"P" versus "Q": Differences and Commonalities between the Two Areas of Quantitative Finance

Attilio Meucci[1]
www.symmys.com

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Abstract

There exist two separate branches of finance that require advanced quantitative techniques: the "$\mathbb{Q}$" area of derivatives pricing, whose task is to "extrapolate the present"; and the "$\mathbb{P}$" area of quantitative risk and portfolio management, whose task is to "model the future".

We briefly trace the history of these two branches of quantitative finance, highlighting their different goals and challenges. Then we provide an overview of their areas of intersection: the notion of risk premium; the stochastic processes used, often under different names and assumptions in the $\mathbb{Q}$ and in the $\mathbb{P}$ world; the numerical methods utilized to simulate those processes; hedging; and statistical arbitrage.

JEL Classification: C1, G11

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1 Derivatives pricing: the "$\mathbb{Q}$" world

In this section we provide a brief overview of the "$\mathbb{Q}$" world of derivatives pricing. Refer to Figure 1 below for a summary.

The goal of derivatives pricing is to determine the fair price of a given security in terms of more liquid securities whose price is determined by the law of supply and demand. Examples of securities being priced are exotic options, mortgage backed securities, convertible bonds, structured products, etc. Once a fair price has been determined, the sell-side trader can make a market on the security. Therefore, derivatives pricing is a complex "extrapolation" exercise to define the current market value of a security, which is then used by the sell-side community.

Goal:	"**extrapolate the present**"
Environment:	risk-neutral probability $\mathbb{Q}$
Processes:	continuous-time martingales
Dimension:	low
Tools:	Ito calculus, PDE's
Challenges:	calibration
Business:	sell-side

Figure 1: The $\mathbb{Q}$ world of derivatives pricing

Quantitative derivatives pricing was initiated by Bachelier (1900) with the introduction of the most basic and most influential of processes, the Brownian motion, and its applications to the pricing of options. The theory remained dormant until Merton (1969) and Black and Scholes (1973) applied the second most influential process, the geometric Brownian motion, to option pricing. The next important step was the fundamental theorem of asset pricing by Harrison and Pliska (1981), according to which the suitably normalized current price P_0 of a security is arbitrage-free, and thus truly fair, only if there exists a stochastic process P_t with constant expected value which describes its future evolution:

$$P_0 = \mathbb{E}\{P_t\}, \quad t \geq 0. \tag{1}$$

A process satisfying (1) is called a "martingale". A martingale does not reward risk. Thus the probability of the normalized security price process is called "risk-neutral" and is typically denoted by the blackboard font letter "$\mathbb{Q}$". The relationship (1) must hold for all times t: therefore the processes used for derivatives pricing are naturally set in continuous time.

The quants who operate in the $\mathbb{Q}$ world of derivatives pricing are specialists with deep knowledge of the specific products they model. Securities are priced individually, and thus the problems in the $\mathbb{Q}$ world are low-dimensional in nature.

Calibration is one of the main challenges of the $\mathbb{Q}$ world: once a continuous-time parametric process has been calibrated to a set of traded securities through a relationship such as (1), a similar relationship is used to define the price of new derivatives.

The main quantitative tools necessary to handle continuous-time $\mathbb{Q}$-processes are Ito's stochastic calculus and partial differential equations (PDE's). Throughout the past decades, these advanced techniques have attracted mathematicians, physicists and engineers to the field of derivatives pricing.

2 Risk and portfolio management: the "$\mathbb{P}$" world

In this section we provide a brief overview of the "$\mathbb{P}$" world of risk and portfolio management. Refer to Figure 2 below for a summary.

Risk and portfolio management aims at modelling the probability distribution of the market prices at a given future investment horizon. This "real" probability distribution of the market prices is typically denoted by the blackboard font letter "$\mathbb{P}$", as opposed to the "risk-neutral" probability "$\mathbb{Q}$" used in derivatives pricing. Based on the $\mathbb{P}$ distribution, the buy-side community takes decisions on which securities to purchase in order to improve the prospective profit-and-loss profile of their positions considered as a portfolio.

