Next Generation Local Volatility

The Thalesians
Online
February 2021

Jesper Andreasen
Saxo Bank, Copenhagen
kwant.daddy@saxobank.com

Outline

- Local volatility in continuous time.
- Discrete time local volatility.
- Calibration by Monte-Carlo.
- Stochastic volatility and VIX options.
- Multi dimensional case: positive definiteness and arbitrage.
- Minimal models for foreign exchange and interest rates.
- Beyond minimal models.

- Numerical examples: Foreign exchange and interest rates.
- Conclusion.

References

- Andreasen, J (2005): “Back to the Future.” *Risk* Sept.
- Andreasen, J (2020): “Multi-Factor Cheyette Monte-Carlo Calibration.” *Saxo WP*.
- Andreasen, J (2020): “Local Volatility in Multi Dimensions.” *WBS, BBQ, QM, MF*.
- Andreasen, J and B Huge (2011a): “Volatility Interpolation.” *Risk* March.
- Andreasen, J and B Huge (2011b): “Random Grids.” *Risk* June.
- Austing, P (2011): “Repricing the Cross Smile: An Analytic Joint Density.” *Risk* July.

- Austing, P (2020): “Finite Difference Schemes with Exact Recovery of Vanilla Option Prices.” *Risk* Nov.
- Dupire, B (1994): “Pricing with a Smile.” *Risk* July.
- Guyon, J (2014): “Local Correlation Familes.” *Risk* February.
- Guyon, J (2020): “The Joint S&P 500/Vix Smile Calibration Puzzle Solved.” *Risk* April.
- Gyöngy, I (1986): “Mimicking the One-Dimensional Marginal Distributions of Processes having and Ito Differential.” *Probability Theory and Related Fields* 71.
- Harms, P (2019): “Strong Convergence Rates for Markovian Representations of Fractional Brownian Motion.” *University of Freiburg WP*.

- Hutchings, N et al (2020): “Lifted Fractional Volatility.” *QM*.

- McCloud, P (2011): “The CMS Triangle Arbitrage.” *Risk* January.

- Piterbarg, V (2011): “Farka’s Lemma, Spread Options, and Linear Programming“. *Risk* August.

- Shelton, D (2015): “Interpolating the Smile with Path-Dependent Local Volatility.” *ICBI Global Derivatives*.

Local Volatility in Continuous Time

- Suppose the underlying stock evolves according to a continuous Markov process

$$ds = \sigma(t,s)\,dW \qquad (1)$$

- Then initial European option prices

$$c(t,k) = E[(s(t)-k)^{+}] \qquad (2)$$

- … satisfy the Dupire (1994) forward equation

$$0 = -c_t + \frac{1}{2}\sigma(t,k)^2 c_{kk} \quad , c(0,k) = (s(0)-k)^{+} \qquad (3)$$

- This is beautiful mathematics but how do you actually get this to work on a computer?

Your Dad's Local Volatility

- A naïve "academic" approach is to set

$$\sigma(t,k)=[2\frac{c_t(t,k)}{c_{kk}(t,k)}]^{1/2} \qquad (4)$$

- … and combine with Euler simulation

$$\Delta s(t_h)=\sigma(t_h,s(t_h))\Delta W(t_h) \quad ,\Delta W(t_h)\sim N(0,\sqrt{t_{h+1}-t_h}) \qquad (5)$$

- … relying on numerical $\Delta t \to dt$ convergence and computational efforts for this to work in practice.

Local Volatility on a Computer

- There is ***a lot*** left for interpretation here. For example:
  - Interpolation and extrapolation of discrete option prices in (4). How?
  - Differentials and finite differences in (4): implicit, explicit, CN?
  - Arbitrage: is the resulting number in (4) on the real axis?
  - Discretisation: how small do we need Δt to be in (5)?
- So in essence a lot easier to state in equations (4-5) than to do in practice.

Local Volatility on a Finite Difference Grid

- Andreasen and Huge (2011a-b) and Austing (2020), consider discretely consistent, and arbitrage consistent, finite difference implementation of local volatility models.

- … including the simulation of such models.

