The Volatility Surface: Statics and Dynamics

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(Including joint work with Antoine Jacquier)

Outline

- History of SVI
- Static arbitrage
- Equivalent SVI formulations
- SSVI: Simple closed-form arbitrage-free SVI surfaces
- SSVI fits to historical SPX options data

History of SVI

- SVI was originally devised at Merrill Lynch in 1999 and subsequently publicly disseminated in [4].
- SVI has two key properties that have led to its subsequent popularity with practitioners:
  - For a fixed time to expiry t, the implied Black-Scholes variance $\sigma_{\mathrm{BS}}^2(k,t)$ is linear in the log-strike k as $|k| \to \infty$ consistent with Roger Lee's moment formula [11].
  - It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.
- The consistency of the SVI parameterization with arbitrage bounds for extreme strikes has also led to its use as an extrapolation formula [9].
- As shown in [6], the SVI parameterization is not arbitrary in the sense that the large-maturity limit of the Heston implied volatility smile is exactly SVI.

Previous work

- Calibration of SVI to given implied volatility data (for example [12]).
- [2] showed how to parameterize the volatility surface so as to preclude dynamic arbitrage.
- Arbitrage-free interpolation of implied volatilities by [1], [3], [8], [10].
- Prior work has not successfully attempted to eliminate static arbitrage.
- Efforts to find simple closed-form arbitrage-free parameterizations of the implied volatility surface are widely considered to be futile.

Notation

- Given a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, the forward price process $(F_t)_{t\geq 0}$ is $F_t := \mathbb{E}\left(S_t|\mathcal{F}_0\right)$.
- For any $k \in \mathbb{R}$ and $t > 0$, $C_{\mathrm{BS}}(k, \sigma^2 t)$ denotes the Black-Scholes price of a European Call option on S with strike $F_t \mathrm{e}^k$, maturity t and volatility $\sigma > 0$.
- $\sigma_{\mathrm{BS}}(k, t)$ denotes Black-Scholes implied volatility.
- Total implied variance is $w(k, t) = \sigma^2_{\mathrm{BS}}(k, t)t$.
- The implied variance $v(k, t) = \sigma^2_{\mathrm{BS}}(k, t) = w(k, t)/t$.
- The map $(k, t) \mapsto w(k, t)$ is the volatility surface.
- For any fixed expiry $t > 0$, the function $k \mapsto w(k, t)$ represents a slice.

Characterisation of static arbitrage

Definition 2.1

A volatility surface is free of static arbitrage if and only if the following conditions are satisfied:

(i) *it is free of calendar spread arbitrage;*

(ii) *each time slice is free of butterfly arbitrage.*

Calendar spread arbitrage

Lemma 2.2

If dividends are proportional to the stock price, the volatility surface w is free of calendar spread arbitrage if and only if

$$\partial_t w(k,t) \geq 0, \quad \text{for all } k \in \mathbb{R} \text{ and } t > 0.$$

- Thus there is no calendar spread arbitrage if there are no crossed lines on a total variance plot.

Butterfly arbitrage

Definition 2.3

A slice is said to be free of butterfly arbitrage if the corresponding density is non-negative.

Now introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(k) := \left(1 - \frac{k w'(k)}{2 w(k)}\right)^2 - \frac{w'(k)^2}{4} \left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

Lemma 2.4

A slice is free of butterfly arbitrage if and only if $g(k) \geq 0$ for all $k \in \mathbb{R}$ and $\lim_{k \to +\infty} d_+(k) = -\infty$.

The raw SVI parameterization

For a given parameter set $\chi_R = \{a, b, \rho, m, \sigma\}$, the *raw SVI parameterization* of total implied variance reads:

Raw SVI parameterization

$$w(k; \chi_R) = a + b \left\{ \rho \, (k - m) + \sqrt{(k - m)^2 + \sigma^2} \right\}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\sigma > 0$, and the obvious condition $a + b \, \sigma \sqrt{1 - \rho^2} \geq 0$, which ensures that $w(k, \chi_R) \geq 0$ for all $k \in \mathbb{R}$. This condition ensures that the minimum of the function $w(\cdot, \chi_R)$ is non-negative.

Meaning of raw SVI parameters

Changes in the parameters have the following effects:

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

The natural SVI parameterization

For a given parameter set $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$, the *natural SVI parameterization* of total implied variance reads:

Natural SVI parameterization

$$w(k; \chi_N) = \Delta + \frac{\omega}{2}\left\{1 + \zeta\rho\,(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\},$$

where $\omega \geq 0$, $\Delta \in \mathbb{R}$, $\mu \in \mathbb{R}$, $|\rho| < 1$ and $\zeta > 0$.

- This parameterization is a natural generalization of the time ∞ Heston smile explored in [6].

