



OpenGamma Quantitative Research

**Convexity adjustment for
inflation derivatives**

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Abstract

The purpose of this document is to present analytical formulas for the convexity adjustment within the year-on-year inflation swap and the zero-coupon inflation swap with payment delay. This adjustment is also used in the calculation of the forward for inflation-linked options. The framework used in this note is the forward price index market model. This technical note is complementary to [Zine-eddine \(2013\)](#), and consequently reuses the same notation and definition, particularly for the price index and the forward price index.

Contents

1	Introduction	1
2	Model description: the forward price index market model	1
3	The convexity adjustment for an upfront Year-on-Year cash flow	2
4	The convexity adjustment for zero-coupon cash flow with a payment delay	3
5	The convexity adjustment for a Year-on-Year cash flow with a payment delay	4
6	The volatility of the bond forward	4
7	Implementation	5
7.1	Market data	5
7.2	Implementation in Opengamma Analytics Library	5
7.3	Numerical example	5

1 Introduction

The nominal rates refer to the habitual rates, and real rates refer to the rates adjusted with the inflation. In the real economy, prices are defined in terms of purchasing power (real price) and not in nominal price.

A relation between nominal rates (n), real rates (r), and inflation (i) is given by the well-known Fisher equation:

$$1 + n = (1 + r)(1 + i)$$

An analogy can be made with the foreign currency market: real rates can be interpreted as a foreign asset, nominal as domestic asset and exchange rate as the inflation price index.

We will use the subscript n (respectively r) when a quantity refers to nominal economy (respectively the real economy).

We assume there is a discount curve in both economies $t \mapsto P_x(t_0, t)$ for $x \in \{n, r\}$ and $t_0 \leq t$. For $i = 0 \dots N$, we define a schedule T_i such as $T_0 < T_1 < \dots < T_N$. The related forward Libor rates for the two economies can be computed from the related discount bond prices for $i = 1 \dots N$:

$$L_{t,x}^i = L_x(t, T_{i-1}, T_i) = \frac{1}{\tau_i} \left(\frac{P_x(t, T_{i-1})}{P_x(t, T_i)} - 1 \right), \quad x \in \{n, r\},$$

where τ_i is the day count fraction between T_{i-1} and T_i . The nominal and real risk-neutral measures are denoted by $\mathbb{Q}_n, \mathbb{Q}_r$, and the price index is noted $I(t)$.

The forward price index at time t for maturity T_i is defined by

$$\mathcal{I}(t, T_i) = I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)} \quad (1)$$

In this note, we assume we have a forward price index curve ($T \mapsto \mathcal{I}(t, T), T \geq t$). This curve is constructed with standard instruments (usually inflation-linked zero-coupons without payment delay), and we want to price non-standard instruments using this curve. These non-standard instruments are year-on-year swaps (with or without payment delay) and zero-coupon swaps with payment. The computation of forwards for inflation-linked options (caps and floors) uses the same methodology.

2 Model description: the forward price index market model

Here we will give a brief description of the market model used for inflation-linked instruments. A more complete description and a study of this model can be found in [Belgrade et al. \(2004\)](#). By definition, the forward CPI is a martingale under its natural measure, the T_i -forward neutral measure $\mathbb{Q}_n^{T_i}$. Observing that, we also make the assumption of a lognormal distribution under $\mathbb{Q}_n^{T_i}$:

$$d\mathcal{I}(t, T_i) = \sigma_i^I \mathcal{I}(t, T_i) dW_i^I(t),$$

where the volatility σ_i^I is a positive constant and where $W_i^I(t)$ is a standard Brownian motion under $\mathbb{Q}_n^{T_i}$.

For $i, j = 1 \dots N$ we also define $\rho_{i+1,i}^I$ as the instantaneous correlation between $\mathcal{I}(t, T_i)$ and $\mathcal{I}(t, T_j)$, so we have by definition the following relationship

$$\rho_{i,j}^I dt = dW_i^I(t) dW_j^I(t)$$

Of course for each i, j , we have the following relation $\rho_{i,i}^I = 1$ and $\rho_{i,j}^I = \rho_{j,i}^I$.

