



OpenGamma Documentation

Libor Market Model with Displaced Diffusion: Implementation

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Abstract

The Libor Market Model (LMM) with displaced diffusion is described. The details of the implementation are provided.

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1 Introduction

The Libor Market Models (LMM) were introduced at the end of the 1990s. First by [Miltersen et al. \(1997\)](#) and almost simultaneously and not completely independently by [Brace et al. \(1997\)](#), and [Jamshidian \(1997\)](#) (see the introduction in [Gařtarek et al. \(2006\)](#)). They are nowadays very popular and almost standard interest rate models. The document provides the results necessary for the implementation, the theoretical developments are not provided. The reader is referred to the original papers and books mentioned in the text for the theoretical justifications. The notations of the multi-curves framework are the one of [Henrard \(2010\)](#).

The note is divided in several sections. The second section described the notation for Ibor rates and swaps. The third section describes the theoretical background of the model. The fourth section described the pricing of European swaptions. This part is important for the calibration. The fourth section describes the Monte-Carlo implementation required for some exotics.

2 Ibor and Swaps

Let $P^D(t, u)$ be the discounting curve. Given the forward curve $P^j(t, u)$, the Ibor fixing in t_0 for the period $[t_1, t_2]$ is

$$I_{t_0}^j(t_1, t_2) = \frac{1}{\delta} \left(\frac{P^j(t_0, t_1)}{P^j(t_0, t_2)} - 1 \right) \quad (1)$$

The forward Ibor rate is defined in t by

$$F_t^j(t_0, t_1) = \frac{1}{\delta} \left(\frac{P^j(t, t_0)}{P^j(t, t_1)} - 1 \right) \quad (2)$$

where δ is the accrual fraction between t_0 and t_1 in the day count fraction associated to the Ibor rate.

An swap (IRS) exchanges a set of n fixed cash-flows against floating (Ibor) rates paid in their natural day count convention on their natural periods. An IRS is described by a set of fixed coupons or cash flows c_i at dates \tilde{t}_i ($1 \leq i \leq \tilde{n}$). For those flows, the discounting is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ ($1 \leq i \leq n$)¹.

The value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^n P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right). \quad (3)$$

Let $\tilde{\delta}_i$ be the accrual fraction associated to the fixed coupons paid in \tilde{t}_i . The level or present value of a basis point of the swap is given by

$$\text{PVBP}_t = \sum_{i=1}^n \tilde{\delta}_i P^D(t, \tilde{t}_i).$$

The (forward) swap rate is given by

$$S_t = \frac{\sum_{i=1}^n P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right)}{\text{PVBP}_t}.$$

⁰First version: March 2012; this version: October 2012.

¹In practice, due to weekends and holidays, the periods used for the fixings can be slightly different from the payment dates. We will not make that distinction here.

We define

$$\beta_t^j(u, u+j) = \frac{P^j(t, u)}{P^j(t, u+j)} \frac{P^D(t, u+j)}{P^D(t, u)}. \quad (4)$$

With that definition, a floating coupon price is

$$P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right) = \beta_t^j(t_{i-1}, t_i) P^D(t, t_{i-1}) - P^D(t, t_i).$$

Throughout this note we assume the constant spread hypothesis **S0** of [Henrard \(2010\)](#).

The swap is often represented by its cash-flow equivalent $(t_i, d_i)_{i=0, \dots, n}$. The date t_0 is the settlement date and $(t_i)_{i=1, \dots, m}$ the payment dates (fixed and floating legs). The cash flow amounts are $c_0 = -\beta^j(t_0, t_1)$, d_i ($i = 1, \dots, n-1$) the coupons including the adjustments for floating coupons and $d_n = 1 + c_n$ the final coupon plus 1 for the notional.

Note that the swap and Ibor rates can be computed using only the re-based discount factors in any numeraire as both involve only ratios.

