Arbitrage Free Dynamics of the Volatility Surface Part I

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Outline

- Introduction: pricing vs hedging, computational considerations.
- Absence of dynamic arbitrage continuous maturity and strike case.
- Absence of dynamic arbitrage discrete maturity and strike case.
- Discrete time simulation.
- Model specification: Volatility of forward volatility.
- Structure of volatility of options, VIX options and minimum variance delta.
- Conclusion.

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Introduction

- In this talk we'll consider the models that diffuse the full surface of forward volatility.
- Similar to HJM where a curve of forward rates are diffused but here we'll move forward a surface (expiry x strike) of forward volatility.
- We'll identify restrictions on the drift of forward volatility.
- Including the discrete expiry and strike case as well as efficient ways of computing the drift of forward volatility and the calibration to options on variance.
- There will be overlap with classical literature of Dupire (1994, 1996), Derman and Kani (1997), and Wissel (2007) – but also a few new things.

Pricing versus Hedging

- Why look at these models?

- The models we use are generally decent for pricing but not realistic or sufficient for hedging analysis.

- If volatility is driven by a single factor then volatility risk can be hedged with any option – which is very unrealistic.

- In practice, due to transactions costs, traders are very careful with choosing option hedges.

- If we want find the best dynamic hedge we need to use many factors and introduce transaction costs.

- Solving such a problem involves a trade-off between risk and return (or costs) and since this is the case we need models with absence of arbitrage.

- If we train a hedging robot on potentially arbitragable dynamics it may get fooled…

- … and then we might get fooled.

- On the other hand, if we have a statistically estimated (or learned or trained) model of the volatility surface, a la Buehler et al (2019) and Chirikhin (2019), what does it take to make it arbitrage free?

- Or what are the estimated risk premia (with standard error) in the volatility surface?

Computational Considerations

- This type of model is generally non-Markovian and as such simulations will be slow.

- But we note that thanks to modern computational techniques such as:

<u>Adjoint differentiation</u>, multithreading, vectorization, TensorFlow, GPUs, …

- … the need and greed for speed *should* be less than what it used to be.

- This is not quantum computing, cold fusion or science fiction – it is actually stuff you can do today. See for example Huge and Savine (2019).

- That said, this will be a relatively dry presentation -- with the numerical work left for Part II.

Absence of Static Arbitrage

- Assume zero rates and dividends and consider a full continuum of European option prices

$$C(t;T,K)=E_t[(S(T)-K)^+] \tag{1}$$

- Absence of static arbitrage requires maturity and butterfly spreads to be positive (non-negative is sufficient):

$$C_T(t;T,K)>0 \ \ ,C_{KK}(t;T,K)>0 \tag{2}$$

- This in turn is equivalent to the existence of a surface of positive local variance $\{\vartheta(t;T,K)\}$ so that

$$0=-C_T+\vartheta(t;T,K)C_{KK} \quad ,C(t;t,K)=(S(t)-K)^+ \tag{3}$$

- If the underlying evolves continuously

$$dS(t)=\sigma(t)\cdot dW \tag{4}$$

- … where W is a vector Brownian motion and σ is a general vector process, then the local variance is the conditional expectation of the future variance

$$\vartheta(t;T,K)=\frac{1}{2}E_t[\|\sigma(T)\|^2|S(T)=K]=\frac{1}{2}\frac{E_t[\|\sigma(T)\|^2\,\delta(S(T)-K)]}{E_t[\delta(S(T)-K)]} \tag{5}$$

- Proof: Ito expand $H=(S-K)^+$, integrate, and take expectations.

- Particularly, if volatility is only a function of spot, $\sigma(t)=\sigma(t,S(t))$, then (3) is a relation for directly backing out a spot price process that supports the observed option prices.

- However, (3) can also be used the other way around: to generate option prices from the local variance.

- We will now assume that the surface of forward variances is evolving according to a family of diffusions

$$d\vartheta(t;T,K)=\mu(t;T,K)dt+\psi(t;T,K)\cdot dW \qquad (6)$$

- … and investigate what restrictions are put on μ dictated by absence of arbitrage.