- In this talk we consider discretely consistent Monte-Carlo implementation of local volatility models.

- … including models that do not allow finite difference implementation.

- Including for example: rough volatility, interest rates and high dimensions in general.

Local Volatility in an Euler Scheme

- Suppose we Euler discretise the continuous time local volatility model

$$s(t_{h+1}) = s(t_h) + \sigma(t_h, s(t_h))\Delta W(t_h) \quad ,0 = t_0 < t_1 < \ldots \tag{6}$$

- A European option expiring at time t_{h+1} has the time 0 price

$$c(t_{h+1}, k) = E[E_{t_h}[(s(t_{h+1}) - k)^+]] = E[\underbrace{b(s(t_h) - k, \sigma(t_h, s(t_h))\sqrt{\Delta t_h})}_{Bachelier's\ \ formula}]$$

$$b(x, v) = x\Phi(\frac{x}{v}) + v\phi(\frac{x}{v}) \tag{7}$$

- This is so due to normality of Δs over the time step $[t_h, t_{h+1}]$.

Calibration by Monte-Carlo

- In Monte-Carlo implementation over simulations $\{\omega\}$, equation (7) can be written as

$$c(t_{h+1},k)=\frac{1}{N}\sum_{\omega} b(s(t_h)-k,\sigma(t_h,s(t_h))\sqrt{\Delta t_h})(\omega) \qquad ,N=\#\{\omega\} \tag{8}$$

- If we wish to hit the strikes $k_1,\ldots,k_I$ we can make the volatility function $\sigma(t_h,\cdot)$ dependent on I parameters.

- For example by linear interpolation between the strike points $k_1,\ldots,k_I$ and flat extrapolation.

- Here, a Levenberg-Marquardt solver can be used fitting $\sigma(t_h,\cdot)$ to hit the option prices.

- We note that the simulated points $\{s(t_h,\omega)\}$ are kept constant for each update of the volatility function $\sigma(t_h,\cdot)$.

- We do not need to re-simulate. Paths to t_h are kept in memory.

- We also note that for any parameter a of the volatility function, the derivative of the target is

$$\frac{\partial c(t_{h+1},k)}{\partial a}=\frac{1}{N}\sum_{\omega}[\underbrace{\frac{\partial b}{\partial v}(s(t_h)-k,z(t_h)\sigma(t_h,s(t_h))\sqrt{\Delta t_h})}_{Bachelier\,vega}\cdot\sqrt{\Delta t_h}\frac{\partial \sigma(t_h,s(t_h))}{\partial a}](\omega) \tag{9}$$

- We note that the vega can be produced at very little cost in each pricing.

- Bootstrap in expiry/time: After calibration to expiry t_{h+1} we move on to expiry t_{h+2} and so forth.

Notes on Calibration and Implementation

- The model can use a finer time grid for pricing than for calibration.
- We only need the calibration time grid to be a subset of the pricing time grid.
- We only require discrete option price quotes. The model will interpolate/extrapolate to a continuum.
- Absence of arbitrage is good though not strictly necessary.
- The model will catch-up calibrate at a later expiry if we need to chop volatility at a prior one.
- … as opposed to naïve implementation of local volatility as discussed earlier.

Stochastic Local Volatility

- It's straightforward to extend the model to include stochastic volatility.

- For example

$$s(t_{h+1}) = s(t_h) + z(t_h)\sigma(t_h, s(t_h))\Delta W \tag{9}$$

- Where z is some discrete time stochastic process.

- In this case we still have an option price expression of the form

$$c(t_{h+1}, k) = \frac{1}{N}\sum_{\omega} b(s(t_h) - k, z(t_h)\sigma(t_h, s(t_h))\sqrt{\Delta t_h})(\omega) \tag{10}$$

- … to be calibrated over the local volatility function $\sigma(t_h,\cdot)$.

- So exactly the same procedure as in (8).

- Over the time step $[t_h,t_{h+1}]$, the volatility of z is also a free parameter.

- So we should be able to calibrate this parameter to options on variance.

- This, however, is not trivial and some abracadabra is needed.

