Local Volatility in Multi Dimensions

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References

- Andreasen, J (2020): “Multi-Factor Cheyette Monte-Carlo Calibration.” *Saxo WP*.
- Andreasen, J (2005): “Back to the Future.” *Risk* September.
- Austing, P (2011): “Repricing the Cross Smile: An Analytic Joint Density.” *Risk* July.
- Dupire, B (1994): “Pricing with a Smile.” *Risk* July.
- Guyon, J (2014): “Local Correlation Familes.” *Risk* February.
- Gyöngy, I (1986): “Mimicking the One-Dimensional Marginal Distributions of Processes having and Ito Differential.” *Probability Theory and Related Fields* 71.

- McCloud, P (2011): “The CMS Triangle Arbitrage.” *Risk* January.

- Piterbarg, V (2011): “Farka’s Lemma, Spread Options, and Linear Programming“. *Risk* August.

- Shelton, D (2015): “Interpolating the Smile with Path-Dependent Local Volatility.” *ICBI Global Derivatives*.

Prelude

- To simplify the exposition and save time I will work with the model in its simplest form.
- It is relatively straightforward to generalise the simple model to FX and equities.
- … but interest rates are more complicated so I will go in more detail with this in the last section of this talk.

Multi Asset Arbitrage

- Consider a market with stocks $s_1, \ldots, s_I$ and bank account s_0.
- Assume interest rates and dividends are zero, and set the start prices to be $s_i(0) = 0$.
- Assume you can trade variance contracts on all spreads

$$v_{ij}(t) = PV[(s_i(t) - s_j(t))^2] \tag{1}$$

- We note that covariance matrix is given by

$$g_{ij} = PV[s_i s_j] = \frac{1}{2} PV[s_i^2 + s_j^2 - (s_i - s_j)^2] = \frac{1}{2}(v_{i0} + v_{j0} - v_{ij}) \tag{2}$$

- Absence of arbitrage implies that the covariance matrix

$$G(t) = \{g_{ij}(t)\} \qquad (3)$$

- … must be ***positive semi definite*** for all t.
- If not, there exist non-zero weights $\{w_i\}$ so that

$$PV[(\sum_i w_i s_i(t))^2] = \sum_i \sum_j w_i w_j PV[s_i(t) s_j(t)] = w' G(t) w < 0 \qquad (4)$$

- This is contradicting absence of arbitrage since:

$$(\sum_i w_i s_i(t))^2 \geq 0 \qquad (5)$$

- The arbitrage portfolio is in this case given by

$$\{ \underbrace{(w_i w_j)}_{\substack{portfolio\\ weight\\ ij}} \cdot \underbrace{g_{ij}}_{\substack{\mathrm{cov}\,ij\\ contract}} \} \tag{6}$$

Multi Asset Arbitrage -- Notes

- We can sharpen a bit: Positive definiteness has to hold for

$$\{g_{ij}(t_2)-g_{ij}(t_1)\} \tag{7}$$

- … for ***all*** pairs $t_1<t_2$.

- Identification of arbitrage: any symmetric matrix G can be written as

$$G=O\Lambda O' \tag{8}$$

- …. where $\Lambda=Diag(\lambda_1,\ldots,\lambda_N)$ is a diagonal matrix of eigenvalues and O is an orthogonal matrix of eigenvectors, i.e. $OO'=I$.

- If $\lambda_j < 0$ then $w_i = O_{ij}$ is a set of arbitrage weights.

- … with the arbitrage portfolio given by

$$\{(w_i w_j) \cdot g_{ij}\} \tag{9}$$

Minimal Multi Asset Models

- … is a multi asset local volatility model

$$
\begin{aligned}
ds_i &= \sigma_i(t,s_i)\,dW_i \\
dW_i \cdot dW_j &= \rho_{ij}(t,s_i,s_j)\,dt
\end{aligned} \tag{10}
$$

- … where the local correlation is given from the volatility of the spread

$$
\begin{aligned}
&(d(s_i - s_j))^2 / dt = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j = \sigma_{ij}^2 \\
&\Downarrow \\
&\rho_{ij}(t,s_i,s_j) = \frac{\sigma_i(t,s_i)^2 + \sigma_j(t,s_j)^2 - \sigma_{ij}(t,s_i - s_j)^2}{2\sigma_i(t,s_i)\sigma_j(t,s_j)}
\end{aligned} \tag{11}
$$

- So the model is parameterised from the local spread volatilities $\{\sigma_{ij}(s_i - s_j)\}$ which are given as function of the spread levels.

