Derivatives

Theory and Practice

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To Alfred Stanley
—Keith Cuthbertson

To Emily and Jacqueline
—Dirk Nitzsche

To my parents and Sara
—Niall O'Sullivan

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About the Companion Site

This book is accompanied by a companion website:

`www.wiley.com/go/derivativestheorypractice`

The website includes the following materials for instructors (password protected) and students (open access):

Instructors:

- Additional exercises
- Multiple Choice Questions
- PowerPoint slides
- MATLAB and Excel files

Students:

- Answers to end of chapter exercises
- MATLAB and Excel files (interactive) to reproduce tables/graphs in the text.

Preface

The aim of this book is to present a clear exposition of key results on pricing, hedging, and speculation using derivative securities. The emphasis is on drawing out the practical uses of derivatives. The reader needs only to have undertaken an introductory course in finance, together with some basic statistics and simple algebra, including calculus. The mathematics and statistics have been kept to a minimum, with the emphasis on intuitive explanations and practical applications. For those requiring some revision of basic finance concepts, these can be found in a companion text by K. Cuthbertson and D. Nitzsche, *Investments* (2nd edition, John Wiley, 2008).

The material has been successfully used on non-specialist MBA degrees, 3rd-year undergraduate and MSc courses in derivatives, as well as with participants in executive education courses from banks, law firms and other financial institutions. The topics have been broken down into relatively short chapters so that it is easy for the instructor to set up their own lecture course based on the book and we also provide a substantial amount of complementary material.

Our emphasis on practical aspects has, for example, allowed us to discuss the use of derivatives by hedge funds, the use of strip and stack hedges by corporates, the use of put–call parity to market 'guaranteed bonds' and analysing how risky the stock market can be for long-term investors. In addition, there is an analysis of how collateralised debt obligations (CDOs) and credit default swaps (CDS) played such a prominent role in propagating systemic risk in the 2008 financial crisis. Also the various methodologies used to measure value at risk, have immediate practical implications for financial institutions and regulators.

Linking theoretical and practical aspects is a major aim of the book. After completing a course based on the book, the reader should be in a position to tackle more theoretical approaches (e.g. which use continuous time mathematics), more advanced numerical techniques, as well as understanding some of the debacles involving derivatives, which are reported all too frequently in the financial media.

STUDENT LEARNING

Each chapter has learning objectives, worked examples, finance blogs, technical appendices and end of chapter exercises.

FINANCE BLOGS

In the text we have provided a number of finance blogs. Some of these use light-hearted analogies to illustrate some key theoretical points. These include using 'Dolly' the sheep as a replication portfolio, taking Ken and Barbie to demonstrate the use of swaps and also to illustrate possible payoffs from options. Other blogs include Metallgesellschaft's rolling hedge, Leeson's straddle, carbon trading, the spark-spread swap and ISDA swap agreements.

SLIDES

For lecturers who adopt the book, all PowerPoint slides in the book are available on the Wiley website and we have also provided a set of slides for a possible one-semester course on derivatives and risk management.

EXERCISES

On the Wiley website there are about 350 questions with full answers which are available to students to work on independently. There are an additional 350 questions and answers available to lecturers who adopt the book for their derivatives courses – these can be used to test the students in closed-book exams or in mid-term tests and assignments. There are also a set of multiple choice questions which cover the main topic areas and these are also available to lecturers who adopt the book.

EXCEL AND MATLAB

Many of the tables in the text are based on Excel files (which do not require knowledge of Visual Basic) and a selection of these are available on the website for students and lecturers. These clearly demonstrate how the results in the text have been obtained and can form the basis of more complex calculations. Where more precision is required (e.g. pricing options, measuring value at risk, Monte Carlo simulation), we have provided MATLAB files on the website, which have the advantage that the code is very close to the algebraic formulation used in the text (and in other widely used programming packages).

ACKNOWLEDGEMENTS

We should like to thank many colleagues at CASS Business School for their comments and suggestions and also the many university students and finance professionals on which this material has been tried and tested.

CHAPTER 1

Derivative Securities

Aims

- To explain *forward and futures* contracts, their similarities and differences.
- To examine the basic concepts behind *call and put options* and how their payoffs at maturity can provide ‘insurance’.
- To show how *interest rate swaps* can be used to alter the cash flow profile of uncertain payments or receipts, and hence reduce the risk attached to such cash flows.
- To analyse how derivative securities are used in speculation, hedging and arbitrage.
- To explain short-selling.

There are three main types of derivative securities, namely futures, options and swaps. Derivative securities are assets whose price depends on the price of some other (underlying) asset. Hence the derivatives price is *derived* from the price of this underlying asset.

For example, lets assume a futures contract on AT&T stocks is traded on the Chicago Mercantile Exchange (CME). The underlying asset in the futures contract is the stock of AT&T itself, which is traded on a different exchange, namely the New York Stock Exchange (NYSE). The price of the stock on the NYSE is for immediate delivery of the stock, and is known as the *spot or cash market price*. In contrast, today’s *futures price* (on AT&T stock) quoted in Chicago is a price quote for delivery of AT&T stock (in Chicago) *at a specific date in the future*. However, we can show that the AT&T futures price (in Chicago) is (largely) determined by the stock price of AT&T (quoted on the NYSE). If AT&T’s stock price changes on the NYSE by $1 (in the next few seconds) then the AT&T futures price will also immediately change by about $1 on the Chicago futures exchange – even though the two markets are separated by around 1,000 km. The tight link between the spot/cash market price of AT&T in New York and the AT&T futures price in Chicago is due to a process known as (risk-free) arbitrage.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O’Sullivan.

Derivatives are used by hedgers, speculators and arbitrageurs. Derivatives often receive a 'bad press', in part because there have been some quite spectacular derivatives losses. For example, in 1994 Nick Leeson, who worked for Barings Bank in Singapore, lost $1.4bn when trading futures and options on the Nikkei 225, the Japanese stock index. This led to Barings going bust. In 1998, Long Term Capital Management (LTCM), a hedge fund which leveraged its trades using derivatives, had losses of over $4bn and had to be rescued by a consortium of banks under the imprimatur of the Federal Reserve Board. This was somewhat ironic since Myron Scholes and Robert Merton, two academics who received the Nobel Prize for their work on derivatives, were key players in the LTCM debacle. Derivatives are a bit like nuclear fission – they can be used in 'good ways' (like low cost, low carbon electricity from nuclear power) or in 'bad ways' (like nuclear bombs) – if used incorrectly they may become 'financial weapons of mass destruction' – to quote Warren Buffett (2002). Let us now examine how derivatives are used in practice, so that you can begin to make up your own mind on this issue.

1.1 FORWARDS AND FUTURES

Forward and futures contracts are analytically very similar, although the way the two contracts are traded differ in some respects. A holder of a long (short) *forward contract* has an agreement to buy (sell) an asset at a certain time in the future for a certain price which is fixed today.

The *buyer (seller or short position)* in a *forward contract:*

- acquires a legal obligation to buy (sell) an asset *(the underlying)*
- at some specific future date *(maturity/expiry date)*
- in an amount *(contract size)*
- and at a price (*the forward price)* which is fixed today.

A *forward contract* is an over-the-counter (OTC) instrument, and trades take place directly between the buyer and seller as negotiated between the two parties (usually over the phone) for a specific amount and specific delivery date.

Originally, forward (and futures) markets were introduced to eliminate risk due to changes in the spot (cash market) price of agricultural commodities. For example, a farmer might know in April that he will harvest 5,000 bushels of wheat in September. A wholesaler who purchases grain for use in the food industry might know their requirements for wheat in September, as early as April. The two participants can eliminate (or hedge) risk by negotiating a contract to supply 5,000 bushels of grain in September at a price which is agreed in April – so the *September-forward price* is agreed in April but will not be paid until September, when the grain is delivered.

The forward contract (if held to maturity) eliminates risk for each side of the bargain. Both sides of the deal are 'locked in' to the forward price (of the September contract) quoted in April and thereby remove any 'price risk'. This is a 'natural hedge' since both sides of the deal wish to 'lock in' a known forward price, quoted today, for delivery of wheat at a specific date (and location) in the future. If, when we get to September, the spot/cash market price of wheat (for immediate delivery) is very high or very low, this is of no consequence, since both parties to the forward contract have agreed to exchange wheat in September at the pre-agreed forward price.

If a *futures contract* on wheat is held to maturity then delivery of the underlying asset (wheat) takes place at the pre-agreed futures price and under these circumstances the futures contract is the same as a forward contract. Theoretically the quoted forward and futures prices will be the same if they are entered into at the same time (and with the same maturity date). Although futures contracts are very similar analytically to forward contracts, they do differ in certain practical aspects.

In a forward contract the terms in the contract (e.g. delivery quantity, delivery date, etc.) are 'tailor made' between two traders (a buyer and seller) whereas the terms of a *futures contract* are standardised (e.g. delivery quantity, delivery point, delivery dates, etc.). Also, trading in futures contracts takes place on an organised exchange, rather than over-the-counter as with forward contracts.

Consider wheat. The physical commodity wheat can be bought and sold for (near) *immediate delivery* in the *spot (cash) market* for wheat. When you buy or sell a futures contract on wheat, it is the 'legal right' to the terms in the futures contract that is being purchased or sold – you do not immediately receive the physical commodity 'wheat' when you buy a wheat futures contract. However, as we shall see, there is a close link between the futures price of wheat and the spot price for wheat but they are not the same thing – the spot price and the futures price are quoted in two different markets.

Futures contracts are traded between market makers in a 'pit' on the floor of the exchange, of which the largest are the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME) – which merged to form the CME Group in 2007. However, in recent years there has been a move away from trading by 'open outcry' in a 'pit' towards electronic trading between market makers (and also over the internet). The largest 'pit trading' futures exchange in Europe was the London International Financial Futures Exchange (LIFFE), which has now merged and become NYSE-Euronext – an electronic trading platform. You can also trade 'out-of-hours' in many futures contracts using the GLOBEX electronic trading platform. Some futures contracts that are traded on US exchanges are shown in Table 1.1.

1.1.1 Market Classification

A key feature of a futures contract is that it involves *deferred delivery* of the underlying asset (e.g. wheat, AT&T stocks), whereas spot (cash market) assets are for *immediate delivery* (although in practice, there is usually a delay of a few days). A primary use of derivative securities is to minimise price uncertainty. Therefore, where the underlying assets (e.g. currencies, stocks, oil, agricultural produce) are widely traded and yet their spot prices exhibit great volatility, there is likely to be a large active derivatives market.

TABLE 1.1 Selected futures contracts

Contract	Exchange	Contract size
1. Metals and petroleum		
Gold	CME Group (COMEX)	100 troy oz
Silver	CME Group (COMEX)	5,000 troy oz
Crude oil	CME Group (NYMEX)	1,000 barrels
Natural gas	CME Group (NYMEX)	10,000 mm Btu
2. Agricultural		
Corn	CME Group (CBOT)	5,000 bu
Lean hogs	CME	40,000 lbs
Pork bellies	CME	40,000 lbs
Frozen orange juice	ICE	15,000 lbs
3. Foreign currency		
British pound	CME	£62,500
Swiss franc	CME	SFr125,000
Euro	CME	€125,000
Japanese yen	CME	¥12.5m
4. Stock indices		
S&P 500	CME Group (CBOT)	$250 × Index
FTSE 100	NYSE-Euronext	£10 × Index
Eurotop 100	NYSE-Euronext	€20 × Index
Nikkei 225	CME Group (CBOT)	$5 × Index
5. Interest rates		
Eurodollar – 90 days	CME Group (IMM)	$1,000,000
US T-bills	CME Group (IMM)	$1,000,000
US T-bonds	CME Group (CBOT)	$100,000
UK 3-month Euro LIBOR	NYSE-Euronext	£100,000
UK Long Gilt Futures	NYSE-Euronext	€1,000,000

Note: CBOT = Chicago Board of Trade (part of CME Group), CME = Chicago Mercantile Exchange, IMM = International Money Market (Chicago, part of CME Group), NYSE-Euronext (previously London International Financial Futures Exchange, LIFFE).

Trading in derivative securities can be on a trading floor (or 'pit') or via an electronic network of traders, within a well-established organised market (e.g. with a clearing house, membership rules, etc.). However, many derivatives contracts – for example, all FX-forward contracts and swap contracts – are traded in OTC markets, where the contract details are not standardised but individually negotiated between clients and dealers. Options are traded widely on exchanges but the OTC market in options (particularly 'complex' or 'exotic' options) is also very large.

Today there are a large number of exchanges which deal in futures contracts. Most can be categorised as either agricultural futures contracts (where the underlying 'asset' is, for example, pork bellies, live hogs or wheat), energy futures (e.g. crude oil, natural gas, heating oil), metallurgical futures (e.g. silver, platinum) or financial futures contracts (where the underlying asset could be a portfolio of stocks represented by the S&P 500, currencies, T-bills, T-bonds, Eurodollar Deposits, etc.). Agricultural, energy and metallurgical futures are often generically referred to as 'commodity futures'. There are some futures contracts that do not really fit into any of these definitions, such as weather futures – which we meet later.

Futures contracts in agricultural commodities have been traded (e.g. on the CBOT) for over 100 years. In 1972 the CME began to trade currency futures, the introduction of interest rate futures occurred in 1975 and in 1982 stock index futures (colloquially known as 'pinstripe pork bellies') were introduced. The CBOT introduced a clearing house in 1925, where each party to the contract had to place 'cash deposits' into a margin account. The latter provides insurance if one of the parties defaults on the futures contract.

Analytically, forwards and futures can be treated in a similar fashion. However, they differ in some practical details (see Table 1.2). Forward contracts (usually) involve no 'up front' payment and 'cash' only changes hands at the maturity of the contract. A forward contract is negotiated between two parties and (generally) is not marketable. In contrast, a futures contract is traded in the futures market (in Chicago)[1] and when initiated, traders have to provide a cash deposit known as the *initial margin*. However, the initial margin is merely a form of collateral to ensure both parties can fulfil the terms of the futures contract – it is not a payment for the futures contract itself and it is not the 'futures price'. The margin

TABLE 1.2 Forward and futures contracts

Forwards	Futures
Private (non-marketable) contract between two parties	Traded on an exchange
(Large) trades are not communicated to other market participants	Trades are immediately known by other market participants
Delivery or cash settlement at expiry	Contract is usually closed out prior to maturity
Usually one delivery date	Range of maturity dates
No cash paid until expiry	Cash payments into (out of) margin account, daily
Negotiable choice of delivery dates and size of contract	Standardised contracts

[1] For ease of exposition here, we assume futures and options are traded in Chicago and any delivery at maturity of the underlying asset in these derivative contracts (e.g. gold, oil, live cattle, stocks, T-bonds) is also in Chicago.

usually earns a competitive interest rate so it is not a 'cost'. As the futures price changes then 'payments' (i.e. debits and credits) are made into (or out of) the margin account. Hence a futures contract is a forward contract that is 'marked-to-market', daily.

Because the futures contract is marketable, the contracts have to be standardised – for example, by having a set of fixed delivery dates and a fixed 'contract size' (e.g. 1,000 barrels of oil for the oil futures contract or $100,000 for the US T-bond futures contract). In contrast, a forward contract can be 'tailor made' between the two parties, in terms of size and delivery date. Finally, forward contracts almost invariably involve actual delivery of the underlying asset (e.g. currency) whereas futures contracts can be (and usually are) 'closed out' prior to maturity which cancels any delivery obligations.

Forward contracts can be used for speculation. First, consider a speculator who on 1 April thinks the gold (spot) price will be high in September. On 1 April she therefore 'buys' ('goes long') a September-forward contract on gold, but does not pay any cash on 1 April. If the spot price of gold rises (falls), the speculator will make a gain (loss) on the forward contract *at the time the contract matures in September*.

For example, suppose on 1 April ($t = 0$) the quoted spot ('cash-market') price of gold is $S_0 = \$1{,}000$ per oz and the gold forward price quoted in Chicago (for delivery in September, T) is $F_0 = \$1{,}010$ per oz. Assume the speculator is correct and the spot (cash market) price for gold in September (T) turns out to be higher at $S_T = \$1{,}200$ per oz. The speculator holding the ('long') forward contract can take delivery of the 'physical' gold (in the forward contract, in Chicago) for which she pays $F_0 = \$1{,}010$ in September. But she can now immediately sell the gold in the spot/cash market for $S_T = \$1{,}200$, giving an overall profit of $190 per contract ($= S_T - F_0$).

Now consider speculation using *futures* contracts, which can be closed out at any time before maturity. Most futures contracts are closed out prior to maturity and when they are, the clearing house (CH) in Chicago sends out a cash payment which reflects *the change* in the futures prices between the opening trade and closing out the contract (i.e. a buy followed by a sell, or vice versa). Therefore futures contracts can be used for speculation over very short time horizons. Remember that the futures price moves (approximately) dollar-for-dollar with changes in the price of the underlying spot ('cash-market') asset. If you think the futures price will rise (fall) then today a speculator will buy (sell) a futures contract.

Suppose on 15 June you purchased a September-futures contract on stock-XYZ at a price $F_0 = \$100$ and one week later on 22 June you closed out the contract by selling it at its new price of $F_1 = \$110$.[2] Then the CH 'effectively' sends you a cheque for $10 – the difference between your buying price and selling price. Simplifying a little, the CH obtains this $10 from the person who initially sold the contract to you at $100 and is now buying it back at $110 – if you have gained $10 then the 'other side of the contract' must have lost $10. It also follows that a

[2]A more accurate notation for the September-futures price would be $F_{t,T}$ where today is time $t = 0$ (15 June) and the contract matures at time T (say, 15 September), so $F_{0,T} = \$100$. As we move through time (e.g. $t = 1$ month later = 15 July) then the 'new' quoted futures price for the September futures on 15 July is $F_{1,T} = \$120$ (for September delivery). However, we use the simpler notation F_0 and F_1.

speculator would earn a \$10 profit if she initially sold a contract at F_0 = \$100 and later closed out the contract by *buying it back* at a lower price of F_1 = \$90 (i.e. 'sell high, buy low'). The different types of futures contracts that can be traded is almost limitless but only those which are useful for hedging and speculation will continue to be traded. The exchange will cease to trade any futures contracts where the trading volume is below a certain threshold.

1.2 OPTIONS

Options are a little more difficult to understand than forwards and futures and here we present a quick introductory overview. While futures markets in commodities have existed since the middle of the 1800s, options contracts have been traded for a shorter period of time. There are two types of option, calls and puts:

> *The holder of a call (put) option has the right (but not an obligation) to buy (sell) the 'underlying asset' at some time in the future ('maturity date') at a known fixed price (the 'strike price', K) but she does not have to exercise this right.*

Table 1.3 provides a summary of several types of option contract and the assets underlying these contracts.

TABLE 1.3 Selected option contracts

Options contract	Exchange	Contract size
1. Individual stocks (Stock options)	CBOE, NASDAQ PHLX, NYSE-Euronext	Usually for delivery of 100 stocks
2. Stock indexes (Index options)		
S&P 500 (SPX)	CBOE	\$100 × Index
FTSE 100	NYSE-Euronext	£10 × Index
S&P 100 Index (OEX)	CBOE	\$100 × Index
3. Foreign currency		
British pound	NASDAQ PHLX	£31,250
Japanese yen	NASDAQ PHLX	¥6.25m
Canadian dollar	NASDAQ PHLX	C\$50,000
Swiss franc	NASDAQ PHLX	SFr62,500
4. Futures Options		
S&P 500 Futures Index	CME	
US T-bill Futures	CME Group (IMM)	
US T-bond Futures	CME Group (IMM)	

Note: CBOE = Chicago Board Options Exchange, CME = Chicago Mercantile Exchange, IMM = International Money Market (Chicago, part of CME Group), NYSE-Euronext (previously London International Financial Futures Exchange, LIFFE). PHLX = Philadelphia Stock Exchange (part of NASDAQ).

For the moment we consider stock option contracts, so the underlying asset in the option contract is the stock of a particular company-XYZ which is traded on the NYSE. The option contract itself, we assume is traded in Chicago.

Above we noted that the holder of a long *futures* contract on a stock-XYZ commits herself to buy the stock at a certain price at a certain time in the future and if she does nothing before the maturity date, she will *have to* take delivery of stock-XYZ, at the pre-agreed futures price. In contrast, the holder of a (European) 'call option' on stock-XYZ can *decide* whether to pay the known strike price and take delivery of stock-XYZ on the maturity date of the option contract – this is called 'exercising the option contract by taking delivery'. If it is advantageous *not* to exercise the option (in Chicago) then the holder of the call option will simply do nothing. For the privilege of being able to decide whether or not to take delivery of stock-XYZ (at maturity of the option contract) the buyer of the call option must pay an upfront, non-refundable fee – the option price (or premium).

The holder of a *European* call option can 'exercise the option', that is, take delivery and buy the stock-XYZ in Chicago for K – only on the maturity date (expiration date) of the option contract. But the holder of an *American* call option can 'exercise the option' contract in Chicago on any day (up to and including the maturity date).[3] Below, we deal only with European options.

Note, however, that any option contracts you hold (whether American or European) can be *sold t*o a third party, at any time prior to expiration – this is known as trading and allows *closing out* the option contract. If you 'close out' your call (or put) option contract then delivery (to you) of the underlying asset in the option contract, will not take place.

1.2.1 Call Options

If today you buy a *European call option* and pay the *call premium/price*, then this gives you the right (but not an obligation):

- to purchase the underlying asset
- at a designated delivery point
- on a specified future date (known as the *expiration or maturity date*)
- for a fixed known price (the *exercise or strike price*)
- and in an amount (*contract size*) which is fixed in advance.

For the moment, think of a call option as a 'piece of paper' that contains the contract details (e.g. strike price, maturity date, amount, delivery point, type of underlying asset). You can purchase this contract today in the options market in Chicago if you pay the quoted call premium. There are always two sides to every trade – a buyer and a seller – but we will concentrate on your trade, as a buyer of the option. Note that all transactions in the option contract

[3]There are two basic types of options: calls and puts (which can either be American or European).

TABLE 1.4 Leverage from options

	Chicago (options market)	NYSE (cash market)
15 July	Call premium, C = \$3 Strike price, K = \$80	Spot price, S_0 = \$80
25 October	Payoff = \$8 = (\$88 – \$80) Profit = \$5 = (\$8 – \$3) Returns = \$5/\$3 = 167%	Spot price, S_T = \$88 Profit = \$8 = (\$88 – \$80) Return = \$8/\$80 = 10%

are undertaken in Chicago but the underlying asset, for example a stock, is traded on another exchange (e.g. NYSE).

Suppose the current price of stock-XYZ on the NYSE on 15 July is $S_0 = \$80$. On 15 July you can pay the call premium $C = \$3$ and buy (in Chicago) an October-European call option on the stock-XYZ. The strike price in the contract is $K = \$80$,[4] and the expiry date T is in just over 3 months' time on 25 October. Because the maturity of the call is in October, and the strike is $K = \$80$, it is known as the 'October-80 call' (Table 1.4). Assume each call option is for delivery of one stock of XYZ.

1.2.2 Long Call: Speculation

How might a speculator use this call option contract? As we shall see, the speculator will buy the call option if she thinks stock prices will increase (sufficiently) in the future and end up above the strike price, K (on the option's maturity date). If stock prices do increase (sufficiently) then the speculator will make a profit when she exercises the call option (on 25 October, its maturity date).

For example, if the stock price on 25 October (on the NYSE) turns out to be $S_T = \$88$, then the holder of the call option can 'present' (i.e. exercise) the option contract in Chicago on 25 October (the maturity date of the option), pay the strike price $K = \$80$ and receive one stock. This is exercising the option by taking delivery. She could then immediately sell the stock on the NYSE for $S_T = \$88$, making a cash profit on 25 October equal to $S_T - K = \$88 - \$80 = \$8$. Alternatively, the long call option can be 'cash settled' for $S_T - K = \$8$ which is paid via the clearing house in Chicago (and no stock is delivered). In either of these scenarios (i.e. delivery or cash settlement) the option's speculator has made \$8 on an initial outlay of 'own funds' of $C = \$3$, which is a percentage return of $[(8 - 3)/3 \times 100\%] = 167\%$ (over a 3-month period).

Had the speculator bought the stock itself (with her 'own funds') for \$80 and then sold at $S_T = \$88$, she would have made a percentage return of 10% (i.e. \$8 on an initial outlay of \$80).

[4]When the strike price chosen for the option ($K = \$80$) is the same as the *current* stock price ($S_0 = 80$), then the option is said to be 'at-the-money' (ATM). If $S_0 > K$, the (call) option is currently said to be 'in-the-money' (ITM), and if $S_0 < K$ the option is said to be 'out-of-the-money' (OTM).

The much larger *percentage* return when using the call option arises because you can purchase the option for the relatively small payment of \$3, whereas the stock costs you \$80. The higher *percentage* return from the option (relative to the percentage return from buying the stock with your 'own funds') is called leverage – here, a 10% increase in the stock price gives rise to a 167% return on the option strategy.

If the stock price on 25 October turns out to be $S_T = \$75$ which is less than the strike price $K = \$80$ then the option is not worth exercising – after all, why pay $K = \$80$ for delivery of stock-XYZ in Chicago, when XYZ is only worth $S_T = \$75$ on the NYSE. In this case the option on 25 October is worth zero and the speculator 'throws it away' (i.e. does not present/exercise the option in Chicago). Note, however, that no matter how low the stock price turns out to be on 25 October, the maximum amount the option's speculator can lose is known in advance and is equal to the call premium $C = \$3$.

So a speculator who buys a call option has some rather nice advantages – she can benefit substantially from any upside in the stock market but can never lose more than the (rather small) option premium of \$3 she initially paid, even if stock prices fall to zero. Contrast this with buying the stock on 15 July for $S_0 = \$80$ on the NYSE – this might lead to a maximum loss of \$80, if company-XYZ entered bankruptcy before 25 October.

1.2.3 Closing Out

When a speculator buys a call option she can make a profit if the stock price increases at any time before the maturity date of the option. She does this by selling (shorting) the call option to another options trader, after the stock price has increased – this is called 'closing out' (or 'reversing') her initial long position in the option. The speculator is able to make a profit because when stock-XYZ increases in price on the NYSE then this results in a rise in the call premium (on XYZ) in Chicago. For example, if stock-XYZ increases in price by \$2 over one day then the price of the call option (quoted in Chicago) may increase from say \$3 to $C_1 = \$4$, over one day. Hence the speculator who purchased the October-call for \$3 on 15 July, can now sell the call in Chicago on 16 July (to another options trader) for \$4. The speculator actually receives her \$4 from closing out the contract, via the options clearing house in Chicago. She therefore makes a speculative profit of \$1 (= \$4 − \$3), the difference between the buying and selling price of the call – a return of 33% (= \$1/\$3) over one day.

Conversely, after the speculator purchased the October-call for $C_0 = \$3$, on 15 July, if the stock price falls by \$1 (say) on the NYSE, then the October-call premium will fall to \$2.2 (say) and when she sells it to another trader (i.e. closes out) in Chicago, the options speculator will make a loss of \$0.8 on the deal (but she will never lose more than the initial option premium of \$3). Thus a *naked (or open) position* in a long call is risky.

1.2.4 Put Options

If you *buy* a European put option (in Chicago) this gives you the right *to sell* the underlying asset (in Chicago) at some time in the future, for a price which is fixed in the contract.

If today you buy a *European put option* and pay the *put premium/price,* then this gives you the right (but not an obligation):

- to sell the underlying asset at a
- specified future date (*the expiration/maturity date*) at a
- designated delivery point
- for a known fixed price *(strike/exercise price)*
- in an amount *(contract size)* which is fixed in advance.

1.2.5 Long Put: Speculation

A put option can be used for speculation. In contrast to speculation with a long call, a speculator will buy ('go long') a put option if she expects the stock price *to fall* in the future (and end up below the strike price). Suppose on 15 July the stock price for company-XYZ is $S_0 = \$72$ on the NYSE and a speculator thinks the stock price will fall in the future. Assume the speculator is not going to gamble on the price fall by using stock-XYZ, itself. Instead, on 15 July the speculator buys (in Chicago) a European 'October-put' (expiry date is 25 October) with a strike price $K = \$70$ for a put premium paid of $P = \$2.2$.

Suppose the spot price of stock-XYZ on the NYSE on 25 October is $S_T = \$65$ (i.e. below $K = \$70$). Then on 25 October the speculator can purchase stock-XYZ on the NYSE for $S_T = \$65$ and then immediately 'deliver' stock-XYZ in Chicago along with the put contract and receive $K = \$70$, given the terms in the put contract. The options speculator makes a profit of $(K - S_T) = \$70 - \$65 = \$5$ on 25 October. Alternatively, the speculator can simply 'cash settle' the long put contract, in which case the options clearing house makes a cash payment of $(K - S_T) = \$5$ when the long put is 'cash settled' on 25 October (and the holder of the long put does not have to deliver stock-XYZ).

In either of the above cases the speculator who is long the put, receives \$5 on 25 October. The speculator's initial outlay was the put premium of $P = \$2.2$ paid on 15 July. The speculator by using the (long) put contract to speculate on a future stock price fall, has made a percentage return of 127% (= [(5 – 2.2)/2.2]100%), over a 3-month period. The fall in the stock price is 9.7% $[= (S_T - S_0)/S_0 = (65 - 72)/72]$. But the *percentage* return from investing in the long put (and exercising the put at maturity) is much larger at 127%. Hence, buying the put option for $P = \$2.2$ has provided a leveraged return for the speculator.

If the spot price of stock-XYZ on the NYSE on 25 October turns out to be higher than the strike price (e.g. $S_T = \$73$, $K = \$70$) then the put option will not be exercised. Why would a speculator who holds the put, buy the stock for \$73 on the NYSE on 25 October, if she could then only obtain $K = \$70$ by delivering the stock and exercising her put contract in Chicago? The put option is therefore worth zero on 25 October and the speculator 'throws it away' (i.e. does not exercise the option). But the most the speculator with a long put can lose is the put premium of $P = \$2.2$. So a speculator who is long a put option can benefit from a sufficiently large *fall* in the price of stock-XYZ – but if she guesses wrong and the stock price rises, she can never lose more than the (rather small) put premium initially paid.

1.2.6 Long Put plus Stock: Insurance

Options can also be used to provide *insurance*. For example, suppose you run a pension fund and *already own* stocks whose current price on 15 July (on NYSE) is $S_0 = \$72$. But you are worried about a fall in price of the stocks between now and 25 October when your stocks will be sold to provide lump sum payments to pensioners. Well, you can 'insure' your stocks by buying an October-put option with a strike price of, say, $K = \$70$ (with maturity date 25 October). Note that in this example you hold two assets: the stock-XYZ and a put option (on stock-XYZ).

If stock prices in New York fall to $S_T = \$30$ on 25 October, then instead of selling your stocks in New York at $S_T = \$30$, you can exercise your October-put option in Chicago, which means delivering your stock-XYZ in Chicago and you will receive $K = \$70$ for each stock (from the options clearing house). By buying the put option on 15 July, you have guaranteed a minimum price of $K = \$70$ on 25 October at which you can sell the stocks-XYZ, held by the pension fund. The cost of this 'insurance' is the put premium $P = \$2.2$ paid on 15 July. True, the pension fund has lost \$2 per stock as the initial price of the stock was $S_0 = \$72$ in July since the pension fund can only obtain $K = \$70$ when they deliver the stock and exercise the put option in Chicago – the \$2 is the 'deductible' in the put insurance contract.[5] Losing \$2 per stock because you had the foresight to take out insurance by buying a put option (with $K = \$70$), is a lot better than if you had not purchased the put, since then your stocks-XYZ would have fallen in value by \$42 (= 72 − 30) on the NYSE.

How does this 'options insurance contract' look if prices rise over the next 3 months? Suppose prices rise on the NYSE from $S_0 = \$72$ in July to $S_T = \$80$ in October. Then your long put is worthless as the pension fund would not 'deliver' (sell) its stocks in Chicago for $K = \$70$ (using the stocks and the put) when it can sell its stocks in New York for $S_T = \$80$. Indeed the pension fund will 'throw away' (i.e. not exercise) the put option in Chicago and will sell stocks-XYZ on the NYSE for \$80 – so the pension fund (and the new pensioners) will be very happy.

The insurance policy provided by the 'stock+put' has allowed the pension fund to fix a *minimum selling price* of $K = \$70$ at which it can sell its stock holdings on 25 October, by exercising the October-put contract in Chicago. No matter how low the stock price on the NYSE on 25 October, the pension fund – by exercising the put option – can secure a minimum price of $K = \$70$. But the 'stock+put' also allows the pension fund to benefit from any 'upside' if stock prices rise, because then the pension fund simply sells its stocks-XYZ on the NYSE at the high price. There are many types of situation that can be analysed using an options approach and some of these are discussed in Finance Blog 1.1.

[5]Given the current stock price $S_0 = \$72$, if you had purchased a put with a strike price of $K = \$72$ (i.e. an ATM put) this would guarantee a minimum selling price of \$72 on 25 October – hence you would not lose anything (other than the put premium). But a put with $K = \$72$ would cost you more than the $P = \$2$ you paid for the put with a strike of $K = \$70$ (see later chapters).

Finance Blog 1.1 Hidden Options

Aristotle in Book I of *Politics,* mentions the Greek philosopher Thales who developed a 'financial device' which was in fact an option. One winter he 'read the stars' and decided that next autumn would result in an exceptionally good olive harvest. He therefore quietly went around the owners of olive presses and paid them a small retainer (i.e. the call option premium) to secure the right to be first to use their olive presses in the autumn, for a fixed price (the strike price), if he so wished. Come autumn, the harvest was good and therefore the demand for the olive presses was high and Thales could charge a high price to the olive growers to let them use the olive presses, but Thales only paid the lower strike price to the owners of the olive presses. Even if Thales had been wrong about the harvest, the most he could have lost was the small option premium he initially paid to the owners of the olive presses.

Although some people may not be aware of it, they probably hold options. For example, consider rural bus services whose fares are often subsidised via local government taxes (e.g. sales taxes and community charges). If you live out of town, you have the option to take the bus into town by paying the known fixed fare (= strike price). You will do this if the value of your journey on that day by bus exceeds the fixed fare (strike price). Hence, if you live out of town you are holding an implicit call option and the call premium is that part of your local taxes that goes to subsidise the bus company. You may never use the bus but the option to use the bus (e.g. if your car breaks down) has a positive value to you and hence you may be willing to see the rural bus service subsidised by local taxes.

Next, suppose in January you have been offered a place at one of *several* universities, if you achieve a grade B (or above) in your examinations in June. You will make your final decision about attending a specific university or not, in September. The (implicit) option premium you pay is the time and effort you put into studying between January and June. You have nine months to decide on your choice of 'the best' university for you (i.e. the time to maturity of the option), which is conditional on getting appropriate grade B or above in the June exams.

In September, if you decide to go to a university, you will have to 'pay' the strike price (i.e. tuition and living expenses and income foregone while attending the course). In September you will choose that university with the largest net payoff $S_T - K > 0$ where S_T is the (expected) present value of your *additional* earnings after graduating from a particular university. If $S_T - K > 0$ then you will 'take delivery' of one of the university courses, so the option you have is a 'call option'. Of course if $S_T < K$ then you will choose not to go to university (and instead look for a job) – that is, you will not exercise your 'call option', as your extra post-university earnings do not cover the costs of attending university.

The above is a type of *exotic option* since in September you are allowed to choose that university which maximises the possible payoffs $S_T - K$ across the *different* universities who

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have made you an offer. Because you can choose that university with the highest payoff (i.e. the largest value of $S_T - K$) then this is known as a *rainbow option (or min-max option)*.

Suppose it is January so you do not yet have your June examination results. You are holding a form of exotic option known as a *barrier option*. If your intellectual prowess increases and you achieve the grade B in June, then you cross the exam barrier and the option 'knocks in' (i.e. comes 'alive') and can be exercised later (in September), if you wish.

To be even more precise you are holding a *knock-in, rainbow call option*. If you do not achieve the grade B in June, then all your university offers are void and your implicit option expires worthless in June. All that effort spent studying (i.e. the implicit option premium) will not help you get into any of your chosen 'top' universities in September. The option you hold is a *path dependent rainbow barrier option,* since the final payoff in September (i.e. your choice of the 'best' university) depends on whether you achieved at least a B-grade at an earlier date on your path to success (i.e. in your examinations in June).

Source: Adapted from Cuthbertson and Nitzsche (2001).

1.3 SWAPS

Swaps are another type of derivative contract which first appeared in the early 1980s. They are primarily used for hedging interest rate or exchange rate risk *over many future periods*.

> *A swap is a negotiated (OTC) agreement between two parties to exchange cash flows over a series of pre-specified future dates ('reset dates').*

A plain vanilla interest rate swap involves a periodic exchange of interest payments. One set of future interest payments are at a fixed swap rate, $sp_0 = 3\%$ p.a. (say), which is determined when the swap is initiated. The other set of interest payments are determined by the prevailing level of some 'floating' interest rate (usually LIBOR). The swap will be based on a notional principal amount of \$100m, say.

For example, in July 2018 a US firm 'BigBurger' might have a swap-deal with JPMorgan where BigBurger has agreed to receive annual interest payments from the swap dealer based on (USD) LIBOR rates on 15 July 2019 and on 15 July 2020 (the reset dates). BigBurger also agrees to pay the swap dealer (JPMorgan) a *fixed* swap rate of $sp_0 = 3\%$ p.a., on these dates (on a notional principal amount of \$100m). BigBurger is a 'floating-rate receiver' and a 'fixed-rate payer' in the swap. The payments are based on a \$100m (notional) principal amount, but only the interest payments are exchanged (and not the \$100m principal itself). The maturity of the swap, the reset dates, notional principal, the fixed swap rate and the type of floating rate (usually LIBOR) to be used in the swap deal are set at the outset of the contract.

