# Curves and Term Structure Models.

## Definition, Calibration and Application of Rate Curves and Term Structure Market Models

(Multi-Curve Configuration)

Christian P. Fries
email@christian-fries.de

December 31, 2012
(This version February 28, 2013)

Version 0.5

## Abstract

In this note we discuss the definition, construction, interpolation and application of *curves*. We will discuss discount curves, a tool for the valuation of deterministic cash-flows and forward curves, a tool for the valuation of linear cash-flows of (possibly) stochastic indices.

The aim of this note is to carefully derive the definition of discount and forward curves and work out their relation to market instruments: A curve is tool to value linear products, i.e., product which can be replicated by static hedges.

We will distinguish forward curves from discount curves. Since forward curves are associated with a discount curve (representing the collateralization of the forward), this motivates an alternative interpolation method, namely interpolation of the forward value (the product of the forward and the discount factor).

In addition, treating forward curves as native curves (instead of representing them by pseudo-discount curves) will avoid other problems, like that of overlapping instruments.

We discuss the calibration of the curves for which we give a generic object oriented implementation.

In the last section we will show how to define term-structure models (analog to a LIBOR market model) based on the definition of the performance index of an accrual account associated with a discount curve.

# Contents

# 1 Introduction

In this note we discuss the definition, construction, interpolation and application of *curves*. We will discuss discount curves, a tool for the valuation of deterministic cash-flows and forward curves, a tool for the valuation of linear cash-flows of (possibly) stochastic indices.

With respect to the interpolation of (interest rate) forward curves, a common mistake is to represent a forward curve in terms of (pseudo-)discount factors and apply an interpolation scheme on these discount factors. While this approach is in general not backed by an economic concept it also introduces several (self-made) problems, e.g., the interpolation of (so-called) overlapping instruments. Such complications will not occur if curves are derived (defined) from the associated market instruments.

We derive discount curves and forward curves by considering valuation of the associated market instruments. This will also motivate natural interpolation schemes, e.g., the interpolation of a forward value.[1]

Once the curves and interpolations are defined, we are considering the problem of calibrating a set of curves to given market quotes. The value of an instrument is in general determined by a whole collection of curves, e.g., one or two discount curves and zero or more forward curves.

For the calibration we need the valuation formulas which express the value of a financial product with respect to curves and a calibration algorithm. Often curves are determined by the successive calibration to a single instrument, retaining the calibration two previous calibration instruments, considering a specific ordering of the instruments (e.g., increasing in maturity). Such a method is often called a bootstrap. A bootstrap algorithm may introduce difficulties (like error propagation) and is to some extend not flexible enough since it requires a certain ordering and independence of the calibration products. Instead we consider a global calibration.

We consider a multi-variate optimization of the whole set of curves to a set of calibration instruments. The algorithm is numerical less performant than a classical bootstrap, but more flexible and robust.

In summary the calibration framework can calibrate many different types of curves (discount curves, forward curves, cross-currency discount curves) to a set of different calibration instruments (swaps, tenor-basis swaps, cross-currency swaps) using a simple and generic implementation. A reference implementation is provided.

[1] A forward value is the product of a *forward* and the associated *discount factor*.

# 2 Foundations and Notation

We will give a short review of the mathematical background, that is, risk neutral valuation. This section is mainly to fix notation. For an in depth introduction to mathematical finance see the references and references there in.

## 2.1 Mathematical Background

Under some assumptions the valuation of a future cash flow can be written as an expectation, that is

$$V(t) \;=\; N(t) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(T)}{N(T)} \mid \mathcal{F}_t\right), \tag{1}$$

where $V(T)$ is the time $T$ cash-flow, $N$ is the value process of a traded asset which can serve as a *numéraire* and $\mathbb{Q}^N$ is the equivalent martingale measure associated with $N$.

If the above valuation formula holds, we have that the value of a linear function of future cash-flows is the linear function of the values of the single cash flows. In other words: we can represent the valuation of so called linear products by a *basis* consisting of the values of elementary products. This basis of elementary products is the set of curves.

Curves, like discount curves and forward curves are constructed for two reasons:

- Valuation of linear instruments. This is performed by decomposing instruments into the value of single cash-flows, which then allows to synthesize the valuation of linear function of the individual cash flows.
- Valuation of a time $T$ cash-flows as interpolation of valuations of cash flows at discrete times $\{T_i\}_{i=0}^{n}$.

Thus, curves are simply a methodology to interpolate on the cash-flows with respect to their payment time.[2]

## 2.2 Currencies and Collateralization

### 2.2.1 Valuation in Foreign Currency

In the above valuation formula (1) it is assumed that $V$ and $N$ are expressed in the same currency. If the two are in different currency, one of them has to be converted by an exchange rate, which we will denote by $FX$. Let $V$ be in currency $U_2$ and the numéraire $N$ in currency $U_1$, then the valuation formula is given by

$$V(t) \;=\; FX^{\frac{U_2}{U_1}}(t) \cdot N(t) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(T)}{FX^{\frac{U_2}{U_1}}(T) \cdot N(T)} \mid \mathcal{F}_t\right),$$

[2] This also applies to forward curve, see below, although in these cases there is also an associated fixing time of an index.

where $FX^{\frac{U_2}{U_1}}(t)$ denotes the time $t$ exchange rate for one unit of currency $U_1$ into one unit of currency $U_2$. Furthermore

$$FX^{\frac{U_1}{U_2}} = \left(FX^{\frac{U_2}{U_1}}\right)^{-1}.$$

### 2.2.2 Valuation of Collateralization

As discussed in [2], the valuation of a collateralized claim can be written as an expectation with respect to a specific numéraire, namely the collateral account $N = N^{\mathsf{C}}$. We denote the currency of the collateral numéraire by $[\mathsf{C}]$. Let $U$ denote the currency of the cash flow $V(T)$. Assume that the cash flow $V(T)$ is collateralized by units of $N^{\mathsf{C}}$. In this case the equation (1) holds with the numéraire $N = N^{\mathsf{C}}$, i.e.,

$$V(t) = FX^{\frac{U}{[\mathsf{C}]}}(t) \cdot N^{\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{\mathsf{C}}}}\left(\frac{V(T)}{FX^{\frac{U}{[\mathsf{C}]}}(T) \cdot N^{\mathsf{C}}(T)} \mid \mathcal{F}_t\right) \tag{2}$$

(given that $V(t)$ is the collateral amount in the account $N^{\mathsf{C}}$).