- The model is constructed as to be able to fit the initial option prices

$$c_{ij}(t,k) = E[(s_i(t) - s_j(t) - k)^+] \tag{12}$$

- … through the Dupire equation

$$0 = -\frac{\partial c_{ij}}{\partial t} + \frac{1}{2}\sigma_{ij}(t,k)^2 \frac{\partial^2 c_{ij}}{\partial k^2} \tag{13}$$

- Absence of arbitrage is dictated through the usual conditions

$$\frac{\partial c_{ij}}{\partial t} > 0 \;, \frac{\partial^2 c_{ij}}{\partial k^2} > 0 \qquad (14)$$

- … ***plus*** the correlation matrix given by (11):

$$\{\rho(t, s_i, s_j)\} \qquad (15)$$

- … needs to be bounded in $[-1,1]$ and ***positive definite***.

- The construction through spread volatility rather than correlation is similar to Austing (2011).

Minimal Model and Arbitrage

- If a minimal model exists then there is absence of arbitrage.
- Does absence of arbitrage imply the existence of a minimal model?
- Unfortunately not. Counter example:

$$
\begin{aligned}
ds_1 &= \sigma(s_1,s_2)dW_1 \\
ds_2 &= \sigma(s_1,s_2)dW_2 \\
\sigma(s_1,s_2) &= \underline{\sigma} + (\bar{\sigma} - \underline{\sigma})1_{s_1 - s_2 = k} \quad , dW_1 \cdot dW_2 = 0
\end{aligned}
\tag{16}
$$

- … for some constants $\underline{\sigma} < \bar{\sigma}$.

- Minimal model correlation:

$$
\begin{aligned}
\rho(s_1,s_2) &= \frac{1}{2}\frac{E[(ds_1)^2|s_1]+E[(ds_2)^2|s_2]-E[(ds_1-ds_2)^2|s_1-s_2]}{(E[(ds_1)^2|s_1]E[(ds_2)^2|s_2])^{1/2}} \\
&= \frac{1}{2}\frac{\underline{\sigma}^2+\underline{\sigma}^2-(2\underline{\sigma}^2+2(\bar{\sigma}^2-\underline{\sigma}^2)1_{s_1-s_2=k})}{\underline{\sigma}^2} \\
&= -\frac{\bar{\sigma}^2}{\underline{\sigma}^2}1_{s_1-s_2=k}
\end{aligned}
\tag{17}
$$

- … so $\rho(s_1,s_2)<-1$ on $\{s_1-s_2=k\}$.

- Obviously a quite specific case but it extends to more interesting cases if we replace the indicator function with something like $\exp(-\frac{1}{2}(s_1-s_2-k)^2/(\Delta k)^2)$.

- The example suggests that if the volatility smile is more pronounced in the spread direction s_1-s_2 than in the primal directions s_1, s_2, then the specification $\sigma_i(s_1-s_2)$ may be a better choice than $\sigma_i(s_i)$.

Discrete Time

- For several reasons it is beneficial to consider the discrete time case.
- First, models live in computers and computers live in discrete time.
- Secondly, in real applications the model setup will have to be somewhat modified relative to what we have outlined so far.
- Thirdly, it would be nice to be able to handle various model extensions such as stochastic volatility and stochastic interest rates.
- It turns out that these modifications and extensions are relatively straightforward to handle in discrete time.

Discrete Time Minimal Model

- An Euler discretisation of the model on the time grid $\{t_h\}$ is

$$\Delta s_i(t_h) = \sigma_i(t_h, s_i(t_h))\Delta W_i(t_h)$$

$$\{\Delta W_i(t_h)\} \sim N(0, \{\rho_{ij}(t_h)\}) \tag{18}$$

$$\rho_{ij}(t_h, s_i, s_j) = \frac{\sigma_i(t_h, s_i)^2 + \sigma_j(t_h, s_j)^2 - \sigma_{ij}(t_h, s_i - s_j)^2}{2\sigma_i(t_h, s_i)\sigma_j(t_h, s_j)}$$

- … where we have used the notation $\Delta x(t_h) = x(t_{h+1}) - x(t_h)$.

- As in the continuous time case, the model is specified through spread volatility rather than correlation.

- We require the matrix $P = \{\rho_{ij}\}$ to be positive definite.