The SVI Jump-Wings (SVI-JW) parameterization

- Neither the raw SVI nor the natural SVI parameterizations are intuitive to traders.
- There is no reason to expect these parameters to be particularly stable.
- The *SVI-Jump-Wings (SVI-JW) parameterization* of the implied variance v (rather than the implied total variance w) was inspired by a similar parameterization attributed to Tim Klassen, then at Goldman Sachs.

SVI-JW

For a given time to expiry $t > 0$ and a parameter set $\chi_J = \{v_t, \psi_t, p_t, c_t, \widetilde{v}_t\}$ the SVI-JW parameters are defined from the raw SVI parameters as follows:

SVI-JW parameterization

$$
\begin{aligned}
v_t &= \frac{a + b\left\{-\rho\, m + \sqrt{m^2+\sigma^2}\right\}}{t}, \\
\psi_t &= \frac{1}{\sqrt{w_t}}\,\frac{b}{2}\left(-\frac{m}{\sqrt{m^2+\sigma^2}} + \rho\right), \\
p_t &= \frac{1}{\sqrt{w_t}}\, b\,(1-\rho)\,, \\
c_t &= \frac{1}{\sqrt{w_t}}\, b\,(1+\rho)\,, \\
\widetilde{v}_t &= \left(a + b\,\sigma\,\sqrt{1-\rho^2}\right)/t
\end{aligned}
$$

with $w_t := v_t\, t$.

Interpretation of SVI-JW parameters

The SVI-JW parameters have the following interpretations:

- v_t gives the ATM variance;
- ψ_t gives the ATM skew;
- p_t gives the slope of the left (put) wing;
- c_t gives the slope of the right (call) wing;
- $\widetilde{v}_t$ is the minimum implied variance.

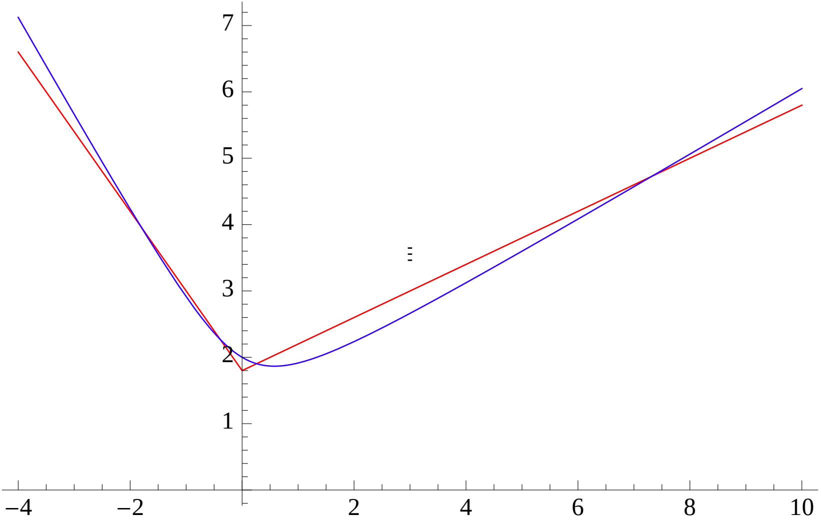
Features of the SVI-JW parameterization

- If smiles scaled perfectly as $1/\sqrt{w_t}$, SVI-JW parameters would be constant, independent of the slice t.
  - This makes it easy to extrapolate the SVI surface to expirations beyond the longest expiration in the data set.
- The choice

$$\psi_t = \left.\frac{\partial \sigma_{\mathrm{BS}}(k,t)}{\partial k}\right|_{k=0}$$

  of volatility skew as the skew measure rather than variance skew for example, reflects the empirical observation that volatility is roughly lognormally distributed.
- Since both features are roughly consistent with empirical observation, we expect (and see) greater parameter stability over time.
  - Traders can keep parameters in their heads.

SVI slices may cross at no more than four points



Condition for no calendar spread arbitrage

Lemma 3.1

Two raw SVI slices admit no calendar spread arbitrage if a certain quartic polynomial has no real root.

Ferrari Cardano

The idea is as follows:

- Two total variance slices cross if

$$
\begin{aligned}
& a_1 + b_1 \left\{ \rho_1 \left(k - m_1\right) + \sqrt{\left(k - m_1\right)^2 + \sigma_1^2} \right\} \\
= \quad & a_2 + b_2 \left\{ \rho_2 \left(k - m_2\right) + \sqrt{\left(k - m_2\right)^2 + \sigma_2^2} \right\}
\end{aligned}
$$

- Rearranging and squaring gives a quartic polynomial equation of the form

$$\alpha_4\, k^4 + \alpha_3\, k^3 + \alpha_2\, k^2 + \alpha_1\, k + \alpha_0 = 0,$$

  where each of the coefficients are lengthy yet explicit expressions in terms of the raw SVI parameters.
  - If this quartic polynomial has no real root, then the slices do not intersect.

