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BRANCHING PROCESSES WITH IMMIGRATION AND RELATED LIMIT THEOREMS

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Introduction. A branching process is a mathematical model for the evolution of a population whose growth or decay is subject to the law of chance. The simplest and the most fundamental branching processes are Galton — Watson branching processes (cf. Harris [3]). If we suppose that, in addition to the ordinary Galton — Watson process, there exists another source of particles from which immigration into the population occurs during each generation, we get a stochastic process called a Galton — Watson branching process with immigration.

The convergence of Galton — Watson processes to a class of diffusion processes was first considered by Feller [1]. Lamperti [9], [11] has studied such limit theorems in detail. Limit processes form a class of Markov processes called continuous state branching processes (CB-processes). CB-processes were first introduced by Jiřina [6] and have been studied by Lamperti [10], Silberstein [15] and Watanabe [17].

The purpose of this paper is to discuss similar problems for branching processes with immigration. In § 1, we shall describe the class of limit processes. They are called CBI-processes and characterized, in terms of the semigroup, by the simple property (1.5). The class of stochastically continuous CBI-processes will be determined completely following a method of Watanabe [17]. This result covers, of course, the case of CB-processes. In § 2, we shall study the limit theorems for Galton—Watson branching processes with immigration. Theorem 2.1 and Theorem 2.2 are natural extension of theorems in Lamperti [11]. In Theorem 2.3, we shall obtain the condition for convergence of a modified sequence formed of a *single* Galton — Watson process with immigration. As examples, the conditioned Galton — Watson processes and upcrossing numbers of Brownian motion in connection with the local time will be discussed.

§ 1. CBI-processes

Definition 1.1. A Markov process $X = (X_t, \mathbf{P}_x)$ on $[0, \infty]$ with ∞ as a trap*** is called a *CBI-process* if for each $t \geqslant 0$ and $\lambda \geqslant 0$, there exist

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*** I. e. $\mathbf{P}_x[X_t = \infty \Rightarrow X_s = \infty \text{ for } \forall s \geqslant t] = 1$ for every $x \in [0, \infty]$.

$\varphi(t, \lambda) \geqslant 0$ and $\psi(t, \lambda) \geqslant 0$ such that

$$\mathbf{E}_x [e^{-\lambda X_t}; t < e_\infty] = \varphi(t, \lambda) e^{-x\psi(t, \lambda)} \tag{1.1}$$

for every $x \in [0, \infty)$ (here $e_\infty = \inf\{t; X_t = \infty\}$, $\inf \phi = + \infty$).

In a particular case when $\varphi \equiv 1$, X is called a *CB-process.* CB-processes were studied by Jiřina [6], Lamperti [10], [11], Silberstein [15] and, for multidimensional case, by Watanabe [17]. Let T_t be the semigroup of X:

$$T_t f(x) = E_x [f(X_t); t < e_\infty]. \tag{1.2}$$

Let Λ and $\widetilde{\Lambda}$ be the subsets of $\widetilde{C}$ $[0, \infty)$ * defined by

$$\Lambda = \{f(x) = e^{-\lambda x}; \lambda \geqslant 0\} \tag{1.3}$$

and

$$\widetilde{\Lambda} = \{f(x) = ce^{-\lambda x}; c \geqslant 0, \lambda \geqslant 0\}. \tag{1.4}$$

If X is a CBI-process, then the semigroup T_t maps Λ into $\widetilde{\Lambda}$:

$$T_t(\Lambda) \subset \widetilde{\Lambda}; \tag{1.5}$$

this is clearly equivalent to $T_t(\widetilde{\Lambda}) \subset \widetilde{\Lambda}$. If X is a CB-process, then the semigroup T_t maps Λ into Λ:

$$T_t(\Lambda) \subset \Lambda. \tag{1.6}$$

Since the linear hull of Λ is dense in $\widetilde{C}$ $[0, \infty)$, T_t is a semigroup on $\widetilde{C}$ $[0, \infty)$ in both cases. Conversely, it is easy to see that, for a non-negative contraction semigroup T_t on $\widetilde{C}$ $[0, \infty)$ with property (1.5) (with property (1.6)), there exists a unique** CBI-process (resp. CB-process) with T_t as its semigroup. By the semigroup property, φ satisfies

$$\varphi(0, \lambda) = 1, \ \varphi(t + s, \lambda) = \varphi(t, \lambda)\, \varphi(s, \psi(t, \lambda)), \tag{1.7}$$

and ψ satisfies

$$\psi(0, \lambda) = \lambda, \ \psi(t + s, \lambda) = \psi(t, \psi(s, \lambda)) \tag{1.8}$$

for $t, s \geqslant 0$.

It is easy to verify the following

Proposition 1.1. *The following conditions are equivalent each other:*

(i) *X is stochastically continuous for every $\mathbf{P}_x$,*

(ii) *$\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are continuous in t for each $\lambda > 0$,*

(iii) *T_t is a strongly continuous semigroup on $C_\infty [0, \infty)$* ***,

(iv) *X is a Hunt process; in particular, it is a right continuous process with left-hand limits and strongly Markov.*

Definition 1.2. Let $\Psi = \{\psi(\lambda) = c_0 + c_1\lambda + \int_0^\infty (1 - e^{-\lambda u})\, n(du)$, where $c_0 \geqslant 0$, $c_1 \geqslant 0$ and $n(du)$ is a non-negative measure on $(0, \infty)$

* $\widetilde{C}$ $[0, \infty) = \{f;$ continuous on $[0, \infty]$ and $\lim_{x \to \infty} f(x)$ exists$\}$.

** Up to equivalence of stochastic processes.

*** $C_\infty [0, \infty) = \{f \in \widetilde{C}$ $[0, \infty)$: $\lim_{x \to +\infty} f(x) = 0\}$.

such that $\int_0^\infty (1 \wedge u)\, n(du) < \infty$. A one-parameter family $\{\psi_t(\lambda)\}_{t\in[0,\infty)}$ of functions in Ψ is called a *Ψ-semigroup* if

$$\psi_0(\lambda) = \lambda \text{ and } \psi_{t+s}(\lambda) = \psi_t(\psi_s(\lambda)). \tag{1.9}$$

It is well known that Ψ is closed under the composition and also that if $\psi_n \in \Psi$, $\psi_n \to \psi$ on a dense set of (λ_1, λ_2) $(0 \leqslant \lambda_1 < \lambda_2 \leqslant \infty)$, then there exists a unique extention $\tilde{\psi} \in \Psi$ of ψ and $\psi_n \to \tilde{\psi}$ everywhere and furthermore the convergence is uniform on each bounded set. $\psi(t, \lambda)$ corresponding to a CB-process forms a Ψ-semigroup and *vice versa*. It will be shown below that $\psi(t, \lambda)$ corresponding to a CBI-process forms also a Ψ-semigroup.

The following theorem characterizes completely the class of stochastically continuous CBI-processes.

Theorem 1.1. *Let $X = (X_t, \mathbf{P}_x)$ be a stochastically continuous CBI-process. Then there exist real constants $\alpha \geqslant 0$, β, $\gamma \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$* and *non-negative measures $n_1(du)$ and $n_2(du)$ on $(0, \infty)$ with*

$$\int_0^\infty \frac{u^2}{1+u^2} n_1(du) < \infty, \quad \int_0^\infty \frac{u}{1+u} n_2(du) < \infty$$

such that, if we set

$$R(\lambda) = -\alpha\lambda^2 + \beta\lambda + \gamma - \int_0^\infty \left(e^{-\lambda u} - 1 - \frac{\lambda u}{1+u^2}\right) n_1(du) \tag{1.10}$$

and

$$F(\lambda) = c + d\lambda - \int_0^\infty (e^{-\lambda u} - 1)\, n_2(du) \tag{1.11}$$

then $\psi(t, \lambda)$ and $\varphi(t, \lambda)$ are given as follows: $\psi(t, \lambda)$ is the solution of

$$\frac{\partial\psi}{\partial t}(t, \lambda) = R(\psi(t, \lambda)), \qquad \psi(0, \lambda) = \lambda. \tag{1.12}$$

and $\varphi(t, \lambda)$ is given by

$$\varphi(t, \lambda) = \exp\left\{-\int_0^t F(\psi(s, \lambda))\, ds\right\}. \tag{1.13}$$

Conversely, given real constants $\alpha \geqslant 0$, β, $\gamma \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$ and non-negative measures n_1 and n_2 on $(0, \infty)$ such that

$$\int_0^\infty \frac{u^2}{1+u^2} n_1(du) + \int_0^\infty \frac{u}{1+u} n_2(du) < \infty,$$

there exists a unique CBI-process X whose $\varphi(t, \lambda)$ and $\psi(t, \lambda)$ are given by (1.12) *and* (1.13).

