A Tale of Three Cities

Comparison between GVV, SVI and IRV

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Outline

- **GVV Cost Framework**
- SVI Parameterization
- IRV Framework
- Link with GVV and SVI

Gamma-Vanna-Volga Cost Framework

Main Idea

The main idea is the following [Arslan et al. 2009]:

For the options on the same underlying with the same maturity

- Each option with different strikes comes with its own set of risks.
- Assuming that on each trading day, **theta** could be written as a linear combination of dollar **gamma, vanna and volga**, and the **relative price of each Greeks is constant**.

Gamma-Vanna-Volga Cost Framework

Smile Construction

- Assume

$$\Theta = \Omega_\Gamma \Gamma + \Omega_{Va} Va + \Omega_{Vo} Vo \tag{1}$$

where Γ, Va, Vo is dollar Gamma, dollar Vanna and dollar Volga respectively

$$(\Gamma, Va, Vo) = \left(\frac{\partial^2 C}{\partial S^2} S^2, \frac{\partial^2 C}{\partial S \partial \sigma} S\sigma, \frac{\partial^2 C}{\partial \sigma^2} \sigma^2 \right). \tag{2}$$

The coefficients $\Omega_\Gamma, \Omega_{Va}, \Omega_{Vo}$ are constant. This identity holds for each option of different strikes.

- If we have three pillar options' implied volatility (IV), we could invert the equation (1) to get $(\Omega_\Gamma, \Omega_{Va}, \Omega_{Vo})$ explicitly.Then, any 4th option's IV σ_4 could be obtained by inverting from equation

$$\Theta(\sigma_4) = \Omega_\Gamma \Gamma(\sigma_4) + \Omega_{Va} Va(\sigma_4) + \Omega_{Vo} Vo(\sigma_4) \tag{3}$$

Thus the whole smile could be constructed in the same way.

Gamma-Vanna-Volga Cost Framework

Smile Construction

- We define implied remaining variance as $\omega = \sigma^2 (T - t)$. Substitute all Gamma, Vanna, and Volga formulas into (3), we end up with

$$\Omega_{Vo} \left(k^2 - \frac{\omega^2}{4} \right) + \frac{\Omega_{Va} + 1}{2} \omega + \Omega_{Va} k + \Omega_{\Gamma} = 0 \tag{4}$$

- This is actually a **quadratic equation** for ω and k, with 3 parameters to be determined.

Outline

- GVV Cost Framework
- **SVI Parameterization**
- IRV Framework
- Link with GVV and SVI

Roger Lee's Moment Formula

- Roger Lee's paper [Lee 2003] offers a bound of the **high/low-strike tail** of the IRV skew using the **moment formula**.
- With $k = \log \frac{K}{F}$, define

$$\begin{aligned} \beta_L &= \limsup_{k \to -\infty} \frac{\sigma_{BS}^2 (k, T)\, T}{|k|} = \limsup_{k \to -\infty} \frac{\omega(k)}{|k|} \\ \beta_R &= \limsup_{k \to +\infty} \frac{\sigma_{BS}^2 (k, T)\, T}{|k|} = \limsup_{k \to +\infty} \frac{\omega(k)}{|k|} \end{aligned} \tag{5}$$

  Under the existence of martingale measure, Lee shows that $\beta_L, \beta_R \in [0, 2]$ for any maturity $T > 0$. Here 2 could be refined by further specification of the distribution of the underlying asset price process by the moment formula. In other words, **if β_L, β_R are out of this range, there will be arbitrage.**
- This implies that the Implied Variance curve at each maturity **could be linear at extreme strike**.

SVI Parameterization of Implied Volatility

Original SVI formula

- Inspired by Roger's result, Merill Lynch devised SVI formula in [Gatheral 2004]

$$\omega = a + b\left\{\rho\left(k-m\right)+\sqrt{\left(k-m\right)^2+\sigma^2}\right\} \tag{6}$$

where $a \in \mathbb{R}, b \geq 0, |\rho| < 1, m \in \mathbb{R}, \sigma > 0$.

- These parameters have the following effects:
  - a governs the **general level** of variance, a vertical translation of the smile;
  - b governs the **slope of both** the put and call wings, tightening or loosening the smile;
  - ρ governs the **rotation** of the smile;
  - m **translates** the smile to the right / left;
  - σ governs **the ATM curvature** of the smile.

Hyperbolic Curve

Curve generated by our quadratic equation

- Coincidently both the SVI formula and GVV method can be transformed into a **quadratic equation**

$$A\omega^2 + B\omega k + Ck^2 + D\omega + Ek + F = 0. \tag{7}$$

This equaiton represent **conic section**.

- Both SVI and GVV formula satisfy $\Delta = B^2 - 4AC > 0$. This leads the curve of (7) to be **one segment of the hyperbolic curve**[2].

[2] The standard form of hyperbolic curve is $y^2 - x^2 = 1$.