The agreed swap rate is $sp_0 = 3\%$ p.a. Suppose LIBOR rates turn out to be $\text{LIBOR}_1 = 5\%$ on 15 July 2019 and $\text{LIBOR}_2 = 2\%$ on 15 July 2020. Then on 15 July 2019 the swap dealer

JPMorgan owes BigBurger, \$5m in interest based on $\text{LIBOR}_1 = 5\%$ and BigBurger owes JPMorgan (the swap dealer) \$3m based on the fixed swap rate of $sp_0 = 3\%$, hence:

$$\textit{Swap dealer's payoff to BigBurger} = \$100\text{m}\ (LIBOR_1 - sp_0) = \$2\text{m}$$

On 15 July 2020 the swap dealer owes BigBurger \$2m based on the out-turn $\text{LIBOR}_2 = 2\%$ and BigBurger owes the swap dealer \$3m (based on $sp_0 = 3\%$). So on 15 July 2020, BigBurger *pays the swap dealer* \$1m:

$$\textit{Swap dealer's payoff to BigBurger} = \$100\text{m}\ (LIBOR_2 - sp_0) = -\$1\text{m}$$

The negative sign indicates that it is actually BigBurger who pays \$1m to the swap dealer (JPMorgan).[6]

Suppose BigBurger has a bank loan of \$100m (say) with Citibank with interest payments (each year) based on future values of LIBOR. If BigBurger, in July 2018, also has a 'receive-fixed, pay-float (LIBOR)' interest rate swap (with JPMorgan), at a fixed swap rate of $sp_0 = 3\%$ p.a. (say) then the 'effective cost' of the bank loan to BigBurger at any future reset date, T, is:

$$\begin{aligned}\textit{Effective cost} &= \textit{Loan interest} - \textit{Payoff to firm-A from swap}\\ &= \$100\text{m}\ LIBOR_T - \$100\text{m}\ (LIBOR - sp_0) = \$100\text{m}\ sp_0\end{aligned}$$

Hence, the net effect is that BigBurger pays the known fixed swap rate of $sp_0 = 3\%$ p.a., regardless of whether the out-turn value for LIBOR is low at 2% p.a. or high at 5% p.a. Of course, with the swap in place, this means that BigBurger cannot take advantage of low LIBOR loan rates in the future, should they occur. On the other hand, BigBurger only has to pay an effective interest rate on the loan of $sp_0 = 3\%$ p.a., even if LIBOR turns out to be high at 5% p.a. So in July 2018 if BigBurger really does want to 'lock in' an effective loan rate equal to $sp_0 = 3\%$ p.a. (on the next two loan interest rate reset dates) then it will take out a 'receive fixed-pay floating (LIBOR)' interest rate swap with JPMorgan.

The intermediaries in a swap transaction are usually large investment banks who act as swap dealers. They are usually members of the International Swaps and Derivatives Association (ISDA) who provide some standardisation in swap agreements via the *master swap agreement,* which can then be adapted where necessary to accommodate most customer requirements. Dealers make profits via the bid–ask spread (on the fixed leg of the swap) and might also charge a small brokerage fee for setting up the swap.

[6] As we see in later chapters this swap is just like two Floating Rate Agreements (FRAs). One FRA applies to the LIBOR rate on 15 July 2019 and the second FRA applies to the LIBOR rate on 15 July 2020 and both FRAs have the same 'fixed' FRA rate, equal to the swap rate of 3% p.a. Therefore the interest rate swap is analytically equivalent to two FRAs which mature on 15 July 2019 and the second on 15 July 2020.

1.4 HEDGING, SPECULATION, AND ARBITRAGE

Part of the reason for the success of both futures and options is that they provide opportunities for hedging, speculation, and arbitrage.

1.4.1 Hedgers

Examples of hedging using the *forward market* in foreign exchange are perhaps most common to the lay person. If a US exporter expects to receive £3,000 in 3 months, then the US exporter can buy dollars today in the forward market at the 3-month forward FX-rate, $F = 1.5$ (\$/£). The key feature is that today, the US company fixes the amount of USD it will receive at \$4,500, in exchange for the £3,000 it provides, in 3 months' time.

Futures contracts *if held to maturity,* are like forward contracts – they fix the price that the hedger will pay or receive at maturity of the futures contract. However, it can be shown that even if the futures contract is closed out before maturity much of the risk can be hedged, but a small amount does remain (this is known as *basis risk*).

Options contracts provide 'insurance'. Investors in options can protect themselves against adverse price movements in the future but they still retain the possibility of benefiting from any favourable price movements. To obtain this insurance, the option's purchaser ('the long') of either a call or a put has to pay the option premium, today.

For example, a US exporter to the UK can 'insure' (i.e. set a lower limit for) her future US dollar receipts in 3 months' time, if today she buys a put option on sterling at a strike price of $K = 2$ (\$/£, USD/GBP), which matures in 3 months. Suppose the put option is for 'delivery' of £3,000. The put option implies she will receive a minimum of $K = 2$ USD/GBP by exercising her put in Chicago in 3 months' time – so the minimum she will receive from exercising the put is \$6,000 (even if the quoted spot-FX rate in 3 months' time is $S_T = 1.5$ USD/GBP, say). But if in 3 months' time, the spot exchange rate is $S_T = 2.1$ USD/GBP, she can 'walk away' from the put option contract (i.e. not exercise the put) and exchange her £3,000 at the higher spot rate (and receive \$6,300 from the spot FX-dealer). For the privilege of having this 'option' to choose the best outcome in the future, she has to pay the put premium, at the outset.

1.4.2 Speculators and Leverage

We have seen that because the call option premium is small relative to the price of the underlying asset, then speculation with calls can provide a high percentage return on the 'own capital' used to purchase the option. In our above examples, buying a call option on stocks gave a return of 167%, whereas buying the stock itself only produced a return of 10% – options therefore provide leverage.

Leverage also applies to futures contracts because a speculator does not have to provide any of her own funds. Suppose on 25 January you ring up Chicago and buy a 'June-futures' contract (on stocks-ABC) at a price of $F_0 = \$90$. Assume the futures matures on 25 June. In January you do not pay any money – here we ignore (so-called) margin requirements which are small and earn a competitive interest rate, and therefore are not a cost. Suppose on 15 March, you close out your June-futures contract in Chicago by selling at the market price of $F_1 = \$100$. You make a cash profit of $F_1 - F_0 = \$10$, on 15 March. Because the futures trade (buying then selling) does not require any 'own funds', the *percentage* return and hence leverage is infinite.

By using futures, speculators can make very large losses as well as very large gains. However, there is a difference between futures and options. In the case of futures the potential loss or gain can be very large. But when call or put options are purchased by speculators, the speculator's loss is limited to the option premium, yet the upside can be very large.

1.4.3 Arbitrageurs

Arbitrage involves 'locking in' a riskless profit by entering into transactions in two or more markets simultaneously. Usually 'arbitrage' implies that the investor does not use any of his own capital when making the trades. Arbitrage plays a very important role in the determination of both futures and options prices as we shall see in later chapters. Arbitrage is often loosely referred to as the 'law of one price' for financial assets. Simply expressed, this implies that identical assets must sell for the same price. We consider a very simple example of arbitrage in Finance Blog 1.2.

Finance Blog 1.2 Arbitrage: Dolly the Sheep

By way of an analogy consider 'Dolly' the sheep. You will remember that Dolly was cloned by scientists at Edinburgh University and was an exact replica of a 'real' Highland sheep. Dolly and real Highland sheep are identical and indistinguishable. Dolly is a form of genetic engineering or 'synthetic' or 'replication' sheep. Assume we could engineer 'Dolly' in Edinburgh at a cost (in terms of wages and equipment) of £200 per Dolly. So you can purchase a Dolly in Edinburgh for £200.

Suppose the current market price of real sheep being sold in the local market in the Highlands is £220. 'Sheep arbitrageurs' sitting in London seeing these prices on their internet screens would (ethical issues aside) buy a 'replication' Dolly in Edinburgh for £200 and simultaneously they would sell Dolly in the local market in the Highlands for £220 making an arbitrage (risk-free) profit of £20. (We ignore any transportation costs of getting Dolly from Edinburgh to the Highlands.)

This increase in demand for a Dolly in Edinburgh and sale of a Dolly in the Highlands (by many arbitrageurs in London doing the same trades) would mean the price of a Dolly would be bid up in Edinburgh and the price of a Dolly in the Highlands would fall – this is 'supply and demand' at work. Arbitrage and 'supply and demand' would ensure that the price of a Dolly in Edinburgh and the Highlands would quickly move to equality at

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say £210 for each – at which point arbitrage activity would stop – since there are now no risk-free profits to be made. Economists would say that £210 is the current *equilibrium price* of a Dolly.

Arbitrage ensures that the price of real Highland sheep must eventually equal the cost of producing an identical 'Dolly replicant'. Dolly is like a 'synthetic' or 'replication' portfolio in financial economics. As we shall see, this is how we price many derivatives – we create a 'synthetic' or 'replication' portfolio, which has identical payoffs to the derivative itself.

Larry Summers, a prominent US economist (and previous US Secretary of the Treasury) rather impishly characterised the difference between economists and traditional finance specialists with the following analogy. He said that economists are interested in why, for example, the price of a bottle of ketchup moves up and down (e.g. because of changes in incomes, relative prices, innovation in production processes, etc.), while finance specialists are only interested in whether a 16 oz bottle of ketchup sells for the same price as two 8 oz bottles. He's only half right.

1.5 SHORT-SELLING

If an investor purchases a security (e.g. stocks) she is said to *go long* and if she sells a security *she owns*, she is said to *go short*. However, if she sells a security *that she does not own* this is known as short-selling. Hedging may involve short-selling, so we outline the main features here.

Suppose a speculator (Ms Short) thinks a particular stock will fall in price in the future but she does not own the stock. She may be able to make a profit by short-selling. Initially, she borrows the stock from her *prime broker* (e.g. Goldman's) for an agreed time period. The prime broker may already hold the stock on behalf of another client in a custodial account or the broker has to 'locate' the stock, which may be borrowed from another bank (JPMorgan) or fund management company (Fidelity) or pension fund (Legal and General [L&G]), who hold stocks 'on behalf of their customers' in a custodial account. Suppose Ms Short sells Apple stock (which her broker has borrowed from L&G) for \$100 (and the stock is purchased by AXA insurance company). If the price falls over the next month to say \$90, then she can repurchase Apple stock in the market at \$90, return the stock to her broker, thus pocketing the difference of \$10.

If the Apple stock pays dividends over the period of the short-sale[7] then Ms Short has to pay an equivalent cash amount to her broker (which is then passed on to L&G). Of course,

[7]This is referred to as the stock going 'ex-dividend' which means that AXA Insurance Company (which currently owns the stock) will receive the next dividend payment. But L&G, the initial owner of the stock is expecting to receive the dividend. Hence, Ms Short has to provide the dividend for L&G from her own funds.

if the stock price rises and Ms Short has to close out, then she will make a loss (which can increase without limit). In the US you can only short-sell on an 'uptick' (i.e. only if the last change in price was positive).

Short-selling is risky for the prime broker (Goldman's), as in the future Ms Short may not 'replace' the stocks she has borrowed and sold in the open market. So the broker requires the cash proceeds from the short-sale to be held at the brokerage firm (e.g. Goldman's)[8] and will also require an additional margin payment (of say 50% of the value of the short-sale) as further 'collateral' (i.e. a 'good faith' deposit).[9] Further margin calls may be made if the stock price subsequently rises. However, if the stock price subsequently falls, Ms Short's short position is worth more and then she may be allowed to withdraw any surplus cash from her margin account.

The calculation of the (percent) rate of return from short-selling is often based on the initial receipts from the sale of the stocks.[10] For example, suppose Ms Short short-sells 100 Apple stocks at \$2 per stock and buys them back later at \$1.50. Assume the \$200 proceeds from the short-sale cannot be used by Ms Short (and are held as collateral by the broker, who usually pays interest on these funds).

Assume the dividend yield on the stocks is $d = 5\%$ and these dividends accrue over the period Ms Short has short-sold the stocks. Any dividend payments (over the period Ms Short borrows the stocks) must be paid to L&G (via her prime broker's account) and are equal to \$10 ($= 5\% \times \$200$). If we ignore the interest and commission costs of short-selling then the return on the short sale is:

$$\begin{aligned}\text{Return} &= \frac{\text{Net profit}}{\text{Proceeds from short-sale}} = \frac{\$200 - \$150 - \$10}{\$200} \\ &= \frac{\$40}{\$200} = 0.20\ (20\%) \qquad (1.1)\end{aligned}$$

$$\begin{aligned}\text{Return short-sale} &= \%\text{Price fall} - \%\text{Dividend yield} = (P_0 - P_1)/P_0 - d = 25\% - 5\% = 20\% \\ &= -(\%\text{Price rise} + \%\text{Dividend yield})\end{aligned}$$

Hence, the (simplified) return to the short-seller is simply the return to a *purchaser* of a stock, but with the sign reversed. Finally, note that the broker usually takes a small

[8]If Ms Short deposits the proceeds of the short-sale with her broker as cash, then the broker usually pays interest (at LIBOR minus a spread) on these funds, which is often referred to as 'short interest rebate'.

[9]If the short-seller borrows funds (from a bank) to provide the margin payment, then the short will incur a 'margin cost' at LIBOR+spread. If the margin payment consists of other securities owned by Ms Short (e.g. T-bills) then she will earn the risk-free T-bill rate on this collateral, so it is not a 'cost' to Ms Short – although her broker may apply a 'haircut' and only allow 90% of the collateral to earn interest.

[10]An alternative is to use 'own funds' in the form of any collateral, lodged with the broker by the short-seller, as the denominator when measuring the return. We do not pursue this here.

(percentage) commission for organising the short-sale, sometimes referred to colloquially as a 'haircut' and this should also be deducted when calculating the above return.[11]

1.6 SUMMARY

- In a *forward contract,* the forward price is agreed today, for delivery of the underlying asset in the contract (e.g. foreign exchange, stocks-XYZ, barrels of oil), at a specific place and specific date in the future (maturity date). Forward contracts are usually held to maturity, when delivery of the underlying asset takes place.
- *Forward contracts* are over-the-counter (OTC) agreements whose terms (e.g. 'size', delivery date, delivery point and forward price) are negotiated between two counterparties, today.
- *Futures contracts* if held to maturity are very similar to forward contracts. However, futures contracts are standardised (e.g. fixed contract size, delivery dates, etc.), are traded on exchanges and the futures price fluctuates and is observable continuously on the exchange (or electronic platform).
- A buyer or seller of a futures contract has to immediately provide a 'good faith deposit' which is known as the initial margin (on which interest is usually paid). The initial margin is not the 'price' of the futures contract.
- Most futures contracts are closed out before maturity. The profit from an initial long (buy) futures position is the difference between the final selling price (when the contract is closed out) and the initial price paid. The profit from an initial short (sell) futures position is the difference between the selling price and the price at which you 'buy-back' the contract. Futures contracts are 'contracts for differences'.
- Futures contracts can be used for hedging or speculation. They only require a small upfront amount of capital placed with the clearing house known as the 'margin payment'. Hence, for a speculator futures contracts provide 'leverage'.
- *Options contracts* can be used to provide 'insurance'. If you *plan* to purchase stocks in the future and are worried their price might rise, then by purchasing a call option today you can fix the *maximum* price at K (the strike price) you will have to pay for the stocks in the future. But you can also take advantage of low stock prices should they occur, by

[11] Often speculators take a long-short position on different stocks. The total return involved to a *long-short* position involves several payments and receipts on both the long (buy) and short (sell) side of the trade. These elements combined give the total return on the long-short position which comprises: Total return = Price return to the long position + Price return to short position + Dividend income from the long position + Interest earned on any excess cash in brokerage account + 'Short' interest rebate – Commission costs ('haircut') of borrowing shares (for the short sale) – Dividend payments on the short position – Margin (interest) costs on collateral borrowed to finance the short position – Interest cost of any borrowed funds to leverage a long position.

not exercising the call option and instead purchasing the stocks at the low cash-market price on the NYSE.

- If you *already own stocks* and you are worried that their price might fall in the future, you can buy a put option which will fix the *minimum* price (= K) you will receive when you deliver your stocks in the futures contract and receive the strike price K at maturity of the put contract. On the other hand, if stock prices rise (above K) you do not exercise the put option but sell your stock at the high cash-market price on the NYSE.
- Options can be used for speculation without having to provide much capital (own funds) since you only pay a 'small' option premium (i.e. 'small', relative to the spot (cash-market) price of the underlying asset). So options provide *leverage*.
- Acting as a speculator, if you think stock prices will rise (fall) in the future you would buy a call (put) option, today. The most a speculator can lose when buying calls or puts are the respective option premia. But the potential upside for calls (puts) can be very large if there is a large rise (fall) in the stock price.
- An interest rate swap allows you to switch from paying an uncertain LIBOR interest rate in the future (e.g. on a bank loan) and instead to end up paying a known fixed interest rate (over the life of the swap contract).
- Short-selling stocks allows speculators to take advantage of a future fall in stock prices, even if they do not currently own the stock – they borrow the stock from their broker and sell it, hoping to buy it back at a lower price in the future.

EXERCISES

Question 1
Why are futures and options contracts generically referred to as 'derivatives'?

Question 2
You are a US exporter ('USam') who will receive €10m in 6 months' time from the sale of Barbie dolls in Euroland. How can you hedge your foreign exchange FX risk using a forward contract?

Question 3
As a speculator, how does going long a futures contract on a stock give you 'leverage' compared with using your own funds to buy the stock? Use $F_0 = \$101$ and $S_0 = \$100$, with out-turn values (3 months later) of $F_1 = \$111$ and $S_1 = \$110$.

Question 4
Under what circumstances would you make a profit at maturity T, from a long position in futures contract on 'hogs'? Assume the futures price is $F_0 = \$100$ (per hog) and at maturity of the futures contract, the spot (cash market) price of hogs is $S_T = 110$.

Question 5

You are a speculator and you think stock prices will increase. Should you buy a call or a put option?

Question 6

If $K = 150$ and the put premium is $P = 5$ should you exercise the put option if the spot price at expiration is $S_T = 148$? What is the payoff and the profit?

Question 7

In what way is a call option to marry Vito Corleone's daughter (Connie, in *Godfather I*) in one year's time different from a (1-year) futures contract? Assume the strike price is K and the current price of the futures is also equal to K. Assume both contracts are held to maturity.

Question 8

You have a (mortgage) loan for \$200,000 which has been in existence for 2 years and has a further 10 years to maturity. Interest on the loan is paid every year, at whatever the (one-year) interest rate is at that time (i.e. it is a floating rate loan at LIBOR). You took out this loan when interest rates were low but now you think interest rates will be permanently higher in the future. How can you use the swaps market to effectively give you a loan with a fixed interest rate over the next 10 years? (Assume it is an 'interest only' loan, so the principal of \$200,000 remains fixed and the latter is paid off using a 'lump sum' from your pension).

Question 9

A bank, BigMoney, raises deposit funds at LIBOR (currently 10% p.a.) in the interbank market and on-lends the funds in fixed interest loans at 11% p.a. What are the risks involved and how might the bank hedge the risk using swaps?

PART I

FORWARDS AND FUTURES

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 2

Futures Markets

Aims

- To examine trading arrangements for futures contracts, including delivery of the underlying asset, margin requirements and closing out.
- To interpret futures quotes, including the settlement price and open interest.
- To discuss the different types of futures traders.

We have already discussed the basic principles behind forward and futures contracts. Forward contracts are analytically easier to deal with than futures contracts and so we often apply mathematical results from forwards (e.g. pricing forward contracts) to futures contracts. However, there are differences in practice between the two types of contract and we discuss the mechanics of both of these contracts in this chapter.

2.1 TRADING ON FUTURES MARKETS

Forward contracts are traded over-the-counter (OTC) whereas most futures are traded on an exchange and the differences between these two approaches are summarised in Table 2.1.

The range of assets on which futures contracts are written is very wide and they are traded on a large number of exchanges around the world (Table 2.2). The basic requirements in buying and selling different types of futures contracts and using them for hedging and speculation are very similar, even though futures contracts are written on a diverse set of underlying assets (e.g. gold, oil, wheat, stocks, stock indices, T-bills, T-bonds, interest rates). The growth in futures contracts is primarily due to the increased volatility of the price of the underlying assets and the need to hedge this risk. Futures contracts which trade in a highly liquid market (with low bid–ask spreads and low commissions) are available on a wide variety of underlying

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

TABLE 2.1 Derivatives markets

Over-the-counter	Exchanges
– Supplied by intermediaries (i.e. banks) – Customised to suit buyer – Can be done for any amount, and settlement date – Credit risk of counterparty and expensive to unwind – Allows anonymity – important for large deals – New contracts do not need approval of regulator	– Traded on exchanges (e.g. NYSE-Euronext, CME) – Available for restrictive set of assets – Fixed contract sizes and settlement dates – Easy to reverse the position – Credit risk eliminated by clearing house margining system ('marking-to-market')

TABLE 2.2 Financial futures

Instruments	Exchanges
– Money market instruments (e.g. 3-month Eurodollar deposit, 90-day US T-bills, 3-month Sterling or Euro deposits) – Bonds (e.g. US T-bonds, German Bund, UK Gilts) – Stock indices (e.g. S&P 500, FTSE 100) – Currencies (e.g. euro, sterling, yen) – Mortgage pools (i.e. GNMA)	– Chicago Board Option Exchange (CBOE) – CME Group (includes CBOT, NYMEX) – Philadelphia Stock Exchange (PHLX) – NYSE-Euronext (formerly known as LIFFE) – Exchanges also located in Singapore, Hong Kong, Tokyo, Osaka, Sydney and many other financial centres

assets, whereas only the OTC forward market for foreign exchange rivals the volume of trading on futures markets.

A futures exchange is usually a corporate entity whose members elect a board of directors, who decide on the terms and conditions under which existing contracts are traded and whether to introduce new contracts (subject usually to the regulatory authority which in the US is the Commodity Futures Trading Commission [CFTC]).

Each futures contract specifies a delivery month. The quoted futures price varies continuously as the spot (cash market) price changes. Consider a futures contract where the underlying asset is the stock of AT&T. Today (15 May), a *buyer* of the September futures contract at a quoted price $F_0 = \$100$ is said to be *long the futures contract*. If she holds the contract to maturity (25 September) then she must purchase the *underlying asset* in the futures contract (i.e. AT&T stock) in Chicago (the delivery point) at a price of $F_0 = \$100$ on 15 September (maturity date), even if the spot price on the NYSE for AT&T stock is, say, $S_T = \$60$ or $S_T = \$120$.

So, if you hold the futures contract to maturity you must take delivery of the underlying asset and pay the futures price $F_0 = \$100$ in Chicago – you have no choice in the matter.

The writer or seller of the futures at a quoted price $F_0 = \$100$ is said to be *short the contract* and if she holds the contract to maturity, she must deliver and *sell* AT&T stock for the price of $F_0 = \$100$ in Chicago. Since each contract always has a buyer and a seller, then if the contracts are held to maturity there will always be a 'short' ready to deliver the underlying asset to the person holding the long position in the futures contract.

Financial futures are written on financial assets (e.g. stock indices, currencies, T-bills, T-bonds) whereas commodity futures are written on, say, wheat, silver, oil, etc. The underlying asset in a futures contract, for example the stock of AT&T, is traded at the 'spot price' in the cash market, which in this case is the NYSE. The futures is a *derivative security,* because changes in futures prices are closely linked to changes in the spot/cash market price (of the underlying asset).

Futures markets can be used for speculation, hedging, or arbitrage. Because there is always a counterparty to a futures contract (i.e. Ms A is long, Ms B is short) then any gains by Ms A are accompanied by losses for Ms B – overall it is a zero sum game (ignoring transactions costs). However, for any individual trader on one side of the market (i.e. either long or short) there are potential cash gains and losses to be made.

Since futures contracts (unlike forward contracts) are traded on an exchange, there needs to be some standardisation of the contracts. Also, to minimise default risk, a clearing house tracks all trades and requires buyers and sellers of futures contracts to post collateral (e.g. cash, T-bills), known as a margin payment. This 'good faith' payment is used to compensate a trader if another trader defaults on the contract.

2.1.1 Standardisation

The futures exchange sets the size of each contract, the units of price quotation, minimum price fluctuations, the 'grade' and place for delivery, any daily price limits and margin requirements as well as opening hours for trading. For agricultural commodities, the type or grade is also fixed in the futures contract. For example, for corn the standardised grade is 'No. 2 Yellow' but other grades can also be delivered – for example 'No 1. Yellow' is deliverable for 1.5 cents per bushel more than 'No. 2 Yellow' because the quality is higher. The futures exchange sets the minimum contract size (e.g. delivery of 5,000 bushels of corn), delivery dates (e.g. specific dates in March, May, June, July, September, and December) and delivery arrangements (e.g. delivery only to towns A, B, and C).

For futures on financial assets such standardisation is easier. For example, a foreign exchange (FX) futures contract on the pound sterling (GBP) is rather a homogenous product and only the delivery dates, settlement price, and contract size need to be organised by the exchange. Some futures contracts are traded with maturity dates of only up to a year or two ahead but some contracts have much longer maturity dates. It depends primarily on the demand for such contracts by hedgers. For example, the Eurodollar futures contract is actively traded with maturities out to 15 years or more because these contracts are used by swap dealers to hedge their interest rate swap positions.

The size of the contract is important. If too small, speculators will not trade the contract because the transactions costs per contract may be relatively high but if the 'size' is too large, then hedgers will not be able to hedge relatively small amounts (e.g. the Eurodollar futures has a contract size of \$1m and you cannot hedge \$500,000 by using half a contract). The *tick size* and *tick value* should be easily understood by market participants. For example, the US T-bill futures contract has a contract size of \$1m and a tick size of 1 basis point (i.e. one-hundredth of 1%) and a minimum price change (tick value) of \$25. Hence, if the futures price changes from $F = 99.00$ to $F = 99.01$, the value of one futures contract (on T-bills) changes by \$25.

As each contract has both a long and a short position, the total number of futures contracts outstanding either long or short, is called the *open interest*.

The way 'open interest' changes, assuming it starts at zero, is shown in Table 2.3. When trader-A (buyer) and trader-B (seller) first enter the market on day-1, then open interest increases by one (i.e. one long and one short). Similarly, for trader-C and trader-D, who increase open interest by three when they buy (= C) and sell (= D) three contracts on day-2.

TABLE 2.3 Open interest

Trading	Open interest
1. Trader-A buys 1 contract and trader-B sells 1 contract	1
2. Trader-C buys 3 contracts and trader-D sells 3 contracts	4
3. Trader-A sells 1 contract and trader-D buys 1 contract (trader-A is offsetting her initial long positions and trader-D is partly offsetting her initial position)	3
4. Trader-C sells 1 contract and trader-E buys 1 contract (trader-C is offsetting part of her initial position)	3

Trader's final position	Long	Short
Trader-A	0	0
Trader-B	0	1
Trader-C	2	0
Trader-D	0	2
Trader-E	1	0
All traders	**3**	**3**

On day-3, trader-A, who is long one contract, sells her contract to trader D who is initially short three contracts – open interest falls by one. Trader-A is out of the market and trader-D is now only short two contracts. If on day-4 trader-C sells one contract to a 'new' trader-E then trader-C closes out one contract (and ends up with being long two contracts). But because trader-E enters the market, open interest is unchanged since trader-E has in effect taken the place of trader-C for this contract.

2.2 FUTURES EXCHANGES AND TRADERS

In some futures markets, traders meet face-to-face in a pit, such as on the International Money Market (IMM) in Chicago and on the trading floor of the Chicago Board of Trade (CBOT) and, until 1999, on the London International Financial Futures Exchange (LIFFE), which has now merged and become the electronic trading platform, 'NYSE-Euronext'. Traders in the pit indicate prices and deal sizes using hand signals and the system is known as open outcry.

All exchanges *settle* trades using computers but now more exchanges are moving away from open outcry and actual trades are conducted electronically. Some of these systems are 'order driven' whereby the buyers and sellers are 'matched' via a computer as, for example, on NYSE-Euronext. Futures trading is 'global'. For example GLOBEX (which is owned by the CBOT and Reuters plc) provides an *after-hours* electronic futures trading system – 'after-hours' being from a US prospective. Note, however, that GLOBEX does not automatically match buyers and sellers and then automate the trades. It merely provides price information to traders.

On the IMM, floor traders can be either commission brokers called futures commission merchants (FCM) or 'locals'. FCMs merely act as middlemen, buying and selling on behalf of clients and they make profits from their commission charges. Locals trade on their own account and hold a book (i.e. take positions in various contracts). Dual trading is also allowed.

2.2.1 Futures Trading

Public orders must be placed through a broker who will then contact a floor broker on the exchange. Trades are monitored by the clearing house (e.g. CME Clearing) and all floor traders must have an account with a member clearing firm. If your floor broker purchases a futures contract on your behalf, then you will have to pay the initial margin to the floor broker, who will then pay this into her clearing firm ABC. The seller may be a local or a broker acting on behalf of a customer 'off the floor'. The seller of the futures also deposits the initial margin with her clearing firm XYZ. Both clearing firms ABC and XYZ each aggregate up their net positions from their customers and place a margin payment with the clearing house, who then guarantees both sides of the contract.

2.2.2 Trading Costs

In the US, brokers' commissions charged to the public include a payment for both buying and closing out the contract. However, commissions are very low, as are bid–ask spreads, making futures contracts very low cost instruments for hedging and speculation. The commissions, bid–ask spreads, and clearing fees for futures are usually less than those for forward contracts, which is perhaps not surprising since many forward contracts are tailor-made to suit the client. Delivery costs for 'the short' (if the contract goes to maturity) vary, depending on what is being delivered (e.g. 'live cattle' versus a cash settled futures contract on the S&P 500).

2.3 MARGINS AND MARKING-TO-MARKET

Margin payments provide financial protection in case one of the counterparties to the futures contract defaults. Consider a futures contract on oil (e.g. on 'West Texas Intermediate WTI, Light, Sweet Crude Oil'). The contract size is for delivery of 1,000 barrels. If in Chicago the oil futures price is $F_0 = \$98$ (per barrel) then the value of one futures contract is \$98,000. If, for example, the futures price increases by \$0.3 (e.g. from 98 to 98.3) the value of one long futures contract changes by \$300 and the value of one short futures position decreases by \$300.

Assume the *initial margin* is \$2,000 per contract and the *maintenance margin* is \$1,500 per contract.[1] The initial margin is not a payment for the futures contract. It is a 'good faith' deposit to ensure that the terms of the futures contract are honoured. Often a competitive interest rate is paid on the balance in the margin account, so funds in the margin account do not represent a cost. If the balance in the margin account falls below the maintenance margin of \$1,500 per contract then the trader has to deposit extra funds known as the *variation margin* to restore the balance to the *initial margin* (of \$2,000 per contract). The variation margin ensures that the balance in the margin account never becomes negative. As a ball-park estimate, the maintenance margin is set at around 75% of the initial margin.

The initial margin can be in cash but T-bills/bonds are often accepted (at 90% of their face value) and sometimes shares are accepted (at about 50% of their market value). The variation margin can usually only be paid in cash. Margin requirements are also referred to as 'performance bonds'. The investor in the futures contract is allowed to withdraw any balance in the margin account in excess of the initial margin.

Suppose at 12 a.m. on 5 June Ms Long buys $N_F = 2$ December-futures contracts on oil at $F_0 = \$98$ and closes out the contracts 4 days later at 1 p.m. on 9 June by selling two December contracts at a price of $F_3 = \$98.3$. Ms Long makes a profit on the two contracts of \$600 ($= 2 \times \$0.3 \times 1000$ barrels). Let us see how the margin account achieves the same result as the above simple 'differences' calculation.

[1] These figures are illustrative. For crude oil futures the current maintenance margin is around \$2,000 to \$2,500, depending on the contract. The initial margin on CME is 110% of the maintenance margin.

TABLE 2.4 Margin account

Day/time	Futures price (value) per contract	Gain/loss per contract	Daily gains and losses $N_F = 2$	Balance in margin account	Margin call
Day-1, 2.10 p.m.	98.0 ($98,000)			$4,000	
Day-1, Close	97.9 ($97,900)	($100)	($200)	$3,800	
Day-2, Close	97.4 ($97,400)	($500)	($1,000)	$2,800	$1,200
Day-3, Close	97.8 ($97,800)	$400	$800	$4,800	
Day-4, 1 p.m.	98.3 ($98,300)	$500	$1,000	$5,800	

Note: N_F = 2 contracts, initial margin = $2,000 per contract, maintenance margin = $1,500 per contract. The first and last entries for 'Futures price' are actual trading prices – the rest are 'settlement prices' at the close of trading each day.

When Ms Long purchases (go long) one futures contract at $98, the initial face value of her futures position is $98,000. If Ms Long buys $N_F = 2$ futures contracts, she pays an initial margin of $4,000 (= 2 × $2,000) – see Table 2.4.

Suppose that by the close of day-1 the ('settlement'[2]) futures price falls from $F_0 = \$98$ to $F_1 = \$97.9$. The 'long' has a loss of $100 per contract since at the end of day-1 she can now only sell each futures contract for $97,900. The loss on two contracts is $200 and the balance in Ms Long's margin account is therefore reduced by $200 to $3,800. This is *marking-to-market*. The balance at the end of day-1 is above the maintenance margin of $3,000 (for two contracts).

At the end of day-2 the futures price has fallen to $97.4 and the loss on two contracts is $1,000 so the value in the margin account is $2,800 which is below the maintenance margin of $3,000. The margin account must be made up to $4,000 (the initial margin) by the *end* of the next day (end of day-3), which requires a variation margin payment of $1,200. By the end of day-3, the futures price is 97.8, an increase of 0.4 which implies that the balance in the margin account is increased by $800, making a final balance of $4,800.

At 1 p.m. on day-4, the two futures contracts are closed out (sold) at 98.3 (to another futures trader) and Ms Long's margin account is credited with $1,000 giving a final balance of $5,800. The $5,800 is paid to Ms Long by the clearing house. Since Ms Long previously paid in $5,200 (=$4,000 initial margin + $1,200 variation margin), her profit over the 4 days is $600. This profit from the operation of the margin account is just the change in the futures price over the holding period of 4 days, that is, $N_F(F_4 - F_0)\ 1{,}000 = [2 \times (\$98.3 - \$98.0)\ 1{,}000] = \$600$.[3]

[2]The amount in the margin account is recalculated at the end of each day. The futures price used to calculate the value of the futures position is the average price of the last few trades of the day and is known as the 'settlement price'.

[3]This ignores any daily interest earned on the margin account.

In the case of a price fall, Ms Long has losses deducted from her margin account and the clearing house credits the margin account of someone who has a short position (i.e. who has previously sold one futures contract). The opposite occurs for a rise in the futures price. A futures trader can withdraw (if she wishes) any balance in her margin account, over and above the initial margin (per contract).

2.3.1 Price and Position Limits

If the balance in the margin account falls below the maintenance margin, then the investor must top up her account (i.e. she pays a 'variation margin') to equal the *initial margin* by the end of the next day. If the investor does not do this, the broker closes out the position by selling/buying the contracts. Clearly it is possible for the futures price to fall (or rise) so dramatically over one day that the margin account is depleted. To prevent this happening the exchange sets daily price limits. If the price falls (or rises) in one day by as much as the 'limit down' ('limit up'), then trading (usually) ceases for that day. These *circuit breakers* limit the daily payments/receipts to and from the margin account so that the balance in the margin account does not fall below zero and can be replenished before trading begins the next day. Often the initial margin is set equal to the value of the contract's daily price limit and therefore the initial margin can be small relative to the value of the futures contract.

The exchange also imposes position limits, that is the maximum number of contracts a speculator may hold (e.g. a maximum of y contracts with no more than $z\ [< y]$ in any one delivery month). This prevents speculators from having undue influence on the market. Bona-fide hedgers are not subject to position limits.

2.3.2 Closing Out

In practice most futures positions are closed out before maturity. You therefore make a gain or loss depending on the difference between the initial futures price $F_1 = \$100$ and the futures price at which you close out the contract, say $F_2 = \$110$ (Figure 2.1). Closing out the contract merely involves reversing your initial trade.

If in January Ms Long purchases a September-futures contract (on stocks) at $F_1 = \$100$ in Chicago and in July she sells a September-futures contract at $F_2 = \$110$, the profit on the trade is $F_2 - F_1 = \$10$. Also, by buying and then later selling the same September-futures contract, any delivery of the underlying asset to Ms Long is nullified. If you are long September-futures in January, then an *increase* in the futures price implies you make a *profit* when you close out – this is a positive relationship.

If in January you buy (go long) a September-futures at $F_1 = \$100$ and the futures price falls to $F_2 = \$90$ in July, then you make a loss of \$10 (i.e. a profit of –\$10). Again, this is a positive relationship – a *fall* in the futures price leads to a *fall* in profits. You bought for $F_1 = \$100$ and you close out by selling at $F_2 = \$90$ – a loss of \$10.

The profit on the long futures position $F_2 - F_1$ is equal to the *cumulative gain* or loss incurred day by day, as the contract is marked-to-market. So, in practice, the profit accrues in daily instalments over the life of the futures contract. Although most futures contracts are closed out, it is the possibility of delivery which links the futures price to the underlying spot price (via riskless arbitrage) and arbitrage implies that the futures price in Chicago (over short time horizons) moves almost dollar-for-dollar with the spot price (in New York).

Now consider an investor Ms Short who in January sells (shorts) a September-futures contract at $F_1 = \$100$ (Figure 2.1). Suppose the futures price then *falls* to $F_2 = \$90$ in July. She can buy back the contract in July (and hence close out her position) at the quoted futures price of $F_2 = \$90$. She initially sold at \$100 and she buys back in July at \$90 – making a profit of +\$10. Conversely, if the futures price *rises* to $F_2 = \$110$ in July, then Ms Short would make a *loss* of \$10 if she closed out. Hence for Ms Short there is a negative relation between her profit and the change in the futures price.

