SVI Model Free Wings

Tahar Ferhati

SVI Model Free Wings

Tahar FERHATI
tahar.ferhati@gmail.com
March 15, 2020

ABSTRACT

In this paper we study the Stochastic Volatility Inspired model (SVI). Until recently, it was not possible to find sufficient conditions that would guarantee the absence of static arbitrage in this SVI model. Recently, we proposed a new numerical method based on Sequential Quadratic Programming (SQP) algorithm to resolve this problem. The main contribution in this paper is that we provide sufficient conditions that guarantee an SVI static arbitrage-free. These conditions ensure that the probability density function will remain positive.Finally, we present several computational synthetic examples with static arbitrage and we show how to fix them.Next, we use real market data coming from 23 indexes to calibrate the SVI model. The calibration method is robust and easy to implement,it guarantees calibration arbitrage free (calendar spread and butterfly arbitrage).

Keywords: Implied/Local Volatility, SVI, Arbitrage-Free,butterfly spread, calendar spread, Calibration, Quadratic Programming.

Contents

Introduction

The implied volatility is an important element used for pricing and hedging in the financial market. It can be obtained by inverting Black-Scholes formula (1973) for a given strike and maturity. These values are explicitly available only for some strikes and maturities. For this reason we need a model that allow to obtain an accurate values close to those observed in the market.

Many interpolation techniques has been proposed in the last decades, we can class them in two categories: mathematical interpolation models and financial interpolation models. The appropriate model should respect some static arbitrage free condition.

Many authors in the past such as Dupire (1994) [1], Derman and Kani (1994) [2], and Rubinstein (1994) tried to model local volatility, we propose here an overview of the most recent methods used to resolve this problem.

Kahalé (2004) [3] presented an interpolation method for implied volatilities in the equity and forex markets using one-dimensional interpolation algorithm with smoothness properties. Kahalé assumes arbitrage-free in the input market volatilities and computes an interpolating surface for all strikes and maturities in three-step procedure: in the first step he interpolates the (call) price for each maturity using piecewise convex polynomials. Hence, the call price function obtained is arbitrage free and after he calculates the implied volatility by inverting the BS formula, in the second step he interpolates linearly the total implied variance. Finally, he makes some adjustments to the call prices to ensure that IVS is globally arbitrage-free.

Jim Gatheral (2004) presented for the first time the Stochastic Volatility Inspired model (SVI) in the Global Derivatives and Risk Management conference in Madrid.Benko et al.(2007) [4] applied non-parametric smoothing methods to estimate the implied volatility (IV). They combine the IV smoothing with the state-price density (SPD) estimation in order to correct the arbitrage condition reflected by the non-positive SPD, for this, they used the local polynomial smoothing technique.

Fengler (2009) [5] proposed an approach for smoothing the implied volatility smile and provides a methodology for arbitrage free interpolation. His methodology consists in using the natural smoothing splines under suitable shape constraints. This method works even when the input data is not arbitrage free.

Andreasen-Huge (2010) [6] presented an interpolation and extrapolation method of European option prices based on a one step implicit finite difference Euler scheme applied to a local volatility parametrization. Glaser and Heider (2012)[7] use locally constrained least squares approximations to construct the arbitrage-free call price surfaces. They calculate derivatives of the call surface to obtain implied volatility, local volatility and transition probability density.Fingler-Hin (2013) [8] use semi-nonparametric estimator for the entire call price surface based on a tensor-product B-spline. Gatheral & Jacquier (2014) [9] presents the stochastic volatility inspired model (SVI) a parametric model of the implied volatility smile. The model fits very well the equity market, however, no condition is known that could guarantee the arbitrage free of the model. This is the main challenge in our project.

Several problems arise in the past regarding the SVI calibration.In this paper, we study deeply the framework the SVI model and calibration of the model.The main result that we provide is a sufficient conditions that guarantee SVI calibration arbitrage free.

The rest of this paper is organized as follows: in the first part we present the SVI model formulation and the different forms such as the natural SVI, Jump-Wings SVI. We also define the characterization of static arbitrage: calendar spread and butterfly.Next, we study the volatility in the wings where We provide for the first time an analytical sufficient conditions that guarantee an SVI model arbitrage-free.In the second part, we present the SVI parameters boundaries and the initial guess. Finally, we illustrate the performance of our calibration method first using two synthetic examples with aribtrage and after that using real market data coming from 23 indexes.

Chapter 1

Stochastic Volatility Inspired SVI

In this section, we present the general framework of the SVI model as presented by Jim Gatheral and Antoine Jacquier in [9].Next, we describe the characterisation of static arbitrage which includes calendar spread and butterfly arbitrage and finally, we present the main result: a sufficient conditions that ensure a positive probability density function using SVI's parameters.

1.1 History of SVI

The stochastic volatility inspired model (SVI) was used for the first time at Merrill Lynch in 1999 by Jim Gatheral, and presented in 2004 at the annual conference of *Global Derivatives* in Madrid. As traders and practitioners require intuitive interpretation of the SVI parameters, Gatheral and Jacquier show in [10] that the SVI model could be the convergence of the total implied variance under Heston model for large enough maturity T, additionally, they provide an intuitive interpretation for SVI parameters.

The success of this model is due to its particular features:

1. For a fixed time to expiry T, the implied Black-Scholes variance $\sigma^2_{BS}(k, T)$ is linear, with respect to the log forward moneyness strike $k := \log(K/F_T)$ as $|k| \to \infty$,where K is the strike and F_T is the forward price of the stock.This linearity is consistent with Roger Lee's moment formula [11].

2. It fits very well listed option prices, and careful choice of parameters allows for an arbitrage free interpolation.

1.2 SVI Model Formulation

The original SVI formulation is the so called the raw SVI presented in 2004. This formulation is very tractable. We will present other formulation such as: the natural SVI parameterization and the SVI Jump-Wings (SVI-JW) parameterization.

1.2.1 The raw SVI parameterization

The raw SVI parameterization is a parametric model with 5 parameters $\chi_R = \{a, b, \rho, m, \sigma\}$, it models **the total implied variance** $w\,(k; \chi_R) := \sigma^2_{imp}(k; \chi_R)T$.

$$w\,(k; \chi_R) = a + b\left\{\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\right\} \tag{1.1}$$

Where $k := \log(\frac{K}{F_T})$ is the log forward moneyness and K is the strike.
For every $k \in \mathbb{R}$, and the set of the models parameters χ_R is given by;

$$
\begin{aligned}
a &\in \mathbb{R} \\
b &\geq 0 \\
|\rho| &< 1 \\
m &\in \mathbb{R} \\
\sigma &> 0 \\
a + b\sigma\sqrt{1-\rho^2} &\geq 0
\end{aligned}
$$

The last condition is calculated using the fact that the minimum of the total implied variance is positive; $w(k;\chi_R) \geq 0$.

This choice of using the log forward moneyness is motivated by scaling reason: dividing the strike by the forward price and taking the logarithm does not change our computation, we get a small numbers on the X axis in the same order with the total variance in the Y axis. We adapt the notation of log forward moneyness strike k.

$$
w\left(k;\chi_R\right) = \sigma^2_{imp}(k;\chi_R)T. = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right\} \tag{1.2}
$$

In practical applications, the model parameters are adopted to each option expiry.For a fixed maturity T, the smile (or slice) is the total variance

$$
w\left(k;\chi_R\right) := \sigma^2_{imp}(k;\chi_R)T
$$

We define also:

- **The total implied variance**: $w(k,T) = T\sigma^2_{BS}(k,T)$
- **The implied variance**: $v(k,T) = \sigma^2_{\mathrm{BS}}(k,T) = w(k,T)/T$
- **The map of the volatility surface:** $(k,T) \longmapsto w(k,T)$
- **The slice function:** for any fixed expiry $T > 0, k \mapsto w(k,T)$

Now we consider the effects and the interpretation of changing the parameters $\chi_R = \{a,b,\rho,m,\sigma\}$ as describes in the figure (1.1).

- a: determines the overall level of variance: an increasing a increases the general level of variance, a vertical translation of the smile.
- b: controls the angle between the left and right asymptotes: increasing b increases the slopes of both the left and right wings, tightening the smile.
- ρ: determines the orientation of the graph: increasing ρ decreases the slope of the left wing, and increases the slope of the right wing, it's a counter-clockwise rotation of the smile.
- m: horizontal translation of the smile: increasing m translates the smile to the right.
- σ: determines curvature of the smile: increasing σ reduces the at-the-money (ATM) curvature of the smile.

The total implied variance $w\left(k;\chi_N\right)$ has the left and right asymptotes that respect the assumption of linear wings, this result is consistent with the Roger Lee's moment formula [11].

$$
\begin{aligned}
W_L(k) &= a + b(\rho-1)(k-m) \quad k \to -\infty \\
W_R(k) &= a + b(\rho+1)(k-m) \quad k \to \infty
\end{aligned} \tag{1.3}
$$

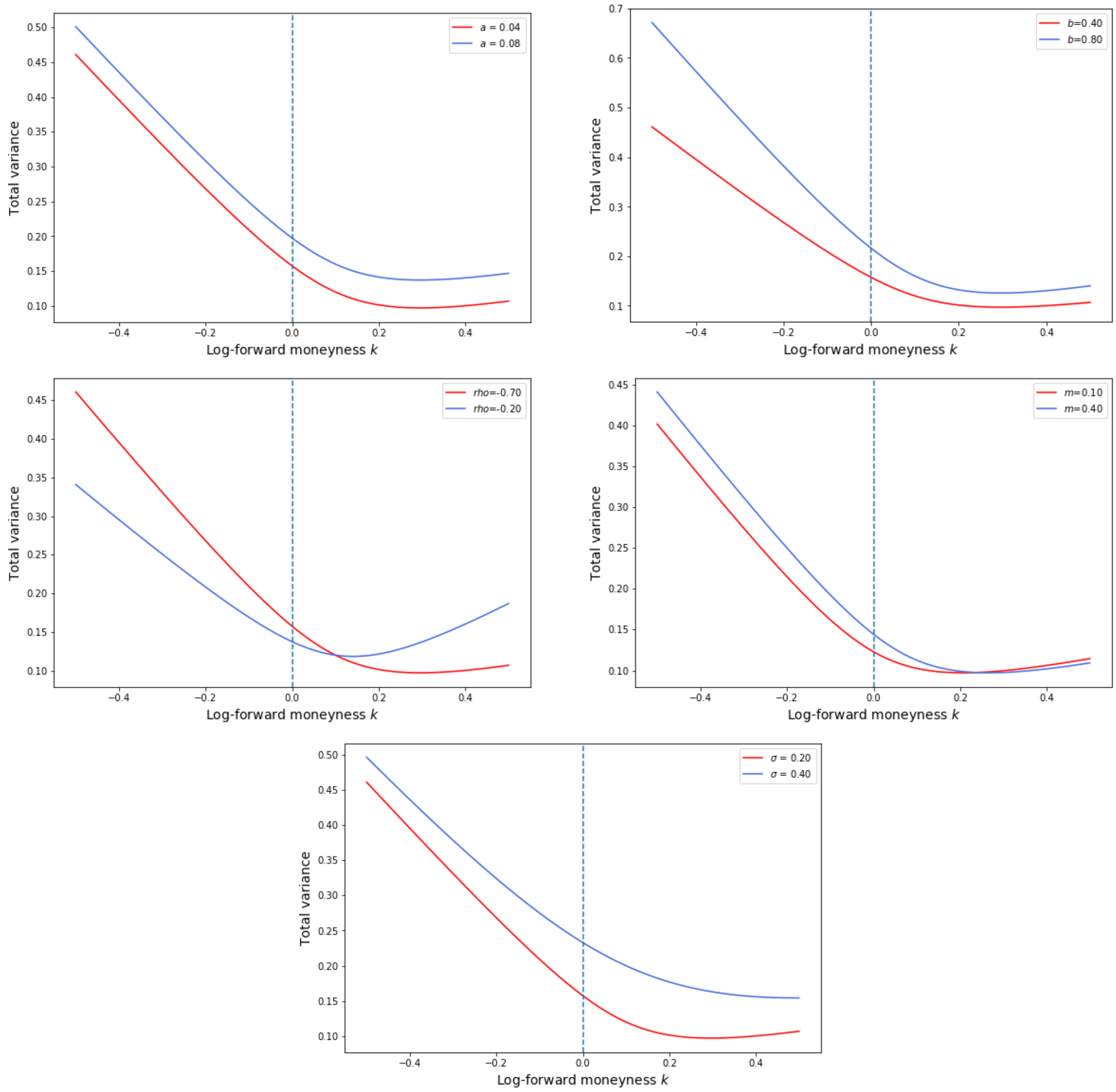

Figure 1.1: The effects of the parameters $\chi_R = \{a, b, \rho, m, \sigma\}$ in SVI model

1.2.2 The natural SVI parameterization

Gatheral and Jacquier propose in [9] another formulation of the natural SVI parameterization, it describes the total implied variance with parameters $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$.

$$w\left(k;\chi_N\right) = \Delta + \frac{\omega}{2}\left\{1 + \zeta\rho(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\} \tag{1.4}$$

with $\omega \geq 0, \Delta \in \mathbb{R}, \mu \in \mathbb{R}, |\rho| < 1$ and $\zeta > 0$

We can show the relationship between the raw and the natural SVI model.