Hyperbolic Curve

Geometric property

- A general hyperbolic curve looks like

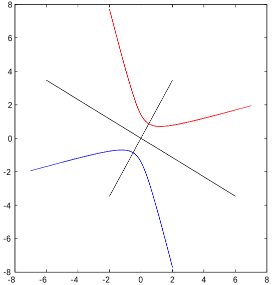


Figure: $x^2 - y^2 - 2\sqrt{3}xy - 2 = 0$

- **This simple beauty in hyperbolic curve was actually studied in ante christum which turns out to be our grail.**

Outline

- GVV Cost Framework
- SVI Parameterization
- **IRV Framework**
  - **Original IRV3 Model**
  - **IRV4 Model**
  - **IRV5 Model**
  - **No Arbitrage**
- Link with GVV and SVI

Original IRV3 Model

The original paper [Carr and Sun 2013] proposed an approach for IRV:

- Under forward measure Q^T, the forward price F is a martingale

$$dF_t = \sqrt{v_t} F_t dW_t, \;\; t \in [0, T] \tag{8}$$

- The instantaneous volatility $\sqrt{v_t}$ is introduced only to help specify the underlying. Considering the IV σ_{BS} is always associated with $T - t$, [Carr and Sun 2013] proposed to bypass the unobserved v_t to specify the risk-neutral dynamics for Implied Remaining Variance (IRV)

$$\omega_t (K) = \sigma^2_{BS,t} (K) (T - t), \;\; K > 0, t \in [0, T]. \tag{9}$$

and model it as

$$\begin{aligned} d\omega_t (K) &= a (\omega_t (K)) v_t dt + b (\omega_t (K)) \sqrt{v_t} dZ_t \\ dW_t dZ_t &= \rho (\omega_t) dt. \end{aligned} \tag{10}$$

- [Carr and Sun 2007] use a similar specification in a variance swap.

Original IRV3 Model

Basic equation

- Denote $C_t(F, K, \omega_t(K))$ as Black's call formula, measured in zero T-Bond:

$$C_t(F, K, \omega_t(K)) = FN(d_1) - KN(d_2). \tag{11}$$

- The martingale property of $C_t(F, K, \omega_t(K))$ implies that at each time t, the curve $\omega_t(K)$ must satisfy

$$\begin{aligned} 0 &= a(\omega_t) v_t \frac{\partial C}{\partial \omega}(F_t, K, \omega_t) + \frac{v_t}{2} F_t^2 \frac{\partial^2 C}{\partial F^2}(F_t, K, \omega_t) \\ &+ \frac{v_t}{2} b^2(\omega_t) \frac{\partial^2 C}{\partial \omega^2}(F_t, K, \omega_t) + v_t \rho(\omega_t) b(\omega_t) F_t \frac{\partial^2 C}{\partial F \partial \omega}(F_t, K, \omega_t) \end{aligned} \tag{12}$$

And if we plug in the Greek letters,

$$a(\omega) + 1 + \frac{\rho(\omega) b(\omega)}{2} - \left(\frac{1}{\omega} + \frac{1}{4}\right) \frac{b^2(\omega)}{4} + \frac{\rho(\omega) b(\omega) k}{\omega} + \frac{b^2(\omega) k^2}{4\omega^2} = 0 \tag{13}$$

with $k = \log \frac{K}{F}$. This **basic equation** plays an important role in our framework.

Original IRV3 Model

Lower order parameterization in IRV3 model

- In the original paper, the authors choose a lower order polynomial parametrization of the functions of $a(\omega)$, $b(\omega)$ and constant ρ as

$$\begin{aligned} a(\omega) &= -a_1\omega + a_0 - 1 \\ b(\omega) &= b\omega \end{aligned} \tag{14}$$

  to get a quadratic equation of ω and k.

$$\frac{1}{16}\omega^2 + \left(\frac{1}{4} - \frac{\rho}{2b} + \frac{a_1}{b^2}\right)\omega - \left(\frac{1}{4}k^2 + \frac{\rho}{b}k + \frac{a_0}{b^2}\right) = 0 \tag{15}$$

  This equation represent a **hyperbolic curve**!
- Uniformalize the parameters by setting $b = 1$, the whole implied volatility surface becomes solution to **quadratic equation with 3 unknown parameters** $\{\rho, a_0, a_1\}$.
- But we will take b back in later sections.

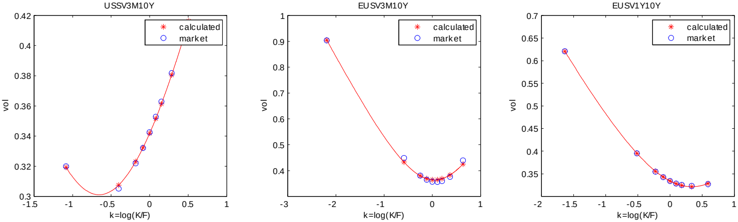
Original IRV3 Model

- Using the quadratic root formula, we have

$$\omega = 8\left(-\beta + \sqrt{\beta^2 + \frac{1}{4}\alpha}\right) \tag{16}$$

$$\beta = \frac{1}{4} - \frac{\rho}{2} + a_1, \quad \alpha = \frac{1}{4}k^2 + \rho k + a_0$$

- **The model fit the market data reasonably well**, for example, in the swaption market



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IRV3 Specification

Worst fitting days in calibration

- In most days, the data fits very well. But in some days, the fitting is not perfect (1%~4% in time series data).