2.3.3 Delivery and Settlement

Although most futures are closed out before maturity it is important to understand the delivery process. The futures contract is referred to by its delivery month. The delivery period set by the Chicago exchange varies from contract to contract. For many contracts the delivery period is the whole month (but could also be for specific days in the delivery month).

The decision on when to deliver is made by the party with the short position. So, a person holding a short September wheat-futures can issue a notice of intention to deliver, to the clearing house. For commodities, the notice will contain the number of contracts to be delivered, the grade, delivery point and specific delivery date. The clearing house then selects someone with a long position in September wheat-futures and lets them know that they *must* accept delivery within the next few days. Selection of the long may be done randomly or by taking the 'traders' with the oldest outstanding long position. The long must accept the delivery notice but if the notice is 'transferable', she has about 30 minutes to find another person with a long position to accept the notice from them.

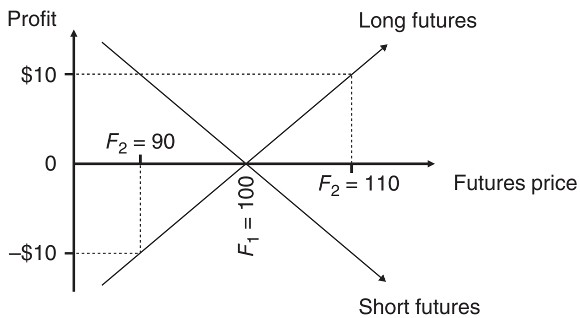

FIGURE 2.1 Speculation with futures

For 'commodities', delivery is usually in the form of a warehouse receipt and 'the long' then pays the most recent settlement price. The precise time and place for delivery is determined by the party with the short position. The long is then responsible for any 'warehousing costs' (e.g. for wheat, silver, gold) or in the case of livestock futures, care of the animals. Delivery of oil and natural gas is at a specific point along the 'pipeline' – for example, in the US at Cushing, Oklahoma for oil and at Henry Hub in Louisiana for natural gas.

The 'first notice day' is the first day on which the short can issue a notice to deliver. Hence a person with a long position should close out before the first notice day if she does not want to take delivery. The 'last notice day' is the last day the short can issue the notice to deliver. Trading in the futures contract usually ceases a few days before the last notice day.

Delivery (for non-cash settled contracts) is usually about a 3-day process. Two days before someone with a short position intends to make delivery, she notifies the clearing house. The next business day, the exchange selects the party with a long position to take delivery. On the third day, the *delivery day*, the short delivers the underlying asset and the long pays the short (and the deal is confirmed).

Some financial futures contracts involve the delivery of the underlying asset (e.g. T-bills, T-bonds) while others, such as stock index futures, are settled in cash. Often cash-settled contracts use the settlement price of the spot asset on the last trading day – the positions of the long and the short are then 'closed out' by the clearing house and the contract no longer exists. The settlement price may be the opening or closing price on the settlement day.

One of the key differences between forwards and futures is that forward contracts cannot be easily closed out. Usually Ms A can only close out her long *forward contract* (initially entered into with counterparty B) by selling her contract to Ms C. In general Ms B will have to agree the 'new' contract between Ms A and Ms C because of the credit risk to Ms B who is short the contract. In contrast, if Ms A has a long position in a 'hog' *futures* contract,[4] then she can easily offset it by selling a *new* futures contract *with the same delivery date*. For example, if on 1 February Ms A purchased one September-futures on hogs at $F_0 = \$100$ and she later sold one September-futures on hogs on 1 June for $F_1 = \$110$, then Ms A makes a cash profit of $10 but there will be no delivery of hogs *to Ms A* in September. As far as Ms A is concerned this contract is now nullified.

2.3.4 Newspaper Quotes

An illustrative example of futures quotes for gold on 10 October (say) is given in Table 2.5. The size of the contract is for delivery of 100 troy oz of gold and the price is in US dollars per troy oz. There are various maturity dates. The nearby contract is the October-futures and the table shows contracts with maturities out to the next August.

The opening futures price (for the trade immediately after the opening bell), and the 'high' and 'low' daily futures prices during the trading day are given in columns 2 to 4. The fifth

[4] A hog is a type of pig.

TABLE 2.5 Futures quotes, 10 October, Gold (GC), 100 troy oz ($ per troy oz)

Month	Open	High	Low	Settle	Change	Open interest
October	1236.1	1236.5	1235.5	1237.4	4.6	633
December	1239.0	1246.5	1232.2	1243.1	4.4	299,488
February	1244.8	1252.3	1242.7	1249.4	4.4	33,161
April	1251.3	1251.3	1249.3	1255.5	4.5	18,513
June	1253.0	1262.1	1253.0	1261.6	4.5	18,037
August	1261.0	1267.6	1261.0	1267.4	4.5	18,244

column ('Settle') gives the settlement price which is an average of prices just before the closing bell and the sixth column shows the *change* in the settlement price, since the previous day. The settlement price is used to calculate daily changes in margin requirements. For example, if between two trading days the settlement price for December-futures increases by $4.5 and an investor is long one gold futures contract (for 100 troy oz), she would have her margin account credited by $450 (= 100 × $4.5). The trader who is short the December futures would have her margin account debited by $450.

If we look down the column marked 'Settle' we see that on 10 October, the long-dated futures contracts on gold have a higher quoted futures price, than the more near-dated contracts. As we shall see in Chapter 3, this is due to interest costs and storage costs for gold, which both increase as the time to maturity of the futures contract increases.

Open interest is the number of futures contracts currently outstanding and most traders are currently holding the December contract with open interest of around 300,000 contracts (i.e. 300,000 trades who are long the December contract and 300,000 counterparties who are short the December contracts). On 10 October the far-dated contracts, with maturity from February onwards, are not very actively traded. This is probably because investors who wish to hedge their current holdings of (spot) gold are more interested in hedging over the October to December period, rather than over longer horizons.

2.3.5 Types of Traders

Some futures traders are *scalpers,* who hold their positions for less than a few minutes. *Day traders* hold their positions for less than a day, for fear of overnight news changing the price by a large amount. *Position traders* may hold their positions much longer since they base their investment decisions on the economic fundamentals driving the price of the underlying asset. *Spread traders* buy one futures contract and simultaneously sell another futures contract in the hope of making a profit. For example, in an *intracommodity spread* the trader buys a contract with one expiry month and sells an otherwise identical contract with a different expiry month, hoping to benefit from the differential move in the two futures prices. An *intercommodity* spread consists of a long position in a contract on one commodity (e.g. coffee) and a short position in a contract on a different commodity (e.g. cocoa) where the speculator is trying to exploit an 'abnormal' difference between the two prices, which she hopes will be corrected in the near future.

Floor brokers who have a seat on the exchange merely buy and sell futures contracts on behalf of their customers and usually earn a commission on each trade. They do not hold a book and they are not obliged to 'make a market' in the futures.

Instructions to your broker might involve a *market order* which is for immediate execution at the best available price. A *limit order* specifies a maximum price for a purchase and a minimum price to sell. The limit order may be a day order (i.e. it is only valid until the end of the day) or a *good-till-cancelled* order. If you already hold a futures contract you may institute a *stop-loss order*, which instructs your broker to sell the futures if the price falls a fixed amount below the current price.

2.4 SUMMARY

- A futures contract is *marked-to-market,* so its value changes each day.
- The clearing house keeps track of all trades and sets the amounts for the initial margin and the maintenance margin. If the amount in the trader's margin account falls below the maintenance margin, additional variation margin payments have to be made to bring the balance in the margin account up to the 'initial margin'. This minimises counterparty (credit) risk.
- The clearing house sets the terms for each futures contract, such as the contract size, tick value, delivery dates, the precise definition of the settlement price and the margin requirements.
- Trading can be via *open outcry* in a trading pit (e.g. in Chicago) but there has been a major shift towards electronic trading which is 24 hours a day (e.g. on the CME-Globex platform).

EXERCISES

Question 1
Explain how a long one-year forward contract on a stock, entered into at $F_0 = \$100$ can be used for speculation. Assume the forward contract is held to maturity.

Question 2
What are the key differences between forward contracts and futures contracts?

Question 3
Briefly explain 'open interest' and 'trading volume' for futures markets.

Question 4
Explain an 'initial margin' of \$5,000 (per futures contract) and a 'variation margin' of \$2,000. Why are these concepts useful?

Question 5
On Monday, you are *already long* 100 contracts at a settlement price of \$50,000 per contract. Next day (Tuesday) at 11 a.m. you acquire an additional 20 contracts at a price of \$51,000 per contract. The initial margin is \$2,000 per contract. The settlement price at the end of Tuesday is \$52,000 per contract. By how much does the margin account change between Monday and Tuesday?

Question 6
At the end of day-1 you *purchase* 100 contracts of the September-futures at a settlement price of $F_0 = \$50{,}000$ per contract. The initial margin on *any* contracts you *initially* purchase is \$2,000 per contract. Hence, at the end of day-1 you have paid \$200,000 into your margin account. On day-2, at 11 a.m. you acquire an additional 20 (September) contracts at a price of $F_2 = \$51{,}000$ per contract. The *settlement price* at the end of the day is $F_3 = \$50{,}200$ per contract.

(a) What is the net gain or loss on the value of *all of your contracts* at the end of day-2? Express this gain or loss algebraically.
(b) What happens if you close out all your contracts at the end of day-2 at the settlement price of £50,200? What is *the value* of your margin account at the end of day-2?

Question 7
You buy stock index futures SIF at an index price of $F = 452$ at 11 a.m. on 1 August. The initial margin is \$9,000 and the maintenance margin is \$6,000. You close out the contract at 11 a.m. on 4 August at 450. The settlement prices at the end of the day on 1, 2, 3 August are 453, 430, 445. You do not withdraw any excess cash out of your margin account.

What is the profit (loss) on the SIF contract? Construct a table to show inflows and outflows from the margin account. (Assume the value of an index point on the futures contract is \$250.)

CHAPTER 3

Forward and Futures Prices

Aims

- To construct a synthetic or replication portfolio whose cash flows mimic the cash flows in an actual forward contract.
- To show how cash-and-carry *arbitrage* can be used to determine the correct (no-arbitrage) 'fair' price for a forward contract on a (non-dividend paying) stock.
- To interpret cash-and-carry arbitrage in terms of the implied repo rate and the actual repo rate.
- To extend our arbitrage approach to determine the fair forward price when the underlying asset pays an income stream (e.g. dividends on a stock) or has storage costs (e.g. oil, wheat) or can be used as an input to the production process (e.g. the convenience yield of owning oil).
- To distinguish between the price of a forward contract at inception and the value of a forward contract that has been in existence for some time.

3.1 PRICING FORWARD CONTRACTS

In this section we analyse how forward and future prices are determined as functions of known variables such as the current market price of the underlying asset and the risk-free interest rate. Forward contracts are easier to analyse than futures contracts, since the latter have the added complication of daily settlement (i.e. marking-to-market) whereas for the forward contract there is one payment at the maturity date. However, it can be shown that the *futures* price

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

closely follows that of the forward price (for contracts with the same maturity date). Therefore, for the most part the reader can consider the analysis of 'forward' and 'futures' prices as being interchangeable. In general we assume:

- zero transactions costs
- zero tax rates
- agents can borrow or lend unlimited amounts at the risk-free rate of interest
- short-selling of the underlying asset is possible
- risk-free arbitrage opportunities are instantaneously eliminated.

For the most part we use (risk-free) arbitrage to price futures contracts and this method is also referred to as the *cost of carry* method.

3.1.1 Non-income Paying Security

Consider the futures price on a stock (which pays no dividends). The futures contract is for the delivery of a single stock in 3 months' time (Figure 3.1). We set up a situation where initially it is possible to make a risk-free arbitrage profit because the *quoted* futures price is incorrect. Consider the following 'prices':

$$\begin{aligned} F_q &= \$102 \text{ is the } \textit{quoted} \text{ futures price in Chicago on 25 June } (t=0), \text{(with maturity} \\ &\quad \text{date of 25 September)} \\ T &= 1/4 \text{ is the time to maturity of futures contract (in years, or fraction of a year)} \\ S &= \$100 \text{ is the spot (cash market) price of the stock on the NYSE on 25 June} \\ r &= 0.04 \text{ is the risk-free interest rate } (= 4\% \text{ p.a., simple interest}) \end{aligned}$$

Stock price	$S = \$100$
Risk-free rate	$r = 4\%$ p.a.
Quoted future price	$F = \$102$

Strategy today (25 June)
Sell (short) futures contract at \$102 (receive no cash today)
Borrow \$100 from bank and buy the stock (= synthetic future)
Use no 'own funds'

3 months' time : 25 September ($T = 1/4$)

Bank loan outstanding = \$100(1 + 0.04/4)	= \$101
Deliver stock, receipt from futures contract	= \$102
Riskless profit	= \$1

FIGURE 3.1 Risk-free arbitrage

3.1.2 Overpriced Futures Contract

Below, we see that the correct ('no-arbitrage' or 'fair') futures price is $F = \$101$, which with a quoted price $F_q = \$102$ implies the futures contract is overpriced and it is possible to earn a risk-free (arbitrage) profit. Here's how. The arbitrageur Ms Arb on 25 June (at 11 a.m.) borrows \$100 from bank-A and purchases stock-ABC on the NYSE. She therefore knows for certain that she can deliver the stock in 3 months on 25 September (since it is 'in her pocket' and will stay there – she 'carries' the stock). This strategy involves no 'own capital' by Ms Arb, since the money is borrowed.

Ms Arb on 25 June (at 11 a.m.) also sells a futures contract (on stock-ABC) in Chicago at $F_q = \$102$ (to another trader in Chicago, who may be a hedger or a speculator – we don't care – we only care about *our own* trades). Note that Ms Arb's 'sell order' is noted by the futures clearing house in Chicago and Ms Arb does not pay any money today (we ignore margin payments). By selling the September-futures contract on 25 June *and holding it to maturity,* Ms Arb is legally obliged to deliver one stock on 25 September in Chicago, when she will receive today's quoted futures price, $F_q = \$102$.

What is the cost between 25 June and 25 September to Ms Arb of 'carrying' and then 'delivering' stock-ABC in Chicago to fulfil the conditions of her short futures position? On 25 September, Ms Arb owes the bank the initial loan of \$100 plus interest, which equals \$101 [$= \$100(1 + 0.04/4) = S(1 + rT)$]. By purchasing stock-ABC in the cash-market in June with borrowed money, Ms Arb can (with certainty) deliver it in 3 months' time in Chicago on 25 September – she has created a 'synthetic' or 'replication' futures contract at a cost of $S(1 + rT) = \$101$ (which accrues on 25 September).

At maturity of Ms Arb's (short) futures contract on 25 September, she delivers one stock and receives $F_q = \$102$ in Chicago. The cost of having the stock 'in her pocket' ready for delivery in September is \$101 – therefore Ms Arb makes a profit of $(102 - 101) = \$1$ on 25 September. This strategy is risk-free since Ms Arb knows for certain what the outcome will be on 25 September, *when she makes her trades on 25 June*. The reason there is a 'cash-and-carry' arbitrage opportunity is because the quoted futures price F_q is greater than the correct ('fair') price of the futures contract $F = S(1 + rT)$ that is:

$$F_q > F = S(1 + rT) \tag{3.1}$$

Many arbitrageurs will take advantage of this risk-free profit opportunity by 'buying low' and 'selling high'. Buying stock-ABC in the cash market (NYSE) and borrowing money on 25 June tends to increase S and r, while selling futures contracts in Chicago on 25 June will tend to depress the quoted futures price F_q – this is just supply and demand and will tend to quickly move the quoted futures price to equal the fair futures price: $F_q = F = S(1 + rT)$. So the 'correct', 'no arbitrage' or 'fair' futures price is:

$$F = S(1 + rT) \tag{3.2}$$

If the quoted futures price on 25 June had been $F = \$101$, then there would be no risk-free arbitrage opportunities available. Above, we have implicitly assumed that *all* futures traders are quoting a price $F_q = \$102$ which makes the cash-and-carry arbitrage possible for *all* traders like Ms Arb. However, there is another way all traders will quickly move their *quoted* futures prices to equal the no-arbitrage futures price, $F = S(1 + rT)$.

If trader-A is quoting $F_A = \$102$ and another trader-B is quoting the no-arbitrage price $F_B = \$101$ (for the September contract) then Ms Arb on 25 June will buy a September-futures from trader-B and sell a September-futures to trader-A. She has a long-short position in September-futures. On 25 September, at maturity, she takes delivery from trader-B and pays $F_B = \$101$ and immediately delivers the stock in Chicago to futures trader-A and receives $F_A = \$102$ – making a *risk-free* arbitrage profit of \$1. It is the fact that the futures contracts *mandate* delivery on 25 September that makes this long-short position risk-free. In practice, when lots of Ms Arbs sell futures contracts at $F_A = \$102$, supply and demand ensures that the futures price on 25 June will quickly fall to its no-arbitrage correct value of $F = S(1 + rT) = \$101$.

We can write Equation (3.2) as:

$$\text{“Fair” Futures Price} = \text{Spot Price} + \text{Dollar Cost of Carry} = \$100 + \$1$$

$$F = S + DCC \tag{3.3a}$$

where the *dollar* cost of carry $DCC = SrT = \$1$. Equivalently we have:

$$\text{“Fair” Futures Price} = \text{Spot Price } (1 + \text{Percent Cost of Carry})$$

$$F = S(1 + PCC) \tag{3.3b}$$

where the *percent (proportionate)* cost of carry $PCC = r.T = 1\%$. Determining the futures price using Equation (3.2) involves *cash-and-carry arbitrage*.

The spot price of the stock on the NYSE will change continuously but the actions of arbitrageurs means that $F = S(1 + rT)$ at all times. It is immediately apparent from Equation (3.2) that the futures price and the stock price will move closely together (if r stays constant). In our case $F = (1.01)S$ and hence over a short time horizon, the change in the futures price is approximately equal to the change in the spot price: $F_1 - F_0 = 1.01(S_1 - S_0)$.

Arbitrageurs ensure that the spot price (in New York) and the futures price (in Chicago) move together (over a short time period), almost dollar-for-dollar

However, from Equation (3.2), if interest rates change, then F and S will change by slightly different amounts – this is a source of *basis risk* when hedging with futures contracts.[1] If the futures contract is correctly priced (i.e. $F = \$101$) then it must equal the

[1] Aficionados will note that the ‘correct’ change in the futures price between $t = 0$ and $t = 1$ is $F_1 - F_0 = S_1(1 + r_1T_1) - S_0(1 + r_0T_0) = S_1 - S_0 + (S_1r_1T_1 - S_0r_0T_0)$ and the final term in parentheses is the source of basis risk (which is usually small relative to the change in S).

cost of creating the 'synthetic (replication) futures' $S(1+rT)$ – at any other quoted futures price, traders could make a risk-free (arbitrage) profit. This 'mispricing' cannot persist in transparent well-functioning markets with many traders looking to exploit any 'risk-free' trading strategies.

Another way of looking at the no-arbitrage approach to determining the fair futures price is to note that the actual futures contract and the 'synthetic (replication) futures' both have the same outcome at T. Hence to prevent arbitrage opportunities, they must have the same value today. To take a *long position* in the actual futures contract, no 'own funds' are required at $t = 0$ (we ignore margin payments). At maturity of the actual futures, you pay out $F = \$101$ and take delivery of one stock. To create the 'synthetic future' you borrow \$100 (@ 4% p.a.) and purchase the stock today – hence you also use no 'own funds'. Also, this 'synthetic portfolio' *replicates* the outcome in the actual futures contract since at T, you owe the bank \$101 and you own (hold) one stock. The synthetic futures *replicates* the outcome at T, for the *actual* futures contract. Hence to prevent arbitrage opportunities both must have the same value today, so the 'no-arbitrage' quoted futures price $F = S(1+rT)$.

3.1.3 Underpriced Futures Contract

Above we showed that if $F_q > S(1+rT)$ this gives rise to 'cash-and-carry' arbitrage profits. Alternatively, when this inequality is reversed, $F_q < S(1+rT)$, risk-free profits can be made by 'reverse cash-and-carry arbitrage'. Suppose (as above) on 25 June, $S = \$100$, $r = 4\%$ p.a., $T = 3$ months so the correct ('fair' or 'no-arbitrage') futures price is $F = \$101$ but the quoted futures price $F_q = \$99$. Hence on 25 June the actual futures contract is underpriced, $F_q < S(1+rT)$. Thus, arbitrage profits can be made by 'selling high, buying low'.

On 25 June, Ms Arb borrows stock-ABC from her prime broker and sells it on the NYSE – she also buys the September (3-month) futures contract. The proceeds from the short-sale of stock-ABC[2] are placed in a (risk-free, 3-month term) deposit at bank-A at an interest rate r. On 25 September, cash in the deposit account has accrued to $\$101 = S(1+rT)$. Ms Arb takes delivery of stock-ABC (in Chicago) from her short futures position and pays $F_q = \$99$, giving her an arbitrage profit of \$2 $(= \$101 - \$99 = S(1+rT) - F_q)$. The stock delivered in the (short) futures contract is then returned to Ms Arb's broker.

Hence, if the quoted futures price F_q is either slightly above or slightly below the 'fair' (no-arbitrage) futures price $F = S(1+rT)$, arbitrageurs will initiate simultaneous trades in the cash market (NYSE) and the futures market (Chicago), which will quickly move quoted prices on these two exchanges so that $F_q = S(1+rT)$ at *almost* every instant of time. But it is possible for (3.2) not to hold at absolutely every instant of time and providing lots of 'Ms Arbs' act quickly enough, they may make a small arbitrage profit – but such opportunities are likely to be very short lived and the 'no-arbitrage' pricing equation $F_q = S(1+rT)$ will be quickly restored.

[2] In practice the cash from the short-sale is held by the prime broker as collateral but interest is usually paid to Ms Arb on this cash deposit and is known as 'rebate interest'.

The above formulas assume r is measured as a simple interest rate. If we use a discrete compound rate or a continuously compounded rate, then the 'fair' futures price is given by the following formulas:

$$F = S(1 + r)^T \quad \text{(here } r = \text{compound interest rate)} \tag{3.4a}$$

$$F = Se^{rT} \text{(here } r = \text{continuously compounded rate)} \tag{3.4b}$$

3.1.4 Futures Price at Maturity

The 'September-futures' matures on 25 September and will be continuously traded from 25 June (when it was initiated) and the futures price (in Chicago) changes from day-to-day (as the stock price on the NYSE changes). On 25 September ($= T$) the maturity date of the futures contract, the quoted futures price F_T in Chicago must equal the quoted stock price on the NYSE, S_T – otherwise risk-free arbitrage profits can be made.

For example, suppose the futures price in Chicago on 25 September for the 'September futures contract' on stock-ABC is quoted as $F_T = \$98$ and the stock price on the NYSE is $S_T = \$98.3$. An arbitrage profit can be made. Ms Arb buys the futures at $F_T = \$98$ on 25 September and takes immediate delivery in Chicago of one stock and pays $F_T = \$98$. Ms Arb can then sell stock-ABC in the cash market (NYSE) for $S_T = \$98.3$ making an instantaneous risk-free profit of \$0.3.

On the maturity date of the September-futures (i.e. 25 September) unless the quoted futures price equals the stock price ($F_T = S_T$), then risk-free arbitrage profits are possible. The actions of Ms Arb will tend to lead to a fall in stock-ABCs price on the NYSE and rise the futures price in Chicago – this is supply and demand again. Arbitrageurs will trade futures and stocks until $F_T = S_T$, which will happen almost instantaneously on 25 September, in well-functioning liquid markets like the NYSE and the Chicago futures exchange.

3.1.5 Impediments to Arbitrage

To establish the 'fair' or 'no-arbitrage' futures price $F = S(1 + rT)$ we have assumed that (a) buying and selling prices for spot market assets (e.g. stocks) are equal, and similarly for futures contracts (b) borrowing and lending rates (from the bank) are the same, (c) there are zero transactions costs (e.g. commission fees), and (d) short-selling of the underlying asset in the futures contract is always possible.

We now relax some of the above assumptions. In practice, 'cash-and-carry arbitrage' involves borrowing funds at r_b and buying the stock at the ask-price S_{ask}. If at T, the total dollar transaction costs[3] by Ms Arb of borrowing funds, buying the stock (at $t = 0$) and

[3]We accrue all cash flows to time T. Hence any *transactions costs* at $t = 0$ of buying the stock, selling the futures and borrowing the funds are compounded forward to time T, using the risk-free interest rate r. Any transactions costs of closing out the futures occurs at time T.

selling the futures (at T) is TC_T, then arbitrage is profitable if the quoted futures bid-price $F_q > S_{ask}(1 + r_b T) + TC_T$. The upper bound for the no-arbitrage futures price is therefore $F_q \leq S_{ask}(1 + r_b T) + TC_T$.

For 'reverse cash-and-carry arbitrage' Ms Arb buys the futures (at the ask-price) and short-sells the stock at the bid-price S_{bid}. If the broker takes a 'haircut' of $(1 - z)$ then Ms Arb will place a proportion z (= 0.99) of the proceeds of the short-sale (zS_{bid}) in a bank deposit, earning interest r_d. At T, Ms Arb has cash of $zS_{bid}(1 + r_d T) - TC^*_T$, where TC^*_T is the (dollar) *transaction costs* at T of placing funds on deposit, short-selling the stock and buying the futures contract. At T, Ms Arb pays the quoted futures ask-price F_q and if $F_q < zS_{bid}(1 + r_d T) - TC^*_T$ there are arbitrage profits to be made. The lower bound for the no-arbitrage futures price is therefore $F_q \geq zS_{bid}(1 + r_d T) - TC^*_T$. Hence, when we take into account these 'real world' factors there is no longer a single correct forward price but the no-arbitrage forward price will lie between an upper and lower bound.

For 'professional traders' operating in highly liquid markets for 'investment assets' such as stocks, bonds, foreign exchange and even commodities like gold and silver, the upper and lower bounds for the futures price will be 'close to' each other. Also, for these investment assets, short-selling the cash market asset is usually not a problem (e.g. gold or silver held as investment assets by traders, can be 'borrowed' by short-sellers). Also, there are always many traders who hold spot/cash market assets like gold and silver who are willing to sell from their own inventory and execute reverse cash-and-carry arbitrage. (Note that if you own gold or silver and are selling from inventory the 'haircut' $z = 0$, so the upper bound for the futures price is different to that for a genuine short-seller who has to borrow the underlying asset from her broker.) In contrast, certain spot market assets (e.g. crude oil, gas, agricultural produce) are used in the production process and for these 'consumption assets' the ease with which short-selling can be undertaken is more questionable – we deal with this below.

3.1.6 Implied Repo Rate

There is another way we can describe the arbitrage opportunities discussed above. Suppose on 25 June, the *quoted* futures price is $F_q = 102$ and $S = 100$, $r = 4\%$ and $T = 1/4$ year. The 'fair' futures price is $F = S(1 + rT) = 101$ – so the quoted futures price is too high and arbitrage profits can be made.

Suppose on 25 June Ms Arb sells the September futures contract at $F_q = 102$ and simultaneously buys stock-ABC at $S = 100$ *using her own funds*. At maturity of the futures T, Ms Arb holds stock-ABC to deliver against her short futures position and she receives $F_q = 102$ in Chicago. The return *per annum* she earns (on her initial outlay of 'own funds' of \$100) is known as the *implied repo rate:*

$$\hat{r} = (1/T)[(F_q/S) - 1] = 0.08 \text{ (8\% p.a.)}$$

Alternatively, if Ms Arb executes the arbitrage strategy by *borrowing* the \$100 (at $t = 0$) in (what is called) the 'repo market', then the actual cost of borrowing is known as the *repo rate*.[4] Suppose the repo cost of 3-month borrowing is $r = 4\%$ p.a. then:

If the 'implied repo rate' $\hat{r}$ is greater or less than the 'quoted repo rate' (i.e. cost of borrowing funds, r) then a (risk-free) arbitrage trade is possible.

If the implied repo rate $\hat{r}$ and actual repo rate r are equal then there are no arbitrage opportunities:

$$\hat{r} = (1/T)[(F_q/S) - 1] = r, \quad \text{which implies } F_q = S(1 + rT)$$

So comparing the implied repo rate with the actual repo rate is just a different way of seeing whether the *quoted* futures price F_q is higher or lower than the fair futures price $F = S(1 + rT)$ and if so, then profitable arbitrage opportunities are possible.

3.2 DIVIDENDS, STORAGE COSTS, AND CONVENIENCE YIELD

In practice, many futures contracts are not as straightforward as the one described above. For example, a stock might pay dividends (over the life of the futures contract). We first consider the case of discrete dividend payments. Suppose (again) on 25 June, $S = 100$, $r = 4\%$ p.a., and the September-futures contract has a maturity of $T = 1/4$ year (3 months) but is written on a *dividend paying stock-ABC, which pays a known dividend of* $D_1 = \$1$ *in one month's time,* on 25 July (Figure 3.2). We also assume (for simplicity of exposition) that the yield curve is flat, so that $r = 4\%$ p.a. for borrowing or lending money at any horizon (e.g. over 1 month, 2 months, etc.).

Consider implementing cash-and-carry arbitrage. For example, on 25 June, Ms Arb borrows cash, buys stock-ABC and holds the stock for 3 months. This is the 'synthetic future'. The actual quoted price for the September-futures must equal the cost of creating this 'synthetic future' (on stock-ABC) – if there are to be no arbitrage profits possible. We repeat our initial example (see above) but in this case Ms Arb borrows funds from the bank over two different time horizons: using 1-month and 3-month interest rates (which are both 4% p.a.).

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[4]For example, suppose on 25 June Ms Arb provides collateral of \$100 of her own T-bills to bank-A, in exchange for receiving a \$100 loan from bank-A. She also agrees on 25 June that in 3 months' time she will pay the bank \$101 and the bank will then return her T-bills. Ms Arb's cost of borrowing from the bank is 1% over 3 months or $r = 4\%$ p.a. This cost of borrowing is known as the repo rate – 'repo' is short for sale and repurchase of the T-bills. At the end of the loan period, bank-A returns the T-bills to Ms Arb. So Ms Arb's T-bills are just held as collateral by bank-A. The amount she agrees to pay bank-A in 3 months' time does not depend on the *market price* of the T-bills at any point in time between 25 June and 25 September.

$PV(D_1) = D_1/(1+rT) = \$1/(1+0.04/12) = \$\ 0.997$

'Fair price' of futures contract : $F = [S - PV(D_1)](1+rT) = 99.97$

FIGURE 3.2 Futures price on dividend paying stock

There is a *single known* dividend payment after 1 month (1/12th year) of $D_1 = \$1$, on 25 July. We have a flat yield curve with $r = 4\%$ p.a. so the present value of this future dividend payment is $PV(D_1) = \$1/\,[(1 + 0.04(1/12)] = \$0.9967$. Hence Ms Arb can borrow \$0.9967 on 25 June from bank-A at the *one-month* interest rate and know that she can pay back the \$1 (owed to bank-A) from the (known) dividend payment on stock-ABC, on 25 July. For Ms Arb there are zero *net cash flows* on 25 July – the dividend received on the stock will be used to pay off the bank loan outstanding of \$1.

Therefore in order to purchase the stock at a price of $S = \$100$, Ms Arb only needs to borrow an additional $S - PV(D_1) = \$99.0033$ on 25 June from bank-A. For this second loan she borrows at the 3-month rate, so that repayment of this loan coincides with the maturity date of the futures contract. In 3 months' time (on 25 September) Ms Arb will owe bank-A, \$99.9933 [= \$99.0033 (1+0.04/4)]. But 25 September is also the maturity date of the actual futures contract and from our earlier discussion we know that the correct ('fair', 'no-arbitrage') futures price F must equal the cost of creating the synthetic futures hence:

$$F = [S - PV(D_1)](1 + rT) = \$99.9933 \tag{3.5a}$$

Notice that *all* the borrowed funds, \$0.9967 + \$99.0033 are used to purchase the stock for \$100, in order to execute the arbitrage strategy. The 'first' \$0.9967 Ms Arb borrows means she owes \$1 after *1 month* – but this debt is paid off with the \$1 dividend payout on 25 July from stock-ABC that Ms Arb holds. The net cost of the cash-and-carry arbitrage is the \$99.9933 owed to the bank after 3 months – and this is the correct (no-arbitrage) price for the September-futures contract, quoted on 25 June.

3.2.1 Several Discrete Dividend Payments

Suppose there are N dividend payments of different dollar amounts D_i on the stock, between today and the maturity date of the futures contract, that are payable at discrete times t_i (from today). Then the correct (no-arbitrage) futures price is:

$$F = [S - PV(D)](1 + rT) \tag{3.5b}$$

where $PV(D) = \sum_{i=1}^{N} D_i/(1 + r_i t_i)$, r_i = spot interest rates (simple rate) between today and t_i and r = interest rate with maturity T (when the futures contract matures).

3.2.2 Bond Futures

In principle, the above formula can also be applied to pricing a futures contract on a coupon paying T-bond. The futures price is $F = [S - PV(\mathrm{C})](1 + rT)$ where S is the current cash-market price of the underlying (deliverable) bond, $PV(\mathrm{C})$ is the present value of the coupon payments on the underlying bond over the life of the futures contract, T is the maturity of the futures and r is the risk-free rate for maturity T. This formula ignores several 'real world' complications (e.g. accrued interest) which are discussed in Chapter 13.

3.2.3 Continuous Dividend Payments

Suppose stocks pay out dividends (evenly) at a rate of $\delta = 2\%$ p.a. Then in the above arbitrage example, when Ms Arb buys the stock for \$100, some of the borrowed money can be paid back from the receipts of future dividends – this reduces the 'cost of carry'. If the rate at which dividends are paid is $\delta = 2\%$ p.a. then over 3 months Ms Arb receives \$0.5 in dividends ($= \$100 \times 0.02 \times 1/4 = S\delta T$) which reduces the overall cost of carry. So the correct (no-arbitrage) futures price is:

$$F = S\ [1 + (r - \delta)T] \tag{3.6a}$$

When using compound and continuously compounded interest rates the formulas are:

$$F = S(1 + r - \delta)^T \ \text{(compound rates)} \tag{3.6b}$$

$$F = Se^{(r-\delta)T} \ \text{(continuously compounded rates)} \tag{3.6c}$$

Using Equations (3.6a)–(3.6c), the quoted futures price (in Chicago) on 25 June (say) will be above the quoted stock price (on the NYSE), if $r > \delta$ and below the quoted stock price if $r < \delta$. These two cases are referred to as:

$F > S \Rightarrow$ The 'basis' $(F - S)$ is positive $\Rightarrow$ Futures market is in contango

$F < S \Rightarrow$ The 'basis' $(F - S)$ is negative $\Rightarrow$ Futures market is in backwardation

Often backwardation is also referred to as an *inverted market*. Contango and backwardation are also used in a slightly different way, to describe a whole series of futures prices for contracts with different maturity dates (but on the same underlying asset). This is the *term structure of futures prices* – which can be represented graphically. For example, the futures market is said to be in contango (on 25 June, say) if the quoted futures prices increase with the maturity dates of the contracts – ($F^{T=5} > F^{T=4} > F^{T=3} > F^{T=1} > S$), so prices of 'far dated' futures contracts are higher than those of 'near dated' contracts.

3.3 COMMODITY FUTURES

Let's consider the determination of the correct (no-arbitrage) future prices on commodities such as grains and oilseed (e.g. wheat, soybeans, sunflower oil), or food (e.g. cocoa, orange juice), or metals and petroleum (e.g. gold, silver, platinum, heating oil), or livestock and meat (hogs, live cattle, pork bellies), which are actively traded on various exchanges around the world – for example, New York Cotton Exchange (NYCE), Kansas City Board of Trade (KCBT), Mid America Commodity Exchange (MCE), New York Mercantile Exchange (NYMEX) and London Metals Exchange (LME).

When discussing futures prices we split commodities into two main types. *Investment commodities* include precious metals such as gold, silver, platinum and copper which are widely held as investment assets (as well as being used in the production of other goods). *Production commodities* are primarily held as inputs to the production process to provide consumer goods (e.g. agricultural and energy commodities) and are not usually held for investment purposes.

3.3.1 Investment Commodities

Investment commodities can be priced using 'cash-and-carry' arbitrage, if we take account of storage costs. If to undertake cash-and-carry arbitrage you purchase gold in the cash market (using borrowed money), then you have to store and insure the gold until the maturity date of the futures contract. You have storage costs on top of the existing interest cost on your borrowed money, so the futures price for gold is given by[5]:

$$F = S[1 + (r + s)T] \tag{3.7}$$

where s = storage cost of gold (per annum). This equation will hold at all points in time since arbitrage strategies including those which require short-selling the spot asset (gold, silver) will always be possible, because many holders of such investment assets do not have an alternative use for them (e.g. they are not jewellery manufacturers). Arbitrage involving purchases of these spot investment assets clearly poses no problems in liquid markets. Equation (3.7) indicates that at any time, the current quoted futures price F for gold (in Chicago) *will always* be above the spot price S for gold (quoted in New York, say).

[5]When interest rates and storage costs are continuously compounded the formula is $F = Se^{(r+s)T}$.

3.3.2 Production Commodities

One might think that a futures contract on a 'production commodity' like oil would be determined by Equation (3.7), which implies the oil futures price (in Chicago in USD) would always be above the spot price for oil (quoted in USD in Texas, say). But for many production commodities such as oil, at certain times it is observed that the quoted futures price is actually below the spot price, (i.e. $F < S$) – we then say that oil futures are in *backwardation*. This outcome is rationalised by adapting Equation (3.7) and introducing a concept known as the *convenience yield* y, for oil. How does this come about?

The convenience yield arises because the holder (Exxon) of a spot commodity (e.g. barrels of crude oil) has the added advantage that they can supply his local customers (in Texas) if the oil goes into short supply (e.g. during abnormally cold winter months) and hence retain customer loyalty. This 'convenient' state of affairs has an intrinsic value for the holder of oil, which is referred to as the convenience yield.

