Poisson Random Measures

Throughout, let $(S, \mathcal{S}, m)$ denote a sigma-finite measure space with $m(S) > 0$, and $(\Omega, \mathcal{F}, \mathrm{P})$ the underlying probability space.

A construction of Poisson random measures

Definition 1. A *Poisson random measure* Π *with intensity* m is a collection of random variables $\{\Pi(A)\}_{A \in \mathcal{S}}$ with the following properties:

(1) $\Pi(A) = \mathrm{Poiss}(m(A))$ for all $A \in \mathcal{S}$;

(2) If $A_1, \ldots, A_k \in \mathcal{S}$ are disjoint, then $\Pi(A_1), \ldots, \Pi(A_k)$ are independent.

We sometimes write "PRM(m)" in place of "Poisson random measure with intensity m."

Theorem 2. PRM(m) *exists and is a.s. purely atomic.*

Proof. The proof proceeds in two distinct steps.

Step 1. First consider the case that $m(S) < \infty$.

Let $N, X_1, X_2, \ldots$ be a collection of independent random variables with $N = \mathrm{Poiss}(m(S))$, and $\mathrm{P}\{X_j \in A\} = m(A)/m(S)$ for all $j \geq 1$ and $A \in \mathcal{S}$. Define,

$$\Pi(A) := \sum_{j=1}^{N} \mathbb{1}_A(X_j) \qquad \text{for all } A \in \mathcal{S}.$$

Clearly, Π is almost surely a purely-atomic measure with a random number [i.e., N] atoms. Next we compute the finite-dimensional distributions of Π.

If we condition first on N, then we find that for every disjoint $A_1, \ldots, A_k \in \mathcal{S}$ and $\xi_1, \ldots, \xi_k \in \mathbf{R}$,

$$\begin{aligned}\mathrm{E}e^{i\sum_{j=1}^k \xi_j \Pi(A_j)} &= \mathrm{E}\left(\prod_{\ell=1}^N \exp\left\{ i \sum_{j=1}^k \xi_j \mathbb{1}_{A_j}(X_\ell)\right\}\right) \\ &= \mathrm{E}\left[\left(\mathrm{E}\exp\left\{ i \sum_{j=1}^k \xi_j \mathbb{1}_{A_j}(X_1)\right\}\right)^N\right].\end{aligned}$$

Because the A_j's are disjoint, the indicator function of $(A_1 \cup \cdots \cup A_k)^c$ is equal to $1 - \sum_{j=1}^k \mathbb{1}_{A_j}$, and hence

$$\begin{aligned}\exp\left\{ i \sum_{j=1}^k \xi_j \mathbb{1}_{A_j}(x)\right\} &= \sum_{j=1}^k \mathbb{1}_{A_j}(x) e^{i\xi_j} + 1 - \sum_{j=1}^k \mathbb{1}_{A_j}(x) \\ &= 1 + \sum_{j=1}^k \mathbb{1}_{A_j}(x)\left(e^{i\xi_j} - 1\right) \quad \text{for all } x \in S.\end{aligned}$$

Consequently,

$$\mathrm{E}\exp\left\{ i \sum_{j=1}^k \xi_j \mathbb{1}_{A_j}(X_1)\right\} = 1 + \sum_{j=1}^k \frac{m(A_j)}{m(S)}\left(e^{i\xi_j} - 1\right),$$

and hence,

$$\mathrm{E}\exp\left(i \sum_{j=1}^k \xi_j \Pi(A_j)\right) = \mathrm{E}\left(\left\{1 + \sum_{j=1}^k \frac{m(A_j)}{m(S)}\left(e^{i\xi_j} - 1\right)\right\}^N\right).$$

Now it is easy to check that if $r \in \mathbf{R}$, then $\mathrm{E}(r^N) = \exp\{-m(S)(1-r)\}$. Therefore,

$$\mathrm{E}\exp\left(i \sum_{j=1}^k \xi_j \Pi(A_j)\right) = e^{-\sum_{j=1}^k m(A_j)\left(1 - e^{i\xi_j}\right)}. \tag{1}$$

This proves the result, in the case that $m(S) < \infty$, thanks to the uniqueness of Fourier transforms.

Step 2. In the general case we can find disjoint sets $S_1, S_2, \ldots \in \mathcal{S}$ such that $S = \cup_{k=1}^\infty S_k$ and $m(S_j) < \infty$ for all $j \geq 1$. We can construct independent PRM's $\Pi_1, \Pi_2, \ldots$ as in the preceding, where Π_j is defined solely based on subsets of S_j. Then, define $\Pi(A) := \sum_{j=1}^\infty \Pi_j(A \cap S_j)$ for all

$A \in \mathcal{S}$. Because a sum of independent Poisson random variables has a Poisson law, it follows that $\Pi = \mathrm{PRM}(m)$. $\square$