Monte-Carlo Pricing

- In a Monte-Carlo simulation over samples $\{\omega\}$, the value of an option that expiries as time t_{h+1} can be written as a sum over Bachelier's formula

$$
\begin{aligned}
c_{ij}(t_{h+1},k) &= \frac{1}{N}\sum_{\omega} E_{t_h}[(\underbrace{s_i(t_{h+1}) - s_j(t_{h+1})}_{Conditional\ Normal\ Distributed} - k)^{+} | \omega] \\
&= \frac{1}{N}\sum_{\omega} \underbrace{b(\Delta t_h, k; s_i - s_j, \sigma_{ij}(t_h, s_i - s_j))(t_h,\omega)}_{Bachelier's\ \ formula}
\end{aligned}
\qquad (19)
$$

- … where $N = \#\{\omega\}$ is the number of samples and Bachelier's formula is

$$b(\tau,k;s,v)=(s-k)\Phi(x)+v\sqrt{\tau}\phi(x) \quad ,x=\frac{s-k}{v\sqrt{\tau}} \tag{20}$$

- This is so because over each time step, $s_i - s_j$ has a conditional normal distribution – due to the Euler discretisation.

- The pricing formula is ***exact*** within the discrete model.

Monte-Carlo Calibration

- If we wish to calibrate the model to the strikes $\{k_{ij}^1,\ldots,k_{ij}^L\}$ at expiry t_{h+1} then we parameterise the volatility function $\sigma_{ij}(t_h;s_i-s_j)$ with L parameters.

- … for example linear interpolation between the L strike points.

- We then solve the minimization problem

$$\inf_{\sigma_{ij}(t_h,\cdot)} \sum_l \big(\underbrace{c(t_{h+1},k_{ij}^l)}_{mc\,\text{model}\,price} - \underbrace{\hat{c}(t_{h+1},k_{ij}^l)}_{market\ price} \big)^2 \tag{21}$$

- Note that the calibration of $\{\sigma_{ij}(t_h,\cdot)\}$ is independent for different pairs (i,j).

- After calibration to the options for each spread pair (i, j) then can we construct the correlation matrix $P = \{\rho_{ij}\}$.

- The methodology can also be used for correlation structures that are not minimal.

- If we for example set

$$\sigma_{ij} = \sigma_{ij}(a_{ij} \cdot s) \tag{22}$$

- … for constant vectors a_{ij}, then the calibration problem is still independent over the different pairs (i, j).

- We do, however, not yet have a methodology for optimal choice of directional vectors $\{a_{ij}\}$.

Positive Definiteness and Bootstrap

- The resulting correlation matrix P is not necessarily positive definite.

- To make it positive definite, decompose into the product $P = O\Lambda O'$, chop negative eigenvalues and rescale to obtain units along the diagonal.

- This procedure is not computationally costless.

- Once done with calibration of the time step $t_h \mapsto t_{h+1}$, we simulate forward to calibrate the model to the time step $t_{h+1} \mapsto t_{h+2}$.

Catch-Up and Discrete Quotes

- If fiddling with the correlation (or covariance) matrix is necessary then the model will not hit the option prices at the particular expiry.

- However, the bootstrap methodology will attempt to ***catch-up*** at the next expiry.

- This is so because the Monte-Carlo pricing/calibration (19) works a bit like updating local volatility according to

$$\sigma(t_h)^2 = 2 \underbrace{\frac{\hat{c}(t_{h+1}) - c(t_h)}{t_{h+1} - t_h}}_{\substack{\textit{maturity spread} \\ \textit{computed as} \\ \textit{market}-\text{model}}} [\delta_{kk} \hat{c}(t_{h+1})]^{-1} \tag{23}$$

- Equation (23) is a trick that has been used with success in finite difference implementation of local volatility models.

- Hence, the model fit will only be broken at the expiry with positive definiteness problems -- not necessarily at subsequent expiries.

- Also, note we only calibrate to a discrete number of option strikes.

- Hence, we do not rely on perfectly smooth and arbitrage free volatility surfaces in all directions.

Timelines

- The calibration time line $\{t_h\}$ is fixed.