This theorem can be stated in the following analytical form:

Theorem 1.1′. *Let T_t be a non-negative strongly-continuous contraction semigroup on C_∞ $[0, \infty)$ with the property* (1.5). *Then there exists $\alpha \geqslant 0$, β, $\gamma \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$ and non-negative measures n_1 and n_2 on $(0, \infty)$ such that*

$$\int_0^\infty \frac{u^2}{1+u^2} n_1(du) + \int_0^\infty \frac{u}{1+u} n_2(du) < \infty$$

*and, if A is the infinitesimal generator in Hille — Yosida sense of T_t with the domain $D(A)$, then $\mathcal{L}(\Lambda_0)$ * $\subset D(A)$ and for $f \in \mathcal{L}(\Lambda_0)$*

$$Af(x) = \alpha x f''(x) + (\beta x + d) f'(x) - (\gamma x + c) f(x) +$$
$$+ x \int_0^\infty \left[f(x+y) - f(x) - \frac{y}{1+y^2} f'(x) \right] n_1(dy) + \int_1^\infty [f(x+y) - f(x)] n_2(dy). \tag{1.14}$$

Conversely, given α, β, γ, c, d, n_1 and n_2 with the above properties, the operator A with domain $\mathcal{L}(\Lambda_0)$ has the smallest closed extention which generates a semigroup T_t on C_∞^* $[0, \infty)$ and T_t is strongly continuous non-negative and contraction with the property (1.5).

In order to see the equivalence of Theorems 1.1 and 1.1′ it is sufficient to remark that $\mathcal{L}(\Lambda_0)$ is a core of A and this is a direct consequence of Lemma 2.2 of [18] **.

As a corollarry of the theorem we see that $\psi(t, \lambda)$ is a Ψ-semigroup to which a stochastically continuous CB-process corresponds. Indeed, as is shown in [15] and [17], every such Ψ-semigroup is obtained as the solution of (1.12).

To prove Theorem 1.1 we need some lemmas.

Lemma 1.1. *Let T_t be a strongly continuous contraction semigroup on C_∞ $[0, \infty)$ and A be the infinitesimal generator with domain $D(A)$. Then for each $x_0 \in (0, \infty)$, there exist $\alpha(x_0) \geqslant 0$, $\beta(x_0)$, $\gamma(x_0) \geqslant 0$, $\delta(x_0) \leqslant 0$ and a bounded measure $\nu_{x_0}(dy)$ on $(0, \infty) - \{x_0\}$ such that, for every $f \in$ $\in D(A) \cap C^2(0, \infty)$, we have*

$$Lf(x_0) = 0$$

where

$$Lf(x_0) = \alpha(x_0) f''(x_0) + \beta(x_0) f'(x_0) - \gamma(x_0) f(x_0) +$$
$$+ \delta(x_0) Af(x_0) + \int_{(0,\infty)-\{x_0\}} \left[f(x) - f(x_0) - \frac{(x-x_0)}{1+(x-x_0)^2} f'(x) \right] \frac{1+(x-x_0)^2}{(x-x_0)^2} \nu_{x_0}(dx). \tag{1.15}$$

Furthermore, if $\alpha(x_0) = \beta(x_0) = \gamma(x_0) = \delta(x_0) = 0$, then $\nu_{x_0} \neq 0$.

This lemma can be proved by the same way as in [19] and so the proof is omitted.

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* $\Lambda_0 = \{f(x) = e^{-\lambda x}; \lambda > 0\} = \Lambda \cap C_\infty [0, \infty)$ and $\mathcal{L}(\Lambda_0)$ is the linear hull of Λ_0.

** Indeed, take $\mathcal{L}(\Lambda_0)$ as D in Lemma 2.2 of [18].

Let $X = (X_t, \mathbf{P}_x)$ be a stochastically continuous CBI-process and T_t be its semigroup on $C_\infty [0, \infty)$. We shall denote by $f_\lambda(x)$ $(\lambda > 0)$ the function $e^{-\lambda x}$. Then $T_t f_\lambda(x) = \varphi(t, \lambda) f_{\psi(t,\lambda)}(x)$ by (1.1).

Lemma 1.2. *$F(\lambda) = -\partial\varphi/\partial t\ (0+, \lambda)$ exists and $F(\lambda)$ is of the form*

$$F(\lambda) = c + d\lambda - \int_0^\infty (e^{-\lambda y} - 1)\, n_2(dy) \tag{1.16}$$

where $c \geqslant 0$, $d \geqslant 0$ and $n_2(dy)$ is a non-negative measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{u}{1+u} n_2(du) < \infty.$$

Proof. Since $0 < \varphi(t, \lambda) = T_t f_\lambda(0) \leqslant 1$ and $\varphi(t+s, \lambda) = \varphi(t, \lambda)\, \varphi(s, \psi(t, \lambda))$, $\varphi(t, \lambda)$ is decreasing in t. Thus, for every $\lambda > 0$, $\partial\varphi/\partial t$ exists a.e. in t. But if $\partial\varphi/\partial t$ exists at $t = t_0$, then $\partial\varphi/\partial t\,|_{t=0}$ exists for $\lambda' = \psi(t_0, \lambda)$. Hence $\partial\varphi/\partial t\,|_{t=0}$ exists for $\lambda \in L$ where L is a dense set in $(0, \infty)$. On the other hand, for each $t > 0$ $(1 - \varphi(t, \lambda))/t \in \Psi$. Thus $\lim_{t \downarrow 0} (1 - \varphi(t, \lambda))/t = F(\lambda)$ exists and $F \in \Psi$. Q.E.D.

Lemma 1.3. *$R(\lambda) = \partial\varphi/\partial t\ (0+, \lambda)$ exists and $R(\lambda)$ is of the form*

$$R(\lambda) = -\alpha\lambda^2 + \beta\lambda + \gamma - \int_0^\infty \left(e^{-\lambda u} - 1 - \frac{\lambda u}{1+u^2}\right) n_1(du) \tag{1.17}$$

where $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $n_1(du)$ is a non-negative measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{u^2}{1+u^2} n_1(du) < \infty.$$

Hence $\psi(t, \lambda)$ is differentiable in t and satisfies (1.12).