3 Model and hypothesis

In general, the HJM framework describes the behavior of $P^D(t, u)$, the price in t of the zero-coupon bond paying 1 in u ($0 \leq t \leq u \leq T$). When the discount curve $P^D(t, \cdot)$ is differentiable in a weak sense, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t, u)$ such that

$$P^D(t, u) = \exp \left(- \int_t^u f(t, s) ds \right). \quad (5)$$

The idea of [Heath et al. \(1992\)](#) was to exploit this property by modeling f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u) \cdot dW_t$$

for some suitable (potentially stochastic) μ and σ and deducing the behavior of P from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The volatility and the Brownian motion are m -dimensional while the drift and the rate are 1-dimensional. The dimension m will be call the model number of factors. The model technical details can be found, among other places, in the original paper or in the chapter *Dynamical term structure model* of [Hunt and Kennedy \(2004\)](#).

The probability space is $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration \mathcal{F}_t is the (augmented) filtration of a m -dimensional standard Brownian motion $(W_t)_{0 \leq t \leq T}$. To simplify the writing in the rest of the note, the bond volatility is denoted by

$$\nu(t, u) = \int_t^u \sigma(t, s) ds.$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0 \leq s \leq T}$ the short rate given by $r_t = f(t, t)$. The equations of the model in the numeraire measure \mathbb{N} associated to N_t are

$$df(t, u) = \sigma(t, u) \cdot \nu(t, u)dt + \sigma(t, u) \cdot dW_t.$$

3.1 Libor Market Model

The idea behind the Libor Market Model² (LMM) is to embed different Black-like equations for the forward (Libor) rates between standard dates $t_0 < \dots < t_n$ into a unique HJM model. The risk-free deposit rates D_s^i for deposits between t_i and t_{i+1} are defined by

$$1 + \delta_i D_s^i = \frac{P^D(s, t_i)}{P^D(s, t_{i+1})}.$$

The factors δ_i are the accrual factors or day count fractions and represent the fraction of the year spanned by the interval $[t_i, t_{i+1}]$ in the selected LMM convention.

The equations underlying the Libor Market Model are

$$dD_t^j = \gamma_j(D_t^j, t) \cdot dW_t^{j+1} \quad (6)$$

in the probability space with numeraire $P^D(., t_{j+1})$. The γ_j ($0 \leq j \leq n-1$) are m -dimensional functions.

The Brownian motion change between the N_t and the $P^D(t, t_{j+1})$ numeraire is given by

$$dW_t^{j+1} = dW_t + \nu(t, t_{j+1}) dt.$$

The difference $\nu(t, t_{j+1}) - \nu(t, t_j)$ can be written as

$$\nu(t, t_{j+1}) - \nu(t, t_j) = \frac{1}{D_t^j + \frac{1}{\delta_j}} \gamma_j(D_t^j, t) \quad (7)$$

The general description of the models can be found in the three original papers mentioned in the introduction or in books on the subject like [Rebonato \(2002\)](#) or [Gątarek et al. \(2006\)](#).

Recursively the change between the different numeraires $P(t, t_{j+1})$ and $P(t, t_n)$ is given by

$$dW_t^{j+1} = - \sum_{i=j+1}^{n-1} \frac{1}{D_t^i + \frac{1}{\delta_i}} \gamma_i(D_t^i, t) dt + dW_t^n.$$

All the rates can be written with respect to the same (last) numeraire

$$dL_t^j = - \left(\sum_{i=j+1}^{n-1} \frac{1}{D_t^i + \frac{1}{\delta_i}} \gamma_i(D_t^i, t) \cdot \gamma_j(D_t^j, t) \right) dt + \gamma_j(D_t^j, t) \cdot dW_t^n. \quad (8)$$

3.2 Displaced diffusion

The model will be analyzed with a separable displaced diffusion. By this we mean that the volatility has the form

$$\gamma_j(D, t) = \alpha(t)(D + a_j) \gamma_j \quad (9)$$

with α a deterministic scalar function, a a constant vector of dimension n and $(\gamma_j)_{j=1, \dots, n}$ vectors of dimension m . The form of α will usually be $\alpha(t) = \exp(a_{\text{HW}} t)$ for some constant a_{HW} to mimic the Hull-White volatility term structure.

²In the multi-curves framework we use here, the model should be called Discounting Market Model or something similar as we do not model the Ibor curve forward rates $F_t^j(t_i, t_i + 1)$ but the risk-free deposit forwards rates D_t^i .

