Looking Forward to Backward-Looking Rates:
Completing the Generalized Forward Market Model

Andrei Lyashenko[1] and Fabio Mercurio[2]

[1]Quantitative Risk Management, Inc.
[2]Bloomberg, L.P.*

Abstract

In this paper, we show how the generalized Forward Market Model (FMM) introduced by Lyashenko and Mercurio (2019a) can be extended to make it a complete term-structure model describing the evolution of all points on a yield curve, and not just of a spanning set of forward term rates. The extended model is both theoretically sound and computationally efficient. Because the FMM is an extension of the popular LIBOR Market Model (LMM), it can be also used to complete the yield curve evolution under existing LMM implementations.

1 Introduction

Lyashenko and Mercurio (2019a) extended the concept of forward term rates to encompass both the traditional forward-looking (LIBOR-like) rates and the new setting-in-arrears backward-looking rates, which are expected to replace LIBORs as the reference rates in derivative contracts. The extended forward term rate: i) is defined for all times, ii) is equal to the realized forward-looking rate at the start of the application period, iii) continues to evolve within the application period, iv) settles to the realized backward-looking rate at the end of the application period, and v) is a martingale under the forward measure associated with the end of the application period. Lyashenko and Mercurio then showed that the classic interest-rate modeling framework can be naturally augmented to model the extended forward term rates. This holds, in particular, for the LIBOR Market Model (LMM), whose extension, called the generalized Forward Market Model (FMM), specifies the dynamics of forward rates also within their application periods, thus providing information about the rates evolution that is not directly available in the LMM. Another important advantage

*The views and opinions expressed in this article are our own and do not represent the opinions of any firm or institution. Bloomberg LP and Bloomberg Professional service are trademarks and service marks of Bloomberg Finance L.P., a Delaware limited partnership, or its subsidiaries.

of the FMM over the LMM is that rates can be modeled in the (continuous) risk-neutral measure.

In this paper, we show that the additional information about the rates dynamics provided by the FMM can be used to infer the evolution of generic term rates as well as the short rate, and not just the given set of spanning term rates. The ability to jointly model and simulate all points on a yield curve is crucial for the applicability of the model to the pricing and hedging of large heterogeneous portfolios such as a banking book consisting of a large variety of complex instruments such as mortgages or mortgage backed securities. Moreover, to be able to price variable-rate instruments such as floating-rate notes and adjustable-rate mortgages indexed by compounded overnight rates, we also need to simulate the bank account value in addition to the forward-rate curve.

To obtain the evolution of all points on the yield curve, we "embed" the FMM into a (finite-dimensional) Markovian Heath, Jarrow and Morton (HJM) model with separable volatility structure by aligning the HJM and FMM dynamics of the forward term rates modeled by the FMM. This FMM-aligned HJM model is effectively a hybrid between an instantaneous forward-rate model and a LMM, and shares the advantages of both approaches, with the caveat that the number of variables to simulate (that is, the given forward rates supplemented by the state variables in the Markovian representation) could be too high.

Alternatively, instead of using the FMM-aligned HJM model directly, one could use the zero-bond pricing formulas available under a separable HJM to derive new formulas expressing the price of an arbitrary zero-bond, as well as the bank account, as a function of the modeled forward term rates and their volatilities. In this FMM-HJM construct, FMM acts as a "coarse" model capturing a "macro" structure of the market such as the covariance structure of the set of modeled rates, while the FMM-aligned HJM serves as a finer modeling environment used to fill the gaps left by the coarser FMM. This two-step approach of modeling forward term rates and then filling the gaps with the help of a finer model has advantages over directly modeling and simulating the overnight rates because term rates are less noisy and have liquid derivative markets directly linked to them, including both linear derivatives such as futures, FRAs and swaps and options such as caps and swaptions. Similar two-level, coarse-fine, modeling approaches are used in other areas of science and engineering, such as macro- and micro-economic modeling.

The problem of recovering the whole yield curve evolution from the modeled set of LIBOR rates has been extensively discussed in the LMM literature, and is often referred to as LIBOR-rate interpolation, or front- and back-stub interpolations. The main references are the works of Schlögl (2002), Piterbarg (2004), Beveridge and Joshi (2009), Werpachowski (2010), and chapter 15 of Andersen and Piterbarg (2010).[1] Some of these approaches, however, are ad-hoc in nature, some may lead to unrealistic interpolations, and some others are not even arbitrage free. By contrast, the approach proposed in this paper is theoretically and practically sound, and arbitrage-free by construction. In fact, the formulas we

[1]Description of more recent attempts to refine LMM dynamics to extract finer information about rate evolution can be found in Fries (2019).

derive are of the same level of complexity as the Gaussian-model interpolation suggested by Andersen and Piterbarg (2010).

As a final point, we want to stress that although the FMM was originally proposed as an extension of the LMM to model backward-looking term rates, it is highly beneficial to switch from LMM to FMM regardless of the transition to the new rates. Indeed: a) FMM embeds LMM by preserving LIBOR dynamics, b) it can be calibrated to LIBOR caps and swaptions using the same (or similar) analytical pricing formulas, c) it provides additional information about the rates evolution within each accrual period, which can be used, as shown in this paper, to complete the curve in a conceptually-sound and efficient way, (d) converting an existing LMM implementation into its FMM extension is quite straightforward and fast, and (e) our tests, including production implementation, show that an FMM is as robust and efficient as an LMM.

2 Definitions and notation

We assume a single-curve framework where interest rates are risk free. The instantaneous risk-free rate at time t is denoted by $r(t)$, and is the the rate at which an associated money-market (or bank) account $B(t)$ accrues continuously starting from $B(0) = 1$, that is:

$$\mathrm{d}B(t) = r(t)B(t)\,\mathrm{d}t \tag{1}$$

so

$$B(t) = e^{\int_0^t r(u)\,\mathrm{d}u}$$

We assume the existence of a risk-neutral measure Q, whose associated numeraire is $B(t)$, and denote by $\mathbb{E}$ the expectation with respect to Q, and by $\mathcal{F}_t$ the sigma-algebra generated by the model risk factors up time t.

We consider a time structure $0 = T_0, T_1, \ldots, T_M$, and denote by τ_j the year fraction for the interval $[T_{j-1}, T_j)$. For each time t, we define $\eta(t) = \min\{j : \; T_j \geq t\}$, which is the index of the element of the time structure that is the closest to time t being equal to or greater than t.

We then denote by $P(t, T)$ the price at time t of the extended zero-coupon bond with maturity T. Introduced by Lyashenko and Mercurio (2019a), the extended bond price $P(t, T)$ is the time-t value of the self-financing strategy that consists of buying a classic zero-coupon bond with maturity T, and reinvesting the proceeds of the bond's unit notional at the risk-free rate $r(t)$ from time T onwards. In particular, we have that $P(t, 0) = B(t)$ for each time t. Denoting the instantaneous forward rate at time t with maturity T by $f(t, T)$, extended bond prices, like classic bond prices, can be written as

$$P(t, T) = e^{-\int_t^T f(t,u)\,\mathrm{d}u} \tag{2}$$

provided that we define $f(t, u) = r(u)$ when $t \geq u$.

As explained by Lyashenko and Mercurio (2019a), an extended zero-coupon bond is a viable numeraire. The probability measure Q^T associated with $P(t,T)$ is called (extended) T-forward measure.