The presence of a convenience yield might therefore prevent cash-and-carry arbitrage operating and this invalidates Equation (3.7). For example, suppose the current spot price of oil (quoted in Texas) is high – an indication that oil is currently in short supply (i.e. low levels of oil stored on-shore in Texas). Ms Arb wants to undertake an arbitrage strategy which requires her to short-sell spot oil at S ($ per barrel) and simultaneously buy futures contracts on oil at F ($ per barrel). Short-selling means Ms Arb has to borrow oil from a holder of oil (Exxon) – which is being stored in onshore oil containers (or maybe sitting in the hold of a tanker in the middle of the ocean). Short-selling may not be possible in practice if Exxon is unwilling to 'loan out' their oil inventory.

To make a risk-free profit Ms Arb has to borrow barrels of oil today and quickly sell them in the spot market (in Texas) and immediately invest the proceeds in a risk-free deposit account. When her long futures contract matures she pays F and takes delivery of oil in the futures contract and 'returns' the 'borrowed oil' to Exxon. But if holders of spot oil (Exxon) will not provide spot oil to Ms Arb (at the required geographical destination) then arbitrage involving short-selling of oil becomes impossible.

Hence at these times Equation (3.7) will not hold for quoted futures prices on oil and $F \neq S\ [1 + (r + s)T]$ based on historical data. But (daily) historical data on F, S, r, s and T are available. We therefore introduce an 'unknown variable', the convenience yield y, to *force an equality*, so that:

$$F = S[1 + (r + s - y)T] \tag{3.8a}$$

We do this by taking quoted (daily) prices from the historical data, on F, S, r, s and T we derive ('back out') the historical 'convenience yield' (p.a.) which is given by:

$$y = r + s - \frac{1}{T}\left(\frac{F - S}{S}\right) \tag{3.8b}$$

Using calculated daily values for y using (3.8b) it must be the case that for the historical data the equality in (3.8a) holds. The right-hand side of Equation (3.8a) now provides the determinants of the *historical* oil futures price, F. This provides a way of trying to determine what will happen to the futures price for oil over the coming days and months. If today, we know what the convenience yield y of holding a spot barrel of oil (over say the next 3 months) is likely to be, then we could use (3.8a) to price a 3-month futures contract. The historical time series for y can be used to try and forecast what y might be *over the next 3 months*. But clearly this is a forecast and could be very inaccurate. However, the market will today set a futures price, which will embody traders' best estimate of the future convenience yield.

If the convenience yield remains constant then *changes* in the oil futures price (over the next 3 months) will be closely linked to *changes* in the spot price of oil. But the convenience yield of oil could itself change dramatically over the next 3 months, as supply and demand for oil in the production process and by consumers, as well as inventories of spot oil, change. Hence there is therefore not such a close link between *changes* in spot and futures prices for 'production commodities' as there is for futures contracts on other 'investment' assets (e.g. stocks, stock indices, currencies, bonds, gold) – so F and S may not move approximately dollar-for-dollar because of changes in y. This makes hedging 'production commodities' more difficult – the technical way of saying this is that 'basis risk' is high for commodity futures where the underlying asset is a 'production commodity'. We will consider these issues further in the chapter on energy derivatives.

3.3.3 Continuous and Discrete Compounded Rates

The above formulas for futures prices using simple rates, compounded rates, and continuously compounded interest rates, look slightly different. However, they all give exactly the same value for F. This is easily verified by recalling the relationship between the various definitions. Let r_s = simple rate, r = discrete compound rate and r_c = continuously compounded rate – all expressed 'per annum' (decimal). If \$1 is invested today, we require that this will result in the same amount at the end of, say, 3 months (i.e. $T = 1/4$), regardless of which interest rate convention is used. Therefore the interest rate conventions are *defined* so that:

$$1 + r_s T = (1 + r)^T = e^{r_c.T} \tag{3.9}$$

For a futures contract on (a non-dividend paying) stock we have $F = S(1 + r_s T)$ – but using (3.9) this is equivalent to $F = S(1 + r)^T$ and $F = Se^{r_c T}$ – hence the alternative formulas give the same value for F. It is also true that all three formulas would give *approximately* the same answer for F even if we used the 'incorrect' interest rate in each formula. This is because for 'small' values of r, the following approximations are quite accurate:

$$e^{r.T} \approx 1 + r\,T \quad \text{and} \quad (1 + r)^T \approx 1 + r\,T \tag{3.10}$$

Finally note that if:

$$F = Se^{r_c T} \quad \text{then} \quad S = Fe^{-r_c T} \tag{3.11}$$

3.3.4 Non-storable Commodities

How can we price a forward/futures contract on a non-storable commodity like electricity and what about any seasonality in the spot prices of commodities such as an agricultural product like wheat? The spot price of wheat tends to rise just before the harvest, usually around August and then falls as the wheat supply comes onto the (spot/cash) market. Seasonality in the spot price will be reflected in seasonality in the futures price because arbitrage is possible within the year, using inventories of wheat. The convenience yield of wheat just before the harvest is likely to be high and just after the harvest, when inventories of wheat are large, the convenience yield is likely to be low. We can use Equation (3.8a) to determine the futures price providing we can obtain an estimate of the market's view of the future availability of the commodity, which influences the convenience yield. We can 'back out' the historical time series of the convenience yield using (3.8b) and then use statistical forecasting methods, to forecast the convenience yield (over the life of the futures contract).

Instead of trying to estimate the convenience yield, a different approach to determine today's futures price involves trying to forecast the future spot price of a commodity. Suppose it is 1 May and we wish to determine today's futures price F_0 for delivery of wheat in 6 months' time on 1 November ($T = 1/2$ year). Suppose the expected spot price of wheat (per bushel) for November is ES_T and the riskiness of wheat prices means that investors require a (simple) return of R p.a. to speculate on the future spot price of wheat. If the futures price quoted today for delivery *and payment* of wheat in 6 months is F_0, then a speculator could borrow cash equal to $F_0/(1 + rT)$ and take a long position in the futures contract. Hence, at T the speculator has funds available to take delivery of wheat in the futures contract. Suppose the speculator believes that the expected price of wheat in 6 months' time is ES_T which has a present value of $ES_T/(1 + R.T)$. If we equate the present value of payment for the wheat delivered in the futures contract with the present value of supplying spot-wheat in 6 months' time, we have:

$$F_0/(1 + rT) = ES_T/(1 + R.T) \tag{3.12a}$$

and

$$F_0 = ES_T[(1 + rT)/(1 + R.T)] \tag{3.12b}$$

So today's futures quote depends on the expected spot price of wheat, which itself is influenced by forecasts of supply and demand for wheat in 6 months' time – this can lead to seasonality in spot and futures prices. Where possibilities for arbitrage are limited then the close link between spot and futures prices is broken and successful hedging becomes more difficult. Note also that the above approach implies that the forward price is not an unbiased forecast of the expected future spot price – unless the required return on the underlying asset R, equals the risk-free rate, r. (This can be shown to be the case using the Capital Asset Pricing Model [CAPM], when the underlying asset and the 'market return' are uncorrelated and hence the asset-beta is zero.)

3.4 VALUE OF A FORWARD CONTRACT

In this section we explicitly deal with forward contracts rather than futures contracts. Because a futures contract is marked-to-market daily and cash is credited or debited to the margin account, a futures contract has a value of zero at the end of each trading day.

But when dealing with forward contracts it is important to distinguish between price and value. Suppose that on 25 January ($t = 0$) you initiate (go long) a *6-month* forward contract ($T = 1/2$ year) on a stock, at a *forward price* $F_0 = \$90$. At $t = 0$ no cash changes hands. The maturity date of the contract is 25 July. On 25 January you have agreed to exchange the underlying assets (stocks) in exchange for a cash payment of $F_0 = \$90$ on 25 July.

At any future date (e.g. $t =$ April) during the life of the forward contract, you can enter into a 'new' forward contract *with the same delivery date, 25 July,* as the initial forward contract. The forward price F_t for a 'new' forward contract entered into on 25 April (with delivery date, 25 July) will be different from the price of the 'old' forward contract $F_0 = \$90$ initiated on 25 January (because of changes in the stock price between 25 January and 25 April). The question we wish to answer is: what determines the value of the 'old' forward contract (initiated on 25 January) each day, as this contract approaches its maturity date of 25 July?

3.4.1 Value When Initiated

Suppose at the initiation of the forward contract on 25 January the forward price is $F_0 = S_0(1 + rT) = \$90$. However, *the value* of the long forward contract is zero, since no arbitrage opportunities exist and no cash changes hands.

At the initiation of a forward contract, the value of the contract is zero.

3.4.2 Value at Maturity

Clearly the *price* of a forward contract on its maturity date must equal the spot price: $F_T = S_T$. If $F_T \neq S_T$ it would be possible to make arbitrage profits. For example, if $F_T = \$90$ and $S_T = \$100$ you could buy a long forward contract, take delivery of the underlying (stock) immediately and pay \$90. You could then immediately sell the underlying (stock) in the spot market (NYSE) for \$100 and pocket the difference of \$10. The deal is virtually risk-free. Hence arbitrage will ensure that $F_T = S_T$ on the maturity date of the forward contract.

Now consider the *value* of a long position in *a* forward contract at maturity. If the forward contract was initially entered into on 25 January at a price $F_0 = \$90$, then at maturity T (25 July), *the value* of this 'old' (maturing) forward contract is the profit you could make at T, hence:

$$V_T = \textit{Value of long position in 'old' forward contract at maturity} = S_T - F_0$$

If you are long the 'old' forward contract at an initial forward price (agreed on 25 January) of $F_0 = \$90$, then at T (on 25 September) you can take delivery of the underlying stock in the forward contract, pay $F_0 = \$90$ and immediately sell the underlying for S_T in the cash market. If $S_T > F_0$ the long position in the forward contract has a positive value (and vice versa).

3.4.3 Value of Forward Contract Prior to Expiration

Suppose on 25 January ($t = 0$) a 6-month forward contract (on a non-dividend paying stock) with maturity on 25 July is purchased at a forward price $F_0 = \$90$. Sometime later on 25 April, the forward price F_t for a 'new' forward contract (with the same maturity date, 25 July) has only 3 months to maturity ($T - t = 0.25$) (see Figure 3.3) and if $S_t = \$100$ and $r = 0.05$ (continuously compounded), then the price of this 'new' forward contract on 25 April is:

$$F_t = S_t e^{r(T-t)} = \$100\ \mathrm{e}^{0.05(0.25)} = \$101.2578$$

The question we wish to answer is: what is the value V_t on 25 April, of the *original* 'old' forward contract?

On 25 January, an investor Ms Player initiated her long position in the 'old' contract at a forward price of $F_0 = \$90$. On 25 April, Ms Player could short the 'new' contract (with the same maturity date of 25 July) at a forward price $F_t = \$101.2578$. Then at maturity of *both* contracts on 25 July Ms Player can (a) take delivery of the stock from her long position in the 'old' forward contract by paying $F_0 = \$90$, and (b) then deliver this stock to fulfil her obligations in the 'new' short forward contract and receive $F_t = \$101.2578$ in cash on 25 July. This cash profit at time T (25 July) is $F_t - F_0$ and is risk-free, because at $t = 25$ April she knows what this profit will be. The value of the *old* contract at time $t = 25$ April must therefore equal the *present value* of $F_t - F_0$ over the remaining period $T - t$:

$$\begin{aligned} V_t(\text{'Old' forward contract}) &= PV\ [\text{'New' forward price at } t\text{–'Old' forward price}] \\ &= [F_t - F_0]\ e^{-r(T-t)} = S_t - F_0 e^{-r(T-t)} \end{aligned} \tag{3.13}$$

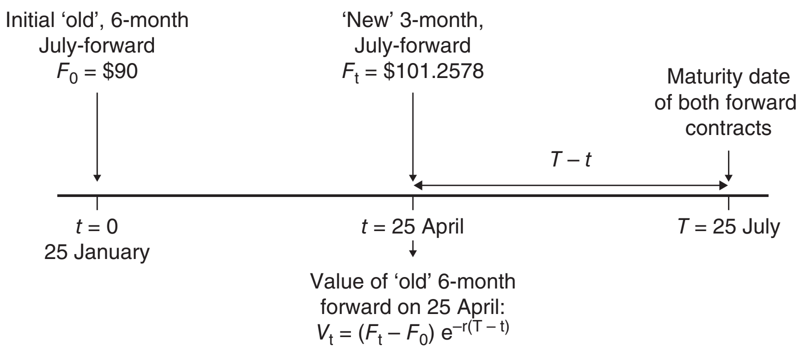

FIGURE 3.3 Value of forward contract

where we have used $F_t = S_t e^{r(T-t)}$. The above expression for the value of the 'old' contract follows from arbitrage arguments.

Ms Player purchased the 'old' July-forward contract on 25 January at a forward price F_0 = \$90 and on 25 April she shorts a new July-forward contract at F_1 = \$101.2578. Shorting the 'new' contract on 25 April will yield receipts of F_1 = \$101.2578 on 25 July and the 'old' contract requires a payment of F_0 = \$90 on 25 July. Hence the profit on 25 July is \$11.2578 [= \$101.2578 – \$90]. Since this strategy is risk-free, another investor would be prepared to pay Ms Player V_t = \$11.118 [= \$11.2578 $e^{-0.05(0.25)}$] on 25 April, for the 'old' forward contract (which was initially negotiated on 25 January). Otherwise, arbitrage profits can be made.[6]

Value of an 'old' forward contract at any time prior to maturity

$$V_t = [F_t - F_0]\, e^{-r(T-t)} = S_t - F_0\, e^{-r(T-t)}$$

The 'mark-to-market' value of the long forward contract on 25 April is V_t = \$11.118. If JPMorgan had initially sold this contract it would have to pay the person holding the long position \$11.118 on 25 April, to nullify the contract.

3.4.4 Replication Portfolio

The value of the 'old' forward contract can also be determined by creating a 'replication portfolio' at t = 25 April. Suppose on 25 April you purchase the stock for S_t and also borrow a cash amount of $F_0 e^{-r(T-t)}$ from the bank at interest rate, r – this is the replication portfolio which will mimic the outcome of the 'old' forward contract at maturity, T.

The amount you owe the bank on 25 July is F_0. The net cost (i.e. 'own funds') to initiate this replication strategy at t = 25 April is $S_t - F_0 e^{-r(T-t)}$. At T you have one stock worth S_T and you owe the bank F_0. But the latter outcome from the replication portfolio at T is exactly the same as if you were long a forward contract (which matures at T) with forward price of F_0. We have therefore 'replicated' the outcome at T, for a forward contract with a forward price of F_0. The cost of setting up this 'replication portfolio' at t = 25 April is $S_t - F_0 e^{-r(T-t)}$. At t, the value of the 'old' forward contract must equal the cost of setting up the replication portfolio (otherwise arbitrage profits can be made) hence, $V_t = S_t - F_0 e^{-r(T-t)}$.

Note that if we use compound interest rates or simple interest (rather than continuously compounded rates), the value of a forward contract (if the underlying asset has no cash flows) is:

$$V_t = \frac{F_t - F_0}{(1+r)^{T-t}} = S_t - \frac{F_0}{(1+r)^{T-t}} \quad \text{(compound rates)} \tag{3.14a}$$

[6]Note that the value on 25 April of \$11.118 applies to the 'old' forward contract. The 'new' forward contract when initiated on 25 April has zero value and is introduced only so we can value the old forward contract.

or

$$V_t = \frac{F_t - F_o}{[1 + r(T - t)]} = S_t - \frac{F_o}{[1 + r(T - t)]} \quad \text{(simple rates)} \tag{3.14b}$$

3.4.5 Underlying Asset Has a Cash Inflow

For a forward contract on a dividend paying stock, which matures on 25 July, the 'new' forward contract (initiated on 25 April) has a *forward price:*

$$F_t = S_t e^{(r-\delta)(T-t)}$$

where δ is the (continuously compounded) dividend yield. On 25 April, the value of an 'old' forward contract (initiated on 25 January, with maturity date 25 July) which pays dividends on the underlying stock is:

$$V_t = [F_1 - F_0]\, e^{-r(T-t)} = S_t e^{-\delta(T-t)} - F_0 e^{-r(T-t)} \tag{3.15}$$

At inception, a forward contract has a *value* of zero which is consistent with the above equation – set $t = 0$ and note that at inception $F_0 = S_0 e^{(r-\delta)T}$ so we obtain $V_0 = 0$. At maturity $t = T$ and (3.15) implies $V_T = S_T - F_0$, which is consistent with the value of the forward contract at maturity, as we found earlier. Note that at $t > 0$ the *value* of the forward contract V_t can be either positive or negative.[7]

Note also that δ could represent any (continuous) cash *inflows* from the underlying asset. For a forward contract on a commodity (e.g. oil), δ = convenience yield less the storage cost ($= y - s$). For a forward contract on foreign currency, δ = risk-free interest rate on foreign bank deposits and r = risk-free rate on domestic deposits.

If the net cash *inflows* on the underlying asset are discrete then the value of the 'old' forward contract at t is

$$V_t = S_t - PV \; (net\ cash\ inflows) - F_0 e^{-r(T-t)}$$

where PV is the present value of any (dollar) net cash inflows between today t and the maturity date of the forward contract, T. The net cash inflows could be 'dollar dividends' on a stock, or receipts of coupons on a bond, or for commodities the 'dollar' convenience yield less storage costs.

[7]Because we need to clearly distinguish between the two forward prices F_t and F_0 in the above expressions, the initial forward price F_0 of the 'old' contract is sometimes referred to as 'the delivery price' and is given the symbol K (i.e. $F_0 \equiv K$). The delivery of the underlying asset in the 'old' forward contract results in a cash payment $F_0 \equiv K$.

3.5 SUMMARY

- Risk-free arbitrage ensures that the correct/fair forward price is determined by the cost of a synthetic or replication portfolio which produces the same outcomes as the forward contract itself – this is cash-and-carry arbitrage. For the simplest forward contract on an underlying asset S (which pays no cash flows) we have: $F = S(1 + rT)$.
- We assume that futures prices are determined by arbitrage in the same way as forward prices – this is usually a good working assumption because margin payments on futures contracts do not have a major impact on the above relationship.
- If the quoted futures price does not equal the 'correct' (no-arbitrage) futures price determined by the cost-of-carry model, then risk-free arbitrage profits can be made.
- Cash-and-carry arbitrage ensures that the futures price on a stock (traded in Chicago) moves almost dollar-for-dollar with the spot (cash-market) price of the stock (on the NYSE), over small time intervals.
- Cash-and-carry arbitrage can be used to determine the correct (no-arbitrage) futures price where the underlying assets involve intermediate cash flows (e.g. futures contract on dividend paying stocks and stock indices) and that involve storage costs (e.g. silver, gold).
- Futures contracts on commodities that are also used in the production of consumer goods (e.g. oil, gas, wheat, live cattle) are more difficult to price by cash-and-carry arbitrage because the futures price depends on the 'convenience yield' which is difficult to forecast.
- The futures price may not move exactly in line with the (underlying) spot price if interest rates, dividends, storage costs or the convenience yield change in unpredictable ways – and the convenience yield is the most unpredictable element.
- The *forward* price (on a non-dividend paying stock) with maturity in T years is $F_0 = S_0 e^{rT}$ determined today ($t = 0$). After initiation, the *value* of this forward contract changes over time and at time $t > 0$ the value of the forward contract is: $V_t = [F_t - F_0]\, e^{-r(T-t)} = S_t - F_0 e^{-r(T-t)}$. Hence V_t varies with the spot price S_t, interest rate and time to maturity.
- On the other hand, the *value of a futures contract* at the end of each day is zero, since the contract is marked-to-market and the margin account is credited or debited daily.

EXERCISES

Question 1
How do futures contracts provide 'leverage'?

Question 2
For a futures contract on a (non-dividend paying) stock, carefully explain how you can make an arbitrage profit if the quoted futures price F_q is less than the 'fair' (or 'synthetic') futures price.

Question 3
For a futures contract on a dividend paying stock, with a single dividend payment D before maturity of the futures, carefully explain how you can make an arbitrage profit if the quoted futures price F_q is above the 'fair' ('no-arbitrage' or 'synthetic') futures price.

Question 4
What is the 'convenience yield' and how does it complicate arbitrage between the spot and futures markets?

Question 5
The stock price is currently at $S = 400$, and the quoted futures price on a T = 4-month contract is $F_q = 405$, $r = 10\%$ p.a. and the dividend yield is 4% p.a. (both continuously compounded). How could you make an arbitrage profit?

Question 6
The spot price of an asset is S_0, the futures price today for delivery or settlement at period T in the future is F_0, the risk-free rate over the period from $t = 0$ to $t = T$ is r (simple rate). Explain the relationship that must hold between S_0, F_0, and r.

Now suppose the asset pays a known income stream C over the period, the present value of which is $PV(C)$. Explain the relationship that must hold between S_0, F_0, $PV(C)$ and r.

CHAPTER 4

Futures: Hedging and Speculation

Aims

- To examine how futures can be used for hedging.
- To explain basis risk, cross hedge and rolling hedge.
- To examine some novel types of futures contracts.
- To show how futures are used for speculation.

4.1 HEDGING USING FUTURES

Futures contracts can be used for hedging an existing position in a spot (cash-market) asset. For example, an oil producer might have a large amount of oil coming 'on stream' in 3 months' time and may fear a fall in the spot oil price over the next 3 months, so will have to sell their oil at a low spot price. Or, a US exporter who is receiving sterling from the sale of goods in the UK in 3 months, may fear a fall in the spot FX rate for sterling – which implies they will receive fewer dollars when they exchange sterling for dollars.

In the above cases the seller (of oil or sterling) does not know what the spot/cash market price will be in 3 months' time and their future (dollar) receipts are therefore subject to price-risk. Hedging with futures can be used to reduce such risk to a minimum. In practice, using futures contracts cannot reduce price-risk to zero so a 'perfect hedge' is usually not possible.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

4.1.1 Short Hedge

The basic idea behind hedging is very simple. If you are holding (i.e. long) the underlying asset (e.g. stock-ABC) then you need to take a futures position (on stock-ABC) such that the monetary gain, after closing out the futures contract, largely offsets any monetary losses in the spot market, over your chosen hedge period. Futures and spot prices tend to rise and fall together (approximately) dollar-for-dollar because arbitrageurs ensure that $F = S(1 + rT)$ at all times. Hence, on 25 June in order to hedge a long position in stock-ABC with price $S_0 = \$100$ (over the next 3 months, $T = 1/4$), you need to short (i.e. sell) September-futures contracts today. If on 25 June the risk-free rate is $r = 1\%$ p.a., and the quoted September-futures price is $F_0 = \$100.25$ $(= S_0(1 + rT))$.

If the spot/cash market price of the stock falls by \$10 (on the NYSE) to $S_1 = \$90$ over the next 3 months, then arbitrageurs will ensure that the futures price (which matures close to 25 September) will also be close to $F_1 = \$90$. You initially sold the futures contract for $F_0 =$ \$100.25 on 25 June and on 25 September you can 'buy back' the futures contract (i.e. close out your futures position) at $F_1 = \$90$. Your profit is $F_0 - F_1 = \$10.25$ – which you obtain from the clearing house in Chicago.

A 'sell' followed by a 'buy' for futures contracts (with the same maturity date and underlying asset) means that any delivery to you of the stock is nullified. Closing out means 'no delivery' – just the cash settlement of \$10.25. Here, the gain on the futures hedge of \$10.25 more than offsets the \$10 loss on your stocks. Without the hedge you would have lost \$10.

A hedge is *designed* to result in no change in the value of your overall position (i.e. stocks plus futures). You therefore forego the prospect of any *overall* gains (or losses) when you use futures contracts for hedging.

Let's examine the above hedge in more detail. On 25 June Ms Long holds one stock-ABC with a spot (cash market) price $S_0 = \$100$. Normally Ms Long is willing to accept the price risk of stock-ABC. But suppose over the next 3 months ($T = 1/4$ year), she thinks the stock market will be more volatile than normal and in particular she fears an exceptionally large fall in stock prices by 25 September, when she *might* want to sell her stock-ABC or her trading position will be assessed.

To remove any price risk she could sell stock-ABC on 25 June for $S_0 = \$100$ and invest the \$100 in an interest bearing bank deposit until 25 September, when she could either (a) repurchase the stock-ABC possibly at a lower price ($S_1 = \$90$) or at a higher price ($S_1 = \$110$), or (b) continue to hold cash in the deposit account which will have accrued to $\$100.25 = S_0(1 + rT)$.

There are problems with this strategy. First, if she invests in the bank deposit on 25 June but later, on 25 September, she decides she wants to hold the stock, then she must repurchase stock-ABC at either the low price of $S_1 = \$90$ which would be advantageous, or at the high price $S_1 = \$110$ which would be expensive (relative to the initial price of $S_0 = \$100$ on 25 June). But on 25 June she does not know which of these outcomes might occur, so this strategy has not removed price risk – if she subsequently wants to hold the stock. Also this strategy requires selling stock-ABC on 25 June and repurchasing stock-ABC on 25 September – so she will incur transactions costs in the form of bid–ask spreads and commissions. However, if she is *certain* she wants to hold cash on 25 September, then investing in the bank deposit guarantees she will have a cash amount of $S_0(1 + rT)$ in her deposit account on 25 September.

Instead of using the above strategy, Ms Long can hedge her existing stock position on 25 June by shorting a futures contract. This means that if stock prices rise or fall dramatically between 25 June and 25 September, Ms Long will continue to hold her stocks-ABC (and hence avoid any transactions costs of buying and selling). In addition, with the futures hedge in place she will have neither gained or lost and hence has removed any price risk (unlike the case above). Although hedging with futures allows Ms Long to offset any losses from a fall in stock prices, it does not allow Ms Long to benefit from a rise in stock prices – but this is the outcome that hedging is designed to produce.

The September-futures price on 25 June is $F_0 = \$100.25$. Can Ms Long who holds stocks-ABC guarantee an outcome of (around) \$100.25 on 25 September? Note that on 25 June, the outcome Ms Long seeks to achieve is an effective value for her stocks of $F_0 = \$100.25$ in 3 months' time – she cannot lock in the current stock price $S_0 = \$100$.

On 25 June Ms Long initiates the hedge by selling (shorting) the September-futures contract at $F_0 = \$100.25$ (but no money changes hands). Suppose that on 25 September the price of stock-ABC has fallen to $S_1 = \$90$ and the futures to $F_1 = \$90$ (Table 4.1).

TABLE 4.1 Hedge (basic data)

25 June	25 September
$S_0 = \$100$	$S_1 = \$90$
$F_0 = \$100.25$ (September delivery)	$F_1 = \$90$

On 25 September Ms Long closes out her futures position by buying (back) the September-futures at the lower price $F_1 = \$90$, making a profit of $(F_0 - F_1) = \$10.25$ – which approximately offsets the loss in the spot market of \$10 (Table 4.2). The value V_1 (on 25 September) of Ms Long's hedged portfolio is:

$$\begin{aligned} V_1 &= \text{Spot price of stock-ABC} + \text{\$-Change in futures position} \\ &= S_1 + (F_0 - F_1) = (S_1 - F_1) + F_0 = \$100.25 \end{aligned} \tag{4.1}$$

TABLE 4.2 Short hedge: outcome on 25 September

Sale of stocks: S_1	= \$90
Gain on futures: $(F_0 - F_1)$	= (\$100.25 – \$90) = \$10.25
Total receipts	= \$90 + \$10.25 = \$100.25 $= S_1 + (F_0 - F_1) = F_0 + (S_1 - F_1)$

F_0 is known at the outset of the hedge on 25 June but S_1 and F_1 are not. Hence the risk in the hedge depends on 'the basis' where:

$$\text{Final basis}: \quad b_1 = S_1 - F_1 \tag{4.2}$$

In our example the value of the hedged portfolio is \$100.25 (on 25 September) and this equals the quoted futures price F_0 on 25 June – so Ms Long on 25 June has fixed the value of her portfolio, even if the stock price falls. This is a perfect hedge because we made the final basis, b_1 equal to zero – as we assume the futures contract matures on (or near) 25 September so that $S_1 = F_1$. In general, if the contract is closed out before the maturity date then b_1 will not be zero (but it will be small) – hence there is always some basis risk in a hedge.

Note that in the hedge, the price that is 'locked in' is the futures price F_0 and not the spot price S_0. In fact, the spot price at the outset $S_0 = \$100$ plays no direct role in the calculation of the *value* of the hedged position on 25 September (see Equation (4.1)). The only transactions cost you incur are in selling then buying back the futures and these costs are very small (and usually much lower in the futures market than on the NYSE).

If the futures contract had been on 'live cattle' and you require your cattle in Boston it should be obvious that it is better not to pay $F_0 = 100.25$ on 25 September and take delivery of cattle in the futures contract near Chicago (the delivery point in the futures contract). It is more convenient to take the cash profit on the futures of \$10.25, buy your cattle on 25 September in the local market in Boston at the spot price $S_1 = \$90$, since then you do not have the additional cost of transporting the cattle between Chicago and Boston. Closing out (just before maturity) rather than delivery (on the maturity date T), also happens with many *financial* futures contracts (e.g. on stocks, bonds, FX). Also some futures contracts can only be cash settled at maturity rather than taking delivery (e.g. futures on the S&P 500 index, where it would be extremely inconvenient and costly to 'deliver' 500 stocks).

An *alternative way* of representing the hedge is to examine the *net profit* Π, that is, the loss (gain) on the spot position relative to the gain (loss) on the short futures position.

$$\begin{aligned}\Pi &= \text{Change in spot price} - \text{Change in futures}\\ &= (S_1 - S_0) - (F_1 - F_0) = (\$90 - \$100) - (\$90 - \$100.25) = +\$0.25\\ &= (S_1 - F_1) - (S_0 - F_0) = b_1 - b_0 \end{aligned} \tag{4.3}$$

In this example, the gain on the futures position is \$10.25 which more than offsets the loss on the spot position of \$10 and hence the hedge is broadly successful. The profit from the hedged position depends on the *change in the basis*. The change in the basis is small because spot and futures prices are highly correlated. Thus the hedge position is much less risky than the unhedged position – since without the hedge the loss (or gain) would be the much larger amount $S_1 - S_0$. Note also that the 'initial basis' b_0 is known and 'fixed' at the outset of the hedge. It is the 'final basis' $b_1 = S_1 - F_1$ that is the source of uncertainty in the net profit.

TABLE 4.3 Long hedge: outcome on 25 September

Sale of stocks: S_1	$= \$90$
Loss on futures: $(F_0 - F_1)$	$= (\$100.25 - \$90) = \$10.25$
Total receipts	$= \$90 + \$10.25 = \$100.25$
	$= S_1 + (F_0 - F_1) = F_0 + (S_1 - F_1)$

4.1.2 Long Hedge

Suppose on 25 June, the portfolio manager of a pension fund, Ms Wait, knows she will receive a cash inflow in 3 months' time on 25 September (from contributors to the pension fund), which she will use to purchase stocks. The stock price on 25 June is $S_0 = \$100$ – but she knows she cannot lock in this price by hedging – she can only lock in today's quoted futures price.

She is worried that stock prices will rise – so her stocks will cost more in 3 months, when the cash arrives to purchase them. To hedge, Ms Wait goes long (i.e. buys) a futures contract on 25 June at $F_0 = \$100.25$ and takes delivery of the stocks in 3 months and pays \$100.25 per stock (in Chicago) on 25 September. Alternatively, Ms Wait could simply close out her position by selling the futures contract on 25 September (see Table 4.3).

Suppose we have the same outcome as before, namely the stock prices falls to $S_1 = \$90$, as does the futures price $S_1 = \$90$. The cost of buying the stocks in the cash market (on the NYSE) is $S_1 = \$90$, but the cash loss from the long futures is $F_0 - F_1 = \$100.25 - \$90 = \$10.25$, making a net cost of buying the stocks on 25 September = \$90 + \$10.25 = \$100.25. Hence, the hedge when cash settled also 'locks in' the futures price $F_0 = \$100.25$. Again, this is because we have assumed the final basis $b_1 = S_1 - F_1$ is zero.

Some might find it confusing that the hedge does not 'lock in' a price of $S_0 = \$100$ on 25 September. But this is impossible since the price $S_0 = \$100$ on the NYSE applies to 25 June – you can only 'lock in' the futures price F_0 (quoted on 25 June for maturity on 25 September).

Note that with hindsight, it would have been better if the hedge had not been undertaken, since then the cost of buying the stocks on 25 September would have been $S_1 = \$90$. However, when the hedge was initiated in June, the hedger could not have known that the price would fall over the next 3 months – hindsight is a wonderful thing.

EXAMPLE 4.1

Long Futures Hedge when Stock Price Increases

Today on 25 June, Ms Wait hedges her future stock purchases by buying (going long) a 3-month futures contract at $F_0 = \$100.25$. If the stock price on the NYSE rises by \$10 to

(continued)

(*continued*)

S_1 = \$110 on 25 September, then $F_1 \approx$ \$110 in Chicago. (Arbitrage ensures that $F_1 \approx S_1$ close to the maturity date of the futures contract.)

Outcome

On 25 September Ms Wait closes out (sells) her futures at $F_1 \approx$ \$110 making a cash profit of $F_1 - F_0$ = \$9.75, credited to her by the clearing house in Chicago. Although the stocks cost \$10 more to buy on the NYSE on 25 September, this higher cost is mostly offset by the \$9.75 profit from the futures trade – so the hedge is successful.

The effective purchase price for the stocks on 25 September is S_1 = \$110 minus the cash profit of \$9.75 from the futures position, that is \$100.25 – which is the same as the quoted futures price on 25 June F_0 = \$100.25. Hence this hedge has locked in the known futures price quoted on 25 June.

In the above examples we assume (for expositional clarity) that on 25 September, $F_1 = S_1$. In practice this equality only holds exactly on the delivery/maturity date of the futures contract which might be a few days later at T = 28 September (say). If S_1 = \$90 on 25 September then F_1 might actually be equal to \$90.1 (say) rather than F = \$90 assumed above. Hence the final basis would be $b_1 = (S_1 - F_1) = -0.1$, so the above numerical results would be slightly different but the general points made are still valid.

4.1.3 Cross Hedge

If the asset underlying the futures contract is not the same as the asset held by the hedger then there is an additional source of basis risk. Let S^{HO} be the cash market price of heating oil and F^{HO} the price of a heating oil futures contract.

On 25 June, Cloud-9, an airline, knows that it is going to purchase jet fuel in 2 months' time on 25 August, at its hub airport (LAX) at the cash market price S^{JF}. But it fears a rise in jet fuel prices. Suppose there is no futures contract available on jet fuel[1] so Cloud-9 hedges by going long *September-heating oil futures* contracts (with a maturity date on 28 September). The effective cost V_1 of the jet fuel for Cloud-9 on 25 August (when the airline closes out its heating oil futures contracts) is:

$$V_1 = \text{Spot price } \textit{jet fuel} - \text{cash profit on } \textit{heating oil} \text{ futures} = S_1^{JF} - (F_1^{HO} - F_0^{HO}) \quad (4.4)$$

[1]Until recently there was not a futures contract on jet fuel. But airlines tend to hedge using heating oil futures because these contracts are more liquid (in the financial meaning of the word!) than the more recently introduced, futures on jet fuel.

which can be written:

$$V_1 = F_0^{HO} - [(F_1^{HO} - S_1^{JF}) + (S_1^{JF} - S_1^{HO})]$$
$$= F_0^{HO} + b_1^{HO} + (S_1^{JF} - S_1^{HO}) \tag{4.5}$$

so there is an additional source of basis risk arising from the difference between the spot prices of jet fuel and heating oil on 25 August. Also since $b_1^{HO} = S_1^{HO} - F_1^{HO}$ equals zero only on the maturity date of the futures (28 September), the choice of the futures delivery month is important.

Clearly if you are going to close out your futures position on 25 August, then on 25 June you must choose a futures contract that matures *after* 25 August. Also, the size of the final basis $b_1^{HO} = S_1^{HO} - F_1^{HO}$ is more uncertain, the longer the gap between the end of the hedge period (t_1= 25 August) and the maturity of the futures contract (here T = 28 September). It is therefore best to choose a *delivery month* that is just after the hedge period – as this futures contract is likely to be heavily traded, with high liquidity and low transactions costs.

For example, suppose on 25 June when Cloud-9 initiates its hedge, heating oil futures contracts are available with maturity dates on 27 June, 28 September, 29 December and (next year on) 25 March. If you intend to close out the hedge on 25 August then you should probably use the September contract – this will ensure the final basis is reasonably small (i.e. $S_1^{HO} \approx F_1^{HO}$) and the hedge will work reasonably well.

4.1.4 Rolling Hedge

Assume it is now 5 April 2019 and Ms Refiner is holding 10,000 barrels of oil (i.e. she is long the underlying) which she will sell on 5 January 2020. She fears a price fall, so to hedge she needs to sell (short) oil futures, today. The contract size for oil futures is for delivery of 1,000 barrels, so to hedge her cash-market position of 10,000 barrels she needs to sell (short) ten futures contracts. Futures contracts are available with maturity dates on the 28th of the month for June-2019, September-2019, December-2019 and March-2020. However, on 5 April she feels that only the 'nearby' June-2019 contract is liquid enough (with low transactions costs) to use in the hedge.

One possible scenario is that on 5 April 2019 she shorts ten June-2019 futures contracts. Towards the end of June she buys back the ten June-2019 contracts which are about to expire and she simultaneously shorts ten September-2019 contracts (which are now the 'nearby contract'). Towards the end of September-2019 she closes out (buys back) these ten September-2019 contracts and then shorts ten December-2019 futures contracts. Finally, at the end of December-2019 she closes out (buys back) these ten December-2019 contracts – leaving her unhedged for a short period until 5 January 2020.

