REPLICATION PRICING FOR LINEAR AND TEC FORMAT CMS

OPENGAMMA QUANTITATIVE RESEARCH

ABSTRACT. Technical details for Constant Maturity Swap pricing by replication on cash-settled swaptions. Valid for linear standard CMS and non-linear TEC-like CMS.

1. INTRODUCTION

Constant Maturity Swaps (CMS) are floating instruments paying coupons related to a swap rate fixing.

The standard linear CMS coupon pays the swap rate over the coupon period (typically three or six months), typically with the rate reset at the start of the period (fixing in advance, paid in arrears): however, fixing in arrears is relatively frequent. CMS cap/floors pay a typical cap/floor pay-off but based on the swap rate instead of an Ibor rate.

Non-linear CMS (swap or cap/floor) are also traded, especially in France. The French government issues floating bonds with coupons linked to the TEC index (French treasury index). The TEC index is similar to a CMS index but linked to the French government 10 year rate. The format of the floating rate is annual rate transformed to quarterly, giving terms of the form

$$(1+S)^{\frac{1}{4}}.$$

in the payment formula.

Some investors like to have CMS products with a similar formula. In particular, CMS caps with pay-off of the form

$$((1+S)^q - (1+K)^q)^+$$

where q is some exponent, are not uncommon.

This note outlines some details related to the pricing of linear and non-linear CMS.

2. REPLICATION BY CASH-SETTLED SWAPTIONS

The payoff described above can be priced in a model-free replication using swaptions, subject to approximations. The document is based on Hagan [2003] for CMS pricing and Henrard [2010a] for the *multi-curve setting*.

The replication analysed here uses euro-like cash-settled swaptions. This approach is suitable mainly for EUR and GBP, as cash-settled swaptions are the most liquid in these currencies. In USD, there are also liquid cash-settled swaptions but the cash amount computation formula used makes them closer to physically-settled swaptions in terms of pricing.

The replication requires the knowledge of swaption prices for swaptions with strikes up to infinity[1]. Therefore, the impact of the tail is not trivial. It means that the impact of the extrapolation beyond market-quoted strikes is important.

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[1]For floors, the prices for strike from 0 to the floor strike.

2.1. **Instrument details.** The analysis framework is a *multi-curves setting*. There is one discounting curve denoted $P^D(s,t)$ and one forward curve $P^j(s,t)$, where j is the relevant Ibor tenor.

The underlying swap has a start date t_0, fixed leg payment dates $(t_i)_{1\leq i\leq n}$ and m payments per annum. The accrual fraction for each fixed period is $(\delta_i)_{1\leq i\leq n}$. The floating leg payment dates are $(\tilde{t}_i)_{1\leq i\leq \tilde{n}}$ and the fixing period start and end dates are (s_i) and (e_i).

The (delivery) *annuity* is

$$A_t = \sum_{i=1}^{n} \delta_i P^D(t,t_i).$$

The swap rate is

$$S_t = \frac{\sum_{i=1}^{\tilde{n}} P^D(t,\tilde{t}_i)\left(\frac{P^j(t,s_i)}{P^j(t,\tilde{e}_i)} - 1\right)}{A_t}.$$

As the CMS rate depends on an existing fixing, the underlying swap is always in practice a plain vanilla, standard convention swap.

Let t_p be the payment date of the coupon fixing in θ.

2.2. **Linear CMS.** The payoff[2] is

(1) $$p(S_\theta) = S_\theta.$$

To standardise the pricing formula we use $K = 0$ for these instruments.

2.3. **Linear CMS cap/floor.** Let K be the strike. The cap payoff is

(2) $$p(S_\theta) = (S_\theta - K)^+.$$

The floor pay-off is

(3) $$p(S_\theta) = (K - S_\theta)^+.$$

2.4. **TEC-like CMS cap.** The payoff is denoted

(4) $$p(S_\theta) = ((1 + S_\theta)^q - (1 + K)^q)^+.$$

2.5. **Replication with cash-settled swaptions.** The value at 0 of a cap is (using the $P^D(.,t_0)$ numeraire)

$$P^D(0,t_0)\mathrm{E}^0\left[p(S_\theta)\frac{P^D(t_0,t_p)}{P^D(t_0,t_0)}\right].$$

If the coupon is fixed in arrears, $t_0 = t_p$ and the ratio of discount factor is 1.