Based on Lyashenko and Mercurio (2019a), for each $j = 1, \dots, M$, we define the forward term rate with application period $[T_{j-1}, T_j)$ as:

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{P(t,T_{j-1})}{P(t,T_j)} - 1\right] \tag{3}$$

Forward rate $R_j(t)$ is a Q^{T_j}-martingale and is defined for each time t: i) for $t \le T_{j-1}$, it coincides with the classic single-curve simply-compounded forward rate associated to $[T_{j-1}, T_j)$; ii) at time T_{j-1}, it gives the fixing of the forward-looking term rate $R_j(T_{j-1})$; iii) at time T_j, it gives the fixing of the backward-looking setting in-arrears spot rate $R_j(T_j)$; iv) for $t \ge T_j$, $R_j(t) = R_j(T_j)$. Notice that inside the application period, that is when $t \in (T_{j-1}, T_j)$, by definition of extended bond price we have:

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{e^{\int_{T_{j-1}}^{t} r(u)\,\mathrm{d}u}}{P(t,T_j)} - 1\right] \tag{4}$$

3 The Generalized Forward Market Model (FMM)

The generalized FMM introduced by Lyashenko and Mercurio (2019a) postulates the following dynamics of each forward rate $R_j(t)$ under the risk-neutral measure Q:

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t)\sum_{i=1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}\,\mathrm{d}t + \sigma_j(t)\gamma_j(t)\,\mathrm{d}W_j(t) \tag{5}$$

where, for each $i, j = 1, \dots, M$, $\sigma_j(t)$ is an adapted process, $W_j(t)$ is a standard Q-Brownian motion such that $\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \rho_{i,j}\,\mathrm{d}t$, and $\gamma_j(t)$ is a (piece-wise) differentiable (deterministic) function such that: $\gamma_j(t) = 1$ for $t \le T_{j-1}$, $\gamma_j(t)$ is monotonically decreasing in $[T_{j-1}, T_j]$ and $\gamma_j(t) = 0$ for $t \ge T_j$.[2]

Since $\gamma_i(t) = 0$ for $t > T_i$, dynamics (5) are equivalent to:

$$\mathrm{d}R_j(t) = \sigma_j(t)\gamma_j(t)\sum_{i=\eta(t)}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}\,\mathrm{d}t + \sigma_j(t)\gamma_j(t)\,\mathrm{d}W_j(t) \tag{6}$$

[2] A more general formulation, covering for instance the case of the Hull-White (1994) two-factor model, could be introduced by replacing $\sigma_j(t)$ with an n-dimensional square matrix, $\gamma_j(t)$ with an n-dimensional vector and $W_j(t)$ with an n-dimensional vector as well. This, however, would excessively burden the notation providing little benefit to the analysis that follows.

3.1 Using a vector notation

Processes $\sigma_j(t)$ and Brownian motions $W_j(t)$ are one-dimensional. A vector notation can be introduced as follows, where we assume that vectors are column vectors, and denote by $A^\intercal$ the transpose of matrix A.

Denoting by ρ the instantaneous correlation matrix of $(W_1(t), \dots, W_M(t))^\intercal$, that is $\rho = (\rho_{i,j})_{i,j=1,\dots,M}$, and denoting by $N \leq M$ the rank of ρ, there exist an N-dimensional Brownian motion $\bar{W}(t) = (\bar{W}_1(t), \dots, \bar{W}_N(t))^\intercal$ and an $M \times N$ matrix C such that $\rho = CC^\intercal$ and

$$(\mathrm{d}W_1(t), \dots, \mathrm{d}W_M(t))^\intercal = C\, \mathrm{d}\bar{W}(t)$$

Therefore, we can write:

$$\mathrm{d}R_j(t) = \gamma_j(t)\sigma_j^R(t)^\intercal \sum_{i=\eta(t)}^{j} \sigma_i^R(t) \frac{\tau_i \gamma_i(t)}{1 + \tau_i R_i(t)}\, \mathrm{d}t + \gamma_j(t)\sigma_j^R(t)^\intercal\, \mathrm{d}\bar{W}(t) \tag{7}$$

where, for each $j = 1, \dots, N$, $\sigma_j^R(t)$ is the N-dimensional adapted process defined by:

$$\sigma_j^R(t)^\intercal = \sigma_j(t) C_j \tag{8}$$

where C_j is the j-th row of C, and where we notice that $\rho_{i,j} = C_i C_j^\intercal$. From (8), it immediately follows that

$$\sigma_i^R(t)^\intercal \sigma_j^R(t) = \sigma_i(t) C_i C_j^\intercal \sigma_j(t) = \rho_{i,j}\sigma_i(t)\sigma_j(t) \tag{9}$$

In particular:

$$\sigma_j^R(t)^\intercal \sigma_j^R(t) = \sigma_j^2(t) \tag{10}$$

Hereafter, with some slight abuse of notation, we will drop the bar in $\bar{W}$, thus writing $W(t)$ instead of $\bar{W}(t)$.

4 An extended Markovian HJM

Heath, Jarrow and Morton (1992) assumed that, for a fixed maturity T, the instantaneous forward rate $f(t, T)$ evolves, under the risk-neutral measure Q, according to:

$$\mathrm{d}f(t, T) = \sigma(t, T)^\intercal \int_t^T \sigma(t, s)\, \mathrm{d}s\, \mathrm{d}t + \sigma(t, T)^\intercal\, \mathrm{d}W(t) \tag{11}$$

with initial condition $f(0, T)$ given by market, and where $\sigma(t, T)$ is an N-dimensional vector of adapted processes, and W is an N-dimensional Q-Brownian motion.

In the classic HJM framework, for each T, volatility $\sigma(t, T)$, and hence process $f(t, T)$, are defined for $t \leq T$. Therefore, for (2) to hold for extended bond prices as well, we extend dynamics (11) to times t after T by zeroing the volatility of forward rate $f(t,T)$ for $t \geq T$. We thus assume:

$$\mathrm{d}f(t,T) = 1_{\{t\leq T\}} \left[\sigma(t,T)^\intercal \int_t^T \sigma(t,s)\,\mathrm{d}s\,\mathrm{d}t + \sigma(t,T)^\intercal\,\mathrm{d}W(t) \right]$$

The indicator function $1_{\{t\leq T\}}$ has the effect of making the process constant after time T: $f(t,T) = r(T)$ when $t \geq T$. This way, $f(t,T)$ is defined for all pairs (t,T), and (2) also holds for $t > T$.

As in the classic HJM framework, the application of Ito's lemma and Fubini's theorem leads to the following risk-neutral dynamics of extended zero-coupon bond prices:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\,\mathrm{d}t - \left(\int_t^T \sigma(t,u) 1_{\{t\leq u\}}\,\mathrm{d}u \right)^\intercal \mathrm{d}W(t) \tag{12}$$

In particular, when $t > T$, this SDE reduces to:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\,\mathrm{d}t$$

which is consistent with the definition of extended bond prices at valuation times t subsequent to the bond's maturity T.

Using (12), we can derive the Q-dynamics of the generalized forward rate $R_j(t)$. We get:

$$\begin{aligned} \mathrm{d}R_j(t) &= \cdots\,\mathrm{d}t + \left[R_j(t) + \frac{1}{\tau_j} \right] \left(\int_{T_{j-1}}^{T_j} \sigma(t,u) 1_{\{t\leq u\}}\,\mathrm{d}u \right)^\intercal \mathrm{d}W(t) \\ &= \cdots\,\mathrm{d}t + \left[R_j(t) + \frac{1}{\tau_j} \right] \left(\int_{T_{j-1}\vee t}^{T_j\vee t} \sigma(t,u)\,\mathrm{d}u \right)^\intercal \mathrm{d}W(t) \end{aligned} \tag{13}$$

4.1 A Markovian HJM matching the FMM

Our goal is to construct a Markovian HJM model that generates dynamics of extended forward rates R_j, $j = 1, \ldots, M$, that are equivalent to the FMM ones. To this end, we first match (13) with (7), obtaining:

$$\gamma_j(t)\sigma_j^R(t)^\intercal = \left[R_j(t) + \frac{1}{\tau_j} \right] \left(\int_{T_{j-1}\vee t}^{T_j\vee t} \sigma(t,u)\,\mathrm{d}u \right)^\intercal \tag{14}$$

for all $j = 1, \ldots, M$, or equivalently:

$$\sigma_j^R(t)\gamma_j(t) = \int_{T_{j-1}\vee t}^{T_j\vee t} \sigma(t,u)\,\mathrm{d}u \left[R_j(t) + \frac{1}{\tau_j} \right] \tag{15}$$

Then, similarly to Cheyette (2001), we assume that the instantaneous forward-rate volatility is given by the following separable form:

$$\begin{aligned}\sigma(t,T) &= \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T)\, 1_{\{T\in(T_{k-1},T_k]\}} \\ &= \varsigma_{\eta(T)}(t)\, g_{\eta(T)}(T)\end{aligned} \tag{16}$$

where, for each $k = 1,\ldots,M$, ς_k is an N-dimensional adapted process and g_k is a deterministic function.