Lemma 1.1 *The parametrisation between the raw and the natural SVI is given by*

$$(a, b, \rho, m, \sigma) = \left(\Delta + \frac{\omega}{2}\left(1-\rho^2\right), \frac{\omega\zeta}{2}, \rho, \mu - \frac{\rho}{\zeta}, \frac{\sqrt{1-\rho^2}}{\zeta}\right) \tag{1.5}$$

and the inverse mapping between the natural and the raw SVI is

$$(\Delta, \mu, \rho, \omega, \zeta) = \left(a - \frac{\omega}{2}\left(1-\rho^2\right), m + \frac{\rho\sigma}{\sqrt{1-\rho^2}}, \rho, \frac{2b\sigma}{\sqrt{1-\rho^2}}, \frac{\sqrt{1-\rho^2}}{\sigma}\right) \tag{1.6}$$

1.2.3 The SVI Jump-Wings (SVI-JW) Parameterization

The SVI-Jump-Wings (SVI-JW) is parameterization of *the implied variance* $v(k, T)$ rather than the implied total variance $w(k, T)$. This kind of parameterization is intuitive for traders, and the parameters have a financial meaning.The model's parameterization was inspired by a similar parameterization attributed by Gatheral to Tim Klassen.

For a fixed time to expiry $t > 0$, and a parameters set $\chi_J = \{v_t, \psi_t, p_t, c_t, \widetilde{v}_t\}$, the (SVI-JW) parameters defined from the row SVI is given by

$$\begin{aligned}
v_t &= \frac{a + b\left\{-\rho m + \sqrt{m^2 + \sigma^2}\right\}}{t} \\
\psi_t &= \frac{1}{\sqrt{w_t}} \frac{b}{2} \left(-\frac{m}{\sqrt{m^2 + \sigma^2}} + \rho \right) \\
p_t &= \frac{1}{\sqrt{w_t}} b(1 - \rho) \\
c_t &= \frac{1}{\sqrt{w_t}} b(1 + \rho) \\
\tilde{v}_t &= \frac{1}{t} \left(a + b\sigma\sqrt{1 - \rho^2} \right)
\end{aligned} \tag{1.7}$$

Setting $w_t := v_t t$, we notice that this parametrization dependency on time to expiration t, hence the (SVI-JW) could be view as a generalisation of the raw SVI.

The SVI-JW parameters interpretation is as following :

- v_t is the ATM variance.
- ψ_t is ATM skew.
- p_t is the slope of the left (put) wing.
- c_t is the slope of the right (call) wing.
- $\tilde{v}_t$ is the minimum implied variance.

1.3 Characterisation of Static Arbitrage

The definition of static arbitrage is an arbitrage that does not require rebalancing of positions between two times.

The total implied variance $w(\cdot, t)$ should satisfy the following conditions to be arbitrage free as presented in the following theorem [12].

Theorem 1.1 If the two-dimensional map $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:
(i) $w(\cdot, t)$ is of class $\mathcal{C}^2(\mathbb{R})$ for each $t \geq 0$
(ii) $w(k, t) > 0$ for all $(k, t) \in \mathbb{R} \times \mathbb{R}_+^*$
(iii) $w(k, \cdot)$ is non-decreasing for each $k \in \mathbb{R}$
(iv) for each $(k, t) \in \mathbb{R} \times \mathbb{R}_+^*$ probability density function $P(k)$, is non-negative.
(v) $w(k, 0) = 0$ for all $k \in \mathbb{R}$
(vi) $\lim_{k\uparrow\infty} d_+(k, w(k, t)) = -\infty$, for each $t > 0$

Then the corresponding call price surface $(K, t) \mapsto \mathrm{BS}(K, w(\log(K), t))$ is free of static arbitrage.

Definition 1.1 Let $w : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}_+$ be a two-dimensional map satisfying Theorem 3.3(i)-(ii)

- w is said to be free of calendar spread arbitrage if condition (iii) in Theorem 3.3 holds;
- w is said to be free of butterfly arbitrage if condition (iv) in Theorem 3.3 holds.

To check the arbitrage as explained in [13] and [14], we have to verify that the risk neutral density (RND) produces a volatility curve that is arbitrage-free.This RND will be used to calculate the price of call/put options that should be also arbitrage-free.

There are two ways to check for arbitrage:

1. The risk neutral density (RND) test for butterfly arbitrage.

2. The test based on option strategies for butterfly and calendar spread arbitrage.

Breeden and Litzenberger [15], derive an expression of the discounted risk neutral density $f_{S_T}(S^*)$ as function of the second derivative of the call price $C(K)$ with respect to the strike K

$$e^{-rT} f_{S_T}(S^*) = \left. \frac{\partial^2 C}{\partial K^2} \right|_{K=S^*} \tag{1.8}$$

1.3.1 RND tests for arbitrage

To satisfy this type of test we have to check the most common feature for a density function:

- Using the RND, we can obtain the call/put prices observed in the market by numerical integration of the RND.

- The RND is a positive function and its integral is equal to the one.

- Using the call price function $C(K)$, with strike K, the RND should produce monotonically decreasing call option prices with respect to the strike K.

  The call price $C(K)$ is

  $$C(K) = e^{-rT} \int_K^\infty (S_T - K) f_{S_T}(s) ds \tag{1.9}$$

  and we should ensure that the first derivative with respect to the strike is negative then;

  $$\left. \frac{\partial C}{\partial K} \right|_{K=K_1} = -e^{-rT} \int_{K_1}^\infty f_{S_T}(s) ds < 0 \tag{1.10}$$

- The last condition is the convexity of the call price $C(K)$ with respect to the strike K. If we consider to strikes $K_1 < K_2$, then the first derivative mentioned above should increases in strike

  $$\left. \frac{\partial C}{\partial K} \right|_{K=K_2} - \left. \frac{\partial C}{\partial K} \right|_{K=K_1} = e^{-rT} \int_{K_1}^{K_2} f_{S_T}(s) ds > 0 \tag{1.11}$$

Defnition 1.2 A volatility surface is free of static arbitrage if and only if both of the following conditions are satisfied:

1. It is free of **calendar spread arbitrage**;

2. Each time slice is free of **butterfly arbitrage**.

Note that the first condition guarantees the monotonicity of European call option prices with respect to theirs maturities. The butterfly arbitrage guarantees the existence of a non negative probability density. Now, we will analyze each condition separately.

1.3.2 Calendar Spread Arbitrage

Calendar spread arbitrage refers to the monotonicity of European call option prices with respect to maturity. In order to obtain no arbitrage condition with respect to implied volatility, we need to transform this condition.

To obtain a calendar spread, we will buy and sell options with the same strike price but for different maturities. For simplify, we assume also that there are no dividend or interest rate.
Let $C(T, K)$ the price of a call with expiry T and strike K. We know that if $T_1 < T_2$ then we have calendar spread arbitrage if $C(T_1, K) > C(T_2, K)$.

Our strategy: we buy (long) call option $C(T_2, K)$ (the cheap) and we sell (short) the call $C(T_1, K)$ (the expensive), then, the difference will be $x = C(T_1, K) - C(T_2, K) > 0$.

At T_1 our position is $x + C(T_2) - \max\{S_{T_1} - K, 0\}$

- If $(S_{T_1} \leq K)$, then the total profit is: $x + C(T_2)$
- If $(S_{T_1} > K)$, and the profit will be: $x + C(T_2) - S_{T_1} + K$

At T_2 :

- If $(S_{T_2} > K)$, we buy the stock by paying the amount K that we had received at T_1, and return the stock that we had short at T_1. The net profit for our trading strategy is $x > 0$.
- If $(S_{T_2} < K)$, we buy the stock by paying T_2 and return the stock that we short at T_1. Note that, we had received k at time T_1, our net profit is then $K - S_{T_2} + x > x > 0$.

Lemma 1.2 *A volatility surface w is free of calendar spread arbitrage if*

$$\partial_T w(k, T) \geq 0, \quad \text{for all } k \in \mathbb{R} \text{ and } T > 0$$

Proof : Let $(X_t)_{t \geq 0}$ be a martingale, $L \geq 0$ and $0 \leq t_1 < t_2$. Then the inequality is true

$$E\left[(X_{t_2} - L)^+\right] \geq E\left[(X_{t_1} - L)^+\right] \tag{1.12}$$

Let consider two options C_1, $C_2$with strikes K_1, K_2 and expiry time t_1,t_2 respectively.
If the two options have the same moneyness (log-moneyness or log-strike := $\log\left(\frac{K}{F_T}\right)$) we get:

$$\frac{K_1}{F_{t_1}} = \frac{K_2}{F_{t_2}} =: \mathrm{e}^k \tag{1.13}$$

Then, the process $(X_t)_{t \geq 0}$ defined by $X_t := S_t / F_t$ is martingale for all $t \geq 0$, and the following relation holds *if the dividends are proportional*:

$$\frac{C_2}{K_2} = \mathrm{e}^{-k} E\left[\left(X_{t_2} - \mathrm{e}^k\right)^+\right] \geq \mathrm{e}^{-k} E\left[\left(X_{t_1} - \mathrm{e}^k\right)^+\right] = \frac{C_1}{K_1} \tag{1.14}$$

This means that if the moneyness is constant, option prices are non-decreasing in time to maturity $(\partial_t C_{BS}(k, \omega(k,t)) \geq 0$. If this is valid for the Black-Scholes prices $C_{BS}(y, \omega(k,t))$, then $\omega(k,t)$ is strictly increasing, i.e $(\partial_t w(k,t) \geq 0)$.
We can interpret the absence of calender spread by the fact that there are no cross lines on the total variance : when the maturity goes up , the SVI slice will be translated up.

1.3.3 Butterfly arbitrage

We consider the butterfly arbitrage for the slice of implied volatility $k \mapsto w(k,t)$.

In the real market the butterfly spread is a strategy with options, which combines simultaneous buying and selling of three similar types options (either calls or puts) with strikes $K - \epsilon < K < K + \epsilon$ and which have the following characteristics:

- The options have the same maturity
- The options are listed on the same underlying
- The strike $K - \epsilon$ and $K + \epsilon$ are equidistant from K as in figure (1.2).

The butterfly consists of the purchase of a call option with strike $K - \epsilon$, the sale of 2 call options with strike K and the purchase of a third option with strike $K + \epsilon$.

In other side, we know that the second derivative of the call price with respect to the strike k is

$$\frac{d^2 C(K,T)}{dK^2} = \lim_{\varepsilon \to 0} \frac{C(K - \varepsilon, T) - 2C(K,T) + C(K + \varepsilon, T)}{\varepsilon^2} \tag{1.15}$$

We consider the following European payoff:

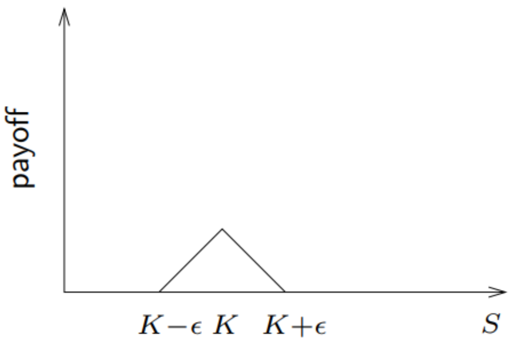

Figure 1.2: Butterfly Spread

- $\frac{1}{\varepsilon^2}$ calls with strike $K-\epsilon$
- $\frac{1}{\varepsilon^2}$ calls with strikes $K+\epsilon$
- $\frac{-2}{\varepsilon^2}$ calls with strikes k

This strategy is called the butterfly spread, and the absence of butterfly arbitrage implies that the price of a butterfly spread is positive and the call price must to be convex with

$$C(K-\epsilon,T) - 2C(K,T) + C(K+\epsilon,T) \geq 0 \tag{1.16}$$

Let's back to the our SVI model, we will find an equivalent condition to the call price convexity defined in (1.15) in terms of the implied total variance $w(k)$.

recall the Black-Scholes formula for a European call option price in terms of the total implied variance $w(k)$

$$C_{\text{BS}}(k, w(k)) = S\left(\mathcal{N}\left(d_+(k)\right) - \mathrm{e}^k \mathcal{N}\left(d_-(k)\right)\right), \quad \text{for all } k \in \mathbb{R} \tag{1.17}$$

with;

$$d_\pm(k) := -k/\sqrt{w(k)} \pm \sqrt{w(k)}/2$$

Let P be the probability density function of S_T, then

$$p(k) = \left.\frac{\partial^2 C(k)}{\partial K^2}\right|_{K=F_t\mathrm{e}^k} = \left.\frac{\partial^2 C_{\text{BS}}(k, w(k))}{\partial K^2}\right|_{K=F_t\mathrm{e}^k}, \quad k \in \mathbb{R} \tag{1.18}$$

$$p(k) = \frac{g(k)}{\sqrt{2\pi w(k)}} \exp\left(-\frac{d_-(k)^2}{2}\right) \tag{1.19}$$

Where the function $g : \mathbb{R} \to \mathbb{R}$

$$g(k) := \left(1 - \frac{k w'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2} \tag{1.20}$$

The condition $g(k) > 0$ is equivalent to the market implied volatility density $P(k)$ is positive.

Defnition 1.3 *A slice is said to be free of buttery arbitrage if the corresponding density define in (1.19) is non-negative*, which is equivalent to $g(k) \geq 0$.

Lemma 1.3 The map $k \mapsto w(k,t)$ is free arbitrage if and only if $g(k) \geq 0$ for all $k \in \mathbb{R}$ and $\lim_{t\to+\infty} d_+(k) = -\infty$

Proof: let P the probability density function that can be calculated from the call price function $C(k)$ using the formula of Breeden-Litzemberger is:

$$p(k) = \left.\frac{\partial^2 C(k)}{\partial K^2}\right|_{K=F_t\mathrm{e}^k} = \left.\frac{\partial^2 C_{\text{BS}}(k, w(k))}{\partial K^2}\right|_{K=F_t\mathrm{e}^k}, \quad k \in \mathbb{R} \tag{1.21}$$

By diferentiating the Black-Scholes formula for any $k \in \mathbb{R}$, we get ;

$$p(k) = \frac{g(k)}{\sqrt{2\pi w(k)}} \exp\left(-\frac{d_-(k)^2}{2}\right) \tag{1.22}$$

We note that the integral of the density function may not all time equal to 1, and we need to impose asymptotic boundary conditions. The limit of the call option $\lim_{k\to+\infty} C_{\mathrm{BS}}(k, w(k)) = 0$, is equivalent to $\lim_{k\to+\infty} d_+(k) = -\infty$.