SVI butterfly arbitrage

Recall the definition:

$$g(k) := \left(1 - \frac{k w'(k)}{2 w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

- The highly nonlinear behavior of g makes it seemingly impossible to find general conditions on the parameters that would eliminate butterfly arbitrage.
- We now provide an example where butterfly arbitrage is violated.

Axel Vogt post on Wilmott.com

AVt
Senior Member

Posts: 971
Joined: Dec 2001

Thu Apr 06, 06 08:37 PM

It works for observables and far beyond for extrapolation.

But for a (theoretical) experiment try the following data

a = -.40998372001772e-1,
b = .13308181151379,
m = .35858898335748,
rho = .30602086142471,
sigma = .41531878803777

The Vogt smile

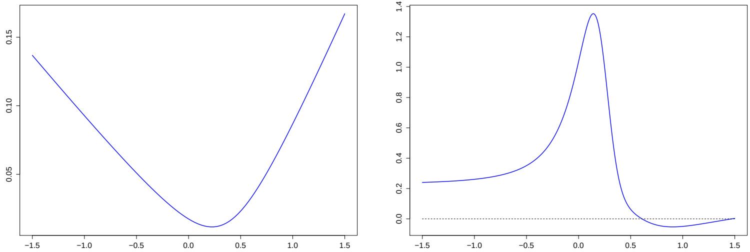


Figure 1: Plots of the total variance smile w (left) and the function g (right), using Axel Vogt's parameters

Is SVI arbitrage-free?

- So it is easy to eliminate calendar spread arbitrage with SVI.
- However, until recently, it was thought impossible to find conditions on the parameters to ensure that SVI is free of butterfly arbitrage.

Surface SVI

Consider now the following extension of the natural SVI parameterization:

Surface SVI (SSVI) parameterization

$$w(k, \theta_t) = \frac{\theta_t}{2} \left\{ 1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1 - \rho^2)} \right\} \quad (1)$$

with $\theta_t > 0$ for $t > 0$, and where φ is a smooth function from $(0, \infty)$ to $(0, \infty)$ such that the limit $\lim_{t \to 0} \theta_t \varphi(\theta_t)$ exists in $\mathbb{R}$.

Interpretation of SSVI

- This representation amounts to considering the volatility surface in terms of ATM variance time, instead of standard calendar time.
- The ATM total variance is $\theta_t = \sigma_{\mathrm{BS}}^2(0,t)\, t$ and the ATM volatility skew is given by

$$\partial_k \sigma_{\mathrm{BS}}(k,t)|_{k=0} = \frac{1}{2\sqrt{\theta_t t}} \partial_k w(k,\theta_t)\bigg|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}}\varphi(\theta_t).$$

- The smile is symmetric around at-the-money if and only if $\rho = 0$, a well-known property of stochastic volatility models.

Conditions on SSVI for no calendar spread arbitrage

Theorem 4.1

The SSVI surface (1) is free of calendar spread arbitrage if and only if

1. $\partial_t \theta_t \geq 0$, *for all* $t \geq 0$;
2. $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1+\sqrt{1-\rho^2}\right)\varphi(\theta)$, *for all* $\theta > 0$,

where the upper bound is infinite when $\rho = 0$.

- In particular, SSVI is free of calendar spread arbitrage if:
  - the skew in total variance terms is monotonically increasing in trading time and
  - the skew in implied variance terms is monotonically decreasing in trading time.
- In practice, any reasonable skew term structure that a trader defines will have these properties.

Conditions on SSVI for no butterfly arbitrage

Theorem 4.2

The volatility surface (1) is free of butterfly arbitrage if the following conditions are satisfied for all $\theta > 0$*:*

1. $\theta\varphi(\theta)\left(1+|\rho|\right) < 4$;
2. $\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4$.

Remark

Condition 1 needs to be a strict inequality so that $\lim_{k\to+\infty} d_+(k) = -\infty$ *and the SVI density integrates to one.*

Are these conditions necessary?

Lemma 4.2

The volatility surface (1) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)\left(1+|\rho|\right) \leq 4, \quad \textit{for all } \theta > 0.$$

Moreover, if $\theta\varphi(\theta)\left(1+|\rho|\right) = 4$, the surface (1) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4.$$

So the theorem is almost if-and-only-if.

The Roger Lee arbitrage bounds

- The asymptotic behavior of the surface (1) as $|k|$ tends to infinity is

$$w(k,\theta_t) = \frac{(1 \pm \rho)\,\theta_t}{2}\varphi(\theta_t)\,|k| + \mathcal{O}(1), \quad \text{for any } t > 0.$$

- Thus the condition $\theta\varphi(\theta)\,(1+|\rho|) \leq 4$ of Theorem 4.2 corresponds to the upper bound of 2 on the asymptotic slope established by Lee [11].
    - Again, Condition 1 of the theorem is necessary.

