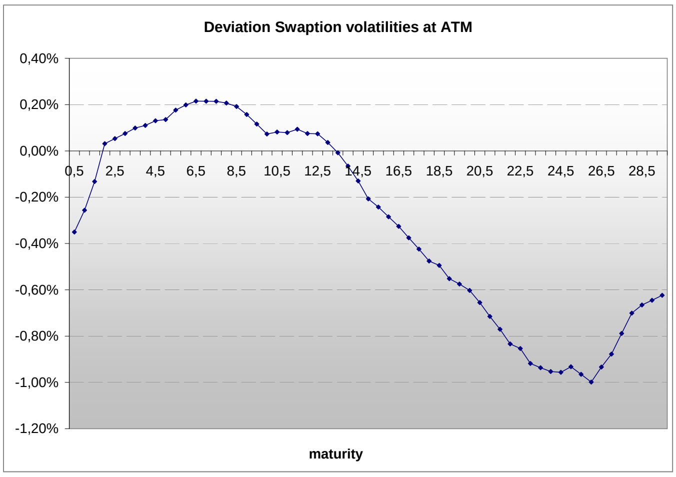


(ii) The deviation we observe in the model:





The deviation of Co-terminal Swaption volatilities after a calibration of the mean reversion parameter a:



5.2 Second Dimension: The FX Calibration

Definition 7 (Standardized Moneyness):

The Standardized Moneyness[12] $\hat{M}$ *for an underlying* S *is defined as*

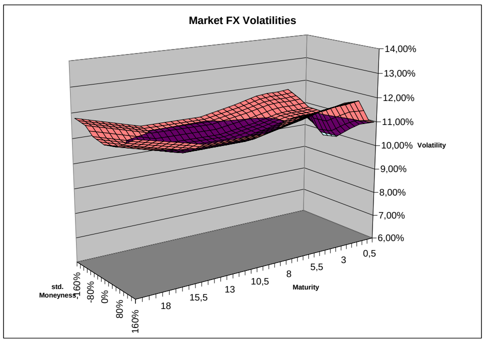
$$\hat{M} := \frac{\ln(\frac{K}{FS(T_i)})}{\hat{\sigma}_i \sqrt{T_i}}$$

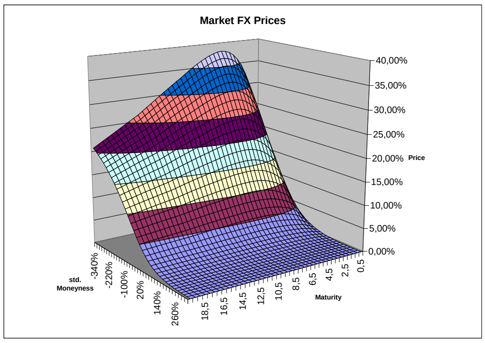
where $\hat{\sigma}_i$ *is the implied Black ATM volatility for a call option with maturity* T_i *on* S *and* $FS(T_i)$ *denotes the forward as seen today of the underlying. K is the value of the corresponding strike at time* T_i.

This moneyness term is close to a denomination of how many standard deviations the strike K is apart from ATM. (If we have smile effects this is not exactly the case but still a sufficient measure.)

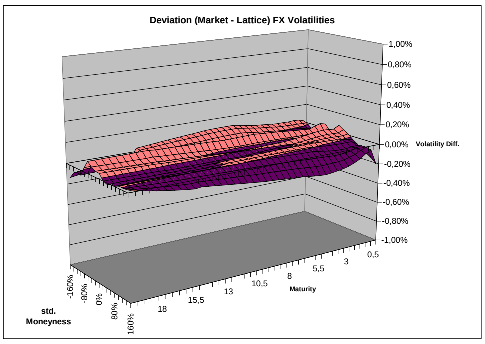
The following charts are plotted against the standardized moneyness.

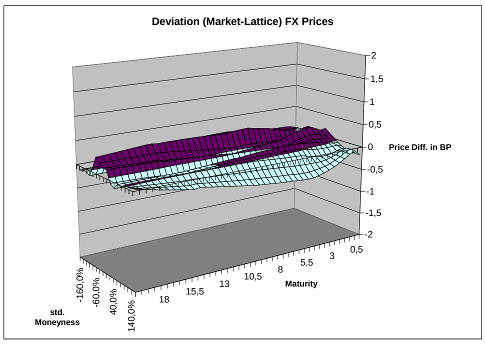
(i) FX sample market data:





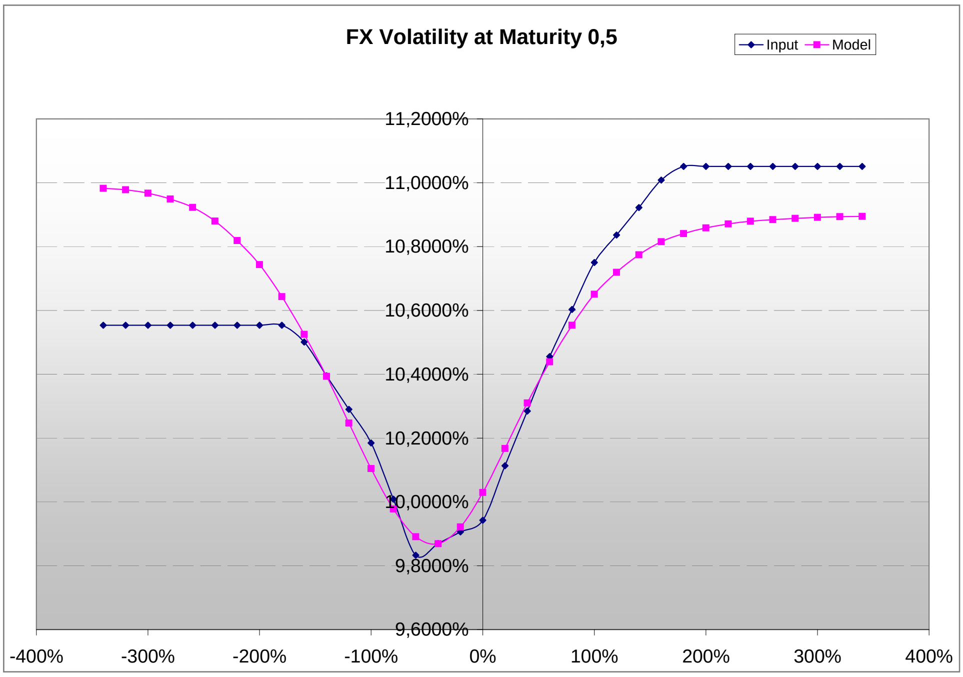
(ii) The deviation we observe in the model:

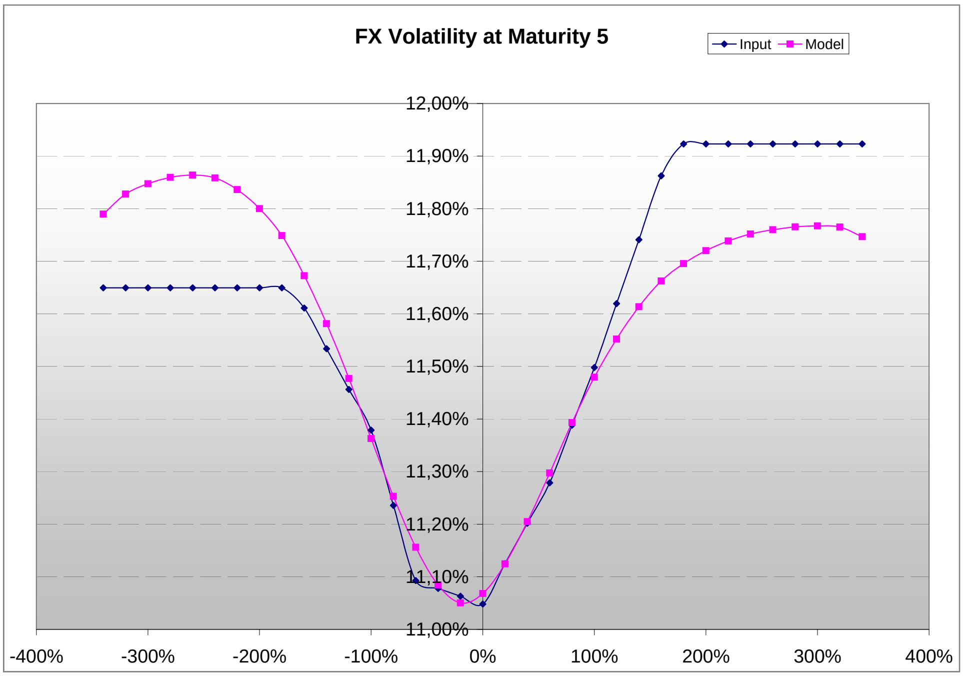


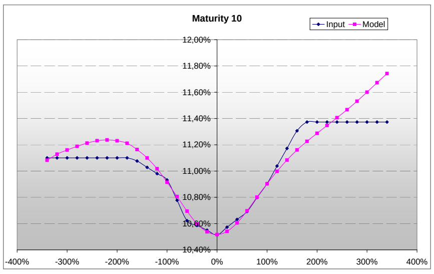


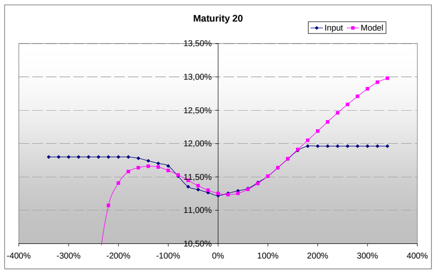
12 A further discussion on the concept can be found in [17].

(iii) The deviation we observe at different maturities:









The extrapolation of the market data is chosen to be constant.

We observe a very good fit to market data in the range of $-160\hat{M}$ to $+160\hat{M}$, which is more than we need in most applications. However one can observe a slight twist in the volatility curve over time, which leads to an unstable calibration (for the sample data after about 22 years). A discussion on the implementation and suggestions of further optimization can be found in [8].

6 Conclusion

In this paper we presented the theory of a Hybrid Markov-Functional Model. As far as we know this is the first attempt of a consistent Hybrid Model with an almost perfect smile fit in two underlings at the same time. As we already pointed out there is still a lot of research to be done. Just one example is that we model the term structure under the terminal measure but an implementation under the rolling spot measure as described in [9] might in fact lead to even better features. It enables for better numerical features as discussed in [8]. In other papers dealing with the Markov-Functional Model we can find several extensions and improvements of the single currency model which can now be combined with or even extended to the Hybrid Model.

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