Absence of Dynamic Arbitrage

- Option prices are martingales, so there exist a family of vector process $\{\alpha(t;T,K)\}$ so that

$$dC(t;T,K) = \alpha(t;T,K) \cdot dW \tag{7}$$

- Let's say Ito to the forward equation (3):

$$\begin{aligned} 0 &= d[-C_T + \vartheta C_{KK}] \\ &= -\alpha_T \cdot dW + (\mu dt + \psi \cdot dW) C_{KK} + \vartheta \alpha_{KK} \cdot dW + \psi \cdot \alpha_{KK} dt \end{aligned} \tag{8}$$

- We obtain for the drift

$$\mu = -\frac{\psi \cdot \alpha_{KK}}{C_{KK}} \qquad (9)$$

- … and for the diffusion of the option prices used in the drift correction, we obtain the vector valued forward PDE

$$0 = -\alpha_T + C_{KK}\psi + \vartheta\alpha_{KK} \quad , \alpha(t;t,K) = 1_{S(t)>K}\sigma(t) \quad , \|\sigma(t)\| = [2\vartheta(t;t,S(t))]^{1/2} \quad (10)$$

- In summary:

  - The options are generated by the forward equation (3).

  - The volatility of the options is generated by (10).

- The forward variances are moved forward by (6) with the drift given by (9).

- The spot is moved forward with the volatility $[2\vartheta(t;t,S(t))]^{1/2}$.

- The model is arbitrage free and specified by the volatility of the forward variances $\{\psi(t;T,K)\}$ and by construction it fits the initial (forward) variance surface.

- This is equivalent to the HJM approach where the model primitives is the initial forward curve and its volatility structure.

- Like the HJM the model will generally be non-Markovian in this case in two dimensions $\{(T,K)\}$ rather than one $\{T\}$.

- The result for the drift (9) appears in Dupire (1996) and Derman and Kani (1997) and is as such a consequence of the measure change

$$\begin{aligned}2\vartheta(t;T,K) &= E_t[\|\sigma(T)\|^2 \frac{\delta(S(T)-K)}{E_t[\delta(S(T)-K)]}] \\ &= E_t[\|\sigma(T)\|^2 \frac{C_{KK}(T;T,K)}{C_{KK}(t;T,K)}] \\ &\equiv E_t^{(T,K)}[\|\sigma(T)\|^2]\end{aligned} \tag{11}$$

- The forward PDE (10) for the option diffusions is to our knowledge new.

- A simpler version appears in Andreasen (1997).

Absence of Static Arbitrage – Discrete Case

- Let $\{x_i\}$ be a discrete axis and define the discrete differential operators

$$\delta_x^{\pm} f(x_i) = \frac{f(x_i) - f(x_{i\pm1})}{x_i - x_{i\pm1}} \quad , \delta_{xx} f(x_i) = 2\frac{(\delta_x^{+} - \delta_x^{-}) f(x_i)}{x_{i+1} - x_{i-1}} \qquad (12)$$

- Absence of static arbitrage for a discrete set of options $\{C(t;T_i,K_j)\}$ corresponds to positive expiry and butterfly spreads

$$\delta_T^{-} C(t;T_i,K_j) > 0 \quad , \delta_{KK} C(t;T_i,K_j) > 0 \qquad (13)$$

- … and is ***equivalent*** to the existence of a discrete surface of positive forward variance $\{\vartheta(t;T_i,K_j\}$ so that

$$0 = -\delta_{T}^{-} C + \vartheta \delta_{KK} C \quad , C(t;t,K_j) = (S(t) - K)^{+} \quad , t = T_0 \tag{14}$$

- Equation (14) is a ***fully implicit finite difference*** discretization of the forward PDE (3):

$$[1 - \Delta T \vartheta \delta_{KK}] C(T) = C(T - \Delta T) \tag{15}$$

- So fully implicit finite difference generates arbitrage free option prices.

- Discrete local volatility is a natural basis for generating discrete arbitrage free option prices.

- Equation (15) is a tri-diagonal matrix system for each maturity step, and the absence of arbitrage is due to the positivity property

$$\vartheta \geq 0 \;\Rightarrow\; [1 - \Delta T \vartheta \delta_{KK}]^{-1} \geq 0 \qquad (16)$$

- Computationally, this is $O(J)$ for each maturity step, where J is the number of points in the strike dimension.

Absence of Dynamic Arbitrage – Discrete Case

- If we Ito expand (14) we obtain the discrete equivalents of the drift (9)

$$\mu(t;T_i,K_j) = -\frac{\psi \cdot \delta_{KK}\alpha}{\delta_{KK}C}(t;T_i,K_j) \tag{17}$$

- … and volatility equation (10)

$$\begin{aligned} &0 = -(\delta_T^- \alpha) + (\delta_{KK}C)\psi + \vartheta(\delta_{KK}\alpha) \\ &\alpha(t;t,K) = 1_{S(t)>K}\sigma(t) \quad , \|\sigma(t)\| = [2\vartheta(t;t,S(t))]^{1/2} \end{aligned} \tag{18}$$

- So the drift condition and the option volatility equation are the same as in the continuous expiry and strike case.

