Arbitrage-free Asset Class Independent Volatility Surface Interpolation on Probability Space using Normed Call Prices

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Abstract

We consider arbitrage-free interpolation of arbitrage-free input data of European option prices. The method derived is independent of the underlying (equity, rates, FX, etc.). A particular contribution of the paper is that for the chosen coordinate system and a wide variety of interpolation methods, we prove that the method results in arbitrage-free interpolations. This result is achieved in two steps: first, we transform option prices to *Normed Call Prices (NCP)*, which are independent of the application. Using NCPs as interpolants, the interpolation problem can be formulated independently of assumptions of deterministic rates. Besides, the constraints necessary for ensuring absence of arbitrage are much simpler in this interpolant system, as they do not involve any scaling at all. Having proposed our choice of the dependent variable, we turn to the independent variables, and transform the strike-maturity or moneyness-maturity coordinate system to a probability-maturity coordinate system, which results in an interpolation method called *probability-space interpolation* in maturity dimension on the proposed NCP system that we prove actually leads to arbitrage-free surfaces in *both* strike and maturity dimensions. The computational tractability is an added advantage of our interpolation scheme.

1 Introduction

The arbitrage-free smooth interpolation of European option prices is a non-trivial problem. Of course, an arbitrage free interpolation is only possible if the input data itself are free of arbitrage. The problem must be reformulated as an arbitrage-free fitting of given European option prices, if the input data are even slightly contaminated with arbitrage violations

Arbitrage-free fitting of option prices is a natural sub-problem in risk management, where we consider “scenarios” applied to current market data. While fitting algorithms cope well in providing interpolations on market data (which mostly exhibit only slight arbitrage violations), artificially created stress scenarios exhibit strong arbitrage violations, and fitting algorithms may take much longer, or even fail to find reasonable fits that ensure smoothness and lack of arbitrage.

The arbitrage free interpolation of European option prices is also numerically more demanding as option prices move further away from “flat” Black-Scholes prices. For example, the credit crunch introduced strong volatility smiles in a situation where interest rates forwards are low. As a consequence, lognormal volatilities are rejected in favor of normal volatilities, and standard volatility interpolation models are discarded.[1]

Having the application of an arbitrage free fitting in mind, we see the arbitrage free interpolation of European option prices lies at the heart of this problem. European option prices are a function of two parameters: strike and maturity. Hence, the option price interpolation problem is to find a suitable two dimensional surface. Interpolation in strike dimension has been studied in detail in previous works such as Coleman et al. [2], Fengler [3] and Kahale [10]. While the interpolation constraints in respect of the strike dimension, i.e., for fixed maturity, is relatively transparent (the price is a convex function of the strike), the full two dimensional interpolation is more subtle. In fact, even if pick an interpolation method (for e.g., linear interpolation) for the maturity dimension, it is not clear which two strikes should be associated for different maturities to ensure both lack of arbitrage in strike and maturity dimension. [2] [3]

That said, it is apparent that a major step in the interpolation of option prices is to provide a good choice for the dependent and independent variables. Instead of considering the interpolation problem in the coordinates strike, maturity and price, we reformulate the problem in a different, more advantageous and more natural coordinate system.

In this paper, we consider arbitrage-free interpolation of arbitrage-free input data. Our first contribution is the introduction of a system of interpolants called *Normed Call Prices (NCP)*. We transform option prices to normed call prices, which are independent of the application. That is, the interpolation methods derived are applicable to equity options, FX options or interest rate options alike. This system is also independent of

[1] An example is the SABR model: In one formulation, the model implies positive interest rates, which may be too restrictive.

[2] Fengler [3] coined the term *calendar arbitrage* to refer to the problem of arbitrage violation in maturity dimension. We use same the terminology.

[3] With respect to strike interpolation, it is clear that three option with same maturities and different strikes must fulfill a simple arbitrage constraint, however, with respect to the interpolation in maturity dimension, it appears not so obvious if one should consider options of same strikes, same moneyness, or other crieria.

scaling parameters like spot and interest rates. A special feature of this NCP system is that the arguments we use are free of the traditional assumptions of deterministic rates. Besides, the constraints necessary for ensuring absence of arbitrage are much simpler in this system, as they do not involve any scaling at all.

Having chosen our choice of the interpolants as dependent variables, we turn to the independent variables, and transform the strike-maturity coordinate system to a probability-maturity coordinate system, which results in an interpolation method called *probability-space interpolation*[4] in maturity dimension on the proposed NCP system that we prove actually leads to arbitrage-free surfaces in *both* strike and maturity dimensions. Roughly speaking, we associate two call options with different maturities, but identical probabilities of a positive terminal payoff for interpolating between these points to obtain the implied volatility or call price for an intermediate maturity. What is more, this interpolation method is akin to market practitioners' belief that interpolation at identical "probabilities" is more "natural" than interpolation based on some other criteria such as moneyness, as comparing moneyness at different maturities is somewhat artificial. While we do not provide any further justification for such beliefs, we draw on it, and justify its use in a suitable form that we formulate in terms of theoretical absence of arbitrage and its computational tractability.

It should be noted that practitioners often use interpolation methods or approximations for which a proof of absence of arbitrage violation is missing, or which are not free of arbitrage. Such an approach is sometimes justified by the use of a post-processing (further smoothing) which further removes arbitrage violations. However, such a two-step approach cannot guarantee that the final result is still an optimal fit to the original input data aside from the computational overhead. Thus, the proof of absence of arbitrage for our interpolation method is a particular achievement of this work. Indeed, an interpolation in maturity dimension that is free of calendar arbitrage does not necessarily guarantee lack of arbitrage in strike dimension. There could also be difficulties in recasting the interpolated point in terms of suitable market parameters (roughly, this may mean conversion from probabilities to strikes). Direct approaches to circumvent this would be to use approximation, however, one should be aware that such approximations may lead to arbitrage violation, even if we assume the original interpolation is arbitrage-free. In fact, built upon the NCP system, the probability space interpolation technique exhibits elegant properties that make it possible to compute in constant time the strike of the interpolated point from the strikes of the interpolating points at identical probabilities, and that the interpolation need not even be linear in maturity to ensure absence of arbitrage which make it possible to use a rich variety of functions of the original maturities for the intermediate maturity.

We note in passing that smoothing is an integral tool for ensuring lack of arbitrage and smoothness in arbitrage-contaminated data. For interpolation of call prices using cubic spline smoothing in the strike dimension, see Fengler [3], and that of normed call prices using cubic spline smoothing in moneyness dimension along with a host of interpolation techniques in maturity dimension including probability space interpolation, see Gope [6]. The later paper by Fengler and Hin [4], has extended cubic spline

[4] Probability as an independent variable has been proposed by Murex [12] before, however, not in connection with NCPs. Instead total variance was used, which in our opinion makes it difficult to prove absence of arbitrage violations.

smoothing to two dimensions by way of fitting a two-dimensional surface. We do not fit surfaces but rather one dimensional smiles, and aim at meaningful interpolation in maturity/time dimension.

Our method may be used to find best fit solutions to arbitrage contaminated input data by combining ours with smoothing techniques. This is a very important application, e.g., when investigating artificial scenarios applied to existing market data (where the "raw" scenario may lead to arbitrage violations which one wishes to eliminate.) While for real market data scenarios, we hardly find cases where existing interpolation methods fall significantly short of expectations compared to the proposed method, we wish to note that it is already a substantial achievement to have an interpolation method which (i) is independent of the application, thus allowing to develop a single version of the interpolation algorithm used consistently in different contexts (ii) comes with a proof of absence of arbitrage (iii) is computationally tractable and exact. Our numerical experiments show, however, that the method compares well. We also observed that the method, if used with standard optimizers, is somewhat more stable.

2 Normed Call Price and Arbitrage Constraints

We consider a single underlying process S_t, and denote by $C_t(K,T)$ the time-t price of a vanilla call option with strike K and maturity T on the underlying S_T. When clear from the context, we take the liberty to drop the subscript t and denote by $C(K,T)$ the time-zero price of the call option. This price can be obtained from the Black-Scholes formula by using the appropriate implied volatility for the option. The implied volatility depends on both K and T. We denote by $\sigma(K,T)$ the implied volatility of this option. By doing so, we are still comfortable denoting by $C(K,T)$, the vanilla call option price, without explicitly incorporating the $\sigma(K,T)$ in the notation for the call price. Of course, the Black-Scholes formula depends on the domestic risk free discounting factor, and dividend yield (or foreign discounting factor), we somehow find ways (which we shall shortly introduce as the normed call price) to avoid explicitly having to work with these entities. Here is the Black's [1] version of the BS option price formula:

$$C(K,T) = B_r(0,T)[F^T N(d_+) - K(N(d_-)] \tag{1}$$

where $B_r(0,T)$, the domestic discount factor, is the price of a zero-bond that pays 1 unit of domestic currency upon maturity T and F^T the T-forward price of the underlying, which, of course, depends on interest rate and dividend yield. More generally, for $0 < t \leq T$, we denote by $B_r(t,T)$, the time-t value of a zero-bond that pays 1 unit of domestic currency upon maturity T, and by F_t^T, the T-forward price of the underlying at time t. Of course, in general, $B_r(t,T)$ and F_t^T are stochastic in nature as opposed to $B_r(0,T)$ or F^T, which are completely known at time 0. $N(x)$ is the normal cdf function, and d_1 and d_2 are given by the following formulas:

$$d_+ = \frac{\ln \frac{F^T}{K} + \frac{\sigma^2(K,T)T}{2}}{\sigma(K,T)\sqrt{T}}, \; d_- = \frac{\ln \frac{F^T}{K} - \frac{\sigma^2(K,T)T}{2}}{\sigma\sqrt{T}} \tag{2}$$

Obviously, $d_- = d_+ - \sigma\sqrt{T}$.