Setting

$$G_k(t,T) = \int_t^T g_k(u)\,\mathrm{d}u$$

we notice that

$$\int_{T_{j-1}\vee t}^{T_j\vee t} \sigma(t,u)\,\mathrm{d}u = \varsigma_j(t)\int_{T_{j-1}\vee t}^{T_j\vee t} g_j(u)\,\mathrm{d}u = \varsigma_j(t)G_j(T_{j-1}\vee t, T_j\vee t)$$

So, (15) becomes:

$$\sigma_j^R(t)\gamma_j(t) = \varsigma_j(t)G_j(T_{j-1}\vee t, T_j\vee t)\left[R_j(t)+\frac{1}{\tau_j}\right] \tag{17}$$

Assuming, with no loss of generality, that $G_j(T_{j-1},T_j)=1$, equality (17) is satisfied, for instance, when the deterministic and stochastic components of both sides match:

$$\begin{aligned}\gamma_j(t) &= G_j(T_{j-1}\vee t, T_j\vee t) \\ \sigma_j^R(t) &= \varsigma_j(t)\left[R_j(t)+\frac{1}{\tau_j}\right]\end{aligned}$$

This leads to

$$\begin{aligned}g_j(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j(t), \quad t\in(T_{j-1},T_j) \\ \varsigma_j(t) &= \sigma_j^R(t)\frac{1}{R_j(t)+\frac{1}{\tau_j}}\end{aligned} \tag{18}$$

Therefore, choosing the instantaneous forward-rate volatilities as in (16), with the deterministic functions g_j and processes ς_j defined by (18), gives the postulated FMM dynamics (7).[3]

[3] The solution (18) is not unique. For any other decay function $\bar{\gamma}_j(t)$ and rate volatility $\bar{\sigma}_j^R(t)$ such that $\bar{\sigma}_j^R(t)\bar{\gamma}_j(t) = \sigma_j^R(t)\gamma_j(t)$, a different solution can be obtained by replacing $\gamma_j(t)$ with $\bar{\gamma}_j(t)$ and $\sigma_j^R(t)$ with $\bar{\sigma}_j^R(t)$ in (18). The choice of (18) is motivated by the particular case where the FMM is derived from a given one-factor Markovian HJM model. In this case, in fact, (18) leads to exactly the same initial HJM dynamics the FMM has been obtained from.

The value of each function $g_j(t)$ outside its corresponding interval (T_{j-1}, T_j) can be set to be zero, so (16) can be simplified to:

$$\sigma(t,T) = \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T) \tag{19}$$

and we can still write that $\sigma(t,T) = \varsigma_{\eta(T)}(t)\, g_{\eta(T)}(T)$.

The same representation in the classic LMM case presents the issue that $\sigma(t,T)$ is only defined for pairs (t,T) such that $t \leq T_{\eta(T)-1}$, and not for all $t \leq T$. This issue is addressed by our generalized FMM. In fact, $\varsigma_j(t)$ is now defined up to the end T_j of the corresponding application interval. Therefore, the HJM volatility $\sigma(t,T)$, as given by (16) is defined for all pairs (t,T) such that $t \leq T$.

4.2 Forward-rate dynamics and zero-coupon bond pricing

As shown in Appendix A, the integrated Q-dynamics of $f(t,T)$ can be expressed as follows:

$$f(t,T) = \begin{cases} f(0,T) + g(T)^\intercal X(t) + g(T)^\intercal Y(t) G(t,T) & \text{if } t < T \\ r(T) = f(0,T) + g(T)^\intercal X(T) & \text{if } t \geq T \end{cases} \tag{20}$$

where

$$\begin{aligned} \mathrm{d}X(t) &= Y(t) g(t)\,\mathrm{d}t + \varsigma(t)\,\mathrm{d}W(t) \\ \mathrm{d}Y(t) &= \varsigma(t)\,\varsigma(t)^\intercal\,\mathrm{d}t \end{aligned} \tag{21}$$

with $X(0) = 0$ and $Y(0) = 0$, and where we denote by:

- $g(T)$ the vector $\{g_1(T), \ldots, g_M(T)\}^\intercal$
- $G(t,T)$ the vector $\{G_1(t,T), \ldots, G_M(t,T)\}^\intercal$
- $X(t)$ the vector $\{X_1(t), \ldots, X_M(t)\}^\intercal$
- $Y(t)$ the $M \times M$-matrix $(Y_{k,h}(t))_{k,h=1,\ldots,M}$
- $\varsigma(t)$ the $M \times N$-matrix $(\varsigma_k(t)^\intercal)_{k=1,\ldots,M}$

To derive a zero-coupon bond pricing formula, we recall that $P(t,T)$ can be written in terms of instantaneous forwards as in (2).Using this and (20) we thus get:

$$P(t,T) = \begin{cases} P(0,t,T) \exp\left\{-G(t,T)^\intercal X(t) - \frac{1}{2} G(t,T)^\intercal Y(t) G(t,T)\right\} & \text{if } t < T \\ \exp\{\int_t^T r(u)\,\mathrm{d}u\} & \text{if } t \geq T \end{cases} \tag{22}$$

where $P(s,t,T)$ denotes the forward discount factor at time s between t and T, that is $P(s,t,T) = \frac{P(s,T)}{P(s,t)}$. When $t < T$, formula (22) could be simplified by noting that the k-th component of vector $G(t,T)$ is zero if $k < \eta(t)$ or $k > \eta(T)$, so:

$$G(t,T) = (0, \ldots, 0, G_{\eta(t)}(t,T), \ldots, G_{\eta(T)}(t,T), 0, \ldots, 0)$$

The bond price dynamics are Markovian in the factors $X_k(t)$ and $Y_{k,h}(t)$, $k, h = 1, \ldots, M$, to which we must add the forward rates $R_j(t)$, $j = 1, \ldots, M$, and any other factor defining the volatilities $\sigma_j^R(t)$. The total number of factors depends on the numbers of forwards R_j being modeled, whereas the FMM-dimensionality parameter N affects the numbers of factors defining the evolution of each process X_k.

To simulate all possible discount factors $P(t, T)$ with $0 < t < T \leq T_M$, we, therefore, need to simulate:

- Each $Y_{k,h}(t)$, $h, k = 1, \ldots, M$, up to time T_h
- Each $X_k(t)$, $k = 1, \ldots, M$, up to time T_k

The number of state variables to simulate thus grows quadratically with the number M of forward rates R_j, which makes formula (22) extremely unappealing and difficult to use in practice, despite its concise and clean representation. This is because the bond price functional $\mathcal{P}$, where $P(t, T) = \mathcal{P}(t, T; X(t), Y(t))$ for $t < T$, depends on forward rates R_j only indirectly through processes $X(t)$ and $Y(t)$. By using (22), we end up repeating some of the calculations that are intrinsic to the simulation of each rate R_j.