The developments described below are based on cash-settled swaptions and thus suitable mainly to EUR and GBP.

To obtain an expectation with respect to only one variable, the following approximation[3] is used on the discount factor ratio:

$$\frac{P^D(t,t_p)}{P^D(t,t_0)} \simeq (1 + \frac{1}{m}S_t)^{-\delta} = h(S_t).$$

with δ the year fraction between the swap settlement date and the payment date.

For a cap, the term to be replicated is written as

$$p(S)h(S) = k(S)G(S)(S - K)^+.$$

A swap can be priced as a cap with strike $K = 0$. For a floor it is

$$p(S)h(S) = k(S)G(S)(K - S)^+.$$

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[2]The actual payoff is the rate multiplied by an accrual fraction and the notional. The accrual fraction and the notional are simply multiplication factors, and are not taken into account in the formulae.

[3]Other approximations are possible, including some that would show some sensitivity to the short term discounting caplet volatility.

The value of k is product-dependent and is described below. The function G is the cash-settled swaption annuity and is introduced to link CMS to swaptions. If the replication was done with swaptions other than cash-settled, it is this part that would change[4]. The product value is

$$G(S) = \sum_{i=1}^{n} \frac{\frac{1}{m}}{(1 + \frac{1}{m}S)^i} = \frac{1}{S}\left(1 - \frac{1}{(1 + \frac{1}{m}S)^n}\right)$$

2.6. **Cap.** A standard replication equality[5] is used:

$$k(S)(S - K)^+ = k(K)(S - K)^+ + \int_K^\infty \left(k''(x)(x - K) + 2k'(x)\right)(S - x)^+ dx$$

which leads to

$$p(S)h(S) = k(K)G(S)(S - K)^+ + \int_K^\infty \left(k''(x)(x - K) + 2k'(x)\right)G(S)(S - x)^+ dx.$$

The price can be rewritten as

$$
\begin{aligned}
P_{\text{Cap}} &= P^D(0, t_0)\text{E}^0\left[p(S_\theta)\frac{P^D(t_0, t_p)}{P^D(t_0, t_0)}\right] \\
&\simeq P^D(0, t_p)h^{-1}(S_0)\text{E}^0\left[p(S_\theta)h(S_{t_0})\right] \\
&\simeq P^D(0, t_p)h^{-1}(S_0)\text{E}^0\left[p(S_\theta)h(S_\theta)\right] \\
(5) \quad &= \frac{P^D(0, t_p)}{P^D(0, t_0)}h^{-1}(S_0)\left[k(K)\text{Swpt}_{\text{Payer}}(K) + \int_K^\infty \left(k''(x)(x - K) + 2k'(x)\right)\text{Swpt}_{\text{Payer}}(x)\right]dx,
\end{aligned}
$$

where $\text{Swpt}_{\text{Payer}}(K)$ refers to the price of a cash-settled payer swaption of strike K. Note that in the second line we have used the approximation twice, once with h and once with h^{-1}, to reduce the approximation impact.

If the cash-settled swaption is priced with a market Black formula[6] with volatility $\sigma = \sigma(S_0, K)$, the price becomes

$$
\begin{aligned}
(6) \qquad P_{\text{Cap}} &= P^D(0, t_p)h^{-1}(S_0)G(S_0)\Big[k(K)\text{BS}_{\text{Call}}(S_0, K, \sigma(S_0, K)) \\
&\quad + \int_K^{+\infty} \left(k''(x)(x - K) + 2k'(x)\right)\text{BS}_{\text{Call}}(S_0, x, \sigma(S_0, x))dx\Big]
\end{aligned}
$$

where we have used

$$\text{Swpt}_{\text{Payer}}(K) = P^D(0, t_0)G(S_0)\text{BS}_{\text{Call}}(S_0, K, \sigma(S_0, K)).$$

For some implementations, we do not use the Black formula with implied volatility everywhere, but use a price extrapolation for very high strikes. We do not modify the notation but some implementations may differ for high strikes. Details can be found in OpenGamma Research [2011].