2.1 Forward Moneyness

Time-t *forward moneyness*, $\kappa_t(K,T)$, for strike K and maturity T, is defined as follows:

$$\kappa_t(K,T) = K/F_t^T \tag{3}$$

When ths subscript-t is ignored, it is assumed that $t = 0$, and this is denoted by $\kappa(K,T)$ for strike K and maturity T. In the rest of this work, we shall write κ in place of $\kappa(K,T)$, whenever the meaning is clear from the context. In particular, when we mention moneyness, we shall mean forward moneyness defined here.

2.2 Foreign Discount Factor or Dividend Discount Factor

$B_\delta(t,T)$, the *foreign time-t discount factor or dividend discount factor*, is defined as follows:

$$B_\delta(t,T) = \frac{F_t^T B_r(t,T)}{S_t} \tag{4}$$

With $B_\delta(t,T)$ defined as above, we can write the forward price F_t^T as follows:

$$F_t^T = S_t B_\delta(t,T)/B_r(t,T) \tag{5}$$

where as mentioned before S_t is the spot at time t and $B_r(t,T)$ the domestic time-t discount factor. We note that $B_r(t,T) = N_t \mathrm{E}^{\mathbb{Q}}[\frac{1}{N_T}\Big|\mathcal{F}_t]$, where $\mathrm{E}^{\mathbb{Q}}$ is the risk neutral expectation operator, and N_t the domestic bank account risk neutral numeraire process. Further, $F_t^T = \frac{N_t}{B_r(t,T)}\mathrm{E}^{\mathbb{Q}}[\frac{S_T}{N_T}\Big|\mathcal{F}_t]$. It follows that $B_\delta(t,T) = \frac{N_t}{S_t}\mathrm{E}^{\mathbb{Q}}[\frac{S_T}{N_T}\Big|\mathcal{F}_t]$. In order for (5) to be valid, we do not require time-t domestic instantaneous interest rate r_t and the foreign instantaneous interest rate or dividend yield δ_t to be deterministic.

2.3 Normed Call Prices

We now define time-t *Normed Call Price (NCP)*, $\widetilde{C}_t(\kappa,T)$, for moneyness κ and maturity T as follows:

$$\widetilde{C_t}(\kappa,T) = \frac{C_t(\kappa F_t^T, T)}{F_t^T B_r(t,T)} \tag{6}$$

$C_t(\kappa F_t^T, T)$ above denotes the time-t price of a vanilla call with strike $K = \kappa F_t^T$ and maturity T. As usual, when clear from the context, we drop the subscript t, and denote by $\widetilde{C}(\kappa,T)$ the time-zero NCP for moneyness κ and maturity T. From BS option pricing formula, it follows

$$\widetilde{C}(\kappa,T) = N(d_+) - \kappa N(d_-) \tag{7}$$

(7) makes sense only through the use of appropriate implied volatility that is implicitly used to evaluate d_+ and d_-. We shall shall extensively work with $\widetilde{C}(\kappa,T)$ as opposed to $C(K,T)$ or even $\sigma(K,T)$. In situations, when we deal with $C(K,T)$ or even $\sigma(K,T)$, we stress that we only do so in order to provide an interface to the existing standard of quotations through implied volatilities and (unnormed) call prices.

2.4 Price, Arbitrage and Numeraire Pairs

This subsection is provided for completeness, and contains some theoretical fine-prints. One may skip this subsection without loss of continuity.

For ensuring absence of arbitrage, we require the existence of numeraire pair $(\mathrm{N}, \mathrm{Q}_{\mathrm{N}})$[5] (See Hunt and Kennedy [8] and Fries [5]). When $N_t = e^{\int_0^t r_u du}$, the domestic bank account value process, $\mathrm{Q}_{\mathrm{N}} = \mathbb{Q}$ is a risk neutral probability measure. We would like to emphasize that we do not, in general, assume that the interest rates (and dividend yields) are deterministic. Although this condition, namely the existence of $(\mathrm{N}, \mathrm{Q}_{\mathrm{N}})$, is sufficient, in general, it is not strictly necessary for absence of arbitrage, and so this condition is more restrictive than it has to be. Yet this is practically useful, as it ensures that in PDE finite difference schemes, the transition probabilities are positive which further ensures convergence of the algorithms. Further, given the existence of a risk neutral measure, completeness of the underlying model would mean that the risk neutral measure is also unique. However, this measure remains incompletely specified through the available market instruments, and the challenge lies in determining it as accurately as possible that preserves the theretical properties of the measure and is consistent with available market data.

In any case, assuming that $(\mathrm{N}, \mathrm{Q}_{\mathrm{N}})$ exists, we take [6] $N_t \mathrm{E}^{\mathrm{Q}_{\mathrm{N}}}[\frac{X_T}{N_T} \Big| \mathcal{F}_t]$, where $\mathrm{E}^{\mathrm{Q}_{\mathrm{N}}}$ is the expectation operator under the probability measure Q_N, as *the* time-t price of any T-measurable [7] payoff X_T for $0 \leq t \leq T$. Although, a risk neutral measure $\mathbb{Q}$ is an obvious choice for Q_{N}, it will turn out that given a maturity T, the T-terminal measure Q^T, with the numeraire $N_t^T = B_r(t,T)$ would be more convenient in the general setup that we are working in. So, we take $B_r(t,T)\mathrm{E}^{\mathrm{Q}^T}[X_T \Big| \mathcal{F}_t]$ as *the* time-t price of the T-payoff X_T, noting that $B_r(T,T) = 1$. The formulation of forward (5) is in line with the following definition: $F_t^T = \mathrm{E}^{\mathrm{Q}^T}[S_T \Big| \mathcal{F}_t]$. Throughout this paper, we assume $\kappa \in [0, \infty)$ and the underlying $S_t > 0$ for all t. An extension can be made relatively easily for cases where $\kappa \in [c, \infty)$, where $c < 0$ is an attachment point, and S_t could be negative for $t > 0$. The NCP curve remains well-defined and asymptotically well-behaved under all these generalizations. Thus our results hold good not only for log-normal dynamics, but also for normal dynamics of the underlying.

[5] $\mathrm{N} = (N_t), t \in [0,T]$, where T is a given time horizon, is the strictly positive value process of a self-financing strategy.

[6] When price is uniquely defined we do not have a choice, otherwise this is a matter of convention.

[7] More precisely, we are assuming that the measure Q_{N} is equipped with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{Q}_{\mathrm{N}})$. X_T is supposed to be $\mathcal{F}_T$ -measurable.

2.5 Strike/moneyness Arbitrage

We first consider arbitrage in strike/moneyness dimension. Following Section 2.4, we have $C(K,T) = B_r(0,T)\mathrm{E}^{\mathrm{Q}^T}[\max(S_T - K, 0)]$. Following (6), we have

$$\begin{aligned}
\widetilde{C}(\kappa,T) &= \mathrm{E}^{\mathrm{Q}^T}[\max(\frac{S_T}{F^T} - \frac{K}{F^T}, 0)] \\
&= \mathrm{E}^{\mathrm{Q}^T}[\max(\frac{S_T}{F^T} - \kappa, 0)] \\
&= \mathrm{E}^{\mathrm{Q}^T}[\max(Y_T - \kappa, 0)],\ Y_t = \frac{S_t}{F^T},\ t \in [0,T] \\
&= \int_\kappa^\infty (y-\kappa) q^T(T,y) dy \qquad (8)
\end{aligned}$$

where $q^T(T,y)$ is the transition probability density of the random variable Y_T under Q^T. Differentiating (8), we obtain

$$\begin{aligned}
\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} &= -\int_\kappa^\infty q^T(T,y) dy \\
&= -\mathrm{Q}^T(Y_T \geq \kappa) \\
&= -\mathrm{Q}^T(S_T \geq \kappa F^T) \\
&= -\mathrm{Q}^T(S_T \geq K) \qquad (9)
\end{aligned}$$

Thus the negative of the first derivative of the normed call price function with respect to moneyness is the Q^T-proability of a terminal positive payoff, with no scaling due to the rates being involved and no underlying assumption of deterministic rates. We note in passing that when rates are deterministic, the T-terminal measure is identical to the risk neutral measure $\mathbb{Q}$, and then the negative of the first derivative of normed call price function is the risk neutral probability of the terminal spot lying above the strike, which is the traditional interpretation (with some scaling due to the rates) of the first derivative of the call price function with respect to strike. From (9) it follows that

$$-1 \leq \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} \leq 0 \qquad (10)$$

This monotonicity requirement is our first strike arbitrage constraint and proves Proposition 2.1 and 2.2 below. We now further differentiate (9) to obtain

$$\frac{\partial^2 \widetilde{C}(\kappa,T)}{\partial \kappa^2} = q^T(T,\kappa) \geq 0. \qquad (11)$$

Note that we do not assume strict convexity here. This convexity requirement (11) is our second strike arbitrage constraint, which also proves Proposition 2.3 below.

2.6 Properties of Normed Call Price Function in Absence of Strike/moneyness Arbitrage

We note that in absence of arbitrage (See Section 2.4 for a precise meaning of this), the normed call price function $\widetilde{C}(\kappa,T)$ has the following properties which are arguably much simpler compared to similar properties that apply to the unnormed call price function $C(K,T)$.

Proposition 2.1 $\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} \leq 0.$

Proof See Section 2.5.

Proposition 2.2 $\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} \geq -1.$

Proof See Section 2.5.

Proposition 2.3 $\frac{\partial^2 \widetilde{C}(\kappa,T)}{\partial \kappa^2} \geq 0.$

Proof See Section 2.5.

Proposition 2.4 *For* n [8]*pairs of moneyness and normed call prices* $(\kappa_i, \widetilde{C}_i)$, $i = 1, ..., n$, *with* $\kappa_i > \kappa_j$ *and* $i > j$, *we have* $-1 \leq \frac{\widetilde{C}_i - \widetilde{C}_{i-1}}{\kappa_i - \kappa_{i-1}} \leq \frac{\widetilde{C}_{i+1} - \widetilde{C}_i}{\kappa_{i+1} - \kappa_i} \leq 0.$

Proof Proof follows from Proposition 2.1, 2.2 and 2.3.