No static arbitrage with SSVI

Corollary 4.1

The SSVI surface (1) is free of static arbitrage if the following conditions are satisfied:

1. $\partial_t \theta_t \geq 0$, *for all* $t > 0$
2. $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1 + \sqrt{1-\rho^2}\right)\varphi(\theta)$, *for all* $\theta > 0$;
3. $\theta\varphi(\theta)\left(1 + |\rho|\right) < 4$, *for all* $\theta > 0$;
4. $\theta\varphi(\theta)^2\left(1 + |\rho|\right) \leq 4$, *for all* $\theta > 0$.

- A large class of simple closed-form arbitrage-free volatility surfaces!

A Heston-like surface

Example 4.1

The function φ defined as

$$\varphi(\theta) = \frac{1}{\lambda\theta}\left\{1 - \frac{1 - \mathrm{e}^{-\lambda\theta}}{\lambda\theta}\right\},$$

with $\lambda \geq (1 + |\rho|)/4$ satisfies the conditions of Corollary 4.1.

- This function is consistent with the implied variance skew in the Heston model as shown in [5] (equation 3.19).

A power-law surface

Example 4.2

The choice

$$\varphi(\theta) = \frac{\eta}{\theta^{\gamma}\,(1+\theta)^{1-\gamma}}$$

gives a surface that is completely free of static arbitrage provided that $\gamma \in (0, 1/2]$ *and* $\eta\,(1+|\rho|) \leq 2$.

- This function is more consistent with the empirically-observed term structure of the volatility skew.

SVI square-root calibration ($\gamma = 1/2$)

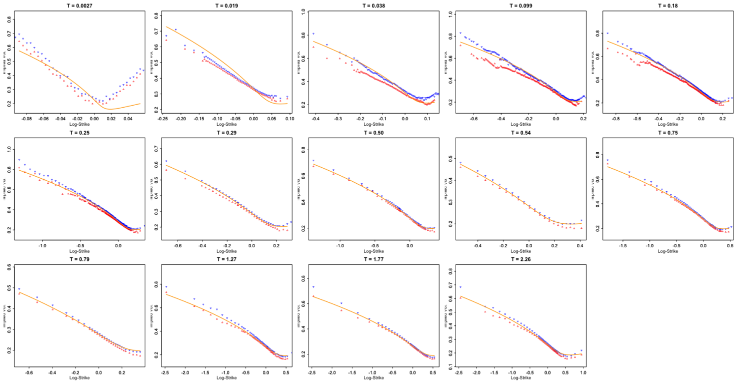


Figure 2: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the square-root SVI fit

SVI square-root calibration: December 2011 detail

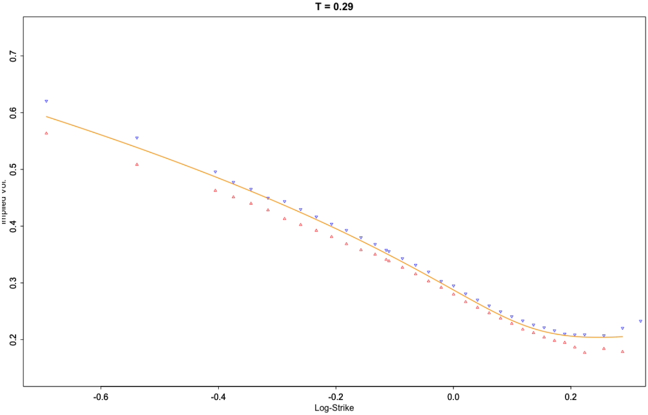


Figure 3: SPX Dec-2011 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the square-root SVI fit

Analysis of historical SPX volatility surface data

Recall that $\theta_t = \sigma_{\rm BS}^2(0,t)\,t$ and that the ATM volatility skew is given by

$$\partial_k \sigma_{\rm BS}(k,t)|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}}\varphi(\theta_t).$$

so that

$$\rho\,\varphi(\theta_t) = \partial_k \log\left(\sigma_{\rm BS}(k,t)^2\,t\right)\Big|_{k=0}.$$

- Thus θ_t and φ_t may be determined empirically from robust estimates of ATM volatility and ATM skew.

An empirical fit

Example 5.1

The choice

$$\varphi_{SPX}(\theta) = \frac{\eta}{\theta^{\gamma_1}\,(1+\beta_1\,\theta)^{\gamma_2}\,(1+\beta_2\,\theta)^{1-\gamma_1-\gamma_2}}$$

gives a surface that is completely free of static arbitrage with

$$\gamma_1 = 0.238;\ \gamma_2 = 0.253;\ \beta_1 = e^{5.18};\ \beta_2 = e^{-3}$$

and $\eta = 2.016048\, e^{\epsilon}$ *where* $\epsilon \in (-1, 1)$.

- We will now show how well this functional form fits historical SPX volatility surface data.