2.7. **Floor.** A standard replication equality is used:

$$k(S)(K - S)^+ = k(K)(K - S)^+ + \int_0^K \left(k''(x)(K - x) + 2k'(x)\right)(x - S)^+ dx$$

which lead to

$$p(S)h(S) = k(K)G(S)(K - S)^+ + \int_0^K \left(k''(x)(K - x) + 2k'(x)\right)G(S)(x - S)^+ dx.$$

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[4]The market approach to go from delivery to cash swaptions is to keep the same Black volatility and change the annuity from the delivery annuity A_0 to the cash annuity $G(S_0)$. This corresponds to what is called the *standard model* in Hagan [2003].

[5]The equality can be proved by showing that both sides are equal in $K = S$, the first derivatives are equal in $K = S$, and that the second derivatives with respect to K are equal.

[6]The standard pricing of cash-settled swaptions by the market standard formula used here is not arbitrage free, as discussed in Mercurio [2008] and Henrard [2010b].

The price can be rewritten as

(7)
$$P_{\text{Floor}} = \frac{P^D(0, t_p)}{P^D(0, t_0)} h^{-1}(S_0) \left[k(K)\text{Swpt}_{\text{Receiver}}(K) + \int_0^K (k''(x)(x - K) + 2k'(x)) \, \text{Swpt}_{\text{Receiver}}(x) \right] dx,$$

where $\text{Swpt}_{\text{Receiver}}(K)$ refers to the price of a cash-settled receiver swaption of strike K.

If the swaption is priced with a market Black formula with volatility $\sigma = \sigma(S_0, K)$, the price becomes

(8)
$$\begin{aligned} P_{\text{Floor}} \quad = \quad & P^D(0, t_p)h^{-1}(S_0)G(S_0) \left[k(K)\text{BS}_{\text{Put}}(S_0, K, \sigma(S_0, K)) \right. \\ & + \left. \int_0^K (k''(x)(x - K) + 2k'(x)) \, \text{BS}_{\text{Put}}(S_0, x, \sigma(S_0, x)) dx \right]. \end{aligned}$$

Note that between the cap and the floor the difference is only in the integration bounds and the replacement of the call by the put.

2.8. Product-dependent details - CMS linear. For the linear CMS,

$$k(S) = \frac{h(S)}{G(S)}.$$

To perform the computations of those results one needs several derivatives:

$$\begin{aligned} k(x) \quad &= \quad \frac{h(x)}{G(x)} \\ k'(x) \quad &= \quad \frac{h'(x)}{G(x)} - \frac{h(x)G'(x)}{(G(x))^2} \\ k''(x) \quad &= \quad \frac{h''(x)}{G(x)} - 2\frac{h'(x)G'(x)}{(G(x))^2} - h(x)\left(\frac{G''(x)}{(G(x))^2} - 2\frac{(G'(x))^2}{(G(x))^3} \right). \end{aligned}$$

Let $\eta = -\delta/\tau$.

$$\begin{aligned} h(x) \quad &= \quad (1 + \tau x)^\eta. \\ h'(x) \quad &= \quad \eta\tau(1 + \tau x)^{\eta-1} = \eta\tau \frac{h(x)}{1 + \tau x} \\ h''(x) \quad &= \quad \eta(\eta - 1)\tau^2(1 + \tau x)^{\eta-2} = (\eta - 1)\tau \frac{h'(x)}{1 + \tau x}. \end{aligned}$$