Proposition 2.4 is an important criteria that is often useful for computational purposes for ensuring absence of strike/moneyness arbitrage violations. Actually, it is a restatement of the convexity requirement in Proposition 2.3 in conjunction with Proposition 2.1 and 2.2.

Proposition 2.5 *As* $\kappa \to 0$, $\widetilde{C}(\kappa,T) \to \widetilde{C}(0,T) = 1$ *and as* $\kappa \to \infty$, $\widetilde{C}(\kappa,T) \to 0$.

Proof We first prove that $\widetilde{C}(\kappa = 0, T) = 1$. Intuitively, this follows immediately from absence of arbitrage and the definition of $\widetilde{C}(\kappa,T)$. However, we give the following formal proof below. $\widetilde{C}(\kappa,T) = \mathrm{E}^{Q^T}[\max(\frac{S_T}{F^T} - \kappa, 0)]$. So, $\widetilde{C}(\kappa = 0, T) = \mathrm{E}^{Q^T}[\frac{S_T}{F^T}] = \frac{F^T}{F^T} = 1$, thus proving our assertion. That as $\kappa \to \infty$, $\widetilde{C}(\kappa,T) \to 0$ follows similarly, and also immediately from arbitrage point of view, as with $\kappa \to \infty$, the option becomes worthless.

Proposition 2.6 $\max(1-\kappa, 0) \leq \widetilde{C}(\kappa,T) \leq 1$

Proof We first prove the lower bound. We note that in absence of arbitrage, we have the T-terminal probability measure Q^T. $\widetilde{C}(\kappa,T) = \mathrm{E}^{Q^T}[\max(\frac{S_T}{F^T} - \kappa, 0)]$. Now by Jensen's inequality, $\widetilde{C}(\kappa,T) \geq \max(\mathrm{E}^{Q^T}[(\frac{S_T}{F^T} - \kappa)], 0) = \max(1-\kappa, 0)$, as the max function is convex. The upper bound follows from Proposition 2.1 and Proposition 2.5.

It is important to note that $\widetilde{C}(\kappa,T) \geq \max(1-\kappa,0)$ also follows directly from Proposition 2.2 and 2.5. Let $g^{(T)}(\kappa) = \widetilde{C}(\kappa,T) - (1-\kappa)$. We have $\frac{d}{d\kappa} g^{(T)}(\kappa) = \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} + 1 \geq 0$, due to Proposition 2.2. So, $g^{(T)}(\kappa) \geq g^{(T)}(\kappa = 0) = 0$, due to Proposition 2.5, thus proving the assertion.

Proposition 2.1 in conjunction with Proposition 2.5 and 2.6 means that $\widetilde{C}(\kappa,T)$ monotonically decreases from 1 to 0, as κ varies from 0 to ∞.

[8] n could be number is input data-points available.

2.7 Calendar Arbitrage

So far, we have considered the properties of the NCP $\widetilde{C}(\kappa, T)$ as we varied κ, the forward-moneyness, for a fixed maturity T. In this section, we keep κ fixed, and vary maturity T, and consider what constraints are necessary to be placed on a family of NCPs $\widetilde{C}(\kappa, T)$, as we vary T, for absence of any possible arbitrage violation. When the rates are deterministic, the following necessary constraint has been proved in Gope [6], which is based on static replication arguments using put-call parity as in Fengler [3]:

$$\widetilde{C}(\kappa, T_2) \geq \widetilde{C}(\kappa, T_1), \text{ when } T_2 > T_1 > 0 \text{ and } \kappa \in [0, \infty) \tag{12}$$

with equality in (12) holding only when $\kappa = 0$.

In our general setup, when rates are allowed to be stochastic, the constraint (12) may not hold good, and the exact nature of the relationship between $\widetilde{C}(\kappa, T_1)$ and $\widetilde{C}(\kappa, T_2)$ will, in general, depend on the correlation between the processes for underlying S_t, the domestic interest rate (short rate) process r_t and the dividend yield or foreign interest rate process δ_t. We do not address the problem that arises out of this correlations here, but instead give a proof of (12) under the following conditions which are somewhat weaker than (though not substantially different from) assuming that the rates are deterministic:

$$B_\delta(T_1, T_2) = \frac{B_\delta(0, T_2)}{B_\delta(0, T_1)} \tag{13}$$

$$B_r(T_1, T_2) = \frac{B_r(0, T_2)}{B_r(0, T_1)} \tag{14}$$

Equations (13) and (14) mean that the rates have zero volatility on $[T_1, T_2]$, and indeed this is a reasonable approximation to make when the maturities T_1 and T_2 are not far apart. Such approximations are used for calibration in practice. When rates are deterministic, (13) and (14) are automatically satisfied. Note that, in general, $\mathrm{E}^{\mathrm{Q}^{T_2}}[\frac{1}{B_r(T_1,T_2)}\Big|\mathcal{F}_t] = \frac{B_r(t,T_1)}{B_r(t,T_2)}$ for $0 \leq t < T_1 < T_2$, and similar relationship holds for dividend (or foreign) discount factors under the foreign T-terminal measure.

Proposition 2.7 *When* (13) *and* (14) *hold, we have* $\widetilde{C}(\kappa, T_2) \geq \widetilde{C}(\kappa, T_1)$*, where* $T_2 > T_1 > 0$ *and* $\kappa \in [0, \infty)$.

Proof We first note that $\widetilde{C}(\kappa, T_1) = \mathrm{E}^{\mathrm{Q}^{T_1}}[(\frac{S_{T_1}}{F^{T_1}} - \kappa)^+] = \frac{B_r(0,T_2)}{B_r(0,T_1)}\mathrm{E}^{\mathrm{Q}^{T_2}}[(\frac{S_{T_1}}{F^{T_1}} - \kappa)^+ \frac{B_r(T_1,T_1)}{B_r(T_1,T_2)}] = \mathrm{E}^{\mathrm{Q}^{T_2}}[(\frac{S_{T_1}}{F^{T_1}} - \kappa)^+]$, due to (13) and (14). We now observe that, by Jensen's inequality, $\mathrm{E}^{\mathrm{Q}^{T_2}}[(\frac{S_{T_2}}{F^{T_2}} - \kappa)^+\Big|\mathcal{F}_{T_1}] \geq (\mathrm{E}^{\mathrm{Q}^{T_2}}[\frac{S_{T_2}}{F^{T_2}}\Big|\mathcal{F}_{T_1}] - \kappa)^+ = (\frac{F_{T_1}^{T_2}}{F^{T_2}} - \kappa)^+ = (\frac{B_r(0,T_2)}{S_0 B_\delta(0,T_2)}\frac{S_{T_1} B_\delta(T_1,T_2)}{B_r(T_1,T_2)} - \kappa)^+ = (\frac{S_{T_1}}{S_0}\frac{B_r(0,T_1)}{B_\delta(0,T_1)} - \kappa)^+ = (\frac{S_{T_1}}{F^{T_1}} - \kappa)^+$, using (13) and (14).

It follows that $\widetilde{C}(\kappa, T_2) - \widetilde{C}(\kappa, T_1) = \mathrm{E}^{\mathrm{Q}^{T_2}}[(\frac{S_{T_2}}{F^{T_2}} - \kappa)^+] - \mathrm{E}^{\mathrm{Q}^{T_1}}[(\frac{S_{T_1}}{F^{T_1}} - \kappa)^+] \geq \mathrm{E}^{\mathrm{Q}^{T_2}}[(\frac{S_{T_1}}{F^{T_1}} - \kappa)^+ - (\frac{S_{T_1}}{F^{T_1}} - \kappa)^+] = 0$.

The constraint (12) means that for a given level of moneyness, the normed call price must be non-decreasing in maturity. This translates into the property that of one plots the normed call prices $\widetilde{C}(\kappa, T_1)$ and $\widetilde{C}(\kappa, T_2)$ against moneyness κ for two different maturities T_1 and T_2, with $T_2 > T_1$, then the plots never intersect and the plot for T_2-maturity lies above the plot for T_1-maturity.

We note that although we cannot assert the validity of constraint (12) in the full general case of non-deterministic rates, this does not detract from the usability of NCP. In probability space interpolation, which will be introduced in the next section, we show that when constraint (12) is satisfied by the original maturities, the interpolation does not violate it.

We provide the following result to characterize the "gap" $\widetilde{C}(\kappa, T_2) - \widetilde{C}(\kappa, T_1)$ which holds good even when (13) and (14) do not.

Lemma 2.8 *The extrema of $\widetilde{C}(\kappa, T_2) - \widetilde{C}(\kappa, T_1)$, where $T_2 > T_1 > 0$ and $\kappa \in [0, \infty)$ are attained for those κ which correspond to identical T_1 and T_2 probabilities.*

Proof Let $g(T_1, T_2)(\kappa) = \widetilde{C}(\kappa, T_2) - \widetilde{C}(\kappa, T_1)$. For the extrema of $g^{(T_1,T_2)}(\kappa)$, we must have $\frac{d}{d\kappa} g^{(T_1,T_2)}(\kappa) = \frac{\partial}{\partial\kappa}\widetilde{C}(\kappa, T_2) - \frac{\partial}{\partial\kappa}\widetilde{C}(\kappa, T_1) = 0$. Since slopes are negative of probabilities, the assertion in the Lemma follows.

2.8 A Practical Application: Dupire's Formula in Normed Call Price Settings and Calendar Arbitrage

In this subsection, we assume the rates to be deterministic as is the practice with local volatility models used to price equity/FX options. The assumption of deterministic rates is restricted to this subsection which is provided as an aside. Here is a version of Dupire's formula that is used in practice in local volatility based pricing engines:

$$\overline{\sigma}^2(K,T) = \frac{2(\frac{\partial C(K,T)}{\partial T} + \delta_T C(K,T) + (r_T - \delta_T) K \frac{\partial C(K,T)}{\partial K})}{K^2 \frac{\partial^2 C(K,T)}{\partial K^2}} \tag{15}$$

The function $\overline{\sigma}(x,t)$ is called the local volatility function. As usual, $C(K,T)$ is the price of a vanilla call with strike K and maturity T, and r_t and δ_t are deterministic instantaneous time-t risk-free domestic interest rate and dividend yield (foreign interest rate in FX world), respectively.