A more efficient algorithm for the generation of bond prices will be outlined in the next section where we will show that the costly dependence of each $X_k(t)$ on off-diagonal elements $Y_{h,k}(t)$, $h \neq k$, can be eliminated by noticing that, for $k = \eta(t)$, the difference $X_k(t) - X_k(T_{k-1})$ only depends on $Y_{k,k}(t)$. By leveraging the simulated values of forward rates at valuation time t as well as at the beginning of their application period, $R_{\eta(t)}(T_{\eta(t)-1})$, we can extend the price functional $\mathcal{P}$ so that the number of state variables needed for its valuation only grows linearly.[4]

5 Completing the curve using the FMM-fitted HJM

Many interest-rate contracts require the simulation of rates that do not lie on the given grid of times $T_0, \ldots, T_M$. Therefore, in addition to rates $R_j(T_{j-1})$ or $R_j(T_j)$, for each j, we may want to generate fixings of general forward-looking as well as backward-looking rates, using respectively the following formulas:

$$\begin{aligned} F(t, T) &= \frac{1}{\tau(t, T)} \left[\frac{1}{P(t, T)} - 1 \right] \\ R(t, T) &= \frac{1}{\tau(t, T)} \left[e^{\int_t^T r(u)\,\mathrm{d}u} - 1 \right] \\ &= \frac{1}{\tau(t, T)} \left[\frac{B(T)}{B(t)} - 1 \right] \end{aligned} \tag{23}$$

for general $t < T$, and where $\tau(t, T)$ denotes the year fraction between t and T.

[4] To be precise, the extended price functional will also depend on $Y_{\eta(t),\eta(t)}(T_{\eta(t)-1})$. Therefore, by giving up on the Markovianity of bond price dynamics, and making formulas mildly path-dependent, we can be much more efficient in the simulation of general discount factors.

Rates $F(t,T)$ can be calculated using the bond price formula (22), whereas rates $R(t,T)$ can be obtained by integrating over paths of $r(t)$ generated using (20). These calculations, however, are typically too expensive to be implemented in a production code. We thus propose a much more efficient algorithm that leverages the value of simulated rates $R_j(t)$.

Explicit dynamics for same-tenor off-grid rates are derived in Appendix C.

5.1 An efficient representation of bond prices and bank account

For every pair (t,T) with $T > T_{\eta(t)}$, following Schlögl (2002), we can write:

$$P(t,T) = P(t,T_{\eta(t)}) \prod_{j=\eta(t)+1}^{\eta(T)-1} \frac{1}{1+\tau_j R_j(t)}\, P(t,T_{\eta(T)-1},T) \tag{24}$$

with the convention that the product is equal to one when the set of indexes j is empty. Similarly to the LMM, the FMM can generate values of the central term in the RHS of the equation above, which depends on simulated forward rates, but can not generate, in general, the discount factors $P(t,T_{\eta(t)})$ and $P(t,T_{\eta(T)-1},T)$. Likewise, both the LMM and FMM can not generate values of $P(t,T)$ when $t < T < T_{\eta(t)}$.

As per the bank account $B(t)$, we notice that we can write:

$$B(t) = P(t,T_{\eta(t)}) \prod_{j=1}^{\eta(t)} [1+\tau_j R_j(t)] \tag{25}$$

which can be simulated by calculating the value of the discount factor $P(t,T_{\eta(t)})$.

In the LMM literature, the calculation of $P(t,T)$ for $t < T \leq T_{\eta(t)}$ is commonly referred to as "front-stub interpolation", whereas the calculation of $P(t,T_{\eta(T)-1},T)$ for $T > T_{\eta(t)}$ is referred to as "back-stub interpolation". In the following, we will derive back and front-stub formulas using the FMM-fitted Markovian HJM introduced before. We first show how to calculate $P(t,T_{\eta(T)-1},T)$, because we will use the back-stub result to derive the front-stub interpolation $P(t,T_{\eta(t)})$, and more generally $P(t,T)$ with $t < T < T_{\eta(t)}$.

5.2 The back-stub interpolation

Assume that $t \le T_{k-1}$ where we set $k = \eta(T)$. Using (22) and the definition of $G(t,T)$, we have:

$$\begin{aligned}
P(t,T_{k-1},T) &= \frac{P(t,T)}{P(t,T_{k-1})} \\
&= P(0,T_{k-1},T)\exp\Big\{-G(T_{k-1},T)^\intercal X(t) - \frac{1}{2}\big[G(t,T)^\intercal Y(t)G(t,T) - G(t,T_{k-1})^\intercal Y(t)G(t,T_{k-1})\big]\Big\} \\
&= P(0,T_{k-1},T)\exp\Big\{-G(T_{k-1},T)^\intercal X(t) \\
&\quad - \frac{1}{2}\big[\big(G(t,T_{k-1}) + G(T_{k-1},T)\big)^\intercal Y(t)\big(G(t,T_{k-1}) + G(T_{k-1},T)\big) - G(t,T_{k-1})^\intercal Y(t)G(t,T_{k-1})\big]\Big\} \\
&= P(0,T_{k-1},T)\exp\Big\{-G(T_{k-1},T)^\intercal\big[X(t) + Y(t)G(t,T_{k-1})\big] - \frac{1}{2}G(T_{k-1},T)^\intercal Y(t)G(T_{k-1},T)\Big\}
\end{aligned} \tag{26}$$

where we used that $G(t,T_{k-1})^\intercal Y(t)G(T_{k-1},T) = G(T_{k-1},T)^\intercal Y(t)G(t,T_{k-1})$ since Y is symmetrical.

Now notice that, since $T_{k-1} < T \le T_k$, all the components of $G(T_{k-1},T)$ are zero with the exception of the k-th one, so we can write:

$$P(t,T_{k-1},T) = P(0,T_{k-1},T)\exp\Big\{-Z_k(t)G_k(T_{k-1},T) - \frac{1}{2}Y_{k,k}(t)G_k^2(T_{k-1},T)\Big\} \tag{27}$$

where the scalar stochastic process $Z_k(t)$ is defined by

$$Z_k(t) = X_k(t) + \sum_{i=\eta(t)}^{k-1} Y_{k,i}(t)G_i(t,T_{k-1}) \tag{28}$$

Setting $T = T_k$ and recalling that $G_k(T_{k-1},T_k) = 1$, we get:

$$P(t,T_{k-1},T_k) = \frac{1}{1+\tau_k R_k(t)} = P(0,T_{k-1},T_k)\exp\Big\{-Z_k(t) - \frac{1}{2}Y_{k,k}(t)\Big\} \tag{29}$$

which, solving for $Z_k(t)$, yields:

$$Z_k(t) = \ln\big(1+\tau_k R_k(t)\big) + \ln P(0,T_{k-1},T_k) - \frac{1}{2}Y_{k,k}(t) \tag{30}$$

Plugging (30) into (27), we get the following back-stub formula for the forward bond $P(t,T_{k-1},T)$ with $t \le T_{k-1} < T \le T_k$:[5]

$$\begin{aligned}
P(t,T_{k-1},T) = P(0,T_{k-1},T)\big(1+\tau_k R_k(t)\big)^{-G_k(T_{k-1},T)}P(0,T_{k-1},T_k)^{-G_k(T_{k-1},T)} \\
\cdot\exp\Big\{\frac{1}{2}G_k(T_{k-1},T)G_k(T,T_k)Y_{k,k}(t)\Big\}
\end{aligned} \tag{31}$$

[5] Andersen and Piterbarg (2010) derived a similar interpolation inspired by the Hull-White one-factor model. We here show that their formula is in fact general in structure, and that the G and Y terms are model dependent.

which can be written in log form as:

$$
\begin{aligned}
&\ln P(t,T_{k-1},T)-\ln P(0,T_{k-1},T)\\
&\quad=G_k(T_{k-1},T)\big[\ln P(t,T_{k-1},T_k)-\ln P(0,T_{k-1},T_k)\big]+\frac{1}{2}G_k(T_{k-1},T)G_k(T,T_k)Y_{k,k}(t)
\end{aligned}
\tag{32}
$$

This representation has an interesting interpretation in terms of a time change. In fact, let us define the G-time forward rate $R_G(t,T_{k-1},T)$ as:

$$
P(t,T_{k-1},T)=\exp\big(-R_G(t,T_{k-1},T)G_k(T_{k-1},T)\big) \tag{33}
$$

Then, equation (32) can be written as:

$$
R_G(t,T_{k-1},T)-R_G(0,T_{k-1},T)=R_G(t,T_{k-1},T_k)-R_G(0,T_{k-1},T_k)-\frac{1}{2}G_k(T,T_k)Y_{k,k}(t) \tag{34}
$$

meaning that the difference between time-t and time-0 values of the G-time forward rate $R_G(t,T_{k-1},T)$ is linear in G-time $G_k(T_{k-1},T)$. Analogously, as per equation (32), the log-difference between time-t and time-0 zero-bond prices over the k-th accrual period is given by a linear interpolation in terms of G-time plus a convexity adjustment that vanishes at both endpoints of the corresponding application interval.