Options on Variance

- Consider a continuous time stochastic local volatility model

$$ds = z\sigma dW \quad , dz = \varepsilon dW \quad , dW \cdot dZ = \rho dt \qquad (11)$$

- We see that the (log-normal) variance of volatility is

$$\begin{aligned}\beta^2 &\equiv \frac{(d(z\sigma))^2}{(z\sigma)^2 dt} = \frac{(z^2\sigma_s \sigma dW + \sigma\varepsilon dZ)^2}{(z\sigma)^2 dt} = \frac{z^4\sigma_s^2\sigma^2 + 2z^2\sigma_s\sigma^2\varepsilon\rho + \sigma^2\varepsilon^2}{(z\sigma)^2} \\ &= z^2\sigma_s^2 + 2\sigma_s\rho\varepsilon + \frac{\varepsilon^2}{z^2}\end{aligned} \qquad (12)$$

- For given $(\beta(t_h),\sigma_S(t_{h+1}))$, the idea is to solve (12) for $(\rho(t_h),\varepsilon(t_h))$ so that the model hits both prices of options on the underlying stock and on its variance expiring at t_{h+1}.

- However, $(\beta(t_h),\sigma_S(t_{h+1}))$ can not directly be observed so we need a methodology to approximate these quantities.

Forward Volatility of Volatility β

- Let $\sigma(t_h, t_i, s(t_h))$ be the local volatility function if we, with a conditional stop at t_h, fitted a model to expiry t_i options.

- I.e. if we calibrate the volatility function $\sigma(t_h, t_i, s(t_h))$ in

$$c(t_i, k) = \frac{1}{N}\sum_{\omega} b(s(t_h) - k, z(t_h)\sigma(t_h, t_i, s(t_h))\sqrt{t_i - t_h})(\omega) \quad, h < i \tag{13}$$

- An option on variance can then be approximated as

$$g(t_{h+1},K)=E[E_{t_h}[(z(t_{h+1})\sigma(t_{h+1},t_i,s(t_{h+1}))-K)^+]]\approx\frac{1}{N}\sum_{\omega}\underbrace{B(m(t_h)/K,\beta(t_h,m(t_h))(\omega)}_{Black\ formula}$$

$$m(t_h)=z(t_h)\{[\sigma(t_h,t_i,s(t_h))^2(t_i-t_h)-\sigma(t_h,t_{h+1},s(t_h))^2(t_{h+1}-t_h)]/(t_i-t_{h+1})\}^{1/2} \qquad (14)$$

$$B(X,V)=F\Phi(\frac{\ln X}{V}+\frac{1}{2}V)-K\Phi(\frac{\ln X}{V}-\frac{1}{2}V)$$

- Calibrate the local volatility function $\beta(t_h,\cdot)$ to match the VIX option smile.

Forward Skew σ_s

- We approximate the forward skew as

$$\sigma_s(t_{h+1},t_i)\approx\frac{\partial}{\partial s}[\{[\sigma(t_h,t_i,s)^2(t_i-t_h)-\sigma(t_h,t_{h+1},s)^2(t_{h+1}-t_h)]/(t_i-t_{h+1})\}^{1/2}] \tag{15}$$

- Insert in (12) and solve for $\varepsilon(t_h)$.

- Once done we move on to next expiry t_{h+1} and so forth…

Summary of Calibration Methodology

- For each t_h:

  0. Calibrate local volatility to S&P $\sigma(t_h, t_{h+1}, \cdot)$.

  1. Calibrate local volatility to S&P $\sigma(t_h, t_i, \cdot)$.

  2. Calibrate local volatility for ViX $\beta(t_h, t_{h+1}, \cdot)$.

  3. Approximate forward skew $\sigma_s(t_{h+1}, t_i, \cdot)$ and solve for $\varepsilon(t_h, t_{h+1})$ to hit $\beta(t_h, t_{h+1}, \cdot)$.

  4. Simulate forward to next maturity t_{h+1}.

Slam Dunk …?