Every 3 months, she hedges using the 'next nearby contract'. This is a rolling hedge as she periodically 'rolls over' into the next nearby contract. Spot and futures prices move approximately dollar-for-dollar. So, for example, if the spot price of oil falls continuously over the coming year, she receives less from selling her barrels of oil, but this is offset by cash receipts ($F_{sell} - F_{buy} > 0$) when she closes out the ten short futures contracts by buying them back at a lower price, when each contract matures. She therefore executes a rolling hedge, over each 3-month period, always using the next nearby (liquid) contract.

An example of a rolling hedge with energy futures undertaken by the German firm Metallgesellschaft can be found in Finance Blog 4.1 – it illustrates some of the practical difficulties of trying to maintain a hedge by rolling over short-dated futures contracts.

Finance Blog 4.1 Metallgesellschaft's Rolling Hedge

Perhaps the most famous calamitous rolling hedge was that undertaken by MGRM, a US subsidiary of the German firm Metallgesellschaft. Unfortunately, in 1993/94, this rolling hedge using various commodity futures resulted in losses of around \$1.3bn, with the result that the firm had to be bailed out by a consortium of banks.

MGRM offered US firms long-term fixed-price purchase contracts on oil products (e.g. diesel fuel, heating oil and gasoline) for up to 10-year horizons, at prices around \$4 below current spot prices. MGRM had to deliver these oil products in the future at the agreed fixed price – and hence was vulnerable to a future rise in spot (cash market) oil prices. It therefore hedged (part) of its position by *going long* in the appropriate commodity futures contracts (e.g. contracts on unleaded gasoline, New York Harbour No. 2 heating oil and light-sweet crude oil – all traded on NYMEX, the New York Mercantile Exchange).

However, because of illiquidity in the far-dated futures contracts, MGRM used short maturity contracts, which it then had to roll over. By September 1993 MGRM had *sold* futures contracts covering 80 million barrels of oil (equivalent to about 90 days of Kuwait's output).

A problem arose near the end of 1993 when spot prices fell substantially. Therefore MGRM was making a *potential notional gain* on its future spot oil purchases but this was a gain it could not realise – since the negotiated fixed-price sale contracts applied to delivery in future years so MGRM was not currently purchasing all its oil requirements in the cash market – as storage costs would have been prohibitive.

However, futures prices fell along with spot prices and as MGRM was long futures, it incurred margin calls. Remember that when hedging, there is nothing unusual about making a loss on futures contracts, since you should then be making a (notional) gain in the spot (oil) market – this is how hedging works. But the 'immediate' cash drain (of \$900m in 1993) from the parent company, to pay variation margin calls on all its futures positions, was not offset by *actual* cash profits by buying spot oil at low prices and selling at high fixed prices to its customers.

Because of the cash drain from margin calls, the parent company eventually chose to close out their long futures contracts by selling them at a lower price and at a huge loss. Metallgesellschaft was saved from liquidation by a consortium of banks in January 1994, who lent the firm $1.2bn. Six months later, spot oil prices increased so that MGRM's now *unhedged* position, incurred further potential losses.

The lesson here is that in a hedge, there must be sufficient cash resources to meet margin calls. Also, had a uniform fall in futures prices of all maturities occurred, then MGRM would have continued to 'roll over' at a profit, but the move from backwardation to contango in the futures market in November 1993 meant it was rolling over its futures contracts at a loss.

Source: Adapted from Cuthbertson and Nitzsche (2001).

4.1.5 Rules for Hedging

- If you are *long* the cash-market asset (e.g. stocks) and hence fear a price fall, then hedge by taking a *short* futures position today (i.e. sell futures contracts).
- If you are planning to purchase the asset in the future (i.e. you are *short* in the cash market) and hence fear a price rise, then hedge by going *long* today (i.e. buy a futures contract).
- Choose a futures contract which is written on the spot (cash-market) asset in question. This maximises the correlation between F and S. Otherwise use a cross-hedge where there is a high correlation between the cash market asset you are trying to hedge (e.g. jet fuel) and the futures contract used in the hedge (e.g. heating oil futures).
- For hedging over a relatively short period of time, choose a futures contract with an expiration date *just longer* than the hedge period. This minimises basis risk since you close out just before the maturity date (T) of the futures, so $F_1 \approx S_1$.
- Suppose the hedge is for a long period and futures contracts with long maturity dates are either not available or illiquid with high transactions costs (bid–ask spreads, etc.). Then choose liquid, short-dated futures contracts (i.e. the next nearby contract) and 'roll over' the hedge as each 'short-dated' futures contract approaches its maturity date.
- Losses on the futures position may involve payments of 'variation margin' and therefore a hedger must have lines of credit or other assets available to meet these margin calls.
- You need to choose the appropriate number of futures contracts to use in the hedge – this is discussed in Chapter 5.

4.2 NOVEL FUTURES CONTRACTS

This book is primarily concerned with hedging positions in financial assets (and to a lesser extent in tradable commodities such as oil and gas) but it is worthwhile reflecting on the possibility of hedging more general risks that society faces. For example, almost all major countries such as the USA, UK, Japan, Australia, Brazil, and Asia have experienced booms and slumps in the housing and commercial property markets. If you have recently purchased a house only to find that its value has plummeted, this can be devastating. Conversely, if first-time buyers fail to buy before a major housing boom, they can be excluded from homeownership for a long time (and maybe forever). Take another even more powerful example. Individuals cannot adequately insure against major changes in their real incomes – through no fault of their own (e.g. booms and slumps either nationally or regionally and in particular industries). There seems to be no adequate mechanism for individuals to insure against these social risks. Could derivatives markets be used to provide such insurance and hence 'spread' these risks across a wide number of participants – just like when we use futures contracts to hedge changes in commodity and stock prices?

Robert Shiller of Yale University, who won the Nobel Prize in economics in 2013 (along with Eugene Fama), favours the establishment of futures markets based on house (and other real estate) price indices and on 'perpetual claims' on income. These contracts could be used to hedge the price risk involved in owning a home and the risk of unforeseen changes in income (e.g. your own occupational income or the real income (GDP) of the whole domestic economy). Shiller (1993) provides a provocative analysis of these issues, some of which are outlined in Finance Blog 4.2.

Finance Blog 4.2 Future Futures? Hedging House Prices

In principle, the idea here is a straightforward application of hedging with futures. Suppose there is a futures contract based on your *own house price*. If you own a house, then to hedge you would short 'house-price' futures. If your house price falls then so would the futures price and you could close out your long futures position by buying back the futures contract at a profit – this would compensate for the loss in the (spot) market value of your house. Of course, if your house price increased you would lose on your short futures position, but this is exactly counterbalanced by the increase in the value of your property.

If you are thinking of entering the market as a first-time buyer in the future then you could hedge any potential rise in house prices by today purchasing (i.e. going long) 'house-price' futures. If house prices rise, then the cash profit after closing out your long futures position, could be used to offset the increase in the purchase price of the house.

Shiller notes that the matter is not so simple in practice. Because people are relatively unfamiliar with futures markets then savings and loans associations (building societies, banks) would have to sell 'standard' house price insurance contracts (rather like fire and theft insurance) to their customers – and it would be the financial institutions that would hedge their position using futures. Also, the futures contracts would have to be written on house price *indices*, not on the price of an individual's house – this is to avoid 'moral hazard' problems. For example, if the futures contract is on your *own* house price and you have shorted futures, then you may have an incentive to allow your house to go into disrepair, since its spot price would then fall leading to a fall in the futures price. You could then close out your short futures position at a profit.

Using futures contracts on house price *indices* would be a cross-hedge, with some basis risk remaining. Of course, if this type of futures contract became popular then they could be written on *regional/local* house price indices as well on indices of state-wide or economy-wide house prices.

Cash and carry arbitrage is needed to establish the correct (no-arbitrage) futures price but 'delivery' of a 'homogeneous house' (i.e. the underlying asset in the futures contract) is somewhat difficult! But if *stock* price indices based on the performance of real estate companies are available, arbitrage becomes less problematic.

In the actual coffee futures market (on CME-NYMEX) 'expert tasters' are used (along with more objective criteria) to establish the 'quality' of the delivered coffee. However, since the quality of housing depends on such factors as site, neighbourhood, proximity of transport, schools, shops etc., it is unlikely that experts could agree on what constitutes a 'homogenous house'. Hence the futures contract would have to be cash settled, based on a house price index that takes account of quality differences.

The firm *Case-Shiller-Weiss* has developed such price indices for US property, using repeat sales prices and incorporating characteristics of the quality of the house, (known as hedonic price indices). The basic idea is to regress observed house prices $p_{it}(i = \text{house}, t = \text{time})$ on a set of characteristics (e.g. square metres of floor space, whether the house has air conditioning units or not, number and quality of schools in the area, distance from retail centre, etc.). One can then estimate a predictive equation for house price indices (for a geographical area) or even obtain an estimate of the value of individual houses. Predictions of individual house prices can be 'tested' against actual sales prices. The futures contract can then be based on these 'underlying' hedonic house price indices.

Another practical problem is margin payments. If you are short housing futures contracts and house prices rise, you may receive margin calls. But you may not have sufficient liquid assets to pay the margin calls. Although the value of your house has risen, you cannot sell part of it to obtain the cash to cover your margin calls. This problem is mitigated if it is a financial intermediary that is undertaking the short hedge (and has sold a 'house price' insurance policy to the homeowner).

Source: Adapted from Cuthbertson and Nitzsche (2001).

Shiller also considered the risks to individuals from changing household incomes as relative earnings change over long periods of time (for example, incomes in the industrial sector may fall relative to those in the information technology [IT] sector over a 10-year period).

Hedging using futures contracts based on an underlying index of national, regional or occupational incomes, works in much the same way as for house prices. (But note that Shiller's actual proposal is more subtle being based on 'long-run' or perpetual income, rather than current or next year's income). If your 'own-income' is positively correlated with regional income, then you (or more likely a financial institution who sold you an 'income insurance policy') would hedge by shorting 'regional income futures' contracts. If your income fell and along with it regional income, then you would receive margin payments (from the futures clearing house). These margin receipts could then be reinvested in a portfolio of income claims in a wide variety of other countries, which would have less variability than your own income stream – hence reducing your overall income risk. The problem of a futures-hedger meeting margin payments remains a problem and requires banks to lend against future income – for the purpose of covering these margin payments.

There are many other practical and provocative ideas in Shiller's books *Macro Markets* (1994) and *Finance and the Good Society* (2013). Who knows, in the future, your children may use these types of futures contract to hedge some of their key long-term 'lifetime risks'.

4.3 SPECULATION

Speculation with futures is relatively straightforward. Consider speculation using a futures contract on stock-ABC. (We deal with speculation using 'stock *index* futures contracts' in Chapter 5.) You are using *futures* contracts to gamble on forecasts for the price of *stock-ABC*.

If you think the price of stock-ABC will rise (fall) in the future (on the NYSE) then today you will speculate by buying (selling) a futures contract (on stock-ABC) at F_0 in Chicago. If you forecast the direction of change in stock prices and hence future prices, then you will be able to close out your futures position at a profit – if you forecast incorrectly you will close out your futures position at a loss. Speculation is risky.

If you close out a *long* futures position before maturity then the profit/loss on each futures contract is $F_1 - F_0$ (i.e. $F_1 - F_0 > 0$ if $F_1 > F_0$ and $F_1 - F_0 < 0$ if $F_1 < F_0$). On the maturity date (T) of the futures, the stock price and futures price must be equal (because of arbitrage) $S_T = F_T$. Hence, if a speculator holds the long futures contact to maturity ($=T$), she can pay F_0, at T, take delivery of the stock (in Chicago) and immediately sell it on the NYSE for S_T. Hence her speculative profit at T on her long futures position at maturity is $S_T - F_0$ (which can be either positive or negative).

Similar arguments to the above apply if a speculator initially sells a futures contract, because she thinks the stock price and hence the futures price will fall, on or before the maturity date of the futures contract.

4.3.1 Speculation with Stocks Versus Futures Contracts

What are the institutional differences between speculating on changes in the price of stock-ABC on the NYSE and speculating with futures (on stock-ABC) in Chicago. If you speculate by buying stock-ABC on the NYSE you (may) have to use your 'own funds' to buy the stock and pay any transactions costs (bid–ask spreads and brokers' commissions). If you use futures contracts on the stock, the long futures position provides leverage, since you do not have to use any 'own funds' when you initiate the futures position (and transactions costs in futures markets are generally much lower than those on the NYSE).

When using futures you have to place a relatively small 'good faith deposit' in your margin account. Of course, if you are long (short) the futures and the futures price falls (rises) then you may have to 'top up' your margin account and speculators therefore need ready cash or collateral (e.g. T-bills) available.

For a speculator holding stock-ABC to benefit from a forecast rise in stock prices, she can (in principle) wait indefinitely for the price to rise. In contrast, when taking a speculative long position in a futures contract, the stock price must rise before the maturity date in the futures contract. There is a 'time limit' when using a long futures position for speculation.

When using stocks to bet on a fall in stock prices a speculator has to short-sell stocks. However, if she cannot 'borrow' the stock from her brokers for a long enough time period she may not be able to 'hold' her short position long enough to benefit from the predicted fall in stock prices. Similarly, when using a short futures position to gain from a fall in stock (and futures) prices, the stock price must fall before the maturity date in the futures contract. In practice there is a 'time limit', either when short-selling a stock or when speculating using a short futures position – the former time limit is somewhat uncertain (depending on your relationship with your broker) and for the futures, the time limit is fixed by the maturity of the contract.

Speculators take an 'open' or 'naked' position and hence the transaction is risky. There are three broad types of speculator: scalpers/algorithmic traders, day traders and position traders.

- *Scalpers (Algorithmic traders, High Frequency Traders, HFT)* transact (mainly) on electronic platforms and may close out their positions within milli-seconds or at most minutes. Traders who use graphs of tick-by-tick prices to predict changes in futures prices come under the generic term 'chartists' of 'technical analysis'. Algorithmic traders buy and sell based on complex mathematical formulas often based on very short-term price movements and volume/turnover in the market often helps predict future short-term price movements.
- *Day Traders* usually close out their positions within a few hours, or at most within the trading day. They often trade, based on their views about 'news items' (e.g. future company dividend outcomes, forecasts of changes in interest rates or FX rates) that may affect futures prices.
- *Position Traders* may hold their positions for as little as one day to one month, or more. An outright open (naked) position in futures is extremely risky so 'position traders' often engage in spread trading.
  - *Spread trading* involves long-short trades in two different futures contracts, so that there is a negative correlation between the two futures positions and this reduces the risk exposure. The margin requirements for spread trades are usually less than for open/naked positions. The two main types of spread trading in futures markets are:
    - *Intracommodity spread*: a long and short position in two futures contracts on the same underlying asset – but with different maturity dates.
    - *Intercommodity spread*: a long and short position in two different futures contracts (e.g. on Microsoft and Apple stocks) – but with the same maturity date.

An intracommodity spread relies on the fact that futures prices on *long maturity dated contracts,* move more than futures prices on nearby contracts, when there is a change in the underlying asset price.[2] For example, if on 25 January you think a large rise in oil prices is likely over the next month, you would take a long position in a long-maturity oil-futures contract (e.g. the December-futures) and a short position in a short-maturity contract (e.g. the March-futures). Even if the spread trader forecasts incorrectly and it turns out that prices of oil fall, the large loss on the long-futures is partly offset by the gain on the short position.

An example, of an intercommodity spread is when a US investor takes a long position on 25 January in a sterling futures contract (with December maturity date) and a short position in yen futures (also with a December maturity date). The speculator is gambling on a sterling appreciation (against the US dollar) which exceeds that for the yen (against the US dollar) then the spread trader after closing out both contracts will make an overall profit. If the US investor's forecast is incorrect, some (or all) of the losses on the long sterling contract may be offset by the gains on the short yen contract.

4.4 SUMMARY

- If an investor is long the underlying cash-market asset (e.g. holds stocks, gold, oil, wheat, etc.) then to hedge, they should sell (short) futures contracts today (and vice versa), and close out the futures contracts at the end of the hedge period. Any change in price of the spot asset should be largely offset by the profit/loss on the futures position.
- There is always some *basis risk* in a hedge – the futures and spot prices may not move exactly dollar-for-dollar because of changes in interest rates, the yield on the underlying asset (e.g. dividend yield), storage costs and the convenience yield (for commodity futures).
- The maturity of the futures contract used in the hedge is usually greater than the maximum horizon over which you want to hedge (e.g. 3 months, 1 year) but this will also depend on liquidity and transactions costs associated with the specific futures contract chosen.
- The most conventional hedge is to use a futures contract that matures just after the end of the chosen hedge period. However, when hedging over a long period, long-dated futures may be illiquid and then short-dated (nearby) futures contracts can be used and 'rolled over' into the next short dated contract – this strategy has 'roll-over risk' and more basis risk than a conventional hedge.
- Speculation with futures allows almost unlimited *leverage* (since any margin payments you have to provide are small and usually earn interest). Hence the proportionate gain (on any own-funds used) is higher for speculation with futures, than speculation by purchasing the underlying cash-market asset using your 'own funds'.

[2]Simplifying a little this can be seen by noting that $F_{T_1} = S(1 + rT_1)$ and $F_{T_2} = S(1 + rT_2)$ so that $dF_{T_1} = dS(1 + rT_1)$ and $dF_{T_2} = dS(1 + rT_2)$ and hence the longer the time to maturity $T_2 > T_1$ the greater the change in the futures price. (For simplicity we have assumed the yield curve is flat and changes in S occur over a small time interval.)

EXERCISES

Question 1
Explain how a futures contract on a stock-X can be used for hedging your current holdings in stock-X.

Question 2
What is basis risk in a hedge and is it ever zero?

Question 3
Why might you need to 'roll over the hedge' when using futures contracts?

Question 4
Why would an oil consumer (Haulage Plc) use a 'futures hedge' to remove 'price risk' on its future purchases of oil in the cash market?

Question 5
You hold a stock worth S_0. Does a 'perfect hedge' using futures, involve locking in the current spot price, so the value of your portfolio in the future will equal its current value? Consider closing out the futures either before maturity or at maturity.

Question 6
Discuss the basic idea behind Shiller's approach to hedging house price changes by using 'novel' future contracts.

Question 7
Suppose you hedge a long position in stocks (currently worth S_0), using futures contracts (with current price F_0). Do you 'lock in' a known value for your portfolio in the future? Alternatively, are any losses/gains in the value of your stocks exactly compensated by gains/losses, when you close out your futures contracts? Assume you close-out your futures contracts just before maturity at F_1 (and your stocks are worth S_1). Explain.

CHAPTER 5

Index Futures

Aims

- To provide details of contract specification, settlement procedures and quotes for stock index futures contracts.
- To demonstrate index arbitrage and program trading using stock index futures.
- To price a stock index futures contract, when stocks which make up the cash market index (e.g. S&P 500), pay discrete dividends.
- To determine the number of futures contracts to use when hedging various cash market positions (e.g. a portfolio of stocks or holding barrels of oil).
- To outline the risks which remain, after hedging with index futures.
- To demonstrate 'tailing the hedge'.

Stock index futures (SIF) are contracts whose price depends on an underlying stock market index such as the S&P 500, FTSE 100 or Nikkei 225 indices. Such futures contracts are widely used in hedging, speculation, and index arbitrage.

In a well-diversified portfolio of stocks, specific (idiosyncratic) risk of individual stocks has been largely eliminated and only 'market' ('systematic' or 'non-diversifiable') risk remains. Stock index futures can be used to eliminate this 'market risk' of the portfolio of stocks. In 'normal times' a fund manager might be quite happy to hold a diversified portfolio of stocks even though they are risky. But if a fund manager believes the market will become increasingly volatile over the next 3 months (i.e. could rise or fall substantially) and she wished to remove this uncertainty, she could sell stock index futures to eliminate the market risk. At the end of the 3-month period, she would have neither gained nor lost.[1] If she believes the stock market

[1]Without the hedge then after 3 months her portfolio could be worth much more or much less – but she does not know in advance which outcome will occur.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

has then returned to a 'normal level of volatility', she would close out her futures position and her stock portfolio would again be subject to market risk.

Alternatively, suppose you plan to invest cash that you will receive in 3 months' time in a diversified portfolio of stocks but you are worried that the stock market will rise over the next 3 months. By buying stock index futures today you can fix the effective future purchase price of the stocks.

Also, speculation on a bear or bull market is easier to achieve with stock index futures rather than incurring the cost of buying or selling a large number of stocks that make up your prospective speculative stock portfolio. Finally, if the quoted stock index futures price is not equal to the 'correct' (no arbitrage) futures price then you can make a (near) risk-free arbitrage profit by simultaneously trading in the spot and futures markets. We elaborate on these ideas in this chapter and consider further strategies in later chapters.

5.1 STOCK INDEX FUTURES (SIF)

Stock index futures contracts are written on aggregate stock market indices and are settled in cash. In the US index futures contracts traded on the CME (Chicago) include those on the S&P 500, the Mini-S&P 500, the Russel 1000 (large cap index), the Dow Jones Industrial Average (DJIA) and the Nikkei 225 (Japanese index), as well as several other indices. Trading volume is highest in the S&P 500 futures contract and taking all of these contracts together, the dollar volume of the underlying stocks traded in these index futures contracts exceeds the dollar volume on the stock market itself.

On LIFFE[2] (pronounced 'life'), futures contracts are available on various UK stock indices including the FTSE 100 and FTSE 250 indices, on European indices such as the FTSE Eurotop 100 and FTSE Eurotop 300 (including and excluding the UK), and several others. The reason for so many different index futures contracts is so that fund managers can effectively hedge their existing equity portfolios, using a futures contract which has an underlying stock index that most closely matches the composition of their own cash market stock portfolio. Details of some of these contracts are given in Table 5.1.

5.1.1 Contract Multiple

All stock index futures are settled in cash (with settlement procedures differing between the different contracts) and nearly all contracts are closed out prior to their maturity date. Cash settlement is based on the *value of an index point* $\$z$ (or the '*contract multiple*'), which is set by the exchange (for the S&P 500, $z = \$250$). For the S&P 500 index futures contract, the smallest price change, a *tick,* is 0.1 index points, representing a \$25 change in value of one futures contract. Hence, a 1-point change in the S&P 500 futures index implies a change in the

[2]Since 2007, LIFFE is part of NYSE-Euronext.

TABLE 5.1 Stock index futures contract

	FTSE 100	S&P 500	FTSE Eurotop 300	Nikkei 225	DJIA
Unit of trading	£10 × index	$250 × index	€20 × index	$5 × index	$10 × index
Expiration	Mar/Jun/ Sep/Dec	Mar/Jun/ Sep/Dec	Mar/Jun/ Sep/Dec	Mar/Jun/ Sep/Dec	Mar/Jun/ Sep/Dec
Exchange	NYSE-Euronext	CME Group (IMM)	NYSE-Euronext	CME Group (IMM)	CME Group (IMM)

dollar value of one futures contract of $250. If the *futures index* on the S&P 500 is currently $F_0 = 2000$, then:

$$\textit{Dollar value of one futures contract}, V_F = 250F_0 = \$500{,}000.$$

5.1.2 Stock Indices

The S&P 500 and FTSE 100 stock indices are value (market capitalisation) weighted indices. The S&P 500 and FTSE 100 indices ignore dividend payments and therefore do not truly reflect the 'total returns' from the constituent shares (which include dividend payments). Often a stock index will have both a 'price version' and a 'total return' version.

5.1.3 Newspaper Quotes

Newspaper quotes for different stock index futures contracts are very similar. An illustrative example of price quotes (on 27 July) for futures contracts on the S&P 500 (traded on CME) with different maturities are given in Table 5.2.

TABLE 5.2 Quotes, index futures (27 July)

S&P 500 Index Futures (CME)						
	Open	Settle	Change	High	Low	Open interest
September	2,484.60	2,469.00	–15.50	2,485.20	2,466.00	677,803
December	2,499.00	2,491.30	–15.70	2,503.00	2,490.00	17,481
March	2,520.20	2,513.30	–16.20	2,526.00	2,514.00	1,094
June	2,537.20	2,561.80	–16.20	2,549.50	2,537.20	929
September	2,562.20	2,561.80	–16.30	2,574.10	2,562.10	146

The settlement price ('Settle') is an average of futures prices for trades at the end of the trading day – the settlement price is used in 'marking-to-market' and establishing margin payments. 'Open interest' is the number of contracts held (long or short) and is highest for the nearby September index futures contract and tails off rapidly for longer maturity contracts. Note that for S&P 500 futures contracts, the longer maturity contracts have higher quoted settlement prices today, than the nearby maturing contracts (see the 'Settle' column, Table 5.2). There are also index futures available on many other stock indices (e.g. on US 'high tech' stocks in the NASDAQ-100 (CME); the CAC-40 French stock index (traded in Paris on the MATIF exchange), and the DAX-30 German index (traded on the EUREX exchange in Frankfurt).

5.2 INDEX ARBITRAGE

In Chapter 3 we used 'cash-and-carry arbitrage' to derive the correct (fair, no-arbitrage) futures price on a stock, which also applies to the futures price on a stock index:

$$F = S[1 + (r_s - \delta_s)T] \quad \text{(using simple rates)} \tag{5.1a}$$

$$F = S(1 + r - \delta)^T \quad \text{(using discrete compound rates)} \tag{5.1b}$$

$$F = Se^{(r_c - \delta_c)T} \quad \text{(using continuously compounded rates)} \tag{5.1c}$$

where

S = S&P 500 stock index (quoted on the NYSE)
F = S&P 500 futures index (quoted in Chicago)
T = time to maturity of the futures contract, number of years (or fraction of a year)
r = (domestic) risk-free interest rate (proportionate, p.a.)
δ = dividend yield (proportionate, p.a)

If the current risk-free rate r is greater than the dividend yield (i.e. $r > \delta$) then the current quoted futures price (in Chicago) will be above the current stock price (on the NYSE), that is $F > S$, and vice versa. In applying the 'cost-of-carry' model we have to make the following assumptions:

- Investors can purchase or short-sell stocks, to 'replicate' the performance of all the stocks in the *stock index*.
- Actual *discrete* dividends paid out from stocks in the S&P 500 index (over the life of the futures) can be approximated by a constant (value weighted) dividend yield, δ, or actual *discrete* dividend payments from the stocks in the index are known with certainty (see below).

5.2.1 Program Trading

If the quoted futures price is not given by the equalities in (5.1), then index arbitrage enables (almost) risk-free profits to be made. In large financial institutions real time data on r, δ, T, S and the quoted futures price F_q are fed into computers and when $F_q \neq F$ in Equation (5.1), an arbitrage strategy is applied.

Borrowing or lending cash at r to undertake the arbitrage transaction is usually done via the repo market. Suppose an arbitrageur is faced with the following data:

$S = 2{,}000$ (S&P 500, index points)	$r_c = 0.10$ (10% p.a.)
$\delta_c = 0.03$ (3% p.a.)	$T = 1/3$ year (i.e. 4 months)

'Fair' futures price:

$$F = Se^{(r_c-\delta_c)T} = 2{,}000e^{(0.10-0.03)(1/3)} = 2,047.2 \text{ (index points)}$$

$$\text{Quoted futures price (all traders), } F_q = 2{,}100 \text{ (index points)}$$

The quoted futures price F_q is above the fair futures price F, so an arbitrage opportunity (i.e. 'sell high, buy low') is possible. You sell the futures contract today at $F_q = 2{,}100$, borrow $S = 2{,}000$ at $r_c = 0.10$ for 4 months ($T = 1/3$) and purchase the underlying stocks today.[3] After receiving dividend payments at the (continuously compounded) rate $\delta_c = 0.03$ on your stocks, you owe 2,047.2 after 4 months. You deliver the stocks against the short futures position after 4 months and receive $F_q = 2{,}100$ (in Chicago, from the long futures counterparty), making an arbitrage profit of 52.8(= 2,100 − 2,047.2).

Conversely, suppose the quoted futures price is $F_q = 2{,}020 < F = 2{,}047.2$. Today, the arbitrage strategy is to buy the futures contract 'low' at 2,020, simultaneously short-sell a portfolio of stocks for $S = 2{,}000$ and invest the proceeds in a bank deposit at the risk-free rate. At maturity of the long futures position you pay out $F_q = 2{,}020$, take delivery of the stocks (in Chicago), which you return to your broker. You also have to pay the broker any dividends due on the borrowed stocks. Hence your net receipts from investing the $S = 2{,}000$ at the risk-free rate would be $Se^{(r_c-\delta_c)T} = 2,047.2$ and the arbitrage profit at T, is 27.2 (receipts of 2047.2 minus payments of 2020).

5.2.2 Discrete Dividend Payments

In practice, dividend payments are clustered around certain months (particularly February and March in the US) so that the assumption of a constant dividend yield is an approximation.

[3]To simplify the exposition, we use index points and do not convert to dollars using the value of an index point of $250 – the latter complexity is introduced later in the chapter.

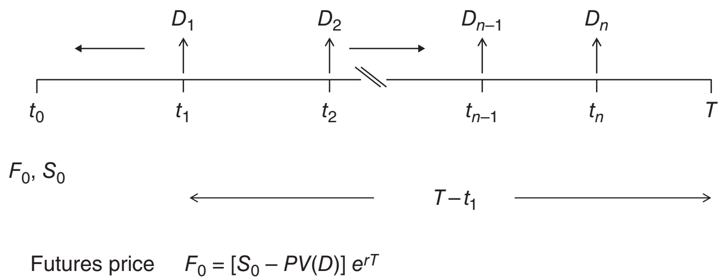

FIGURE 5.1 Futures price: dividend paying stock

However, the arbitrage relationship is similar when dividends are paid in discrete amounts. Assume there are N-dividend payments of D_j, each of which is paid t_j years (or fraction of a year) from today $t = 0$ (Figure 5.1).

All of the individual discrete dividend payments at time $t = 0$ are worth:

$$PV(D) = \sum_{j=1}^{N} D_j \; e^{-r_c t_j} \tag{5.4}$$

Each dividend payment D_j payable t_j years from today is discounted, which puts all dividend payments at $t = 0$, the same point in time as the stock price S_0. Hence the *net* cost of a replication portfolio at $t = 0$ is $S_0 - PV(D)$, which accrues to $[S_0 - PV(D)]e^{r_c T}$ at T the maturity of the futures, so today's no-arbitrage futures price is:

$$F_0 = [S_0 - PV(D)]e^{r_c T} \tag{5.5}$$

Above we assumed r_c is constant for all maturities (i.e. yield curve is flat over $t = 0$ to T) – but if not, this is easily taken care of by using $r_{c,j}$ in place of r_c.

There is *basis risk* in the arbitrage strategy for several reasons. First, traders might only buy (or sell) a subset of the 500 stocks which make up the S&P 500 index. Hence the change in value of the arbitrageur's portfolio of stocks might not exactly match the change in value of the S&P 500 stock index. Second, there are the arbitrageur's transactions costs to consider and any uncertainty in the timing of dividend payments over the life of the futures contract. Third, it might not be possible to execute trades in the index futures contract and all the stocks (in the index) at exactly the same time – in the meantime the prices of some of the underlying stocks might change.

Because index arbitrage opportunities are calculated using computerised real-time data, it is also known as *program trading* and is the subject of some controversy when the markets move rapidly (see Finance Blog 5.1).

Academic studies suggest that index arbitrage profits are difficult to exploit but tend to occur more often in those contracts with a longer time to maturity. This may be because these are the less liquid contracts where the 'fair price' and the quoted futures price might diverge because traders cannot obtain credit lines for these longer periods and therefore cannot instigate the arbitrage transaction.

Over the past few years, replicating the movement in a stock market index has been made much simpler by the introduction of Exchange Traded Funds (ETFs). These are 'funds' that closely track changes in a specific index and can be bought and sold (at all times of the day) in the 'cash market', usually on electronic platforms at low transactions costs. This means that it is nearly always the case that index arbitrage ensures that the index futures price is given by $F_0 = [S_0 - PV(D)]\, e^{r_c T}$.

Finance Blog 5.1 Program Trading and the 1987 Crash

There was much discussion, particularly after the 1987 crash, of whether program trading caused increased volatility in the stock market. If program traders are all selling stock via on-the-close orders on the maturity date of the futures contract then this may increase volatility due to order imbalances. Other traders may also be trading stock index *options,* and options on stock index futures. If these options contracts expire at the same time as the futures contracts, then volatility in the cash market for stocks could be large – such periods are known as the *triple witching hour*.

Stoll and Whaley (1986) find that the volatility on the stock market is greater on futures maturity dates (by a factor of 3) than on other days – particularly in the last hour of trading. The Brady Commission (1988) also felt that program trading exacerbated problems in the 1987 crash – but it is not clear that program trading results in major problems in more normal periods. The policy suggestions at the time included imposing higher margin on futures trading and more use of circuit breakers where the market is closed for 'cooling off' periods. Also, to some extent this problem has recently been overshadowed by the possibility that very high frequency trading using automated computer trades is causing excess volatility in the stock market.

Source: Adapted from Cuthbertson and Nitzsche (2001).

5.3 HEDGING

A well-diversified stock portfolio dramatically reduces 'specific (idiosyncratic) risk', while stock index futures contracts allow investors to hedge against changes in (overall stock) 'market risk' which is usually assumed to be adequately measured by some broad stock market index, which we take to be the S&P 500 index.

Hedging relies on the positive correlation between spot and futures prices. This implies that if you hold a portfolio of stocks ('long in the cash market'), then shorting stock index futures contracts 'creates' a negative correlation – what you lose on your cash market stock portfolio, you hope to gain on your index futures position. If a hedged portfolio entirely eliminates market risk then we might expect it to earn the risk-free rate of return.

Suppose that on 28 December Ms Midas holds a \$10m diversified portfolio of US stocks which exactly mirrors the composition of the S&P 500 index, so the beta of her stock portfolio $\beta_p = 1$. She feels the stock market is likely to be more volatile (than normal) over the next 3 months. This presents both opportunities for large price rises and large price falls. But let us assume that Ms Midas is very worried about a *general fall* in the stock market over the next 3 months, between 28 December and 28 March. This may be because she has already achieved the target return on her portfolio and is worried that her past gains are likely to be eroded over the next 3 months – which is the end of the period over which her performance (and bonus) will be evaluated by her superiors.

She can use stock index futures to eliminate this market risk (and earn the risk-free rate of interest) over the 3-month period. Of course, she could also earn the risk-free rate by selling all her stocks and holding T-bills. However, this would involve high transactions costs, as she would have to sell and then buy-back the stocks 3 months later. Continuing to hold the portfolio of stocks and hedging using stock index futures is far cheaper.

To hedge over 3 months she requires a futures contract which has at least 3 months to maturity. Suppose on 28 December Ms Midas decides to use the June-futures (which has a maturity date towards the end of June). Stock index futures (on the S&P 500) are cash settled with each index point ('tick') on the futures contract is assigned a value of \$250.

Ms Midas can hedge on 28 December by selling (shorting) the June-index futures (on the S&P 500). For the moment, assume a *perfect correlation* between the underlying market *index* S and the futures *index* F (i.e. correlation coefficient of +1) and that both indices move by the same amount. If the S&P 500 index on the NYSE on 28 December is $S_0 = 2{,}000$ and the market falls by 10% over the next 3 months, the S&P index will fall to 1,800 (i.e. by 200 points). Ms Midas's stock portfolio will fall in value by \$1m(= \$10m × 10%). If we assume F and S move by the same *absolute* amount then the change in $F = 200$ points and the change in value of *one* futures contract is \$50,000(= 200 × \$250). Hence the number of futures contracts to hedge your stock portfolio is:

$$N_F = \frac{\textit{Change in value of spot portfolio}}{\textit{Change in value of one futures contract}} = \frac{\$1\text{m}}{\$50{,}000} = 20 \text{ contracts} \tag{5.6}$$

So Ms Midas would hedge by shorting 20 contracts on 28 December. The outcome of the hedged position on 28 March is therefore:

$$\text{Loss on stock portfolio} = \$10\text{m } (10\%) = \$1\text{m}$$

$$\begin{aligned} \text{Gain on futures} &= \text{Change in futures index} \times \text{Tick value} \times 20 \text{ contracts} \\ &= 200 \times \$250 \times 20 = \$1\text{m} \end{aligned}$$

The reader can verify that a similar hedged outcome would ensue if stock prices *rise* by 10%. Here, the capital gain on the stocks would be offset by the loss on the short futures position – after F increases Ms Midas has to buy back the futures at a higher price (i.e. at a loss). In this case, with hindsight it would have been better not to have hedged – but 'hindsight' is not 'available' on 28 December. In any case the whole idea of a hedge is to remove price risk and in doing so, you also forego any favourable outcomes if stock prices rise. 'You can't have your cake and eat it' – you can be a hedger or a speculator in any given deal, but not both.

If the beta of Ms Midas's portfolio of stocks is $\beta_p = 1.5$ and the market falls by 10%, the return on her portfolio falls by 15% ($= R_p = \beta_p R_m$). Hence, she needs more futures contracts for an effective hedge. She would in fact need to short 30 futures contracts. This is because the change in the *dollar value* of her stock portfolio is now expected to be \$10m $\times$ 0.15 = \$1.5m, so $N_F = (1.5\text{m}/50,000) = 30$.

The above example looks a bit too good to be true – a perfect hedge. Yes, this is too optimistic! The reason for this is that we assumed the *absolute* change in the two indices, F and S are *exactly* the same. In practice this is not the case. First, F and S do not move exactly together, 'index point-for-index point' – they do so only approximately. Second, because interest rates might change over the life of the hedge, this slightly affects the relationship between changes in F and S – recall that $F = S(1 + rT)$ – this is basis risk. Third, because the beta of your portfolio of stocks is estimated with error, when the market moves by 10% the value of your stock portfolio might not move by exactly 15% – this is probably the biggest source of error in the hedge.