- The idea would now be to use equations (14, 17, 18) to simulate spot and discrete option prices $\{S(\cdot), C(\cdot; T_i, K_j)\}_{i=0,\ldots,I;\, j=0,\ldots,J}$ on a discrete time grid $\{t_h\}_{h=0,\ldots,H}$.

- The hope would here be that we could get away with a limited number of options, say $I \times J \approx (10 \times 10)$, independent of the number of time points H.

- The result for absence of static arbitrage, the equivalence between (13) and (14), is found in Andreasen and Huge (2011a).

- Wissel (2007) finds both the result for static arbitrage (13-14) and the restriction for absence of dynamic arbitrage (17-18).

- The latter was unknown to me until very recently.

Discrete Time Simulation

- Consider discrete simulation times $\{t_h\}$, maturities $\{T_i\}$, and strikes $\{K_j\}$.

- Since $(\delta_T^- C)$ and $(\delta_{KK} C)$ need to be positive martingales, a discrete simulation scheme that preserves this is

$$\vartheta(T_{h+1};T_i,K_j)=\frac{(\delta_T^- C)\cdot\xi_h((\delta_T^-\alpha)/(\delta_T^- C)))}{(\delta_{KK}C)\cdot\xi_h((\delta_{KK}\alpha)/(\delta_{KK}C))}(T_h;T_i,K_j) \quad ,i\geq h+2$$

$$\xi_h(\gamma)=\exp(-\frac{1}{2}\|\gamma\|^2+\gamma\cdot(W(T_{h+1})-W(T_h))) \tag{19}$$

- … where $\{(C,\alpha)(T_h;T_i,K_j)\}_{i=h,\ldots,I;\, j=0,\ldots,J}$ are generated from the discrete forward equations (14, 18).

- The spot now needs to be simulated to hit a distribution consistent with the one time step option prices $\{C(T_h;T_{h+1},K_j)\}_{j=0,\ldots,J}$.

- To this end there is a number of different methods:

  - Euler simulation with fine time stepping.

  - Discrete finite difference simulation, see A&H (2011b).

  - … or copula based simulation.

- In either case, a Brownian bridge needs to be used.

Model Parameterization

- In principle the model volatility ψ can be estimated empirically, but it is maybe good to have a bit of structure to base this estimation on.

- A general trick we can use is to calculate the volatility of forward volatility in a parametric model. For example, SABR or SVI.

- In the SABR model with local volatility

$$ds = z\sigma(s)dW \quad ,dz = \varepsilon z dZ \quad ,dW \cdot dZ = \rho dt \tag{20}$$

- … Andreasen and Huge (2012) find a short maturity expansion for the local variance

$$\vartheta(k)=\frac{1}{2}[J(y)z\sigma(k)]^2 \quad ,J(y)=[1-2\rho\varepsilon y+\varepsilon^2 y^2]^{1/2} \quad ,y=z^{-1}\int_k^s \sigma(u)^{-1}du \tag{21}$$

- From this we get

$$\psi=2\vartheta\begin{bmatrix} f(y)+\varepsilon(1-f(y)y)\rho \\ \varepsilon(1-f(y)y)\bar{\rho} \end{bmatrix} \quad ,f(y)=\frac{J'(y)}{J(y)} \quad ,\bar{\rho}=\sqrt{1-\rho^2}$$
$$y=\frac{(e^{\varepsilon x}(1-\rho)+\rho)^2-1}{2e^{\varepsilon x}(1-\rho)\varepsilon} \quad ,x=\int_k^s(2\vartheta(u))^{-1/2}du \tag{22}$$

- We can approximate rough volatility behavior by using

$$\tilde{\varepsilon}=\varepsilon\cdot(T-t)^{H-1/2} \tag{23}$$

- … in the above formulas. Here H is the Hurst coefficient.

- The above model is specified as a one factor model for volatility but can be expanded to any number of factors.