- Solving the volatility equation $(\beta,\sigma_S)\mapsto(\varepsilon,\rho)$,

$$\beta^2 = z^2\sigma_S^2 + 2\sigma_S\rho\varepsilon + \frac{\varepsilon^2}{z^2} \tag{16}$$

- … can go bad in (at least) two scenarios:

  - The assumed auto-correlation structure of volatility is wrong and we thus end up with a forward volatility of variance β that isn't positive.

  - If the local volatility component is too dominant, ie $|\sigma_S|$ too large, hence the stochastic volatility is too small.

- It is possible to set $\rho = -\operatorname{sgn}(\sigma_s)$ and thereby obtain a solution for ε for all positive β^2.

- However, in this case the correlation ρ and the skew σ_s go in opposite directions, which can lead to an unstable solution that oscillates in the time/expiry direction.

- So quite a tricky balance.

Bergomish Model Specification

- So a realistic auto-correlation structure of volatility is important.
- To that end we can use a Bergomi inspired model

$$
\begin{aligned}
& s(t_{h+1}) = s(t_h) + z(t_h)\sigma(t_h, s(t_h))\Delta W \quad , z(t_h) = e^{\frac{1}{2}\sum_i y_i(t_h)} \\
& \underbrace{y_i(t_{h+1}) = e^{-\theta_i \Delta t_h} y_i(t_h) + \left(\frac{1-e^{-2\theta_i \Delta t_h}}{2\theta_i \Delta t_h}\right)^{1/2} \sum_j \eta_{ij}(t_h)\Delta Z_j}_{Ornstein\text{–}Uhlenbeck\ process}
\end{aligned}
\tag{17}
$$

- Here, the mean-reversion $\{\theta_i\}$ can be chosen for the model to approximate rough volatility, see for example Harms (2019) and Hutchings et al (2020).

- For a given set of mean-reversions $\{\theta_i\}$, we have for variance contracts

$$d\ln v(t,T) \approx \sum_i e^{-\theta_i(T-t)} \sum_j \eta_{ij}(t)\, dZ_j + O(dt) \quad , v(t,T) = E_t[(ds(T))^2 / dt] \tag{18}$$

- The approximation is exact if σ is independent of spot.
- For a given set of tenors $\{\tau_i\}$ we can thus specify the volatility structure of the $\eta = (\eta_{ij})$ through the instantaneous covariance structure of the variance contracts

$$\eta\eta' = (\Theta^{-1}) C (\Theta^{-1})' \quad , \Theta_{ij} = \{e^{-\theta_j \tau_i}\} \quad , C_{ij} = \mathrm{cov}_t[d\ln v(t, t+\tau_i), d\ln v(t, t+\tau_j)] \tag{19}$$

- … as in Andreasen (2005) for interest rates.

Joint Calibration to S&P Options and VIX Options

- Yes, it can be done and previous slides illustrate how.

- However, it is also clear that it puts some quite strict bounds on the volatility dynamics.

- Specifically, it seems to require:

  - a non-trivial realistic (rough) auto correlation structure

  - significant negative covariance (corr*vol z) between spot and volatility factor

- In some sense good, because tight bounds on dynamics imply that we can actually hedge exotic products quite well with semi static hedging in options and options on variance.

- ***<u>But</u>***: A lot of empirical work is needed before we can draw firmer conclusions.

- ***<u>And</u>***: Liquidity in options on variance is very thin except for short dated VIX/S&P.

Multi Asset Arbitrage

- Consider a market with stocks $s_1,\dots,s_I$ and bank account s_0.
- Assume interest rates and dividends are zero, and set the start prices to be $s_i(0)=0$.
- Assume you can trade variance contracts on all spreads

$$v_{ij}(t)=PV[(s_i(t)-s_j(t))^2] \tag{20}$$

- We note that covariance matrix is given by

$$g_{ij}=PV[s_i s_j]=\frac{1}{2}PV[s_i^2+s_j^2-(s_i-s_j)^2]=\frac{1}{2}(v_{i0}+v_{j0}-v_{ij}) \tag{21}$$

- Absence of arbitrage implies that the covariance matrix

$$G(t)=\{g_{ij}(t)\} \tag{22}$$

- … must be ***positive semi definite*** for all t.