Note that for theoretical fundamental reasons described in the appendix of [Henrard \(2007\)](#), the shift a_i should satisfy $a_i \leq 1/\delta_i$. In practice this is not a restriction.

With the above volatility the equations are

$$dD_t^j = -\alpha^2(t) \left(\sum_{i=j+1}^{n-1} \frac{D_t^i + a_i}{D_t^i + \frac{1}{\delta_i}} \gamma_i \cdot \gamma_j \right) (D_t^j + a_j) dt + \alpha(t) (D_t^j + a_j) \gamma_j \cdot dW_t^n. \quad (10)$$

4 Swaption and bond options price formula

From an option pricing point of view, swaptions and bond options are equivalent. Up to the change of names, all the results are valid both for swaptions and bonds' options.

The swap is described by its cash-flow equivalent (d_i, t_i) ($0 \leq i \leq p$) (see Section 2). The convention of the swap can be different from the one used in the LMM construction and different from the standard swap market convention. The swaption expiry is denoted θ and $\theta \leq s_0$. The swap dates should be included in the model dates with $s_i = t_{k_s+i}$ for some k_s associated to the swap.

To be able to handle extreme roller coasters³, the cash flow equivalent is modified. Let \tilde{d}_0 be the modified strike equal to the instrument maximal notional: $\tilde{d}_0 = -\max(N_i)$. The first modified cash flow equivalent will be the original d_0 compensated by the modification: $\tilde{d}_1 = d_0 - \tilde{d}_0$. This last figure is positive. The other modified cash flows equivalent are the same as the original $\tilde{d}_{i+1} = d_i$ ($1 \leq i \leq n$). The modification described here were not present in the original paper. To simplify the writing, we drop the \sim for the rest of the section.

The forward values of the zero-coupon bonds and swap without the initial notional are given in t by

$$P_t^i = \frac{P^D(t, s_i)}{P^D(t, s_0)} \quad \text{and} \quad B_t = \frac{\sum_{i=1}^{n+1} d_i P(t, s_i)}{P^D(t, s_0)}.$$

Those values are the values re-based by the numeraire $P^D(\cdot, s_0)$

Let $\nu^i(t) = (\nu(t, s_i) - \nu(t, s_0))$ ($0 \leq i \leq p$). In the numeraire measure associated to the numeraire $P^D(\cdot, s_0)$, the rebased prices are martingales satisfying the equations

$$dP_t^i = -P_t^i \nu^i(t) \cdot dW_t^0.$$

The zero-coupon bond prices are not exactly log-normal as the volatility $\nu^i(t)$ is not deterministic but state dependent through Equation (7). The zero-coupon bond volatility is

$$\nu^i(t) = \sum_{j=0}^{i-1} (\nu(s, s_{j+1}) - \nu(s, s_j)) = \alpha(t) \sum_{j=0}^{i-1} \frac{D_t^j - a_j}{D_t^j + 1/\delta_j} \gamma_j.$$

The rebased swap value satisfies

$$dB_t = - \sum_{i=1}^{n+1} d_i P_t^i \nu^i(t) \cdot dW_t^0.$$

Using the notation $\alpha_t^i = d_i P_t^i / B_t$ ($0 \leq i \leq n+1$) and $\sigma(t) = \sum_{i=1}^{n+1} \alpha_t^i \nu^i(t)$, the equation becomes

$$dB_t = -B_t \sigma(t) \cdot dW_t^0.$$

³A roller-coaster swap is a swap with notional going up and down

This is formally a log-normal equation but the σ coefficient is state dependent. Note also that the bond dynamic is multi-factor.

The values of the different rates and zero-coupon bond prices at the strike or at an intermediary point are not unique as the model is multi-factors. The exact way the strike information is transformed into a coefficient information need to be decided. There is one constraint for m parameters.