3 The convexity adjustment for an upfront Year-on-Year cash flow

Let's consider the following case: a year-on-year inflation-linked flow $\frac{I(T_{i+1})}{I(T_i)}$ with no payment delay, which means being paid at T_{i+1} . According to the classic pricing theory, the price of this cash flow at the date t is

$$P_n(t, T_{i+1}) \mathbb{E}_n^{T_{i+1}} \left[\frac{I(T_{i+1})}{I(T_i)} | \mathcal{F}_t \right]$$

The problem is that $\frac{\mathcal{I}(t, T_{i+1})}{\mathcal{I}(t, T_i)}$ is not a martingale under the measure naturally linked to the payment date, the T_{i+1} -forward neutral measure. So we have the following result

$$\mathbb{E}_n^{T_{i+1}} \left[\frac{I(T_{i+1})}{I(T_i)} | \mathcal{F}_t \right] = \frac{\mathcal{I}(t, T_{i+1})}{\mathcal{I}(t, T_i)} \exp \{ \theta_{i+1}(t) \}, \quad (2)$$

where

$$\theta_{i+1}(t) = \sigma_i^I \left(\sigma_i^I - \sigma_{i+1}^I \rho_{i+1,i}^I - \rho_{B^f, I} \sigma_{T_i, T_{i+1}}^{B^f} \right) (T_i - t),$$

where $\sigma_{T_i, T_{i+1}}^{B^f}$ is the volatility of $B_i^{fwd} = \frac{P_n(t, T_{i+1})}{P_n(t, T_i)}$, and where $\rho_{i,i}^{B^f, I}$ is the instantaneous cross correlation between the forward price index $\mathcal{I}(T_i)$ and B_i^f .

Demonstration 1 *Some direct calculation (using the tower property) gives us*

$$\mathbb{E}_n^{T_{i+1}} \left[\frac{I(T_{i+1})}{I(T_i)} | \mathcal{F}_t \right] = \mathbb{E}_n^{T_{i+1}} \left[\frac{\mathcal{I}(T_i, T_{i+1})}{\mathcal{I}(T_i, T_i)} | \mathcal{F}_t \right] \quad (3)$$

We must determine the drift of the forward $\mathcal{I}_i(t)$ under the T_{i+1} -forward neutral measure. We will use the Radon Nikodym derivative

$$\begin{aligned} \frac{d\mathbb{Q}_n^{T_{i+1}}}{d\mathbb{Q}_n^{T_i}} \Big|_t &= \frac{P_n(t, T_{i+1}) P_n(0, T_i)}{P_n(t, T_i) P_n(0, T_{i+1})} \\ &= \frac{1 + \tau_i L_n(t, T_i, T_{i+1})}{1 + \tau_i L_n(0, T_i, T_{i+1})}. \end{aligned}$$

We remember that the Radon Nykodym derivative is the only stochastic process such that for each stochastic process $(X_t)_{(t \geq 0)}$ adapted to the filtration $(F_t)_{t \geq 0}$, for every $T \geq t$

$$\mathbb{E}_n^{T_{i+1}} [X_T | \mathcal{F}_t] = \mathbb{E}_n^{T_i} \left[X_T \frac{d\mathbb{Q}_n^{T_{i+1}}}{d\mathbb{Q}_n^{T_i}} \Big|_T \Big| \mathcal{F}_t \right]$$