Let's return to the case where $\beta_p = 1$ and consider a more realistic scenario where there would be some hedging error. On 28 December the S&P 500 index $S_0 = 2{,}000$ and the futures index price $F_0 = 2{,}040$. Over the next 3 months the portfolio manager's worst fears are met and the S&P 500 index falls by 10% (200 index points) to $S_1 = 1{,}800$, so Ms Midas's stock portfolio falls in value by \$1m. On 28 March, the June-futures index falls to $F_1 = 1{,}818$ – a fall of 222 index points[4] – the futures contracts are closed out (i.e. buy-back) and the outcome of the hedged position on 28 March is:

$$\text{Loss on \$10m, stock portfolio (10\%)} = \$1\text{m}$$

$$\text{Gain on futures} = 222 \times \$250 \times 20 \text{ contracts} = \$1.11\text{m}$$

Hence the hedge position has produced a small profit of \$110,000 (1.1% of the portfolio value of \$10m). But most importantly, it has averted a possible large loss of \$1m had the

[4]The futures prices are calculated using $F_0 = S_0(1 + rT_0)$ and $F_1 = S_1(1 + rT_1)$, where on 28 December the June-futures has $T_0 = 6/12$ years to maturity and on 28 March the June-futures has $T_1 = 3/12$ years to maturity. The risk-free rate is $r = 4\%$ p.a. and remains constant.

position remained unhedged. In practice hedge positions result in small profits or losses but these average out to zero[5] for a large number of hedges, implemented over time.

5.3.1 Minimum Variance Hedge Ratio

Instead of working out the number of futures contracts you need when you consider a *particular* fall (e.g. 10%) in the market index, perhaps the 'best' formula to use (known as the minimum variance hedge ratio) is to calculate the number of futures contracts using:

$$\begin{aligned} N_F &= -\left(\frac{\$\,\textit{Value of cash market position}}{\$\,\textit{Value of one futures contract}}\right)\beta_p \\ &= -\frac{V_p}{V_F}\beta_p = -\frac{\$10\text{m}}{(\$250)(2{,}040)} = -19.6\ (= -20\ \text{contracts}) \end{aligned} \tag{5.7}$$

where, $\beta_p = 1$, $F_0 = 2{,}040$ and $V_F = (\$250F_0) = \$510{,}000$. V_F is the 'dollar value of one futures contract' and is also sometimes referred to as the 'invoice price' or 'contract price' of the stock index futures. But note that it is a purely notional amount, since this money is not paid at inception of the futures contract (only a small margin payment is made).

What is important in hedging is that the change in the futures index is (approximately) matched by the change in the (underlying) S&P 500 stock index. The (arbitrary) use of \$250 simply converts a 1 point change in the futures index to a dollar amount.[6]

The negative sign means Ms Midas needs to short 20 contracts. The above formula gives an answer very close to our 'simple' method described above. But it also has the advantage that you can just 'plug in' the known value for the (index) futures, $F_0 = 2{,}040$, currently quoted in Chicago (or on your trading screen). A slight variant to Equation (5.7) is to use the *dollar value* of the S&P 500 stock index ($= \$250 \times S_0$) in the denominator, where S_0 is the current level of the S&P 500 *stock index*, then:

$$N_F = -\frac{V_p}{(\$250)S_0}\beta_p = -\frac{\$10\text{m}}{(\$250)(2{,}000)} = -20\ \text{contracts} \tag{5.8}$$

Because F_0 and S_0 do not differ greatly, the above two formulas usually give similar answers for N_F. Notice that in the above we used stock index futures contracts to hedge a *portfolio* of stocks – and this is usually the case in practice. But you might also wish to hedge your holdings of *individual* stocks (e.g. Microsoft). Here you have an alternative which is to use a futures contract on Microsoft stock, itself. The above formula still applies but of course here the futures contract on Microsoft has a beta of unity (with respect to Microsoft's stock return). You can also use the above formulas (with slight modifications) to set up a hedge position for oil, wheat, foreign exchange, etc. (see Appendix 5).

[5]More accurately, on average, a sequence of hedges earns a return equal to the risk-free rate – see below.

[6]This '\$250' can be any dollar amount and the formula in (5.8) will ensure the correct number of futures contracts needed for the hedge. Try using '\$500' in place of '\$250' and you will get a different value for N_F but the same outcome, when the futures are closed out.

EXAMPLE 5.1

Hedging, Key Decisions

- If you are long (i.e. own) the spot asset (e.g. portfolio of stocks, oil) then to hedge you sell (short) futures contacts, today.
- If you are planning to purchase the underlying asset (e.g. portfolio of stocks, oil) in the future (i.e. you are short in the cash market) then to hedge you purchase (go long) futures contracts, today.
- Hedging works because the actions of arbitrageurs ensures that F and S move together, almost dollar-for-dollar. Hence a long position in the spot asset (e.g. stocks) and a short position in the futures contract creates a (near perfect) negative correlation between the spot and futures prices – so the net change in the value of the hedged position is (close to) zero.
- Note that any losses on your (long or short) futures position may involve additional margin calls. Hence a hedger must have lines of credit or other collateral available to meet possible margin calls.

5.3.2 Hedging in Practice

Let us now consider a slightly more complex hedging example using SIF, where we explicitly calculate futures prices, use the CAPM equation to work out the expected return on the stock portfolio and where the stocks in the portfolio earn dividends. It is 28 December and Ms Midas holds a $10m portfolio of stocks and she wishes to hedge over the next 3 months (to 28 March). She uses the June-futures index to hedge.

28 December, Ms Midas Portfolio (Hedge period = 3 months)

Portfolio value: $V_p = \$10\text{m}$ $\quad\beta_p = 1.5$

Stock index: $S_0 = 2{,}000$ (index points) $\quad r = 4\%$ p.a. (continuously compounded)

28 December: Stock Index Futures

Maturity of June-forward contract = 6 months (i.e. $T_0 = 1/2$ year)

Dividend yield on market index $\delta = 6\%$ p.a. (continuously compounded)

$$F_0 = S_0 e^{(r-\delta)T_0} = 2000e^{-0.02/2} = 1980$$
$$V_F = (\$250 F_0) = \$495{,}000 \tag{5.9}$$

$$N_F = -\frac{V_p}{V_F}\beta_p = -\frac{\$10\text{m}}{\$495{,}000}(1.5) \quad \text{(i.e. short 30 contracts)} \tag{5.10}$$

28 March (3 months later)
Assume the S&P 500 has fallen by 10% to $S_1 = 1{,}800$ and Ms Midas closes out her June-futures contracts, which now have 3 months to maturity ($T_1 = 1/4$ year):

$$F_1 = S_1\, e^{(r-\delta)T_1} = 1800\, e^{-0.02/4} = 1{,}791\ (F_0 - F_1) = 189 \text{ points}$$

$$\textit{Gain on futures} = N_F(F_0 - F_1)\$250 = 30(189)\ \$250 = \mathbf{\$1.4175m}$$

The percentage return on the *market portfolio (S&P 500)* R_m is the capital loss plus the dividend yield (over 3 months):

$$\begin{aligned} R_m &= [(S_1 - S_0)/S_0] + (\delta/4) \\ &= ((1{,}800 - 2{,}000)/2{,}000) + 0.015 = -0.085\ (8.5\%) \end{aligned} \tag{5.11}$$

The expected return on your stock portfolio *over 3 months* (using the CAPM)[7]:

$$R_p = r + \beta_p(R_m - r) = 1\% + 1.5(-8.5\% - 1\%) = -13.25\% \tag{5.12}$$

Expected value of the stock portfolio after 3 months = \$10m (1 – 0.1325) = **\$8.675 m**

$$\textit{Hedged position on } 28 \textit{ March} = \textit{Value of stock portfolio} + \textit{Gain on futures}$$
$$= \$8.675\text{m} + \$1.4175\text{m} = \mathbf{\$10.0925m}$$

Percentage change in hedged position (over 3 months) = (\$10.0925m/\$10m – 1) = 0.925%.

The hedged portfolio has gained \$92,500 (0.925% of \$10m) in value. Had Ms Midas not hedged she would have lost about 13.25% on her stock portfolio.

The risks in the hedge are:

- beta of the stock portfolio may be estimated incorrectly or it may change over the hedge period.
- The stock portfolio is not well diversified so that some specific risk is present – hence the return on the stock portfolio will not be correctly measured by its beta and the market return.[8]
- the risk-free rate and the dividend yield – the sources of basis risk – may change over the hedge period (which will alter the correlation between F and S).

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[7]Here, for simplicity, we use the continuously compounded interest rate of 4% p.a., or 1% over 3 months in the CAPM equation. When interest rates are low it makes little difference whether you use the full CAPM equation or the simplified approximation $R_p = \beta_p R_m$.

[8]This is because the equation $R_p = r + \beta_p(R_m - r)$ really represents the *expected* return on the stock portfolio. The actual return on the stock portfolio over the hedge period is $R_p = r + \beta_p(R_m - r) + \varepsilon_{p,t}$ where $\varepsilon_{p,t}$ represents 'portfolio specific risk' – and the latter is relatively small but not zero, even in a well-diversified (unhedged) stock portfolio.

The first point is perhaps the most serious and may arise if beta is really time varying but the regression analysis assumes a constant value for beta, estimated using a long data period. The variability in beta can be analysed using recursive regressions – that is, to re-estimate beta as you add more data or you can simply estimate beta over different sample periods, to see how variable its empirical value appears to be. There are also methods for *forecasting* time varying parameters like beta (e.g. using the Kalman filter). Alternatively, note that a time varying beta can also be directly calculated by using direct estimates of variances and covariances:

$$\beta_p = \frac{\sigma_{R_p,R_m}}{\sigma^2_{R_m}} \tag{5.13}$$

The modelling of time varying variances and covariances using ARCH and GARCH models (see Cuthbertson and Nitzsche 2004, 2008) has been rather successful (especially with 'high frequency', daily data) and these models can also be used to forecast β_p over the hedge period using (5.13). There is some evidence that this may improve the hedge outcome compared with using a fixed value for β_p (see, for example, Baillie and Myers 1991). Also note that the hedge ratio given by Equation (5.7) implies that as stock prices change, we should re-calculate V_p and hence N_F periodically. The return on a stock portfolio is:

$$R_p = \alpha_p + \beta_p R_m + \varepsilon_p \tag{5.14}$$

The SIF hedge removes the market risk $\beta_p R_m$, (see Chapter 6) so one source of the hedging error is specific risk ε_p. The *dollar* hedging error dV and the variance of the hedging error are:

$$dV = V_p \varepsilon_p \tag{5.15a}$$

$$\sigma^2_{dV} = V_p^2 \sigma^2_{\varepsilon_p} \tag{5.15b}$$

It is shown (see Chapter 6) that the variance of the (dollar) hedging error can also be expressed as:

$$\sigma^2_{dV} = V_p^2 \sigma^2_{R_p}(1 - Rsq) \tag{5.15c}$$

where Rsq is the 'R-squared' from the OLS regression (5.14). Equation (5.15c) can be used to calculate the expected error in the hedge.

EXAMPLE 5.2

Hedging Error – Stock Portfolio

BigCapital, a mutual fund, has decided to hedge its $100,000 stock portfolio using SIF over the next month and wants to measure the potential hedging error. The OLS regression of the portfolio monthly return on the market return, has $Rsq = 0.8$ and $\beta_p = 1.4$. The monthly

(*continued*)

(*continued*)

standard deviation of the portfolio return is $\sigma_{R_p} = 10\%$. The variance of the (monthly) dollar hedging error is:

$$\sigma^2_{dV} = (\$100,000^2)(0.10^2)(1 - 0.8) = \$20\text{m}$$

The monthly (dollar) hedging error is therefore $\sigma_{dV} = \sqrt{20\text{m}} = \$4{,}472$ (4.5% of the initial value of the portfolio of $100,000). Over shorter horizons the dollar hedging error will be smaller. For example, the hedging error over 1 week uses the weekly standard deviations in (5.15c), where approximately, $\sigma^2_{dV,week} = \sigma^2_{dV,month}/4$.

5.4 TAILING THE HEDGE

There is one other practical issue we should deal with, namely adjusting the hedge to take account of interest paid or received on the margin account. This is called 'tailing the hedge'. Suppose you hedge with a short futures position. If the futures price rises, your short position will make losses, and you may have to make additional margin payments which will increase the overall cost of your hedge. You could offset this increased cost, if you also held some additional long futures positions which would increase in value, when futures prices increase. Hence, if you are initially short (long) N_F futures, then tailing the hedge will involve going long (short) some additional futures contracts. The 'tail' is always an opposite position to the initial futures position (of N_F futures). One method of calculating this tail adjustment is:

$$N_{tail} = -N_F rT \tag{5.16}$$

where N_{tail} is the number of contracts to tail the hedge, N_F = the initial number of futures contracts (see Equation (5.7)), $r = 8\%$ is the risk-free lending or borrowing rate on variation margin payments (any surplus over the maintenance margin is assumed to remain in the margin account), T = time to maturity of the futures. Suppose initially using Equation (5.7) you require $N_F = -100$ (short) futures contracts on the S&P 500 and the maturity of the futures is $T = 90/360$. If the futures index increases by one point, then the short-futures position will lose value:

$$\text{Increased variation margin} = (+1\text{ point}) \times (100\text{ contracts}) \times \$250 = \$25{,}000$$

$$\text{Loss of interest (on new margin)} = \$25{,}000 \times (0.08) \times (90/360) = \$500$$

$$N_{tail} = -N_F rT = -(-100) \times 0.08 \times 90/360 = +2$$

$$\text{Gain on futures 'tail'} = (+1\text{ point}) \times (2\text{ contracts}) \times \$250 = \$500$$

Hence, with 'the tail' in place, the additional interest cost of the variation margin is offset by tailing the hedge. In practice, r will vary over the life of the futures and hence tailing the hedge will not produce the perfect result given above. Of course, given the initial $N_f = -100$ contracts and $N_{tail} = +2$ contracts, then the actual initial hedge position taken would have been to short 98 contracts:

$$N_{new} = N_F - N_F rT = N_F(1 - rT) \tag{5.17}$$

The value for N_{new}, which 'tails the hedge' has to be recalculated periodically and the hedged position rebalanced as r and T change.

5.5 SUMMARY

- There are a number of different index futures contracts on various stock market indices.
- Index futures prices can be determined using the cost-of-carry approach and violations of this relationship give rise to (risk-free) arbitrage – known as program trading, which today involves high-frequency trading on electronic exchanges, using computer algorithms.
- Stock index futures (SIF) provide a low cost method of hedging an existing diversified equity portfolio or in hedging a future purchase of a diversified portfolio of stocks.
- Hedging using SIF is not perfect, because the changes in the index futures price will not exactly match changes in the stock index (S&P 500) and the portfolio beta is an estimate and may not be accurate over the hedge period. There are also transactions costs in the futures market to consider and potential margin payments.

APPENDIX 5: HEDGE RATIOS

In this appendix we use a common methodology to derive the optimal number of futures contracts required in a variety of practical hedging situations. To calculate the hedge-ratio where the underlying asset to be hedged (e.g. crude oil) is the same as the underlying asset in the futures contract (i.e. futures on crude oil), is straightforward. For example, suppose it is January and you expect to sell 10m barrels of crude oil in 3 months' time in April and you are worried that the oil price in the cash market might fall over the next 3 months. The contract size for one futures on crude oil is for delivery of $z = 1{,}000$ barrels, hence to hedge you would short:

$$N_F = -N/z = -\$10\text{m}/1{,}000 \text{ barrels} = -10{,}000 \text{ contracts (short)}$$

Cross Hedge

Now consider the case where the cash market asset you wish to hedge differs from the asset deliverable in the futures contract – this is a cross hedge. To demonstrate a 'quick method' of deriving the hedge ratio, assume the airline AirOFly wishes to hedge a future cash market purchase of jet fuel using heating oil futures. (We assume there are no highly liquid futures contracts on jet fuel).[9] We have:

$$N = \text{number of barrels of jet fuel required}$$
$$P = \text{cash market price of jet fuel (per barrel)}$$
$$R_{JF} \equiv dP/P, \text{the return on jet fuel}$$
$$V_{JF} = \text{\$-value of cash market position in jet fuel } (= N\,P)$$
$$F = \text{price of heating oil futures (per gallon)}$$
$$z = \text{contract size for futures on heating oil (i.e. delivery of } z = 1,000 \text{ gallons)}$$
$$V_F = \text{\$-value of one futures contract on heating oil} = zF_0$$
$$N_F = \text{number of futures contracts for the hedge (to be determined)}$$

The dollar change in the hedge-portfolio is:

$$\begin{aligned} dV &= \text{\$-change in cash market position} + \text{\$-change in futures position} \\ &= N(P_1 - P_0) + N_F(F_1 - F_0)z \\ &= NP_0(dP/P_0) + N_F(zF_0)(dF/F_0) = V_p(dP/P_0) + N_F V_F(dF/F_0) \\ &= V_{JF}R_{JF} + (N_F V_F)R_F \end{aligned}$$

The proportionate change in cash market jet fuel prices (i.e. the return on jet fuel) is $R_{JF} \equiv dP/P_0$. The return on heating oil futures is defined as $R_F \equiv dF/F_0$. But F depends on the cash market price of heating oil, S and arbitrage ensures futures and spot prices of heating oil move together. Hence $R_F = R_{HO}$ where R_{HO} is the percentage change in the spot price of heating oil. We assume that the return on jet fuel and heating oil (spot) prices are linearly related, $R_{JF} = \beta R_{HO}$. A hedge position should involve neither a gain nor a loss so we set $dV = 0$ and after rearranging:

$$N_F = -\frac{V_{JF}}{V_F}\beta \tag{5.A.1}$$

This is the 'quick method' of obtaining the optimal hedge ratio.[10] It is fine except that the relationship for beta, $R_{JF} = \beta R_{HO}$ will only hold 'on average' – so the hedge will not give $dV = 0$ exactly – but over a number of hedges the dollar change in value dV should be small. As we see below, given that prices are stochastic what we are really trying to do in the hedge

[9]On the CME there are some traded jet fuel futures, for example Gulf Coast Jet Fuel futures but these are not as heavily traded as heating oil futures.

[10]You get the same result if you use calculus (which is valid for small changes in the variables). The value of the hedge portfolio is $V = NP + N_F zF$. Hence differentiating, $dV = NP(dP/P) + N_F zF(dF/F) = V_p R_p + N_F V_F R_F = 0$.

is to choose that value for N_F which minimises the *variance* of dV. The correct expression for beta comes from a regression of spot jet fuel returns on the (spot) returns on heating oil:

$$R_{JF} = \alpha + \beta R_{HO} + \varepsilon \tag{5.A.2}$$

Sometimes for hedging over short time periods, absolute rather than percentage changes in prices are used to obtain an estimate of beta, $dP_{JF} = \alpha + \beta dS_{HO} + \varepsilon$ – but the two methods usually give similar estimates.

Hedging a Portfolio of Stocks

We can use the above analysis with minor re-definitions to hedge a portfolio of stocks.

V_p = \$-value of cash market stock portfolio.

F = index price of futures contract (underlying = S&P 500 stock market index).

V_F = \$250 F_0 = notional \$-value of one futures contract.

\$250 is the value of an index point on the futures contract.

N_F = number of futures contracts in the hedge (to be determined).

Long in the Cash Market

The cash market position consists of n-stocks with $V_i(>0)$ dollars in each stock and a total amount invested of V_p dollars. The change in value of the stock portfolio is:

$$dV_p = \sum_{i=1}^{n} dV_i = V_p \sum_{i=1}^{n} (V_i R_i)/V_p = V_p \sum_{i=1}^{n} w_i R_i = V_p R_p \tag{5.A.3}$$

where $V_p = \sum_{i=1}^{n} V_i, R_p = \sum_{i=1}^{n} w_i R_i, \quad w_i = V_i/V_p, \quad \sum_{i=1}^{n} w_i = 1,$

For example, if you hold V_p = \$1m in a diversified portfolio of stocks and the return on the portfolio is $R_p = -10\%$ over 1 month (say) then your stock portfolio falls in value by \$100,000. The change in the dollar value of a hedged portfolio consisting of V_p dollars held in a stock portfolio and N_F, stock index futures (SIF) is:

$$\begin{aligned} dV &= \text{\$-change in stock portfolio} + \text{\$-change in futures position} \\ &= V_p R_p + N_F (F_1 - F_0)\$250 \\ &= V_p R_p + N_F V_F R_F \end{aligned} \tag{5.A.4}$$

where $V_F = \$250F_0$ and $R_F \equiv dF/F_0$. We assume that the index futures price F and the stock market index (S&P 500) move together (because of arbitrage) so $R_F \equiv R_m$ (and $R_m \equiv dS/S$

where S = S&P 500 index). If we assume the return on our portfolio of stocks is (exactly) linearly related to the market return, $R_p = \beta_p R_m$ and setting $dV = 0$ in (5.A.4) then:

$$N_F = -\frac{V_p}{V_F}\beta_p \tag{5.A.5a}$$

If you are long a portfolio of stocks then $V_i > 0$ and hence $V_p > 0$ and as the betas of nearly all stocks are positive then $\beta_p > 0$. Hence to hedge a long position in a portfolio of stocks, the minus sign in (5.A.5a) indicates that you will short stock index futures.

Short-sell in the Cash Market

Let a short-sale of stock-j be represented by a positive (dollar) value $V_j > 0$. So a short-sale of say \$100 of stock-j implies a positive value for $V_j = +100$. If you short-sell m-stocks then the total value of the short sales is $V_S = \sum_{j=1}^{m} V_j > 0$. As we always define the return on a stock as positive (negative) when the price rises (falls) then $dV_S = -V_S R_S$ implies that your cash market short position will *increase* in value if stock prices *fall* that is $R_S < 0$ (and vice-versa). The number of futures contracts to hedge your short position in stocks is then:

$$N_F = \frac{V_S}{V_F}\beta_S \tag{5.A.5b}$$

Hence to hedge a short position in stocks, you go long (buy) stock index futures. (This is discussed further in Chapter 6.)

Minimum Variance Hedge Ratio (MVHR)

The correct (and slightly more involved) way to obtain the above expression for N_F given that prices are stochastic, is to derive the minimum variance hedge ratio MVHR. From (5.A.4):

$$\sigma^2_{dV} = V_p^2\sigma^2_{R_p} + (N_F V_F)^2\sigma^2_{R_F} + 2V_p V_F N_F \sigma_{R_p,R_F} \tag{5.A.6}$$

where $\sigma^2_{R_p}$ is the variance of the portfolio return, σ_{R_p,R_F} is the covariance between the portfolio return and the index futures return. The value of N_F which minimizes the variance of the hedge portfolio is the solution to $\partial\sigma^2_{dV}/\partial N_F = 0$, that is:

$$N_F = -\frac{V_p}{V_F}\beta_p \text{ where } \beta_p \equiv \frac{\sigma_{R_p,R_F}}{\sigma^2_{R_F}} \tag{5.A.7}$$

It can be shown that the above expression for β_p equals the slope coefficient in the OLS regression:

$$R_p = \alpha + \beta_p R_F + \varepsilon_p \tag{5.A.8}$$

where R_p is the proportionate return on the stock portfolio and ε_p is a random error. For SIF we assume (because of arbitrage) that the *index futures* return R_F (in Chicago) equals the return on the S&P 500 *market index* R_m (on the NYSE). The 'market return' is simply the percentage change in the stock market index S, so that $R_m \equiv dS/S$ where S = level of S&P 500 index.

A time series regression of *portfolio* returns R_p on the market return (S&P 500) gives a direct estimate of β_p. An alternative is to calculate the portfolio beta from the individual betas of the constituent stocks in the portfolio using $\beta_p = \sum_{i=1}^{n} w_i \beta_i$. The individual stock betas[11] are obtained from a 'beta book' purchased from an investment bank. The $w_i = V_i/V_p$ are the proportions held in each stock, where V_i is the dollar amount in stock-i and V_p is the total value held in the stock portfolio. Note that replacing $V_F = \$250F_0$ in (5.A.7) by the dollar value of the S&P 500 *index* = $\$250S_0$ gives:

$$N_F = -\frac{V_p}{\$250S_0}\beta_p \tag{5.A.9}$$

which if $F_0 \approx S_0$ gives approximately the same value for N_F as (5.A.7).

EXERCISES

Question 1

You own a stock. A perfect (futures) hedge locks in the current spot price of the stock, after you close out your futures position at some later date (before the maturity of the futures). True or false?

Question 2

In January an investor has a \$1.2m long position in a share. The beta of the share is 1.1. The investor intends to close out her position in the share around mid-March and wants to hedge the price-risk using futures. The March Mini S&P 500 future is currently trading at 1,450 (for delivery of \$50 times the index). Outline the investor's hedging strategy.

Question 3

You are a US resident. It is 30 January and you plan to purchase a \$10m portfolio of around 30 US stocks in 6 months' time with a portfolio beta of 0.5. You are worried about a rise in prices over the next 6 months. The S&P 500 index futures is currently at 1,000.

How could you hedge your future purchases using stock index futures? Would you take a long or short futures hedge?

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[11] These individual betas for each stock-i are obtained from a regression of the excess return on stock-i on the excess market return: $R_i - r = \alpha_i + \beta_i(R_m - r) + \varepsilon_i$. Often adjustments are made to the 'raw estimate' of beta, for example, using Bayesian shrinkage techniques, where the actual stock beta used is a weighted average of the OLS regression beta and the 'market return beta' of 1.

Question 4

It is 8 November and the stock market index is $S_0 = 441.15$. The dividend yield is 3% p.a. and the risk free rate $r = 3.2\%$ p.a. (both continuously compounded). The December index-futures contract which expires in 40 days (at $T = 18$ December) has a quoted price of $F_q = 444$. The contract multiple for the index-futures contract is $z = \$250$.

(a) Calculate the fair (no-arbitrage) price of the December index-futures.

(b) Work out an arbitrage strategy with your maximum allowed credit line from the bank of $V_s = \$20$ m.

(c) Calculate the risk-free profits, if the stock market index on 18 December (i.e. at maturity of the index futures contract) is $S_T = 439$.

Assume any stocks you buy or sell 'mimic' the movement in the stock market index, that is your stock portfolio has a 'beta' of unity.

Question 5

You are the manager of a pension fund with \$50m in a diversified portfolio of stocks. You think the stock market over the next 6 months will be more volatile than normal so the potential for losses (and gains) are more extreme – but you believe the stock market will return to 'normal' after that. You want to hedge your stock position over the next 6 months and then return to your initial unhedged stock portfolio.

What are the relative merits of selling the stocks and moving into risk-free assets (e.g. T-bills) over the next 6 months, versus using futures contracts to hedge your portfolio?

Question 6

What is the minimum variance hedge ratio, when hedging a portfolio of stocks? Derive the optimal number of futures contracts in the hedge.

Question 7

Suppose you hold a diversified portfolio of stocks with a market value of \$10m and a beta of unity. A futures contract on the S&P 500 has 3 months to maturity. Each index point on the futures contract is worth \$250. The current interest rate is $r = 4\%$ p.a. (simple rate) and the current value of the S&P 500 stock index is $S_0 = 1{,}420$.

(a) Show that the (no arbitrage) 'fair' futures price $F_0 = 1434.2$ (use simple, not compound interest).
You want to hedge your \$10m stock portfolio *over the next month*, using stock index futures.

(b) How many futures contracts are required to hedge your portfolio of stocks?

(c) What is the outcome of your hedged position if in *1 month's* time the S&P 500 has fallen by 8% and the futures index stands at $F_1 = 1315.1$? Compare the hedged and unhedged positions.

(d) How do we know that the futures index will be $F_1 = 1,315.1$ (in 1 month's time) after the fall in the S&P 500? Briefly comment on the relationship between the change in the futures and stock indices.

CHAPTER 6

Strategies: Stock Index Futures

Aims

- To show how stock index futures can be used to protect an active 'stock picking' strategy from general movements in the overall stock market return (e.g. S&P 500 index).
- To demonstrate how an investor (e.g. long-short hedge fund) holding a well-diversified portfolio of stocks can benefit from speculating on underpriced or overpriced stocks, while also using stock index futures (SIF) contracts to hedge the impact of any unexpected movements in the market return (S&P 500) on the stock portfolio.
- To show how the 'effective beta' of a stock portfolio can be altered using stock index futures. This enables investors (mutual funds and hedge funds) to implement a 'market timing' strategy, using stock index futures.
- To show how hedge funds engaged in 'merger arbitrage' can slowly purchase the stocks of a prospective takeover target but also protect themselves from rising prices for the target firm caused by an unexpected increase in the overall market return (S&P 500 index).

6.1 UNDERPRICED STOCKS: HEDGING MARKET RISK

Stock index futures (SIF) can be used to protect your speculative 'stock picks' from general changes in the *overall* stock market (e.g. S&P 500 index). For example, the hedge fund *MoneyPlus* takes on risky bets that are potentially profitable – but it also tries to hedge any risks it feels it does not have the information and skill to handle. Suppose MoneyPlus thinks a drug company called MaxPill will soon be getting a favourable clearance from the regulator for its new drug (which reduces the incidence of heart attacks). On announcement of the

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

successful licensing of the drug, MoneyPlus estimates that the stock price of MaxPill will rise by 10% (say) from its current price of $P = \$10$ to its 'true value' $P(\text{true}) = \$11$.

This rise of 10% which is solely due to the mispricing is called the 'alpha' of the MaxPill stock. So MoneyPlus thinks MaxPill is currently underpriced and has a positive (expected) alpha of 10%. Suppose all other market participants think MaxPill will *not* obtain a licence from the regulator and therefore believe the current price of \$10 is correct (and the alpha of the stock is zero).

As a speculator or 'stock picker', MoneyPlus should buy ('go long') the stock *today* at $P_0 = \$10$. If the hedge fund is correct, then when the licence is granted other investors will realise that the true value of the stock is \$11. These investors will then purchase MaxPill, this will cause its price to rise to $P_1 = \$11$, to reflect its new 'true value' (agreed by all traders). MoneyPlus can now sell its MaxPill stocks at a profit of \$1 per stock because of its skill in predicting the success of the licence application – well before the other investors.

Suppose MoneyPlus buys 10,000 stocks of MaxPill at $P_0 = \$10$ so that its initial investment is $V = \$100{,}000$. After the granting of the licence, it sells at \$11 per share and makes a speculative profit of \$10,000.[1] This assumes MoneyPlus is correct about the underpricing and that the market as a whole (i.e. S&P 500 index) remains unchanged.

But MoneyPlus is aware that even if it is correct about the underpricing of MaxPill its price could fall if the market as a whole falls (e.g. due to the onset of recession) – this is 'market risk'. If the beta of MaxPill is $\beta = 2$ and the market return (S&P 500) falls by 3% (say), then MaxPill's price will fall by 6% ($= 2 \times 3\% = \beta R_m$). Hence, even if the mispricing of 10% is corrected, the net change in the price of MaxPill will be:

$$\begin{aligned} \textit{Net \% change in MaxPill} &= \textit{Increase due to underpricing} - \textit{Fall due to market movements} \\ &= 10\% - 6\% = +4\% \end{aligned}$$

The final value of the hedge fund's MaxPill stocks will therefore be \$104,000 – it has only made 4% rather than the 10% underpricing it thought it would make. This is because it has not hedged the market risk. Furthermore, had the S&P 500 market index fallen by more than 5% then MoneyPlus would have made an overall loss on its purchase of MaxPill *even if it was truly underpriced* by 10%. How can MoneyPlus protect itself from the impact of a fall in 'the market', on the underpriced stocks it has purchased, so that it earns the full 10% of the underpricing?

We have already solved this problem! MoneyPlus should sell (short) stock index futures contracts. If the market falls by 3%, then we know that the gain on the short futures position will offset any loss on the MaxPill stocks consequent on the fall in the market. If the hedge is successful, the cash profit on the futures will provide an 'extra' return of 6%, so that when added to the 4% increase in value of the MaxPill stocks, then MoneyPlus will have earned the

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[1] We assume the hedge fund uses its 'own funds' of \$100,000 to purchase the stock – it therefore does not 'buy on margin' and leverage its purchases – see Appendix 6.A for more details.

full 10% 'underpricing'. If $V = \$100{,}000$, $\beta = 2$ and $F_0 = 200$ (say)[2] then the number of futures contracts to short is:

$$N_F = -\frac{V}{V_F}\beta = -\frac{\$100{,}000}{(\$250)(200)}(2) = -4 \text{ (short futures contracts)} \tag{6.1}$$

To illustrate what is happening assume the S&P 500 index S (i.e. 'the market index') and index futures F have the following values:

$$S_0 = 198 \qquad F_0 = 200$$

$$S_1 = 192 \qquad F_1 = 194$$

The S&P 500 has fallen by 6 index points (about 3%) and the futures index has also fallen by 6 points. Hence:

$$\text{Profit from the futures position: } dV_F = N_F(F_0 - F_1)\ \$250 = 4(6)\ \$250 = \$6{,}000$$

The \$6,000 profit is 6% of the initial investment in MaxPill of \$100,000, so the profit on the futures is equivalent to an extra 6% return – which when added to the actual 4% increase in the value of MaxPill stock makes a total return of 10% – the same as the initial mispricing. Alternatively, the MaxPill shares end up being worth \$104,000, but the \$6,000 profit from the futures position gives a total of \$110,000 – that is, 10% of the initial investment of $V = \$100{,}000$. The hedge fund's 'total return' of 10% occurs because it has hedged the market risk and only has exposure to the underpricing of MaxPill (based on its consummate skill in forecasting the outcome of the licensing application for the new drug).

We can also demonstrate the above calculations algebraically:

$$dV_F = N_F\ \$250(F_0 - F_1) = N_F V_F(dF/F_0) = N_F V_F R_m$$

where $V_F = \$250F_0$. We assume the futures index (written on the S&P 500) moves by the same percentage amount (in Chicago) as the (S&P 500) 'market index', on the NYSE (i.e. $dF/F_0 = R_m$) and the simplified CAPM relationship is $R_p = \beta R_m$. The change in value of MoneyPlus hedge portfolio (i.e. long MaxPill and short SIF) is:

$$\begin{aligned} dV &= V(\beta R_m) + N_F V_F R_m \\ &= \$100{,}000(2)R_m - 4(\$250 \times 200)R_m = [\$200{,}000 - \$200{,}000]R_m \end{aligned} \tag{6.2}$$

Hence, for each 1% fall in the market return, the long position *in* MaxPill *falls* by \$200,000 – which is exactly offset by a gain of \$200,000 on the short SIF's position in four

[2]We assume the SIF on the S&P 500 is $F_0 = 200$, so the numerical results are clear. At the time of writing in June 2019, the futures index on the S&P 500 stood at around 2,400.

contracts. The \$200,000 gain on the short futures occurs when the futures are closed out (by buying them back at a lower price).

Hence MoneyPlus holdings of MaxPill stocks are now hedged against rises (or falls) in the market return – as expected. This implies that if there is a correction in the underpricing of 10% (while the futures hedge is maintained), then MoneyPlus will make an overall profit of \$10,000. Note that the index-futures hedge must remain in place until the mispricing is corrected – the timing of the latter is uncertain and hence the futures hedge may have to be 'rolled over' (i.e. when one hedge terminates another must be put in place). Hence this strategy may involve 'roll over' risk.

6.1.1 Futures on MaxPill Stocks

Note that, above, MoneyPlus hedges by using index futures (on the S&P 500 index) and not futures contracts written on the MaxPill stocks, themselves. Let's see why. MoneyPlus is long the underpriced MaxPill stocks. Again, suppose the net rise in MaxPill stock is $R = 4\%$ (due to a fall in the market return of 3% which causes a fall in MaxPill's price of 6%, coupled with a 10% gain from the mispricing).

If MoneyPlus had shorted futures contracts on MaxPill stocks, the MaxPill futures price will increase (approximately) by 4% – so the short futures position loses 4%, when closed out. The overall result is a net 4% rise (10% – 6%) in the cash price of MaxPill stocks but a loss of 4% on the short futures position – giving a zero net gain from the hedged position – rather than the 10% gain from the underpricing that we expect!

Clearly, this is not what MoneyPlus wants to do. Instead, MoneyPlus wants to offset the impact of changes in the market return (S&P 500) on the value of its MaxPill stocks and for this MoneyPlus must use *stock index futures*. At the same time, MoneyPlus wants to 'take a gamble' (i.e. be unhedged) on the prospective underpricing of MaxPill – so it must *not* hedge the underpricing of MaxPill, with futures contracts on MaxPill. Hence when MoneyPlus hedges using *index* futures, it offsets any effects due to overall stock market movements but it remains exposed to the mispricing of MaxPill, itself.

6.2 OVERPRICED STOCKS: HEDGING MARKET RISK

6.2.1 Overpriced Stocks: Long SIF

If MaxPill is viewed as an overpriced stock then MoneyPlus should short-sell $V = \$100{,}000$ (say) of MaxPill stocks and it should hedge the market risk by going *long* four stock index futures contracts:

$$N_F = -\frac{V}{V_F}\beta = -\frac{(-\$100,000)}{\$250(200)}2 = 4 \text{ (long futures contracts)} \tag{6.3}$$

For example, if the market return increases by 3%, this results in a loss of 6% on the short-position in MaxPill (as $\beta = 2$) since you would have to close out (buy back) the stock (on the NYSE) at a higher price. However, correction of the overpricing gives a gain of 10% on the short position in MaxPill – implying a net gain on the MaxPill short position of 4%.

But an increase in the market return of 3% implies an increase in value of four long stock index futures contracts of $6,000, which provides an additional return of 6% (on the initial $100,000) when the long-futures are closed out (i.e. sold). Hence, overall, MoneyPlus again makes 10%, with a net 4% coming from short-selling the overpriced stock in the cash market and 6% from going long stock index futures.