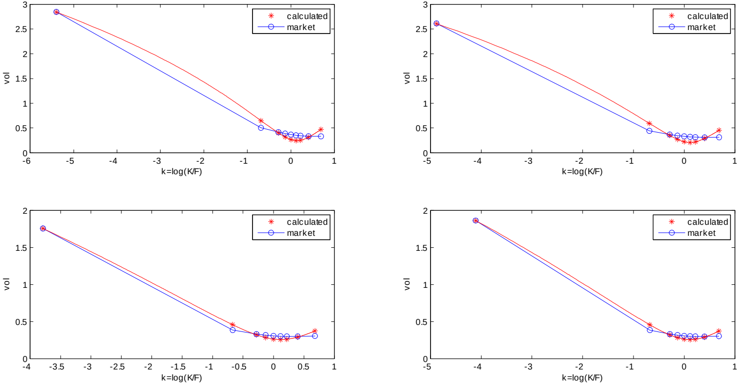


Figure: EUSV1Y10Y 27-Nov-2012 16-Jan-2013 04-Apr-2014 24-Apr-2014

IRV3 Specification

Enforceable symmetric and fixed wings

- The fitting usually fails at **near the money** strikes in order to compensate for better fitting at in / out of money strikes. According to Roger's criteria, we should check the asymptotic behavior of IRV3. In fact

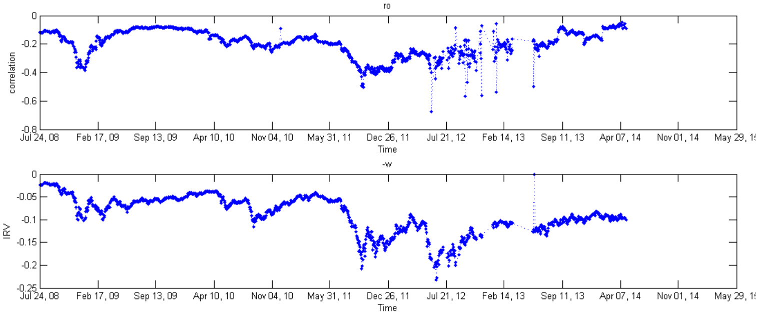
$$\beta_R \triangleq \limsup_{k \to -\infty} \frac{\omega}{|k|} = 2, \quad \beta_L \triangleq \lim_{k \to +\infty} \frac{\omega}{|k|} = 2 \tag{17}$$

- This is **exactly the boundary** in the permitted interval.
- The **enforced symmetry** is interesting.
- We now are inspired to find a suitable refined parameterization of IRV3. Our goal is
  - Firstly, try to **break the enforced symmetry.**
  - Secondly, control the **asymptotic behavior at the boundary**.

Empirical Result

Correlation between IRV and underlying process

- First, see the timeseries data of ρ and ω



EUSV1Y10Y

We could see some correlation between ρ and ω.

- So let us **relax the constant assumption of** ρ.

IRV4 Construction

- We introduce an **affine structure** in ρ

$$\begin{aligned} a(\omega) &= -a_1\omega + a_0 - 1 \\ b(\omega) &= b\omega \\ \rho(\omega) &= c\omega + d \end{aligned} \tag{18}$$

- Then we still have a **hyperbolic curve**[3]

$$\omega = \frac{-(\beta - ck) + \sqrt{(\beta - ck)^2 + 2\gamma\left(a_0 + dk + \frac{1}{4}k^2\right)}}{\gamma}. \tag{19}$$

where $\beta = a_1 - \frac{1}{2}d + \frac{1}{4}$, $\gamma = \frac{1}{8} - c$.

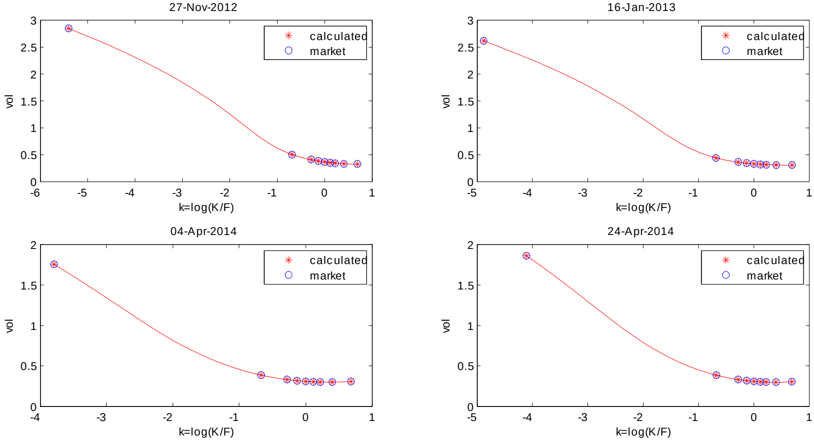
- Since now we have 4 parameters (c, d, a_0, a_1) under IRV Framework, we would like to call it **IRV4 model**. Let's see the data's behavior.