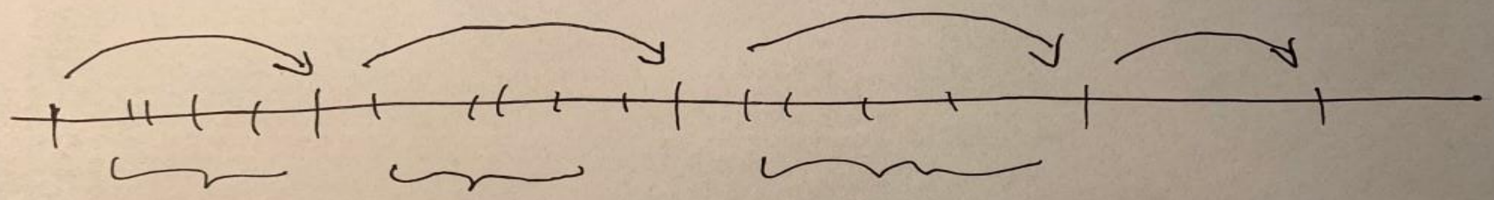
- But we can insert extra simulation time points as we wish inside each calibration time bucket $[t_h, t_{h+1}]$.

- As long as we keep the volatilities and correlations constant over these extra time points.

- In that sense, the model looks a bit like the model in Shelton (2015).

- Here, we use Monte-Carlo rather than numerical integration and this makes our model applicable to high dimensions.

- Bootstrap calibration by Monte-Carlo
- Using Normality of Euler Stepping.



- Any simulation time line after Calibration.
- Freezing Volatility & Correlation over each calibration time Bucket.

Applications and Extensions

- Foreign exchange: Note that log-normal form and currency translations are necessary.

- Equities: Calibrate to basket rather than spread options. Potentially, using notions of average correlation.

- Note that non-trivial dividend models can also be handled this way.

- Interest rates: Non-trivial but interesting. Both multifactor Cheyette and LMM type models can be constructed.

- The interest rate models can potentially calibrate simultaneously to cap/swaption smiles ***and*** smiles of spread and/or mid-curve options.

- Stochastic volatility and even rough volatility is straightforward.

- It is also possible to do models that simultaneously calibrate to SP500 and VIX smiles.

- … and more.

Numerical Implementation

- So far, we have implemented a multi factor Cheyette model for interest rates and a multi price model for FX and equities.
- Both with multi factor stochastic volatility.
- The intention is to combine the two model types to a Next Gen Beast.
- Both are implemented with extensive use of multi threading on CPUs.
- Adjoint differention (AAD) risk has been implemented for the interest rate model.

Numerical Performance

- Hardware is a standard 4 core CPU machine.

- 5 Ccy FX model calibration to 5 strikes in all crosses on expiries 1m, 2m, … 12m:

    - 8,192 paths: 0.46s
    - 65,536 paths: 3.32s

- 4 factor interest rate model. Calibration to 5 strikes in all crosses (spread options and mid curves) in tenors 3m, 2y, 10y, 30y on expiries 1y, 2y, … , 10y.

    - 8,192 paths: 0.45s
    - 65,536 paths: 3.44s

- 6 factor interest rate model. Calibration to 5 strikes in all crosses in tenors 3m, 2y, 5y, 10y, 20y, 30y on expiries 1y, 2y, … , 10y.

  - 8,192 paths: 1.00s
  - 65,536 paths: 7.13s

- Pure simulation times (without calibration) are roughly half.

- AD risk around a factor 10 slower than calibration/pricing.

- Numerical performance is definitely ok, but a tad slower than my audience is used to.

- The approach lends itself very well to TensorFlow/GPU acceleration.

Discrete Model Summary

- The method does not require full continuous surfaces of arbitrage free option prices. Discrete points are sufficient.

- If arbitrage or positive definiteness is broken at a particular time point, the methodology will attempt to catch-up at subsequent time-steps.

- It applies to cases without non-trivial forward equations such as interest rates and commodities.

- Calibration is discretely consistent with discrete Euler stepping. No approximation error.

- Non-minimal correlation can also be handled but we don’t know yet how to optimally choose the directions $\{a_{ij}\}$.

- Long time steps in the calibration, short time steps in pricing.

Interest Rate Options

- Swaption pricing in a Monte-Carlo model

$$\begin{aligned} E[(\underbrace{s_i(t_{h+1})}_{\substack{swap\\rate}} - k)^+ \underbrace{a_i(t_{h+1})}_{annuity}] &= E[E_{t_h}[(s_i(t_{h+1}) - k)^+ a_i(t_{h+1})]] \\ &= E[a_i(t_h) \underbrace{E^a_{t_h}}_{\substack{annuity\\measure}} [(s_i(t_{h+1}) - k)^+]] \\ &\approx \frac{1}{N}\sum_{\omega} a_i(t_h,\omega) E^a_{t_h}[(s_i(t_{h+1}) - k)^+ |\omega] \\ &\approx \frac{1}{N}\sum_{\omega} a_i(t_h,\omega) b(s_i(t_h,\omega) - k, \sigma_i(t_h,\omega)) \end{aligned} \tag{24}$$

- … where $\sigma_i(t_h)$ is the (approximate) volatility of the swap rate.