This formula involves the partial derivatives with respect to strike and maturity of the call price function and involves domestic interest rate and dividend yield (or foreign interest rate), and we now investigate what form the Dupire's formula will take if we switch to normed call price settings, i.e., use moneyness (3) instead of strike and normed call price (6) instead of the call price (unnormed). Indeed, Dupire's formula is remarkably simpler in normed call price settings. We start by

$$\begin{aligned} C(K,T) &= S_0\, B_\delta(0,T)\, \widetilde{C}(\kappa,T), \qquad \text{following (6)} \\ &= S_0\, e^{-\int_0^T \delta_u du}\, \widetilde{C}(\kappa,T) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial C(K,T)}{\partial T} &= \frac{\partial}{\partial T} S_0\, e^{-\int_0^T \delta_u du} \widetilde{C}(\kappa,T) \\ &= S_0\, e^{-\int_0^T \delta_u du} [-\delta_T\, \widetilde{C}(\kappa,T) \\ &+ \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} \frac{\partial \kappa}{\partial T} + \frac{\partial \widetilde{C}(\kappa,T)}{\partial T}] \end{aligned} \tag{17}$$

From (3) and (5),

$$
\begin{aligned}
\kappa &= \frac{K}{F^T} = \frac{K}{S_0\, e^{\int_0^T (r_u - \delta_u) du}} \\
\frac{\partial \kappa}{\partial T} &= -\kappa (r_T - \delta_T)
\end{aligned} \tag{18}
$$

Now using (17) and (18),

$$
\begin{aligned}
\frac{\partial C(K,T)}{\partial T} &= S_0\, e^{-\int_0^T \delta_u du} [-\delta_T\, \widetilde{C}(\kappa, T) - \kappa\, (r_T - \delta_T)\, \frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \\
&+ \frac{\partial \widetilde{C}(\kappa, T)}{\partial T}]
\end{aligned} \tag{19}
$$

Further we also have

$$
\frac{\partial C(K,T)}{\partial K} = \frac{S_0\, e^{-\int_0^T \delta_u du}}{F_T} \frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \tag{20}
$$

and

$$
\frac{\partial^2 C(K,T)}{\partial K^2} = \frac{S_0\, e^{-\int_0^T \delta_u du}}{{F_T}^2} \frac{\partial^2 \widetilde{C}(\kappa, T)}{\partial \kappa^2} \tag{21}
$$

We now put (16), (19) and (20) together and obtain

$$
\begin{aligned}
L &= \frac{\partial C(K,T)}{\partial T} + \delta_T C(K,T) + (r_T - \delta_T) K \frac{\partial C(K,T)}{\partial K} \\
&= S_0\, e^{-\int_0^T \delta_u du} \frac{\partial \widetilde{C}(\kappa, T)}{\partial T}
\end{aligned} \tag{22}
$$

Using (21) and (22), we now rewrite (15) as

$$
\begin{aligned}
\overline{\sigma}^2(K,T) &= \frac{2L}{K^2 \frac{\partial^2 C(K,T)}{\partial K^2}} \\
&= (2\, S_0\, e^{-\int_0^T \delta_u du} \frac{\partial \widetilde{C}(\kappa, T)}{\partial T}) / (K^2 \frac{S_0\, e^{-\int_0^T \delta_u du}}{{F_T}^2}\, \frac{\partial^2 \widetilde{C}(\kappa, T)}{\partial \kappa^2}) \\
&= \frac{2\, \frac{\partial \widetilde{C}(\kappa,T)}{\partial T}}{\kappa^2 \frac{\partial^2 \widetilde{C}(\kappa,T)}{\partial \kappa^2}} \\
&= \overline{\sigma}^2(\kappa, T)
\end{aligned} \tag{23}
$$

Equation (23) is not only nicer and simpler than (15), but is also more useful in terms of calendar arbitrage constraint (12). First, the rates disappear completely. Second, if we have a continuous two-dimensional (moneyness and maturity) normed call price surface that is free of strike/moneyness arbitrage, then if we can ensure that the local volatility-squared, $\overline{\sigma}^2(\kappa, T)$, is positive at all points, then by virtue of (11), (12) and (23), we have a surface that is calendar arbitrage free. In addition, by bounding $\overline{\sigma}^2(\kappa, T)$ at all points by

$$
0 < \overline{\sigma}^2_{min} \leq \overline{\sigma}^2(\kappa, T) \leq \overline{\sigma}^2_{max} \;\; \text{for all } \kappa \in [0, \infty), T > 0 \tag{24}
$$

we put in place some *extraneous* smoothness criteria.

3 Interpolation on Probability Space

3.1 Strict Convexity Requirement

Before we introduce the probability space interpolation technique, we shall assume strict convexity of an arbitrage-free NCP curve. As usual, we denote by $\widetilde{C}(\kappa, T)$ the NCP curve for moneyness κ and maturity T. For this curve to be free of strike/moneyness arbitrage, we strengthen our strike/moneyness arbitrage constraints (10) and (11) described in Section 2.5 as follows:

$$-1 \leq \frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} < 0 \tag{25}$$

$$\frac{\partial^2 \widetilde{C}(\kappa, T)}{\partial \kappa^2} > 0 \tag{26}$$

We shall remind the reader of this requirement from time to time, but unless otherwise mentioned, this *strict* convexity requirement for an NCP curve to be strike/-moneyness arbitrage-free remains in place throughout this section.

We further note that although necessary for the proofs that would establish that the probability space interpolation is both strike/moneyness and calendar arbitrage-free, this strict convexity requirement can be conveniently relaxed. Alternatively, this constraint can be enforced by way of smoothing relatively easily. We provide an algorithm at the end of this section with the details.

3.2 Probability Space Interpolation Method

Given two arbitrage-free strictly convex NCP curves for maturities T_1 and T_2, with $T_2 > T_1$, interpolation on moneyness space performs linear interpolation of $\widetilde{C}(\kappa, T_1)$ and $\widetilde{C}(\kappa, T_2)$ for a fixed κ. Unlike this, in probability space interpolation, we shall do the interpolation by associating different points.

Indeed, we consider points from two existing NCP curves having same slopes. Let $\alpha(T)$ be an increasing fucntion of T, with $T_1 < T < T_2$, satisfying $\alpha(T_1) = 0$ and $\alpha(T_2) = 1$ and $0 < \alpha(T) < 1$ for $T_1 < T < T_2$. A particular choice of $\alpha(T)$ would be as follows:

$$\alpha(T) = \frac{T - T_1}{T_2 - T_1}, \; T_1 < T < T_2 \tag{27}$$

[9] We consider points $(\kappa_1, \widetilde{C}(\kappa_1, T_1))$ and $(\kappa_2, \widetilde{C}(\kappa_2, T_2))$ lying on the T_1 and T_2 maturity NCP curves, respectively, satisfying the following criteria:

$$\left.\frac{\partial \widetilde{C}(\kappa, T_1)}{\partial \kappa}\right|_{(\kappa=\kappa_1)} = \left.\frac{\partial \widetilde{C}(\kappa, T_2)}{\partial \kappa}\right|_{(\kappa=\kappa_2)} \tag{28}$$

We just note that the slopes are negative of probabilities in the respective terminal measures. For completeness, we recall that we do not assume that the rates are deterministic, and that when they are indeed deterministic, the Q^T measure is identical to

[9] In general, $\alpha(T)$ need not be a linear function of T. However, we consider linear interpolation, although all the results we derive in this section hold good even when $\alpha(T)$ is not linear in T. The choice (27) of $\alpha(T)$ that is linear in T works well for probability space interpolation.

the risk-neutral measure $\mathbb{Q}$, and then the absolute value of slope of the $\widetilde{C}(\kappa, T)$ curve as a function of κ, the forward-moneyness, for all maturities $T > 0$, is the risk neutral probability of respective terminal positive payoff under absence of arbitrage.

Proposition 3.1 *If an NCP curve $\widetilde{C}(\kappa, T)$, for any given maturity $T > 0$, is free of strike/moneyness arbitrage in the sense that it satisfies (10) and (11), and in addition is strictly convex, then for this maturity, there is a unique moneyness for a given Q^T-probability $q = -\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa}$, and therefore we have a unique one-to-one mapping $\kappa : (0, 1] \times (0, \infty) \mapsto [0, \infty)$ that maps to a unique $\kappa(q, T)$ for a given (q, T).*

Proof The proof follows immediately from absence of strike/moneyness arbitrage and strict convexity.

Following Proposition 3.1, for any strike/moneyness arbitrage-free strictly convex NCP curve $\widetilde{C}(\kappa, T)$, we can write $\widetilde{C}(\kappa, T) = \widetilde{C}(\kappa(q, T), T)$, with which we have the new coordinate system (q, T), where $q = -\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} = \mathrm{Q}^T(S_T \geq \kappa F^T)$. We thus have $\kappa_1 = \kappa(q, T_1)$ and $\kappa_2 = \kappa(q, T_2)$, and we put together the relationship between the moneyness, strikes and probabilities of the points that we associate for probability space interpolation as follows:

$$\begin{aligned} -q &= -\mathrm{Q}^{T_1}(S_{T_1} \geq K_1) = \frac{\partial \widetilde{C}(\kappa, T_1)}{\partial \kappa}\Big|_{(\kappa=\kappa_1)} \\ &= -\mathrm{Q}^{T_2}(S_{T_2} \geq K_2) = \frac{\partial \widetilde{C}(\kappa, T_2)}{\partial \kappa}\Big|_{(\kappa=\kappa_2)} \end{aligned} \tag{29}$$

where K_1 and K_2 are related by

$$\kappa_1 = \frac{K_1}{F^{T_1}},\ \kappa_2 = \frac{K_2}{F^{T_2}} \tag{30}$$

Thus in probability space interpolation, we interpolate between normed call prices $\widetilde{C}(q, T_1)$ and $\widetilde{C}(q, T_2)$ for the original maturities T_1 and T_2, respectively, having same probability q under the probability measures Q^{T_1} and Q^{T_2}, respectively, to obtain a quantity that is denoted by $\widetilde{C}(q, T)$ according to the following formula:

$$\widetilde{C}(q, T) = (1 - \alpha(T))\widetilde{C}(q, T_1) + \alpha(T)\widetilde{C}(q, T_2) \tag{31}$$

where q is as defined in (29). We assign the quantity $\widetilde{C}(q, T)$ to the normed call price for maturity T that corresponds to a Q^T probability q for that maturity, and obtain a probability space interpolated NCP curve in (q, T) coordinate system from the original T_1 and T_2 maturity arbitrage-free strictly convex NCP curves.