Proof. First we shall show that if we set $L = \{\lambda > 0;\ \partial\psi/\partial t$ exists at $t = 0\}$ then L is dense in an interval of the form $[\lambda_0, \infty)$. Since $\partial\varphi/\partial t|_{t=0}$ exists and $T_t f_\lambda(x) = \varphi(t, \lambda) e^{-x\psi(t,\lambda)}$, it is easy to see that $\lambda \in L$ if and only if $f_\lambda \in D(A)$. Let $\bar\lambda > 0$ and $\bar t > 0$ be fixed. Set

$$u_1(x) = \int_0^{\bar t} e^{-\psi(t, \bar\lambda) x} \varphi(t, \bar\lambda)\, dt = \int_0^{t} T_t f_{\bar\lambda}(x)\, dt.$$

Then, clearly, $u_1 \in D(A) \cap C^\infty (0, \infty)$ and $x \rightsquigarrow u_1(x)$ defines a diffeomorphism of $[0, \infty)$ onto $(0, \eta)$ where $\eta = \int_0^{\bar t} \varphi(t, \bar\lambda)\, dt$. Hence, for every $\lambda > 0$, there exists a C^∞-function $F_\lambda(u)$ on $(0, \eta]$ such that $f_\lambda(x) = F_\lambda[u_t(x)]$. Since

$$F'_\lambda[u_1(x)] = \frac{\lambda e^{-\lambda x}}{\int_0^{\bar t} e^{-\psi(t, \lambda) x} \psi(t, \bar\lambda)\, \varphi(t, \bar\lambda)\, dt}$$

and

$$F''_\lambda[u_1(x)] = \frac{\lambda e^{-\lambda x} \int_0^{\bar t} e^{-\psi(t, \bar\lambda) x} \{\psi^2(t, \bar\lambda)\, \varphi(t, \bar\lambda) - \psi(t, \bar\lambda)\, \varphi(t, \bar\lambda)\, \lambda\}\, dt}{\left[\int_0^{\bar t} e^{-\psi(t, \bar\lambda) x} \psi(t, \bar\lambda)\, \varphi(t, \bar\lambda)\, dt\right]^3},$$

if $\lambda > 3 \max_{0\leqslant t\leqslant \bar{t}} \psi(t, \lambda) \equiv \lambda_0$, then it is easy to verify that $F'_\lambda[u_1(x)]$ and $F''_\lambda[u_1(x)]$ are bounded in $x \in [0, \infty)$. Let $x \in (0, \infty)$ be fixed. Then*

$$M_t = u_1(X_t) - u_1(X_0) - \int_0^t Au_1(X_s)\,ds$$

is a square-integrable martingale ($\mathbf{P}_x$) and it is decomposed as

$$M_t = M_t^c + M_t^d.$$

By a formula on stochastic integrals (cf. [13], Théorème 4), we have

$$f_\lambda(X_t) - f_\lambda(X_0) = F_\lambda[u_1(X_t)] - F_\lambda[u_1(X_0)] = \int_0^t F'_\lambda[u_1(X_s)]\,dM_s +$$

$$+ \int_0^t F'_\lambda[u_1(X_s)]\,Au(X_s)\,ds + \frac{1}{2}\int_0^t F''_\lambda[u_1(X_s)]\,d\langle M^c\rangle_s +$$

$$+ \sum_{\substack{s\leqslant t\\ \Delta X_s\neq 0}} \{F_\lambda[u_1(X_s)] - F_\lambda[u_1(X_{s-})] - [u_1(X_s) - u_1(X_{s-})]\,F'_\lambda[u_1(X_{s-})]\}.$$

The first term is a martingale and hence it is of mean 0. Other three terms are processes with bounded variation. Indeed, as for the last term, we have

$$\mathbf{E}_x\Big[\sum_{\substack{s\leqslant t\\ \Delta X_s\neq 0}} |F_\lambda[u_1(X_s)] - F_\lambda[u_1(X_{s-})] - [u_1(X_s) - u_1(X_{s-})]\,F'_\lambda[u_1(X_{s-})]|\leqslant$$

$$\leqslant \frac{1}{2} \sup_{x\in[0,\infty)} F''_\lambda[u_1(x)]\cdot\mathbf{E}_x\Big[\sum_{\substack{s\leqslant t\\ \Delta X_s\neq 0}} |u_1(X_s) - u_1(X_{s-})|^2\Big] =$$

$$= \frac{1}{2} \sup_{x\in[0,\infty)} F''_\lambda[u_1(x)]\,\mathbf{E}_x[\langle M^d\rangle(t)] \leqslant \frac{1}{2}\sup F''_\lambda[u_1(x)]\,\mathbf{E}_x[\langle M\rangle(t)] < \infty.$$

Hence $\mathbf{E}_x[f_\lambda(X_t)] = T_t f_\lambda(x)$ is a function of bounded variation in t. By the similar argument as in the proof of Lemma 1.2, the set L_0 $\{\lambda;\ \partial\psi/\partial t$ exists at $t = 0\}$ is dense in (λ_0, ∞). Now we shall determine the expression of $R(\lambda) = \partial\psi/\partial t\,(0, \lambda)$ for $\lambda \in L_0$. Since $f \in D(A) \cap C^2(0, \infty)$ for $\lambda \in L_0$, we can apply Lemma 1.1 and see that, for each $x_0 \in (0, \infty)$, there exist $\alpha(x_0) \geqslant 0$, $\beta(x_0)$, $\gamma(x_0) \geqslant 0$, $\delta(x_0) \geqslant 0$, and a bounded measure $\nu_{x_0}(dx)$ on $(0, \infty) - \{x_0\}$ such that, for every $\lambda \in L_0$,

$$Lf_\lambda(x_0) = e^{-\lambda x_0}[\alpha(x_0)\lambda^2 - \beta(x_0)\lambda - \gamma(x_0)] +$$

$$+ \int_{(0,\infty)-\{x_0\}} \Big\{e^{-\lambda(x-x_0)} - 1 - \frac{\lambda(x-x_0)}{1+(x-x_0)^2}\Big\}\frac{1+(x-x_0)^2}{(x-x_0)^2}\,\nu_{x_0}(dx) + \delta(x_0)\,Af_\lambda(x_0) = 0.$$

If $\delta(x_0) = 0$, then

$$h_{x_0}(\lambda) \equiv \alpha(x_0)\lambda^2 - \beta(x_0)\lambda - \gamma(x_0) +$$

$$+ \int_{(0,\infty)-\{x_0\}} \Big\{e^{-\lambda(x-x_0)} - 1 - \frac{\lambda(x-x_0)}{1+(x-x_0)^2}\Big\}\frac{1+(x-x_0)^2}{(x-x_0)^2}\,\nu_{x_0}(dx) = 0$$

* For the general theory of martingales and notations used here, we refer to Meyer [13].

for all $\lambda \in L_0$ and, since $h_{x_0}(\lambda)$ is analytic in $\lambda > 0$, this would imply that $h_{x_0}(\lambda) \equiv 0$. Then, as is well known, this would imply that $\alpha(x_0) = \beta(x_0) = \gamma(x_0) = \nu_{x_0} = 0$ and we get a contradiction. Hence $\delta(x_0) \neq 0$ and we may assume $\delta(x_0) = -1$. Thus, $Af_\lambda(x_0) = e^{-\lambda x_0} h_{x_0}(\lambda)$, $\lambda \in L_0$.

On the other hand,

$$Af_\lambda(x_0) = \frac{\partial}{\partial t}[e^{-\psi(t,\lambda)x_0}\varphi(t,\lambda)]|_{t=0} = [-x_0 R(\lambda) - F(\lambda)]\, e^{-\lambda x_0}$$

where $R(\lambda) = \partial\psi/\partial t_{t=0}$ and $F(\lambda) = -\partial\varphi/\partial t|_{t=0}$. Hence $h_{x_0}(\lambda) = -x_0 R(\lambda) - F(\lambda)$. Putting $x_0 = 1$, we have from (1.16),

$$-R(\lambda) = h_1(\lambda) + F(\lambda) = \alpha(1)\lambda^2 - \beta(1)\lambda - \gamma(1) +$$
$$+ \int_{(-1,\infty)} \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n(dy) + c + d'\lambda - \int_0^\infty \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n_2(dy)$$

where $n(dy) = \frac{1+y^2}{y^2}\nu_1(dy \oplus 1)^*$ and $d' = d - \int_0^\infty \frac{y}{1+y^2} n_2(dy)$.