**Remark 1 (Collateralization in Other Currency):** From the above we see that collateralization in a different currency can be interpreted twofold:

1. We may consider a payment converted to collateral currency and valued with respect to the collateral numéraire $N^{\mathsf{C}}$, or, alternatively,

2. we may consider a payment in the currency $U$ collateralized with respect to the collateral account
$$N^{U,\mathsf{C}} := FX^{\frac{U}{[\mathsf{C}]}} \cdot N^{\mathsf{C}}.$$

We will adopt the latter interpretation, which will also make the valuation look more consistently[3]

$$V(t) = N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{U,N^{\mathsf{C}}}}\left(\frac{V(T)}{N^{U,\mathsf{C}}(T)} \mid \mathcal{F}_t\right). \tag{3}$$

**Remark 2 (Funding and Collateralization):** For an uncollateralized product the role of the collateral account is taken by the funding account and the corresponding numéraire is the funding account. Since the valuation formulas are identical to the case of a "special" collateral account $\mathsf{D}$ (agreeing with the funding account), we will consider an uncollateralized product as a product with a different collateralization.

## 2.3 Notation

In the following we use the notation $U$ for the currency unit of a cash flow, i.e., you may consider $U = 1$€ or $U = 1\$$. We will need this notation only when we consider cross-currency basis swaps. The symbols $V$, $N$ and $P$ will denote value processes including the corresponding currency unit, e.g., $V(t_0) = 0.25$€. The symbol $V$ refers

[3] As has been noted in [2], the measures agree, i.e., $\mathbb{Q}^{U,N^{\mathsf{C}}} = \mathbb{Q}^{N^{\mathsf{C}}}$.

to the value of the product under consideration, while $N$ denotes the numéraire, e.g., the OIS accrued collateral account. The symbols $X$ denotes a real number while $I$ denotes a real valued stochastic process, both can be considered as rates, i.e., unit-less indices, e.g., $X = 2.5\%$. For example $X$ is will denote the fix rate in a swap, $I$ will denote the floating rate index in a swap, $U$ denotes the currency unit of the two legs, $N$ will be used to define the discount factor and the value of the swap. The value of the swap is then denoted by $V$.

## 2.4 Valuation of Linear Products

### 2.4.1 Discount Curves

Consider a fixed constant cash flow $X$, paid in currency $U$ in time $T$, collateralized by an account $\mathsf{C}$. The value of the cash flow is

$$V(t) \;=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left( \frac{X \cdot U}{N^{U,\mathsf{C}}(T)} \mid \mathcal{F}_t \right).$$

Since $X$ is a constant and the expectation operator is linear, we can express the value $V(t)$ as

$$V(t) \;=\; X \cdot P^{U,\mathsf{C}}(T;t), \tag{4}$$

where

$$P^{U,\mathsf{C}}(T;t) \;=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left( \frac{1 \cdot U}{N^{U,\mathsf{C}}(T)} \mid \mathcal{F}_t \right)$$

defines the value of a theoretical zero coupon bond. Note that equation (4) can be used in two ways. First, for given market prices we may determine the $P^{U,\mathsf{C}}(T;t)$ - that is we calibrate the curve

$$T \mapsto P^{U,\mathsf{C}}(T;t).$$

Second, for given $P^{U,\mathsf{C}}(T;t)$ we may value a cash flow.

### 2.4.2 Forward Curves

The same approach can now be applied to a payoff of a cash flow $X \cdot I(T_1)$, paid in currency $U$ in time $T_2$, collateralized by account $\mathsf{C}$, where $X$ is a constant and $I$ is a stochastic index. Its value is

$$V(t) \;=\; N^{U\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left( \frac{X \cdot I(T_1) \cdot U}{N^{U,\mathsf{C}}(T_2)} \mid \mathcal{F}_t \right).$$

Since $X$ is a constant and the expectation operator is linear, we can express the value $V(t)$ as

$$V(t) \;=\; X \cdot F_I^{U,\mathsf{C}}(T_1,T_2;t) \cdot P^{U,\mathsf{C}}(T_2;t), \tag{5}$$

where

$$F_I^{U,\mathsf{C}}(T_1,T_2;t) \;=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left( \frac{I(T_1) \cdot U}{N^{U,\mathsf{C}}(T_2)} \mid \mathcal{F}_t \right) / P^{U,\mathsf{C}}(T_2;t).$$

This definition allows us to derive $F_I^{U,\mathsf{C}}(T_1, T_2; t)$ from given market prices. Conversely, given $P^{U,\mathsf{C}}(T_2; t)$ and $F_I^{U,\mathsf{C}}(T_1, T_2; t)$ we may value all linear payoff functions of in $I(T_1)$ paid in $T_2$.

In (5) the forward depends on the fixing time $T_1$ and the payment time $T_2$. However, the offset of the payment time from the fixing time $d = T_2 - T_1$ can be viewed as a property of the index (a constant) and hence, the forward represents a curve

$$T \mapsto F_I^{U,\mathsf{C}}(T, T+d; t).$$

# 3 Discount Curves and Forward Curves

We now take the derivation to give formal definitions for the discount curve and forward curves.

## 3.1 Discount Curves

**Definition 3 (Discount Curve):** ⌝
Let $P^{U,\mathsf{C}}(T;t)$ denote the time $t$ value expressed in currency unit $U$ of a unit cash-flow of 1 unit of the currency $U$ in $T$, collateralized by a collateral account $\mathsf{C}$.

In this case we call $T \mapsto P^{U,\mathsf{C}}(T;t)$ the discount curve for cash flows in currency $U$ collateralized by the account $C$. ⌟

**Remark 4 (Discount Curve):** We assume that the value of a fixed future cash-flow $X$ is a linear function of its amount. In this case, we have that the time $t$ value of a cash flow $X$ in $T$ and currency $U$, collateralized with an account $C$ is

$$X \cdot P^{U,\mathsf{C}}(T;t).$$

In other words, the discount curve allows us to valuate all fixed cash flows in a given currency, collateralized by a given account.