Now we will summarize the characterisation of static arbitrage that we will use in practice in the next section of calibration.

AOA Characterisation	Call criterion	equivalent criterion	equivalent criterion
Butterfly arbitrage (convexity)	$\frac{\partial^2 C(k)}{\partial K^2}\Big\vert_{K=F_t e^k} \geq 0$	$p(k) \geq 0$	$g(k) \geq 0$
Limit Price function	$\lim_{K\to+\infty} C_{\mathrm{BS}}(T,k) = 0$	$\lim_{t\to+\infty} d_+(k) = -\infty$	$b(1+\rho) < 2$
Calendar Spread (monotonicity)	$\mathrm{T} \longrightarrow C_{\mathrm{BS}}(T,K) \nearrow$	$\frac{\partial C_{\mathrm{BS}}(T,x)}{\partial T} \geq 0$	$\partial_T w(k,T) \geq 0$

Table 1.1: Summary of the static arbitrage

Axel Vogt example of arbitrage: generated by a parametrisation of SVI with T=1

$$(a, b, m, \rho, \sigma) = (-0.0410, 0.1331, 0.3586, 0.3060, 0.4153) \tag{1.23}$$

These parameters give a positive total variance function w,however, the density function $p(k)$ and the $g(k)$ function defined in (2.20), are negative. We will show explicitly, in the next section, how to tackle this problem.

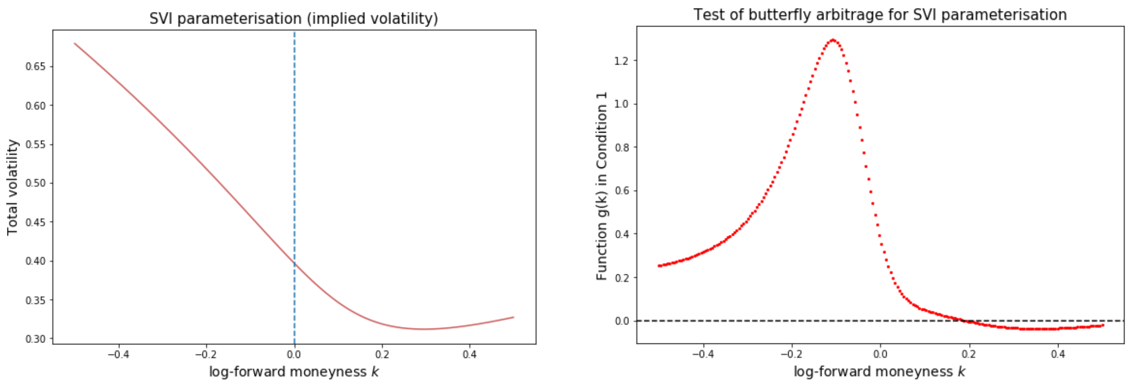

Figure 1.3: Plots of the total variance smile w (left) and the function g (right), using Axel Vogt's parameters

1.4 Volatility on the wings (main result)

Inspired by the work done by Jäckel in [16], in this section we will present the main result: an asymptotic study of the function $g(k)$, for large strikes, in order to get sufficient conditions that ensure a probability density function $P(k)$ positive and hence, an SVI butterfly arbitrage free.

We recall the $g : \mathbb{R} \to \mathbb{R}$

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2} \tag{1.24}$$

In the wings we know that the total implied variance $w(k;\chi_N)$ has the left and right asymptotes that respect the assumption of linear wings, this result is consistent with the Roger Lee's moment formula mentioned in [11].

$$\begin{cases} W_R(k) & = b(\rho+1)k + [a - bm(\rho+1)] \quad k \to +\infty \\ W_L(k) & = b(\rho-1)k + [a - bm(\rho-1)] \quad k \to -\infty \end{cases} \tag{1.25}$$

In the general case (left and right wing), we consider the total implied variance $w\left(k;\chi_N\right)$ for large strikes

$$w(k) = \alpha + \beta k, \quad and \qquad w'(k) = \beta, \quad w''(k) = 0$$

We replace in (1.24) and we get

$$g(k) = \left(1 - \frac{k}{2}\frac{\beta}{(\alpha+\beta k)}\right)^2 - \frac{\beta^2}{4}\left(\frac{1}{\alpha+\beta k} + \frac{1}{4}\right) \tag{1.26}$$

After simplification we get the following quadratic equation

$$g(k) = \frac{\beta^2}{4}\left(1 - \frac{\beta^2}{4}\right)k^2 + \frac{1}{4}\left(4\alpha\beta - \beta^3 - \frac{\alpha\beta^3}{2}\right)k + \frac{1}{4}\left(4\alpha^2 - \alpha\beta^2 - \frac{\alpha^2\beta^2}{4}\right) \tag{1.27}$$

We calculate the discriminant Δ of the quadratic function $g(k)$ and we get

$$\Delta = \frac{\beta^4}{16}\left(\beta^2 + \alpha^2 - 4\alpha\right) \tag{1.28}$$

In order to guarantee that the function $g(k)$ is positive, we will be interested only to the case where the discriminant Δ, of the quadratic function $g(k)$, is strictly negative ($\Delta < 0$) in both wings sides and the first coefficient of the k^2 is strictly positive.

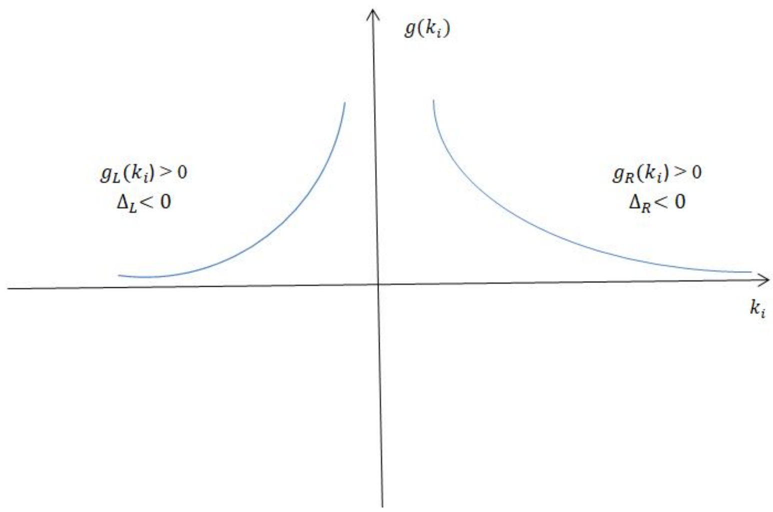

Figure 1.4: Large strike asymptotics for the $g(k)$ function

We study both wings, right and left, to find sufficient conditions for the butterfly arbitrage.

1.4.1 The right wing

We avoid the arbitrage in the right wing by considering that the discriminant Δ_R is negative in the right side of the quadratic function $g_R(k)$.

We recall the asymptotic formula, for large strikes, for the total implied variance

$$W_R(k) = b(\rho+1)k + [a - bm(\rho+1)] = \alpha + \beta k \quad k \to +\infty$$

Where for the right wing we have;

$$\alpha = a - bm(\rho+1), \quad \beta = b(\rho+1)$$

The discriminant Δ_R is calculated by replacing α and β in (1.28)

$$\Delta_R = \frac{b^4(\rho+1)^4}{16}\left[b^2(\rho+1)^2 + (a - mb(\rho+1))^2 - 4(a - bm(\rho+1))\right] \tag{1.29}$$

If we consider, as first constraint, that the term $\frac{b^4(\rho+1)^4}{16}$ is strictly positive, the sign of Δ_R will be decided by the second term, and we get

$$\begin{aligned}
\big(\Delta_R < 0\big) &\iff \Big(b^2(\rho+1)^2 + (a - mb(\rho+1))^2 - 4(a - bm(\rho+1)) < 0\Big) \\
&\iff b^2(\rho+1)^2 < \big(a - mb(\rho+1)\big)\big(4 - a + bm(\rho+1)\big) \\
&\iff \frac{\big(a - mb(\rho+1)\big)\big(4 - a + bm(\rho+1)\big)}{b^2(\rho+1)^2} > 1
\end{aligned}$$

Where; $b^2(\rho+1)^2 > 0$

Δ_R is negative, hence, the quadratic function $g_R(k)$ is strictly positive if the first coefficient of this quadratic function is positive, i.e

$$\frac{\beta^2}{4}\left(1 - \frac{\beta^2}{4}\right) > 0 \iff \frac{b^2(\rho+1)^2}{4}\left(1 - \frac{b^2(\rho+1)^2}{4}\right) > 0$$

We combine with the first constraint and we get

$$0 < b^2(\rho+1)^2 < 4$$

Finally, for the right wing, $g_R(k)$ is strictly positive if the following two conditions holds

$$\boxed{\begin{cases} \dfrac{\big(a - mb(\rho+1)\big)\big(4 - a + mb(\rho+1)\big)}{b^2(\rho+1)^2} > 1 \\ 0 < b^2(\rho+1)^2 < 4 \end{cases}} \tag{1.30}$$

1.4.2 The left wing

We fellow the same analysis as the previous one, the calculus is the same we change only the term $(\rho+1)$ by $(\rho-1)$.

We can avoid the arbitrage in the left wing by considering that the discriminant Δ_L is negative in the left side of the function $g_L(k)$.

We recall the total implied variance left asymptotic formula for the large strikes

$$W_L(k) = b(\rho-1)k + [a - bm(\rho-1)] = \alpha + \beta k \quad k \to +\infty$$

Where;

$$\alpha = a - bm(\rho-1), \quad \beta = b(\rho-1)$$

The discriminant Δ_L is calculated by replacing α and β in (1.28)

$$\Delta_L = \frac{b^4(\rho-1)^4}{16}\left[b^2(\rho-1)^2 + (a - mb(\rho-1))^2 - 4(a - bm(\rho-1))\right] \tag{1.31}$$

We consider the term $\frac{b^4(\rho-1)^4}{16}$ is strictly positive, the sign of Δ_L will be decided by the second term, and we get

$$\begin{aligned}
\big(\Delta_L < 0\big) &\iff \Big(b^2(\rho-1)^2 + (a - mb(\rho-1))^2 - 4(a - bm(\rho-1)) < 0\Big) \\
&\iff b^2(\rho-1)^2 < \big(a - mb(\rho-1)\big)\big(4 - a + bm(\rho-1)\big) \\
&\iff \frac{\big(a - mb(\rho-1)\big)\big(4 - a + bm(\rho-1)\big)}{b^2(\rho-1)^2} > 1
\end{aligned}$$

Where; $b^2(\rho-1)^2 > 0$

Δ_L is negative, hence, the quadratic function $g_L(k)$ is strictly positive if the first coefficient of this quadratic function is positive, i.e

$$\frac{\beta^2}{4}\left(1-\frac{\beta^2}{4}\right) > 0 \iff \frac{b^2(\rho-1)^2}{4}\left(1-\frac{b^2(\rho-1)^2}{4}\right) > 0$$

We combine with the term $\frac{b^4(\rho-1)^4}{16}$ is strictly positive, and we get

$$0 < b^2(\rho-1)^2 < 4$$

Finally, for the right wing, $g_R(k)$ is strictly positive if the following two conditions are satisfied

$$\boxed{\begin{cases} \dfrac{\big(a-mb(\rho-1)\big)\big(4-a+mb(\rho-1)\big)}{b^2(\rho-1)^2} > 1 \\ 0 < b^2(\rho-1)^2 < 4 \end{cases}} \tag{1.32}$$

We summary the necessary conditions that guarantee a butterfly arbitrage free for SVI model in the following theorem.

Theorem 1.2

The total implied variance $w(k,t)$ in the SVI model is said to be *free of butterfly arbitrage* if the following conditions holds

$$\begin{cases} \dfrac{\big(a-mb(\rho+1)\big)\big(4-a+mb(\rho+1)\big)}{b^2(\rho+1)^2} > 1 \\ \dfrac{\big(a-mb(\rho-1)\big)\big(4-a+mb(\rho-1)\big)}{b^2(\rho-1)^2} > 1 \\ 0 < b^2(\rho+1)^2 < 4 \\ 0 < b^2(\rho-1)^2 < 4 \end{cases}$$

Chapter 2

SVI Calibration Arbitrage Free

One of the desirable features of the SVI model is that it fits the input data very well especially in the equity market.The problem that faced this model in the last decay is that arbitrage-freeness in strike and maturity is not guarantee and it's not automatically verified during the calibration step. Usually, we calibrate the SVI model and next we check the arbitrage condition via the risk neutral density(RND) if it's positive.In practice, this is a quiet inconvenience and most practitioners require a more robust approach in order to be implemented in the pricing library.

We provide solution for this problem by implementation of robust calibration method with sufficient constraints that allow to calibrate the SVI model and also to eliminate both arbitrage type automatically during the calibration step.We no longueur need to check the positivity of the density function $p(k)$. This calibration method is very important in practice: it allows to implement the model in the pricing library and also to calculate the local volatility using Dupire formula.

Finally, we give some synthetic examples with arbitrage and we show in practice how our algorithm could perform in this case. Also, we test the performance of our calibration method using the implied volatility extracted from call/put options price listed on 23 indexes.