Thus,

$$-R(\lambda) = \alpha(1)\lambda^2 - (\beta(1) - d')\lambda - (\gamma(1) - c) +$$
$$+ \int_{(-1,\infty)-\{0\}} \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n_1(dy) \tag{1.18}$$

where $n_1(dy) = -I_{(0,\infty)}(y)\, n_2(dy) + n(dy)$. Hence

$$h(x_0) = -x_0 R(\lambda) - F(\lambda) = \alpha(1)x_0\lambda^2 - (\beta(1) - d')x_0\lambda - (\gamma(1) - c)x_0 +$$
$$+ x_0 \int_{(-1,\infty)-\{0\}} \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n_1(dy) - c - d'\lambda +$$
$$+ \int_0^\infty \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n_2(dy)$$

and since L_0 is dense in (λ_0, ∞), we have

$\alpha(x_0) = \alpha(1)x_0$, $\beta(x_0) = (\beta(1) - d')x_0 + d'$, $\gamma(x_0) = (\gamma(1) - c)x_0 + c$ and

$$\frac{1+y^2}{y^2}\nu_{x_0}(dy \oplus x_0) = x_0 n_1(dy) + I_{(0,\infty)}(y)\, n_2(dy).$$

By letting $x_0 \to 0$, we see at once that $n_1(dy)$ has no mass for $y < 0$ and by letting $x_0 \to \infty$, we see that $\gamma(1) - c \geqslant 0$ and $n_1(dy) \geqslant 0$. Hence we have

$$R(\lambda) = -\alpha\lambda^2 + \beta\lambda + \gamma + \int_0^\infty \left(e^{-\lambda y} - 1 - \frac{\lambda y}{1+y^2}\right) n_1(dy) \tag{1.19}$$

where $\alpha = \alpha(1) \geqslant 0$, $\beta = \beta(1) - d'$, $\gamma = \gamma(1) = c \geqslant 0$ and $n_1(dy)$ is a non-negative measure on $(0\ \infty)$ such that $\int_0^\infty \frac{y^2}{1+y^2} n_1(dy) < \infty$. Since

* $E \oplus 1 = \{x + 1;\ x \in E\}$.

$Af_\lambda(x) = [-xR(\lambda) - F(\lambda)]\, e^{-\lambda x}$ for $\lambda \in L_0$ and A is a closed operator, it is clear that L_0 is closed and hence it contains $[\lambda_0, \infty)$. In particular, $\partial\psi/\partial t|_{t=0}$ exists for all $\lambda \in [\lambda_0, \infty)$. Hence if $\lambda_2 > \lambda_1 > \lambda_0$, then there exists $t_0 > 0$ such that $\{\psi(t, \lambda), \lambda \in [\lambda_1, \lambda_2], t \in [0, t_0]\}$ is a solution of

$$\frac{\partial\psi}{\partial t}(t, \lambda) = R(\psi(t, \lambda)), \qquad \psi(0, \lambda) = \lambda. \tag{1.20}$$

But the solution of (1.20) exists for every $\lambda > 0$ and analytic in λ. Therefore, in coincides with $\psi(t, \lambda)$ for $t \leqslant t_0$. This implies $\left.\frac{\partial\psi}{\partial t}\right|_{t=0}$ exists for every $\lambda > 0$ and it is given by (1.19). The proof of the lemma is now complete.

P r o o f o f T h e o r e m 1.1. The first part of the theorem has been proved. Suppose we are given $\alpha \geqslant 0$, β, $\gamma \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$ and n_1, n_2 such that

$$\int_0^\infty \frac{u^2}{1+u^2}\, n_1(du) + \int_0^\infty \frac{u}{1+u}\, n_2(du) < \infty.$$

Define $R(\lambda)$ and $F(\lambda)$ by (1.10) and (1.11) respectively. Now solve the equation (1.12). It is known that the solution $\psi(t, \lambda)$ defines a Ψ-semigroup (cf. [15], [17]). Define $\varphi(t, \lambda)$ by (1.13). Since $\int_0^t F(\psi(s, \lambda))\, ds \in \Psi$, it is clear that for every $t \geqslant 0$ and $x \in [0, \infty)$, there exists a substochastic measure $P(t, x, dy)$ on $[0, \infty)$ such that $\int_{0-}^\infty e^{-\lambda y} P(t,x,dy) = e^{-x\psi(t,\lambda)}\varphi(t, \lambda)$. Clearly, φ and ψ satisfy (1.7) and (1.8); hence, by a simple calculation, it is easy to see that $P(t, x, dy)$ satisfies the Chapman — Kolmogorow equation:

$$\int_{0-}^\infty P(t, x, dy)\, P(s, y, E) = P(t+s, x, E).$$

Hence, it defines a unique Markov process on $[0, \infty)$ with ∞ as the trap. By the continuity of φ and ψ in t, this process is stochastically continuous. Thus, theorem is completely proved.

D e f i n i t i o n 1.3. A CBI-process is called *conservative* if for every $t > 0$ and $x \in [0, \infty)$ $\mathbf{P}_x[X_t < \infty] = 1$.

Theorem 1.2. *Let $X = (X_t, \mathbf{P}_x)$ be a stochastically continuous CBI-process. X is conservative if and only if $\gamma = c = 0$ and*

$$\int_{0+}^\infty R^*(\lambda)^{-1}\, d\lambda = +\infty \tag{1.21}$$

where $R^*(\lambda) = R(\lambda) \vee 0$.

We omit the proof (cf. Ikeda — Watanabe [4]).

Example 1.1. Let X be a stochastically continuous CBI-process corresponding to $R(\lambda) = -c\lambda^{\alpha+1}$ and $F(\lambda) = d\lambda^{\alpha}$ ($c \geqslant 0$, $d \geqslant 0$ and $0 < \alpha \leqslant 1$). Then $\psi(t, \lambda)$ is the solution of

$$\frac{\partial\psi}{\partial t}(t, \lambda) = -c\{\psi(t, \lambda)\}^{\alpha+1},$$
$$\psi(0, \lambda) = \lambda.$$

Hence

$$\psi(t, \lambda) = \frac{\lambda}{[1 + t\,\alpha c\lambda^{\alpha}]^{1/\alpha}}. \tag{1.22}$$

Then $\varphi(t, \lambda) = \exp\left(-d\int_0^t \psi^{\alpha}(s, \lambda)\,ds\right)$ is given by

$$\varphi(t, \lambda) = (1 + t\,\alpha c\lambda^{\alpha})^{-d/\alpha c}. \tag{1.23}$$

We notice also that the infinitesimal generator of this CBI-process is given, for $f \in C^2(0, \infty)$, by

$$\begin{aligned} Af(x) &= \frac{\alpha(1+\alpha)c}{\Gamma(1-\alpha)}\, x\int_0^{\infty} [f(x+y) - f(x) - yf'(x)]\, y^{-\alpha-2}dy + \\ &\quad + \frac{\alpha d}{\Gamma(1-\alpha)}\int_0^{\infty} [f(x+y) - f(x)]\, y^{-\alpha-1}\,dy, \quad 0 < \alpha < 1, \\ Af(x) &= cxf''(x) + df'(x), \qquad \alpha = 1. \end{aligned} \tag{1.24}$$

Thus X is a diffusion process only for $\alpha = 1$. They are all conservative processes.

Remark 1.1. Lamperti proved that if X_t is a CB-process such that $\psi(t, \lambda)$ is measurable in t and $\mathbf{P}_x[X_t = 0] < 1$ for some t, $x > 0$ then X_t is stochastically continuous (cf. [11], Lemma 2.3). This is not necessarily true for CBI-process. In fact, the following $\psi(t, \lambda)$ and $\varphi(t, \lambda)$ correspond to a conservative CBI-process such that $\mathbf{P}_x[X_t = 0] = 0$ for all t, $x > 0$:

$$\psi(t, \lambda) = \begin{cases} \lambda, & t = 0, \\ 0, & t > 0, \end{cases}$$
$$\varphi(t, \lambda) = \begin{cases} 1, & t = 0, \\ e^{-\alpha}, & t > 0. \end{cases}$$

We remark also that the above Lamperti's result is not necessarily true for a non-conservative CB-process as is seen in the following example of a Ψ-semigroup:

$$\psi(t, \lambda) = \begin{cases} \lambda, & t = 0, \\ 1, & t > 0. \end{cases}$$