[3] We set $b = 1$ for the same reason of SC3

IRV4 Specification

IRV4 could overcome the bad case in IRV3

- **Works much better than IRV3**. There are nearly no bad fitting days. See the bad case of IRV3 in swaption market

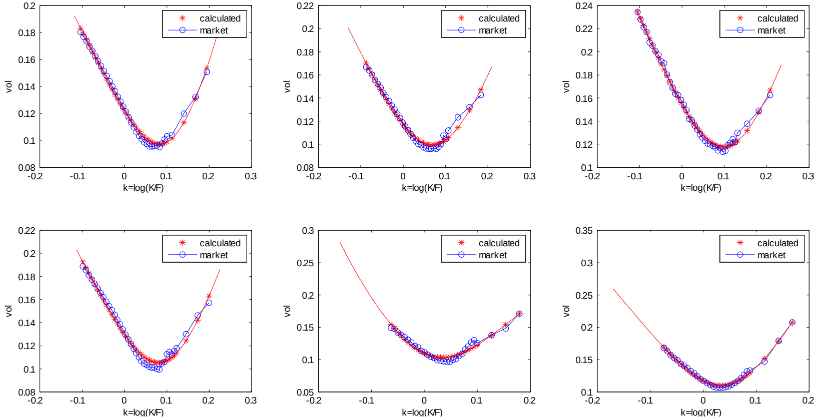


EUSV1Y10Y

IRV4 Specification

Option market

- IRV4 also works very well in **SNP option market**



SNP option market, maturity:2013/6/20

IRV4 Specification

Semi-free wings in IRV4

- Let us check the extreme strike behavior of IRV4, we find

$$\beta_L = 2,\ \beta_R = \frac{1}{\frac{1}{2} - c}. \tag{20}$$

$$\text{where } \frac{\omega}{|k|} \rightarrow \beta_R \text{ or } \beta_L \text{ as } k \rightarrow \pm\infty \tag{21}$$

- The left wing of IRV4 is still fixed to 2, while **the right wing could be adjusted by changing** c.
- There is **still a missing degree of freedom!** This inspired the next generation – IRV5. (Future we will see that there **may not exist IRV6**.)

IRV4 Specification

Symmetry

- Before we go to IRV5, let us consider an interesting phenomenon
  - The curve generated by IRV3(i.e. in IRV4 $c = 0$) is forced to be symmetric.
  - In IRV4, $\omega(k)$ is not symmetric if $c \neq 0$.
- **Conjecture**: Our $\rho(\omega) = c\omega + d$ is defined by $dWdZ = \rho(\omega)\,dt$.
  - Assume the instantaneous variance v_t is driven by $\tilde{W}$.
  - According to [Carr and Lee 2009], this symmetry is equivalent to W and $\tilde{W}$ being independent $\left(corr\left(W, \tilde{W}\right) = 0\right)$.
  - **Thus we guess that c may play the role of $corr\left(W, \tilde{W}\right)$!**

IRV5 Construction

- Let us introduce a more generalized parameterization

$$\begin{aligned} a(\omega) &= a_2\omega^2 - a_1\omega + a_0 - 1 \\ b(\omega) &= b\omega \\ \rho(\omega) &= c\omega + d \end{aligned} \tag{22}$$

which still lead to a **hyperbolic curve**[4]

$$\omega = \frac{-(\beta - ck) + \sqrt{(\beta - ck)^2 + 2\gamma\left(a_0 + dk + \frac{1}{4}k^2\right)}}{\gamma} \tag{23}$$

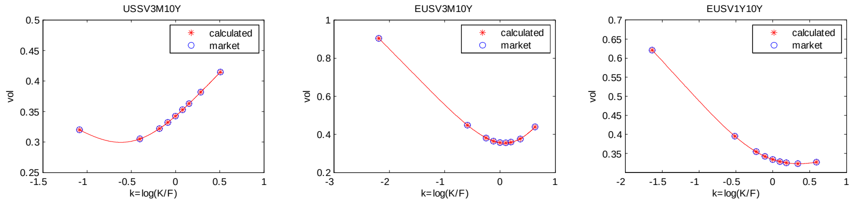
$$\begin{aligned} \alpha &= a_1 + \frac{1}{4},\ \beta = \alpha - \frac{1}{2}d \\ \gamma &= \frac{1}{8} - c - 2a_2,\ c^2 + \frac{1}{2}\gamma = \left(c - \frac{1}{4}\right)^2 - a_2. \end{aligned}$$

[4] The same with SC3, we uniformalize the parameters by setting $b = 1$.