Volatility Equation

- Without loss of generality we can split the volatility of the forward volatility ψ in two: a part that is perfectly correlated with spot and one that locally independent

$$\psi(t;T,K) = \underbrace{\underbrace{\delta(t;T,K)}_{\in\mathbb{R}}\,\sigma(t)}_{\in span(\sigma)} + \underbrace{\eta(t)}_{\in\mathbb{R}}\,\underbrace{\xi(t;T,K)}_{\in span(\sigma)^{\perp}} \qquad (24)$$

- Option volatility splits in two $\alpha = \alpha^0 + \eta\alpha^1$ where

$$\begin{aligned} 0 &= -\alpha^0_T + C_{KK}\delta\sigma + \vartheta\alpha^0_{KK} \quad &&, \alpha^0(t;t,K) = 1_{S(t)>K}\sigma(t) \\ 0 &= -\alpha^1_T + C_{KK}\xi + \vartheta\alpha^1_{KK} \quad &&, \alpha^1(t;t,K) = 0 \end{aligned} \qquad (25)$$

- The first part, α^0, is option volatility due to the local volatility and stochastic volatility fully correlated with the stock.

- The second part, $\eta\alpha^1$, is uncorrelated stochastic volatility.

- We note that $\alpha^0 \cdot \alpha^1 = 0$.

Minimum Variance Delta

- The minimum variance delta is the position in the stock that minimizes the local variance of the portfolio of option and stock

$$\min_{\Delta} \mathrm{var}[dC - \Delta dS] \;\Rightarrow\; \Delta = \frac{dS \cdot dC}{(dS)^2} = \frac{\sigma \cdot \alpha}{\|\sigma\|^2} = \frac{\|\alpha^0\|}{\|\sigma\|} \qquad (26)$$

- Using the split of the volatility of options, this leads to a minimum-variance

$$0 = -\Delta_T + C_{KK}\delta + \vartheta\Delta_{KK} \quad , \Delta(t;t,K) = 1_{S>K} \qquad (27)$$

- … so the min var delta provides a way of estimating the component of volatility of forward volatility that is correlated with the stock.

- If at specific point in time $\{\Delta(t;T,K)\}$ is known (or guesstimated) then (27) can be used for backing out $\{\delta(t;T,K)\}$.

- We note the relation $\alpha^0 = \Delta\sigma$.

VIX Options

- Consider a contract on the integrated variance

$$v(t;T_1,T_2)=\frac{1}{2}E_t[\int_{T_1}^{T_2}\|\sigma(u)\|^2\,du]=\int_{T_1}^{T_2}[\int_{-\infty}^{+\infty}(\vartheta C_{KK})(t;T,K)dK]dT \qquad (28)$$

- Using the forward equation we get

$$v(t;T_1,T_2)=\int_0^{\infty}(C(t;T_2,K)-C(t;T_1,K))dK \qquad (29)$$

- From which we obtain that the volatility of the integrated variance splits in two independent components

$$
\begin{aligned}
dv(t;T_1,T_2) &= [\int_0^\infty (\alpha(t;T_2,K)-\alpha(t;T_1,K))dK]\cdot dW \\
&= [\int_0^\infty (\Delta(t;T_2,K)-\Delta(t;T_1,K))dK]\sigma(t)\cdot dW \\
&\quad +\eta(t)[\int_0^\infty (\alpha^1(t;T_2,K)-\alpha^1(t;T_1,K))dK]\cdot dW \\
&\equiv [\underbrace{D(t;T_1,T_2)\sigma(t)}_{Stock\,hedgeable}+\underbrace{\eta(t)A(t;T_1,T_2)}_{Stock\,orthogonal}]\cdot dW
\end{aligned} \tag{30}
$$

- Let options on the variance be denoted

$$
G(T_0,L)=E[(v(T_0,T_1,T_2)-L)^+] \tag{31}
$$

- Let G^M and G^O be respectively *Model* and *Observed* prices of options on variance.

- For the model to be able to fit prices of options on variance over $[T_1, T_2]$ with expiry T_0, we need there to exist a local volatility function $\beta(L)$ for the variance so that

$$\frac{1}{\Delta T}\left(G^O(T_0, L) - G^M(T_0 - \Delta T, L)\right) = \frac{1}{2}\beta(L)^2 \delta_{LL} G^O(T_0, L) \qquad (32)$$

- For a small expiry step ΔT we must have

$$D^2 \|\sigma\|^2 + \eta^2 \|A\|^2 = \beta^2 \;\Rightarrow\; \eta^2 = \frac{\beta^2 - D^2 \|\sigma\|^2}{\|A\|^2} \qquad (33)$$

- This defines a way of bootstrap calibrating the model to options on variance without iteration.

- It does, however, require Monte-Carlo simulation to obtain the values for the option at the previous time step:

$$G^M(T_0 - \Delta T, L) \tag{34}$$

- This can go wrong in two cases: either positivity in (33) is violated or the maturity spread is violated

$$G^O(T_0, L) - G^M(T_0 - \Delta T, L) > 0 \tag{35}$$

- The positivity of the maturity spread can be violated if ξ doesn't decay fast enough with $(T-t)$.