Inspecting (31) or (32), we see that to simulate $P(t,T_{k-1},T)$ for all possible choices of t and T (and hence k), besides forward rates $R_k(t)$, we need to also simulate each process $Y_{k,k}(t)$ up to time T_{k-1}. So, the back-stub interpolation only comes at the cost of simulating the diagonal elements of $Y(t)$ up to their fixing times:

$$
Y_{k,k}(t)=\int_0^t \varsigma_k(s)^{\intercal}\varsigma_k(s)\,\mathrm{d}s=\int_0^t\left[\frac{\sigma_k(s)}{R_k(s)+\frac{1}{\tau_k}}\right]^2\mathrm{d}s \tag{35}
$$

for $t\le T_{k-1}$, and where we used (18) and (10). We also notice that, because of (18), we can write that $G_k(t,T)=\gamma_k(t)-\gamma_k(T)$. In particular:

$$
\begin{gathered}
G_k(T_{k-1},T)=1-\gamma_k(T)\\
G_k(T,T_k)=\gamma_k(T)
\end{gathered}
\tag{36}
$$

which can be used to make formulas (31) and (32) more explicit.

5.3 The front-stub interpolation

We now consider the front-stub case where $T_{k-1}<t<T\le T_k$ with $k=\eta(t)=\eta(T)$. Following Appendix B, and formulas (48) and (49) in particular, we can write:

$$
P(t,T)=P(T_{k-1},t,T)\exp\left\{-G_k(t,T)x_k(t)-\frac{1}{2}G_k^2(t,T)y_k(t)\right\} \tag{37}
$$

where, for $T_{k-1} \leq t \leq T_k$,

$$\mathrm{d}x_k(t) = g_k(t)y_k(t)\,\mathrm{d}t + \frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}\,\mathrm{d}W_k(t) \tag{38}$$

with $x_k(T_{k-1}) = 0$, and

$$\mathrm{d}y_k(t) = \mathrm{d}Y_{k,k}(t) = \left[\frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}\right]^2 \mathrm{d}t \tag{39}$$

with $y_k(T_{k-1}) = 0$.[6]

Therefore, the simulation of $P(t, T)$, when $T_{k-1} < t < T \leq T_k$, can be achieved (exactly, in an HJM sense) at a little extra computational cost, that is the simulation of processes $x_k(t)$ and $y_k(t)$ inside their corresponding application period. The latter can be calculated by leveraging the simulated paths of $Y_{k,k}(t)$ since $y_k(t) = Y_{k,k}(t) - Y_{k,k}(T_{k-1})$, whereas the former can be simulated using (38).

It is relatively easy to show that $x_k(t)$ only depends on the realized paths of $\sigma_k(t)$ and $R_k(t)$. In fact, using (4), (48) and $r(s) = f(s, s)$, we have:

$$\begin{aligned} 1 + \tau_k R_k(t) &= \frac{1}{P(T_{k-1}, T_k)} \exp\left\{\int_{T_{k-1}}^{t} g_k(s)x_k(s)\,\mathrm{d}s + G_k(t, T_k)x_k(t) + \frac{1}{2}G_k^2(t, T_k)y_k(t)\right\} \\ &= \frac{1}{P(T_{k-1}, T_k)} \exp\left\{\int_{T_{k-1}}^{t} G_k(s, T_k)\,\mathrm{d}x_k(s) + \frac{1}{2}G_k^2(t, T)y_k(t)\right\} \end{aligned}$$

which leads to

$$x_k(t) = \int_{T_{k-1}}^{t} \frac{1}{G_k(s, T_k)}\,\mathrm{d}\Big(\ln(1 + \tau_k R_k(s)) - \frac{1}{2}G_k^2(s, T_k)y_k(s)\Big) \tag{40}$$

This dynamics is a viable alternative to (38) to generate values of the local process $x_k(t)$. When using an Euler scheme, in fact, (40) allows us to calculate $x_k(t)$ just using the known values and increments of $R_k(t)$ and $y_k(t)$.

The forward discount factor $P(T_{k-1}, t, T)$ in (37) can be calculated by taking the ratio of $P(T_{k-1}, T)$ and $P(T_{k-1}, t)$, which are in turn calculated using the back-stub formula (31) with $t = T_{k-1}$. After some minor simplification, we finally get:

$$\begin{aligned} P(t, T) = P(0, t, T)P(0, T_{k-1}, T_k)^{-G_k(t,T)}\big(1 + \tau_k R_k(T_{k-1})\big)^{-G_k(t,T)} \exp\Big\{ &- G_k(t, T)x_k(t) \\ - \frac{1}{2}G_k^2(t, T)y_k(t) + \frac{1}{2}Y_{k,k}(T_{k-1})\left[G_k(T, T_k)G_k(T_{k-1}, T) - G_k(t, T_k)G_k(T_{k-1}, t)\right] &\Big\} \end{aligned} \tag{41}$$

[6]The front-stub methodology proposed by Werpachowski (2010) relies on the introduction of a fictitious rate that evolves stochastically after a forward-looking (LIBOR) rate $L_k(t)$ is fixed at the beginning of its application period T_{k-1}. In the FMM, this can be avoided thanks to our definition of forward term rate $R_k(t)$, which is equal to the fixing of the forward-looking term rate at T_{k-1} and continues to evolve stochastically until T_k.

In particular, for $T = T_k$:

$$P(t, T_k) = P(0, t, T_k) P(0, T_{k-1}, T_k)^{-G_k(t,T_k)} \big(1 + \tau_k R_k(T_{k-1})\big)^{-G_k(t,T_k)} \\ \cdot \exp\left\{ -G_k(t, T_k) x_k(t) - \frac{1}{2} G_k^2(t, T_k) y_k(t) - \frac{1}{2} G_k(t, T_k) G_k(T_{k-1}, t) Y_{k,k}(T_{k-1}) \right\} \tag{42}$$

5.4 Summary of the simulation steps

We assume that the FMM is simulated on a grid of times $0 = t_0, t_1, \ldots, t_m = T_M$, which contains the FMM dates $T_0, \ldots, T_M$ as well as all the fixing and maturity dates that are relevant for given valuation purposes. The steps to take to simulate general rates are as follows. Starting from time-0 values, for $i = 1, \ldots, m$:

- Simulate rates $R_k(t_i)$ and volatilities $\sigma_k(t_i)$ for each $k = \eta(t_i), \ldots, M$.
- Calculate $Y_{k,k}(t_i)$ for each $k = \eta(t_i), \ldots, M$, using (35).
- Calculate $y_k(t_i) = Y_{k,k}(t_i) - Y_{k,k}(T_{k-1})$ for each $k = \eta(t_i), \ldots, M$.
- Simulate $x_k(t_i)$ for each $k = \eta(t_i), \ldots, M$, using (38) or (40) leveraging already-calculated values and increments of $y_k(t)$ and $R_k(t)$.
- Calculate $P(t_i, T)$ for all relevant times $t_i < T \leq T_{\eta(t_i)}$ using the front-stub form ula (41).
- Calculate $P(t_i, T)$ for all relevant times $T > T_{\eta(t_i)}$ using (24), that is:

$$P(t_i, T) = P(t_i, T_{\eta(t_i)}) \prod_{j=\eta(t_i)+1}^{\eta(T)-1} \frac{1}{1 + \tau_j R_j(t_i)} \; P(t_i, T_{\eta(T)-1}, T)$$

  where $P(t_i, T_{\eta(t_i)})$ is calculated using the front-stub formula (42), and $P(t_i, T_{\eta(T)-1}, T)$ using the back-stub formula (31).[7]
- Calculate the bank account $B(t_i)$ using (25), that is:

$$B(t_i) = P(t_i, T_{\eta(t_i)}) \prod_{j=1}^{\eta(t_i)} [1 + \tau_j R_j(t_i)]$$

  where $P(t_i, T_{\eta(t_i)})$ is calculated using the front-stub formula (42).