Specification of IRV5

Calibrate IRV5 to market

- In swaption market, the IRV5 **fit the data perfectly**. For example, see



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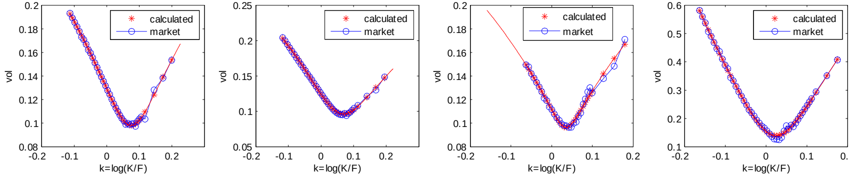
- The SNP option market is **also fitted perfectly**, see some



Option market maturity:2013/6/20

Specification of IRV5

Freedom of wings

- Now we will examine the asymptotic behavior of IRV5, denote $u = c, \; v = \frac{1}{16} - \frac{c}{2} - a_2,$

$$
\begin{aligned}
\beta_R &= \frac{u + \sqrt{u^2 + v}}{2v} \\
\beta_L &= \frac{-u + \sqrt{u^2 + v}}{2v} \\
u &= \frac{\beta_R - \beta_L}{4\beta_R \beta_L}, \; v = \frac{1}{4\beta_R \beta_L}
\end{aligned}
\tag{24}
$$

  This implies that **we could obtain any value of** (β_R, β_L) except for $\beta_R = 0$ or $\beta_L = 0$ by choosing suitalbe c and a_2. Now, we could state that both left and right wings in our IRV5 model are **totally free!**

- This is critical for **no arbitrage requirement**, which will soon become apparent.

No Static Arbitrage

Definition

- **No static arbitrage[5]:** A call price surface C is free of static arbitrage if there exists a non-negative martingale X such that $C(K,\tau) = E\left((X_\tau - K)^+ \,|\mathcal{F}_0\right)$ with $X_0 = S_0$,
- This means, if such a martingale and probability space exists, then we say that the call price surface is free of static arbitrage, thus there is no arbitrage opportunities trading in the surface.
- Denote $\omega(k,\tau) = \sigma_{BS}^2(k,\tau)\,\tau$, with $k = \log\frac{K}{F}$ and $\tau = T - t$, we will give **a sufficient and nearly necessary(truly necessary for IRV models) conditions for an IRV surface** to be free from *static arbitrage*.

[5] The explicit definition and theorems could be found in [Roper 2010]

No Arbitrage Condition

Conditions for call price surface

- It is well known that a **call price surface** $C(K,\tau)$ is free of static arbitrage (exists consistent martingale) if and only if
  1. $C(K,\tau)$ is a non-decreasing and convex function w.r.t K.
  2. $C(K,\tau)$ is non-decreasing w.r.t τ.
  3. $\lim\limits_{K\to\infty} C(K,\tau)=0$.
  4. $C(K,0)=(S_T-K)^+$
  5. $(S_t-K)^+\leq C(K,\tau)\leq S_t$.
- Then, we will translate these conditons to **IRV surface.**

No Arbitrage Condition

Conditions for IRV surface

We translate those condition for price into IRV language:

- Consider **a smooth IRV surface** $\omega(k,\tau)$, it is free of static arbitrage **if and only if**
  1. Inequality $0 \leq \left(1 - \frac{k\partial_k\omega}{2\omega}\right)^2 - \frac{1}{4\omega}\left(\partial_k\omega\right)^2 - \frac{1}{16}\left(\partial_k\omega\right)^2 + \frac{\partial_{kk}\omega}{2}$ always hold.
  2. $\omega(k,\tau)$ is non-decreasing w.r.t. τ.
  3. $\lim_{k\to\infty} d_1\left(k,\omega(k,\tau)\right) = -\infty$.
  4. $\omega(k,0) = 0$.
- Note that each item is correspond to the same index in last slides for price.

Lognormal Variance and Proportional Volatility Model

- We now use this result to examine the:
  - LNV model, proposed in [Carr and Wu 2011] where the implied volatility is given by

$$\sigma_{BS}^2 = -a + \sqrt{a^2 + b + \frac{8\rho_t \sigma_t k}{w_t e^{-\eta_t \tau} \tau^2} + \frac{4k^2}{\tau^2}} \qquad (25)$$

$$\text{with } a\left(\kappa_t, w_t, \eta_t, \rho_t, \sigma_t, \tau\right), b\left(\kappa_t, w_t, \eta_t, \theta_t, \sigma_t, \tau\right)$$

    with six parameters $\left(\rho_t, \sigma_t, \kappa_t, w_t, \eta_t, \theta_t\right)$, $\tau = T - t$.
  - PV model, proposed in [Carr and Wu 2013] where the implied volatility is given by

$$\sigma_{BS}^2 = a_t + \frac{2}{\tau}\sqrt{\left(k - b_t\right)^2 + c_t} \qquad (26)$$

$$\text{with } a\left(\eta_t, m_t, w_t, v_t, \rho_t, \tau\right), b\left(\eta_t, w_t, v_t, \rho_t, \tau\right), c\left(\eta_t, m_t, w_t, v_t, \rho_t, \tau\right)$$

    with five parameters $\left(m_t, w_t, \eta_t, v_t, \rho_t\right)$, $\tau = T - t$.