Approximation 1 (Predictor-Corrector implied volatility) *In the local volatility LMM, the price, with corrector approximation, of a receiver swaption is given at time 0 by*

$$R_0 = P(0, s_0) (B_0 N(\kappa_K + \bar{\sigma}_K) - K N(\kappa_K))$$

where

$$\kappa_K = \frac{1}{\bar{\sigma}(K)} \left(\ln \left(\frac{B_0}{K} \right) - \frac{1}{2} \bar{\sigma}(K)^2 \right),$$

$$\bar{\sigma}(K)^2 = \int_0^\theta |\sigma(K, t)|^2 dt$$

and $\sigma(K, t)$ can take one of the two forms described below (in formula 11 and 12).

The price of a payer swaption is

$$P_0 = P(0, s_0) (K N(-\kappa_K) - B_0 N(-\kappa_K - \bar{\sigma}(K)))$$

Construction: The swap price is *at-the-money* for a bond price B at expiry when

$$\sum_{i=1}^{n+1} d_i P_\theta^i = B$$

The prices that interest us the most are the initial price B_0 , the strike price K and the middle point $M = (B_0 + K)/2$.

The discounted value of the zero-coupon bond can be approximated (initial freeze) by

$$P_\theta^i = \frac{P(0, s_i)}{P(0, s_0)} \exp \left(-\tau_i X_i - \frac{1}{2} \tau_i^2 \right)$$

with

$$\tau_i^2 = \int_0^\theta |\nu_0^i(t)|^2 ds$$

and the X_i standard normally distributed random variables. By choosing (arbitrarily) to have all the stochastic variables X_i equal at the required price the (one dimensional) equation to solve is

$$\sum_{i=1}^{n+1} d_i P_0^i \exp \left(-\tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right) = B.$$

Obtaining the solution to the above equation requires to solve a one dimensional equation equivalent to the one solved in the swaption price for the Gaussian HJM model ([Henrard \(2003\)](#)). For

numerical reasons one may prefer not to have to solve this type of equation. The above equation can be replaced by its first order approximation

$$\sum_{i=1}^{n+1} d_i P_0^i \left(1 - \tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right) = B,$$

the solution of which is explicit:

$$\bar{x} = \frac{\sum_{i=1}^{n+1} d_i P_0^i - B - \frac{1}{2} \sum_{i=1}^{n+1} d_i P_0^i \tau_i^2}{\sum_{i=1}^{n+1} d_i P_0^i \tau_i}.$$

The zero-coupon prices, in the exponential case and the approximated first order case, are given by

$$P_B^i = P_0^i \exp \left(-\tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right), \quad \text{respectively} \quad P_B^i = P_0^i \left(1 - \tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right).$$

The rates and bond prices are

$$\prod_{j=0}^{i-1} (1 + \delta_j L_B^j) = (P_B^i)^{-1} \quad \text{and} \quad B = \sum_{i=1}^{n+1} d_i P_B^i.$$

We define

$$\alpha_B^i = \frac{c_i P_B^i}{B}, \quad \nu_B^i(t) = \alpha(t) \sum_{j=0}^{i-1} \frac{D_B^j + a_j}{D_B^j + 1/\delta_j} \gamma_j \quad \text{and} \quad \sigma_B(t) = \sum_{i=1}^n \alpha_B^i \nu_B^i(t).$$

The swaption can be priced with an (option strike dependent) approximated volatility which is the average between the initial price B_0 volatility and the strike $K = -d_0$ volatility

$$\sigma(K, t) = \frac{1}{2} (\sigma_{B_0}(t) + \sigma_K(t)) \quad (11)$$

or with the volatility at the mid point

$$\sigma(K, t) = \sigma_M(t). \quad (12)$$

With the approximation, the solution is

$$B_\theta \simeq \exp \left(\int_0^\theta \sigma(K, s) \cdot dW_s^0 - \frac{1}{2} \bar{\sigma}^2(K) \right) = \exp \left(\bar{\sigma}(K) X - \frac{1}{2} \bar{\sigma}^2(K) \right)$$

for some standard normal distribution X . The standard Black formula approach can be used and gives

$$R_0 = P(0, t_0) \mathbb{E}^0 \left[P^{-1}(\theta, t_0) (B_\theta - K P(\theta, t_0))^+ \right] \simeq P(0, t_0) \mathbb{E}^0 \left[\left(B_0 \exp \left(\bar{\sigma}(K) X - \frac{1}{2} \bar{\sigma}^2(K) \right) - K \right)^+ \right]$$