Log-log plot of empirical SPX $\rho\varphi$ vs θ

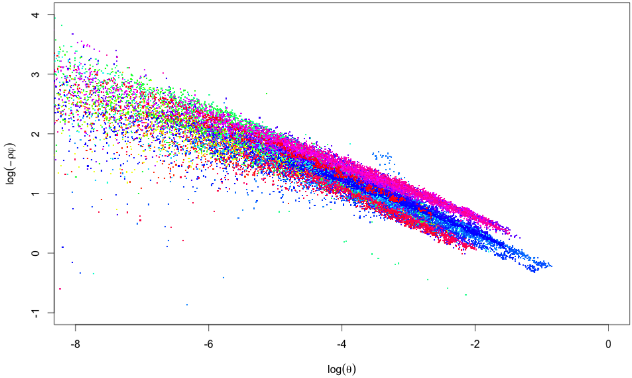


Figure 4: $\log(\varphi)$ vs $\log(\theta)$ for each of the 2,616 days in the sample superimposed. Points are color-coded with dates ranging from red via blue to violet.

Log-log plot of empirical SPX $\rho\varphi$ rescaled by φ_{SPX} vs θ

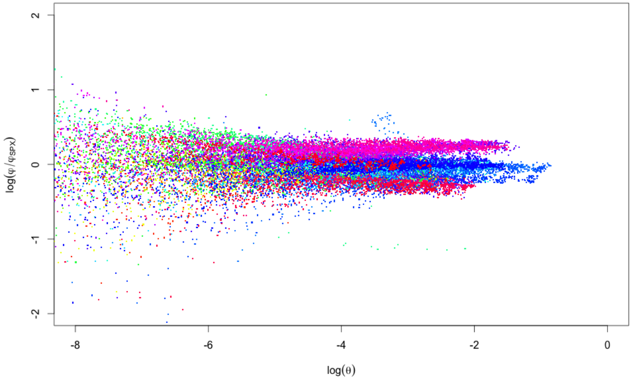


Figure 5: $\log(\varphi/\varphi_{SPX})$ vs $\log(\theta)$ for each of the 2,616 days in the sample superimposed.

Remarks on Figure 5

- The scatter plot looks flat
- On average, it looks as if the empirical φ is equal to φ_{SPX}.

- Let's check with kernel regression...

Kernel regression

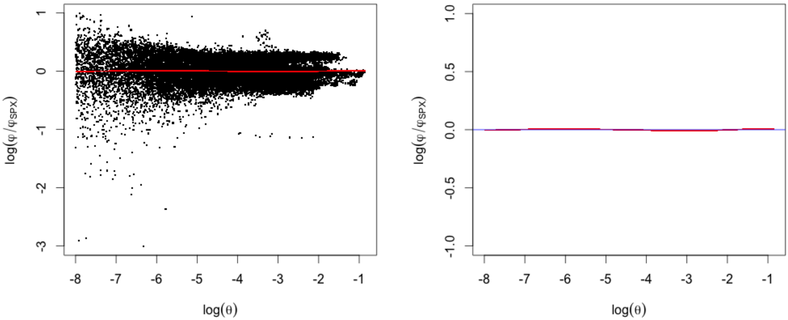


Figure 6: Kernel regression of Figure 5 data. The LH plot shows the data and regression line. The RH plot shows how flat the line is and that it is mean zero.

SSVI fits to historical SPX options data

- Our dataset consists of roughly 2,600 days of SPX option closing prices from OptionMetrics™.
- We fit the function

$$w(k,\theta_t) = \frac{\theta_t}{2}\left\{1 + \rho\,\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1-\rho^2)}\right\}$$

  with

$$\varphi(\theta) = \frac{\eta}{\theta^{\gamma_1}\,(1+\beta_1\,\theta)^{\gamma_2}\,(1+\beta_2\,\theta)^{1-\gamma_1-\gamma_2}}$$

  and $\theta_t = \sigma_{\rm BS}(0,t)^2\,t$.
- For each day, we find the combination of η and ρ that minimizes the mean squared error in variance, weighted by the implied volatility spread.

Plot of η_t

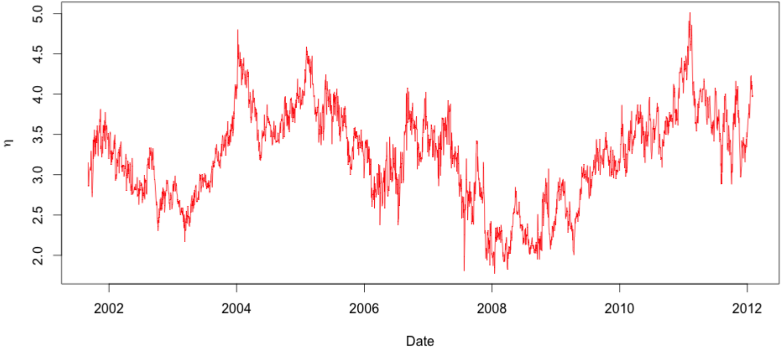


Figure 7: Time series of fitted η_t. The high was on 25-Jan-2011 and the low on 09-Sep-2008.