Arbitrage in LNV and PV

- [Andersson 2014] pointed out that the LNV and PV can't fulfil the condition necessary for no static arbitrage.
  - In LNV $\sigma_{BS}\sqrt{\tau} = \sqrt{2k} + O\left(k^{-\frac{1}{4}}\right)$ as $k \to \infty$ and hence $\lim_{k\to\infty} d_1\left(k, \sigma_{BS}^2\tau\right) = \lim_{k\to\infty}\left(-\frac{k}{\sigma_{BS}\sqrt{\tau}} + \frac{\sigma_{BS}\sqrt{\tau}}{2}\right) = 0.$
  - In PV $\sigma_{BS}\sqrt{\tau} = \sqrt{2k} + O\left(k^{-\frac{1}{4}}\right)$ as $k \to \infty$ and hence $\lim_{k\to\infty} d_1\left(k, \sigma_{BS}^2\tau\right) = \lim_{k\to\infty}\left(-\frac{k}{\sigma_{BS}\sqrt{\tau}} + \frac{\sigma_{BS}\sqrt{\tau}}{2}\right) = 0.$

  Where in NA condition 3, we need $\lim_{k\to\infty} d_1\left(k, \omega\left(k,\tau\right)\right) = -\infty$.
- It seems that the problem comes from the same limit in both LNV and PV that

$$\beta_{R/L} = \lim_{k\to\pm\infty} \frac{\sigma^2\tau}{|k|} = 2. \tag{27}$$

- This is also the case in our IRV3: both left and right wings are fixed to 2 which must be avoided.

Large Moneyness Behavior of IRV5

- Luckily, as we've discussed before, the limit β_R, β_L in IRV5 could **obtain any value** by choosing appropriate c and a_2.
- We could easily prove that
  - if $2a_2 + c < -|c|$,then $\beta_L \in (0,2)$, $\beta_R \in (0,2)$;
  - if $\lim_{k\to\infty} \frac{\sigma^2(k,T)T}{k} \in (0,2)$ (strictly less than 2), we must have $\lim_{k\to\infty} d_1(k, \sigma(k,T)) = -\infty$.
  - **It is sufficient to choose a_2 and c so that $2a_2 + c < -|c|$, then NA condition 3 would be satisfied.**
- **The problem that [Andersson 2014] proposed has been solved.**

No Arbitrage Constraint

- In our IRV5 model
  - we can choose parameters which satisfy NA condition 2, 3, 4.
- **So, the remaining problem for us is the $1st$ condition.**

No Arbitrage Constraint

- The no-arbitrage condition 1 for IV is the following inequality

$$g(k) \triangleq \left(1 - \frac{k\partial_k \omega}{2\omega}\right)^2 - \frac{1}{4\omega}(\partial_k \omega)^2 - \frac{1}{16}(\partial_k \omega)^2 + \frac{\partial_{kk}\omega}{2} \geq 0 \qquad (28)$$

This is equivalent to $\frac{\partial^2 C}{\partial K^2} > 0$.

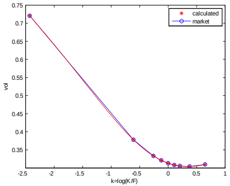
- In fact, in our IRV5

$$\begin{aligned} \lim_{k \to +\infty} g(k) &= \frac{1}{4} - \frac{1}{16}\beta_R^2 > 0 \\ \lim_{k \to -\infty} g(k) &= \frac{1}{4} - \frac{1}{16}\beta_L^2 > 0 \end{aligned} \qquad (29)$$

So we only need to consider the middle value k.

No Arbitrage Constraint

- Consider $(c, d, a_0, a_1, a_2) = (-1.95, 0.007818, 0.431, 3.222, -8.4)$ satisfying NA condition 2,3,4, which generate the well fitting curve



Swaption EUSV1Y10Y 31-Jan-2014

If we plot the function $g(k)$, we will see

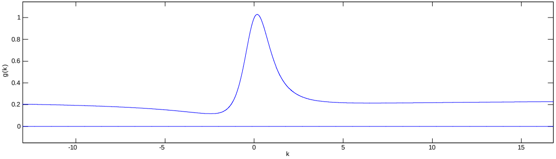
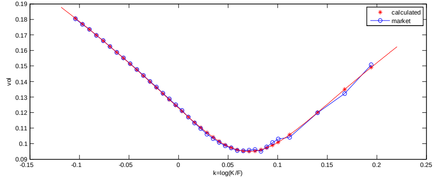


Figure: $g(k)$ of EUSV1Y10Y 31-Jan-2014

No arbitrage constraint

$$(c, d, a_0, a_1, a_2) = (-0.00255, -0.0415726, 0.00306, 0.1775513, -83.584)$$



SNP option 02-Apr-2013

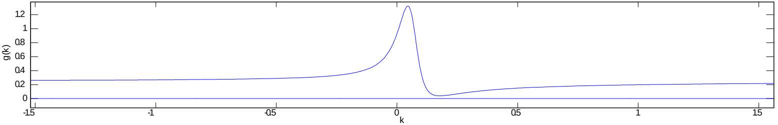


Figure: $g(k)$ of EUSV1Y10Y 02-Apr-2013, Expiration: 19-Jun-2013

where (c, d, a_0, a_1, a_2) satisfy the NA condition 1-4.