- If not, there exist non-zero weights $\{w_i\}$ so that

$$PV[(\sum_i w_i s_i(t))^2]=\sum_i \sum_j w_i w_j PV[s_i(t) s_j(t)]=w'G(t)w<0 \tag{23}$$

- This is contradicting absence of arbitrage since:

$$(\sum_i w_i s_i(t))^2 \geq 0 \tag{24}$$

- The arbitrage portfolio is in this case given by

$$\{ \underbrace{(w_i w_j)}_{\substack{\textit{portfolio}\\ \textit{weight}\\ ij}} \cdot \underbrace{g_{ij}}_{\substack{\mathrm{cov}\,ij\\ \textit{contract}}} \} \tag{25}$$

- This can be sharpened to $\{g_{ij}(t_2)-g_{ij}(t_1)\}$ needs to be positive semi definite for all pairs $t_1<t_2$.

- So positive definiteness is not just a technical condition but indeed an arbitrage condition.

Minimal Multi Asset Models

- One way of modelling multi assets is to let

$$\Delta s_i(t_h) = \sigma_i(t_h, s_i(t_h))\Delta W_i(t_h)$$

$$\{\Delta W_i(t_h)\} \sim N(0, \{\rho_{ij}(t_h)\}) \tag{26}$$

$$\rho_{ij}(t_h, s_i, s_j) = \frac{\sigma_i(t_h, s_i)^2 + \sigma_j(t_h, s_j)^2 - \sigma_{ij}(t_h, s_i - s_j)^2}{2\sigma_i(t_h, s_i)\sigma_j(t_h, s_j)}$$

- Here correlation is specified through local volatility of the stocks and spread of these.

- This fits nicely with the Monte-Carlo calibration approach (8).

- All stocks are simulated jointly.
- For each expiry, local volatilities are calibrated separately for each (spread) option smile.
- Local volatilities are combined to produce correlations according to the formula in (26).
- Negative eigenvalues and vectors will need to be chopped in case they occur.
- However, due to the catch-up calibration built in here, the methodology doesn’t collapse if negative definiteness occurs. There are limits of course.
- Applications include foreign exchange and interest rates.

Interest Rate Models

- In interest rates there are three main types of option products traded:
  - Swaptions (and caplets): options to enter spot starting swaps.
  - CMS spread options: cash-settled options on the spread between two swap rates.
  - Mid curve options: options to enter forward starting swaps.
- The latter two are usually thought of as correlation products.
- In all of these cases it is usually possible to come up with approximations for one period option prices. Which is what we need for the calibration approach to work.

Swaption Pricing

- For example, swaption pricing in a Monte-Carlo model

$$
\begin{aligned}
E[(\underbrace{s_i(t_{h+1})}_{\substack{swap\\ rate}}-k)^+\underbrace{a_i(t_{h+1})}_{annuity}] &= E[E_{t_h}[(s_i(t_{h+1})-k)^+ a_i(t_{h+1})]] \\
&= E[a_i(t_h)\underbrace{E^a_{t_h}}_{\substack{annuity\\ measure}}[(s_i(t_{h+1})-k)^+]] \\
&\approx \frac{1}{N}\sum_{\omega} a_i(t_h,\omega) E^a_{t_h}[(s_i(t_{h+1})-k)^+|\omega] \\
&\approx \frac{1}{N}\sum_{\omega} a_i(t_h,\omega) b(s_i(t_h,\omega)-k,\sigma_i(t_h,\omega))
\end{aligned}
\tag{27}
$$

Multi Smile Interest Rate Models

- … where $\sigma_i(t_h)$ is the (approximate) volatility of the swap rate which can be computed in most models.
- For spread and mid curve options we have similar expressions.
- This allows us to construct models that simultaneously fit smiles in both caplets/swaptions and spread and/or mid curve options.
- In principle at least.

Practical Experiences

- In FX, the minimal models seem to work reasonably well up to a handful of dimensions.
- Beyond that there seem to be too many violations of the positive definiteness.
- In interest rates, our experiences are similar or slightly worse.
- That said, it is possible to calibrate models in high dimensions if we fix correlation to only fit at-the-money.
- Yield curves and their (lack of) dynamics are extreme, at the moment.
- They always are, though.