Plot of ρ_t

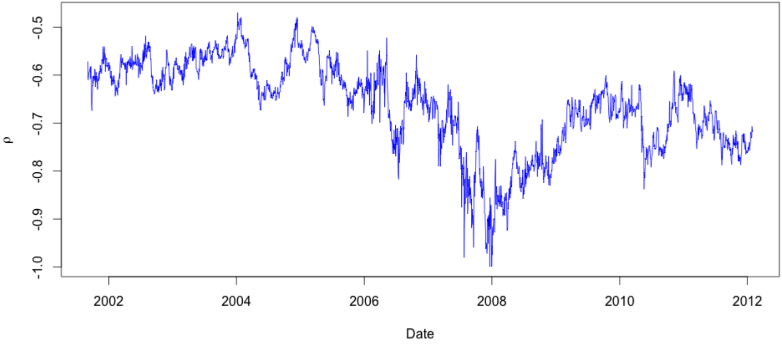


Figure 8: Time series of fitted ρ_t. Note how ρ_t goes to -1 in late 2007.

Does the ρ_t plot make sense?

Figure 9: We see that the realized correlation between VIX and SPX moves was in fact much greater in late 2007 than recently.

Realized correlation and ρ_t

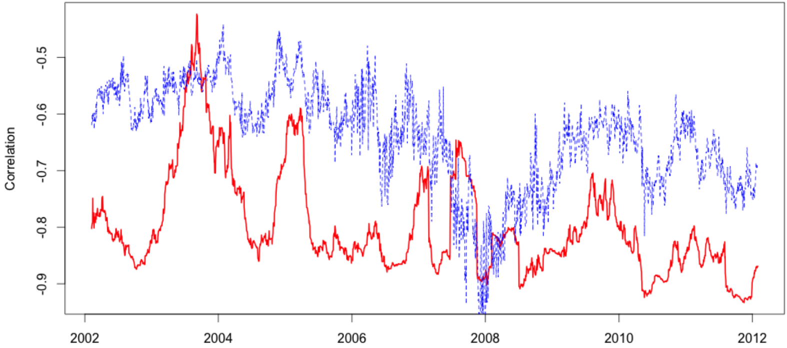


Figure 10: The red line is 100-day realized correlation between log-differences of SPX and 1-month ATM volatilities respectively; the blue dashed line is the time series of ρ_t

Plot of $\rho_t \, \eta_t$

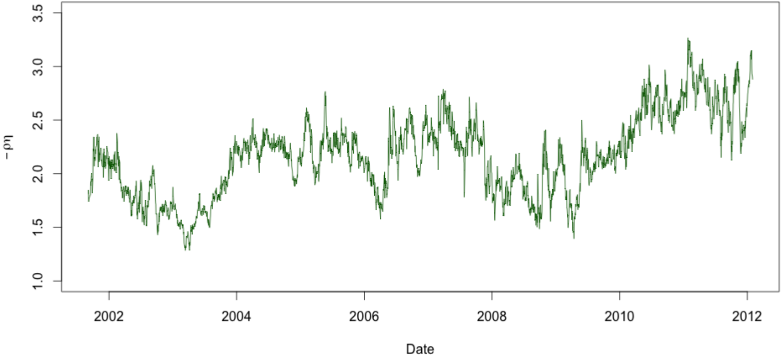


Figure 11: The product $\rho_t \, \eta_t$ is a measure of ATM volatility skew.

Plot of worst fit (2009-12-30)

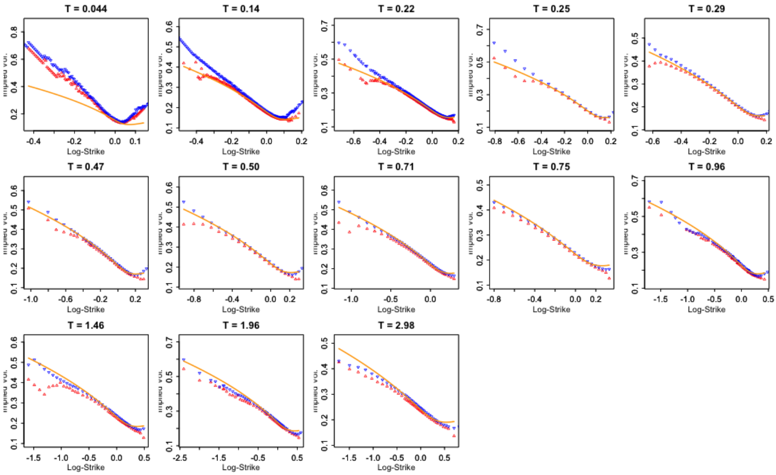


Figure 12: The worst fit in the dataset (30-Dec-2009).