Theorem 3. *Let $\Pi :=$ PRM(m), and suppose $\varphi : S \to \mathbf{R}^k$ is measurable and satisfies $\int_{\mathbf{R}^d} \|\varphi(x)\|\, m(\mathrm{d}x) < \infty$. Then, $\int_{\mathbf{R}^d} \varphi\, \mathrm{d}\Pi$ is finite a.s., $\mathrm{E}\int_{\mathbf{R}^d} \varphi\, \mathrm{d}\Pi = \int \varphi\, \mathrm{d}m$, and for every $\xi \in \mathbf{R}^k$,*

$$\mathrm{E}e^{i\xi\cdot\int\varphi\,\mathrm{d}\Pi} = \exp\left(-\int\left(1-e^{i\xi\cdot\varphi(x)}\right)\, m(\mathrm{d}x)\right). \tag{2}$$

The preceding holds also if m is a finite measure, and φ is measurable. If, in addition, $\int_{\mathbf{R}^d} \|\varphi(x)\|^2\, m(\mathrm{d}x) < \infty$, then also

$$\mathrm{E}\left(\left\|\int_{\mathbf{R}^d} \varphi\,\mathrm{d}\Pi - \int_{\mathbf{R}^d} \varphi\,\mathrm{d}m\right\|^2\right) \le 2^{k-1}\int_{\mathbf{R}^d} \|\varphi(x)\|^2\, m(\mathrm{d}x).$$

Proof. By a monotone-class argument it suffices to prove the theorem in the case that $\varphi = \sum_{j=1}^n c_j \mathbb{1}_{A_j}$, where $c_1, \ldots, c_n \in \mathbf{R}^k$ and $A_1, \ldots, A_n \in \mathcal{S}$ are disjoint with $m(A_j) < \infty$ for all $j = 1, \ldots, n$. In this case, $\int \varphi\,\mathrm{d}\Pi = \sum_{j=1}^n c_j \Pi(A_j)$ is a finite weighted sum of independent Poisson random variables, where the weights are k-dimensional vectors $c_1, \ldots, c_n$. The formula for the characteristic function of $\int \varphi\,\mathrm{d}\Pi$ follows readily from (1). And the mean of $\int \varphi\,\mathrm{d}\Pi$ is elementary. Finally, if φ^j denotes the jth coordinate of φ, then

$$\mathrm{Var}\int \varphi^j\,\mathrm{d}\Pi = \sum_{i=1}^n c_i^2 \mathrm{Var}\Pi(A_i) = \sum_{i=1}^n c_i^2 m(A_i) = \int |\varphi^j(x)|^2\, m(\mathrm{d}x). \tag{3}$$

The L^2 computation follows from adding the preceding over $j = 1, \ldots, k$, using the basic fact that for all random [and also nonrandom] mean-zero variables $Z_1, \ldots, Z_k \in L^2(\mathrm{P})$,

$$|Z_1 + \cdots + Z_k|^2 \le 2^{k-1}\sum_{j=1}^k |Z_j|^2. \tag{4}$$

Take expectations to find that $\mathrm{Var}\sum_{j=1}^k Z_j \le 2^{k-1}\sum_{j=1}^k \mathrm{Var}(Z_j)$. We can apply this in (3) with $Z_j := \int \varphi^j\,\mathrm{d}\Pi$ to finish. $\square$

The Poisson process on the line

In the context of the present chapter let $S := \mathbf{R}_+$, $\mathcal{S} := \mathcal{B}(\mathbf{R}_+)$, and consider the intensity $m(A) := \lambda|A|$ for all $A \in \mathcal{B}(\mathbf{R}_+)$, where $|\cdots|$ denotes the one-dimensional Lebesgue measure on $(\mathbf{R}_+\,, \mathcal{B}(\mathbf{R}_+))$, and $\lambda > 0$ is a fixed finite constant. If Π denotes the corresponding $\mathrm{PRM}(m)$, then we can define

$$N_t := \Pi((0\,, t]) \qquad \text{for all } t \geq 0.$$

That is, N is the cumulative distribution function of the random measure Π. It follows immediately from Theorem 2 that:

(1) $N_0 = 0$ a.s., and N has i.i.d. increments; and
(2) $N_{t+s} - N_s = \mathrm{Poiss}(\lambda t)$ for all $s, t \geq 0$.

That is, N is a classical Poisson process with intensity parameter λ in the same sense as in Math. 5040.

Problems for Lecture 4

Throughout let N denote a Poisson process with intensity $\lambda \in (0\,,\infty)$.

1. Check that N is cadlag and prove the following:

(1) $N_t - \lambda t$ and $(N_t - \lambda t)^2 - \lambda t$ define mean-zero cadlag martingales;
(2) (The strong law of large numbers) $\lim_{t\to\infty} N_t/t = \lambda$ a.s.

2. Let $\tau_0 := 0$ and then define iteratively for all $k \geq 1$,

$$\tau_k := \inf\{s > \tau_{k-1} : \; N_s > N_{s-}\}.$$

Prove that $\{\tau_k - \tau_{k-1}\}_{k=1}^\infty$ is an i.i.d. sequence of $\mathrm{Exp}(\lambda)$ random variables.

3. Let τ_k be defined as in the previous problem. Prove that $N_{\tau_k} - N_{\tau_k -} = 1$ a.s.