2.1 The Raw SVI calibration

The raw SVI model represents the total variance or the implied volatility of the call or put options observed in the market. Therefore, we define a loss function to be optimized. This function is the difference between the value of variance given by SVI model and the value of implied volatility observed in the market.The minimization of this function will permit to calibrate SVI to observed implied volatility.

Most algorithms require an initial guess of the model parameters, as well as boundaries conditions and an objective function. The resulting parameters should respect the arbitrage free conditions: butterfly arbitrage and calendar spread arbitrage.
Recall the expression of the raw SVI model with 5 parameters $\chi_R = \{a, b, \rho, m, \sigma\}$.

$$w\left(k; \chi_R\right) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\} \tag{2.1}$$

For every strike $k \in \mathbb{R}$, and

$$\begin{aligned} a &\in \mathbb{R} \\ b &\geq 0 \\ |\rho| &< 1 \\ m &\in \mathbb{R} \\ \sigma &> 0 \\ a + b\sigma\sqrt{1-\rho^2} &\geq 0 \end{aligned} \tag{2.2}$$

2.2 SVI's Parameters Boundaries

We determine lower and upper boundaries for each SVI parameters (a, b, ρ, m, σ) in order to run an efficient calibration that respect conditions related to the model and also to avoid arbitrage.

We have some restrictions on the parameters that follow from the parameterization of the model in (1.1), such as;

$$b \geq 0; \qquad |\rho| < 1; \qquad \sigma > 0.$$

Zeliade [17] present some parameter constraints and limiting cases.They incorporate also some boundaries that are deduced the no arbitrage conditions.

- **Parameter a and SVI Minimum**
  In this part we define the lower and upper bound of the parameter a.
  $w\left(k; \chi_R\right)$ has a unique minimum if $\rho^2 \neq 1$ and it's value is

$$W_{\min}(k) = a + b\,\sigma\,\sqrt{1-\rho^2} \tag{2.3}$$

  This minimum is located at the point: $k^* = m - \frac{\rho\sigma}{\sqrt{1-\rho^2}}$.
  We replace the positivity condition: $a + b\,\sigma\,\sqrt{1-\rho^2} > 0$ by the restriction $a > 0$ which is stronger as both parameters b and σ are positive and obviously $\sqrt{1-\rho^2}$ also. Finally, we can conclude that the conditions

$$(a > 0) \Longrightarrow W_{\min}(k) = a + b\,\sigma\,\sqrt{1-\rho^2} > 0 \tag{2.4}$$

  If we impose the condition $a > 0$,the question that comes up in mind: which lower bound could a take. This is a very important question that we will discuss it in the calibration section.

  At the moment,we take an arbitrary small value $a_{\min} = 10^{-5}$.

  For the upper bound, and as the parameter a represents the overall level of total variance or the vertical translation, hence, it could not suitable to get a value of a greater than the largest value of the observed total variance.Therefore we impose for a the following boundaries

$$0 < a_{\min} = 10^{-5} \leq a \leq max(W_{SVI}^{market}) \tag{2.5}$$

- **Parameter b and left wing**
  We can estimate the left and the right SVI asymptotes for large strikes;

$$\begin{aligned} W_L(k) &= a + b(\rho - 1)(k - m) \\ W_R(k) &= a + b(\rho + 1)(k - m) \end{aligned} \tag{2.6}$$

  For the right wing the slop is $b(\rho + 1)$ and it should not exceed to 2 which is consistent with Roger Lee formula value of implied volatility for the large strike tail [11].

  We can also take the advantage from the others arbitrage constraints such as

$$\lim_{K \to +\infty} C_{\mathrm{BS}}(T, k) = 0 \iff \lim_{t \to +\infty} d_+(k) = -\infty \tag{2.7}$$

  We show that the condition $\lim_{k \to \infty} d_1(k, w(k)) = -\infty$, i.e :

$$\lim_{k \to \infty} d_1\left(k, w_{\mathrm{SVI}}(k)\right) = \lim_{k \to \infty} \left(-\frac{k}{\sqrt{w_{\mathrm{SVI}}(k)}} + \frac{1}{2}\sqrt{w_{\mathrm{SVI}}(k)} \right) = -\infty \tag{2.8}$$

  Is satisfied for a function $w(k)$ if:

$$\limsup_{k \to \infty} \frac{w(k)}{2k} < 1. \tag{2.9}$$

Or we have :

$$\limsup_{k\to\infty} \frac{w(k)}{k} = b(\rho+1) \tag{2.10}$$

Finally, to be consistent with the Roger Lee's moment formula, the right slope condition should satisfy the inequality:

$$b(\rho+1) < 2. \tag{2.11}$$

Using this last inequality we can conclude that for $\rho \neq -1$

$$b < \frac{2}{(\rho+1)} \tag{2.12}$$

For any $\mid \rho \mid < 1$, the minimum of the function $f(\rho) = \frac{2}{(\rho+1)}$ is $f(1) = 1$, (with $\mid \rho \mid < 1$).

For the b lower bound, we can take a small value around $b_{min} = 10^{-3}$.

Finally we obtain that the b boundaries are:

$$0 < b_{min} = 10^{-3} < b < 1 \tag{2.13}$$

- **Correlation Parameter ρ**
  The correlation coefficient between the Brownian motion of the underlying and the implied variance process is ρ, hence, it's value will be in the interval $]-1,1[$

- **Translation Parameter m**
  By the same way, m is the horizontal translation of the smile to the right, and as the smile could not more out side the zone limited by the log forward moneyness of our input data, hence, we can cap it at some reasonably translation level as following.

  $$2\min_i k_i \leq m \leq 2\max_i k_i \tag{2.14}$$

  Where k_i is the log forward moneyness or log strike: $k_i := \log\left(\frac{K_i}{F_T}\right)$.

- **Curvature Parameter σ**
  The positive value of sigma ($\sigma > 0$) means that the total implied variance has a positive at-the-money curvature.

  As σ represents the curvature, it's rare to see σ takes large value,in general, σ takes small values around 0.01 or 0.02 for the short maturities and it goes more large for long maturities, like 0.2 or 0.3 for 10 years time to expiry.

  For this reason, we can cap σ by any arbitrary reasonable value σ_{max}, a constant value such as 1, 2 or other number.

  $$0 < \sigma_{min} = 0.01 \leq \sigma \leq \sigma_{max} = 1 \tag{2.15}$$

We can summarize the obvious boundaries for the SVI raw parameters model as following:

$$\begin{cases} 0 < a_{\min} = 10^{-5} \leq a \leq \max(W_{SVI}) \\ 0 < b_{\min} = 10^{-3} < b < 1 \\ -1 < \rho < 1 \\ 2\min_i k_i \leq m \leq 2\max_i k_i \\ 0 < \sigma_{\min} = 0.01 \leq \sigma \leq \sigma_{\max} = 1 \end{cases} \tag{2.16}$$

2.2.1 The Initial Guess

In order to run our calibration algorithm, we need to define the initial guess parameter's values, this values are very important and we should to care when we choose these right values. Also, the initial point should below to the boundaries intervals for each variable.

For the first parameter a that represent the vertical translation, one of the reasonable value is to be close to the minimum of the market total variance value, hence, we can take it

$$a = \frac{1}{2} min(W_{SVI})$$

The parameter b could be taken as $b = 0.1$, in general this value is the middle between the b value for the short maturities of one weeks to the large maturities of 10 years.

For the correlation parameter ρ, it's obvious to guess it's initial value for the equity market, we are sure that it takes negative value and the initial guess should be

$$\rho = -0.5$$

Note that the extreme values of ρ are -1 and 1, for $\rho = 1$, the SVI model is graphically similar to the payoff of the call, and for $\rho = -1$, the ρ = -1 , the SVI model (the smile) is similar to the put payoff.

Therefore $\rho = -0.5$, is a reasonable value for the smile in the equity market. We will see after calibration that ρ vary with respect to the time to expiry: more the time to expiry is large, more will decrease taking small negative values.

Regarding the horizontal translation parameter m: we notice that m can take as positive or negative values: more the m value is high , more the smile will be translated to the right and vice versa. The initial value $m = 0.1$, is a resealable value between the short and long maturities.

The last parameter is the curvature parameter σ which is strictly positive in general it's values are small around 0.01 for the short maturities (weekly or monthly) and 0.2 for the large maturities of 10 years, hence, taking $\sigma = 0.1$ will be acceptable value valid for the short and the long maturities.

Note that these values are arbitrary and any other values close could be work also, finally, we summarize the initial guess values for the SVI calibration in the box bellow.l

The Initial Guess

$$\begin{cases} a = \frac{1}{2} min(W_{SVI}) \\ b = 0.1 \\ \rho = -0.5 \\ m = 0.1 \\ \sigma = 0.1 \end{cases} \tag{2.17}$$

2.2.2 The Butterfly Arbitrage Constraints

Let's recall the expression of P, the probability density function that can be calculated from the call price function $C(k)$ using the formula of Breeden-Litzemberger [15].

$$p(k) = \left.\frac{\partial^2 C(k)}{\partial K^2}\right|_{K=F_t e^k} = \left.\frac{\partial^2 C_{\text{BS}}(k, w(k))}{\partial K^2}\right|_{K=F_t e^k}, \quad k \in \mathbb{R} \tag{2.18}$$

By diferentiating the Black-Scholes formula for any $k \in \mathbb{R}$, we get;

$$p(k) = \frac{g(k)}{\sqrt{2\pi w(k)}} \exp\left(-\frac{d_-(k)^2}{2}\right) \tag{2.19}$$

The limit of the call option $\lim_{k\to+\infty} C_{\text{BS}}(k, w(k)) = 0$, which is equivalent to:

$$\lim_{k\to+\infty} d_+(k) = -\infty$$

The SVI model is butterfly arbitrage free if the SVI parameters could guarantee that the probability density function P is positive for any log forwad moneyness k,hence, $p(k)$ is positive if and only if the function $g(k) > 0$ is also positive.

Where the function $g : \mathbb{R} \rightarrow \mathbb{R}$

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2} \tag{2.20}$$

Finally, we can summarize all the constraints that could guarantee to get SVI model with arbitrage free following these conditions:

$$\boxed{\begin{cases} 0 < a_{min} = 10^{-5} \leq a \leq max(W_{SVI}) \\ 0 < b_{min} = 10^{-3} \leq b < 1 \\ -1 < \rho < 1 \\ 2\min_i k_i \leq m \leq 2\max_i k_i \\ 0 < \sigma_{min} = 0.01 \leq \sigma \leq \sigma_{max} = 1 \\ g(k) > 0 \end{cases} \tag{2.21}}$$

We replace the condition $g(k) > 0$ by sufficient, butterfly arbitrage free, conditions presented on (1.4.2)

$$\begin{cases} \dfrac{\big(a - mb(\rho+1)\big)\big(4 - a + bm(\rho+1)\big)}{b^2(\rho+1)^2} > 1 \\ \dfrac{\big(a - mb(\rho-1)\big)\big(4 - a + bm(\rho-1)\big)}{b^2(\rho-1)^2} > 1 \\ 0 < b^2(\rho+1)^2 < 4 \\ 0 < b^2(\rho-1)^2 < 4 \end{cases}$$

Let's define the least-Squares objective function $f\,(k; \chi_R)$ to optimize, where $\chi_R = \{a, b, \rho, m, \sigma\}$ is the set of the parameters model, for an expiry time fix T.

$$f\,(k; \chi_R) = \sum_{i=1}^{n} \left(\omega_{SVI(i)}^{model} - \omega_{Total(i)}^{market}\right)^2 \tag{2.22}$$

$$f\,(k; \chi_R) = \sum_{i=1}^{n} \left[a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\} - \omega_{Total(i)}^{market}\right]^2 \tag{2.23}$$

Where $k_i := \log\left(\frac{K_i}{F_T}\right)$

The problem reduced to find the optimal model's parameters $\chi_R = (a^*, b^*, \rho^*, m^*, \sigma^*)$.

2.3 SVI Summary

In this section, we redefine the SVI model framework and the parameters associate.We summary all the necessary arbitrage free conditions and the parameters boundaries.It's the recipe that we need to use for the calibration.

SVI Model

the model: The SVI model represents the total implied variance with 5 parameters $\chi_R = \{a, b, \rho, m, \sigma\}$

$$w\,(k;\chi_R) = \sigma^2_{imp}(k;\chi_R)T = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right\} \tag{2.24}$$

1. **Parameters:** $a > 0, \quad 0 < b < 1, \quad -1 < \rho < 1, \quad m \in \mathbb{R}, \quad \sigma > 0$
2. **Parameters Boundaries**
   - **a: verticall translation** with $0 < a_{min} < a \leq max(W^{Market}_{SVI})$
   - **b: wings slope:** $0 < b < 1$
   - ρ**: counter-clockwise rotation:** $-1 < \rho < 1$
   - m**: horizontal translation:** $2\min k_i \leq m \leq 2\max k_i$
   - σ**: the smile curvature:** $0 < \sigma_{min} \leq \sigma \leq \sigma_{max}$
3. **Butterfly arbitrage constraints**
   - $\big(a - mb(\rho+1)\big)\big(4 - a + mb(\rho+1)\big) - b^2(\rho+1)^2 > 0$
   - $\big(a - mb(\rho-1)\big)\big(4 - a + mb(\rho-1)\big) - b^2(\rho-1)^2 > 0$
   - $0 < b^2(\rho+1)^2 < 4$
   - $0 < b^2(\rho-1)^2 < 4$
4. **Calendar spread arbitrage constraints**
   - $\partial_T w(k,T) > 0 \iff w(k,T_i) > w(k,T_{i-1}) \quad for \quad 1 \leq i \leq n$

Remark: In the calibration algorithm, it's better to avoid the division as operation in the arbitrage constraints, represented in (1.4.1 and 1.4.2), as the term $b^2(\rho+1)^2$ or $b^2(\rho-1)^2$ could be very small or close to zero and hence, the optimizer will diverge to infinity.