Plot of best fit (2002-08-06)

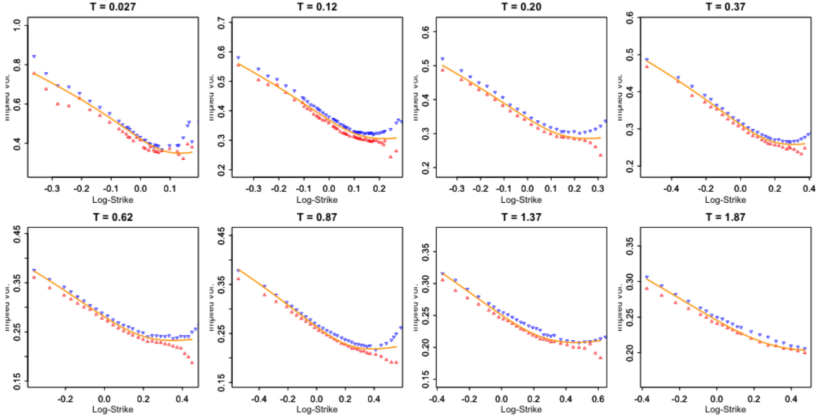


Figure 13: The best fit in the dataset (06-Aug-2008).

Plot of most impressive fit (2008-09-05)

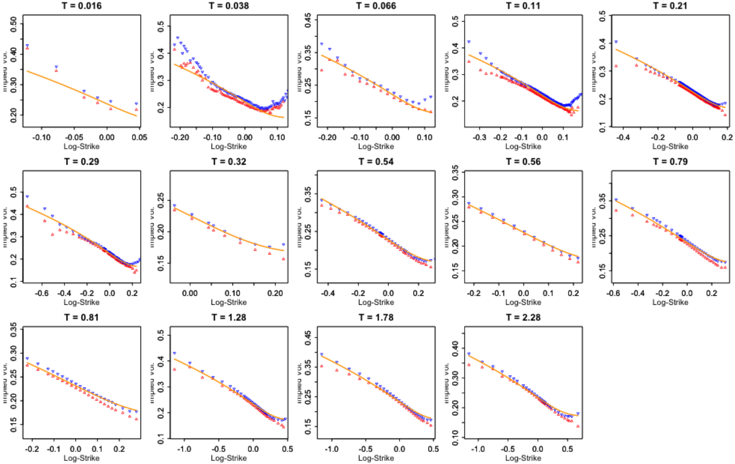


Figure 14: The “most impressive” fit in the dataset (05-Sep-2008).

SSVI fits to 15-Sep-2011 data

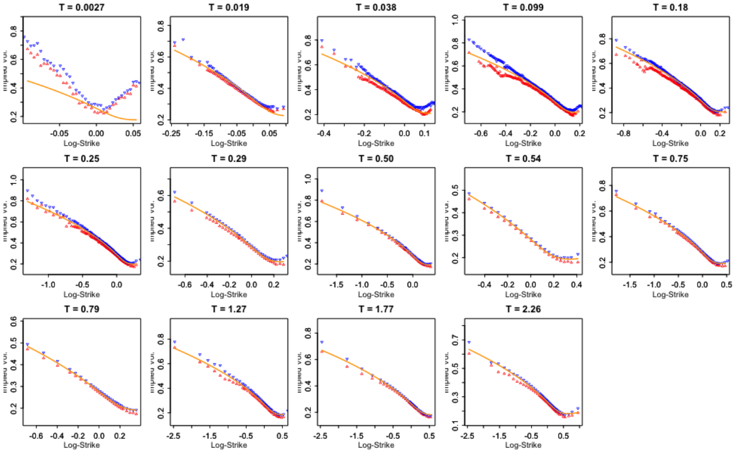


Figure 15: SSVI fit to SPX options as of 15-Sep-2011 (the day before triple-wtiching).