6.2.2 Portfolio of Stocks

Now consider the case where MoneyPlus buys (goes long in) several underpriced stocks (e.g. Apple, AT&T, Exxon-Mobile, Cisco Systems, Ford, Starbucks, etc.) with $V_i > 0$ (dollars) in each underpriced stock. To hedge the market risk of this portfolio, the number of stock index futures to short is given by Equation (6.1) where β_p is the beta of the underpriced *portfolio* of stocks:

$$\beta_p = \sum_{i=1}^{n} w_i \beta_i \tag{6.4a}$$

where

$$\sum_{i=1}^{n} V_i = V_p, \qquad w_i \equiv V_i / V_p, \qquad \sum_{i=1}^{n} w_i = 1 \tag{6.4b}$$

EXAMPLE 6.1

Hedge a Long Stock Portfolio

MoneyPlus invests $V_1 = \$100{,}000$ of its own funds in underpriced stock-1 with $\beta_1 = 1.5$ and $V_2 = \$200{,}000$ in underpriced stock-2 with $\beta_2 = 2$. The total amount invested is $V_p = V_1 + V_2 = \$300{,}000$ and the portfolio beta is:

$$\beta_p = (1/3)1.5 + (2/3)2 = 1.8333.$$

If the stock index futures $F_0 = 200$ points, the number of short contracts required to hedge the market risk of this speculative position in underpriced stocks is:

$$N_F = -\frac{V_p}{V_F}\beta_p = -\frac{\$300{,}000}{(\$250)(200)}(1.833) = -11 \ (\textit{short, futures contracts})$$

(*continued*)

(*continued*)

Also, taking positions in a portfolio of many underpriced stocks reduces the specific risk of the hedge fund's stock portfolio.

Conversely if stock-1 and stock-2 are thought to be overpriced then MoneyPlus would short-sell \$100,000 and \$200,000 (i.e. $V_s = \$300{,}000$) of these two stocks and go long $N_F = 11$ stock index futures contracts.

6.3 MARKET-NEUTRAL HEDGE FUND

Here we initially consider holding only a portfolio of stocks – initially we do not hold SIF contracts. MoneyPlus can construct a long-short, stock-picking strategy which is 'dollar neutral' ('market neutral') by altering the dollar amounts it has in its long ($V_{L,i} > 0$) and short ($V_{S,i} > 0$) positions in stocks. Note that for a short position of say \$50 in a stock we assign a positive value for $V_{S,i} = +\$50$. The change in the dollar value of a long-short stock portfolio (comprising n-stocks held long and m-stocks which have been short-sold), due to a change in the market return, is[3]:

$$dV_{LS} \equiv dV_L - dV_S = \sum_{i=1}^{n} V_{L,i}R_{L,i} - \sum_{i=1}^{m} V_{S,i}R_{S,i} = \left[\sum_{i=1}^{n} V_{L,i}\beta_{L,i} - \sum_{i=1}^{m} V_{S,i}\beta_{S,i}\right] R_m \tag{6.5}$$

Using the (simplified) CAPM (single index model), Equation (6.5) can be written in terms of *portfolio* betas (see Appendix 6.A), $R_L = \beta_L R_m$ and $R_S = \beta_S R_m$:

$$dV_{LS} = V_L R_L - V_S R_S = [V_L\beta_L - V_S\beta_S]R_m \tag{6.6}$$

where $V_L = \sum_{i=1}^{n} V_{L,i}$ and $V_S = \sum_{i=1}^{m} V_{S,i}$ are the *total* dollar amounts held long and short, respectively and $\beta_L = \sum_{i=1}^{n} w_{L,i}\beta_{L,i}$, and $\beta_S = \sum_{i=1}^{m} w_{S,i}\beta_{S,i}$ are the long and short *portfolio* betas, respectively. The individual stock-betas β_i are fixed. But the hedge fund can choose the dollar amounts $V_{L,i} > 0$ held long and $V_{S,i} > 0$ which are short-sold, to ensure that the term in brackets in (6.6) is zero – then any change in the market return will result in no change in the dollar-value of its long-short stock portfolio.

[3]Some texts use the convention that buying stocks implies $V_i > 0$ and short-selling implies $V_i < 0$ then $dV_{LS} = \sum_{i=1}^{n} V_i R_i$ but we choose to keep the long and short positions explicit.

For example, a *market-neutral* position might involve the hedge fund going long $V_L =$ \$200,000 in a portfolio of underpriced stocks with portfolio beta of $\beta_L = 1.5$ and short-selling $V_S =$ \$150,000 of overpriced stocks with portfolio beta $\beta_S = 2$, giving a 'long-short, dollar-beta', $[V_L\beta_L - V_S\beta_S] = 0$.[4]

Clearly in this case, MoneyPlus does not have to take a position in futures to hedge its market risk – and it simply has to wait to see if the over-pricing and under-pricing are corrected. However, it is worth noting that in practice, a hedge fund which describes itself as 'market neutral' may not always have its stock portfolio *completely insulated* from movements in the market return. In this case there may be some residual market risk in the stock portfolio and we should more accurately describe this fund as a 'long-short' hedge fund.

6.4 LONG-SHORT HEDGE FUND

In practice, a 'long-short' hedge fund will often find that its portfolio of overpriced and underpriced stocks will not have a portfolio dollar-beta of exactly zero. In this case MoneyPlus might choose to hedge its remaining market risk using stock index futures as this is less costly than altering the stocks in its portfolio to generate a zero dollar-beta. As we shall see, this just involves treating the long and short positions separately. The number of futures contracts required to hedge the long-short position is just the 'difference' between the number required to hedge the long positions and the number required to hedge the short positions, hence:

$$N_L = -(V_L/V_F)\beta_L \text{ (short futures)}, \quad V_L > 0 \tag{6.7a}$$

$$N_S = (V_S/V_F)\beta_S \text{ (long futrures)}, \quad V_S > 0 \tag{6.7b}$$

The net position in futures is:

$$N_F = N_L + N_S \tag{6.8}$$

which could be a net-long or net-short (depending on the size of the long and short positions and the portfolio betas). Suppose the fund manager has two underpriced stocks held long and also has short-sold two overpriced stocks – the calculation of the long and short portfolio betas are given in Table 6.1.

[4]This is more accurately described as a 'dollar neutral' position since a change in the market return has zero impact on the *dollar* value of the long-short stock portfolio. An alternative definition of 'market neutral' is when $\beta_L = \beta_S$ but this will only give a 'dollar neutral' outcome if it is also the case that $V_L = V_S$. In practice, some hedge funds do take (a) *exactly offsetting* long-short dollar positions and (b) have the same long-beta and short-betas – they are then simultaneously 'market (beta) neutral' and 'dollar neutral'. We will use 'market neutral' to mean 'dollar neutral'.

TABLE 6.1 Calculation of long and short portfolio beta

	Long		
	Asset 1	**Asset 2**	**Portfolio**
Value, $V_{L,i}$	\$120	\$80	\$200
Weights, $w_{L,i}$	0.6	0.4	1
Beta, $\beta_{L,i}$	1.8	1.05	1.5
	Short		
	Asset 1	**Asset 2**	**Portfolio**
Value, $V_{S,i}$	\$30	\$20	\$50
Weights, $w_{S,i}$	0.6	0.4	1
Beta, $\beta_{S,i}$	2.1	1.85	2

Note: Amounts are in thousands of dollars. $w_{L,1} = V_{L,1}/V_L$, $w_{L,2} = V_{L,2}/V_L$, $\beta_L = \Sigma w_{L,i}\beta_{L,i}$, $\beta_S = \Sigma w_{S,i}\beta_{S,i}$, are the long and short portfolio betas.

EXAMPLE 6.2

Hedge a Net-Long Stock Portfolio

Using the data in Table 6.1, if $F_0 = 200$ (and each index point is worth \$250) then:

$$N_F = N_L + N_S = -\frac{200{,}000}{(\$250)(200)}1.5 + \frac{\$50{,}000}{(\$250)(200)}2 = -6 + 2 = -4 \text{ contracts}$$

The change in value of the futures position is $dV_F = N_F V_F R_m$ where we assume $dF/F_0 = R_m$. The dollar change in value of the long-short stock portfolio plus the futures contracts is:

$$\begin{aligned} dV &= \$200{,}000\ \ \beta_L R_m - \$50{,}000\ \ \beta_S R_m + N_F V_F R_m \\ &= \$200{,}000\ \ (1.5)R_m - \$50{,}000\ \ (2)R_m - 4(\$250 \times 200)R_m \\ &= [\$300,000 - \$100,000 - 200,000]R_m \end{aligned}$$

For the net-long portfolio in the above example, if the market return rises by 1% the long stock position increases in value by $300,000, the short stock position loses $100,000, making a net gain on the long-short *stock* position of $200,000, which is then exactly offset by a loss of $200,000 on the short futures position in four contracts. The $200,000 loss on the short futures occurs when the futures are closed out by buying them back at a higher price, consequent on the rise in the index futures price. Hence the initial (non-market neutral) long-short stock portfolio is now hedged against rises (or falls) in the market return – as required.

Whether you go long or short in stock index futures to hedge the market risk of your long-short stock portfolio depends on the sign of $[V_L\beta_L - V_S\beta_S]$. For example, if the dollar amount MoneyPlus decides to go short is relatively large and the stocks which it shorts have relatively high betas, then $[V_L\beta_L - V_S\beta_S]$ may be negative. The hedge fund would then hedge by having a net long position in SIF (as in Example 6.3).

EXAMPLE 6.3

Hedge a Net-Short Stock Portfolio

MoneyPlus is long $V_L = \$200{,}000$ in underpriced stocks with $\beta_L = 1.5$ and short-sells $V_S = \$300{,}000$ in overpriced stocks with $\beta_S = 2$. Hence, $[V_L\beta_L - V_S\beta_S] = -\$300{,}000$. If $F_0 = 200$, then $V_F = (\$250 \times 200) = \$50{,}000$ and to hedge the market risk of this net-short stock portfolio requires:

$$N_F = -\frac{V_L}{V_F}\beta_L + \frac{V_S}{V_F}\beta_S = -6 + 12 = +6 \text{ (long futures contracts)}$$

The outcome of the hedge portfolio due to changes in the market return is:

$$\begin{aligned} dV &= \$200{,}000(1.5)R_m - \$300{,}000(2)R_m + 6(\$250 \times 200)R_m \\ &= [\$300,000 - \$600,000 + \$300,000]R_m \end{aligned}$$

Hence if the market return increases by 1% the long stock position increases by $300,000, the short position loses $600,000, a net loss of $300,000 on the 'cash market' stock portfolio – which is exactly compensated by the $300,000 profit when the initial six long futures are closed out (by selling them at a higher price).

Note that in all the above examples we hedge the market risk of a long-short stock portfolio consisting of both overpriced and underpriced stocks. If the hedge fund has correctly identified stocks which are genuinely overpriced (e.g. a biotech firm that will have relatively few successful patents in the future) and stocks which are genuinely underpriced (e.g. a media company that will produce exceptionally popular TV shows in the future), then the hedge fund will earn speculative profits when the mispricing of the two stocks is corrected – regardless of what happens to the market return (S&P 500) – since changes in the market return are hedged by the position in N_F futures contracts.

6.5 CHANGING STOCK MARKET EXPOSURE

6.5.1 Reducing Stock Market Exposure

Suppose you hold a \$1m diversified portfolio of stocks with $\beta_p = 2$ but there are no underpriced stocks in the portfolio – you merely hold a 'passive' diversified portfolio. This portfolio is subject to substantial market risk because if the market falls 10%, the value of the 'passive' portfolio will fall by 20% – that is by \$200,000. Normally, you are willing to tolerate this risk since there is an equivalent 'up-side' – should the stock market rally by 10%, your portfolio would increase in value by 20%. However, if your wish to *completely hedge* your stock portfolio using futures (when $F_0 = 200$) you require:

$$N_F = -(V_p/V_F)\beta_p = -(\$1\text{m}/(\$250 \times 200))2 = -40 \text{ contracts} \tag{6.9a}$$

By shorting 40 stock index futures contracts you have made the 'effective beta' of your 'long stocks + short futures' portfolio equal to zero – the hedge has reduced your 'effective beta' from 2 to zero.

Suppose you currently feel the market over the next 2 months is likely to be more volatile than normal and therefore you want to reduce *but not eliminate* your market exposure. For example, suppose your *desired beta* is $\beta_d = 0.5$ – a reduction of 75% compared to its current value of 2. To achieve a desired beta $\beta_d = 0.5$, you would short 30 futures contracts today ($= 0.75 \times 40$) and close out the position in 2 months' time (when the effective beta of your stock portfolio would again revert to 2). This is cheaper than reducing your portfolio beta by selling some high beta stocks today and buying them back after 2 months (when you want to reinstate your portfolio beta of 2). The formula for the number of futures required to achieve your desired beta is:

$$N_F = \frac{V_p}{V_F}(\beta_d - \beta_p) = \frac{\$1\text{m}}{\$250(200)}(0.5 - 2) = -30 \text{ contracts} \tag{6.9b}$$

By shorting 30 futures contracts today you have reduced your effective beta from 2 to 0.5. Hence if the market return moves up or down by 10% the value of your hedged portfolio will change by only plus or minus 5% – that is, by plus or minus \$50,000. Without the futures hedge it would have changed by plus or minus \$200,000. This 'partial hedge' has substantially reduced the volatility of your portfolio, compared with not being hedged – but (by construction) the volatility has not been reduced to zero – you are not fully hedged.

6.5.2 Market Timing Using Index Futures

Suppose the current beta of (a passive) stock portfolio is $\beta_p = 2$, so the return on your stock portfolio is $R_p = \beta_p R_m$. If you think the market return is going to increase by a large amount in the near future then you could increase your return by selling some of your low-beta stocks and using the funds to buy high-beta stocks – thus increasing your overall portfolio beta to, say, $\beta_p = 4$. If the market return subsequently increases by 10% as predicted, then you will make a return of 40% (rather than 20%). You may then wish to revert back to having a desired portfolio beta of 2 and to achieve this you will have to reverse your previous trades. This method of increasing your beta just before a market rise and then returning to your initial beta may involve high transactions costs of buying and selling stocks (e.g. brokerage fees, bid–ask spreads, price impact effects, etc.). An alternative and cheaper method of increasing (decreasing) your exposure when you think the market will rise (or fall) is to use stock index futures.

Suppose you hold a $1m diversified portfolio of stocks with $\beta_p = 2$. Above, we noted that you can *completely hedge* this position and effectively move to a beta of zero by *shorting* 40 futures contracts. By shorting 40 contracts your effective beta is reduced from 2 to zero. Now, suppose you think the stock market is going to rise (more than average) over the next two months. You could double the beta of your portfolio to a 'desired beta' of $\beta_p = 4$ by *buying 40* futures contracts today. Again, the formula required to achieve your desired beta is:

$$N_F = \frac{V_p}{V_F}(\beta_d - \beta_p) = \frac{\$1\text{m}}{\$250(200)}(4 - 2) = +40 \text{ contracts} \tag{6.9c}$$

If your prediction is correct and the stock market rises by say 10% over the next 2 months, the value of your stock portfolio plus the cash profits after closing out your long futures contracts, will result in the equivalent of a 40% increase $(= \beta_d R_m = 40\%)$ in value, from $1m to $1.4m. Without the long futures, you would have made only 20% on your stock portfolio. With $N_F = 40$ the change in value of the 'stock+futures' portfolio, using $R_p = \beta_p R_m$, $dF/F = R_m$ and $R_m = 0.10$ (10%) is:

$$\begin{aligned} dV &= V_p R_p + N_F V_F (dF/F) = V_p(\beta_p R_m) + N_F V_F R_m \\ &= \$1\text{m}\,(2)0.10 + 40(50{,}000)\,0.10 = (\$200,000 + \$200,000) \end{aligned}$$

Hence your new portfolio value is $1.4m, made up of $200,000 from the increase in value of your stock portfolio and a $200,000 cash profit after closing out your index futures. You started with a $1m stock portfolio and ended up being worth $1.4m, an increase of 40%. The stock market increased by 10% but your wealth increased by 40% – that is, you achieved an effective (desired) beta of 4.

Here you have used SIF to *leverage up* the returns on your stock portfolio – this is exactly how some hedge funds leverage their returns (particularly those using 'global macro-strategies' on domestic and foreign stock markets). Of course, leverage is a 'two-way street', if you guess wrong and the stock market falls by 10%, then you lose on both your stock portfolio and the long futures contracts and your total losses will be 40%, ($400,000) – that is, a $200,000 fall in the value of your stocks and a $200,000 cash loss when you close out (i.e. sell) your long futures contract at a lower price.

6.6 MERGER ARBITRAGE

Suppose you want to purchase 1 million shares of firm-A because after doing careful research, you believe it will be announced in 1 month's time[5] that firm-A is a *takeover target*. However, other traders do not yet realise it is a takeover target. When the takeover is finally announced or finally agreed (after 'due diligence'), you hope to make a speculative profit – as takeovers usually result in a rise in the target firm's price by over 30%, when the takeover is announced/completed. This 30% potential profit would be referred to as 'the alpha from merger arbitrage'. Precisely what actions you take to 'capture' this potential 30% profit depends on what risks you are willing to take and what risks you want to hedge.

Clearly, to gain from the 30% increase in price you need to purchase the stock before the takeover is announced (or information about the takeover leaks to the market). If you delay your purchases of stock-A, two things could happen. First, the stock market as a whole might increase, thus increasing the cost of purchasing stock-A in the future; and second, information about the takeover could become widely available and that would also increase the price of stock-A, before you have time to complete your speculative purchases.

6.6.1 Using Stock Index Futures Contracts

Consider the scenario whereby you do not think any information about the takeover will become available for at least 1 month. Suppose you want to 'stagger' your purchases over several days of the month using several brokers, so you do not alert other traders to your actions and put possible upward price pressure on target-A's stock price. However, you also fear a general rise in the stock market over the next month (as the economy recovers) which would increase the purchase price of your target stock-A (this is 'beta' again) and hence reduce your speculative profits.

You can hedge this market risk by today going long *stock index futures* contracts (and closing them out when you are ready to purchase stock-A). Any gain on your index futures will offset the higher cost of your purchases of stock-A over the next month – *consequent on any rise in the market return*. This is the 'standard' anticipatory hedge when you fear a price rise due to an increase in the market return.

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[5]We initially assume that you are certain that the takeover announcement (or any affirmative news of the takeover) will not occur for at least 1 month.

Assume firm-A's stock price is \$3 per share and $\beta_A = 2$. You want to buy 1m shares (in 1 month's time) at a cost of $V_A = \$3\text{m}$. The number of index futures contracts (with $F_0 = 200$) you need to purchase today, to hedge the impact of a (possible) rise in the market as a whole is:

$$N_F = \frac{V_A}{V_F}\beta_A = \frac{\$3\text{m}}{\$250(200)}2 = 120 \text{ contracts} \tag{6.10}$$

Suppose the stock market rises $R_m = 10\%$ over the next month. The net increase in cost of purchasing stock-A in 1 month's time, after closing out your long index futures positions, is:

$$\begin{aligned} Net\ increase\ in\ cost &= Increase\ cost\ of\ stocks - Gain\ from\ long\ index\ futures \\ &= V_A R_A - N_F V_F (dF/F) = V_A(\beta_A R_m) - N_F V_F R_m \\ &= \$3\text{m}(2)0.10 - 120(\$250 \times 200)0.10 = 0 \end{aligned}$$

If the market return increases by 10% then stock-A's price rises by 20% (\$600,000) but this extra cost of buying 1m stocks in firm-A is completely offset by the \$600,000 profit from closing out the long position in index futures. You have hedged the market risk of a future purchase of stock-A.[6] The success of this hedge depends on an accurate measure of stock-A's beta.

A position in index futures protects you from *market risk*. But, if information about the takeover target 'leaks out' during the month, then stock-A's price will move more than that given by its 'beta'. Your index futures position only hedges the increase in the purchase price due to market risk – but any increase in price (during the month) because other traders push up the price due to a leak of information about the takeover, will not be hedged. (This is specific risk.)

If you really believe information about the potential takeover announcement will become generally known to the market *during* the next month, then you need to purchase firm-A's shares immediately or before this information becomes widely known (and hence is incorporated in a higher market price of stock-A). However, if you purchase stock-A today you are also exposed to market risk – the market might fall substantially and this could wipe out (or substantially reduce) the 30% profit, when the takeover is eventually announced. So once you have purchased the stocks you need to hedge this market risk by *shorting* (stock) index futures today – while waiting for the official announcement of the takeover.

Hence, your *future purchases* of the stock can be hedged by going long SIF today. But once you actually purchase the stocks, you need to short index futures to hedge the stocks you now hold against a potential fall in the market return (while you wait for the merger announcement).

[6] This also means that if the market should fall, causing a fall in the price of stock-A, you will not benefit from this lower price in the future – because the loss on the long futures means the effective price of stock-A remains unchanged. If the market falls (prior to you purchasing stock-A) you will regret having hedged the market risk – but you could not know in advance that this would happen. 'Hedging' protects you from the 'downside' (i.e. higher purchase price) but does *not* allow you to benefit from the 'upside' (i.e. lower purchase price).

6.6.2 Using Futures Contracts on Stock-A

Alternatively, you can speculate on the takeover being announced/completed during the next month by today, purchasing *futures contracts on stock-A*. Note that you are purchasing futures on stock-A, not stock index futures.[7] Also, here we assume you only take a position in *futures contracts on stock-A* and you do not buy stock-A on the NYSE. When the takeover is announced/completed then you can close out your futures positions on stock-A at a profit.[8] As futures only require a small margin payment, you do not require much 'own capital' to set up your long futures position.

Note, however, that if you are long futures contracts on stock-A, you are also exposed to market risk over the next month. If the market (S&P 500) falls then the price of stock-A will fall, as will the futures price of stock-A (because of arbitrageurs) – and you may have to close out your long futures at a loss. Clearly, your long futures position on stock-A will earn (net) profits if the rise in price of stock-A, due to the takeover announcement, is not offset by a fall in the price of stock-A due to a fall in the market – but there is no guarantee that this will happen.

When you buy futures contracts on stock-A you are gambling on both the takeover being announced/completed and 'the market' not falling substantially. However, you can remove the market risk of your long position in the futures contract on stock-A by shorting *index futures* contracts. You then have a long position in futures on stock-A and a short position in stock index futures contracts – the former provides the profits from the takeover announcement/completion and the latter provides protection against falls in the market return, while you wait for the takeover to be announced/completed. Just after the takeover is announced/completed you close out both the futures on stock-A and your index-futures contracts.

There are other variants on the above scenarios. Hedge funds may not gamble on the future *announcement* of a takeover target because this is difficult to predict (unless you act on inside information, which is illegal). However, it has been noted that after the announcement of a takeover target, the stock price of the target tends to increase over the next 5–10 days and will rise further if the takeover is completed (after due diligence). The hedge fund can take advantage of this 'post-announcement price drift' by either buying stock-A or buying futures contracts *on stock-A,* on the day of the merger announcement. It could hedge either of these long positions by also shorting *index* futures to protect against falls in the price of stock-A due to falls in the market.

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[7]The choice between using purchases of stock-A or purchasing futures contracts *on stock-A* are the usual generic ones – for example, relative transactions costs, whether you can purchase the stocks 'on margin', the finite life of the futures contract and the possibility of margin calls.

[8]Assume the futures contract you choose a maturity date which occurs after the date of the 'takeover announcement'. As the latter is probably not known with certainty, you would be wise to choose a contract with a long rather than a short maturity date – otherwise you may have to 'roll over' your short-dated futures contracts.

In addition, hedge funds often go long the target company's stock and short-sell the acquirer's stock because the latter tends to fall slightly after the takeover announcement – this long-short position also provides some (but not total) protection against falls in the market return as it lowers the effective beta. If the hedge fund simultaneously implements several 'merger arbitrage' strategies then it will have a more diversified portfolio and this will reduce the specific risk of these gambles. Finally, note that if the hedge fund relies on 'post-announcement price drift' and the proposed takeover is eventually abandoned, then the price of the target company tends to fall dramatically and the hedge fund's long position in the target company will be closed out at a loss.

6.7 SUMMARY

- A hedge fund can implement a market timing strategy by altering the 'effective beta' of its existing stock portfolio by buying or selling index futures. A hedge fund can then achieve its desired level of exposure to changes in the market return (S&P 500).
- If a hedge fund has a long position in a portfolio of stocks, it can increase its 'effective beta' by buying index futures when it thinks the market will rise in the future. Alternatively it can reduce its effective beta by selling index futures. If it sells enough index futures, the hedge fund can have an effective beta which is negative and hence profit if there is a fall in the market return in the future.
- A 'market-neutral' hedge fund short-sells overpriced stocks, buys underpriced stocks and also holds some stocks that it thinks are correctly priced (to reduce specific risk). It may also alter the composition of its stock portfolio to ensure that the overall (dollar) beta is zero. The value of the market-neutral hedge fund's stock portfolio is then protected from changes in the market return (e.g. S&P 500), but it can still benefit when any over- or underpricing are corrected. The hedge fund can then close out its under- and overpriced stocks at a profit.
- Some 'long-short' hedge funds short-sell overpriced stocks and buy underpriced stocks but the resulting overall (dollar) beta of the stock portfolio may be non-zero – hence, it is not 'market neutral'. Stock index futures can then be used to protect the value of this 'active portfolio' from changes in the market return (S&P 500) while the hedge fund waits for the mispricing in its 'stock picks' to be corrected.
- A hedge fund that correctly identifies a future takeover target (before other merger arbitrage funds) may stagger its purchases of the target's stock, to avoid an adverse price impact which would raise the purchase price. Alternatively, it may have to wait until funds are available to purchase the stock. If it delays its purchases of the stock, a rise in the market will increase the cost of buying the stocks in the future. It can hedge the risk of the target's stock price increasing, solely because of an increase in the market return, by buying stock index futures, today. Once the merger arbitrage hedge fund has purchased the target company's stock then it can hedge the market risk by shorting index futures.

APPENDIX 6.A: STOCK PICKING AND MARKET RISK

In this section we assume a hedge fund MoneyPlus only takes long positions in a portfolio of underpriced stocks – this makes the initial algebra fairly straightforward. Next we discuss short-selling. Then we show how short-selling can be incorporated in the analysis of hedging the market risk of a long-short portfolio of both underpriced and overpriced stocks.

Suppose we hold a (long) stock portfolio with $V_i(>0)$ dollars in each stock, some of which we assume are potentially underpriced (and others are 'correctly priced'). The total portfolio value is V_p. Assume the portfolio return R_p can be represented using the Single Index Model, SIM[9]

$$R_p = \alpha_p + \beta_p R_m + \varepsilon_p \tag{6.A.1}$$

where

$$R_p = \sum_{i=1}^{n} w_i R_i, \quad w_i = V_i/V_p, \quad V_p = \sum_{i=1}^{n} V_i, \quad \sum_{i=1}^{n} w_i = 1$$

and

$$\alpha_p = \sum_{i=1}^{n} w_i \alpha_i, \quad \beta_p \equiv \sum_{i=1}^{n} w_i \beta_i, \quad \varepsilon_p = \sum_{i=1}^{n} w_i \varepsilon_i$$

ε_i is a random error which represents the specific risk of a stock and has an expected value of zero as does the specific risk on the portfolio of stocks $\varepsilon_p = \sum_{i=1}^{n} w_i \varepsilon_i$. The portfolio alpha α_p is determined by the dollar amount held long ($V_i > 0$) in underpriced stocks ($\alpha_i > 0$) and the dollar amounts $V_i > 0$ held in 'correctly priced' or 'passive stocks' ($\alpha_i = 0$) – the latter are included in the stock portfolio to reduce specific risk and reap the benefits of (naïve) diversification.

MoneyPlus buys individual 'underpriced' stocks to produce an expected positive value for α_p, and this would be referred to as an 'active portfolio strategy' or 'stock picking'. For example, if MoneyPlus takes long positions in underpriced stocks which it forecasts will add 1% over the next month to its portfolio return when their mispricing is corrected, then $\alpha_p = 1\%$, per month.[10] Note that α_p is independent of movements in the market return and depends only on the mispricing being corrected in the future. But the overall return on the (unhedged) stock portfolio R_p, depends on the market return, via the portfolio beta. To eliminate the market risk over the next month, the number of index futures contracts to short is[11]:

$$N_F = -(V_p/V_F)\beta_p \tag{6.A.2}$$

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[9]The algebra can be done more simply by assuming $R_p = \beta_p R_m$, although this omits any discussion of the impact of alpha and specific risk.

[10]In this case the returns in (6.A.1) are monthly returns.

[11]If MoneyPlus only wants to hedge the underpriced stocks (and not the 'passively held' stocks) then all 'metrics' (e.g. dollar amount held long, beta and alpha) would correspond only to those for the underpriced stocks themselves (and not for the whole portfolio of underpriced and passively held stocks). In this case the passively held stocks would be subject to market risk.

The change in value of the hedged-portfolio is:

$$\begin{aligned} dV &= V_p R_p + N_F V_F\,(dF/F) = V_p[\alpha_p + \beta_p R_m + \varepsilon_p] - \frac{V_p}{V_F}\beta_p(V_F R_m) \\ &= V_p(\alpha_p + \varepsilon_p) \end{aligned} \tag{6.A.3}$$

where $dF/F \approx R_m$, as over short horizons the futures index moves with the stock market index (e.g. S&P 500). Portfolio specific risk ε_p should be relatively small (i.e. close to zero) for a well-diversified portfolio – however, it will not be zero, for any single hedge. The variability in the (dollar) outcome of a series of hedges (using Equation (6.A.3)) is:

$$\sigma_{dV} = V_p \sigma_{\varepsilon_p} \tag{6.A.4}$$

and the variability in the hedged *return* $R_H \equiv dV/V_p$ is $\sigma_{R_H} = \sigma_{\varepsilon_p}$. From (6.A.4) and using results from OLS regression, it can be shown that the variance of the (dollar) hedging error is:

$$\sigma^2_{dV} = V_p^2 \sigma^2_{R_p}(1 - Rsq) \tag{6.A.5}$$

Rsq is the regression 'R-squared' from the OLS regression of Equation (6.A.1)[12]: In moving from (6.A.4) to (6.A.5) we use $\sigma^2_{\varepsilon_p} = \sigma^2_{R_p} - \beta_p^2\sigma^2_{R_m}$ (from 6.A.1) and the (OLS) definition, $Rsq \equiv \beta_p^2\sigma^2_{R_m}/\sigma^2_{R_p}$.

If the overall stock portfolio is 'passive' (i.e. contains only 'correctly priced' stocks) then $\alpha_p = 0$ and from (6.A.3) the *expected* return on the *hedged* portfolio is zero. Note that this zero outcome for the hedged portfolio also depends on the estimated beta of the stock portfolio being an accurate (unbiased) estimate of the 'true portfolio beta'. On the other hand, if the diversified portfolio contains genuine underpriced stocks then the *expected* return on the hedge portfolio containing these 'positive alpha-stocks' is $E(dV/V_p) = \alpha_p > 0$.

For any stock portfolio, the *out-turn value* for portfolio specific risk ε_p over a particular hedge period may not be zero. For example, if the portfolio consists of long positions in underpriced stocks but is hedged against market risk, there may still be some residual portfolio specific risk remaining. The impact on stock returns of environmental or regulatory risks, labour relations, IT failures, legal disputes, patents granted, etc. may vary across firms in the portfolio but may not average out to zero, over the hedge period – that is, the out-turn for ε_p may be positive or negative. This implies that even if MoneyPlus successfully picks underpriced stocks so that the out-turn value is $\alpha_p > 0$, the actual outcome for the hedged portfolio $dV = V_p(\alpha_p + \varepsilon_p)$ is uncertain because the outcome for ε_p is not known in advance.

The hedge strategy relies on two elements. First, that the stock portfolio is well diversified so that the out-turn value for ε_p is 'small' (relative to α_p), over the 1-month period it expects the

[12]In the two variable regression of (6.A.1), it is also the case that the square root of the regression R-squared is equal to the correlation coefficient ρ between the portfolio return R_p and the market return, R_m. Hence, $\rho = \sqrt{Rsq}$.

mispricing to be corrected.[13] However, the latter cannot be guaranteed. Second, over repeated 'stock picks' in different months, the hedge fund expects that portfolio specific risk will average out to be relatively small over time.[14]

The more diversified the portfolio, the smaller is specific risk. Usually a reasonably small amount of specific risk may be achieved by holding around 25 'passive stocks' (e.g. that could be chosen randomly from all stocks on the NYSE or more likely are chosen by sector, to match a particular passive index, such as a large cap or small cap index). These passive stocks could be 'equally weighted' which means that if you have a total of \$1,000 invested in 25 'passive stocks' then you initially hold \$40 ($= \$1,000/25$) in each stock. As the value of your 'passive stocks' changes (some rise in price and others fall in price) then their total value might be \$1,100 (say) after one month. To maintain 'equal weights' you must now rebalance your 'passive stocks' so you have \$44 ($= \$1,100/25$) in each stock. Also, you have to choose the frequency with which you rebalance the portfolio back to equal weights and this depends on transactions costs of buying and selling stocks.

Of course, the 'passive component' of your stock portfolio need not be equally weighted to achieve diversification benefits (e.g. you might choose to use market capitalisation weights, or global minimum variance weights, or ('fundamental') weights based on the relative dollar earnings paid by each company, etc.). The 'optimal choice' for your passive weighting scheme is a matter of ongoing research (see, for example, DeMiguel et al. 2009) and some methods of choosing the appropriate weights are often referred to as 'smart beta' by professional investors. If you hold positions in *several* underpriced stocks that are not too large relative to your holdings in passive stocks, then the specific risk of your 'active plus passive portfolio' may remain relatively small – but some specific risk is always present.

APPENDIX 6.B: MARKET TIMING

Suppose MoneyPlus holds a diversified long position $V_p > 0$ in a portfolio of stocks with an overall portfolio beta $\beta_p > 0$. But it wants to increase or decrease its exposure to market

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[13] Here we refer to the *monthly out-turn* value for $\varepsilon_p = \sum_{i=1}^{n} w_i \varepsilon_i$ across many stocks, each of which have positive or negative values for ε_i. If in any month, the specific risks across firms have low correlations (e.g. they may be 'independent', with correlation coefficient of zero) then the sum of the ε_i should be 'small' when averaged over many stocks *in any given month*. This is the result of (naïve) diversification across stocks (at a point in time), where the specific risks of different stocks have very low *contemporaneous* correlations.

[14] In any one month ε_p will be non-zero. But if specific risks of firms are not correlated *over time* (e.g. are not autocorrelated) then we expect positive and negative values for $\varepsilon_{p,t}$ over a large number of (monthly) time periods to be random around zero (partly because each firm is assumed to experience both 'good' and 'bad' luck over time, with equal probability). Hence $\varepsilon_{p,t} = \sum_{i=1}^{n} w_i \varepsilon_{i,t}$ is also expected to average out to zero over many time periods, $\sum_{t=1}^{T} \varepsilon_{p,t} \approx 0$ and its variance $\sigma^2_{\varepsilon_p}$ will also be small relative to the market risk $\beta_p^2 \sigma_m^2$.

(systematic) risk and hence achieve a new 'desired beta'. This can be done using stock index futures (on the S&P 500). We need to determine how many futures contracts N_F are required to achieve a 'desired beta' β_d.

MoneyPlus continues to hold its initial stock portfolio and now takes a position in N_F index futures contracts. The dollar change in value of MoneyPlus stocks plus N_F futures (SIF) contracts is:

$$dV = V_p R_p + N_F V_F \ (dF/F) = V_p(\beta_p R_m) + N_F V_F R_m \tag{6.B.1}$$

where $dF/F \approx R_m$, $V_F = (\$250)F$ and the return on MoneyPlus initial stock portfolio is $R_p = \beta_p R_m$. If MoneyPlus desired beta is β_d, then its desired portfolio return is $R_p^d = \beta_d R_m$ and hence, the desired dollar change in MoneyPlus stock portfolio is:

$$dV^d = V_p R_p^d = V_p(\beta_d R_m) \tag{6.B.2}$$

Equating (6.B.1) and (6.B.2) and solving for (the unknown) N_F gives the required number of futures contracts for MoneyPlus to achieve its desired beta:

$$N_F = (V_p/V_F)(\beta_d - \beta_p) \tag{6.B.3}$$

For example, assume MoneyPlus currently holds a portfolio which is (net) long ($V_p > 0$) with $\beta_p > 0$, then alternative desired betas give rise to the following futures positions:

If $\beta_d > \beta_p$ then $N_F > 0$ (buy, go long futures)
If $\beta_d < \beta_p$ then $N_F < 0$ (sell, go short futures)

Hence, if MoneyPlus wants a larger (smaller) effective beta than its current portfolio beta, then it should buy (sell) index futures today. Of course, even when MoneyPlus achieves its desired beta (using Equation (6.B.3)), the (ex-post) return on its hedged portfolio will also depend on α_p and ε_p.

Special Cases

Equation (6.B.3) subsumes our initial hedging result of Equation (6.B.1) above. When MoneyPlus wants to hedge *all* the market risk of its *existing* stock portfolio, then this implies that after taking a position in N_F futures it wants a desired $\beta_d = 0$. Setting $\beta_d = 0$ in (6.B.3) gives:

$$N_F = -(V_p/V_F)\beta_p$$

For our second 'special case' suppose MoneyPlus holds no stocks (so $\beta_p = 0$). But by using *only* stock index futures contracts it wants to achieve an exposure to the market return, with a desired beta of β_d (=2 say), on a notional dollar amount of $V_p = \$1$m (say).[15] Then MoneyPlus'

[15]This is a 'notional amount' as MoneyPlus does not have to use any of its 'own capital' when using futures – it just has to meet any margin payments on the futures.

required position in stock index futures is:

$$N_F = (V_p/V_F)\beta_d$$

This may be a cheaper way of gaining exposure to the stock market than directly using MoneyPlus' own funds to purchase an index tracker fund or an ETF.