No Arbitrage Constraint

- In conclusion, there exists a class of parameters satisfying no arbitrage conditions 1-4 in our model. **Thus our model is able to be totally free of both static and dynamic arbitrage!**

Greeks Under IRV5 Model

- The Delta of call price under IRV5 should be calculated as

$$\begin{aligned}\frac{\partial C}{\partial F} &= \frac{\partial C^{BS}}{\partial F} + \frac{\partial C^{BS}}{\partial \omega} \cdot \frac{\partial \omega}{\partial F} \\ &= N\left(d_1\right) - \frac{N'\left(d_1\right)}{2\sqrt{\omega}} \cdot \frac{c}{\gamma}\left(1 + \frac{\left(ck-\beta\right)}{\sqrt{\left(\beta-ck\right)^2 + 2\gamma\left(a_0 + dk + \frac{1}{4}k^2\right)}}\right)\end{aligned} \tag{30}$$

  As a consequence, this shows the true Delta should be adjusted by an additional term.

- **The updated delta of IRV5 calibrated from the market is very reasonable**
  - Fisrt, they all belong to $[0, 1]$
  - Second, the adjustment is in the range of $\pm 0.001\tilde{\ } \pm 0.04$.

Outline

- GVV Cost Framework
- SVI Parameterization
- IRV Framework
- **Link with GVV and SVI**
  - **Link with GVV**
  - **Link with SVI**

Link with GVV

- Remind that we've introduced an emprical method named **Gamma-Vanna-Volga Cost Framework**.
- Assume theta cost of one dollar of gamma, vanna or volga and denoted by $\Omega_\Gamma, \Omega_{Va}$,and Ω_{Vo},repectively – is constant among all options in a given maturity. More explicitly

$$\Theta = \Omega_\Gamma \Gamma + \Omega_{Va} Va + \Omega_{Vo} Vo \tag{31}$$

where $\Omega_\Gamma, \Omega_{Va}, \Omega_{Vo}$ is **constant among different strikes for given maturity**.

Link with GVV

- If we consider the formalized Greek Letters as

$$(\Theta, \Gamma, Va, Vo) = \left(\frac{\partial C}{\partial t}(T-t), \frac{\partial^2 C}{\partial F^2}F_t^2, \frac{\partial^2 C}{\partial F \partial \sigma}F_t\sigma, \frac{\partial^2 C}{\partial \sigma^2}\sigma^2\right) \tag{32}$$

IRV framework actually implies **the relative price of the Greek letters** is

$$(\Omega_\Gamma, \Omega_{Va}, \Omega_{Vo}) = \frac{1}{2\left(\frac{a(\omega_t)}{\omega_t} - \frac{b^2(\omega_t)}{4\omega_t^2}\right)}\left(1, \frac{\rho(\omega_t)\, b(\omega_t)}{\omega_t}, \frac{b^2(\omega_t)}{4\omega_t^2}\right). \tag{33}$$

Link with GVV

IRV 3G construction

- If we let $c = 0, \; a_2 = 0, \; a_0 = 1$ in IRV5, with

$$\begin{aligned} a(\omega) &= a\omega \\ b(\omega) &= b\omega \\ \rho(\omega) &= \rho \end{aligned} \tag{34}$$

we will find that **the relative price of the Greek letters are independent with ω, then with strike**

$$\tilde{\Theta}(\sigma) = \frac{\frac{1}{b^2}}{2\left(\frac{a}{b^2} - \frac{1}{4}\right)} \tilde{\Gamma}(\sigma) + \frac{\frac{\rho}{b}}{2\left(\frac{a}{b^2} - \frac{1}{4}\right)} \tilde{V}a(\sigma) + \frac{\frac{1}{4}}{2\left(\frac{a}{b^2} - \frac{1}{4}\right)} \tilde{V}o(\sigma) \tag{35}$$

This is exactly what GVV want to achieve.

- GVV is actually a subset of our IRV models!

Link with SVI

- Remind that we've introduced another empirical method named **SVI parameterization**.
- SVI constructs parameters to describe the geometric property of the curve

$$\omega = a + b\left\{\rho\left(k - m\right) + \sqrt{\left(k - m\right)^2 + \sigma^2}\right\} \tag{36}$$

.