□

In the particular case of the displaced diffusion LMM with mean reversion time dependent factor, the integral defining $\bar{\sigma}(K)$ can be computed explicitly with (for the middle point)

$$\bar{\sigma}^2(K) = \frac{1}{2a} (\exp(2a\theta) - 1) \left| \sum_{i=1}^n \alpha_M^i \sum_{j=0}^{i-1} \frac{L_M^j + a_j}{L_M^j + 1/\delta_j} \gamma_j \right|^2.$$

TO DO

TO DO: Add cap/floor prices.

5 Monte Carlo implementation

To simplify the writing, we use the notation

$$dD_t^j = \alpha_t^2 \mu_j(D_t)(D_t^j + a_j) dt + \alpha(t)(D_t^j + a_j) \gamma_j \cdot dW_t^n.$$

Note that the drift depend on (potentially) all the variables D_t^i and not only the one of the current equation j .

Let $Y_t^j = \ln(D_t^j + a_j)$. With that new variable the geometric equation becomes a arithmetic one:

$$dY_t^j = \alpha_t^2 (\mu_j(Y_t) - \gamma_j^2) dt + \alpha(t) \gamma_j \cdot dW_t^n.$$

Obviously the required values D_t^j are given by

$$D_t^j = \exp(Y_t^j) - a_j.$$

5.1 Predictor-Corrector or Runge-Kutta

The difficult part in the simulation of the above equation is the state dependent drift μ_j . We implement a predictor corrector approach to simulate the equation. The description of the method in this framework can be found in (Rebonato, 2002, Section 5.3).

The time is divided in time steps (not necessarily small) $0 = s_0 < s_1 < \dots < s_k$. The method consists in first (predictor) applying a Euler type approximation.

$$YE_{k+1}^j = Y_k^j + \alpha^2(s_k) (\mu_j(Y_k) - \gamma_j^2) (s_{k+1} - s_k) + \alpha(s_k) \gamma_j \cdot X \sqrt{s_{k+1} - s_k}$$

where X is a normally distributed random variable of dimension m . In the second step (corrector) one uses for the state dependent drift the output of the first step:

$$Y_{k+1}^j = Y_k^j + \alpha^2(s_k) ((\mu_j(Y_k) + \mu_j(YE_{k+1}))/2 - \gamma_j^2) (s_{k+1} - s_k) + \alpha(s_k) \gamma_j \cdot X \sqrt{s_{k+1} - s_k}$$

with the same random variable X .

This methods allows long steps (between relevant dates). It is typically used with dates three or six months apart. On the other side at each step two computations are required and made the method twice slower (for the same number of steps).

5.2 Predictor-corrector impact

The impact of the *correction* may appear small. Rebonato (2002) shows, in his Table 5.1 and Figure 5.4, that it reduces the error significantly on FRA. Our Figure 1 was done using Predictor-Corrector approach and Euler scheme for a 10Yx5Y swaption and a two factors LMM with $a_j = 0.05$. A series of 100 prices, each with 50,000 paths was computed with one (very) long jump. The Euler bias appears clearly. The exact price (with three months jumps and 10 millions paths) is the plain vertical line. The dotted vertical lines are the 10 and 90 percentiles of the different schemes. This example shows that with a predictor-corrector scheme a unique jump between dates is enough.

The error on the predictor-corrector is coming from the uncertainty of the Monte-Carlo simulation. The bias is negligible. Both scheme results are computed with the same random draws. The difference between the two is a fair measure of the bias for the Euler scheme.