So using the following notation, ρ_{I, B_i^f} is the cross correlation between the forward CPI $\mathcal{I}(t, T_i)$ and the value of the forward bond $B_i^{fwd} = \frac{P_n(t, T_{i+1})}{P_n(t, T_i)}$ and where $\sigma_{T_i, T_{i+1}}^{B^f}$ is the instantaneous volatility

of the forward bond. This last volatility comes from the fact that we assume that the forward bond B_i^{fwd} is lognormal under $\mathbb{Q}_n^{T_i}$. Some calculations give the following diffusions under $\mathbb{Q}_n^{T_{i+1}}$:

$$\begin{aligned} d\mathcal{I}(t, T_{i+1}) &= \mathcal{I}(t, T_{i+1}) \sigma_{i+1}^I dW_{i+1}^I(t) \\ d\mathcal{I}(t, T_i) &= \mathcal{I}(t, T_i) \sigma_i^I \sigma_{T_i, T_{i+1}}^{B^f} \rho_{i,i}^{I, B^f} dt + \mathcal{I}_i(t) \sigma_i^I dW_{i+1, i}^I(t) \end{aligned}$$

where $W_{i+1, i}^I(t)$ is a brownian motion under $\mathbb{Q}_n^{T_{i+1}}$. Then the Ito derivative of the ratio of the forward prices index is the following:

$$\begin{aligned} d \left(\frac{\mathcal{I}_{i+1}(t)}{\mathcal{I}_i(t)} \right) &= \frac{\mathcal{I}_{i+1}(t)}{\mathcal{I}_i(t)} \left[\sigma_{I, i+1} dW_{I, i+1}(t) - \sigma_i^I dW_{I, i}(t) \right. \\ &\quad \left. + \left(-\sigma_i^I \sigma_{T_i, T_{i+1}}^{B^f} \rho_{i,i}^{I, B^f} + (\sigma_i^I)^2 - \sigma_i^I \sigma_{i+1}^I \rho_{i, i+1}^I \right) dt \right], \end{aligned}$$

Then we just have to take the expectation to have the result.

4 The convexity adjustment for zero-coupon cash flow with a payment delay

Let's consider the following case: a zero-coupon inflation-linked flow $\frac{I(T_i)}{I(T_0)}$ with a payment delay, which means being paid at $T > T_i$. The problem is that $\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_0)}$ is not a martingale under the measure naturally linked to the payment date, the T -forward neutral measure. So we have the following result

$$\mathbb{E}_n^T \left[\frac{I(T_i)}{I(T_0)} | \mathcal{F}_t \right] = \frac{\mathcal{I}(t, T_i)}{I(T_0)} \exp \{ \theta_i(t) \}, \quad (4)$$

where

$$\theta_i(t) = \sigma_i^I \sigma_{T_i, T}^{B^f} \rho_{i,i}^{B^f, I} (T_i - t)$$

Demonstration 2 The method is the same as for the year-on-year swap; we will have to use change of numeraire techniques. More precisely, we know the diffusion of $\mathcal{I}(t, T_i)$ under the T_i -forward neutral measure $\mathbb{Q}_n^{T_i}$ and we have to determine it under the T -forward neutral measure \mathbb{Q}_n^T . Once again, we will use the Radon Nikodym derivative

$$\begin{aligned} \frac{d\mathbb{Q}_n^T}{d\mathbb{Q}_n^{T_i}} \Big|_t &= \frac{P_n(t, T) P_n(0, T_i)}{P_n(t, T_i) P_n(0, T)} \\ &= \frac{1 + \tau_i F_n(t, T_i, T)}{1 + \tau_i F_n(0, T_i, T)}. \end{aligned}$$

Then some calculation give us the diffusions of $\mathcal{I}(t, T_i)$ under the T -forward neutral measure

$$d\mathcal{I}(t, T_i) = \mathcal{I}_i(t) \sigma_i^I \sigma_{T_i, T}^{B^f} \rho_{i,i}^{B^f, I} dt + \mathcal{I}_i(t) \sigma_i^I dW_{T, i}^I(t)$$

where $W_{T, i}^I(t)$ is a brownian motion under \mathbb{Q}_n^T .

Then we simply have to take the expectation of this previous diffusion to obtain the result.