$$\begin{aligned} G(S) \quad &= \quad \frac{1}{S}\left(1 - \frac{1}{(1 + \tau S)^n}\right) \quad \text{and} \quad G(0) = \frac{n}{m} \\ G'(S) \quad &= \quad -\frac{1}{S^2}\left(1 - \frac{1}{(1 + \tau S)^n}\right) + \frac{\tau n}{S}(1 + \tau S)^{-(n+1)} \quad \text{and} \quad G'(0) = -\frac{n(n+1)}{2m^2} \\ G''(S) \quad &= \quad \frac{2}{S^3}\left(1 - \frac{1}{(1 + \tau S)^n}\right) - \frac{2\tau n}{S^2}(1 + \tau S)^{-(n+1)} - \frac{(n+1)n\tau^2}{S}(1 + \tau S)^{-(n+2)} \\ & \quad \text{and} \quad G''(0) = \frac{1}{m^3}\frac{n(n+1)}{2}\left(1 + \frac{n+2}{3}\right) \end{aligned}$$

2.9. Product dependent details - CMS TEC-like. For TEC-like CMS,

$$k(S) = \frac{p(S)h(S)}{(S - K)G(S)}.$$

The ratio $p(x)/(x - K)$ is not well defined at $x = K$; we extend it to that point by continuity. For this we use

$$\frac{h(S)p(S)}{G(S)} = \frac{p(S)h(S)}{G(S)(S - K)}(S - K)^+ = k(S)(S - K)^+.$$

To perform the actual computations of those results one needs several derivatives:

$$k(x) = \frac{p(x)h(x)}{(x-K)G(x)} = \frac{k_1(x)}{k_2(x)}$$

$$k'(x) = \frac{k_1'(x)}{k_2(x)} - \frac{k_1(x)k_2'(x)}{(k_2(x))^2}$$

$$k''(x) = \frac{k_1''(x)}{k_2(x)} - 2\frac{k_1'(x)k_2'(x)}{(k_2(x))^2} - k_1(x)\left(\frac{k_2''(x)}{(k_2(x))^2} - 2\frac{(k_2'(x))^2}{(k_2(x))^3}\right).$$

For x close to K, the ration $p(x)/(x-K)$ can be approximated (to avoid numerical instability). We use a third order approximation (to cope with second order derivative required). Let $m(x) = p(x)/(x-K)$. The approximation is $m(x) \simeq p'(K) + \frac{1}{2}p''(K)(x-K) + \frac{1}{2}\frac{1}{3}p'''(K)(x-K)^2$.

$$k_1(x) = p(x)h(x)$$

$$k_1'(x) = p'(x)h(x) + p(x)h'(x)$$

$$k_1''(x) = p''(x)h(x) + 2p'(x)h'(x) + p(x)h''(x).$$

$$k_2(x) = (x-K)G(x)$$

$$k_2'(x) = G(x) + (x-K)G'(x)$$

$$k_2''(x) = 2G'(x) + (x-K)G''(x).$$

$$p(x) = (1+x)^q - (1+K)^q$$

$$p'(x) = q(1+x)^{q-1} = qp(x)/(1+x)$$

$$p''(x) = q(q-1)(1+x)^{q-2} = (q-1)p'(x)/(1+x).$$

2.10. **Derivative (Delta).** The delta can be computed by differentiating Equation 5. The curve impacts the price only through $P^D(0,t_p)$ and S_0. Let $n(S) = h^{-1}(S)G(S)$. The derivative with respect to S_0 is given by

$$\Delta_{S_0} = P^D(0,t_p)\Big[k(K)\left(n'(S_0)\mathrm{BS}(S_0,K) + n(S_0)(\mathrm{BS}'(S_0,K))\right)$$
$$+ \int_K^\infty \left(k''(x)(x-K) + 2k'(x)\right)\left(n'(S_0)\mathrm{BS}(S_0,x) + n(S_0)\mathrm{BS}'_S(S_0,x)\right)dx\Big]$$

where BS' has to be understood as the total derivative with respect to S, i.e. $\mathrm{BS}'(S,K) = D_S\mathrm{BS}(S,K,\sigma) + D_\sigma\mathrm{BS}(S,K,\sigma)D_S\sigma$ in the Black-Scholes with implied volatility case.