§ 2. Limit theorems

Let $f(s) = \sum_{k=0}^{\infty} p_k s^k$ and $h(s) = \sum_{k=0}^{\infty} q_k s^k$ be probability generating functions. By a *Galton — Watson branching process corresponding to* $f(s)$ we mean a time-discrete Markov chain Z_n on $\{0, 1, 2, \ldots\}$ with (one-step) transition matrix P_{ij} given by

$$\sum_{j=0}^{\infty} P_{ij}s^j = [f(s)]^i, \quad i = 0, 1, 2, \dots . \tag{2.1}$$

By a *Galton — Watson branching process with immigration corresponding to* $[f(s), h(s)]$, we mean a time-discrete Markov chain Z_n on $\{0, 1, 2, \dots\}$ with transition matrix P_{ij} given by

$$\sum_{j=0}^{\infty} P_{ij}s^j = [f(s)]^i h(s), i = 0, 1, 2, \dots . \tag{2.2}$$

The n-step transition matrix $P_{ij}^{(n)}$ is given, in the case of Galton — Watson process, by

$$\sum_{j=0}^{\infty} P_{ij}^{(n)} s^j = [f_n(s)]^i, i = 0, 1, 2, \dots \tag{2.3}$$

and, in the case of Galton — Watson process with immigration, by

$$\sum_{j=0}^{\infty} P_{ij}^{(n)} s^j = f_n(s)^i \prod_{k=0}^{n-1} h(f_k(s)), \tag{2.4}$$

where $f_n(s)$ is the n-th functional iteration of $f(s) = f_1(s)$.

Let $\{Z_n^{(m)}\}_{m=1,2,\dots}$ be a sequence of Galton—Watson branching processes with immigration corresponding to $[f^{(m)}(s), h^{(m)}(s)]_{m=1,2\dots}$ We suppose that

$$\lim_{m\to\infty} f_{[mt]}^{(m)}(e^{-\frac{\lambda}{b_m}})^{c_m} = \varphi_t^{(1)}(\lambda), \tag{2.5}$$

$$\lim_{m\to\infty} \prod_{k=0}^{[mt]-1} h^{(m)}(f_k^{(m)}(e^{-\frac{\lambda}{b_m}})) = \varphi_t^{(2)}(\lambda) \tag{2.6}$$

exist for $t > 0$ and $\lambda \geqslant 0$, the convergence being compact uniform in $\lambda \in \in [0, \infty)$ for each fixed $t \geqslant 0$, where b_m and c_m are sequences of positive integers such that $b_m \uparrow \infty$ and $c_m \uparrow \infty$ when $m \uparrow \infty$. These assumptions imply that the finite dimensional joint distributions of $\{Z_{[mt]}^{(m)}/b_m\}$, given the condition $Z_0^{(m)} = c_m$, converge to those of a certain stochastic process X_t on $[0, \infty)$ with $\mathrm{E}(e^{-\lambda x_t}) = \varphi_t^{(1)}(\lambda)\ \varphi_t^{(2)}(\lambda)$.

Theorem 2.1. *Suppose that* (2.5) *and* (2.6) *hold. Suppose further that* $\varphi_t^{(1)} < 1$ *for some* $t > 0$ *and* $\lambda > 0$. *Then the limit process is a stochastically continuous and conservative CBI-process.*

P r o o f. Under these assumptions, Lamperti [11] proved that $\lim_{m\to\infty} \frac{c_m}{b_m} = c > 0$ exists and if we write $\varphi_t^{(1)}(\lambda) = e^{-c\psi_t(\lambda)}$ then

$$\{f_{[mt]}^{(m)}(e^{-\lambda/b_m})\}^{[xb_m]} \to e^{-x\psi_t(\lambda)}.$$

Also he showed that $\psi_t(\lambda)$ is a Ψ-semigroup and continuous in t. We shall show that $\varphi_{t+s}^{(2)}(\lambda) = \varphi_t^{(2)}(\lambda)\,\varphi_s^{(2)}(\psi_t(\lambda))$. In fact,

$$\varphi_{t+s}^{(2)}(\lambda) = \lim_{m\to\infty} \prod_{k=0}^{[mt]-1} h^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) \prod_{k=0}^{[ms]-1} h^{(m)}(f_{[mt]}^{(m)}(e^{-\lambda/b_m})) =$$
$$= \varphi_t^{(2)}(\lambda)\,\varphi_s^{(2)}(\psi_t(\lambda))$$

since $f_{[mt]}^{(m)}(e^{-\lambda/b_m}) = \exp\left(-\frac{1}{b_m}[\psi_t(\lambda)+o(1)]\right)$, and the convergence in (2.6) is compact uniform in $\lambda \geqslant 0$. Thus X_t is a CBI-process such that

$$\mathbf{E}_x(e^{-\lambda x_t}) = \varphi^{(2)}(t,\lambda)\, e^{-x\omega_t(\lambda)}.$$

$\varphi^{(2)}(t, \lambda)$ is continuous in t. In fact, since $\varphi^{(2)}(t, \lambda)$ is decreasing, it is continuous in t except for at most countable points. Hence $\lim_{t\downarrow 0} \varphi^{(2)}(t, \lambda) = 1$ on a dense λ-set in $(0, \infty)$ * and this implies that $\lim_{t\downarrow 0} \varphi^{(2)}(t, \lambda) = 1$ for all $\lambda \geqslant 0$. Therefore $\varphi^{(2)}(t, \lambda)$ is continuous in t. Thus X_t is stochastically continuous. Q.E.D.

Remark 2.1. If we suppose only (2.5) and (2.6), the limit process is not necessarily stochastically continuous even if we assume $\mathbf{P}[X_t = 0] < 1$ $(t > 0)$. For example, if $f^{(m)}(s) \equiv f(s) = \frac{1}{2} + \frac{s}{2}$, $h^{(m)}(s) = s^m$, $b_m = m$ and $c_m = [mx]$, (2.5) and (2.6) hold with

$$\varphi_t^{(1)}(\lambda) = \begin{cases} e^{-\lambda x}, & t = 0, \\ 1, & t > 0, \end{cases}$$

and

$$\varphi_t^{(2)}(\lambda) = \begin{cases} 1, & t = 0, \\ e^{-2\lambda}, & t > 0, \end{cases}$$

i.e., the limit process is a degenerate CBI-process of Remark 1.1.

Now we will show the converse of Theorem 2.1. For this, we use the following result of Lamperti [11].

Lemma 2.1. *Any infinitely divisible distribution on* $[0, \infty)$ *is the weak limit of a sequence of distributions of the form*

$$G^{(d)}(x) = F^{(d)}(dx)^{*d}, \quad d = 1, 2, \ldots, \tag{2.7}$$

where $F^{(d)}$ *is a distribution concentrated on non-negative integers and exponent means convolution. Let* $\exp(-\psi(\lambda))$ *be the Laplace transform of the infinitely divisible distribution and* $f^{(d)}(s)$ *be the generating function of the distribution* $F^{(d)}$. *Then*

$$\lim_{d\to\infty} f_k^{(d)}(e^{-\lambda/d})^d = e^{-\psi_k(\lambda)}, \quad \lambda \geqslant 0, \tag{2.8}$$

where $f_k^{(d)}$ *and* ψ_k *are k-th functional iterations of* $f^{(d)}$ *and* ψ *respectively. The convergence in* (2.8) *is compact uniform in* $\lambda \geqslant 0$.

* We use here a similar argument as in the proof of Lemma 1.2.

Theorem 2. 2. *Let $X = (X_t, \mathbf{P}_x)$ be a stochastically continuous and conservative CBI-process such that $\mathbf{E}_x(e^{-\lambda x_t}) = \varphi(t, \lambda)\, e^{-x\psi(t,\lambda)}$. Then, there exists a sequence of Galton — Watson branching processes with immigration $\{Z_n^{(m)}\}$ corresponding to $[f^{(m)}(s), h^{(m)}(s)]$ and a sequence b_m of positive integers such that*

$$\lim_{m\to\infty} f_{[mt]}(e^{-\lambda/b_m})^{[b_m x]} = \exp(-x\psi(t,\lambda)),\ \lambda \geqslant 0, t \geqslant 0, x \geqslant 0, \quad (2.9)$$

$$\lim_{m\to\infty} \prod_{k=0}^{[mt]-1} h^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) = \varphi(t,\lambda),\ \lambda \geqslant 0,\ t \geqslant 0. \quad (2.10)$$

Thus, the finite dimensional distributions of $\{Z_{mt}^{(m)}/b_m\}$ given the initial condition $Z_0^{(m)} = [b_m x]$ converge to those of $(X_t, \mathbf{P}_x)$.