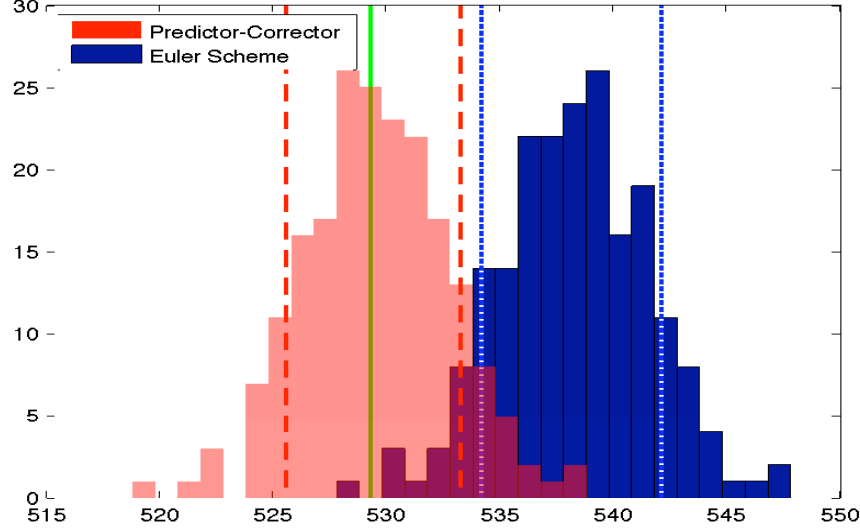


Figure 1: Predictor-corrector bias improvement over Euler scheme.

5.3 Efficient implementation

Due to the special structure of the drift term, it is possible to reduced the computation burden drawback. For this we solve the equations recursively starting by the last one. As the numeraire is the natural numeraire of the last Libor, its drift is not state dependent:

$$Y_{k+1}^n = Y_k^n + \alpha^2(s_k) (-\gamma_j^2) (s_{k+1} - s_k) + \alpha(s_k) \gamma_j \cdot X \sqrt{s_{k+1} - s_k}$$

No corrector is needed for that term. Suppose now that all the variables have been computed from rate n to rate $j-1$. The important point to notice is that only the value Y_k^i with $i > j$ are required to compute the drift and they have already been computed. The rate j is given by

$$Y_{k+1}^j = Y_k^j + \alpha^2(s_k) \left((\mu_j(Y_k) + \mu_j(Y_{k+1}^{j+1}, \dots, n)) / 2 - \gamma_j^2 \right) (s_{k+1} - s_k) + \alpha(s_k) \gamma_j \cdot X \sqrt{s_{k+1} - s_k}$$

With this recursive approach, the computation of the intermediary value YE is not required anymore at the (organizational) cost of working from the last rate. The computation of the drifts are still required and so the time saved is not huge but at the same time not negligible. The improvement presented in this paragraph is, to our knowledge, original.

5.4 Random number generator

TO DO

TO DO: Random number generator

5.5 Pricing

In the Monte-Carlo approach the price is computed as the expected value over different paths. As we use the last date discount factor as numeraire, the price of an instrument with pay-off H viewed from u in s is

$$P^D(0, t_n) E^n \left[P^{D^{-1}}(u, t_n) H P(u, s) \right].$$

The pay-off H can be fixed at any date before or on s . Due to the discrete implementation of the model we require that $s = t_i$ for some $0 \leq i \leq n$. The pay-off H can be fixed at any date, even different from the $\{t_i\}_{0 \leq i \leq n}$. In general this will be the case as the fixing take place two days before the period start. The fixing date should be in the list of dates for which the rates are simulated $u = s_j$ for some $0 \leq j \leq k$.

The discount factor ratio $P^D(s_j, t_i)/P^D(s_j, t_n)$ need to be written as function of the factors simulated in the implementation. The discount factor can be written as a composition of rates

$$\frac{P^D(s_j, t_i)}{P^D(s_j, t_n)} = \prod_{l=i+1}^n (1 + \delta_l D_{s_j}^l).$$

Note that it is possible to evaluate products with fixing dates not in the period dates t_i thanks to the simplification above.

6 Calibration

An important part of the implementation is the calibration. In the LMM case, the calibration part include also the *choice* on dates for the Libor. The model will be product dependent.

We consider a product for which the relevant payment and fixing start and end dates are in the set $s_0 < s_1 < \dots < s_p$. Those dates can be the dates of Libor periods, the payment dates or the dates linked to a CMS fixing.

Those dates should be incorporated into the regular set of dates $t_0 < \dots < t_n$. The relation between those two set of dates will be product dependent. It is described below for some products.

Then one need to choose the calibration instruments. There again the choice is not unique and product dependent.