As a final remark, we notice that each $Y_{k,k}(t)$ is simulated from time zero to its last fixing time T_k, whereas each $x_k(t)$ evolves only in the corresponding application period $[T_{k-1}, T_k]$. For each k, the simulation of both $Y_{k,k}(t)$ and $x_k(t)$, if an Euler scheme is used, does not require the generation of additional stochastic variables but only the valuation of sums (integrals) based on already-calculated quantities.

[7] The formula for the bond price $P(t_i, T)$ can be slightly simplified by expressing (42) in terms of processes $\bar{x}_k(t) = X_k(t) - X_k(T_{k-1})$ and $Y_{k,k}(t)$ instead of $x_k(t)$ and $Y_{k,k}(t)$.

6 Numerical results

We tested the performance of both our back- and front-stub interpolation formulas using the following:

- *Proof-of-concept (round-trip) test*
  We first run an analytically-tractable one-factor separable HJM model, equivalent to a one-factor Hull-White (1990) model, to generate forward term-rate paths and use our back- and front-stub formulas to derive the corresponding values for other off-grid rates. We then compare the results to those given by the original HJM model.
- *Implementation robustness test*
  When running the proof-of-concept test, we make some simplifying assumptions, which are appropriate for a practical implementation of the FMM, to see whether the fit errors remain small.
- *Production implementation test*
  In one of our organizations, we converted the existing LMM implementation into an FMM, including back- and front-stub formulas, and compared the FMM performance to that of the LMM it replaced.

In the remainder of this section, we discuss some of the results obtained when running the tests described above.

6.1 Proof-of-concept test

We used a one-factor Hull-White (HW-1F) model, where the instantaneous rate $r(t)$ evolves according to the following Q dynamics:

$$\mathrm{d}r(t) = a[\theta(t) - r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$$

where a and σ are positive constants, and θ is a deterministic function. As is well known, HW-1F is equivalent to a one-factor separable HJM model with the instantaneous forward-rate volatility given by

$$\sigma(t,T) = \sigma\, e^{-a(T-t)}$$

It is then easy to show that we can express this volatility $\sigma(t,T)$ in the form (16), that is:

$$\sigma(t,T) = \sum_{k=1}^{M} \varsigma_k(t)\, g_k(T)$$

where we set

$$\varsigma_k(t) = \sigma \frac{e^{-a(T_{k-1}-t)} - e^{-a(T_k-t)}}{a}$$
$$g_k(T) = \frac{a}{e^{-a(T_{k-1}-T)} - e^{-a(T_k-T)}}\, 1_{\{T\in[T_{k-1},T_k]\}}$$

so the constraint $G_k(T_{k-1}, T_k) = 1$ is satisfied.

For a given $k = 1, \ldots, M$, we simulated values of the forward rate $R_k(t)$ and used back-stub and front-stub formulas to compute values $P(t, T)$ and $B(t)$ for $T_{k-1} \le t < T \le T_k$. We set the volatility parameter σ to a typical value of 0.01, and tested our formulas for T_{k-1} ranging from 1 year to 30 years, and for the mean reversion parameter a ranging from 0.0 to 0.5. We used six-month term rate periods and weekly time steps. In all our tests, the error for the back-stub formulas, when compared to the exact values given by HW-1F, was zero because the back-stub formula (31) depends on $R_k(t)$, which was calculated using HW-1F, as well as on other quantities that can be expressed in exact closed form. The front-stub formulas produced slightly bigger (but still negligible) errors around 0.01-0.02 basis points, because of the numerical integration involved.

6.2 Implementation robustness test

In a practical implementation of the FMM, especially when converting an existing LMM implementation, we are likely to use a constant function $g_k(T) = 1/(T_k - T_{k-1})$, $T \in [T_{k-1}, T_k]$, which corresponds to a linear volatility decay function for the term rate R_k, and have the volatility $\varsigma_k(t)$ flat-extrapolated in the application period: $\varsigma_k(t) = \varsigma_k(T_{k-1})$, $t \in [T_{k-1}, T_k]$. These formulas for $\varsigma_k(t)$ and $g_k(T)$ are exact in the case of zero mean reversion, $a = 0$. For $a > 0$, there is an approximation error that grows with the value of the mean reversion parameter a.

We tested the effects of these practical simplifications using the HW-1F described in the previous section. Our tests showed that the back- and front-stub errors increase for increasing values of a, as expected, with most of the error coming from the approximation of function $g_k(T)$ that enters the back- and front-stub formulas in the form of the function $G_k(t, T)$. We also saw that the largest part of the error in the front-stub interpolation comes from the back-stub term $P(T_{k-1}, t, T)$ in the front-stub formula (37). We can easily verify this by looking at the instantaneous forward rates. By differentiating the back-stub formula (32) with respect to T, we get for $t \le T_{k-1} < T < T_k$:

$$\begin{aligned} f(t,T) &= f(0,T) \\ &\quad + g_k(T)\left[\ln P(0, T_{k-1}, T_k) - \ln P(t, T_{k-1}, T_k) + \frac{1}{2}\big(G_k(T_{k-1}, T) - G_k(T, T_k)\big)Y_{k,k}(t)\right] \end{aligned} \tag{43}$$

Doing the same to the front-stub formula (37), see also formula (48) in Appendix B, yields for $T_{k-1} < t < T < T_k$:

$$f(t,T) = f(T_{k-1}, T) + g_k(T)\big[x_k(t) + G_k(t,T)y_k(t)\big] \tag{44}$$

The square-bracket term in the back-stub formula (43) gives the sensitivity to function $g_k(T)$. In the front-stub formula (44), the first term $f(T_{k-1}, T)$ is given by the back-stub formula (43) with $t = T_{k-1}$. The additional sensitivity to $g_k(T)$ in (44) comes from the second term where the value in the brackets is relatively small because terms $x_k(t)$ and $y_k(t)$ are local and both zero at T_{k-1}.

6.3 Production implementation test

We have converted the existing LMM implementation into an FMM, including back- and front-stub interpolations, by flat-extrapolating volatility parameters into the application periods and by using linear volatility decay functions, as described in the previous section. The model was calibrated to LIBOR-based caps and swaptions using the same analytical formulas as the LMM ones. The back- and front-stub approaches used within the existing LMM implementation were based on the assumption, originally proposed by Brace et al. (1997), that the volatility of the front-stub bond $P(t, T_{\eta(t)})$ is zero, which implies a deterministic forward-rate evolution within each application period: $P(t,T) = P(T_{\eta(t)-1}, t, T)$ for $t < T \leq T_{\eta(t)}$. Under this assumption: 1) the LMM becomes a specific case of the FMM with $\gamma_k(t) = 1_{\{t \leq T_{k-1}\}}$; 2) the discrete (spot-LIBOR) and continuous (money-market) risk-neutral measures coincide, and 3) the realized backward-looking term rates are the same as the corresponding forward-looking ones.