## 3.2 Forward Curves

**Definition 5 (Forward Curve):** ⌝
Let $t \mapsto I(t)$ denote an index, that is $I$ is an adapted stochastic real valued process. Let $V_I^{U,\mathsf{C}}(T, T+d;t)$ denote the time $t$-value of a payment of $I(T)$ is paid in $T+d$ in currency $U$, collateralized by an account $\mathsf{C}$ (where $d \geq 0$).

Then we define the *forward of a payment of $I(T)$ paid in $T+d$ in currency $U$, collateralized by an account $C$* as

$$F_I^{U,\mathsf{C}}(T;t) \ := \ \frac{V_I^{U,\mathsf{C}}(T, T+d;t)}{P^{U,\mathsf{C}}(T+d;t)}.$$

⌟

**Remark 6 (Forward Curve):** The forward curve allows us to value a future payment of the index $I$. In fact it is

$$V_I^{U,\mathsf{C}}(T, T+d;t) \ = \ F_I^{U,\mathsf{C}}(T;t) \cdot P^{U,\mathsf{C}}(T+d;t).$$

We assume that the value of a floating future cash-flow $X \cdot I(T_1)$, where $X$ is a real valued constant, is a linear function of its amount $X$. In this case, we have that the time $t$ value of a cash flow $X \cdot I(T)$ paid in $T+d$ and currency $U$, collateralized with an account $\mathsf{C}$ is

$$X \cdot F_I^{U,\mathsf{C}}(T;t) \cdot P^{U,\mathsf{C}}(T+d;t).$$

In other words, the forward curve allows us to valuate all linear cash flows of an index in a given currency, collateralized by a given account.

### 3.2.1 Performance Index of a Discount Curve (or "self-discounting")

The OIS swap pays the performance of an account, accruing with the overnight rate, that is:

**Definition 7 (Overnight Index Swap):**
Let $N^{\mathsf{C}}(t)$ denote the account accruing at the overnight rate, $N^{\mathsf{C}}(0) = 1$, i.e.

$$N(t) \;:=\; \prod (1 + r(t_i)\Delta t_i) \;\approx\; \exp\left(\int_0^t r(s)\mathrm{d}s\right).$$

The overnight index swap pays fix and receives the performance $I_i$ of the accrual accrual account, that is

$$I_i \;:=\; \frac{N^{\mathsf{C}}(T_{i+1})}{N^{\mathsf{C}}(T_i)} - 1.$$

in $T_{i+1}$ with a quaterly tenor $T_0, T_1, \ldots$.

The OIS swap is collateralized with respect to the account $N^{\mathsf{C}}$. Due to this, it is sometimes called "self-discounted". However, an alternative natural interpretation is that the discount factor is given by the accrual account and that the swap pays a special index, namely one derived from the discount factors. In other words: we may define a special type of forward curve from discount curve (and not the other ways around).

The forward of the index

$$I_i \;:=\; \frac{N^{\mathsf{C}}(T_{i+1})}{N^{\mathsf{C}}(T_i)} - 1$$

is

$$\frac{P^{\mathsf{C}}(T_i;t) - P^{\mathsf{C}}(T_{i+1};t)}{P^{\mathsf{C}}(T_{i+1};t)}.$$

Hence, this is the same situation as for single curve interest rate theory swaps.

Let us consider a discount factor curve $P^{U,\mathsf{C}}(T;t)$ as seen in time $t$. The curve allows the definition of a special index, namely the performance rate of the collateral account $\mathsf{C}$ in currency $U$ over a period of period length $d$: Let

$$I(T_i) \;:=\; \frac{1 - P^{U,\mathsf{C}}(T_i+d;T_i)}{P^{U,\mathsf{C}}(T_i+d;T_i)},$$

where $P^{U,\mathsf{C}}(T_i+d;T_i)$ is the discount factor for the maturity $T_i+d$ as seen in time $T_i$. The index $I(T_i)$ is the payment we have to receive in $T_i+d$ collateralized with respect to the collateral account $\mathsf{C}$, such that $1 + I(T_i)$ in $T_{i+1}$ has the same value as $1$ in $T_i$. This index has a special property, namely that its forward can be expressed in terms of the discount factor curve $P^{U,\mathsf{C}}$ too: The time $t$ forward of $I(T_i)$ is $F(T_i;t)$ where

$$\begin{aligned}
F(T_i;t) \cdot P^{U,\mathsf{C}}(T_i+d;t) \;&=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left(\frac{I(T_i) \cdot U}{N^{U,\mathsf{C}}(T_i+d)} \mid \mathcal{F}_t\right) \\
&=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left(\frac{I(T_i) \cdot P^{U,\mathsf{C}}(T_i+d;T_i)}{N^{U,\mathsf{C}}(T_i)} \mid \mathcal{F}_t\right) \\
&=\; N^{U,\mathsf{C}}(t) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left(\frac{1 \cdot U - P^{U,\mathsf{C}}(T_i+d;T_i)}{N^{U,\mathsf{C}}(T_i)} \mid \mathcal{F}_t\right) \\
&=\; P^{U,\mathsf{C}}(T_i;t) - P^{U,\mathsf{C}}(T_i+d;t).
\end{aligned}$$

Thus

$$F(T_i;t) \;=\; \frac{P^{U,\mathsf{C}}(T_i;t) - P^{U,\mathsf{C}}(T_i+d;t)}{P^{U,\mathsf{C}}(T_i+d;t)}.$$

Consequently this index has the property that its forward can be expressed by the same discount factors evaluated at a different time.

**Definition 8 (Forward associated with a Discount Curve):**
Let $P^{U,\mathsf{C}}(T_i+d;t)$ denote a discount curve. For a given period length $d$ we define the forward $F^{d,U,\mathsf{C}}(T_i;t)$ as

$$F^{d,U,\mathsf{C}}(T_i;t) \;:=\; \frac{P^{U,\mathsf{C}}(T_i;t) - P^{U,\mathsf{C}}(T_i+d;t)}{P^{U,\mathsf{C}}(T_i+d;t)\cdot d}. \tag{6}$$

$F^{d,U,\mathsf{C}}(T_i;t)$ is the forward associated with the performance index of $P^{U,\mathsf{C}}$ over a period of length $d$.