- For spread options we have

$$E^{t_{h+1}}[(\underbrace{s_i(t_{h+1})-s_j(t_{h+1})}_{swap\,rate\,spread}-k)^+]=E^{t_{h+1}}[E^{t_{h+1}}_{t_h}[(s_i(t_{h+1})-s_j(t_{h+1})-k)^+]]$$

$$\approx\frac{1}{N}\sum_{\omega}E^{t_{h+1}}_{t_h}[(s_i(t_{h+1})-s_j(t_{h+1})-k)^+|\omega] \qquad (25)$$

$$\approx\frac{1}{N}\sum_{\omega}b(s_i(t_h,\omega)-s_{ij}(t_h,\omega)-k,\sigma_{ij}(t_h,\omega))$$

- … where $\sigma_{ij}(t_h)$ is the (approximate) volatility of the swap rate spread with

$$\sigma_{ij}^2=\sigma_i^2+\sigma_j^2-2\sigma_i\sigma_j\rho_{ij} \qquad (26)$$

- In (25) we have ignored a small one-period convexity adjustment.

- For mid curve swap we have

$$
\begin{aligned}
&s_{ij} = \frac{a_i s_i}{a_i - a_j} - \frac{a_j s_j}{a_i - a_j} \\
&\Rightarrow \\
&(ds_{ij})^2 / dt = \bar{\sigma}_{ij}^2 \approx \left(\frac{a_i}{a_i - a_j}\right)^2 \sigma_i^2 + \left(\frac{a_j}{a_i - a_j}\right)^2 \sigma_j^2 - 2\frac{a_i}{a_i - a_j}\frac{a_j}{a_i - a_j}\rho_{ij}\sigma_i\sigma_j
\end{aligned}
\tag{27}
$$

- Swap rate and spread volatility can be approximated in most yield curve models.

Multi Factor Cheyette Model

- … is a Markov version of the general HJM model with the following $n+n(n+1)/2$ dimensional representation

$$
\begin{aligned}
dx_i &= (-\kappa_i x_i + \sum_j y_{ij})dt + \sum_j \eta_{ij} dW_j \\
dy_{ij} &= (-(\kappa_i + \kappa_j)y_{ij} + \sum_k \eta_{ik}\eta_{jk})dt \\
P(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-\sum_i G_i x_i + \sum_{ij} G_i y_{ij} G_j} \qquad , G_i = G(t,T;\kappa_i) = \frac{1-e^{-\kappa_i (T-t)}}{\kappa_i}
\end{aligned}
\tag{28}
$$

- In this type of model we have that the swap rate evolves according to

$$ds_i = \underbrace{\frac{\partial s_i}{\partial x}}_{\in \mathbb{R}^{1\times n}} \underbrace{\eta}_{\in \mathbb{R}^{n\times n}} \underbrace{dW}_{\in \mathbb{R}^{n\times 1}} + O(dt) \tag{29}$$

- If we stack up the swap rates and invert the system we get that

$$\begin{aligned} \eta &= \underbrace{\{\frac{\partial s_i}{\partial x_j}\}^{-1}}_{\in \mathbb{R}^{n\times n}} \underbrace{\{\sigma_i \sigma_j \rho_{ij}\}^{1/2}}_{\in \mathbb{R}^{n\times n}} \\ \rho_{ij} &= \frac{\sigma_i^2 + \sigma_j^2 - \sigma_{ij}^2}{2\sigma_i \sigma_j} \quad ,\sigma_i = \sigma_i(s_i) \quad ,\sigma_{ij} = \sigma_{ij}(s_i - s_j) \end{aligned} \tag{30}$$

- … so model is specified though local volatilities of swap rates and spreads of these.

- … in a way similar to the price model case.

- So we can construct multi factor yield curve models that potentially fit smiles in all directions: caps, swaptions, CMS spread options and mid-curve options.

- Note: we do not rely on forward equations to do this trick.

Conclusion

- We have presented an approach to multi factor local volatility with associated Monte-Carlo calibration methodology that is performing, flexible and general.

- Next steps:

  - Combining interest rate and price models in a 5G Beast.

  - Non-minimal correlation structures.

  - GPU/TensorFlow acceleration.

- The future is bright.