5 The convexity adjustment for a Year-on-Year cash flow with a payment delay

Let's consider the following case: a year-on-year inflation-linked flow $\frac{I(T_{i+1})}{I(T_i)}$ with a payment delay, which means being paid at $T > T_{i+1}$.

This case is a combination of the two previous case, so we have the following result

$$\mathbb{E}_n^T \left[\frac{I(T_{i+1})}{I(T_i)} | \mathcal{F}_t \right] = \frac{\mathcal{I}(t, T_{i+1})}{\mathcal{I}(t, T_i)} \exp \{ \theta_{i+1}(t) \}, \quad (5)$$

where

$$\begin{aligned} \theta_{i+1}(t) &= \sigma_i^I \left(\sigma_i^I - \sigma_{i+1}^I \rho_{i+1,i}^I - \rho_{i,i}^{B^f,I} \sigma_{T_i,T}^{B^f} \right) (T_i - t) \\ &+ \sigma_{i+1}^I \sigma_{T_{i+1},T}^{B^f} \rho_{i+1,i+1}^{B^f,I} (T_{i+1} - t) \end{aligned}$$

6 The volatility of the bond forward

The bond forward volatility is estimated using Libor ATM Caplet market volatilities. Our goal is to compute the volatility of the bond forward $B^{fwd}(t, T_1, T_2) = \frac{P_n(t, T_2)}{P_n(t, T_1)}$. We denote δ the tenor of the standard caplet in the currency considered (for example 6 month for EUR), we denote n the biggest integer such that $n\delta \leq T_2 - T_1$, and for $i = 0 \dots n$, we denote $t_i = T_1 + i\delta$.

So we can express the bond forward as the following quantity:

$$\begin{aligned} B^{fwd}(t, T_1, T_2) &= \frac{P_n(t, T_2)}{P_n(t, T_{n\delta})} \frac{P_n(t, T_{n\delta})}{P_n(t, T_1)} \\ &= \frac{1}{(1 + \tau'_n L_n(t, t_n, T_2))} \prod_{i=0}^{n-1} \frac{1}{(1 + \tau_i L_n(t, t_i, t_{i+1}))} \end{aligned}$$

where τ'_n is the day count fraction between t_n and T_2 . Notice that in most practical cases $T_2 = t_n$, in which case the term $\frac{1}{(1 + \tau'_n L_n(t, t_n, T_2))}$ is equal to 1 so we don't need to take it into account. Assuming that the volatility component of the zero-coupon bond is lognormal, we have the following relation:

$$d \ln B^{fwd}(t, T_1, T_2) = d \ln \left(\frac{1}{(1 + \tau'_n L_n(t, t_n, T_2))} \right) + \sum_{i=0}^{n-1} d \ln \left(\frac{1}{(1 + \tau_i L_n(t, t_i, t_{i+1}))} \right)$$

We assume we are working in a black framework for libor caplet and we denote σ_j the black volatility of the standard atm libor caplet of maturity t_j , so by taking the square and integrating the previous relation, we have the following relation

$$\begin{aligned} (\sigma^{B^f})^2 (T_2 - t) &= \int_t^{T_2} (l'_n(u) \sigma_n)^2 \mathbb{1}_{u \leq T_2} du \\ &+ \int_t^{T_2} 2 l'_n \sigma_n \sum_{i=0}^{n-1} \rho_{j,n} \sigma_i l_i(u) \mathbb{1}_{u \leq t_i} du \\ &+ \int_t^{T_2} \sum_{i,j=0}^{n-1} \rho_{i,j} \sigma_i \sigma_j l_i(u) l_j(u) \mathbb{1}_{u \leq t_i} \mathbb{1}_{u \leq t_j} du \end{aligned}$$

where $l_j(t) = \frac{\tau_j L_n(t, t_j, t_{j+1})}{1 + \tau_j L_n(t, t_j, t_{j+1})}$, $l'_n(t) = \frac{\tau'_n L_n(t, t_n, T_2)}{1 + \tau'_n L_n(t, t_n, T_2)}$, and $\rho_{j,k}$ is the correlation between the two libor forward rates $L_n(t, t_j, t_{j+1})$ and $L_n(t, t_k, t_{k+1})$.