P r o o f. Set $\alpha_m(\lambda) = \min\{\psi(\frac{k}{m}, \lambda);\ 1 \leqslant k \leqslant m^2\}$. Let $P^{(m)}(dx)$ be the probability distribution on $[0, \infty)$ defined by $\int_0^\infty e^{-\lambda x} P^{(m)}(dx) = \exp(-\psi(\frac{1}{m}, \lambda))$. $P^{(m)}$ is infinitely divisible and hence, by Lemma 2.1, for given constants $a > 0$, $0 < \lambda_1 < \lambda_2$ and $m > 0$, we can choose $d_0 = d(m)$ such that for every $d \geqslant d_0$ and $\lambda \in [\lambda_1, \lambda_2]$, we have

$$-\psi\left(\frac{k}{m}, \lambda\right) - a\,\alpha_m(\lambda)\, e^{-m} \leqslant d \log \bar{f}_k^{(d)}(e^{-\lambda/d}) \leqslant$$
$$\leqslant -\psi\left(\frac{k}{m}, \lambda\right) + a\,\alpha_m(\lambda)\, e^{-m},\ k = 1, 2, \ldots, m^2 \quad (2.11)$$

for some probability generating functions $\bar{f}^{(d)}(s)$. Take $b_m \geqslant d_0$ such that $\frac{b_m}{m} \uparrow \infty$ when $m \uparrow \infty$. We write $f^{(m)} \equiv \bar{f}^{(b_m)}(s)$. Hence,

$$f_k^{(m)}(e^{-\lambda/b_m}) = \exp(-(\psi(k/m, \lambda) + \alpha_m(\lambda)\, o(e^{-m}))/b_m) \quad (2.12)$$

for $\lambda \in [\lambda_0, \lambda_1]$. Thus (2.9) holds for $\lambda \in [\lambda_0, \lambda_1]$ and since $f_{[mt]}(e^{-\lambda/b_m})^{[b_m x]}$ and $\exp(-x\psi(t, \lambda))$ are Laplace transforms of probability measures on $[0, \infty)$, the convergence in (2.9) is valid for every $\lambda \geqslant 0$.

Next, set

$$H^{(m)}(s) = \mathbf{E}_0(s^{b_m X_{1/m}}) = \varphi\left(\frac{1}{m}, -b_m \log s\right),\ 0 \leqslant s \leqslant 1. \quad (2.13)$$

We will show that

$$\lim_{m\to\infty} \prod_{k=0}^{[mt]-1} H^{(m)}(f_m^{(m)}(e^{-\frac{\lambda}{b_m}})) = \varphi(t, \lambda),\ \lambda \geqslant 0, t \geqslant 0. \quad (2.14)$$

In fact,

$$\varphi(t, \lambda) = \lim_{m\to\infty} \varphi\left(\frac{[mt]}{m}, \lambda\right) = \lim_{m\to\infty} \prod_{k=0}^{[mt]-1} \varphi\left(\frac{1}{m}, \psi\left(-\frac{k}{m}, \lambda\right)\right) \quad (2.15)$$

and hence by (2.12),

$$\Big|\prod_{k=0}^{[mt]-1} H^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) - \varphi(t,\lambda)\Big| \leqslant \sum_{k=0}^{[mt]-1} |H^m(f_k^{(m)}(e^{-\lambda/b_m})) - \varphi\left(\tfrac{1}{m}, \psi_{\frac{k}{m}}(\lambda)\right)| \leqslant$$

$$\leqslant mt \max_{0\leqslant k\leqslant [mt]-1} \Big| \mathbf{E}_0\left(\exp\left\{-\left[\psi\left(\frac{k}{m},\lambda\right) + \alpha_m(\lambda)\, o(e^{-m})\right] X_{\frac{1}{m}}\right\}\right) -$$

$$- \mathbf{E}_0\left(\exp\left\{-\psi\left(\frac{k}{m},\lambda\right) X_{\frac{1}{m}}\right\}\right) \leqslant$$

$$\leqslant mtO(e^{-m})\, \mathbf{E}_0\,[\exp\{-\alpha_m(\lambda)(1-O(e^{-m}))X_{\frac{1}{m}}\}\, \alpha_m(\lambda)\, X_{\frac{1}{m}}] \leqslant$$

$$\leqslant mtO(e^{-m}) \to 0, \quad \lambda \in [\lambda_0, \lambda_1],$$

and by the same argument as above the convergence is valid every $\lambda \geqslant 0$.

Finally, we set

$$h^{(m)}(s) = \mathbf{E}_0\left(s^{[b_m X_{\frac{1}{m}}]}\right), \quad m = 1, 2, \ldots . \tag{2.16}$$

Then $h^{(m)}(s)$ is a probability generating function. Now, if $\lambda \in [\lambda_0, \lambda_1]$,

$$\Big|\prod_{k=0}^{[mt]-1} h^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) - \varphi(t,\lambda)\Big| \leqslant$$

$$\leqslant \Big|\prod_{k=0}^{[mt]-1} h^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) - \prod_{k=0}^{[mt]-1} H^{(m)}(f_k^{(m)}(e^{-\lambda/b_m}))\Big| +$$

$$+ \Big|\prod_{k=0}^{[mt]-1} H^{(m)}(f_k^{(m)}(e^{-\lambda/b_m})) - \varphi(t,\lambda)\Big| \leqslant$$

$$\leqslant mt \max_{0\leqslant k\leqslant [mt]-1} (1 - f_k^{(m)}(e^{-\lambda/b_m})) + mtO(e^{-m}) =$$

$$= mt \max_{0\leqslant k\leqslant [mt]-1} \left(1 - \exp\left\{-\left(\psi\left(\frac{k}{m},\lambda\right) + \alpha_m(\lambda)\, O(e^{-m})\right) \Big/ b_m\right\}\right) + mtO(e^{-m}) \to 0$$

as $m \to \infty$, since $m/b_m \to 0$. Thus (2.10) is proved. Q. E. D.

Now we shall discuss the case when $Z_n^{(m)}$ is independent of m. Let Z_n be a Galton — Watson branching process with immigration corresponding to $[f(s), h(s)]$ and consider a sequence of stochastic processes $X_t^{(n)} = Z_{[nt]}/b_n$, $n = 1, 2, \ldots$. Suppose $X_t^{(n)}$, given the initial condition $X_0^{(n)} = c_n/b_n$*, converges to a stochastic process X_t on $[0, \infty)$. We suppose further that

$$\lim_{n\to\infty} f_{[nt]}(e^{-\lambda/b_n})^{b_n} = \varphi_t^{(1)}(\lambda), \tag{2.17}$$

$$\lim_{n\to\infty} \prod_{k=0}^{[nt]-1} h(f_k(e^{-\lambda/b_n})) = \varphi_t^{(2)}(\lambda) \tag{2.18}$$

exist such that $\varphi_t^{(1)}(\lambda) < 1$ for some t, $\lambda > 0$, the convergence being compact uniform in $\lambda \in [0, \infty)$. Then the limit process X_t is a stochastically conti-

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* b_n and c_n are sequences of positive integers such that $b_n \uparrow \infty$ and $c_n \uparrow \infty$.

nuous and conservative CBI-process such that $\mathbf{E}_x[e^{-\lambda x_t}] = \varphi_t^{(2)}(\lambda) \times \times \exp(x \log \varphi_t^{(1)}(\lambda))$.