In order for interpolation (31) to make sense in practice, there must be a unique T-moneyness, κ^* that should correspond to the Q^T-probability q. We do not know yet if this is so. However, we shall soon see that strict convexity of the interpolating strike/moneyness arbitrage-free NCP curves ensures that the interpolated NCP curve is also strike/moneyness arbitrage-free and strictly convex. We shall prove this in Lemma 3.4, but for now, through Proposition 3.1, assuming that a unique T-moneyness

$\kappa^* = \kappa(q, T)$ corresponds to a unique Q^T-probability q, we rephrase (31) in terms of moneyness as follows:

$$\widetilde{C}(\kappa^*, T) = (1 - \alpha(T))\widetilde{C}(\kappa_1, T_1) + \alpha(T)\widetilde{C}(\kappa_2, T_2) \tag{32}$$

Even when the equivalence of (31) and (32) holds good and there exists a unique T-moneyness for a given Q^T-probability for the interpolated NCP curve, we are at a disadvantage here, since we are implicitly evaluating the slope of the interpolated curve at an arbitrary moneyness κ^*, while the curve is specified explicitly only in terms of probability and *not* moneyness. Instead of brute force approximations or heuristics that are likely to introduce arbitrage violations, we shall deal with the problem by exploiting properties of NCPs under absence of arbitrage that would lead to an analytically and computationally exact approach for doing this conversion that does not introduce arbitrage violation due to errors in conversion from probability to moneyness. Indeed, we shall demonstrate that there is a very simple linear relationship holding between κ_1, κ_2 and κ^* (Lemma 3.7) which would be the fulcrum of our approach.

3.3 Absence of Strike/moneyness Arbitrage with Probability Space Interpolation

Proposition 3.2 *Our first strike/moneyness arbitrage constraint (25): $-1 \leq \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} < 0$ is equivalent to $0 < q \leq 1$, and is satisfied automatically by considering positive Q^T probabilities. Here, $\widetilde{C}(\kappa, T)$ denotes any strike/moneyness arbitrage-free NCP curve under strict convexity constraint.*

Proof The proof follows immediately from the definition of q.

Proposition 3.3 *Our second strike/moneyness constraint (26): $\frac{\partial^2 \widetilde{C}(\kappa,T)}{\partial \kappa^2} > 0$ is equivalent to $\frac{\partial \widetilde{C}(q,T)}{\partial q} > 0$. Here, $\widetilde{C}(\kappa, T)$ denotes any strike/moneyness arbitrage-free NCP curve under strict convexity constraint.*

Proof We first note that

$$q = -\frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \tag{33}$$

$$\frac{\partial q}{\partial \kappa} = -\frac{\partial^2 \widetilde{C}(\kappa, T)}{\partial \kappa^2} < 0. \tag{34}$$

It follows that

$$\begin{aligned}
\frac{\partial \widetilde{C}(q, T)}{\partial q} &= \frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \frac{\partial \kappa}{\partial q} = \frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \Big/ \frac{\partial q}{\partial \kappa} \\
&= -\frac{\partial \widetilde{C}(\kappa, T)}{\partial \kappa} \Big/ \frac{\partial^2 \widetilde{C}(\kappa, T)}{\partial \kappa^2} \\
&> 0, \text{ using (25) and (26)}
\end{aligned}$$

We note that on probability space, as opposed to the moneyness space, we have rather a simplified and more intuitive version of the strike/moneyness arbitrage constraints. We just need to consider meaningful ($\in (0,1]$) probabilities and ensure that the first derivative of the normed call prices with respect to the probability is positive! This certainly makes sense as one would expect that the normed call price (and hence the call price) would increase with the increasing probability of a positive terminal payoff. This is arguably more natural (though equivalent) than requiring that the second derivative of normed call price with respect to the moneyness (3) be positive.

Proposition 3.4 *Interpolation (31) or equivalently (32), where κ_1 and κ_2 are implicitly determined by (29), does not admit strike/moneyness arbitrage. Further, strict convexity of the original T_1 and T_2 maturity NCP curves implies strict convexity of the interpolated T-maturity NCP curve as well.*

Proof We shall work with the version (31) here. In view of Proposition 3.3, we need to show that $\frac{\partial \widetilde{C}(q,T)}{\partial q} > 0$. We note that as the original T_1 and T_2 NCP curves are arbitrage free and strictly convex, we have $\frac{\partial \widetilde{C}(q,T_1)}{\partial q} > 0$ and $\frac{\partial \widetilde{C}(q,T_2)}{\partial q} > 0$. With $\alpha(T)$ satisfying $0 < \alpha(T) < 1$, it follows $0 < \widetilde{C}(q,T) \leq 1$ with $q \in (0,1]$, and that $\frac{\partial \widetilde{C}(q,T)}{\partial q} = (1-\alpha(T))\frac{\partial \widetilde{C}(q,T_1)}{\partial q} + \alpha(T)\frac{\partial \widetilde{C}(q,T_2)}{\partial q} > 0$.

Now that Proposition 3.4 guarantees absence of strike/moneyness arbitrage and strict convexity of the interpolated NCP curve obtained using probability space interpolation, we can use Proposition 3.1, and write the κ_1, κ_2 in (29) and κ^* in (32) as follows:

$$\kappa_1 = \kappa(q,T_1), \kappa_2 = \kappa(q,T_2), \kappa^* = \kappa(q,T) \tag{35}$$

We may also rewrite (32) as follows:

$$\widetilde{C}(\kappa(q,T),T) = (1-\alpha(T))\widetilde{C}(\kappa(q,T_1),T_1) + \alpha(T)\widetilde{C}(\kappa(q,T_2),T_2), \tag{36}$$

which makes the equivalence of (31) and (32) explicit. We further note that since the mapping $(q,T) \mapsto \kappa(q,T)$ is defined only for strike/moneyness arbitrage-free and strictly convex NCP curves, $\frac{\partial q}{\partial \kappa} < 0$ (inequality (34)) as $\kappa \in [0,\infty)$ and $q \in (0,1]$. It immediately follows that for any maturity T when the corresponding NCP curve is strike/moneyness arbitrage-free and strictly convex,

$$\kappa(1,T) = 0 \tag{37}$$
$$\kappa(0,T) = \infty \tag{38}$$

Lemma 3.5 *Under strict convexity (26) of the original NCP curves, in probability space interpolation (31) or equivalently (32), where κ_1 and κ_2 are given by (29), $\kappa^* = 0$ if and only if $\kappa_1 = 0$ and $\kappa_2 = 0$.*

Proof We first let $\kappa^* = 0$ and let $q = q_0$ be the probability that corresponds to this moneyness for the T-maturity. Now it must be that $q_0 = 1$, as interpolation (31) or equivalently (32) is strike/moneyness arbitrage-free, and the original and the interpolated NCP curves are strictly convex (Proposition 3.4), and hence $\frac{\partial q}{\partial \kappa} < 0$ with

$q \in (0,1]$ and $\kappa \in [0,\infty)$. Now from (37), it follows that $\kappa_1 = \kappa(q=1,T_1) = 0$ and $\kappa_2 = \kappa(q=1,T_2) = 0$. For the other side, we employ the same reasoning, and note that $\kappa_1 = \kappa(q,T_1) = \kappa_2 = \kappa(q,T_2) = 0$ means $q = 1$, and therefore $\kappa^* = \kappa(q=1,T) = 0$.

3.4 Absence of Calendar Arbitrage with Probability Space Interpolation

We shall now present a formal proof that probability space interpolation does not violate the calendar arbitrage constraint in the sense that it satisfies (12), whenever the original maturities satisfy (12). We refer to Gope [6], containing insights on the mechanics of probability space interpolation for an informal discussion that motivates the formal constructs used here.

Proposition 3.6 *Under strict convexity (26) of the arbitrage-free original NCP curves which satisfy the calendar arbitrage constraint* (12), *interpolation (32), where κ_1 and κ_2 are implicitly given by (29), is free of calendar arbitrage.*

Before we give a formal proof of Proposition 3.6, we need a couple of lemmas.