**Remark 9 (Forwards from Discount Factors):** The above definition relates a forward curve and discount factor curve. Note however, that we define a forward from a discount factor curve and that this definition is backed by a clear interpretation of the underlying index. Conversely, we may define a discount curve from a forward curve "implicitly" such that the relation (6) holds. Note however, that a generalization of this relation should be considered with care, since the associated product may not exist.

**Remark 10 (Self Discounting):** A product like the OIS swaps are sometimes called "self-discounting" since the discounting is performed on a curve corresponding to the index they fix. From the above, we find an alternative (and maybe more natural) interpretation, namely that the swap pays the performance index of its collateral account, i.e., it pays the index associated with the discount curve.

# 4 Interpolation of Curves

In this section we consider a discount curve $P^{U,\mathsf{C}}$ and an associated forward curve $F^{U,\mathsf{C}}$. To simplify notation we set $D(T) := P^{U,\mathsf{C}}(T;t)$ and $F(T) := F^{U,\mathsf{C}}(T;t)$.

## 4.1 Introduction

Forwards and discount factors are linked together by Definition 5, which says that the value of a forward contract $V(T)$ is the product of the forward $F(T)$ and the associated discount factor $D(T+d)$, i.e., we have

$$V(T) \ = \ F(T) \cdot D(T+d). \tag{7}$$

Note that $V$ and $D$ are *value curves*, i.e., for a fixed $T$ the quantities $V(T)$ and $D(T)$ are values of financial products. However, $F(T)$ is a derived quantity, the forward.

Since $V$ and $D$ represent values of financial products, there is a natural interpretation for a linear interpolation of different values $V(T_i)$ and of different values of $D(T_i)$, since this would correspond to a portfolio of such products. Note that defining an interpolation method for $V$ and $D$ implies a (possible more complex) interpolation method of $F$.

On the other hand, it is common practice to define an interpolation method for a rate curve (both forward curve and discount factor curve) via zero rates, sometimes even regardless of the nature of the curve, which then implies the interpolation of the value curves $D$ and $V$. Some of these interpolations will result in natural interpolations on the value process $V$, others not. Examples are:

- log-linear interpolation of the forward, log-linear interpolation of the discount factor: the case is equivalent to log-linear interpolation of the value.
- linear interpolation of the forward, log-linear interpolation of the discount factor: the case is equivalent with a linear interpolation of the value, with an interpolation weight being a function of the discount factor ratio.

For an in depth discussion of interpolation methods for rate curves see [5].

## 4.2 Interpolation Method and Interpolation Entities

The two basic ingredients for an interpolation are the *interpolation method*, e.g., linear interpolation of interpolation points $\{(T_i, x_i)\}$, and the *interpolation entity*, that is, a (bijective) transformation from $(T,x)$ to the actual curve. For example, for discount curves one might consider a linear interpolation of the zero rate. In this case the interpolation method is *linear interpolation* and the interpolation entity is for $T > 0$

$$(T, x(T)) \ = \ (T, \frac{\log(D(T))}{T}),$$

where $D$ denotes the discount curve. Given $0 < T_i \leq T \leq T_{i+1}$ and discount factors $D(T_j)$, a linear interpolation of the zero rates would then imply the interpolation

$$D(T) \ := \ \exp\left( \left( \frac{T - T_i}{T_{i+1} - T_i} \frac{\log(D(T_{i+1}))}{T_{i+1}} + \frac{T_{i+1} - T}{T_{i+1} - T_i} \frac{\log(D(T_i))}{T_i} \right) \cdot T \right).$$

### 4.2.1 Interpolation of Forward Curves

For forward curves, a common approach is to consider an interpolation of the forward as an independent entity (like for the discount curve). For interest rate forwards, a popular interpolation scheme (coming from the single curve interpretation of interest rates forwards) is to represent the forward in terms of synthetic discount factors. That is, if $d$ denotes a period length associated with the forward and if $F(T_i)$ is given for $T_i = i \cdot d$, then one might consider interpolation of (pseudo-)discount factor

$$D^F(T_i) \;:=\; \prod_{k=0}^{i-1} (1 + F(T_k) \cdot p)^{-1},$$

possibly considering another transformation on $D^F(T)$ to define the actual interpolation entity.

It is obvious that this definition of the interpolation entity for forward curve is complex, results in problems for non-equidistant interpolation points and is - without further assumptions - not backed by a meaningful interpretation.

### 4.2.2 Interpolation of Forward Curve via Value Curve

The definition of the forward curve suggests an appealing alternative for the creation of an interpolated forward: Like a discount factor curve, the curve $V(T) = F(T) \cdot D(T+d)$ represents the value of a financial product. Hence, we may consider the interpolation of $V$ like we did for the curve $D$. For example, if we consider linear interpolation of the value curve $V$, we interpolate the forward curve $F$ by considering the interpolation entity $F(T) \cdot D(T+d)$ with a given discount curve $D$, i.e., we have

$$F(T) \;:= \\ \frac{1}{D(T+d)} \left( \frac{T - T_i}{T_{i+1} - T_i} F(T_{i+1}) D(T_{i+1} + d) + \frac{T_{i+1} - T}{T_{i+1} - T_i} F(T_i) D(T_i + d) \right).$$

for $T_i \leq T \leq T_{i+1}$ and given points $F(T_j)$.

# 5 Calibration of Curves

## 5.1 Global Optimization

A curve (discount curve or forward curve) is used to encode values of market instruments. A forward curve together with it's associated discount curve, allows to value all linear products (linear payoffs) in the corresponding currency under the corresponding collateralization.

The standard way to calibrate a curve is, hence, to obtain given market values of (linear) instruments (e.g., swaps). For each market value a single "point" in a single curve is calibrated. Hence the total number of curve calibrated interpolation points (aggregated across all curves) equals the number of market instruments.

By "'sorting"' and combining the calibration instruments, the corresponding equations can be brought into the form of a system of a equations with a triangular structure, i.e., the value of the $n$-th calibration instruments only depends on the first $n$ curve points. This allows for an iterative construction of the curve.

However, in [3] we favor to calibrate the system of equations using a multi-variate optimization algorithm, like the Levenberg-Marquard algorithm. This approach brings serval advantages, e.g., the freedom to specify the calibration instruments and the ability to extend the approach to over-determined systems of equations. This comes at the cost of slower performance in terms of required calculation time.