2.4 Input Data

We download Market data for example $Call_{\text{BS}}(k, w(k))$ or $Put_{\text{BS}}(k, w(k))$, for an expiry time T fix with strikes K, and next we can extract the implied volatility from this market call or put prices by inverting the black Scholes formula, finally, we get what we call the implied variance $\sigma^2_{BS}(k,T)$.

Our data comes from an external American financial services company called SuperDerivatives.The data is the prices of index options Call and Put listed on 23 indexes such as: EURO STOXX 50, CAC 40, NIKKEI 225, FTSE Mid 250 Index, SWISS MARKET IND, Hang Seng, NASDAQ 100, FTSE 100, MSCI world TR Index, Sao Paulo SE Bovespa Index,...etc.

We calculate the total implied variance: $\omega^{market}_{Total(i)} = T\sigma^2_{BS}(k,T)$
The log forward moneyness or log strike: $x := \log\left(\frac{K}{F_T}\right)$, where F_T is the forward price.
We display a sample of data call and put prices listed in the EURO STOXX 50 index in the figure (2.1).
We note in the figure (2.1) that our data respect some proprieties such positivity of both prices call and put and secondly monotonicity: put options price are increasing, and call options price are decreasing with respect to the strike. These proprieties are the basic requirements that our data should respect.

2.5 Numerical Applications

We present in this section some synthetic numerical examples with arbitrage, theses examples are our creation by variate the SVI parameters in order to get an arbitrage in the input data model

Strikes	Implied volatility %	Call	Put
2068.48	24.93	1268.59	7.25
2413.23	22.3	936.83	21.56
2757.98	19.39	621.99	52.79
3016.54	17.17	407.05	97.39
3585.37	12.86	77.18	338.53
3964.59	12.22	15.1	657.11
4481.71	12.98	1.87	1162.98
4998.83	14.17	0.35	1680.55
5688.33	15.65	0.05	2372.38
6033.07	16.32	0.02	2718.41
6377.82	16.96	0.01	3064.46
6722.57	17.56	0.01	3410.52
6894.94	17.86	0	3583.55

Table 2.1: Call and Put Price on 2019 /04/05 with, T =1.01 Y, from EURO STOXX 50

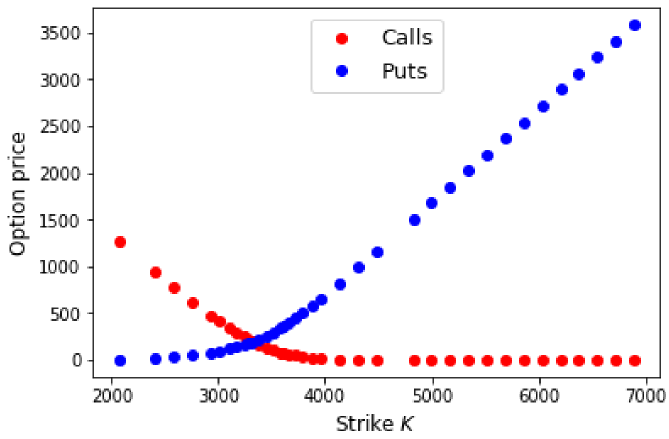

Figure 2.1: Call and Put option prices from Eurostoxx50, (T : 2019/04/05 to 2020/04/06)

and next apply the calibration method to show its performance and eliminate the arbitrage.

We generate the log forward moneyness vector that taking values in the range [-1, 1] with 40 points.

2.5.1 Some Arbitrage Examples

Example arbitrage 01

We consider the raw SVI parameters

$$(a, b, m, \rho, \sigma) = (0.002, 0.8, -0.8, 0.05, 0.35) \tag{2.25}$$

After calibration we get the SVI parameters that garantee an SVI butterfly arbitrage free (i.e positivity of the $g(k)$ function and hence, the density function also).

$$(a, b, m, \rho, \sigma) = (0.182, 0.563, -0.99, 0.145, 0.03) \tag{2.26}$$

We can see clearly in the figure (2.2) how the calibration method using sufficient conditions, for butterfly arbitrage, performs to eliminate the butterfly arbitrage.

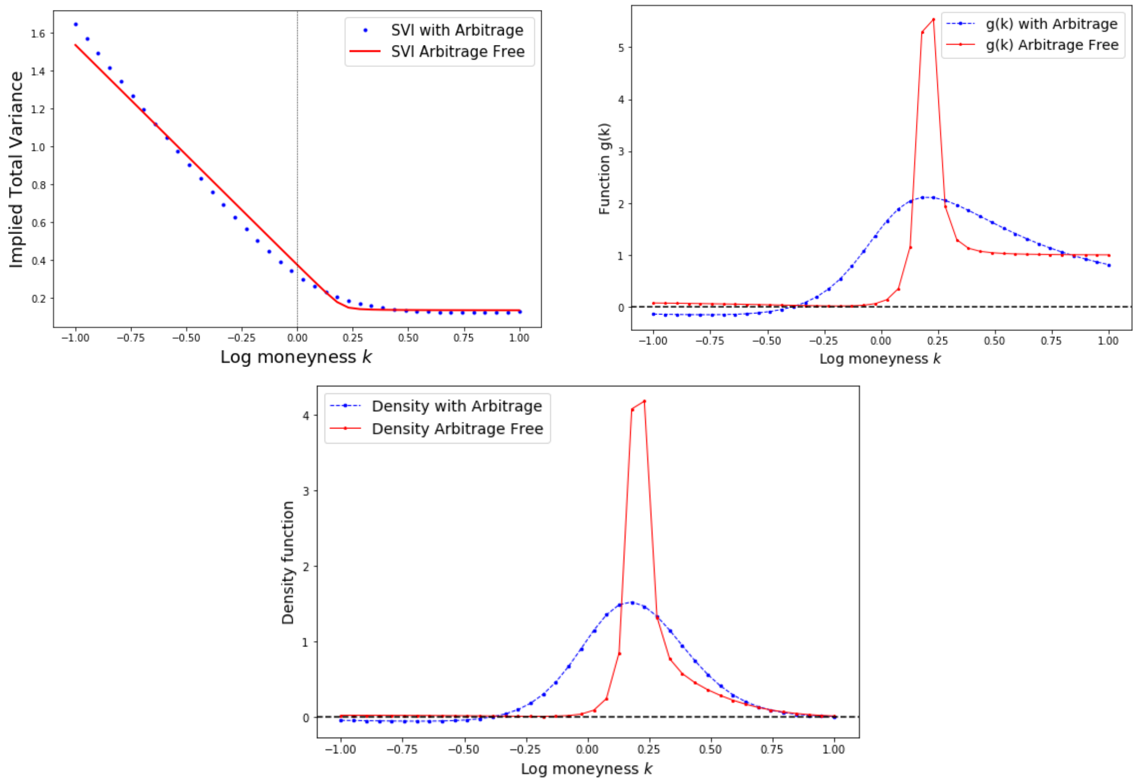

Figure 2.2: arbitrage example 01: plots of the total variance (left), the function g(k) (right), and the density function (down) with and without arbitrage.

Example arbitrage 02
We present another example of SVI with arbitrage and we show how our algorithm can perform to eliminate the butterfly arbitrage in this case. In fact, we can found many others examples with arbitrage: we can just fix a positive minimum of SVI and next to move up the right or the left wing by changing in the value of b and ρ.

Let's consider the following example

$$(a, b, m, \rho, \sigma) = (0.07, 0.95, 0.25, 0.4, 0.25) \tag{2.27}$$

We display in the figure (2.3) below both: SVI with arbitrage using the parameters above. The new SVI parameters with arbitrage free after calibration are

$$(a, b, m, \rho, \sigma) = (0.26, 0.638, -0.0164, 0.2535, 0.0328) \tag{2.28}$$

By analyzing the two examples, we can see the performance of our algorithm to eliminate the butterfly arbitrage, and to keep the SVI model as close as possible (best fit) to the input data.

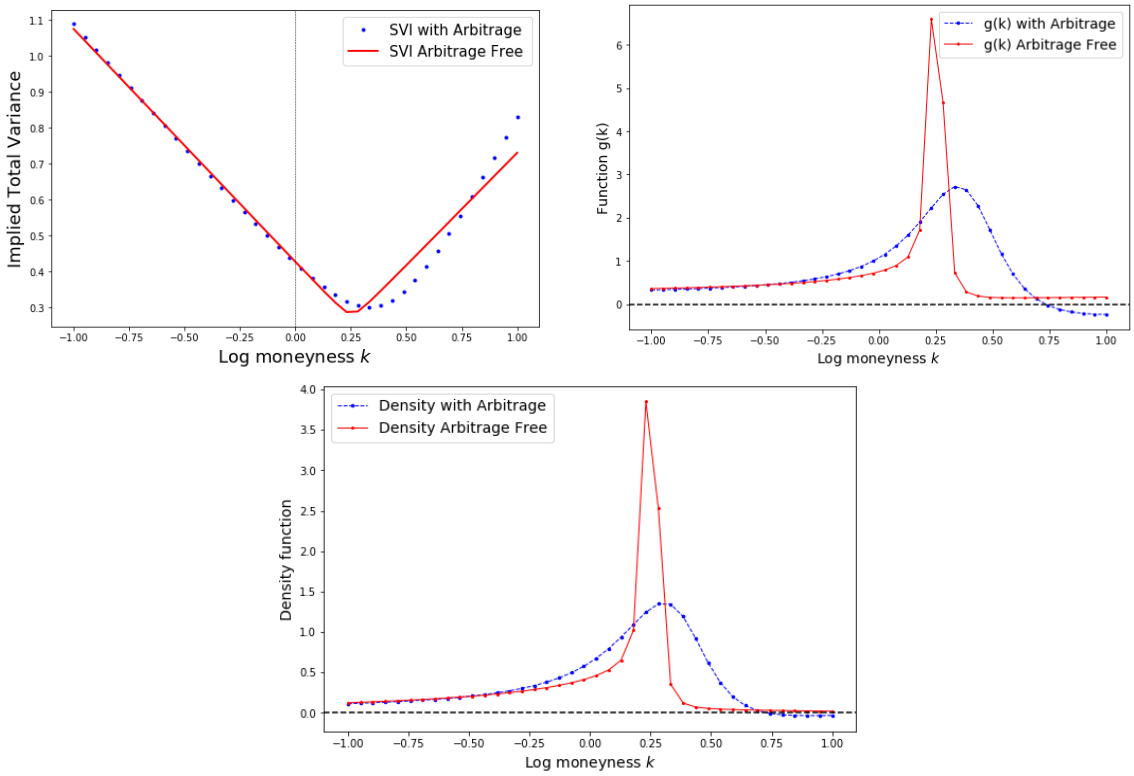

Figure 2.3: arbitrage example 02: plots of the total variance (left), the function g(k) (right), and the density function (down) with and without arbitrage

2.5.2 Equity Indexes Calibration

After testing our algorithm in two synthetic arbitrage examples, in this section we will test the performance of our calibration method using real market data.The input data is the options prices (call and put) listed on 23 indexes such as: EURO STOXX 50, CAC 40, NIKKEI 225, FTSE Mid 250 Index, SWISS MARKET IND, Hang Seng, NASDAQ 100, FTSE 100, MSCI world TR Index, Sao Paulo SE Bovespa Index,...etc. Our algorithm performs well with the 23 indexes, using the same starting initial guess for the SVI parameters, and keeping the same boundaries as described in (2.2.2).
We display the calibration result for one slice with

$$\chi_R = (a^* = 0.01, b^* = 0.07, \rho^* = 0.43, m^* = 0.11, \sigma^* = 0.12)$$

. After calibration, we obtained the total variance model in the figure (2.4).

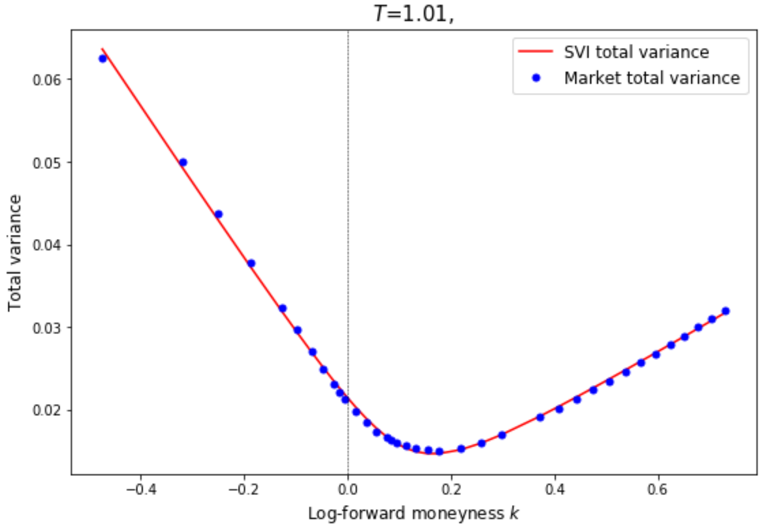

Figure 2.4: SVI fit from least-squares (T=1.01Y : 2019 04 05 to 2020 04 06)

Strike k	Call	Market Total Variance	SVI Model Total variance
2068.48	1268.59	0.06249	0.06361
2413.23	936.83	0.050	0.04935
2757.98	621.99	0.03780	0.03720
3016.54	407.05	0.02964	0.02932
3585.37	77.18	0.01662	0.01674
3964.59	15.1	0.01501	0.01470
4481.71	1.87	0.01694	0.01700
4998.83	0.35	0.02018	0.02037
5688.33	0.05	0.02462	0.02479
6033.07	0.02	0.02678	0.02688
6377.82	0.01	0.02892	0.02887
6722.57	0.1	0.0310	0.03077
6894.94	0	0.03207	0.03169

Table 2.2: SVI fit for EURO STOXX 50 implied volatility (T=1.01Y : 2019 04 05 to 2020 04 06)

Now we display in the figure (2.5) above the calibration of our 14 slices maturities, using the implied volatility of the call options listed on date of 05, April 2019, from the Euro Stoxx 50 index. We calculate for each slice calibration the optimal parameters model and the mean square error MSE.
The analyze of the SVI model fit (slices) shows that our calibration method is robust and it fits very well as for the short maturity (one week) or to the large maturities (8-10 years).