- **A totally empirical method with no consistent dynamic directly:** Although Gatheral makes some motivational linkages between these coefficients and SV modes (Large maturity Heston model, see[Gatheral and Jacquier 2011]), no SV models have been proposed that lead exactly to his SVI form.

Link with SVI

- But, if we transfrom IRV5 with

$$
\begin{aligned}
\tilde{a} &= \frac{\frac{1}{2}\alpha + cd - \frac{1}{4}d}{\left(c-\frac{1}{4}\right)^2 - a_2}, \quad \tilde{b} = \frac{\sqrt{\left(c-\frac{1}{4}\right)^2 - a_2}}{\frac{1}{8} - c - 2a_2} \\
\tilde{\rho} &= \frac{c}{\sqrt{\left(c-\frac{1}{4}\right)^2 - a_2}}, \quad \tilde{m} = \frac{\beta c - \gamma d}{c^2 + \frac{1}{2}\gamma} \\
\tilde{\sigma}^2 &= \frac{2a_0\gamma + \beta^2}{c^2 + \frac{1}{2}\gamma} - \frac{(\gamma d - \beta c)^2}{\left(c^2 + \frac{1}{2}\gamma\right)^2}.
\end{aligned}
\tag{37}
$$

Link with SVI

- But, if we transfrom IRV5 with

$$\begin{aligned}\tilde{a} &= \frac{\frac{1}{2}\alpha + cd - \frac{1}{4}d}{\left(c-\frac{1}{4}\right)^2 - a_2}, \quad \tilde{b} = \frac{\sqrt{\left(c-\frac{1}{4}\right)^2 - a_2}}{\frac{1}{8} - c - 2a_2} \\ \tilde{\rho} &= \frac{c}{\sqrt{\left(c-\frac{1}{4}\right)^2 - a_2}}, \quad \tilde{m} = \frac{\beta c - \gamma d}{c^2 + \frac{1}{2}\gamma} \\ \tilde{\sigma}^2 &= \frac{2a_0\gamma + \beta^2}{c^2 + \frac{1}{2}\gamma} - \frac{(\gamma d - \beta c)^2}{\left(c^2 + \frac{1}{2}\gamma\right)^2}.\end{aligned} \tag{37}$$

- **we could then rewrite IRV5 into**

$$\omega = \tilde{a} + \tilde{b}\left\{\tilde{\rho}\left(k - \tilde{m}\right) + \sqrt{\left(k-\tilde{m}\right)^2 + \tilde{\sigma}^2}\right\}. \tag{38}$$

Link with SVI

- These two parameterizations are totally equivalent, since conversely:

$$
\begin{aligned}
c &= \frac{\tilde{\rho}}{2\tilde{b}\left(1-\tilde{\rho}^2\right)}, \quad d = \tilde{a}c - \frac{\tilde{m}}{2} \\
a_2 &= \frac{1}{16} - \frac{\tilde{\rho}}{4\tilde{b}\left(1-\tilde{\rho}^2\right)} - \frac{1}{4\tilde{b}^2\left(1-\tilde{\rho}^2\right)} \\
a_1 &= \frac{\tilde{m}cd - \frac{1}{4}\tilde{m}d - \frac{1}{2}\tilde{a}cd - \frac{1}{8}\tilde{a}d + 2\tilde{a}a_2 d}{\tilde{a}c - \frac{1}{2}\tilde{m}} \\
a_0 &= \frac{1}{2}\left[\frac{\left(c-\frac{1}{4}\right)^2 - a_2}{\frac{1}{8} - c - 2a_2}\tilde{\sigma}^2 - \frac{\frac{1}{2}\left(\alpha^2 - \alpha d\right) + 2\alpha cd + 2a_2 d^2}{\left(c-\frac{1}{4}\right)^2 - a_2}\right]
\end{aligned}
\tag{39}
$$

- Now, we could state that SVI formula is fully equivalent with our model!

IRV Framework

Conclusion

- The reason we call this approach IRV Framework is quiet apparently:
  - Under this framework, we **generated several models**: IRV3, IRV4, **IRV5**.
  - **Consistent with SVI** and their relative conclusions
  - **Consistent with GVV** and their cost framework, which could be useful to traders
  - Easy to **exclude arbitrage**
  - we could use the model to calculate **better Delta** and other Greeks.
- The consistency implies the intuitive relationship: IRV $\omega = \sigma^2 (T - t)$ represent the uncertainty, log-moneyness $k = \log \frac{K}{F}$ represent the intrinsic value.

IRV Framework

Future Research

- There are still a lot of interesting things left to us to be studied, since this is really a powerful framework.
- Future research could be as following
  - Proposed new statistical method to get more accurate b.
  - Study the geometric character of parabolic curve.
  - Find simple close form conditions for no-arbitrage
  - **Add-in volatility term structure to do pridiction**
  - **From implied volatility to spot volatility.**
  - The same conclusion in Normal Case.
  - Unscented Kalman Filter and other statistical method
  - Quasi explicit calibration in IRV framework.

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