We tested the consistency of the LMM and FMM implementations by comparing the generated rate paths and the values of instrument prices and sensitivities. As expected, the FMM and LMM are very close in terms of the rate path distribution they generate and the instrument prices and sensitivities they produce. For instance, the following are the observed differences in the Option Adjusted Spread (OAS), Option Adjusted Duration (OAD) and Weighted Average Life (WAL), for a very liquid 30-year fixed-rate UMBS and most liquid coupons:

Coupon	OAS (basis points)	OAD (years)	WAL (years)
2.50	0.0500	0.0013	0.0036
3.00	0.0900	0.0011	0.0037
3.50	0.1100	0.0013	0.0035
4.00	0.1300	0.0017	0.0034
4.50	0.1200	0.0019	0.0027

7 Conclusions

We showed that the generalized FMM introduced by Lyashenko and Mercurio (2019a) can be efficiently extended to make it a complete term-structure model describing the evolution of all possible bond prices $P(t,T)$ as well as of the continuously-accruing money-market account $B(t)$, which are needed to compute both the realized forward-looking and backward-looking term rates. This is accomplished by matching the FMM dynamics using a Markovian HJM with separable volatility parameters, which is used to derive back-stub and front-stub formulas to fill the gaps left by the FMM.

We also showed that, unlike existing methods used to complete the LMM, the derived formulas are not only theoretically sound and arbitrage-free but also numerically efficient. Numerical efficiency is achieved by relaxing the expensive dependence on the typically large number of Markovian state variables, and by allowing for the knowledge, on each simulation time, of quantities known at the beginning of the corresponding FMM period.

We finally stress that our FMM extension results in a model that is effectively a hybrid between an LMM and a Markovian HJM, and combines the flexibility of the former with the fine resolution of the latter, while preserving computational efficiency. In fact, this approach can be easily used to extend current LMM implementations at a little development cost.

References

[1] Andersen, L., and Piterbarg, V. (2010) *Interest Rate Modeling.* Atlantic Financial Press.

[2] Beveridge, C., and Joshi, M. (2009). Interpolation Schemes in the Displaced-Diffusion LIBOR Market Model and the Efficient Pricing and Greeks for Callable Range Accruals. Available online at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1461285.

[3] Brace, A., Gatarek D., and Musiela, M. (1997) The Market Model of Interest Rate Dynamics. *Mathematical Finance* 7, 127-155.

[4] Cheyette, O. (2001) Markov Representation of the Heath-Jarrow-Morton Model. Available online at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=6073.

[5] Fries, C. (2019) Critical Review of Interest Rates Models and Time-Homogeneous Discrete Tenor Modelling. Talk at the 15th WBS Quantitative Finance Conference, Rome, Italy.

[6] Hull, J., and White, A. (1990) Pricing Interest-Rate-Derivative Securities. *Review of Financial Studies* 3(4), 573-592.

[7] Hull, J., and White, A. (1994) Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models. *The Journal of Derivatives* 2, 37-47.

[8] Lyashenko, A., and Mercurio, F. (2019a) Libor Replacement: A Modeling Framework for In-arrears Term Rates. *Risk* July, 72-77.

[9] Lyashenko, A., and Mercurio, F. (2019b) Looking Forward to Backward-Looking Rates: A Modeling Framework for Term Rates Replacing LIBOR. Available online at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3330240.

[10] Piterbarg, V. (2004). Computing Deltas of Callable LIBOR Exotics in Forward LIBOR Models. *Journal of Computational Finance* 7, 107-144.

[11] Schlögl, E. (2002). Arbitrage-Free Interpolation in Models of Market Observable Interest Rates. In K. Sandmann and P. Schönbucher, editors, Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann. Springer Verlag, Heidelberg.

[12] Werpachowski, R. (2010). Arbitrage-Free Rate Interpolation Scheme for Libor Market Model with Smooth Volatility Term Structure. Available online at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1729828.

Appendix A: The integrated instantaneous forward-rate dynamics

Under the separable volatility assumption (19), the integrated Q-dynamics of $f(t,T)$ can be obtained as follows:

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \sigma(u,T)^\intercal \int_u^T \sigma(u,s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sigma(s,T)^\intercal\,\mathrm{d}W(s)\\
&= f(0,T) + \int_0^t \sum_{k=1}^M g_k(T)\,\varsigma_k(u)^\intercal \int_u^T \sum_{k=1}^M \varsigma_k(u)\,g_k(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sum_{k=1}^M g_k(T)\,\varsigma_k(s)^\intercal\,\mathrm{d}W(s)\\
&= f(0,T) + \sum_{k,h=1}^M g_k(T) \int_0^t \varsigma_k(u)^\intercal \varsigma_h(u) \int_u^T g_h(s)\,\mathrm{d}s\,\mathrm{d}u + \sum_{k=1}^M g_k(T) \int_0^t \varsigma_k(s)^\intercal\,\mathrm{d}W(s)\\
&= f(0,T) + \sum_{k,h=1}^M g_k(T) \left[\int_0^t \varsigma_k(u)^\intercal \varsigma_h(u)\,\mathrm{d}u \int_t^T g_h(s)\,\mathrm{d}s + \int_0^t g_h(u) \int_0^u \varsigma_k(s)^\intercal \varsigma_h(s)\,\mathrm{d}s\,\mathrm{d}u\right]\\
&\quad + \sum_{k=1}^M g_k(T) \int_0^t \varsigma_k(s)^\intercal\,\mathrm{d}W(s)\\
&= f(0,T) + \sum_{k=1}^M g_k(T) X_k(t) + \sum_{k,h=1}^M g_k(T) \int_t^T g_h(s)\,\mathrm{d}s\,Y_{k,h}(t)
\end{aligned}
$$

where we set, for $k,h = 1,\dots,M$:

$$
\begin{aligned}
\mathrm{d}X_k(t) &= \sum_{h=1}^M g_h(t) Y_{k,h}(t)\,\mathrm{d}t + \varsigma_k(t)^\intercal\,\mathrm{d}W(t)\\
\mathrm{d}Y_{k,h}(t) &= \varsigma_k(t)^\intercal \varsigma_h(t)\,\mathrm{d}t
\end{aligned}
$$

with $X_k(0) = Y_{k,h}(0) = 0$ for all $k,h = 1,\dots,M$. In particular, we have that:

$$
r(t) = f(0,t) + \sum_{k=1}^M g_k(t) X_k(t) \tag{45}
$$

The above equation can be expressed in a more compact form using the vector and matrix notation introduced in Section 4.2. We can then write:

$$
\begin{aligned}
f(t,T) &= f(0,T) + g(T)^\intercal X(t) + g(T)^\intercal Y(t) G(t,T)\\
r(t) &= f(0,t) + g(t)^\intercal X(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{d}X(t) &= Y(t) g(t)\,\mathrm{d}t + \varsigma(t)\,\mathrm{d}W(t)\\
\mathrm{d}Y(t) &= \varsigma(t)\,\varsigma(t)^\intercal\,\mathrm{d}t
\end{aligned}
$$

Appendix B: The local property

Let us consider three times $0 \le S \le t \le T$. We know that the instantaneous forward rate $f(t,T)$ can be represented as:

$$f(t,T) = f(0,T) + g(T)^\intercal X(t) + g(T)^\intercal Y(t) G(t,T)$$

In particular, when $t = S$,

$$f(S,T) = f(0,T) + g(T)^\intercal X(S) + g(T)^\intercal Y(S) G(S,T)$$

Subtracting side by side, we get:

$$f(t,T) = f(S,T) + g(T)^\intercal [X(t) - X(S)] + g(T)^\intercal [Y(t)G(t,T) - Y(S)G(S,T)] \tag{46}$$

In particular, setting $T = t$, we obtain the following short-rate representation:

$$\begin{aligned} r(t) &= f(S,t) + g(t)^\intercal [X(t) - X(S)] - g(t)^\intercal Y(S) G(S,t) \\ &= f(S,t) + g(t)^\intercal [X(t) - X(S) - Y(S)G(S,t)] \end{aligned}$$

Inspired by this, equation (46) can be rewritten as:

$$\begin{aligned} f(t,T) &= f(S,T) + g(T)^\intercal [X(t) - X(S)] + g(T)^\intercal [Y(t)G(t,T) - Y(S)G(S,T)] \\ &= f(S,T) + g(T)^\intercal [X(t) - X(S)] + g(T)^\intercal [Y(t)G(t,T) - Y(S)G(t,T)] \\ &\quad + g(T)^\intercal [Y(S)G(t,T) - Y(S)G(S,T)] \\ &= f(S,T) + g(T)^\intercal [X(t) - X(S) - Y(S)G(S,t)] \\ &\quad + g(T)^\intercal [Y(t) - Y(S)] G(t,T) \\ &= f(S,T) + g(T)^\intercal X^S(t) + g(T)^\intercal Y^S(t) G(t,T) \end{aligned} \tag{47}$$

where we set:

$$\begin{aligned} X^S(t) &= X(t) - X(S) - Y(S)G(S,t) \\ Y^S(t) &= Y(t) - Y(S) \end{aligned}$$

for $t \ge S$, so

$$\begin{aligned} \mathrm{d}X^S(t) &= Y^S(t) g(t) \,\mathrm{d}t + \varsigma(t) \,\mathrm{d}W(t) \\ \mathrm{d}Y^S(t) &= \varsigma(t)\, \varsigma(t)^\intercal \,\mathrm{d}t \end{aligned}$$

with initial conditions $X^S(S) = 0$ and $Y^S(S) = 0$. In particular, $X^0(t) = X(t)$ and $Y^0(t) = Y(t)$.