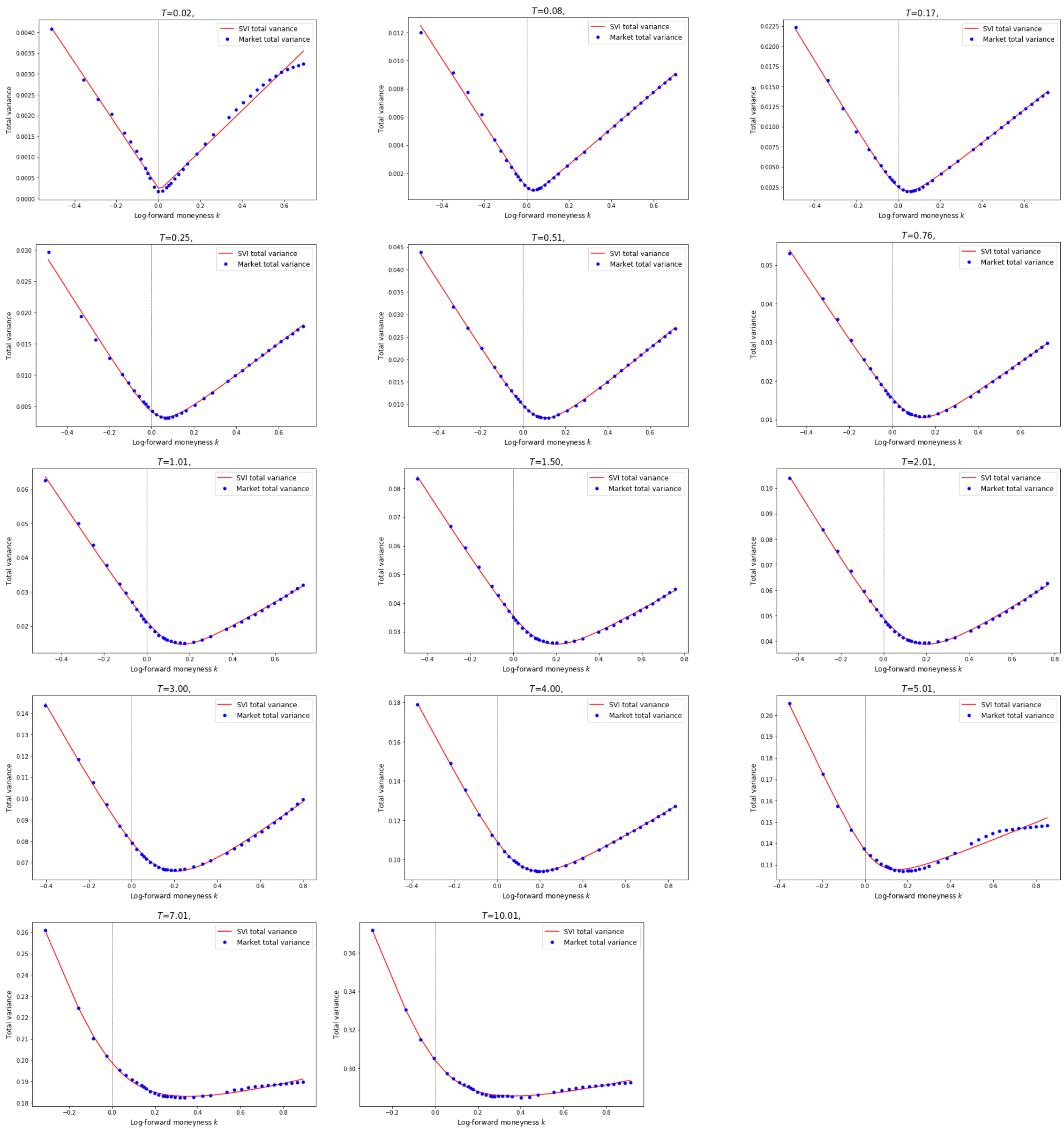

Figure 2.5: Plots of the SVI fits for the Euro Stoxx 50 implied volatility on April 05, 2019.

2.5.3 Multi-Slices SVI Calibration

In this section, we present the SVI calibration for multi slices. We apply this method to calibrate the SVI model in both axes: strikes and maturities. This calibration respect the SVI's boundaries and mostly both type of arbitrage: butterfly (for the strike axis) and calendar spread (in the maturity axis).

We present the objective function $f(k;\chi_R)$ as previously, where $\chi_R = \{a, b, \rho, m, \sigma\}$ is the set of the model's parameters, for an expiry time fix T.

$$f(k;\chi_R) = \sum_{i=1}^{n} \left(\omega_{SVI(i)}^{model} - \omega_{Total(i)}^{market}\right)^2 \tag{2.29}$$

$$f(k;\chi_R) = \sum_{i=1}^{n} \left[a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\} - \omega_{Total(i)}^{market}\right]^2 \tag{2.30}$$

Where $k_i := \log\left(\frac{K_i}{F_T}\right)$

The non linear problem (NLP) is defined as

$$\begin{cases} (\textbf{NLP}) : \min_{x \in R^5} f(k;\chi_R) \\ a_d \le a \le a_u \\ b_d \le b < b_u \\ \rho_d < \rho < \rho_u \\ m_d \le m \le m_u \\ \sigma_d \le \sigma \le \sigma_u \\ g(k;\chi_R) > 0 \\ \partial_T w(k,T) > \epsilon, \quad \forall k \in \mathbb{R}, T > 0 \end{cases} \tag{2.31}$$

Like for the slice, the *butterfly arbitrage constraint* $g(k;\chi_R) > 0$ should to be remplaced by the following conditions;

- $\big(a - mb(\rho+1)\big)\big(4 - a + mb(\rho+1)\big) - b^2(\rho+1)^2 > 0$
- $\big(a - mb(\rho-1)\big)\big(4 - a + mb(\rho-1)\big) - b^2(\rho-1)^2 > 0$
- $0 < b^2(\rho+1)^2 < 4$
- $0 < b^2(\rho-1)^2 < 4$

In fact this calibration is similar to the previous one, however, we incorporate another constraint that insure to avoid calendar spread arbitrage.In practice, we start the calibration with SVI slice corresponding to the lowest maturity and we move up to the next maturity respecting the constraint of non crossing slices as below;

$$\begin{cases} w(k,T_0) > 0 \\ w(k,T_1) > w(k,T_0) \\ \ldots \\ w(k,T_i) > w(k,T_{i-1}) \quad 1 \le i \le n \end{cases} \tag{2.32}$$

Fallowing this procedure, we can guarantee to get an SVI calibration with butterfly and calendar spread arbitrage free in the same time by running only one calibration for all the slices. Moreover, even if there is an arbitrage (butterfly or calendar spread) in our input data, the calibration method can avoid these arbitrages and correct the input data.

Let's show how our calibration method perform in the real market. We calibrate the SVI model using the implied volatility extracted from options listed in three indexes: *SP ASX 200* listed on the Australian Securities Exchange, the index *DJ Stoxx 600 Utilities Rt Inde* and Swiss Market Index *Swiss Market Ind* on date of April 5,2019.

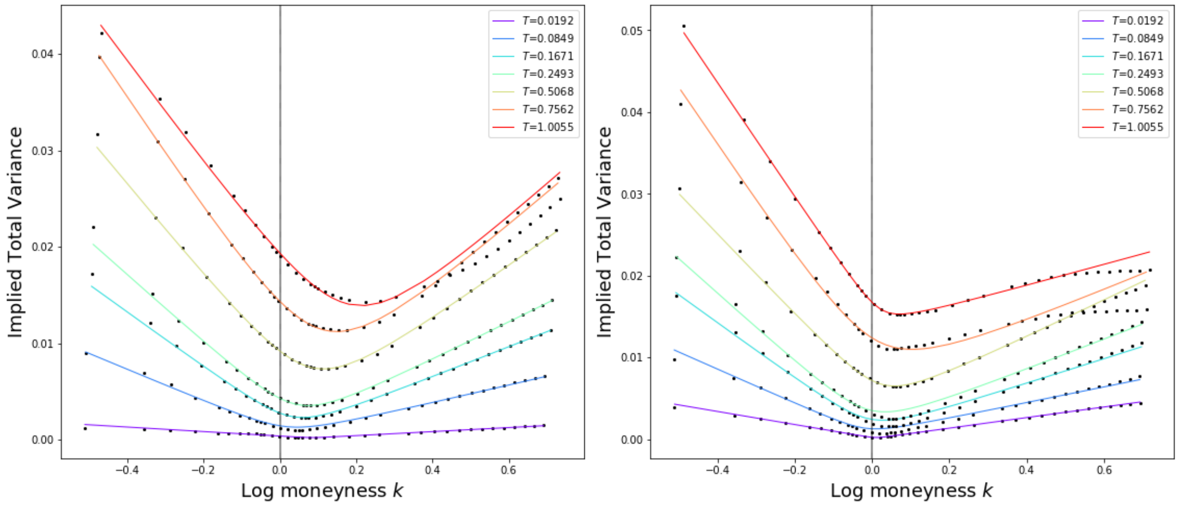

Figure 2.6: SVI calibration arbitrage free (butterfly and calendar spread) for (left: DJ Stoxx 600 Utilities Rt Inde) and (right: SPA SX200 index) on April 5th,2019.Dots are implied total variance input data and continue line is SVI model

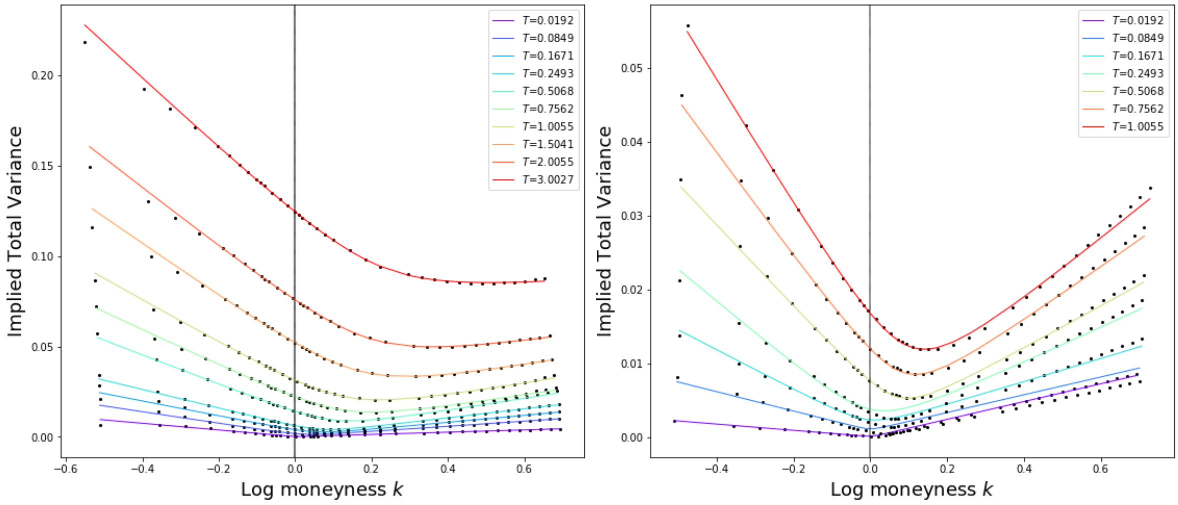

Figure 2.7: SVI calibration for the index (left: NASDAQ 100) and (right: Swiss Market Ind), with both butterfly and calendar spread arbitrage free.Dots are implied total variance input data and continue line is SVI model

We note in both figures above (2.6) ,for (left: DJ Stoxx 600 Utilities Rt Inde), (right: SPA SX200 index) and (down right: Swiss Market Ind), the crossing in the input data with dots which is mean that our input data incorporate the calendar spread arbitrage for many maturities.

We succeed to eliminate both arbitrages: calendar spread and butterfly during the calibration step. Our SVI's parameters are arbitrage free. The continues lines represent the fit of the SVI model, and we note that this lines are separated.

Conclusion

In this paper we studied the Stochastic Volatility Inspired model (SVI) as implied volatility model: we started by the analytic part of the SVI model and next, we established the characterization of static arbitrage (calendar spread and butterfly).

The SVI model had the arbitrage problem: no calibration method was able to grantee an SVI calibration arbitrage-free. The main contribution of this paper is that we provided sufficient conditions that guarantee an SVI static arbitrage-free. These conditions ensure that the probability density function will remain positive. that allows automatic elimination of arbitrage (butterfly and calendar spread) during the calibration which was not possible until recently.

We provided the SVI's parameters boundaries and the initial guess. We illustrated the performance of our algorithm in some synthetic numerical examples with arbitrage and we showed how to fix them. Moreover, in order to test the performance of our calibration method, we applied the method to calibrate the implied volatility for 23 indexes with 14 maturities each (322 slices).

The calibration method is robust and easy to implement. It performs even when the input data contain arbitrage. The model's calibration corrects the input data coming from the market and it fits very well the variance of the equity market.

Using this calibration method, we can interpolate on the maturity axis in order to get the implied volatility surface, and next we can use Dupire's formula to get the local volatility surface in terms of the total variance SVI (the implied volatility). This is very useful to well price different types of path dependant options such as barrier options, American options.