Lemma 3.7 *Under strict convexity (26) of the arbitrage-free original NCP curves, in probability space interpolation (32), where κ_1 and κ_2 are implicitly given by (29), it holds that*

$$\kappa^* = (1-\alpha(T))\,\kappa_1 + \alpha(T)\,\kappa_2 \tag{39}$$

Proof We first recall Lemma 3.5, which states that if $\kappa^* = 0$, then $\kappa_1 = \kappa_2 = 0$. Interpolation (32) is equivalent to (31), which is stated in terms of probability q. We vary κ^*, and note that a given value of κ^* corresponds to a unique probability q, which in turn corresponds to a unique T_1-moneyness κ_1 and a unique T_2 moneyness κ_2. As κ^* varies, κ_1 and κ_2 vary as well. We capture the effect of these variations on the NCP as follows:

$$\widetilde{C}(\kappa^*,T) \quad = \quad (1-\alpha(T))\,\widetilde{C}(\kappa_1,T_1) + \alpha(T)\,\widetilde{C}(\kappa_2,T_2) \tag{40}$$

Differentiating, noting that all the three NCP curves are smooth functions of κ, we have

$$\begin{aligned} \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa}\Big|_{\kappa=\kappa^*} \quad &= \quad (1-\alpha(T))\,\frac{\partial \widetilde{C}(\kappa,T_1)}{\partial \kappa}\Big|_{\kappa=\kappa_1}\,\frac{\partial \kappa_1}{\partial \kappa^*} \\ &+ \quad \alpha(T)\,\frac{\partial \widetilde{C}(\kappa,T_2)}{\partial \kappa}\Big|_{\kappa=\kappa_2}\,\frac{\partial \kappa_2}{\partial \kappa^*} \end{aligned} \tag{41}$$

Now

$$-\,q = \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa}\Big|_{\kappa=\kappa^*} = \frac{\partial \widetilde{C}(\kappa,T_1)}{\partial \kappa}\Big|_{\kappa=\kappa_1} = \frac{\partial \widetilde{C}(\kappa,T_2)}{\partial \kappa}\Big|_{\kappa=\kappa_2} \tag{42}$$

We note that under strict convexity requirement (26) of the original NCP curves, which in turn, by Proposition 3.4, ensures strict convexity of the interpolated NCP curve, we have $q > 0$. So, from (41) it follows that

$$\begin{aligned} 1 \quad &= \quad (1-\alpha(T))\,\frac{\partial \kappa_1}{\partial \kappa^*} + \alpha(T)\,\frac{\partial \kappa_2}{\partial \kappa^*} \\ \Delta\kappa^* \quad &= \quad (1-\alpha(T))\,\Delta\kappa_1 + \alpha(T)\,\Delta\kappa_2 \end{aligned} \tag{43}$$

Due to the fact that when $\kappa_1 = \kappa_2 = 0$, $\kappa^* = 0$ (Lemma 3.5), we now have

$$\kappa^* = (1 - \alpha(T))\,\kappa_1 + \alpha(T)\,\kappa_2 \tag{44}$$

In view of (36),

$$\kappa(q, T) = (1 - \alpha(T))\,\kappa(q, T_1) + \alpha(T)\,\kappa(q, T_2) \tag{45}$$

(45) is a remarkable property of probability space interpolation. It essentially means that as we use the points $(\kappa_1, \widetilde{C}(\kappa_1, T_1))$ and $(\kappa_2, \widetilde{C}(\kappa_2, T_2))$ lying on the original NCP curves to obtain the interpolated point $(\kappa^*, \widetilde{C}(\kappa^*, T))$, the interpolated point necessarily lies on the line segment joining the two original points in the (κ, T) coordinate system. This is enough to ensure compliance with the calendar arbitrage constraint.

Lemma 3.8 *Under strict convexity (26) of the arbitrage-free original NCP curves which satisfy the calendar arbitrage constraint* (12)*, in probability space interpolation (32), where κ_1 and κ_2 are implicitly given by (29), the intermediate T-maturity NCP curve never goes above the upper T_2-maturity NCP curve.*

Proof Define the "gap" function $g^{(T,T_2)}(\kappa^*) = \widetilde{C}(\kappa^*, T_2) - \widetilde{C}(\kappa^*, T)$. We need to show that for all $\kappa^* \in [0, \infty)$, $g^{(T,T_2)}(\kappa^*) \geq 0$ with the equality holding only at $\kappa^* = 0$. We first note that interpolation (32) is equivalent to (31), which is stated in terms of probability q. We now vary κ^*, and note that a given value of κ^* corresponds to a unique probability q, which in turn corresponds to a unique T_2-moneyness κ_2. As κ^* varies, κ_2 varies as well, and we can define the mapping $\kappa_2 = f_2(\kappa^*)$. Similarly, we have $\kappa_1 = f_1(\kappa^*)$. This mappings $f_1 : [0, \infty) \mapsto [0, \infty)$ and $f_2 : [0, \infty) \mapsto [0, \infty)$ are necessarily one-to-one. We note that

$$\frac{d}{d\kappa^*} g^{(T,T_2)}(\kappa^*) = \frac{\partial}{\partial \kappa^*} \widetilde{C}(\kappa^*, T_2) - \frac{\partial}{\partial \kappa^*} \widetilde{C}(\kappa^*, T) \tag{46}$$

From the definition of q,

$$\frac{\partial}{\partial \kappa^*} \widetilde{C}(\kappa^*, T) = \frac{\partial}{\partial \kappa} \widetilde{C}(\kappa, T)\Big|_{\kappa=\kappa^*} = -q = \frac{\partial}{\partial \kappa} \widetilde{C}(\kappa, T_2)\Big|_{\kappa=\kappa_2=f_2(\kappa^*)} \tag{47}$$

We, therefore, rewrite (46) as follows:

$$\frac{d}{d\kappa} g^{(T,T_2)}(\kappa)\Big|_{\kappa=\kappa^*} = \frac{\partial}{\partial \kappa} \widetilde{C}(\kappa, T_2)\Big|_{\kappa=\kappa^*} - \frac{\partial}{\partial \kappa} \widetilde{C}(\kappa, T_2)\Big|_{\kappa=f_2(\kappa^*)} \tag{48}$$

It follows that the $g^{(T,T_2)}$ as a continuous and bounded function of κ^* attains all its local extrema in $(0, \infty)$ for those κ^* for which $\kappa^* = \kappa_2$. With $\kappa^* = \kappa_2$, we have from Lemma 3.7 that $\kappa^* = \kappa_1 = \kappa_2$. However, since the original T_1 and T_2 NCP curves are calendar arbitrage-free, we have from our calendar arbitrage constraint (12) that for these extrema of $g^{(T,T_2)}$ it holds that $g^{(T,T_2)}(\kappa^*) = \widetilde{C}(\kappa^*, T_2) - \widetilde{C}(\kappa^*, T) = (1 - \alpha(T))\,(\widetilde{C}(\kappa^*, T_2) - \widetilde{C}(\kappa^*, T_1)) > 0$. If none of the local minima is attained, we still have $g^{(T,T_2)}(0) = 0$ and $\lim_{\kappa^* \to \infty} g^{(T,T_2)}(\kappa^*) = 0$. Since the function $g^{(T,T_2)}$ of κ^* is continuous and bounded, we thus have $g^{(T,T_2)}(\kappa^*) \geq 0$, with the equality holding only for $\kappa^* = 0$.

Lemma 3.9 *Under strict convexity (26) of the arbitrage-free original NCP curves which satisfy the calendar arbitrage constraint (12), in probability space interpolation (32), where κ_1 and κ_2 are implicitly given by (29), the intermediate T-maturity NCP curve never goes below the lower T_1-maturity NCP curve.*

Proof We could employ the same reasoning using the function $g^{(T_1,T)}(\kappa^*)$ as in the proof of Lemma 3.8. However, we take a different approach here that directly exploits the convexity of the NCP. If possible, let $\widetilde{C}(\kappa^*,T) \leq \widetilde{C}(\kappa^*,T_1)$, where $\widetilde{C}(\kappa^*,T) = (1-\alpha(T))\,\widetilde{C}(\kappa_1,T_1) + \alpha(T)\,\widetilde{C}(\kappa_2,T_2)$ and $\kappa^* \in (0,\infty)$. In light of Lemma 3.7, $\kappa^* = (1-\alpha(T))\,\kappa_1 + \alpha(T)\,\kappa_2$. So it follows that

$$
\begin{aligned}
\widetilde{C}(\kappa^*,T) &= (1-\alpha(T))\,\widetilde{C}(\kappa_1,T_1) + \alpha(T)\,\widetilde{C}(\kappa_2,T_2) \leq \widetilde{C}(\kappa^*,T_1), \qquad \text{by assumption} \\
\widetilde{C}(\kappa^*,T_1) &= \widetilde{C}((1-\alpha(T))\,\kappa_1 + \alpha(T)\,\kappa_2, T_1) \\
&\geq \widetilde{C}(\kappa^*,T) = (1-\alpha(T))\,\widetilde{C}(\kappa_1,T_1) + \alpha(T)\,\widetilde{C}(\kappa_2,T_2) \\
&> (1-\alpha(T))\,\widetilde{C}(\kappa_1,T_1) + \alpha(T)\,\widetilde{C}(\kappa_2,T_1)
\end{aligned} \tag{49}
$$

where in the last step of (49) we have used the calendar arbitrage constraints (12) on the original NCP curves. (49), however, violates the convexity of the T_1-maturity NCP curve as $0 < \alpha(T) < 1$. It follows that the intermediate T-maturity NCP curve never goes below the lower T_1-maturity NCP curve.

In light of Proposition 3.4, Lemma 3.8 and Lemma 3.9, it is obvious that given the strike/moneyness arbitrage-free NCP curves, for maturities T_1 and T_2, which are strictly convex with $T_1 < T_2$ and satisfy the calendar arbitrage constraint (12), if we obtain an intermediate T-maturity NCP curve using probability space interpolation with $T_1 < T < T_2$, the intermediate T-maturity NCP curve along with the original T_1 and T_2 maturity NCP curves belong to a family of NCP curves that are strictly convex and free from strike/moneyness arbitrage as well as satisfy the calendar arbitrage constraint (12). We now need the following lemma to formalize this notion, and conclude that probability space interpolation is arbitrage-free (in particular calendar arbitrage-free).