Above, we have assumed that $L_n(t, t_n, T_2)$ and $L_n(t, t_n, t_n + \delta)$ have the same volatility (notice that if $T_2 = t_n + \delta$, this assumption is not necessary).

Now, we will freeze in t each $l_j(t)$, which means $l_j(t) \approx l_j(0)$, we will denote them l_j . So the lognormal volatility of B^{fwd} is (approximately) equal to:

$$\left(\sigma^{B^f}\right)^2 (T_2 - t) = (l'_n \sigma_n)^2 (T_2 - t) + 2 l'_n \sigma_n \sum_{i=0}^{n-1} \rho_{i,n} \sigma_i l_i (t_i - t) + \sum_{i,j=0}^{n-1} \rho_{i,j} \sigma_i \sigma_j l_i l_j (t_i \wedge t_j - t)$$

And if we use the following notation for $i = 0 \dots n - 1$, $h_i = l_i$ and $h_n = l'_n$, we have this more compact expression

$$\left(\sigma^{B^f}\right)^2 (T_2 - t) = \sum_{i,j=0}^n \rho_{i,j} \sigma_i \sigma_j h_i h_j (t_i \wedge t_j - t)$$

where $t_i \wedge t_j = \min(t_i, t_j)$.

7 Implementation

7.1 Market data

Volatilities (ibor and price index) are retrieved from caps/floors market quotes; here we are here using only ATM volatilities. Correlations (between price indices and between price index and libors) are retrieved from historical data.

A parametrisation can be used to have smoother correlations surfaces. This parametrisation is usually the same as the one used for correlation in the libor market model.

7.2 Implementation in Opengamma Analytics Library

The convexity adjustment is implemented in the Opengamma analytics library in the class `InflationMarketModelConvexityAdjustment`, this implementation is then used within the pricing in each instruments.

7.3 Numerical example

The first example concerns zero-coupon inflation coupons. The first graph (figure 1) denotes the value of the convexity adjustment ($\theta_i(t)$ as defined in section 4) for a zero-coupon with different lags (ie payment delays). The second one is also the convexity adjustment but it is the relative value (ie the difference between the price with and without convexity adjustment) and the third is the price of the same coupons (see 4 for the formula)

We can see that this adjustment is always smaller than a basis point.

The following three graphs are the same but concern the year-on-year coupon without payment delays. Here we have calculated the convexity adjustment for coupons with different maturities (and not with different payment delays). This adjustment is also always smaller than a basis point. But concerning year-on-year swaps, we have to sum coupons, so the convexity adjustment cannot be negligible.

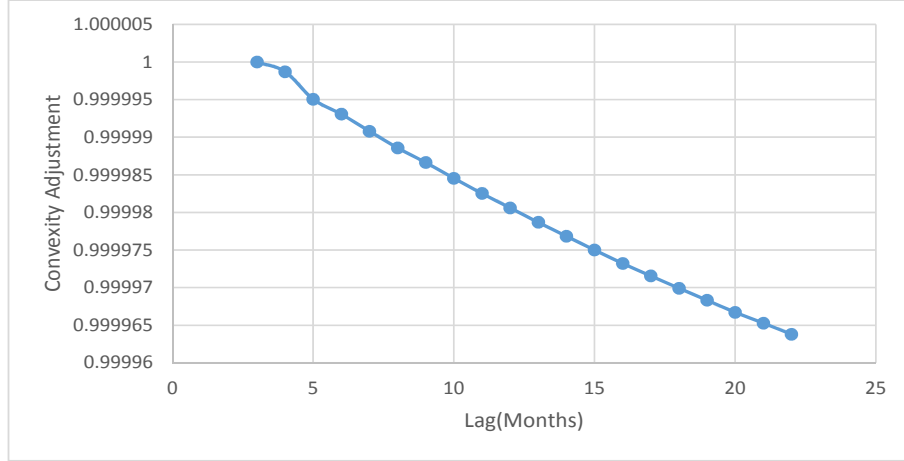


Figure 1: Convexity Adjustment for Zero-coupon inflation swap (in relative value).