The price of the zero-coupon bond $P(t,T) = P^S(t,T)$, when $S \le t \le T$, is then given by:

$$\begin{aligned} P^S(t,T) &= e^{-\int_t^T f(t,u)\,\mathrm{d}u} \\ &= \exp\left\{ -\int_t^T [f(S,u) + g(u)^\intercal X^S(t) + g(u)^\intercal Y^S(t) G(t,u)] \,\mathrm{d}u \right\} \\ &= P(S,t,T) \exp\left\{ -G(t,T)^\intercal X^S(t) - \frac{1}{2} G(t,T)^\intercal Y^S(t) G(t,T) \right\} \end{aligned}$$

The issue with the above representations is that both $X^S(t)$ and $Y^S(t)$ are S dependent, so new processes $X^S(t)$ and $Y^S(t)$ have to be considered (and simulated) when changing the time S. However, a much simpler and useful formulation can be derived by assuming that $\eta(t) = \eta(T) = k$, for some k, and setting $S = T_{k-1}$. In this case, the above formulas simplify as follows:

$$
\begin{aligned}
f(t,T) &= f(T_{k-1},T) + g_k(T)x_k(t) + g_k(T)y_k(t)G_k(t,T) \\
P(t,T) &= P(T_{k-1},t,T)\exp\left\{-G_k(t,T)x_k(t) - \frac{1}{2}G_k^2(t,T)y_k(t)\right\}
\end{aligned} \tag{48}
$$

where we set:

$$
\begin{aligned}
x_k(t) &= X_k^{T_{k-1}}(t) = X_k(t) - X_k(T_{k-1}) - G_k(T_{k-1},t)Y_{k,k}(T_{k-1}) \\
y_k(t) &= Y_{k,k}^{T_{k-1}}(t) = Y_{k,k}(t) - Y_{k,k}(T_{k-1})
\end{aligned}
$$

for $T_{k-1} \leq t \leq T_k$, so

$$
\begin{aligned}
\mathrm{d}x_k(t) &= g_k(t)y_k(t)\,\mathrm{d}t + \varsigma_k(t)^\intercal\,\mathrm{d}W(t) \\
\mathrm{d}y_k(t) &= \varsigma_k(t)^\intercal\,\varsigma_k(t)\,\mathrm{d}t
\end{aligned}
$$

and $x_k(T_{k-1}) = y_k(T_{k-1}) = 0$.

For each $k = 1,\dots,M$, processes x_k and y_k are local in that they are defined in their corresponding intervals $[T_{k-1}, T_k]$. The advantage of this local formulation is that processes $x_k(t)$ and $y_k(t)$ are scalar, so easier to simulate. Moreover, dynamics are effectively one-factor because, using (18) and (35), we have:

$$
\begin{aligned}
\mathrm{d}x_k(t) &= g_k(t)y_k(t)\,\mathrm{d}t + \frac{\sigma_k(t)}{R_k(t)+\frac{1}{\tau_k}}\,\mathrm{d}W_k(t) \\
\mathrm{d}y_k(t) &= \mathrm{d}Y_{k,k}(t) = \left[\frac{\sigma_k(t)}{R_k(t)+\frac{1}{\tau_k}}\right]^2\mathrm{d}t
\end{aligned} \tag{49}
$$

We notice that, as already derived by Lyashenko and Mercurio (2019b), dynamics (49) can be directly obtained by matching a one-factor Cheyette model to the Q-dynamics of $R_k(t)$ inside its application interval $[T_{k-1}, T_k]$.

Appendix C: forward-rate dynamics for general same tenor rates

We here derive the dynamics of a general τ-tenor forward rate under the assumption that the initial time structure $T_0,\dots,T_M$ defines spanning rates with same tenor τ. Even though off-grid rates can be efficiently generated by using the interpolation method summarized in

Section 5.4, having explicit dynamics for off-grid rates can be beneficial when pricing path-dependent contracts such as range accruals, or when using deal-aligned dates as opposed to the market-aligned dates used upon calibration.

A general forward rate $F(t; t_1, t_2)$ is defined by

$$F(t; t_1, t_2) = \frac{1}{\tau(t_1, t_2)} \left[\frac{P(t, t_1)}{P(t, t_2)} - 1 \right] \tag{50}$$

where $t \leq t_1 < t_2$, $t_2 - t_1 = \tau$, and where we recall that $\tau(t_1, t_2)$ denotes the year fraction between t_1 and t_2.

Setting $k = \eta(t_1)$, so $\eta(t_2) = k + 1$, application of (22) leads to the following:

$$\begin{aligned} \mathrm{d}F(t; t_1, t_2) &= \cdots \,\mathrm{d}t + \left[F(t; t_1, t_2) + \frac{1}{\tau(t_1, t_2)} \right] G(t_1, t_2)^\intercal \varsigma(t) \,\mathrm{d}W(t) \\ &= \cdots \,\mathrm{d}t + \left[F(t; t_1, t_2) + \frac{1}{\tau(t_1, t_2)} \right] \\ &\quad \cdot \left[G_k(t_1, T_k) \frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}} \,\mathrm{d}W_k(t) + G_{k+1}(T_k, t_2) \frac{\sigma_{k+1}(t)}{R_{k+1}(t) + \frac{1}{\tau_{k+1}}} \,\mathrm{d}W_{k+1}(t) \right] \\ &= \cdots \,\mathrm{d}t + \left[F(t; t_1, t_2) + \frac{1}{\tau(t_1, t_2)} \right] \sigma_{1,2}(t) \,\mathrm{d}W_{1,2}(t) \end{aligned} \tag{51}$$

where we set

$$\begin{aligned} \sigma^2_{1,2}(t) = G^2_k(t_1, T_k) &\frac{\sigma^2_k(t)}{\left[R_k(t) + \frac{1}{\tau_k} \right]^2} + G^2_{k+1}(T_k, t_2) \frac{\sigma^2_{k+1}(t)}{\left[R_{k+1}(t) + \frac{1}{\tau_{k+1}} \right]^2} \\ &+ 2\rho_{k,k+1} G_k(t_1, T_k) G_{k+1}(T_k, t_2) \frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}} \frac{\sigma_{k+1}(t)}{R_{k+1}(t) + \frac{1}{\tau_{k+1}}} \end{aligned} \tag{52}$$

and the Brownian motion $W_{1,2}(t)$ is defined accordingly.

Equations (51) and (52) define an arbitrage-free interpolation scheme for the volatility of off-grid rates with the same tenor as the given forwards. When t_1 approaches T_{k-1}, then $\sigma_{1,2}(t)$ approaches $\sigma_k(t)/[R_k(t) + 1/\tau_k]$, whereas when t_1 approaches T_k, $\sigma_{1,2}(t)$ approaches $\sigma_{k+1}(t)/[R_{k+1}(t) + 1/\tau_{k+1}]$.