Lemma 3.10 *We consider two original maturities T_1 and T_2 with $T_2 > T_1$ for which we have the arbitrage-free strictly convex NCP curves $\widetilde{C}(q,T_1)$ (or equivalently $\widetilde{C}(\kappa(q,T_1),T_1)$) and $\widetilde{C}(q,T_2)$ (or equivalently $\widetilde{C}(\kappa(q,T_2),T_2)$), respectively, which satisfy the calendar arbitrage constraint (12). Further, we consider intermediate maturities T_1' and T_2' with $T_1 < T_1' < T_2' < T_2$. For intermediate maturities T_1' and T_2', we use probability space interpolation between the original maturities T_1 and T_2, and obtain the NCP curves $\widetilde{C}(q,T_1')$ and $\widetilde{C}(q,T_2')$ using (31), or equivalently $\widetilde{C}(\kappa(q,T_1'),T_1')$ and $\widetilde{C}(\kappa(q,T_2'),T_2')$ using (32). Now we consider an identical moneyness $\kappa = \kappa(q_1',T_1') = \kappa(q_2',T_2')$ for maturities T_1' and T_2', respectively, where q_1' and q_2' are T_1' and T_2' probabilities, respectively. It necessarily holds that $\widetilde{C}(\kappa,T_1') \leq \widetilde{C}(\kappa,T_2')$.*

Proof We employ the same reasoning using the "gap" function $g^{(T_1',T_2')}(\kappa)$ as in the proof of Lemma 3.8. $g^{(T_1',T_2')}(\kappa) = \widetilde{C}(\kappa,T_2') - \widetilde{C}(\kappa,T_1')$. We now recall Lemma 2.8. For extrema of $g^{(T_1',T_2')}(\kappa)$, $\frac{\partial}{\partial\kappa}\widetilde{C}(\kappa,T_2') - \frac{\partial}{\partial\kappa}\widetilde{C}(\kappa,T_1') = 0$. This means that extrema are attained for those $\kappa = \kappa(q_1',T_1') = \kappa(q_2',T_2')$, for which $q_1' = q_2'$, which in turn means that identical pairs of points $(\kappa_1(q,T_1), \widetilde{C}(\kappa_1(q,T_1),T_1))$ and $(\kappa_2(q,T_2), \widetilde{C}(\kappa_2(q,T_2),T_2))$, where $q = q_1' = q_2'$, have been used in interpolation to obtain the points $(\kappa, \widetilde{C}(\kappa, T_1'))$ and $(\kappa, \widetilde{C}(\kappa, T_2'))$ of the intermediate maturities, for the extrema of $g^{(T_1',T_2')}(\kappa)$. Now we have from Lemma 3.7 that $\kappa = \kappa_1(q,T_1) = \kappa_2(q,T_2)$, from which the assertion in the Lemma follows.

Proof of Proposition 3.6 The proof follows from Lemma 3.5 and Lemma 3.8 though 3.10.

3.5 Algorithm for Construction of an Intermediate NCP Curve using Probability Space Interpolation

In this section, we present an algorithm for construction of the intermediate T-maturity NCP curve from existing T_1 and T_2 maturity arbitrage-free strictly convex NCP curves with $T_1 < T < T_2$. At the end of this section, we shall show how this algorithm can be made to work even when this strict convexity requirement is relaxed.

Algorithm 1: Algorithm for construction of an intermediate NCP curve using probability space interpolation

Input : Arbitrage-free, strictly convex NCP curves for maturities T_1 and T_2 with $T_2 > T_1$ on identical discrete moneyness grid $[\kappa_1, \kappa_n]$. This means that we necessarily want the curves to be defined for each κ_i for $i = 1, \dots, n$.

Output : Intermediate T-maturity ($T_1 < T < T_2$) NCP curve that is obtained from the input NCP curves using probability space interpolation (31) or equivalently (32)

1. Set $\alpha = \alpha(T) = \frac{T-T_1}{T_2-T_1}$.
2. If the original NCP curves are specified on the discrete moneyness grid $[\kappa_1, \kappa_n]$, i.e. if they are not defined for $\kappa \in (\kappa_i, \kappa_{i+1})$ for $i = 1, \dots, n-1$, smooth the original T_1 and T_2 maturity NCP curves using techniques described in Fengler [3] or Gope [6]. For each T_2-moneyness $\kappa_i^{(T_2)} = \kappa_i$, for $i = 1 \dots, n$, of the original moneyness grid of the higher T_2 maturity NCP curve, obtain the respective probabilities p_i from the slopes of the smoothed NCP curve at these grid points.
3. For each p_i, for $i = 1 \dots, n$, using a numerical root solver, obtain the moneyness $\kappa_i^{(T_1)}$ for the lower T_1 maturity curve, so that the slope of the T_1-curve at moneyness $\kappa_i^{(T_1)}$ is $-p_i$. Now $\kappa_i^{(T_2)}$, on the T_2-curve, and $\kappa_i^{(T_1)}$, on the T_1-curve, correspond to same probability p_i.
4. Use Lemma 3.7 and set $\kappa_i^{(T)} = (1-\alpha)\,\kappa_i^{(T_1)} + \alpha\,\kappa_i^{(T_2)}$ for $i = 1, \dots, n$
5. Use interpolation (32) to obtain the intermediate T-maturity NCP for the moneyness grid $[\kappa_1^{(T)}, \kappa_n^{(T)}]$ using $\widetilde{C}(\kappa_i^{(T)}, T) = (1-\alpha)\,\widetilde{C}(\kappa_i^{(T_1)}, T_1) + \alpha\, C(\kappa_i^{(T_2)}, T_2)$ for $i = 1, \dots, n$.
6. If necessary, smooth the intermediate T-maturity NCP curve. Note that the T-maturity NCP curve is arbitrage-free even without smoothing by Proposition 3.6.

Note that Algorithm 1 needs to be suitably modified if it were to work with input NCP curves that are arbitrage-free, but not strictly convex. For this, we first note that we are smoothing the original curves, if they are not already smoothed. The purpose of smoothing here is to obtain the NCP values on a continuous moneyness range from a discrete set of NCP values corresponding to a given discrete set of moneyness $\{\kappa_1, \dots, \kappa_n\}$, so that we can compute the slopes of the NCP curves reasonably accurately. Strict convexity can also be ensured as part of smoothing. If a set of points for either maturity T_i with $i = 1, 2$, $(\kappa_j, \widetilde{C}(\kappa_j, T_i))$'s lies on a straight line, we still can ensure while smoothing that the smoothed NCP curves are strictly convex in addition to ensuring that the curves are arbitrage-free.

However, in the event that the input NCP curves corresponding to the discrete moneyness grid $[\kappa_1, \kappa_n]$ are arbitrage-free, but not strictly convex, we may not want to ensure strict convexity artificially by way of smoothing. In such cases, we can resort to the following technique in order to obtain an arbitrage-free T-maturity intermediate NCP curve using probability space interpolation from the input T_1 and T_2 maturity NCP curves. Of course, strict convexity will be violated for the T-maturity NCP curve like the input curves. Let $\{s_i,\ i = 1, \ldots, k\}$ be the set of discrete moneyness, that has been sorted in ascending order, corresponding to probability $p^{(c)}$ for maturity T_1 and likewise $\{t_i,\ i = 1, \ldots, l\}$ the discrete moneyness set, that has been sorted in ascending order, corresponding to the same probability $p^{(c)}$ for maturity T_2. Violation of strict convexity typically means that $\max(l, k) > 1$. We first fix

$$\begin{aligned} u_l &= (1-\alpha)s_1 + \alpha t_1 \\ u_r &= (1-\alpha)s_k + \alpha t_l \end{aligned} \tag{50}$$

Then we obtain

$$\begin{aligned} \widetilde{C}(u_l, T) &= (1-\alpha)\,\widetilde{C}(s_1, T_1) + \alpha\,\widetilde{C}(t_1, T_2) \\ \widetilde{C}(u_r, T) &= (1-\alpha)\,\widetilde{C}(s_k, T_1) + \alpha\,\widetilde{C}(t_l, T_2) \end{aligned} \tag{51}$$

and for $u_l < u < u_r$, we do a linear interpolation and set

$$\begin{aligned} \beta &= \frac{u - u_l}{u_r - u_l} \\ \widetilde{C}(u, T) &= (1-\beta)\,\widetilde{C}(u_l, T) + \beta\,\widetilde{C}(u_r, T) \end{aligned} \tag{52}$$

We first note

$$\frac{\partial \widetilde{C}(u, T)}{\partial u} = \frac{\widetilde{C}(u_r, T) - \widetilde{C}(u_l, T)}{u_r - u_l}, \qquad u_l < u < u_r \tag{53}$$

Now

$$\begin{aligned} u_r - u_l &= (1-\alpha)(s_k - s_1) + \alpha(t_l - t_1), \qquad \text{from (50)} \\ \widetilde{C}(u_r, T) - \widetilde{C}(u_l, T) &= (1-\alpha)\,[\widetilde{C}(s_k, T_1) - \widetilde{C}(s_1, T_1)] \\ &+ \alpha\,[\widetilde{C}(t_l, T_2) - \widetilde{C}(t_1, T_2)], \qquad \text{from (51)} \\ &= -p^{(c)}(1-\alpha)(s_k - s_1) - p^{(c)}\alpha(t_l - t_1) \end{aligned} \tag{54}$$

We have used the fact that $p^{(c)}$ is the probability corresponding to the set of moneyness $\{s_i,\ i = 1, \ldots, k\}$ for maturity T_1 and $\{t_i,\ i = 1, \ldots, l\}$ for maturity T_2. Now from (53) and (54) we have

$$\frac{\partial \widetilde{C}(u, T)}{\partial u} = -p^{(c)}, \qquad \text{for } u_l < u < u_r \tag{55}$$

It follows that along these lines (we skip the minute details here), Algorithm 1 can be made to work even when strict convexity is violated for the input NCP curves, and the resultant T-maturity NCP curve will still be arbitrage-free (strict convexity would be violated for the interpolated T-maturity NCP curve as well).

4 Empirical Results

We use EUR-USD FX vanilla options volatility data set as on May 11, 2011. For simplicity, we have kept domestic (USD) and foreign (EUR) interest rates fixed at 0.35% and 1.33% respectively. The NCPs for the original maturities are both strike/moneyness and calendar arbitrage-free. A series of plots follows.