Goal:	"**model the future**"
Environment:	real probability $\mathbb{P}$
Processes:	discrete-time series
Dimension:	large
Tools:	multivariate statistics
Challenges:	estimation
Business:	buy-side

Figure 2: The $\mathbb{P}$ world of risk and portfolio management

The quantitative theory of risk and portfolio management started with the mean-variance framework of Markowitz (1952). Next, breakthrough advances were made with the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) developed by Treynor (1962), Mossin (1966), Sharpe (1964), Lintner (1965) and Ross (1976).

The above theories provide tremendous insight into the markets, but they assume that the probability distribution $\mathbb{P}$ is known. In reality, the probability distribution $\mathbb{P}$ must be estimated from available information. A major component of this information set is the past dynamics of prices and other financial variables, which are monitored at discrete time intervals and stored in the form of time series.

Estimation represents the main quantitative challenge in the $\mathbb{P}$ world of risk and portfolio management. The analysis of the time series requires advanced multivariate statistics and econometric techniques. Notice that in risk and portfolio management it is important to estimate the joint distribution of all the securities in that market, and thus securities cannot be considered individually as in the $\mathbb{Q}$ world of derivatives pricing. Therefore dimension reduction techniques such as linear factor models play a central role in the $\mathbb{P}$ world.

To address the above issues, in recent years a new breed of quants, the $\mathbb{P}$-quants, has started to populate the financial industry and more $\mathbb{P}$-quants are being trained in the same masters programs that were originally designed to prepare $\mathbb{Q}$-quants.

3 Commonalities between $\mathbb{P}$ and $\mathbb{Q}$

From a comparison of Figure 1 and Figure 2 it appears that the $\mathbb{P}$ and the $\mathbb{Q}$ world of quantitative finance are very different. In reality, commonalities between these two worlds abound and interactions occur frequently in different areas, as we proceed to illustrate.

3.1 Risk premium

Mathematically, the risk-neutral probability $\mathbb{Q}$ and the real probability $\mathbb{P}$ associate different weights to the same possible outcomes for the same financial variables. The transition from one set of probability weights to the other defines the so-called risk-premium. Knowledge of the risk-premium allows us in principle to switch from one world to the other. Unfortunately, the correct estimation of the risk premium is a challenging task.

3.2 Stochastic processes

Stochastic processes are the building blocks of any quantitative model, both in the $\mathbb{P}$ world and in the $\mathbb{Q}$ world. Although $\mathbb{Q}$-quants focus on continuous risk-neutral processes and $\mathbb{P}$-quants focus on discrete-time processes, the same models are used in both areas, possibly under different assumptions and names.

Below we provide a brief overview of such processes and of their main features, which we summarize in Figure 3. The interested reader can find in Meucci (2009a) a more exhaustive overview, a thorough theoretical discussion, an empirical analysis, fully documented code, and further references.

The most fundamental discrete-time process is the random walk, which is the cumulative sum of invariants, i.e. variables that behave independently and identically across time. The random walk is the baseline assumption to model interest rates or the log-price of stocks in risk and portfolio management. In continuos time, random walks become Levy processes. The Brownian motion,

	Discrete-Time	Continuous-Time
Base case:	random walk	Levy (Brownian, Poisson)
Autocorrelation:	ARMA	Ornstein-Uhlenbeck
Vol. clustering:	GARCH	stochastic volatility subordination

Figure 3: Fundamental stochastic processes for the $\mathbb{P}$ and the $\mathbb{Q}$ worlds

which is the most notable instance of a Levy process, is the baseline process for option pricing. Similarly, the Poisson process, which is the second simplest Levy process, pervades the pricing of credit products.

A second class of processes models autocorrelation. Autocorrelation occurs when financial series are not the sum of independent increments. The standard tool to model autocorrelation in discrete time are auto-regressive-moving-average (ARMA) processes, a favorite of buy-side econometricians. The continuous-time version of ARMA processes are Ornstein-Uhlenbeck and related processes. In particular, two forms of the Ornstein-Uhlenbeck process, namely the model by Vasicek (1977) and the model by Cox, Ingersoll, and Ross (1985) (CIR), represent the base case sell-side processes to price bonds.