What remains is to specify the valuation equations for the calibration instruments.

## 5.2 Generalized Definition of a Swap

Many of the following calibration instruments (from OIS swaps to cross-currency basis-swaps) fit under a generalized definition of a swap. The swap consists of two legs. Each leg consists of several periods $[T_i, T_{i+1}]$. We do not distinguish between period start time, period end time, fixing time of the index and payment time. We assume that for the period $[T_i, T_{i+1}]$ index fixing is in $T_i$ and payment is in $T_{i+1}$. This is done purely to ease notation, the generalization to distinguished times is straight forward.

**Definition 11 (Swap Leg):** ⌝
A swap leg pays the a multiple $\alpha$ of the index $I$ fixed in $T_i$ plus some fixed payment $X$, both in currency unit $U$ collateralized by the collateral account $\mathsf{C}$ and paid in time $T_{i+1}$. Here $\alpha$ and $X$ are constants (possibly zero). The value of the swap leg can be expressed in terms of forwards and discount factors as

$$V^{U,\mathsf{C}}_{SwapLeg}(\alpha I, X, \{T_i\}_{i=0}^{n}; t) \ = \ \sum_{i=0}^{n-1} \Big( \alpha F^{U,\mathsf{C}}(T_i) + X \Big) \cdot P^{U,\mathsf{C}}(T_{i+1}),$$

where $F^{U,\mathsf{C}}$ denotes the forward curve of the index $I$ paid in currency $U$ collateralized with respect to $\mathsf{C}$ and $P^{U,\mathsf{C}}$ denotes the corresponding discount curve. ⌟

**Definition 12 (Swap Leg with Notional Exchange):** ⌝
A swap leg with notional exchange has the payments as in Definition 11 together with an additional payment of $-1$ in $T_i$ and $+1$ in $T_{i+1}$. The value of the swap leg with

notional exchange can be expressed in terms of forwards and discount factors as

$$V^{U,\mathsf{C}}_{SwapLeg}(\alpha I, X, \{T_i\}_{i=0}^{n}; t) \;=\; \sum_{i=0}^{n-1} \Big( \Big( \alpha F^{U,\mathsf{C}}(T_i) + X \Big) \cdot P^{U,\mathsf{C}}(T_{i+1}) \\ + P^{U,\mathsf{C}}(T_{i+1}) - P^{U,\mathsf{C}}(T_i) \Big) \,,$$

where $F^{U,\mathsf{C}}$ denotes the forward curve of the index $I$ paid in currency $U$ collateralized with respect to $\mathsf{C}$ and $P^{U,\mathsf{C}}$ denotes the corresponding discount curve.

**Definition 13 (Swap):**
A swap exchanges the payments of two swap legs, the receiver leg and the payer leg. We allow that the legs have different indices, different fixed payments, different payment times, different currency units, but are collateralized with respect to the same account $\mathsf{C}$. The swaps receives a swap leg with value $V^{U_1,\mathsf{C}}_{SwapLeg}(\alpha_1 I_1, X_1, \{T^1_i\}_{i=0}^{n_1}; t)$ and pays a leg with value $V^{U_2,\mathsf{C}}_{SwapLeg}(\alpha_2 I_2, X_2, \{T^2_i\}; t)$. Since the currency unit of the two legs may be different, the value of the swap in currency $U_1$ is

$$\begin{aligned} & V_{Swap}(t) \\ & = V^{U_1,\mathsf{C}}_{SwapLeg}(\alpha_1 I_1, X_1, \{T^1_i\}_{i=0}^{n_1}; t) \;-\; V^{U_2,\mathsf{C}}_{SwapLeg}(\alpha_2 I_2, X_2, \{T^2_i\}_{i=0}^{n_2}; t) \cdot FX^{\frac{U_1}{U_2}}(t) \\ & = \sum_{i=0}^{n_1-1} \Big( \Big( \alpha F^{U_1,\mathsf{C}}_1(T^1_i) + X_1 \Big) \cdot P^{U_1,\mathsf{C}}_1(T^1_{i+1}) \Big) \\ & \quad - \sum_{i=0}^{n_2-1} \Big( \Big( \alpha F^{U_2,\mathsf{C}}_2(T^2_i) + X_2 \Big) \cdot P^{U_2,\mathsf{C}}_2(T^2_{i+1}) \Big) \cdot FX^{\frac{U_1}{U_2}}(t). \end{aligned}$$

**Definition 14 (Swap):**
A swap with notional exchange exchanges the payments of two swap legs with notional exchange. The value of the swap in currency $U_1$ is

$$\begin{aligned} & V_{Swap}(t) \\ & = \sum_{i=0}^{n_1-1} \Big( \Big( \alpha F^{U_1,\mathsf{C}}_1(T^1_i) + X_1 + 1 \Big) \cdot P^{U_1,\mathsf{C}}_1(T^1_{i+1}) - P^{U_1,\mathsf{C}}(T^1_i) \Big) \\ & \quad - \sum_{i=0}^{n_2-1} \Big( \Big( \alpha F^{U_2,\mathsf{C}}_2(T^2_i) + X_2 + 1 \Big) \cdot P^{U_2,\mathsf{C}}_2(T^2_{i+1}) - P^{U_2,\mathsf{C}}(T^2_i) \Big) \cdot FX^{\frac{U_1}{U_2}}(t). \end{aligned}$$

Many instruments can be represented (and hence valued) in this form. We will now list a few of them.

## 5.3 Calibration of Discount Curve to Swap Paying the Collateral Rate (aka. Self-Discounted Swaps)

Discount curves can be calibrated to swaps paying the performance index of their collateral account. For example a swap as in Definition 13 where both leg pay in the same currency $U = U_1 = U_2$. In a receiver swap the receiver leg pays a fixed rate $C$, and the payer leg pays an index $I$. Thus the value of the swap can be expressed in terms of the discount factors $P^{U,\mathsf{C}}(T_{i+1};t)$ only, which allows to calibrate this curve using these swaps. Overnight index swaps are an example.