Theorem 2. 3. *Suppose* (2.17) *and* (2.18) *hold. Then the limit CBI-process* X_t *is given by*

$$\mathbf{E}_x[e^{-\lambda x_t}] = (1 + \alpha c \lambda^\alpha t)^{-d/\alpha c} \exp\{-\lambda x/(1 + \alpha c \lambda^\alpha t)^{1/\alpha}\}$$

for some $c \geqslant 0$, $d \geqslant 0$ *and* $0 < \alpha \leqslant 1$, *i. e.* X_t *is a stochastically continuous CBI-process corresponding to* $R(\lambda) = -c^{\alpha+1}$ *and* $F(\lambda) = d\lambda^\alpha$ (cf. Example 1.1). *The convergense occurs if and only if*

$$f(s) = s + c(1-s)^{\alpha+1} L\left(\frac{1}{1-s}\right), \tag{2.19}$$

$$h(s) = 1 - d(1-s)^\alpha L^*\left(\frac{1}{1-s)}\right) \tag{2.20}$$

where $L(\xi)$ *and* $L^*(\xi)$ *are slowly varying functions at* ∞ *such that* $L(\xi) \sim \sim L^*(\xi)$ $(\xi \uparrow \infty)$ *and*

$$b_n \sim [nL(b_n)]^{1/\alpha} (\sim [nL^*(b_n)]^{1/\alpha}). \tag{2.21}$$

P r o o f. We shall omit the proof of the first part *. We shall prove here only that (2.19) and (2.20) imply

$$\lim_{n\to\infty} f_{[nt]}(e^{-\lambda/b_n})^{[b_n x]} = \exp\left\{-\frac{\lambda x}{(1+\alpha c\lambda^\alpha t)^{1/\alpha}}\right\}, \quad x \geqslant 0,\ t \geqslant 0, \tag{2.22}$$

and

$$\lim_{n\to\infty} \prod_{k=0}^{[nt]-1} h(f_k(e^{-\lambda/b_n})) = (1 + \alpha c \lambda^\alpha t)^{-d/\alpha c}, \quad t \geqslant 0, \tag{2.23}$$

when b_n is taken to satisfy (2.21).

By (2.21), it is easy to see that, for each $\lambda > 0$.

$$nc(1 - e^{-\lambda/b_n})^\alpha L(1 - e^{-\lambda/b_n})^{-1} \sim c\lambda^\alpha \quad (n \to \infty) \tag{2.24}$$

and

$$nd(1 - e^{-\lambda/b_n})^\alpha L'(1 - e^{-\lambda/b_n})^{-1} \sim d\lambda^\alpha \quad (n \to \infty). \tag{2.25}$$

Now we shall prove that for every $\lambda > 0$

$$\sup_i \left| f_{[nt]}(e^{-\lambda/b_n})^i \prod_{k=0}^{[nt]-1} h(f_k(e^{-\lambda/b_n})) - \right.$$
$$\left. -(1 + \alpha c\lambda^\alpha t)^{-d/\alpha c} \exp\left\{-\frac{\lambda}{(1+\alpha c\lambda^\alpha t)^{1/\alpha}} \frac{i}{b_n}\right\}\right| \to 0 \quad (n \to \infty). \tag{2.26}$$

(2.26) imples (2.22) by taking $d = 0$, and it implies (2.23) by taking $i = 0$. Let X_t be the CBI-process given in the theorem and T_t be its semigroup on $C_\infty[0, \infty)$. In § 1, we have proved that if A is the infinitesimal generator of T_t with domain $D(A)$, then, for $\lambda > 0$, $f_\lambda \in D(A)$ where $f_\lambda(x) = e^{-\lambda x}$, and $Af_\lambda(x) = e^{-\lambda x}(c\lambda^{\alpha+1}x - d\lambda^\alpha)$. Furthermore, the linear hull $\mathcal{L}(\Lambda_0)$ of $\Lambda_0 = \{f_\lambda;\ \lambda > 0\} \subset C_\infty[0, \infty)$ is a core of $D(A)$. Hence by Trotter's theorem,

* Cf. Lamperti [9] in the case of CB-processes.

(cf. [16], [7]) we have (2.26) if we can prove that for each $\lambda > 0$,

$$\sup_i \left| n\{[f(e^{-\frac{\lambda}{b_n}})]^i h(e^{-\frac{\lambda}{b_n}}) - e^{-\frac{i}{b_n}\lambda}\} - e^{-\frac{i}{b_n}\lambda}\left[c\lambda^{\alpha+1}\frac{i}{b_n} - d\lambda^\alpha\right]\right| \to 0$$

when $n \to \infty$. (2.27)

Now we shall prove (2.27). Let $\lambda > 0$ be fixed. By (2.24) and (2.25), it is easy to see that

$$-\log f(e^{-\frac{\lambda}{b_n}}) = \frac{\lambda}{b_n} - \frac{1}{nb_n}c\lambda^{\alpha+1}(1+o(1)) \quad (n \to \infty)$$

and

$$-\log h(e^{-\frac{\lambda}{b_n}}) = \frac{1}{n}d\lambda^\alpha(1+o(1)) \quad (n \to \infty).$$

Hence

$$\left| n\{[f(e^{-\frac{\lambda}{b_n}})]^i h(e^{-\frac{\lambda}{b_n}}) - e^{-\frac{i}{b_n}\lambda}\} - e^{-\frac{i}{b_n}\lambda}\left[c\lambda^{\alpha+1}\frac{i}{b_n} - d\lambda^\alpha\right]\right| =$$

$$= \left| e^{-\frac{i}{b_n}\lambda} n\left\{\exp\left[\frac{1}{n}\frac{i}{b_n}c\lambda^{\alpha+1}(1+o(1)) - \frac{1}{n}d\lambda^\alpha(1+o(1))\right] - 1\right\} - \right.$$
$$\left. - c\lambda^{\alpha+1}\frac{i}{b_n} + d\lambda^\alpha \right| \leqslant$$

$$\leqslant e^{-\frac{i}{b_n}\lambda}\left|\frac{i}{b_n}o(1) + o(1) + \left[O\left(\frac{1}{n}\left(\frac{i}{b_n}\right)^2\right) + O\left(\frac{1}{n}\right)\right]\exp\left\{O\left(\frac{1}{n}\frac{i}{b_n}\right)\right\}\right|.$$

Thus we have (2.27). Q. E. D.

Example 2.1. (Conditioned branching processes). Let Z_n be the Galton — Watson process corresponding to $f(s)$. Assume $f'(1) = 1$. By *conditioned process* $\hat{Z}_n$ we mean a Markov chain on $\{1, 2, \dots\}$ with (one-step) transition matrix $\hat{p}_{ij}$ given by

$$\hat{p}_{ji} = \frac{j}{i}p_{ij}, \quad i, j = 1, 2 \dots,$$

where p_{ij} is the transition matrix of Z_n. Roughly speaking, $\hat{Z}_n$ is the process Z_n conditioned never to die out (cf. [12]). Now consider a process Z_n^* on $\{0, 1, 2, \dots\}$ defined by

$$Z_n^* = \hat{Z}_n - 1.$$

Z_n^* is a Galton — Watson branching process with immigration corresponding to $[f(s), f'(s)]$; indeed, for the transition matrix p_{ij}^* of Z_n^*, we have

$$\sum_{j=0}^\infty p_{ij}^* s^j = \sum_{j=0}^\infty \hat{p}_{i+1,j+1}s^j = \sum_{j=0}^\infty \frac{j+1}{i+1}p_{i+1,j+1}s^j = \frac{1}{i+1}\left(\sum_{j=0}^\infty p_{i+1,j}s^j\right)' =$$
$$= \frac{1}{i+1}(f(s)^{i+1})' = f(s)^i f'(s).$$

Suppose, for some $0 < \alpha \leqslant 1$ and slowly varying functions L and L^*,

$$f(s) = s + c(1-s)^{\alpha+1}L\left(\frac{1}{1-s}\right)$$

and

$$f'(s) = 1 - d(1-s)^{\alpha} L^*\left(\frac{1}{1-s}\right).$$

Then we have automatically that $d = c(\alpha + 1)$ and $L(\xi) \sim L^*(\xi)$ ($\xi \uparrow \infty$) (cf. [2], Theorem 1, p. 273). Thus, by Theorem 2.3, if we take b_n such that (2.21) holds, the process $\hat{Z}_{[nt]}/b_n$ with $Z_0 = b_n x$ converges to a stochastically continuous CBI-process X_t corresponding to $R(\lambda) = -c\lambda^{\alpha+1}$and $F(\lambda) = c(\alpha + 1)\lambda^{\alpha}$, i. e.

$$\mathbf{E}_x(e^{-\lambda x_t}) = (1 + \alpha c\lambda^{\alpha} t)^{-\frac{\alpha+1}{\alpha}} \exp\left\{-\frac{\lambda x}{(1+\alpha c\lambda^{\alpha} t)}\right\}.$$

In the case when $f(s) = s + \frac{(1-s)^2}{2}(f''(1) + o(1))$ and $f'(s) = 1 - (1-s)(f''(1) + o(1))$, the limit process is the diffusion process X_t on $(0, \infty)$ with the infinitesimal generator $cx\, d^2/dx^2 + 2c d/dx$, where $c = \frac{f''(1)}{2}$. This result was obtained by Lamperti and Ney [2].