Compare again with SVI square-root calibration

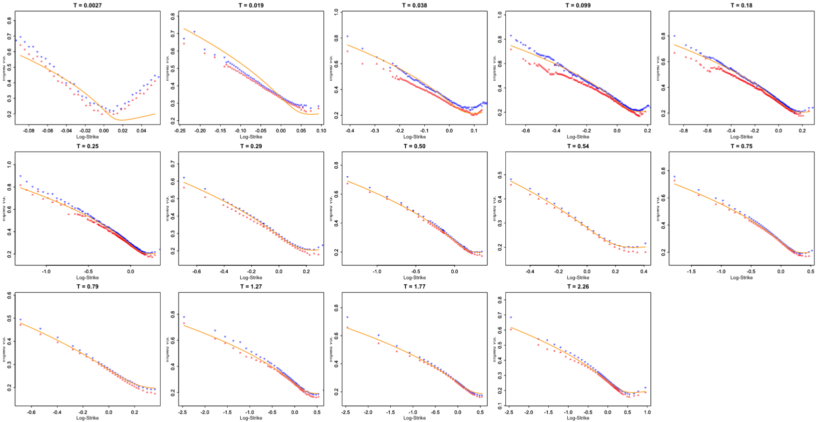


Figure 16: The SSVI fit is obviously superior to the SVI square-root fit

SSVI fit to 15-Sep-2011 data: Mar-2012 detail

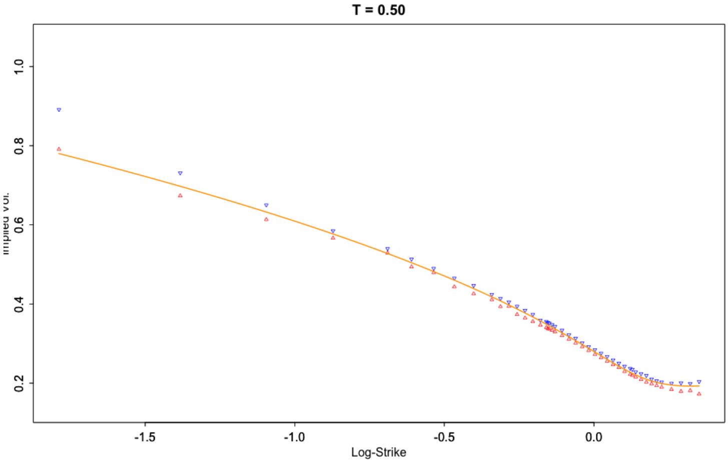


Figure 17: SSVI fit to Mar-2012 smile as of 15-Sep-2011.

Remarks on the SSVI fits

- The 2-parameter (for the whole surface) SSVI fit quality is overall extremely high, only slightly inferior to the full SVI fit with 5 parameters for each slice.
  - For emphasis, the skew term structure function $\varphi(.)$ is assumed to be fixed, independent of time.
  - Only the parameters ρ_t and η_t are allowed to depend on time.
- The SSVI fits clearly beat the square-root fits.
- We see evidence that the shape of the volatility surface is related to the dynamics of the volatility surface:
  - For example, ρ_t seems to be related to realized correlation between index and index volatility movements.

Full SVI calibration

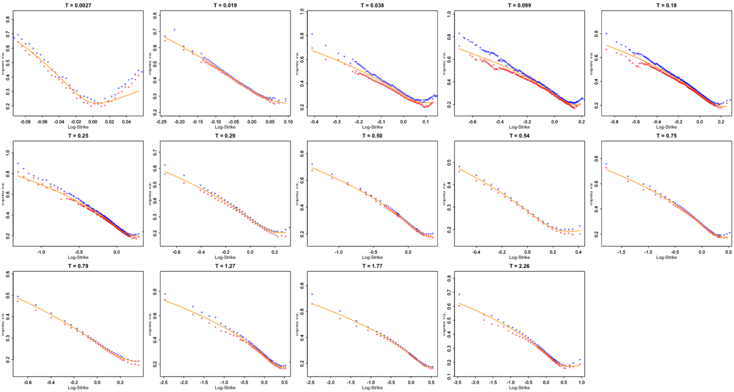


Figure 18: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

Full SVI calibration: March 2012 detail

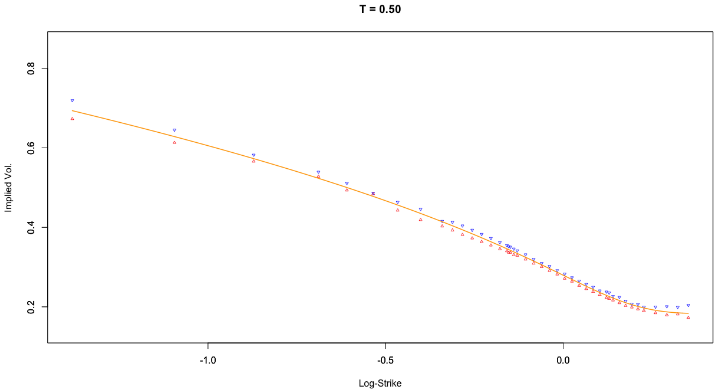


Figure 19: SPX Mar-2012 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

Summary

- We have found and described a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation. We dubbed this class SSVI.
- We estimated from historical SPX option prices a good functional form for the skew term structure function φ.
- We fitted SSVI daily for over 10 years and extracted time series of η_t and ρ_t.
- We checked that the time series of ρ_t was consistent with its interpretation as the correlation between volatility moves and SPX moves.
- We further demonstrated the high quality of typical SVI fits with a numerical example using recent SPX options data.

Acknowledgements

- We are grateful to Baruch MFE students Aliasgar Haji and Yassine Ghalem for generating the implied volatility data.
- We also thank Richard Holowczak of the Subotnick Financial Services Center at Baruch College for supplying the SPX options data as of 3PM on 15-Sep-2011.
- Historical closing prices are from OptionMetrics™ sourced from WRDS.

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