- Positivity in (33) can be violated if δ is not sufficiently negative.

- Minimal variance option price is obtained for a choice of a (negative) δ that reduces Δ^2 to a minimum and sets $\eta = 0$.

- This seems to be in line with findings by Guyon (2019).

Conclusion

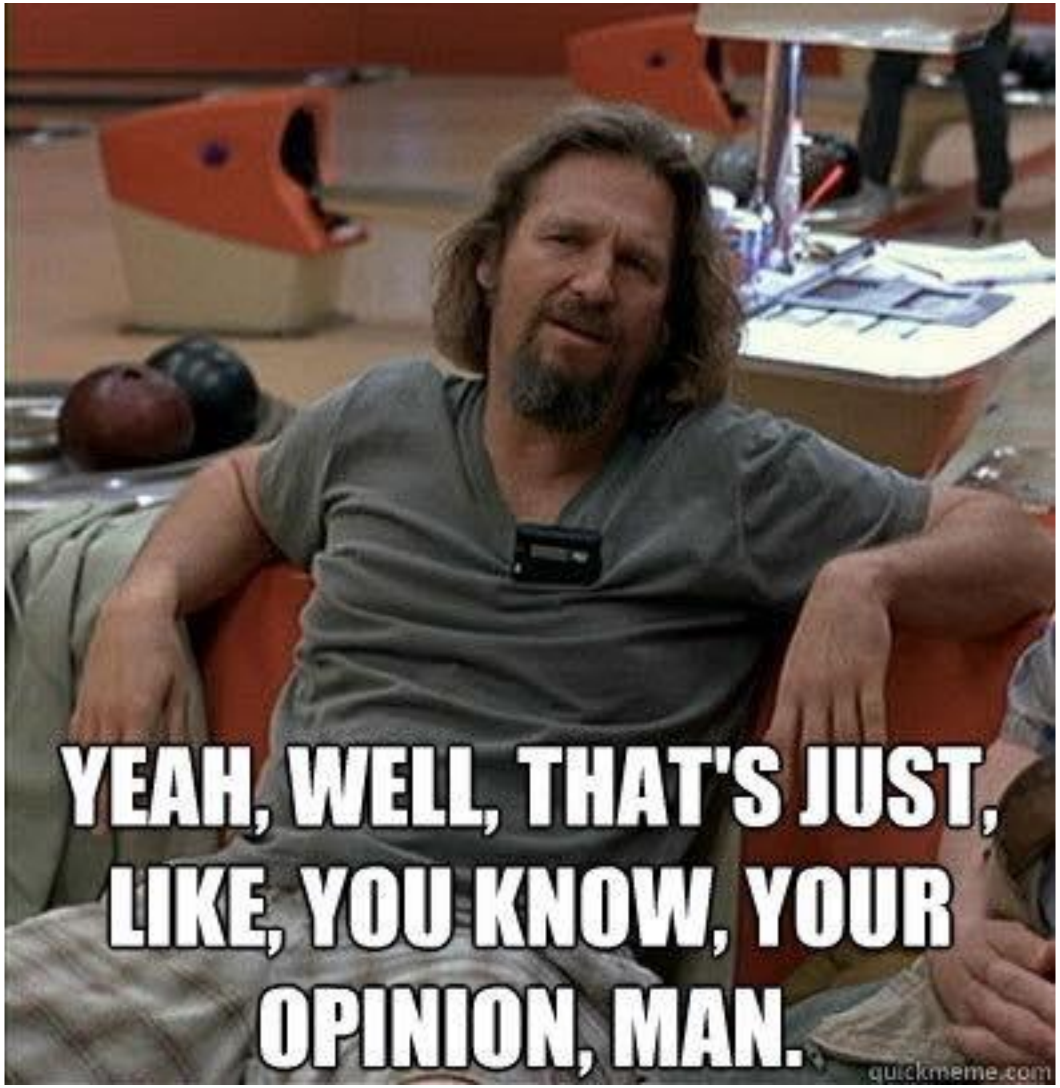
- We have identified a way of computing no-arbitrage drift conditions for the local volatility surface.

- The methodology applies to the continuous as well as the discrete case.

- This provides an alternative for simulation of rough volatility models as well as tying in stochastic local volatility models with options on variance.

- The modeling approach allows a split into forward variance, minimum variance delta, and options on variance.

- Applications of the modeling approach includes exotic option pricing and hedging under transaction costs and automation of trading.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