Example 2.2. (Local time of Brownian motion). Let $(X_t, \mathbf{P}_x)$ be a one-dimensional Brownian motion. It is known (cf. [5]) that there exists a system of random variables $\{\varphi(t, a) : t \geqslant 0,\ a \in (-\infty, \infty)\}$ such that for almost all ω ($\forall \mathbf{P}_x$),

(i) the mapping $(t, a) \rightsquigarrow \varphi(t, a) \in [0, \infty)$ is jointly continuous,

(ii) $2\int_E \varphi(t, a)\, da = \int_0^t I_E(X_s)\, ds$ for every Borel set E.

The simplest way to prove this fact is to show that

$$2\tilde{\varphi}(t, a) = |X_t - a| - |a| - \int_0^t z_a(X_s)\, dX_s, \quad Z_a(x) = \begin{cases} 1, & x \geqslant a, \\ -1, & x < a, \end{cases}$$

has a version $2\varphi(t, a)$ jointly continuous in (t, a) and satisfies the condition (ii) above. $\varphi(t, a)$ is called the *sojourn time density* or the *local time* of the Brownian motion X_t.

In the following, we consider the Brownian motion starting at 0, i. e. the process $\{X_t, \mathbf{P}_0\}$. Let $\Delta > 0$ and $T > 0$. By a *Δ-downcrossing at $x \in (-\infty, \infty)$ before time T*, we mean an interval $[u, v] \subset [0, T)$ such that $X(u) = x + \Delta$, $X(v) = x$ and $x < X(t) < x + \Delta$ for $t \in (u, v)$. Let η be the number of Δ-downcrossings at 0 before $\sigma_{-\Delta}$ where $\sigma_{-\Delta} = \inf\{t; x_t = -\Delta\}$. It is easy to see that

$$\mathbf{E}_0(s^{\eta}) = \frac{1}{2} + \frac{s}{2^2} + \frac{s^2}{2^3} + \ldots = \frac{1}{2-s}.$$

Let $x \geqslant 0$ and define x_i, $i = 0, 1, 2, \ldots$, by $x_i = x + \Delta i$. Let Z_1^{Δ} be the number of Δ-downcrossings at x_i before $\varphi^{-1}(u, x)$, where $\varphi^{-1}(u, x) = \inf\{t;\ \varphi(t, x) = u\}$ *. By a simple consideration, it is easy to see that Z_{i+1}^{Δ} is the sum of Z_i^{Δ} independent copies of η. Hence Z_n^{Δ} is a Galton—Watson

* $u > 0$ is fixed.

branching process corresponding to $f(s) = \frac{1}{2-s}$ with a random initial state ($= Z_0$).

In a similar way, we can define a *Δ-upcrossing at x before time T*: it is an interval $[u, v] \subset [0, T)$ such that $X(u) = x - \Delta$, $X(v) = x$ and $x - \Delta < X(t) < x$ for $t \in (u, v)$. Let $\tilde{\eta}$ be the number of Δ-upcrossings at 0 before σ_Δ. It is easy to see that

$$\mathbf{E}_0(s^{\tilde{\eta}}) = \frac{1}{2-s}.$$

Now let $x > 0$ and define $\tilde{x}_i$, $i = 0, 1, 2, \ldots$ by $\tilde{x}_i = x - \Delta i$. Let $\tilde{Z}_i^\Delta$ be the number of Δ-upcrossings at $\tilde{x}_i$ before $\varphi^{-1}(u, x)$. By a simple consideration, it is easy to see that, if $\tilde{x}_{i+1} \geqslant 0$, then $\tilde{Z}_{i+1}^\Delta - 1$ is the sum of $\tilde{Z}_i^\Delta$ independent copies of $\tilde{\eta}$. Hence $\{\tilde{Z}_n^\Delta;\ n=0, 1, 2, \ldots, [x/\Delta]\}$ is a Galton—Watson branching process with immigration corresponding to $[f(s) = \frac{1}{2-s}$, $h(s) = s]$. It is proved by Ito — McKean ([5], p. 48) that if $Z^\Delta(t)$ is the number of Δ-upcrossings or Δ-downcrossings at x before time t, then

$$\mathbf{P}_0[\lim_{\Delta \downarrow 0} \Delta Z^\Delta(t) = \varphi(t, x) \text{ for all } t \geqslant 0] = 1.$$

Hence, if $x \geqslant 0$,

$$\lim_{n\to\infty} \frac{1}{n} Z^{\frac{1}{n}}_{[nt]} = \varphi(\varphi^{-1}(u, x), x + t) \tag{2.28}$$

and if $x > 0$ and $0 \leqslant x - t$,

$$\lim_{n\to\infty} \frac{1}{n} \tilde{Z}^{\frac{1}{n}}_{[nt]} = \varphi(\varphi^{-1}(u, x), x - t). \tag{2.29}$$

By Theorem 2.3, the limit in (2.28) is a CB-process corresponding to $R(\lambda) = -\lambda^2$, (i. e. diffusion process with generator $x\frac{d^2}{dx^2}$) and the limit process in (2.29) is a CBI-process corresponding to $R(\lambda) = -\lambda^2$ and $F(\lambda) = \lambda$, (i. e. diffusion process with generator $x\frac{d^2}{dx^2} + \frac{d}{dx}$). Thus we arrive at the following results due to Knight [8], Ray [14] and Silberstein [15]: «If we define $\xi^u(t) = \varphi(\varphi^{-1}(u, x), x + t)$ $(x \geqslant 0,\ t \geqslant 0)$ then $[\xi^u(t), \mathbf{P}_0]$ is a diffusion process with infinitesimal generator $x\frac{d^2}{dx^2}$ such that $\xi^u(0) = u$. If we define $\eta^u(t) = \varphi(\varphi^{-1}(u, x), x - t)$, $x > 0$, $0 \leqslant t \leqslant x - u$, then $[\eta^u(t), \mathbf{P}_0]$ is a diffusion process with infinitesimal generator $x\frac{d^2}{dx^2} + \frac{d}{dx}$ such that $\eta^u(0) = u$». In particular,

$$\mathbf{E}_0(e^{-\lambda\xi^u(t)}) = \exp\left[-\frac{\lambda u}{1+\lambda t}\right]$$

and

$$\mathbf{E}_0(e^{-\lambda\eta^u(t)}) = \exp\left[-\frac{\lambda u}{1+\lambda t}\right]\frac{1}{1+\lambda t}.$$

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2.9.69

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ВЕТВЯЩИЕСЯ ПРОЦЕССЫ С ИММИГРАЦИЕЙ И СВЯЗАННЫЕ С НИМИ ПРЕДЕЛЬНЫЕ ТЕОРЕМЫ

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(Резюме)

В работе вводятся ветвящиеся процессы с иммиграцией с непрерывным множеством состояний и показывается, что они и только они могут быть предельными для обычных ветвящихся процессов с иммиграцией.