A third class of processes models volatility clustering: periods of increased activity tend to occur in bulks. In discrete time, GARCH and generalizations thereof capture this feature for the buy-side audience. On the sell-side, volatility clustering is modeled in two ways: stochastic volatility and subordination. In particular, the model by Heston (1993), derived from the CIR model, is the most popular stochastic volatility model to price derivatives.

3.3 Numerical methods

The theoretical stochastic processes discussed above must be implemented in practice. In order to do so, the most popular numerical techniques are "trees" and Monte Carlo simulations.

Trees represent a process as a sequence of an ever-expanding set of potential outcomes: the state of the world today will give rise to multiple possible outcomes tomorrow; each of these in turn will give rise to multiple possible outcomes the day after tomorrow, and so on. As we see, with trees the number of potential outcomes grows with the horizon.

With Monte Carlo simulations, the number of possible outcomes that represent a stochastic process, known as "paths", is kept constant throughout the evolution of the process.

The computationally more costly trees are used when it is important to make decisions along the trajectory of the stochastic process, whereas Monte Carlo is used when only the process distribution is required. Therefore, in the $\mathbb{P}$ world of risk and portfolio management, trees are used to design dynamic strategies, whereas Monte Carlo is used for risk monitoring purposes such as value-at-risk computations. In the $\mathbb{Q}$ world, trees are used for instance to price "American" options, which can be exercised earlier than at expiry, whereas Monte Carlo is used to price "Asian" options, i.e. options on the average price of an underlying instrument over a pre-specified period of time.

3.4 Hedging

Hedging is a clear example where the $\mathbb{P}$ world and the $\mathbb{Q}$ world interact directly.

Hedging aims at protecting the future p&l of a given position from a set of risk factors. Therefore, hedging is a $\mathbb{P}$-world concept.

In order to determine the amounts of the hedging instruments to buy or sell, we need to compute the sensitivity of the given position and of the hedging instruments to those risk factors.

Such sensitivities are known as the "Greeks". The most basic "Greek" is the "delta" of an option written on a given security, which is the sensitivity of the option to the underlying security. The delta of an option tells the trader how much underlying to sell in order to protect the option from swings of the underlying.

The "Greeks" are computed using pricing models from the $\mathbb{Q}$ world and then applied in the $\mathbb{P}$-world for hedging. Interestingly, those very same $\mathbb{Q}$-world pricing models can be derived based on the $\mathbb{P}$-world concept of hedging.

3.5 Statistical arbitrage

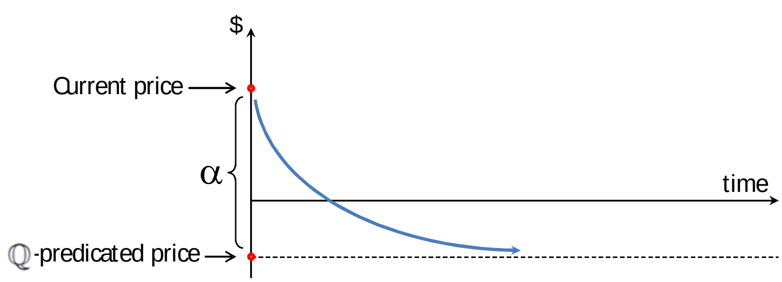

Figure 4: Statistical arbitrage short position

The $\mathbb{Q}$ world has moved into the $\mathbb{P}$ world also in the area of statistical arbitrage algorithms. The general steps of this interaction are as follows.

First, $\mathbb{Q}$ models are used to identify misalignments among security prices today. Second, one assumes that the misaligned security prices will eventually converge to the prices predicated by the $\mathbb{Q}$ models. Therefore, a potential expected return in the real $\mathbb{P}$ world, or "alpha", is identified as the difference between the $\mathbb{Q}$-predicated prices and the current misaligned prices. Third, if the alpha is positive, a long position is set up, i.e. the misaligned securities are bought; if the alpha is negative, a short position is set up, i.e.the misaligned securities are sold, see Figure 4 and refer to Meucci (2009b) for more details.

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