For the swap paying the performance of the collateral account we have

$$\begin{aligned} X_1 &= C = \text{const.} = \text{given}, & X_2 &= 0, \\ F_1^{U_1,\mathsf{C}}(T_i^1) &= 0, & F_2^{U_2,\mathsf{C}}(T_i^2) &= \frac{P^{U,\mathsf{C}}(T_i^2;t) - P^{U,\mathsf{C}}(T_{i+1}^2;t)}{P^{U,\mathsf{C}}(T_{i+1}^2;t)(T_{i+1}^2 - T_i^2)}, \\ P_1^{U_1,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{calibrated}, & P_2^{U_2,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{calibrated}. \end{aligned}$$

From one such swap we calibrate the time $T$ discount factor $P^{U,\mathsf{C}}(T;t)$ with $T = \max(T_n^1, T_n^2)$ (the last payment time).

## 5.4 Calibration of Forward Curves

### 5.4.1 Fix-versus-Float Swaps

Given a calibrated discount curve $P^{U,\mathsf{C}}$ we consider a swap with payments in currency $U$ collateralized with respect to the account $\mathsf{C}$, paying some index $I$ and receiving some fixed cash flow $C$. An example are swaps paying the 3M LIBOR rate. For such a swap we have

$$\begin{aligned} X_1 &= C = \text{const.} = \text{given}, & X_2 &= 0, \\ F_1^{U_1,\mathsf{C}}(T_i^1) &= 0, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F^{U,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ P_1^{U_1,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{given}, & P_2^{U_2,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{given}. \end{aligned}$$

From one such swap we calibrate the time $T$ forward $F^{U,\mathsf{C}}(T)$ of $I(T)$ with $T = T_{n-1}^2$) (the last fixing time).

### 5.4.2 Float-versus-Float Swaps (Tenor Basis Swaps)

Given a calibrated discount curve $P^{U,\mathsf{C}}$ and a calibrate forward curve $F_1^{U,\mathsf{C}}$ belonging to the index $I_1$, both in currency $U$ and collateralized with respect to the account $\mathsf{C}$, we consider a swap collateralized with respect to the account $\mathsf{C}$, paying some index $I_2 = I$ in currency $U$, receiving the index $I_1$ in currency $U$. An example are tenor basis swaps paying the 6M LIBOR rate, receiving the 3M LIBOR rate. For such a swap we have

$$\begin{aligned} X_1 &= C_1 = \text{const.} = \text{given}, & X_2 &= C_2 = \text{const.} = \text{given}, \\ F_1^{U_1,\mathsf{C}}(T_i^1) &= F_1^{U,\mathsf{C}}(T_i^1) = \text{given}, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F_2^{U,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ P_1^{U_1,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{given}, & P_2^{U_2,\mathsf{C}} &= P^{U,\mathsf{C}} = \text{given}. \end{aligned}$$

From one such swap we calibrate the time $T$ forward $F_2^{U,\mathsf{C}}(T)$ of $I(T)$ with $T = T_{n-1}^2$ (the last fixing time of index $I_2$).

## 5.5 Calibration of Discount Curves when Payment and Collateral Currency Differ

### 5.5.1 Fixed Payment in Other Currency

Given a calibrated discount curve $P^{U_1,\mathsf{C}}$ we consider a swap collateralized with respect to the account $\mathsf{C}$, paying some index $I_1$ in currency $U_1$, and receiving some fixed cash flow $C_2$ in currency $U_2$. An example for such a swap is a cross-currency swap paying floating index $I$ in collateral currency and receiving fixed $C_2$ in a different currency.[4] For such a swap we have

$$
\begin{aligned}
X_1 &= C_1 = \text{const.} = \text{given}, & X_2 &= C_2 = \text{const.} = \text{given},\\
F_1^{U_1,\mathsf{C}}(T_i^1) &= F_1^{U_1,\mathsf{C}}(T_i^1) = \text{given}, & F_2^{U_2,\mathsf{C}}(T_i^2) &= 0,\\
P_1^{U_1,\mathsf{C}} &= P^{U_1,\mathsf{C}} = \text{given}, & P_2^{U_2,\mathsf{C}} &= P^{U_2,\mathsf{C}} = \text{calibrated}.
\end{aligned}
$$

We calibrate the discount factor $P^{U_2,\mathsf{C}}(T;t)$ with $T = T_n^2$ (last payment time in currency $U_2$).

### 5.5.2 Float Payment in Other Currency

If instead of a fixed payment we have that an index $I_2$ is paid in an other currency $U_2$ we may encounter the problem that the swap has two unknowns, namely the discount curve $P^{U_2,\mathsf{C}}$ for payments in currency $U_2$ collateralized with respect to $\mathsf{C}$ and the forward curve $F_2^{U_2,\mathsf{C}}$ of the index $I_2$ paid in currency $U_2$ collateralized with respect to $\mathsf{C}$. The two curves can be obtained jointly from two different swaps: first a fix-versus-float swaps in currency $U_2$ collateralized by $\mathsf{C}$ (as in Section 5.4.1), and second a cross-currency swap exchanging the index $I_2$ with an index $I_1$ in currency $U_1$ for which the forward $F_1^{U_1,\mathsf{C}}$ is known. For the first instrument we denote the fixed payment by $C_1$, $C_2$. For the second instrument we denote the fixed payment by $s_1$, $s_2$ (usually a spread). For the first instrument we have

$$
\begin{aligned}
X_1 &= C_1 = \text{const.} = \text{given}, & X_2 &= C_2 = \text{const.} = \text{given},\\
F_1^{U_1,\mathsf{C}}(T_i^1) &= 0, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F_2^{U_2,\mathsf{C}}(T_i^2) = \text{calibrated},\\
P_1^{U_1,\mathsf{C}} &= P_2^{U_2,\mathsf{C}} = \text{calibrated}, & P_2^{U_2,\mathsf{C}} &= \text{calibrated}.
\end{aligned}
$$

for the second swap we have

$$
\begin{aligned}
X_1 &= s_1 = \text{const.} = \text{given}, & X_2 &= s_2 = \text{const.} = \text{given},\\
F_1^{U_1,\mathsf{C}}(T_i^1) &= F_1^{U_1,\mathsf{C}}(T_i^1) = \text{given}, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F_2^{U_2,\mathsf{C}}(T_i^2) = \text{calibrated},\\
P_1^{U_1,\mathsf{C}} &= \text{given}, & P_2^{U_2,\mathsf{C}} &= \text{calibrated}.
\end{aligned}
$$

We calibrate the discount factor $P^{U_2,\mathsf{C}}(T;t)$ with $T = T_n^2$ and the forward $F_2^{U_2,\mathsf{C}}(\mathrm{T})$ with $T = T_{n-1}^2$.