The derivative with respect to the discount factor is, up to the discount factor, the price.

$$\Delta_{P^D} = \frac{1}{P^D(0,t_p)}P.$$

The value of BS' is the derivative of the Black formula with respect to the forward rate S_0. The value of $n'(x)$ can be computed from the values described above:

$$n'(x) = \frac{G'(x)}{h(x)} - \frac{G(x)h'(x)}{h^2(x)}.$$

The bucketed delta (zero-coupon) is, for the discounting curve,

$$\frac{\partial P}{\partial r_i^D} = \Delta_{S_0}\frac{\partial S_0}{\partial r_i^D} + \Delta_{P^D}\frac{\partial P^D(0,t_0)}{\partial r_i^D}$$

and for the forward curve

$$\frac{\partial P}{\partial r_i^F} = \Delta_{S_0}\frac{\partial S_0}{\partial r_i^F}.$$

Class	Underlying	SABR extrapolation
`CouponCMSReplicationSABRMethod`	CMS coupon	No
`CouponCMSSABRExtrapolationRightReplicationMethod`	CMS coupon	Yes
`CapFloorCMSReplicationSABRMethod`	CMS cap/floor	No
`CapFloorCMSSABRExtrapolationRightReplicationMethod`	CMS cap/floor	Yes

TABLE 1. Implementation classes.

2.11. **Derivative (Vega).** The volatility parameter sensitivity can be computed by differentiating Equation 5. We suppose that the only dependance to the model parameters is through σ. Using the same notation as in the previous section, the derivative with respect to a model parameter p is

$$\Upsilon_p \;=\; P^D(0,t_0)n(S_0)\Big[k(K)D_\sigma \mathrm{BS}(S_0,K,\sigma)D_p\sigma(S_0,K)$$

$$+ \int_K^\infty \left(k''(x)(x-K)+2k'(x)\right)D_\sigma \mathrm{BS}(S_0,x)D_p\sigma(S_0,x)dx\Big]$$

2.12. **Derivative (strike).** The derivative of the price with respect to the strike can be computed by differentiating 6 (cap) or 8 (floor).

The derivative with respect to the strike is, for the cap,

$$\frac{\partial}{\partial K}P_{\mathrm{Cap}} \;=\; P^D(0,t_p)h^{-1}(S_0)G(S_0)\Big[-k'(K)\mathrm{BS}_{\mathrm{Call}}+k(K)\left(D_K\mathrm{BS}_{\mathrm{Call}}+D_\sigma \mathrm{BS}_{\mathrm{Call}}D_K\sigma\right)$$

$$- \int_K^{+\infty} k''(x)\mathrm{BS}_{\mathrm{Call}}(S_0,x,\sigma(S_0,x))dx\Big]$$

and, for the floor,

$$\frac{\partial}{\partial K}P_{\mathrm{Floor}} \;=\; P^D(0,t_p)h^{-1}(S_0)G(S_0)\Big[3k'(K)\mathrm{BS}_{\mathrm{Put}}+k(K)\left(D_K\mathrm{BS}_{\mathrm{Put}}+D_\sigma \mathrm{BS}_{\mathrm{Put}}D_K\sigma\right)$$

$$- \int_0^K k''(x)\mathrm{BS}_{\mathrm{Put}}(S_0,x,\sigma(S_0,x))dx\Big]$$

3. Implementation

The methods described here are implemented in OpenGamma analytic library. Table 1 contains the name of the classes and the implementation characteristics.

The computation performance for some instruments and results are provided in Table 2.

References

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Result	Underlying	SABR extrapolation	Time (ms)
Price	CMS coupon	No	??
Price	CMS coupon	Yes	??
Price+delta	CMS coupon	No	??
Price+delta	CMS coupon	Yes	??
Price+delta+SABR risk	CMS coupon	No	??
Price	CMS caplet	No	0.06
Price	CMS floorlet	No	0.06
Price+delta	CMS caplet	No	0.29
Price+delta	CMS floorlet	No	0.24
Price+delta+SABR risk	CMS caplet	No	0.76
Price+delta+SABR risk	CMS floorlet	No	0.55

TABLE 2. Computation times.

CONTENTS

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