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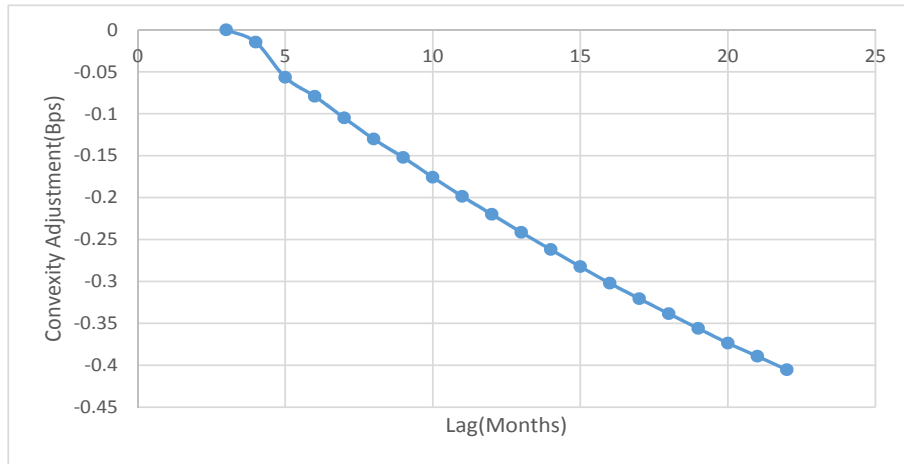


Figure 2: Convexity Adjustment for zero-coupon inflation swap (in Bps).

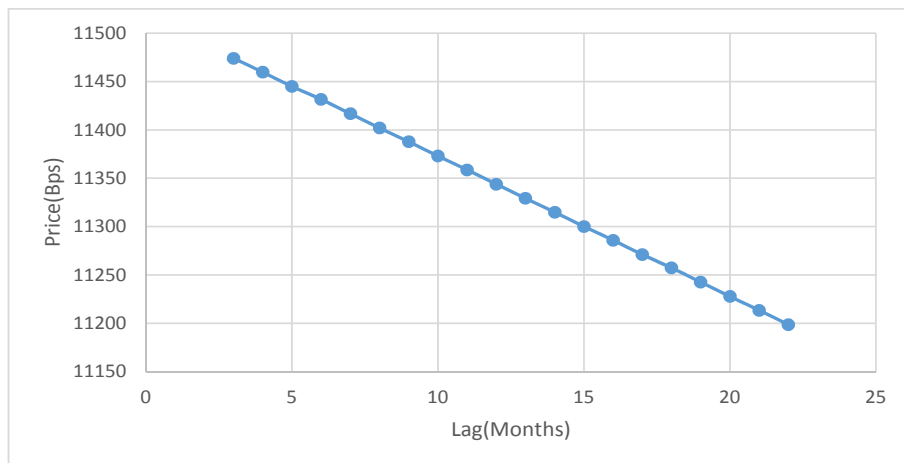


Figure 3: Price of zero-coupon swaps with adjustments.