Often market data is not available to calibrate the forward $F_2^{U_2,\mathsf{C}}$, but the forward $F_2^{U_2,\mathsf{C}_2}$ collateralized with respect to a different account $\mathsf{C}_2$ is available. The two forwards differ by a possible convexity adjustment. One possible approximation (which

[4] Usually cross-currency swaps exchange two floating indices, we will consider this case below.

would follow from the assumption that forwards are independent of their collateralization) is to use $F_2^{U_2,\mathsf{C}} \approx F_2^{U_2,\mathsf{C}_2}$.

The joint calibration of the two curves can be decomposed into two independent calibration steps, which would then allow to re-use a traditional bootstrap algorithm, see, e.g., [1].

## 5.6 Calibration of Discount Curves as Spread Curves

We consider a swap leg with notional exchange and tenor $\{T_i\}_{i=0}^n$, paying an index $I$ plus some constant $X = s(T_n) = const.$ Here $s(T_n)$ has the interpretation of an maturity dependent spread. If this leg is in currency $U$ and with respect to a collateral account (here funding account) $\mathsf{D}$, then its value is

$$\begin{aligned} V^{U,\mathsf{D}}_{SwapLeg}(\alpha I, X, \{T_i\}_{i=0}^n; t) \ = \ \sum_{i=0}^{n-1} \Big( \Big( \alpha F^{U,\mathsf{D}}(T_i) + X \Big) \cdot P^{U,\mathsf{D}}(T_{i+1}) \\ + P^{U,\mathsf{D}}(T_{i+1}) - P^{U,\mathsf{D}}(T_i) \Big) . \end{aligned}$$

An example of such an instrument is an (uncollateralized) floating rate bond, paying an 3M rate plus some spread. If we assume that the forward $F^{U,\mathsf{D}}(T_i)$ is known, this instrument can be used to calibrate the discount curve $P^{U,\mathsf{D}}$. In fact $I + X$ represents the performance of the funding account associated with $P^{U,\mathsf{D}}$.

If the forward $F^{U,\mathsf{D}}(T_i)$ is not known, we encounter the same problem as for cross-currency swaps, namely that the forward curve $F^{U,\mathsf{D}}(T_i)$ and the discount curve $P^{U,\mathsf{D}}$ need to be calibrated jointly to two instruments. The first one is a swap which is collateralized with respect to the funding account $\mathsf{D}$, i.e., it is an uncollateralized swap. The second is the funding floater.

For first instrument, the uncollateralized swap, we have

$$\begin{aligned} X_1 \ &= \ C_1 \ = \ \text{const.} \ = \ \text{given}, & X_2 \ &= \ C_2 \ = \ \text{const.} \ = \ \text{given}, \\ F_1^{U,\mathsf{D}}(T_i^1) \ &= \ 0 \ = \ \text{given}, & F_2^{U,\mathsf{D}}(T_i^2) \ &= \ F^{U,\mathsf{D}}(T_i^2) \ = \ \text{calibrated}, \\ P_1^{U,\mathsf{D}} \ &= \ P^{U,\mathsf{D}} \ = \ \text{calibrated}, & P_2^{U,\mathsf{D}} \ &= \ P^{U,\mathsf{D}}. \end{aligned}$$

for the second instrument, the funding floating rate bond (uncollateralized swap leg with notional exchange) we have

$$\begin{aligned} X_1 \ &= \ S \ = \ \text{const.} \ = \ \text{given}, \\ F_1^{U,\mathsf{D}}(T_i^1) \ &= \ F^{U,\mathsf{D}}(T_i^1) \ = \ \text{calibrated}, \\ P_1^{U,\mathsf{D}} \ &= \ P^{U,\mathsf{D}} \ = \ \text{calibrated}. \end{aligned}$$

**Remark 15 (Cross Currency Analogy of Funding):** The calibration of the funding curve $P^{U,\mathsf{D}}$ is analog to the calibration of the cross-currency discount curve $P^{U_2,\mathsf{C}}$.

### 5.7 Lack of Calibration Instruments (for Difference in Collateralization)

The calibration of cross-currency curves (forward curve and discount curves for currency $U_2$ with collateralization in currency $U_1$, see Section 5.5) and the calibration of un-collateralized curves (forward curves and discount curves for un-collaterlized products, see Section 5.6) may require market data which is not available, e.g., the forward of an index $I$ paid in currency $U_2$ collateralized in a different currency or by a different account.

In this case the curve can be obtained by adding additional model assumptions. Two simple examples are:

- the market rates are assumed to be independent of the type of collateralization, or
- the forward rates are assumed to be independent of the type of collateralization.

The two assumptions lead to different results, since they imply different correlations which will lead to different (convexity) adjustments. Figure 1 shows the difference in the forward rates assuming identical market rates for 3M swaps collateralized in USD-OIS or EUR-OIS, which is around 1 or 2 basis points (0.01%). For details on the example see [3].

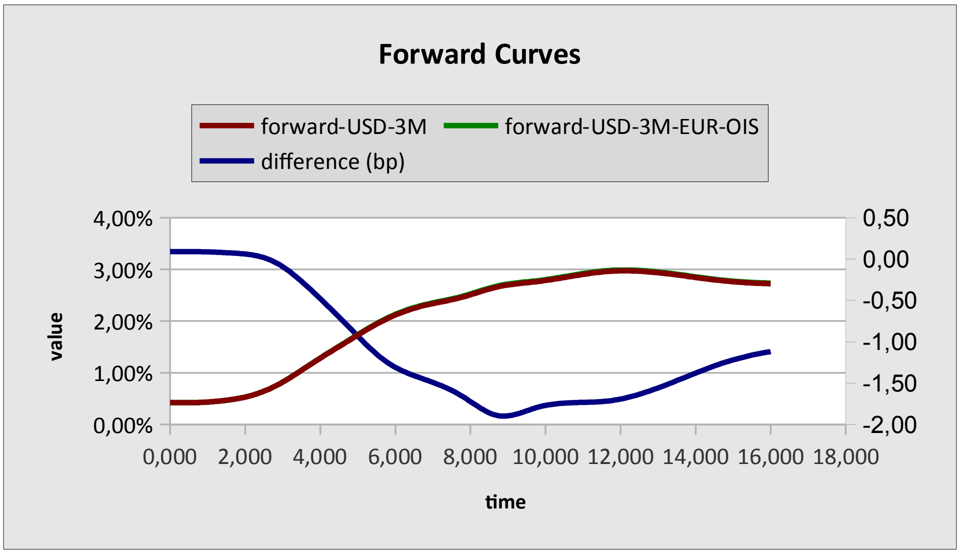


**Figure 1:** Forward curve (USD-3M) calibrated from swaps with with different collateralization (USD-OIS and EUR-OIS) assuming independence of the market rates of from the type of collateralization.