We can also use the method to calibrate the SVI model in the FX market and also to price different interest rates derivatives such as swaptions, cap, and floor. The advantage of this SVI calibration is that our result is guarantee arbitrage-free, and it fits well the input data comparing to SABR model.

Appendix A

SVI Calibration Using Indexes

Remark:the calibration method is tested for 23 indexes, each index with 14 slices. We display here only some examples that we selected to show the performance of the method.

DJ Stoxx 600 Utilities Rt Inde

Forward	Maturity (Year)	$\chi_R = \{a, b, \rho, m, \sigma\}$	MSE
317.91	0.01918	(0.00017, 0.0022, -0.057, 0.084, 0.018)	$7.126373\ 10^{-9}$
317.28	0.08493	(0.000195, 0.013, -0.285, 0.01884, 0.088)	$2.521449\ 10^{-8}$
312.42	0.16712	(0.00059, 0.022, -0.291, 0.03397, 0.0798)	$5.740408\ 10^{-8}$
311.08	0.2493	(0.000988, 0.0276, -0.2845, 0.04493, 0.0959)	$9.921849\ 10^{-8}$
307.99	0.50685	(0.00252, 0.03868, -0.2557, 0.08039, 0.1286)	$6.084056\ 10^{-8}$
306.32	0.75616	(0.00397, 0.0474, -0.2664, 0.1056, 0.1581)	$5.838878\ 10^{-8}$
304.86	1.00548	(0.006786, 0.045, -0.204, 0.18, 0.161)	$1.800535\ 10^{-6}$

Table A.1: SVI fit for DJ Stoxx 600 Utilities Rt Inde

SP ASX200

Forward	Maturity (Year)	$\chi_R = \{a, b, \rho, m, \sigma\}$	MSE
6179.10	0.01918	(0.0001195, 0.00724, -0.106, 0.0095, 0.01)	$4.929172\ 10^{-9}$
6184.80	0.08493	(0.000421, 0.01532, -0.3705, 0.01483, 0.059)	$1.242112\ 10^{-7}$
6158.70	0.16712	(0.000508, 0.0248, -0.3878, -0.00615, 0.0792)	$1.96188\ 10^{-7}$
6153.80	0.2493	(0.00072, 0.03068, -0.39244, -0.00623, 0.0929)	$1.98044\ 10^{-8}$
6098.80	0.50685	(0.00370, 0.03628, -0.3727, 0.02393, 0.08299)	$5.999724\ 10^{-8}$
6081.50	0.75616	(0.005814, 0.04559, -0.559, 0.01207, 0.1367)	$2.38178\ 10^{-6}$
6031.80	1.00548	(0.013815, 0.04155, -0.688, 0.02323, 0.049)	$4.317087\ 10^{-7}$

Table A.2: SVI fit for SP ASX200

NASDAQ 100

Forward	Maturity (Year)	$\chi_R = \{a, b, \rho, m, \sigma\}$	MSE
7576.20	0.01918	(.0000867, 0.01237, -0.4957, -0.00006, 0.01)	$2.58162\ 10^{-7}$
7588.40	0.08493	(0.000343, 0.023149, -0.3847, 0.02053, 0.05065)	$4.351337\ 10^{-7}$
7592.70	0.16712	(0.0010962, 0.030378, -0.35853, 0.05138, 0.056567)	$5.556376\ 10^{-7}$
7603.20	0.2493	(0.0018707, 0.03853, -0.34163, 0.06739,0.0596)	$1.505823\ 10^{-7}$
7633.50	0.50685	(0.00375, 0.05806, -0.41078, 0.1012, 0.0949)	$2.456222\ 10^{-8}$
7663.90	0.75616	(0.00622, 0.06635, -0.495, 0.1250, 0.1292)	$3.056964\ 10^{-7}$
7688.20	1.00548	(0.010716, 0.07854, -0.5305, 0.12891, 0.145812)	$7.034514\ 10^{-7}$
7741.10	1.50411	(0.02077, 0.09305, -0.590, 0.16763, 0.16816)	$3.553784\ 10^{-6}$
7790.60	2.00548	(0.03656, 0.099372, -0.6740, 0.1933, 0.17998)	$4.490482\ 10^{-6}$
7889.20	3.00274	(0.072781, 0.1087, -0.8217, 0.21717, 0.145812)	$7.034514\ 10^{-7}$

Table A.3: SVI fit for NASDAQ 100

SWISS MARKET IND

Forward	Maturity (Year)	$\chi_R = \{a, b, \rho, m, \sigma\}$	MSE
9535.40	0.01918	(0.01918, 0.000092891,0.008128150, 0.48069,0.00471,0.01)	1.74004 10^{-8}
9469.90	0.08493	(0.000946983,0.012566,-0.04130, 0.00013, 0.016016)	1.447122 10^{-6}
9410.20	0.16712	(0.00107,0.02143,-0.26564,-0.00578, 0.06189)	5.02672 10^{-7}
9403.60	0.2493	(0.00026,0.03399,-0.28963,0.00492,0.10285)	5.877006 10^{-7}
9373.50	0.50685	(0.002297,0.042255,-0.314,0.072,0.0747)	1.72667 10^{-7}
9352.50	0.75616	(0.003515,0.054161,-0.314,0.0853,0.09741)	2.388377 10^{-7}
9219.70	1.00548	(0.00467,0.06437,-0.335,0.09863,0.119623)	2.61608 10^{-7}

Table A.4: SVI fit for SWISS MARKET IND

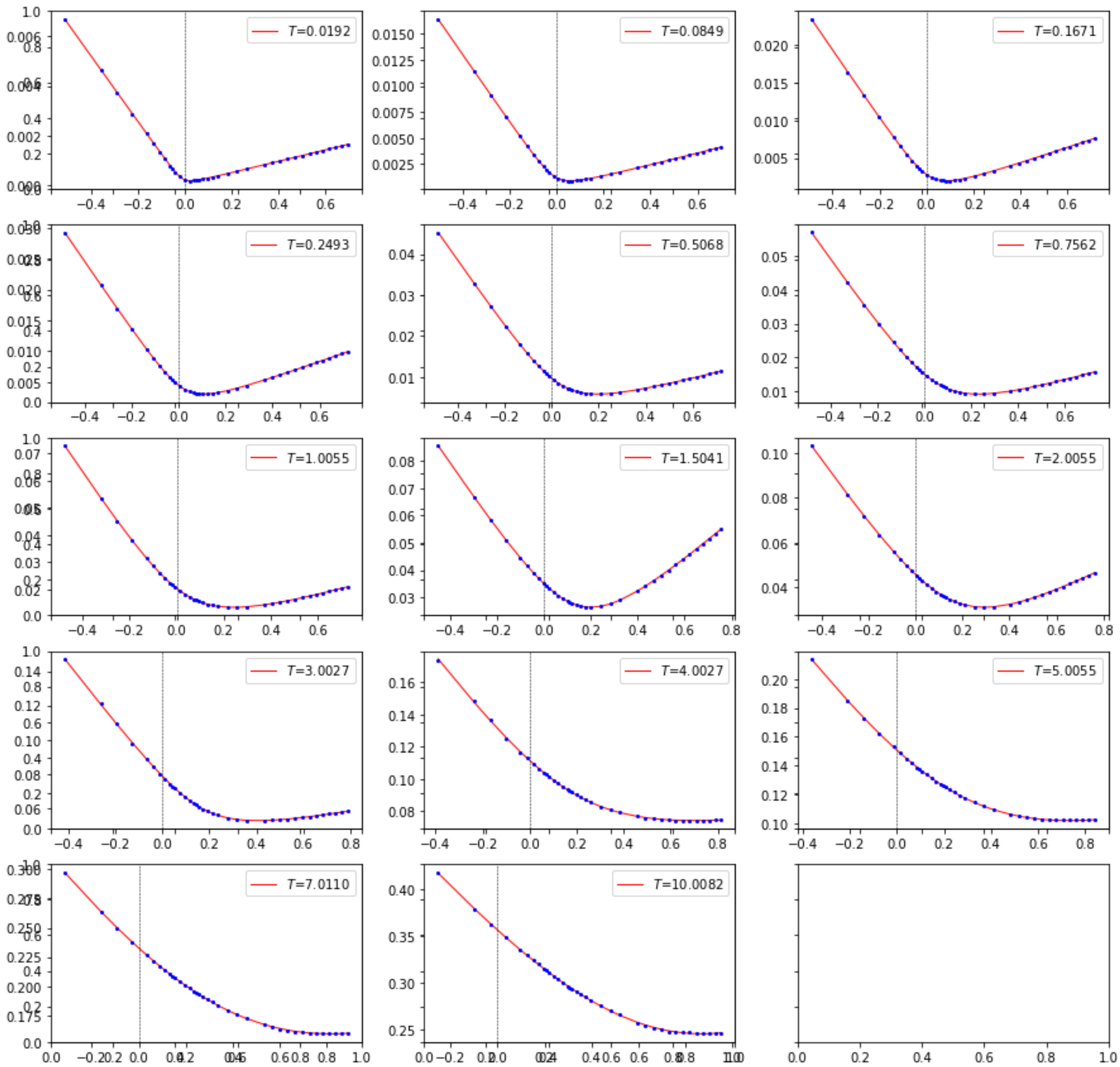

Figure A.1: SVI calibration for the CAC 40 Index implied volatility listed on date April 05, 2019

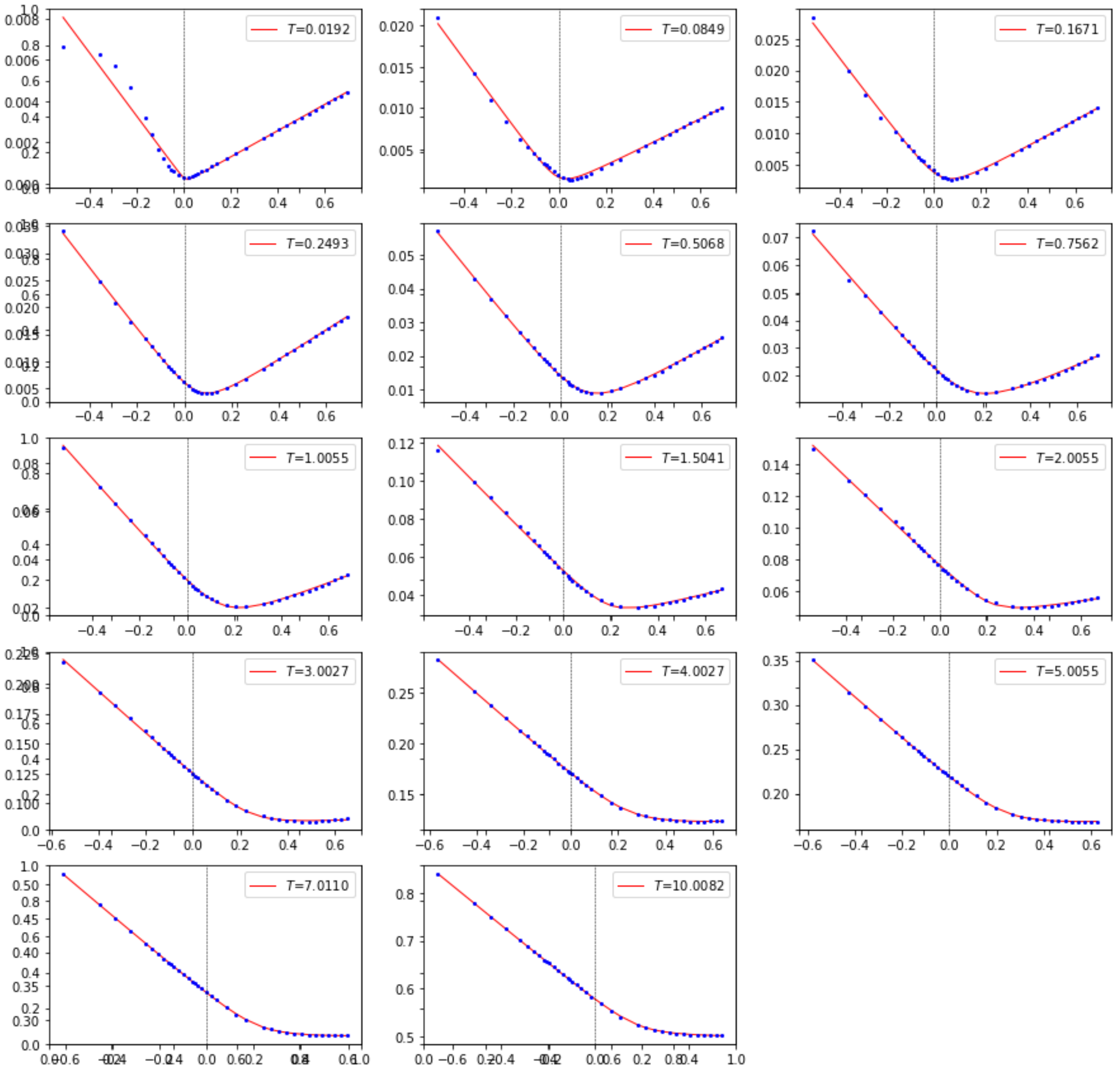

Figure A.2: SVI fits for the NASDAQ 100 Index implied volatility listed on date April 05, 2019