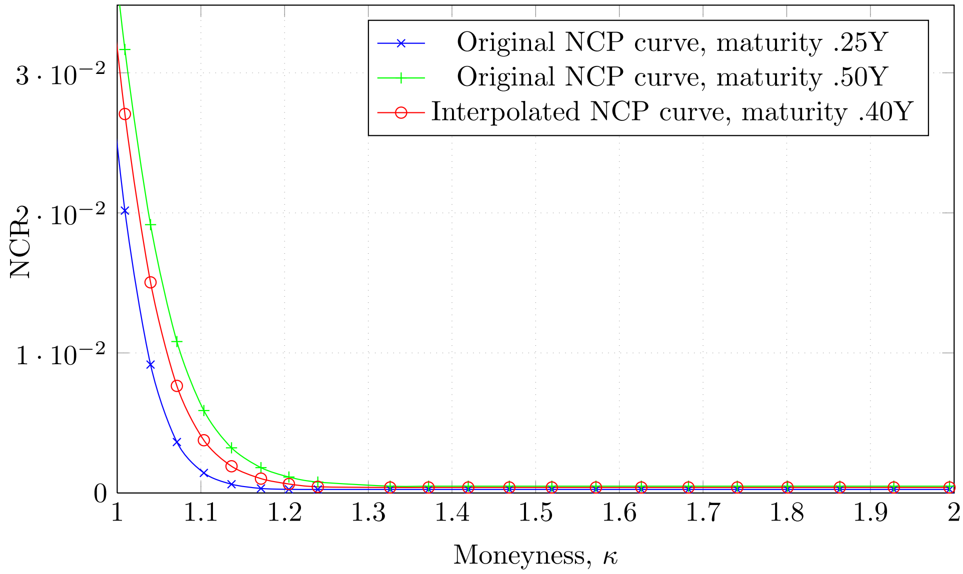

Figure 1: Plot of NCP curve for original maturities .25Y and .50Y and an intermediate maturity .40Y obtained using probability space interpolation (see Section 3.2). The plots indicate no calendar arbitrage violation for the interpolated NCP curve.

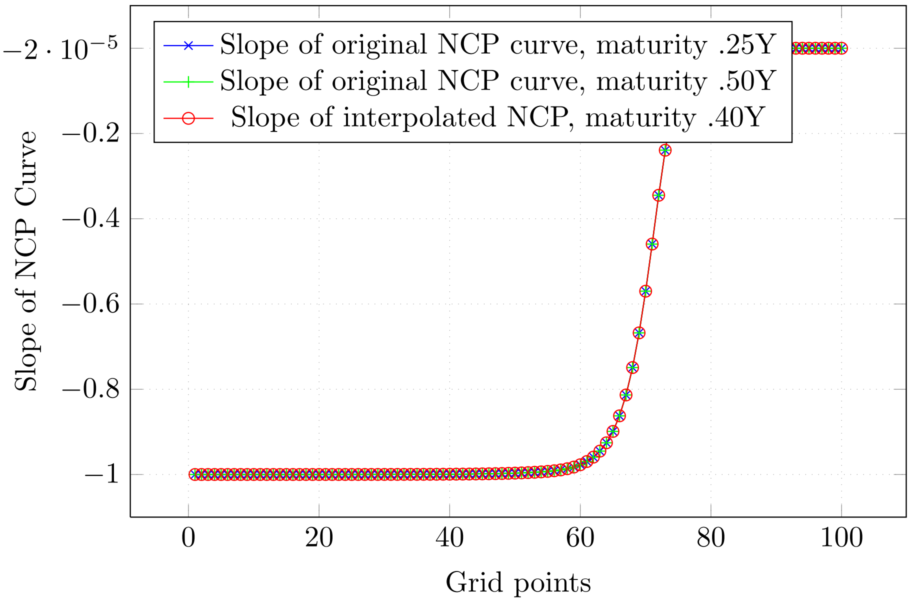

Figure 2: Plot of slopes of NCP curves for original maturities .25Y and .50Y and an intermediate maturity .40Y obtained using probability space interpolation (see Section 3.2) against the grid point indices. Following Algorithm 1, we first obtain the slopes corresponding to the original moneyness grid $[\kappa_1^{(T_2)}, \kappa_n^{(T_2)}] = [\kappa_1, \kappa_n]$ of the higher T_2 (=.50Y) maturity original NCP curve. Then we compute the moneyness grid points $[\kappa_1^{(T_1)}, \kappa_n^{(T_1)}]$ for the lower T_1 (=.25Y) maturity original NCP curve having the same slopes (i.e. same probabilities). Finally, we perform probability scale interpolation using those grid points, and obtain moneyness $[\kappa_1^{(T)}, \kappa_n^{(T)}]$ for the intermediate T (=.40Y) maturity NCP curve using Lemma 3.7, and then use the intermediate NCP values and moneyness to compute *numerically*, using finite difference quotients, the slopes of the intermediate maturity NCP curve at the moneynss grid points $[\kappa_1^{(T)}, \kappa_n^{(T)}]$, computed using Lemma 3.7. Slope that are plotted for the T_1 and T_2 maturities are also evaluated using finite difference quotients. A typical point i on the horizontal axis corresponds to three different moneyness $\kappa_i^{(T_2)}$ for maturity T_2, $\kappa_i^{(T_1)}$ for maturity T_1, and $\kappa_i^{(T)}$ for maturity T for which the slopes of the T_2, T_1 and T maturity NCP curves, respectively, are the vertical coordinates of the three plots. That all the slopes at the respective moneyness grid points seem to coincide is no accident, and is an empirical verification of Lemma 3.7. As before, no smoothing has been applied for the interpolated maturity.

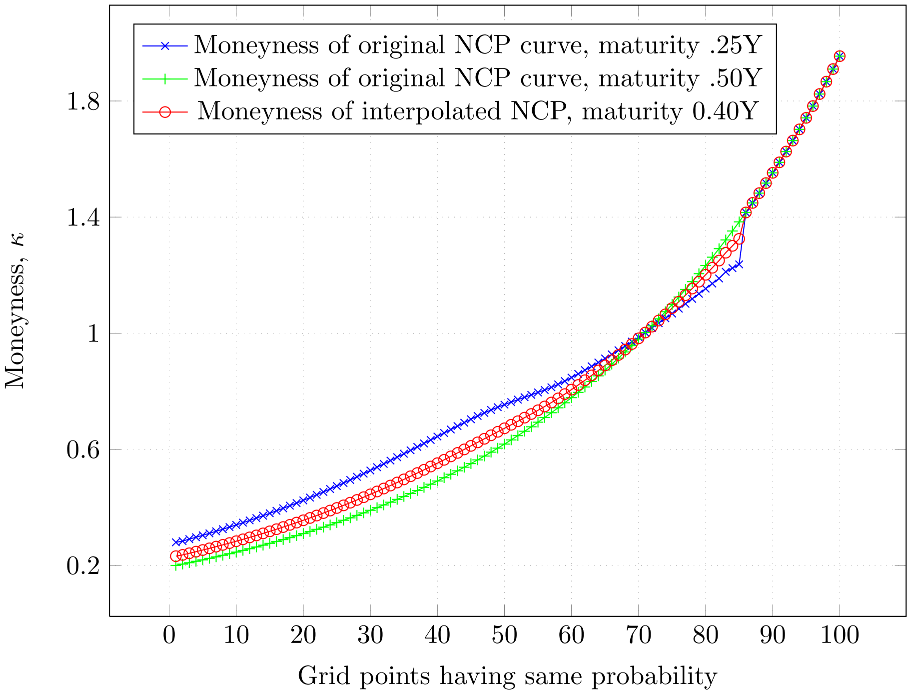

Figure 3: Plot of moneyness of NCP curves for original maturities .25Y and .50Y and an intermediate maturity .40Y obtained using probability space interpolation (see Section 3.2) against the grid point indices. As in Algorithm 1, we first obtain the slopes corresponding to the original moneyness grid $[\kappa_1^{(T_2)}, \kappa_n^{(T_2)}] = [\kappa_1, \kappa_n]$ of the higher T_2 (=.50Y) maturity original NCP curve. Then we compute the moneyness grid points $[\kappa_1^{(T_1)}, \kappa_n^{(T_1)}]$ for the lower T_1 (=.25Y) maturity original NCP curve having the same slopes (i.e. same probabilities). Finally, we perform probability scale interpolation using those grid points, and obtain moneyness $[\kappa_1^{(T)}, \kappa_n^{(T)}]$ for the intermediate T (=.40Y) maturity NCP curve using Lemma 3.7. A typical point i on the horizontal axis corresponds to three different moneyness $\kappa_i^{(T_2)}$ for maturity T_2, $\kappa_i^{(T_1)}$ for maturity T_1, and $\kappa_i^{(T)}$ for maturity T that have been plotted here. The slopes (i.e. probabilities) at moneyness $\kappa_i^{(T_2)}$ and $\kappa_i^{(T_1)}$ for original T_2 and T_1 maturity NCP curves, respectively, are identical by construction, and this is the same slope (probability) that corresponds to moneyness $\kappa_i^{(T)}$ for the intermediate T-maturity smile by implication. As before, no smoothing has been applied for the intermediate maturity.

5 Conclusions

In this paper, we considered normed call prices as a representation of the volatility surface, and derived some new results related to the arbitrage free interpolation of normed call prices, and hence, the construction of an interpolating arbitrage free volatility surface.

Since the normed call prices are considered in their respective T-terminal probability measure Q^T, the results apply to a wide variety of asset classes and are independent of specific assumptions on the numeraire. In particular, it applies to the case of stochastic interest rates.[10]

In addition to this analytic result, smoothing algorithms (Fengler [3] and Gope [6]) work better in terms of robustness and stability of the quadratic program in normed call price settings. For interpolation of volatility on the maturity time/dimension, we have introduced probability space interpolation, and shown that it does not admit strike/moneyness arbitrage or calendar arbitrage, and is built upon the belief that probabilities are more meaningful to compare than moneyness for different maturities when it comes to interpolation. Finally, we have derived results that solve the problem of the interpolated smile being implicitly specified in terms of probability, and lead to conversion from probability to moneyness in an analytically and computationally arbitrage-free way. Simplicity of this analytical relationship further makes it possible to do this conversion in constant time.

[10] Of course, when the rates are deterministic, the measure Q^T is identical to the risk neutral measure $\mathbb{Q}$.

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Notes

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