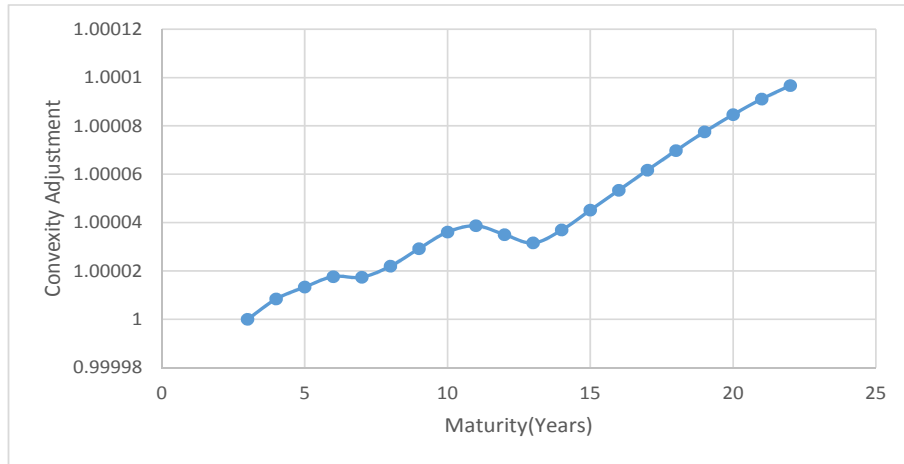


Figure 4: Convexity Adjustment for year-on-year inflation swap (in relative value).

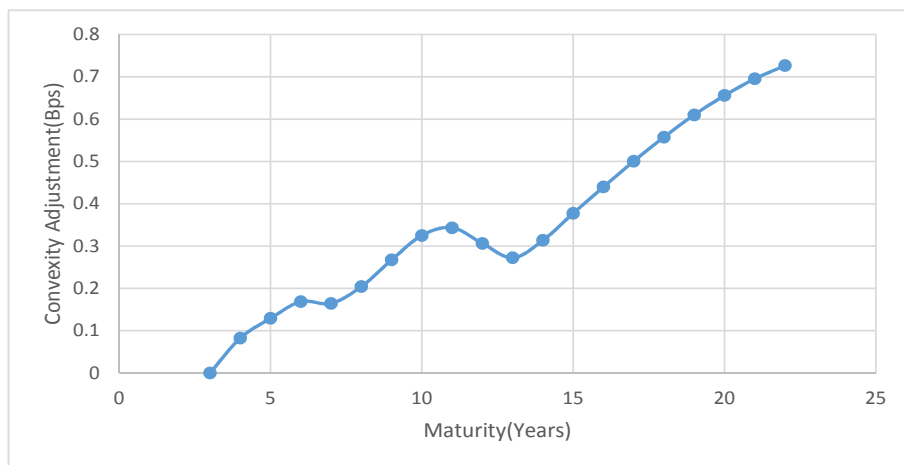


Figure 5: Convexity Adjustment for year-on-year inflation swap (in Bps).

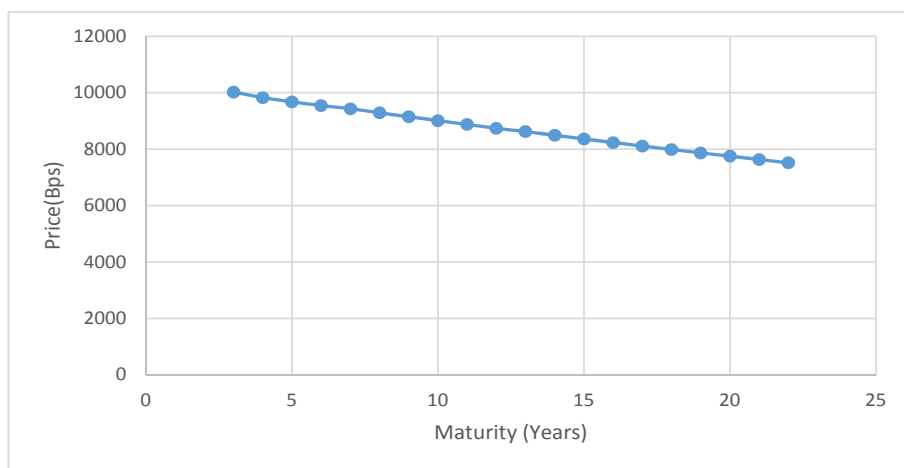


Figure 6: Price of zero-coupon swaps with adjustments.

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About OpenGamma

OpenGamma helps financial services firms unify their calculation of analytics across the traditional trading and risk management boundaries.

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