# 6 Sample Code for Curve Calibration

A reference implementation is available via [3] (source code and demo spreadsheet).

We implement discount curves and forward curves in separate classes. This is necessary since

- the forward curve has an associated discount curves which describes the collateralization of the index,
- the interpolation entity of the two curves may be different.

Both classes inherit from a base class which may perform several interpolation methods on the interpolation entities.

The curve represent the independent parameters of the calibration solver. The dependent values are represented by the generalized swap as in Definition 14. For each such calibration instrument the curve to calibrate and the time to calibrate are specified. The solver performs a global optimization. Thus resolving interdependencies among the products (e.g., through a specific ordering) is not necessary.

# 7 Redefining Forward Rate Market Models

Let $N^{\mathsf{C}}$ denote an accrual account, i.e., $N^{\mathsf{C}}$ is a process with $N^{\mathsf{C}}(t_0) = 1$ (e.g., a collateral account). Then $N^{\mathsf{C}}$ defines a discount curve, namely the discount curve $P^{\mathsf{C}}(T_{i+1}; t)$ of fixed payments made in $T$, valued in $t$ and collateralized by units of $N^{\mathsf{C}}$.

Let $\{T_i\}$ denote a given tenor discretization. As shown in Section 3.2.1 the period-$[T_i, T_{i+1}]$ performance index $I(T_i, T_{i+1})$ of the an accrual account, i.e.,

$$I(T_i, T_{i+1}; T_i) \ := \ \frac{N^{\mathsf{C}}(T_{i+1})}{N^{\mathsf{C}}(T_i)} - 1$$

has the property that it's time $t$ *forward* (of a payment of $I(T_i, T_{i+1})$ made in $T_{i+1}$, collateralized in units of $N^{\mathsf{C}}$) (following the definition of a forward from Section 3.2) is given as

$$I(T_i, T_{i+1}; t) \ := \ \frac{P^{\mathsf{C}}(T_i; t) - P^{\mathsf{C}}(T_{i+1}; t)}{P^{\mathsf{C}}(T_{i+1}; t)}$$

This relation allows us to create a term-structure model for the curve $P^{\mathsf{C}}$ which has the same structural properties as a standard single curve (LIBOR) market model. This model is given by

- A joint modeling of the processes $L_i(t) := \frac{I(T_i, T_{i+1}; t)}{T_{i+1} - T_i}$, e.g., as log-normal processes under the measure $\mathbb{Q}^{N^{\mathsf{C}}}$.
- The additional assumption that the process $P^{\mathsf{C}}(T_i; t)$ is deterministic on it's short period $t \in (T_{i-1}, T_i]$.

From these two assumptions it follows that the processes $L_i$ have the structure of a standard LIBOR market model and $\mathbb{Q}^{N^{\mathsf{C}}}$ corresponds to the spot measure. Indeed we have

$$\begin{aligned}\prod_{j=0}^{i-1} 1 + L_j(T_j) \cdot (T_{j+1} - T_j) \ &= \ \prod_{j=0}^{i-1} 1 + I(T_j, T_{j+1}; T_j) \\ &= \ \prod_{j=0}^{i-1} \frac{N^{\mathsf{C}}(T_{j+1})}{N^{\mathsf{C}}(T_j)} \ = \ N^{\mathsf{C}}(T_i).\end{aligned}$$

What we have described is how to use the standard LIBOR market model as a term structure model for the collateral account $N^{\mathsf{C}}$ (e.g., the OIS curve). Now, modeling all other rates (including LIBOR) can be performed by modeling (possibly stochastic) spreads over this curve. This is analog to a defaultable market model.

## References

*The literature on multi-curve modeling, discounting, funding and collateralization is huge. We just give a few references related to the construction of curves.*

[1] CLARK, JUSTIN: Constructing a Non-USD Swap Discounting Curve with USD Collateral.
http://www.edurisk.ie

[2] FRIES, CHRISTIAN P.: Funded Replication: Fund Exchange Process and the Valuation with Different Funding-Accounts (Cross-Currency Analogy to Funding Revisited). July 2012.
http://papers.ssrn.com/abstract=2115839.

[3] FRIES, CHRISTIAN P.: Curve Calibration. Object oriented reference implementation.
http://www.finmath.net/topics/curvecalibration.

[4] FRIES, CHRISTIAN P.: LIBOR Market Model. Object oriented reference implementation.
http://www.finmath.net/topics/libormarketmodel.

[5] HAGAN, PATRICK S.; WEST, GRAEME: Interpolation Methods for Curve Construction. *Applied Mathematical Finance*, Vol. 13, No. 2, 89-129, June 2006

[6] MERCURIO, FABIO: LIBOR Market Models with Stochastic Basis, March 2010.

# Notes

## Suggested Citation

FRIES, CHRISTIAN P.: Curves and Term Structure Models.
http://ssrn.com/abstract=2194907
http://www.christian-fries.de/finmath/curves

## Classification

Classification: MSC-class: 65C05 (Primary), 68U20, 60H35 (Secondary).
ACM-class: G.3; I.6.8.
JEL-class: G13.

Keywords: Curves, Discount Curve, Forward Curve, Calibration, Bootstrapping, Multi-Curve, Tenor-Basis, Cross-Currency-Basis, Collateralization, Funding, OIS Discounting, Funding Curve, Spread Curve

22 pages. 1 figures. 0 tables.