6.1 Parameter choice

The parameters a_{HW} (mean reversion) and a (displacement) are not calibrated but selected before the pricing.

The calibration procedure (as described below for several instruments) will usually be done on at most one instrument by period while there are as many parameters by period as model factors. The γ shape should be pre-determine to solve the under-determination. The proposal is to use a two factors model. At the start of the calibration a starting vector will be selected and use as a starting γ in the processes described below. The starting γ selected could consists of the two first component of a principal component analysis of the Libor forward rates.

6.2 European swaption

For a European swaption, the set of relevant dates is the set of fixed coupon payment and the set of dates for the Libor periods. It is the settle date and all the payment dates (fixed and floating

leg). Usually those dates are regularly spaced (every three or six months). The set used is simply $(s_i)_{i=0,\dots,p} = (t_{k_s+i})_{i=0,\dots,p}$. The exact dates of the fixing period can be one or two days away from the payment dates (due to weekends and holidays). We ignore those differences and consider that the fixing start and end dates are on the payment dates. The (very small) differences could be handled by adjustment factors embedded in the multi-curve adjustments if needed.

For non-constant European swaption the suggestion for the choice of the calibration instruments is the following. The payment frequency on the floating leg is usually higher than on the fixed leg, but it is not necessarily the case. For the model construction, the dates associated to the most frequent leg will be used. The non-constant notional swap is decomposed in several constant notional swaps. There is one swap in the decomposition for each period on the less frequent leg. Note that the calibration swaptions should have the same dates as the swaptions to be priced. This constraint is due to the discrete date choice in a LMM. The calibration swap strike is equal to the coupon equivalent of the original swap with the same maturity.

The price of each constant notional decomposed European vanilla swaption is priced with the base methodology (e.g. SABR in multi-curves framework). The coefficients of the model (γ_j and a_j) are calibrated to those prices. The calibration is first done on the decomposition swap with the shortest maturity. The relevant γ_j are calibrated on that price.

For the exact calibration on swaptions, the calibration consists in finding a set of γ which is the multiple of the starting γ such that the calibrated swaption is correctly priced. There can be several γ_j for one calibration swap. In general the fixed leg is annual and the floating leg is semi-annual, so there are two γ_j (with several factors) for one swaption. Then an iterative procedure is used. The swaption with the next maturity is calibrated. The coefficients γ_j which are not yet calibrated and used in the pricing of the swaption are now calibrated. This procedure is used until all the coefficients are calibrated.

For the at-best calibration with swaptions of different strikes, the calibration consists in finding a set of γ and a . The γ are multiple of the starting γ and the a are translated from initial a . The initial set is modified in such a way that the calibrated swaptions pricing error is minimise (in a least-square sense). There can be several γ_j and a_j for one calibration swap set. In general the fixed leg is annual and the floating leg is semi-annual, so there are two γ_j (with several factors) and two a_j for one swaption set. Then an iterative procedure is used. The swaption with the next maturity is calibrated. The coefficients γ_j and a_j which are not yet calibrated and used in the pricing of the swaption are now calibrated. This procedure is used until all the coefficients are calibrated.

6.3 Ratchet on Libor

We suppose that the Ratchet payment dates are the same as the fixing dates. The dates are denoted $(s_i)_{i=0,\dots,p}$ with s_0 the start date of the first fixing period and $(s_i)_{i=1,\dots,p}$ the end of the fixing periods and payment dates. The calibration is done on the caplets that compose the ratchet, i.e. caplet with the same fixing date as the ratchet and periods $[s_i, s_{i+1}]$. The calibration procedure is similar to the one used for European swaptions.

7 Implementation

The Libor Market Model description is in the `LiborMarketModelDisplacedDiffusionParameters`. The pricing of European swaption with physical delivery is in

`SwaptionPhysicalFixedIborLMMDDMethod`. The pricing of cap/floors is in `CapFloorIborLMMDDMethod`.

The generic Monte-Carlo implementation for the LMM model is available through `LiborMarketModelMonteCarloMethod`. The price calculator from the Monte-Carlo scenarios is done with the calculator `MonteCarloIborRateCalculator`.

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