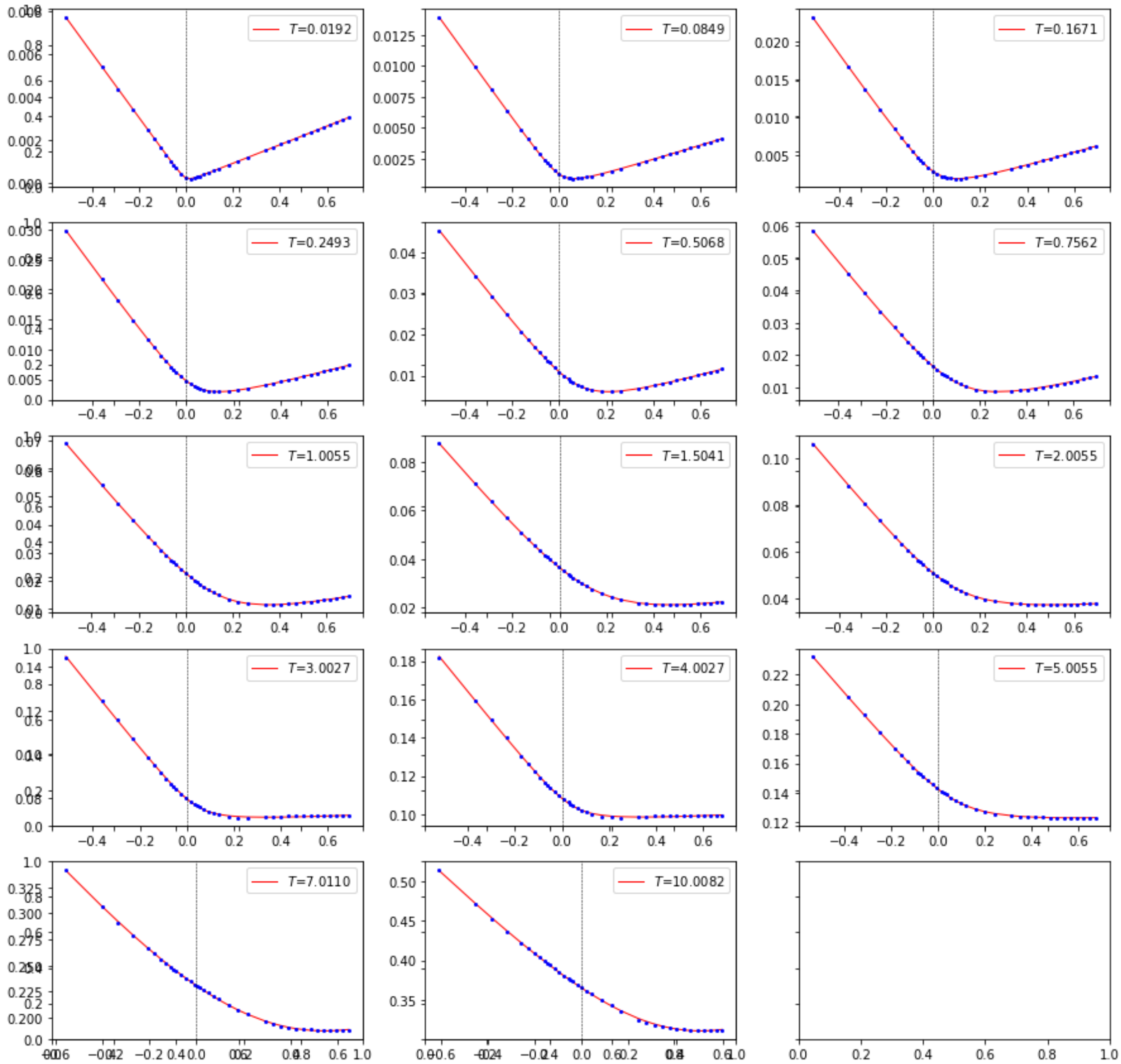

Figure A.3: SVI fits for the DAX 30 Index implied volatility listed on date April 05, 2019

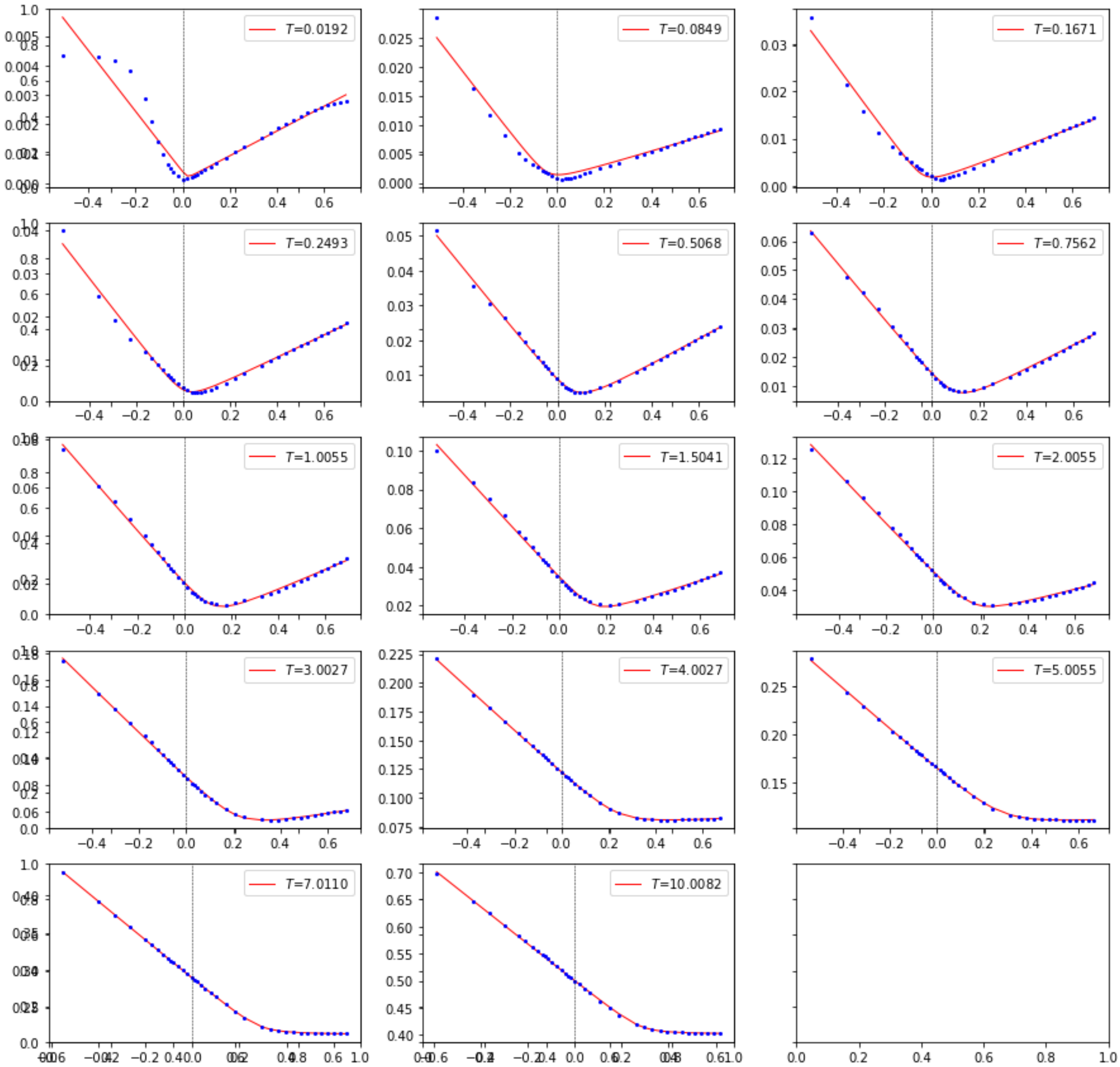

Figure A.4: SVI fits for the S&P500 Index implied volatility listed on date April 05, 2019

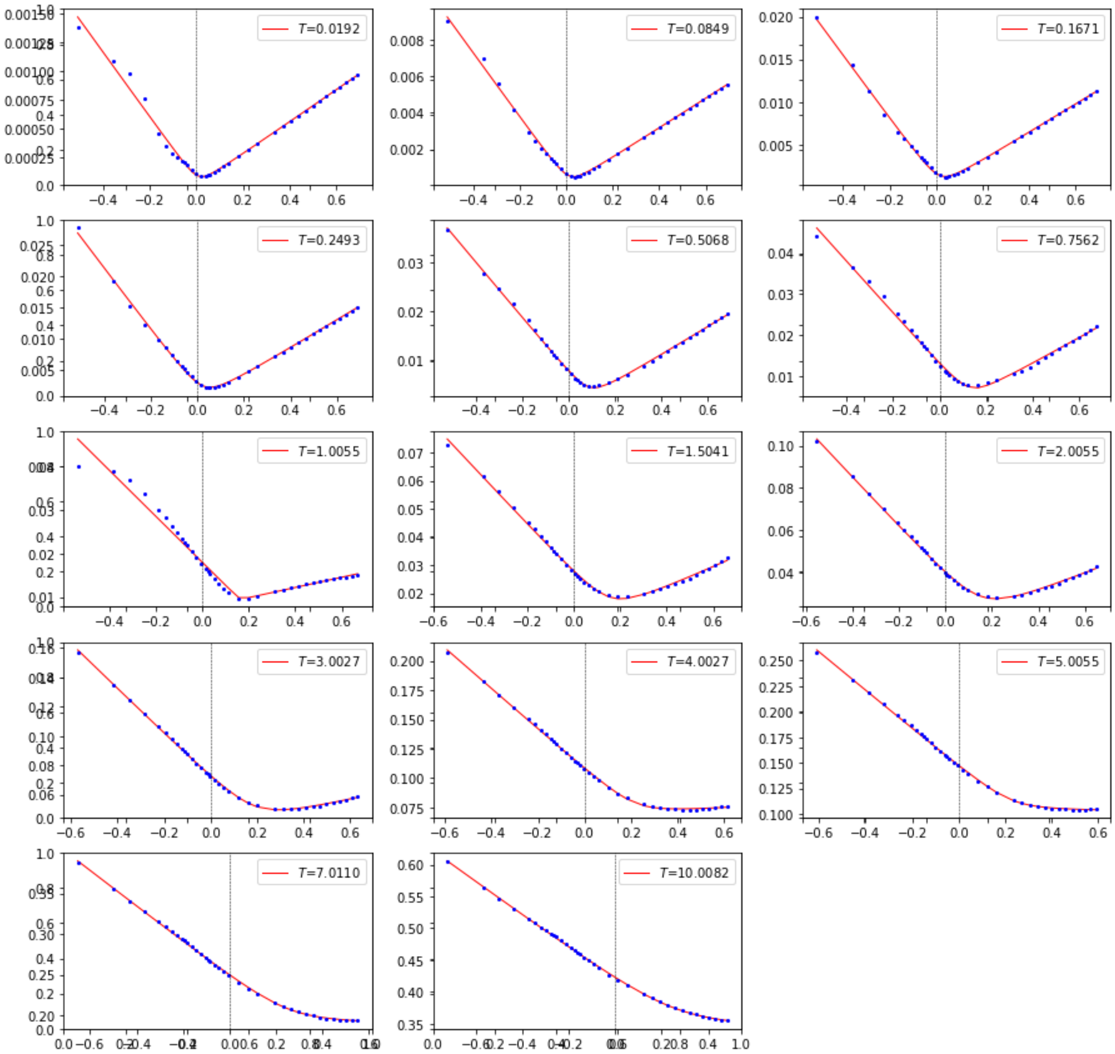

Figure A.5: SVI fits for the MSCI world TR Index implied volatility listed on date April 05, 2019

Bibliography

[1] Bruno Dupire. Pricing with a smile. 1994.

[2] Emanuel Derman and Iraj Kani. Riding on a smile. *Risk*, 7, 01 1994.

[3] Nabil Kahale. An arbitrage-free interpolation of volatilities. *Risk*, 17, 04 2003.

[4] Wolfgang Karl Härdle, M Benko, Matthias Fengler, and Milos Kopa. On extracting information implied in options. *Computational Statistics*, 22:543–553, 02 2007.

[5] Matthias Fengler. Arbitrage-free smoothing of the implied volatility surface. *Quantitative Finance*, 9(4):417–428, June 2009.

[6] Jesper Andreasen and Brian Huge. Volatility interpolation. *Risk Magazine*, 03 2010.

[7] Judith A. Glaser and P. Heider. Arbitrage-free approximation of call price surfaces and input data risk. 2012.

[8] Matthias Fengler and Lin-Yee Hin. Semi-nonparametric estimation of the call price surface under strike and time-to-expiry no-arbitrage constraints. Economics Working Paper Series 1136, University of St. Gallen, School of Economics and Political Science, September 2011.

[9] Jim Gatheral and Antoine Jacquier. Arbitrage-free SVI volatility surfaces. *arXiv e-prints*, page arXiv:1204.0646, Apr 2012.

[10] Jim Gatheral and Antoine Jacquier. Convergence of Heston to SVI. *arXiv e-prints*, page arXiv:1002.3633, Feb 2010.

[11] Roger W. Lee. The moment formula for implied volatility at extreme strikes. volume 14.3, pages 469–480, 2004.

[12] Gaoyue Guo, Antoine Jacquier, Claude Martini, and Leo Neufcourt. Generalised arbitrage-free SVI volatility surfaces. *arXiv e-prints*, page arXiv:1210.7111, Oct 2012.

[13] Peter Carr and Dilip B. Madan. A note on sufficient conditions for no arbitrage. 2005.

[14] Fabrice Douglas Rouah. Using the risk neutral density to verify no arbitrage in implied volatility. *http://www.frouah.com/*.

[15] Douglas T. Breeden and Robert H. Litzenberger. Prices of state-contingent claims implicit in option prices. *The Journal of Business*, 51(4):621–651, 1978.

[16] Peter Jäckel. Clamping down on arbitrage. *Wilmott*, 2014(71):54–69.

[17] Zeliade White and Millard B. Stahle. Quasi-explicit calibration of gatheral ' s svi model. 2009.