Beyond Minimal Models

- Here's an example of a model with no arbitrage but where a minimal model doesn't exist

$$
\begin{aligned}
ds_1 &= \sigma(s_1, s_2)\, dW_1 \\
ds_2 &= \sigma(s_1, s_2)\, dW_2 \qquad\qquad (28) \\
\sigma(s_1, s_2) &= \underline{\sigma} + (\bar{\sigma} - \underline{\sigma}) 1_{s_1 - s_2 = k} \quad , dW_1 \cdot dW_2 = 0
\end{aligned}
$$

- … for some constants $\underline{\sigma} < \bar{\sigma}$.

- Minimal model correlation:

$$
\begin{aligned}
\rho(s_1,s_2) &= \frac{1}{2}\frac{E[(ds_1)^2|s_1]+E[(ds_2)^2|s_2]-E[(ds_1-ds_2)^2|s_1-s_2]}{(E[(ds_1)^2|s_1]E[(ds_2)^2|s_2])^{1/2}} \\
&= \frac{1}{2}\frac{\underline{\sigma}^2+\underline{\sigma}^2-(2\underline{\sigma}^2+2(\bar{\sigma}^2-\underline{\sigma}^2)1_{s_1-s_2=k})}{\underline{\sigma}^2} \qquad (29) \\
&= -\frac{\bar{\sigma}^2}{\underline{\sigma}^2}1_{s_1-s_2=k}
\end{aligned}
$$

- … so $\rho(s_1,s_2)<-1$ on $\{s_1-s_2=k\}$.

- Obviously, a quite specific case but it suggests that if the volatility smiles are more pronounced in the spread directions than in the primal directions, then it may be that the spread is the overall volatility driver …

- When we run into definiteness problems with the minimal models in practice, it probably means trouble with the minimal model approach rather than arbitrage in the market.

- In reality, the minimal model specification is probably just a numerically (too) convenient way of representing dynamics of multiple assets.

- Work is needed to figure out how to specify local volatility models in high dimensions.

- Good news is that we now know how to handle the high dimensional calibration problem numerically.

Numerical Performance

- Hardware is a standard 4 core CPU machine. All models with 2F stochastic volatility.

- Single asset model calibration to 5 strikes on expiries 1m, 2m, …, 12m:

  - 8,192 paths: 0.09s
  - 65,536 paths: 0.66s

- 5 Ccy FX model calibration to 5 strikes in all crosses on expiries 1m, 2m, …, 12m:

  - 8,192 paths: 0.46s
  - 65,536 paths: 3.32s

- 4 factor interest rate model. Calibration to 5 strikes in all crosses (spread options and mid curves) in tenors 3m, 2y, 10y, 30y on expiries 1y, 2y, … , 10y.

    - 8,192 paths: 0.45s
    - 65,536 paths: 3.44s

- 6 factor interest rate model. Calibration to 5 strikes in all crosses in tenors 3m, 2y, 5y, 10y, 20y, 30y on expiries 1y, 2y, … , 10y.

    - 8,192 paths: 1.00s
    - 65,536 paths: 7.13s

- Pure simulation times (without calibration) are roughly half.

- AD risk around a factor 5-10 slower than calibration/pricing.

- Numerical performance is definitely ok, but a tad slower than my audience is used to.
- The approach lends itself very well to AVX or GPU/TF acceleration.

Conclusion

- We have presented an approach to calibration of high dimensional models in Monte-Carlo that is performing, flexible and general.

- We do not rely on forward equations and approximations.

- The approach is exact within the discrete time Euler scheme.

- In some cases like interest rates and VIX, a single time step approximation is necessary.

- We have shown that it has many applications. Including

    - Single asset stochastic local volatility.

- Simultaneous calibration to S&P options and VIX options.

- Multi asset problems in foreign exchange and interest rates.

- Minimal correlation model is too simplistic for high dimensional problems.

- Next step: non-minimal correlation structures.