APPENDIX 6.C: HEDGING: LONG-SHORT PORTFOLIO

Long Position in Stocks

The dollar change in value of long positions in n-stocks with $V_{L,i} > 0$ held in each stock is:

$$dV_L = \sum_{i=1}^{n} V_{L,i}R_{L,i} = \sum_{i=1}^{n} V_{L,i}(\beta_{L,i}R_m) = (\$\beta_L)R_m \tag{6.C.1}$$

where we assume (a simplified Single Index Model/CAPM) $R_{L,i} = \beta_{L,i}R_m$. The 'dollar (long) beta' is defined as $\$\beta_L \equiv \sum_{i=1}^{n} V_{L,i}\beta_{L,i}$ – it is a *dollar weighted* sum of the individual stock betas. For example, if the dollar-beta $\$\beta_L = \$300{,}000$ then a 1% change in the market return will lead to a \$3,000 change in the value of the portfolio of (long) stocks. In finance it is often convenient to express the dollar change in value using the portfolio return R_L and the (standard) portfolio beta β_L. An equivalent expression to (6.C.1) for dV_L in terms of the *portfolio* return R_L is:

$$dV_L = \sum_{i=1}^{n} V_{L,i}R_{L,i} = V_L \sum_{i=1}^{n} \frac{V_{L,i}}{V_L} R_{L,i} = V_L R_L \tag{6.C.2}$$

where $V_L = \sum_{i=1}^{n} V_{L,i}$, $\quad R_L = \sum_{i=1}^{n} w_{L,i}R_{L,i}$, $\quad w_{L,i} = \sum_{i=1}^{n}(V_{L,i}/V_L)$, $\quad \sum_{i=1}^{n} w_{L,i} = 1$,

If we assume $R_{L,i} = \beta_{L,i}R_m$ then $R_L = \beta_L R_m$ where $\beta_L = \sum w_{L,i}\beta_{L,i}$ is the beta of the (long) portfolio then:

$$dV_L = V_L R_L = V_L(\beta_L R_m) \tag{6.C.3}$$

Note that portfolio return R_L and the beta of the portfolio β_L are defined using V_L in the denominator for the portfolio weights $w_{L,i}$, which then sum to one – consistent with the usual textbook presentation. Also $R_L = \beta_L R_m$ and therefore the portfolio beta β_L represents the relationship between the *percentage* return on the stock portfolio and the *percentage* return on the market index – this is the usual textbook representation of beta (and differs from the 'dollar beta' defined earlier, which relates the *dollar* change in portfolio value to the *percentage* change in the market return). However, both (6.C.1) and (6.C.3) give exactly the same value for dV_L – they are just different ways of calculating dV_L.

Short-sell Stocks

Consider the change in value of a portfolio of m-stocks when you have short-sold $V_{S,i} > 0$ in each stock – note that we represent a short position as a positive value for $V_{S,i}$. The change in value of the short position can be expressed in a similar fashion to the long position (but incorporating a minus sign where appropriate). First, using the 'dollar beta':

$$dV_S = -\sum_{i=1}^{m} V_{S,i}R_{S,i} = -\sum_{i=1}^{m} V_{S,i}\beta_{S,i}R_m = -(\$\beta_S)R_m \tag{6.C.4}$$

where the 'dollar (short) beta' is defined as $\$\beta_S \equiv \sum_{i=1}^{m} V_{S,i}\beta_{S,i}$. In terms of the portfolio return and (standard) portfolio beta we have:

$$dV_S = -\sum_{i=1}^{m} V_{S,i}R_{S,i} = -V_S\sum_{i=1}^{m} \frac{V_{S,i}}{V_S}R_{S,i} = -V_SR_S \tag{6.C.5}$$

where $V_S = \sum_{i=1}^{m} V_{S,i}$, $\quad R_S = \sum_{i=1}^{m} w_{S,i}R_{S,i}$, $\quad w_{S,i} = \sum_{i=1}^{m}(V_{S,i}/V_S)$, $\quad \sum_{i=1}^{m} w_{S,i} = 1$,

If we assume $R_{S,i} = \beta_{S,i}R_m$ then $R_S = \beta_S R_m$ where $\beta_S = \sum_{i=1}^{m} w_{S,i}\beta_{S,i}$ is the beta of the (short) portfolio then:

$$dV_S = -V_SR_S = -V_S(\beta_S R_m) \tag{6.C.6}$$

Hedging a Long-Short Stock Portfolio

To hedge the long position or the short position in stocks, using stock index futures we require:

$$dV_L = V_LR_L + N_F^LV_FR_m = V_L(\beta_L R_m) + N_F^LV_FR_m = 0$$
$$dV_S = -V_SR_S + N_F^SV_FR_m = V_S(\beta_S R_m) + N_F^SV_FR_m = 0$$

Hence: $N_L = -(V_L/V_F)\beta_L$ (i.e. short futures) and $N_S = (V_S/V_F)\beta_S$ (i.e. long futures).

The net position in futures is therefore $N_F = N_L + N_S$, which could be a net-long or net-short position depending on the dollar value of the long and short positions and their portfolio betas.

APPENDIX 6.D: MERGER ARBITRAGE AND HEDGING

Long in Cash Market for Stock-A

Assume today, MoneyPlus purchases $V_A = \$300{,}000$ stocks in the takeover target (stock-A) and hedges the market risk using stock index futures. The change in value of the stocks plus index futures position is:

$$\begin{aligned} dV &= V_A R_A + N_F V_F (dF/F) \\ &= V_A[\alpha_A + \beta_A R_m + \varepsilon_A] + N_F V_F R_m \\ &= V_A(\alpha_A + \varepsilon_A) + [V_A\beta_A + N_F V_F]R_m \end{aligned} \tag{6.D.1}$$

where we have used the SIM, $R_A = \alpha_A + \beta_A R_m + \varepsilon_A$ and using stock *index futures* implies $dF/F = R_m$. To hedge the market risk, MoneyPlus shorts N_F stock index futures contracts:

$$N_F = -(V_A/V_F)\beta_A = -[\$300,000/\$250(200)]\,2 = -12 \text{ (short index futures)} \tag{6.D.2}$$

When the merger announcement takes place, MoneyPlus closes out the stock index futures contracts, which compensates for any dollar change in the value of stock-A due to changes in the market return. To see this substitute (6.D.2) in (6.D.1):

$$dV = V_A(\alpha_A + \varepsilon_A) \tag{6.D.3}$$

This leaves MoneyPlus with a potential abnormal percentage return ('bid premium') $\alpha_A = 0.30$ (30% say) due to merger arbitrage (and any positive or negative contribution due to specific risk ε_A).[16]

Long Futures on Stock-A

Instead of purchasing stock-A in the cash market to take advantage of the merger arbitrage underpricing, assume MoneyPlus purchases *futures contracts on* stock-A, today. The contract size for futures on stock-A might be for delivery of $z_A = 100$ stocks so that $V_F^A = z_A F_A$, where $F_A = P_A \exp(rT)$ is the current price of a futures contract on stock-A (we assume the stock pays no dividends, over the speculative horizon considered). Assume the current spot price of stock-A is $P_A = \$3$ and $F_A = \$3.1$.

[16]Note that this analysis applies to any strategy where today MoneyPlus purchases underpriced stocks (e.g. a portfolio of 'underpriced' small stocks or underpriced low book-to-market stocks) and wishes to hedge any resulting market risk.

The futures price (on stock-A) in Chicago moves with the NYSE price of stock-A, $R_A \equiv dP_A/P_A \approx dF_A/F_A$. If MoneyPlus wants to take a long position in futures (on stock-A), *equivalent to a position* of $V_A = \$300{,}000$ in the cash market, then it needs to choose the number of futures contracts N_F^A so the dollar change in the futures position dV_F^A equals that in the cash market dV_A:

$$dV_A = V_A R_A \tag{6.D.4}$$

$$dV_F^A = N_F^A V_F^A (dF_A/F_A) = N_F^A V_F^A R_A \tag{6.D.5}$$

Hence: $N_F^A = V_A/V_F^A = \$300{,}000/(100(3.1)) = 967.74$. The long position in 968 futures contracts on stock-A gives MoneyPlus exposure to the merger announcement *and* to the market return. MoneyPlus may then choose to hedge its exposure to changes in the market return by going short 12 *stock index futures* (see Equation (6.D.2)). MoneyPlus can then close out both futures positions just after the merger announcement is made. It will earn the 'bid premium' on the rise in price of stock-A due to the merger announcement, after closing out its long *futures contracts on stock-A* at a higher price.

Again, the above analysis applies to any strategy based on having 'discovered' (by any method) an underpriced stock-A and taking advantage of the perceived underpricing by using only futures contracts on stock-A (i.e. not stock index futures contracts). For overpriced stocks-B, the analysis is the same but you would *short* $N_F^B = V_B/V_F^B$ futures contracts on stock-B, to benefit from the perceived overpricing.

EXERCISES

Question 1

You are a stock analyst/trader and have just purchased a \$10m position in stock-XYZ which has a beta of 1.5 and based on current information, you believe it is underpriced by 6%. You think the underpricing is likely to be 'corrected' within the year. However, you also think that the stock market may fall by 10% over the next year and you wish to hedge against this fall, while still being able to take advantage of the underpricing of 6%. The current level for index futures on the S&P 500 is 1,000 (points). Each contract is for \$250.

What information do you need in order to determine the optimum number of futures contracts to use in the hedge? What are the risks in your strategy?

What is the outcome if the stock market does fall by 10% and the mispricing of stock-XYZ is corrected? Assume that the stock index futures price also falls by 10%.

Question 2

How can stock index futures be used as a low cost speculative, 'market timing' strategy?

Question 3

You are 'long' a diversified portfolio of stocks a with current value of \$20m. The beta of your portfolio (with respect to the market portfolio) = 1.2. The value of the futures contract on the S&P 500 market index is $F_0 = 400$ points (each contract is for \$250 times the index).

(a) How many futures contracts are required to hedge the portfolio of stocks?
(b) If you want to reduce the 'beta' of the hedged portfolio to 0.6, what should you do?

Question 4

You hold \$20m of *OMG*-stocks on 1 September and the stock index futures on the S&P 500 (stock index) is 1,600. You are forecasting an increase in the S&P 500 over the next 3 months and as your *OMG*-stocks have a beta of 1.5 these will also rise. How can you increase your exposure to the stock market using stock index futures on the S&P 500? What determines the number of futures contracts you need to increase your effective beta to 3?

Question 5

It is 25 July and you are considering a takeover bid for the firm *Hotshots* whose current stock price is \$26 with a beta of 1.8. You are going to purchase 100,000 stocks but not until 25 August. The stocks do not pay dividends. The yield curve is flat at 5% p.a. (continuously compounded).

The nearby September-futures contract on the S&P 500 expires on 25 September (i.e. in 2 months). The price of *Hotshots* stock on 25 August is \$27.9.

(a) If the S&P 500 index $S_0 = 1{,}000$ on 15 July and $S_1 = 1{,}050$ on 25 August, what is the futures index on these two dates?
(b) How might you hedge the cost of your purchases in August, using a futures contract on the S&P 500 (\$250 per index point)?
(c) What is the outcome of your hedge? Explain.

Question 6

In September you decide that the S&P 500 stock index will increase over the next 3 months and hence your £10m portfolio of stocks with a beta of 1.8, will also increase in value.

How can you *increase* your exposure to this rise in the S&P 500 using stock index futures, so that your effective (desired) beta is 2.5? What determines how many futures contracts you will buy or sell in September? The current level for index futures on the S&P 500 is 1,000 (points) and the tick value is \$250.

CHAPTER 7

Currency Forwards and Futures

Aims

- To outline contract specifications, settlement procedures and price quotes for selected foreign exchange (FX) futures contracts – also called 'currency futures'.
- To price a FX-forward contract by creating a replication portfolio using two money market interest rates and the spot exchange rate. This is 'covered interest arbitrage'.
- To show how FX-futures can be used to hedge future payments and receipts in foreign currency.
- To demonstrate how both FX-forwards and futures can be used for speculation.
- To compare the pricing of currency futures with those for forward contracts.

Two major types of 'deal' on the foreign exchange (FX) market involve the spot-rate and the forward-rate. We assume, the spot rate is the exchange rate quoted for immediate delivery of the currency to the buyer (actually, delivery is 2 working days later). The second type of deal involves the forward rate, which is the price agreed today, at which the buyer will take delivery of the currency at some future date. Currency forwards and futures are very similar analytically, even though in practice the contractual arrangements differ.

Both, an FX-forward and an FX-futures contract is an obligation to trade one currency for another at a pre-specified exchange rate on a specific future (delivery) date. The dealing costs in both type of contract are very small. Why then do we need both types of contract? The reasons are slight differences between the two types of contract.

A *currency forward* can be designed to exactly fit the client's requirements as to the principal (e.g. sterling) amount in the trade, the exact delivery date and which currencies are involved. It is an over the counter (OTC) transaction. Most forward deals, on a wide range of currencies, are channelled through London, New York and Tokyo with Hong Kong, Singapore

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

and increasingly Shanghai being influential in the Far East. This OTC market is very efficient with low transactions cost.

Deals are negotiated between large multinational banks (e.g. Barclays, HSBC, Citigroup, Morgan Stanley, JPMorgan-Chase, Goldman Sachs) for delivery of one currency (e.g. Swiss francs) usually against the US dollar (the vehicle currency), at a specific date in the future. It is also possible to negotiate so-called forward cross-rate deals (which do not involve the US dollar), for example, receipt of Swiss francs and payment of euros. For most major currencies, the most actively traded contracts are for maturities between 1 to 6 months and in exceptional circumstances, 3 to 5 years ahead. Use of the forward market eliminates any exchange risk from future changes in spot-FX rates – assuming no credit default risk from the counterparties in the OTC forward trade. This is because the forward rate is agreed today, even though the cash transaction takes place in (say) 1 year's time.

In contrast, a currency futures contract is an exchange traded instrument on a limited number of currencies. The contract sizes are fixed (e.g. delivery of SFr 125,000) as are the set of delivery dates. With a forward contract, delivery usually takes place, whereas with a futures contract you do not necessarily have to take delivery of the currency because you can easily close out your position before maturity. If you are long a currency futures and you decide not to take delivery, then you can close out your position by selling the futures (on the same currency and with the same maturity date). The FX-futures contract is marked-to-market daily (which involves margin payments) and therefore has virtually zero credit risk, whereas the forward contract involves counterparty credit risk.

Currency forwards and futures are used to hedge future cash flows in foreign currency – for example, by exporters and importers. Similarly, they are used to hedge future cash flows from purchases or sales of capital assets such as foreign bonds and stocks as well as future interest cash flows from foreign bank deposits or loans. They also provide leverage in speculative FX-transactions, which are gambles on the future path of exchange rates.

7.1 FX-FUTURES CONTRACTS

7.1.1 Contract Specification

There are a large number of currency futures traded on different exchanges, the major one being the International Money Market (IMM) division of the Chicago Mercantile Exchange (CME). For example, on CME the following currencies are traded against the US dollar: pound sterling, euro, yen, Swiss franc, Canadian dollar, Australian dollar, Mexican peso as well as other currencies. Other notable centres trading FX-futures are the Singapore International Money Exchange (SIMEX) and the Sydney Futures Exchange (SFE). On CME there are also futures on the major cross rates.

The most actively traded futures contracts traded on the CME are in the euro, Canadian dollar, Japanese yen, Swiss franc and sterling. Details of some of these contracts are given in Table 7.1. On the CME there is 24-hour electronic futures trading using the Globex system and

TABLE 7.1 Contract specifications IMM currency futures (CME)

Currency	Contract size	Tick size [value]
Pound sterling	£62,500	0.02 cent per £ [$12.50]
Swiss franc	SFr125,000	cent per SFr [$125]
Japanese yen	¥12,500,000	cent per 100 ¥ [$12.50]
Canadian dollar	C$100,000	cent per C$ [$100]
Euro	€125,000	cent per € [$12.50]

Note: The delivery months for all contracts is January, March, April, June, July, September, October and December and the delivery date is the 3rd Wednesday of the contract months. Last trading day is the second business day before delivery. Initial and maintenance margin are sometimes altered by the exchange. Margins are smaller if the transaction involves either a speculative spread trade or for a hedges position. There are no daily price limits.

Euro futures can be traded at any time if one is willing to trade on different exchanges. Take, for example, the Euro futures contract on the CME (see Table 7.1). If you are long one contract then the convention is:

$$\textit{Long Euro FX Futures} \Rightarrow \textit{Receive}\ €125,000\ \textit{and pay out}\ (\textit{deliver})\ \textit{US dollars}$$

Suppose Ms USA on 1 March wants to receive €125,000 in September. Today she purchases one Euro September-futures contract at $F_0 = 1.0400(\$/€)$. At maturity (in September) she *receives* €125,000 from the futures contract and pays out $130,000. Also, the contract is marked-to-market, daily. So for example, on 2 March, if $F_1 = 1.0450(\$/€)$ then Ms USA has gained, since (forward) euros have appreciated and her margin account will be credited with $625 (i.e. 50 ticks × $12.50 per tick).

7.1.2 Settlement

Although most contracts are closed out before maturity, it is useful to consider what happens on the maturity date of the September-futures. Assume the September spot rate $S_T = 1.05(\$/€)$ – arbitrage ensures this will also be the quoted futures price $F_T = 1.05(\$/€)$.

TABLE 7.2 Settlement of sterling currency futures

1. **Contract requirements: September settlement**
   Initial September-futures price $F_0 = 1.04(\$/€)$
   Ms USA notionally receives €125,000 and pays out $130,000
2. **Settlement convention at maturity**
   Currencies are delivered at the spot rate $S_T = 1.05(\$/€)$, prevailing on the last trading day, at maturity of the futures contract.
   Ms USA's margin account has been credited by:

   Gain on futures:
   $\$1{,}250 = 100 \text{ ticks} \times \$12.50 = (1.05 - 1.04)€125{,}000$

   Net cost to Ms USA
   = FX-Spot market payment–Gain on futures
   $= [S_T–(F_T - F_0)]\ €125{,}000 = F_0€125{,}000 = \$130{,}000$

Ms USA acquires €125,000 in the spot-FX market and pays out $131,250 based on $S_T = 1.05(\$/€)$ (Table 7.2). However, this is offset by profits from her (futures) margin account of $1,250. The net effect of purchasing euros in the spot-FX market and the profit from closing out the long futures contract is $130,000. This implies that she has locked in a rate of $F_0 = 1.04(\$/€) = (\$130,000/€125,000)$. (This calculation ignores any interest earned on the margin account over the life of the contract).

7.1.3 Quotes

Currency futures are quoted as 'USD per unit of foreign currency'. For example on 27 July suppose the September-Euro futures is quoted at $F_0^{Sept} = 0.9416(\$/€)$ and the December futures quote is $F_0^{Dec} = 0.9498(\$/€)$. Given a current spot rate of $S_0 = 0.9420(\$/€)$ then with $F_0^{Sept} = 0.9416(\$/€)$ this implies that you will receive 4 points/ticks less US dollars per euro in September than from the spot-FX rate on 27 July. Hence September-Euro futures are 'at a discount':

$$\textit{September – Euro FX Futures at a discount}$$

$$\Rightarrow \textit{less US dollars per euro in the futures market than in the spot market}, F_0^{Sept} < S_0$$

In contrast, if the quoted futures price $F_0^{Sept} > S_0$ then euros are selling at a forward premium relative to the current spot rate:

$$\textit{September – Euro FX Futures at a premium}$$

$$\Rightarrow \textit{more US dollars per euro in the futures market than in the spot market}, F_0^{Sept} > S_0$$

7.2 PRICING FX-FORWARD CONTRACTS

Pricing an FX-forward contract involves a relationship between the forward rate F and three other variables, the spot FX-rate and the money market interest rates in the two countries – it is known as *covered interest parity (CIP)*. The no-arbitrage forward rate (using simple interest) is:

$$F = S\frac{(1 + r_d T)}{(1 + r_f T)} \tag{7.1}$$

S and F are measured as 'domestic per unit of foreign currency', r_d = domestic interest rate, r_f = foreign interest rate, T = time to maturity (expressed in years or fraction of a year) and we assume the day-count convention is the same in both countries. The quoted forward rate given by CIP ensures that no risk-free arbitrage profits can be made by transacting between the spot currency, the two money markets and the forward market.

The CIP equation is derived as follows. Assume that a UK firm BritArb has £A = £100 which it can invest in the UK or the USA for 1 year. Assume transactions have no credit/default risk. For BritArb to be indifferent as to where the money is invested, the risk-free return from investing in the UK must equal the risk-free return *in sterling* from investing in the USA. Assume the quoted interest rates in the 'domestic' (sterling) money market, the 'foreign' (US) money market and the exchange rates are:

$$\begin{aligned}
r_d &= 0.11\ (11\%) \quad \text{(simple rate)}\\
r_f &= 0.10\ (10\%) \quad \text{(simple rate)}\\
S &= 0.6667\ (£/\$) \quad \text{(equivalent to 1.5 \$/£)}\\
F &= 0.6727\ (£/\$) \quad \text{(equivalent to 1.4865 \$/£)}\\
T &= 1, \text{investment horizon} \quad \text{(years or fraction of a year)}
\end{aligned}$$

Note that the forward rate F and the spot rate S are measured as 'domestic per unit of foreign currency', here £s per \$ (GBP per USD). We show that the above figures give equal returns to investing in either the UK or the US – so F = 0.6727 GBP/USD is the 'correct' no-arbitrage forward rate.

Strategy-1: BritArb invests in UK

In 1 year receive (terminal value): TV_{UK} = £$A(1 + r_d T)$ = £100(1.11) = £111

Strategy-2: BritArb invests in US

(a) Today convert £100 to \$150 (£100/0.6667(\$/£)) at spot-FX rate and invest in US deposit account

(b) Dollar receipts at end-year are: \$150(1.10) = $(A/S)(1 + r_f T)$ = \$165

(c) Enter into a forward contract today at $F = 0.6727$ (£/\$) *to sell* \$165 for delivery of sterling in 1 year's time. Sterling receipts in one year are:

$$TV_{US} = \$165F = £[(A/S)(1 + r_f T)]F = £111 \tag{7.2}$$

All of the above transactions (a)–(c) are undertaken at known 'prices', hence there is no price risk. Since both investment strategies are risk-free, arbitrage will ensure that they give the same terminal value, $TV_{UK} = TV_{US}$, hence:

$$A(1 + r_d T) = [(A/S)(1 + r_f T)]F \tag{7.3}$$

Rearranging, the 'fair', 'correct' or 'no-arbitrage' forward rate is:

$$F = S(1 + r_d T)/(1 + r_f T) \tag{7.4a}$$

or,

$$(F - S)/S = (r_d - r_f)T/(1 + r_f T) \tag{7.4b}$$

The above CIP formulas are exact, but if $r_f T << 1$ then $(1 + r_f T) \approx 1$, hence:

Forward premium/discount $\approx$ *interest rate differential*[1]

$$(F - S)/S \approx (r_d - r_f)T \tag{7.5}$$

7.2.1 Forward Points

In the FX-market, outright forward rates are not quoted, but instead the convention is that the difference between the forward rate and spot rate $F - S$, or the *forward points* or *forward margins* are quoted, where:

$$\begin{aligned} \textit{Forward Points}:\ F - S &= S(r_d - r_f)T/(1 + r_f T) \\ &= 0.6727 - 6667 = 0.0060\ (+60 \text{ points}) \end{aligned}$$

Given forward points of +60 on a dealer's screen, she would quote:

$$\begin{aligned} \text{'Outright' forward rate } F &= S + \text{'forward points'} \\ &= 0.6667 + 0.0060 = 0.6727\ (£/\$) \end{aligned}$$

[1]Note that if interest rates are continuously compounded then the equivalent to Equation (7.1) is $F = Se^{(r_d - r_f)T}$ or $\ln F = \ln S + (r_d - r_f)T$, respectively.

The forward points would usually be quoted on FX-screens for 1, 2, 3, 6, and 12 months and the value dates for the forward contracts would coincide with maturity dates for Eurocurrency deposits and loans. Given that CIP holds, the quoted forward points $F - S$ (approximately) equals the interest differential (multiplied by the spot rate). For example, for $r_d = 11\%$ p.a. and $r_f = 10\%$ p.a., then if you transfer cash from the foreign to the domestic investment you will earn 1% more interest. The only way you *do not* earn a risk-free profit is if the forward domestic currency you receive after 1 year is 1% less than the spot domestic currency you gave up at $t = 0$, that is, if $(F - S)/S$ equals –1%.

As CIP holds at all times, then the 'forward points' could be positive or negative, depending on whether r_d is greater or less than r_f. A further practical complication is that the forward rate calculated from the money market rates will have a bid–ask spread depending on which currency is being borrowed or placed on deposit in the money markets – we ignore this problem here. Two other rearrangements of the CIP condition are worth mentioning, which are:

$$F = S(1 + \mathrm{CC}) \quad \text{where} \quad \mathrm{CC} = (r_d - r_f)/(1 + r_f)$$

$$F = S + \chi \quad \text{where} \quad \chi = S(r_d - r_f)/(1 + r_f)$$

CC is known as the *percentage cost of forward cover* and χ is the *dollar* cost of forward cover. The cost of forward cover depends on the interest differential between the two countries. It is easy to check that our data are consistent with the (exact) algebraic CIP condition:

$$\text{Interest differential} = (r_d - r_f)/(1 + r_f) = 0.0091\ (0.91\%)$$

$$\text{Forward Discount} = (F - S)/S = 0.0091\ (0.91\%)$$

One further 'trick' to note is that the CIP formula looks slightly different if S and F are measured as 'foreign per unit of domestic currency'. However, the following 'rule of thumb' always holds:

If S and F are measured as 'currency X per unit of currency Y' then in the forward rate equation, interest rates are r_X in the numerator and r_Y in the denominator

For example, if S and F are measured as Swiss francs (SFr) per USD then the CIP condition is:

$$F/S = (1 + r_{SF}T)/(1 + r_{US}T)$$

where r_{SF} and r_{US} are the Swiss franc and US dollar interest rates, respectively. In practice, CIP must reflect the day-count conventions used for domestic and foreign interest rates. For example, if the forward currency is for delivery in 90 days and the Swiss interest rate convention is 'actual/365' and the US convention is 'actual/360' then the 3-month forward rate (SFr per USD) would be calculated as:

$$F/S = [1 + r_{SF}(90/365)]/\ [(1 + r_{US}(90/360))] \tag{7.6}$$

7.2.2 Arbitrage Profits

To show how arbitrage profits can be made if the actual forward quote F_q by bank-M does not equal the fair (no-arbitrage) futures price given by the CIP equation (7.1), suppose an investor BritArb is faced with the following data.

- One-year forward quote (from bank M): $F_q = 1.4$ (USD/GBP)
- $r_{UK} = 11\%$ p.a., $r_{US} = 10\%$ p.a., $S_0 = 1.5136$ (USD/GBP)
- Fair forward price $F_0 = S_0(1 + r_{US}T)/(1 + r_{UK}T) = 1.5$ (USD/GBP)

BritArb, can receive sterling in 1 year from bank-M at the forward rate F_q and will have to pay out \$1.40 per £1 in 1 year's time. Since $F_0 > F_q$ then by using the spot rate and the money markets in the two countries, BritArb can borrow £1 today, buy \$1 at a rate S_0, invest in the US money market and will receive \$1.50 per £1. The details are as follows.

Arbitrage strategy for BritArb:

- Enter a forward contract at F_q to receive £100 at $t = 1$ and pay out \$140 to bank-M
- Borrow £90.09 in the UK money market at $t = 0$ (i.e. owe £100 at $t = 1$)
- Convert £90.09 at $S_0 = 1.5136(\$/£)$ and lend \$136.36 in the US money market
- At $t = 1$, BritArb receives \$136.36 (1.10) = \$150
- Risk-free profit for BritArb = \$150–\$140 = \$10

Given this risk-free profit opportunity, many arbitrageurs implement this strategy, but this would result in:

- Borrowing in the sterling money market which raises r_{UK} while lending dollars causes downward pressure on r_{US}
- Buying dollars spot so the dollar appreciates (i.e. S falls)
- The above implies that $S_0(1 + r_{US}T)/(1 + r_{UK}T)$ falls
- Selling dollars forward and buying sterling forward (with bank-M) puts upward pressure on the quoted forward rate so F_q increases).

The above scenario tends to raise F_q and lower $S_0(1 + r_{US}T)/(1 + r_{UK}T)$, hence bringing the quoted forward price into line with the fair futures price $F = S(1 + r_{US}T)/(1 + r_{UK}T)$. This happens very quickly because arbitrage transactions are risk-free, so CIP will hold at all times.

7.3 PRICING FX-FUTURES CONTRACTS

We can price FX-futures contracts using the same formulas we derived for pricing FX forward contracts. Hence, the no arbitrage FX-futures price F(domestic per unit of foreign currency) quoted today for delivery in T years (or fraction of a year), using simple interest r^s, compound rates r and continuously compounded rates r^c are:

$$F = S\frac{(1 + r_d^s T)}{(1 + r_f^s T)}, \tag{7.7a}$$

$$F = S\frac{(1 + r_d)^T}{(1 + r_f)^T}, \tag{7.7b}$$

$$F = Se^{(r_d^c - r_f^c)T} \tag{7.7c}$$

where we have assumed both countries have the same day-count conventions (e.g. $T =$ days/360). All three formulas give the same value for F. In terms of the cost-of-carry approach to futures pricing, the domestic interest rate r_d is the borrowing cost (in GBP) and the income received is the investment in the foreign interest rate r_f. Brokerage fees mean that the above formula needs to take account of bid–ask spreads but the latter are very small for currency futures (and forwards).

7.3.1 Futures Prices

A forward contract involves no funds changing hands until the maturity (delivery) date and hence the forward rate is 'locked in' at the outset of the contract. But, there is a subtle difference between the forward and the futures contract even when both are held to maturity. The futures contract is 'marked-to-market' each day so that payments will be made into the margin account (of the long) if F increases and extra margin calls may be made if F falls. The interest rate applicable to these changes in margin payments is not known at the outset of the futures contract. However, for all practical purposes the interest cash flows from the margin account are not large enough to warrant analysing currency futures prices any differently from currency forwards.

F will be perfectly correlated with S, providing the interest rate differential $r_d - r_f$ remains constant over the hedge period. Hence the outcome from a hedged position will be more uncertain the greater the change in the interest differential over the hedge period.

7.4 HEDGING AND SPECULATION: FORWARDS

7.4.1 Long Hedge

Suppose a US importer, UncleSam, on 1 April knows that it will have *to pay* SFr 500,000 for imports from Switzerland, in 6 months' time on 25 October. (UncleSam has a liability in Swiss francs and hence is 'short' Swiss francs.) UncleSam fears a strengthening of the Swiss franc against the US dollar. If the Swiss franc spot-FX rate appreciates (i.e. the dollar depreciates) over the next 6 months, UncleSam will have to pay out *more dollars* (than at the current spot rate).

To hedge this position UncleSam takes a long position in a Swiss franc forward contract. If the quoted forward rate on 1 April for delivery on 25 October is $F_0 = 0.6620(\$/\text{SFr})$ then he knows today that he will have to pay \$331,000 in 6 months' time. UncleSam has removed uncertainty over the number of US dollars he will receive.

The same principle would apply to a US investor who had Swiss franc liabilities (e.g. had issued bonds denominated in Swiss francs or who had a bank loan denominated in Swiss francs) and was facing future coupon (or bank interest) payments or redemption of the bonds (bank loan) in the future. If the Swiss franc appreciates, the US investor with liabilities in Swiss francs would have to pay out more US dollars. She could hedge by buying Swiss franc forwards.

Ex-post, whether it was better to hedge or not depends on the out-turn for the spot rate on 25 October relative to the quoted forward rate $F_0 = 0.6620(\$/\text{SFr})$. Suppose the Swiss franc depreciates (i.e. USD appreciates) so the spot rate in October is $S_T = 0.6600(\$/\text{SFr})$. With hindsight, UncleSam will wish that on 1 April he had not hedged using the forward market, since he can obtain SFr 500,000 by paying \$330,000 – rather than paying \$331,000 in the forward market. But the crucial point here is that the FX spot rate for October is not known in April. If you choose to hedge then you remove risk, so you also forego any potential gains (as well as losses) which might accrue *ex-post,* when you reach 25 October.

7.4.2 Short Hedge

Suppose on 1 April a US multinational ExportUS expects *to receive* Swiss franc payments on 25 October, either from sales of goods in Switzerland or from Swiss investments. Hence, ExportUS is 'long' Swiss francs in the cash market and may fear a fall in the Swiss franc, thus receiving fewer US dollars in the future. ExportUS would hedge by selling Swiss francs in the forward market on 1 April for delivery on 25 October.

7.4.3 Speculation

Now consider speculation using the US dollar–pound sterling exchange rate (USD-GBP). Suppose the current (1 April) quoted forward rate is $F_0 = 1.50(\$/£)$ for delivery on 25 October. Suppose on 1 April a UK speculator, BritSpec, forecasts the spot rate on 25 October to be $S_T = 1.52(\$/£)$. Hence, BritSpec believes sterling will be worth more in the spot market in October than indicated by the current forward rate. BritSpec therefore buys sterling in the forward market today and sells sterling in the spot market in October ($S_T > F_0$). If BritSpec's informed guess about S_T turns out to be correct then it will make a speculative profit in October.

Today 1 April (Spot GBP to rise, go long forward contract)

- Agree to pay $F_0 = 1.50$ and receive £1 forward on 25 October then:

Maturity 25 October

- Receive $S_T = 1.52$ USDs (and pay £1) in the spot FX market
- Pay $F_0 = 1.50$ USDs (and receive £1) from the forward deal
- Profit $= S_T - F_0 = \$1.52–\$1.50 = \$0.02$ (per £1).

If the principal amount in the transaction is $Q = £100{,}000$ then:

$$\text{Total \$-profit} = \$(S_T – F_0)Q = \$2{,}000 \tag{7.8}$$

Note that this is a highly risky transaction since if the spot rate on 25 October is below 1.50($/£) then BritSpec will make a loss (which could be very large). This is also a levered transaction since BritSpec uses none of its own funds on 1 April.

7.5 HEDGING AND SPECULATION: FUTURES

We see below that hedging and speculation with futures contracts gives a result which is very close to that when using forwards, the main difference being that the futures are often closed out before maturity and futures involves margin payments.

7.5.1 Speculation Using FX-Futures

Suppose BritSpec buys sterling currency futures at $F_0 = 1.04(\$/€)$, which is for delivery of €125,000. If some time later, $F_1 = 1.05(\$/€)$ then BritSpec makes a mark-to-market profit of $1,250 which is credited to its margin account. Alternatively if $F_1 = 1.03(\$/€)$ then BritSpec has $1,250 deducted from her margin account.

If at expiry $F_T = 1.05(\$/€) = S_T$, then the profit would differ slightly from $1,250 because of the interest earned (lost) on BritSpec's additional margin receipts (payments) over the life of the contract. But ignoring interest on margin payments:

$$\begin{aligned} \text{\$-Profit at maturity} &= \$(F_T - F_0)\ €125{,}000 \\ \text{\$-Profit (close-out)} &= \$(F_1 - F_0)\ €125{,}000 \end{aligned} \tag{7.9}$$

7.5.2 Hedging Using FX-Futures

International investors and multinational corporations are vulnerable to *transactions exposure,* namely exchange rate risk of future cash receipts or payments in a foreign currency. Currency futures can be used to hedge this exposure because the correlation between spot and futures prices is high.

7.5.3 Long Hedge (US Importer)

Suppose a US importer, UncleSam (on 1 April) has to pay SFr 500,000 in 6 months' time (on 25 October), for imports from Switzerland. If the spot-FX rate for SFr appreciates (i.e. the dollar depreciates) over the next 6 months then UncleSam will have to pay out more dollars (than at the current spot rate). To hedge this position UncleSam takes a long position in Swiss franc futures (where the maturity date of the futures is after 25 October – so for example, UncleSam could use the December futures). What UncleSam loses in the spot market it hopes to be offset with cash gains in the futures market (and vice versa), when it closes out the futures. Hence:

Need to hedge a payment of SFr in 6 *months* ⇒ *Buy SFr Futures today*

Suppose UncleSam is currently faced with the following rates:

$$S_0 = \text{spot rate} = 0.6700\ (\$/\text{SFr})$$

$$F_0 = \text{futures price (December delivery)} = 0.6738\ (\$/\text{SFr})$$

$$\text{Contract size, } z = \text{SFr } 125{,}000$$

$$\text{Tick size} = 0.0001\ (\$/\text{SFr}) \quad (\text{Tick value} = \$12.50)$$

$$V_s = \text{value of spot exposure in Swiss Francs } (= \text{SFr } 500{,}000)$$

We can use two methods to calculate the number of futures contracts required to hedge the upcoming foreign currency payments. This can be done using either the Swiss franc or the US dollar as the 'common currency' when calculating N_F – but the two methods usually give similar answers. (Which is more useful depends in part on whether we expect the *absolute* change in F and S or their *proportionate* changes to be most closely correlated – see Appendix 7.)

Hedge ratio (1 April, $t = 0$):
The simplest method is to use the SFr amount V_s = SFr 500,000 and divide by the contract size, z = SFr 125,000. This is usually accurate enough and we will concentrate on this method in our example.

$$N_F = \frac{\textit{Cash market position in SFr}}{\textit{Contract size in SFr}} = \frac{V_s}{\text{SFr } 125{,}000} = 4 \text{ contracts} \tag{7.10}$$

A slightly more complex method is to convert both the cash market position and the size of the futures contract into US dollars, the currency of the US importer, then:

$$N_F = \frac{\$V_s}{\$V_F} = \frac{V_s S_0}{z\,F_0} = \frac{500{,}000(0.6700)}{125{,}000(0.6738)} = \frac{\$335{,}500}{\$84{,}225} = 3.98 \text{ (4 contracts)} \tag{7.11}$$

where $\$V_s$ is the total value of spot exposure in US dollars and $\$V_F$ is the value in US dollars of one Swiss franc futures contract ($= zF_0$).

Both methods give $N_F = 4$ long contracts. We can analyse hedging by the US importer either in terms of 'the change in the cash market position less the change in the futures position' or, in terms of 'the dollar value of the final outcome' for the US importer. We discuss each in turn.

In Table 7.3 we show the position on 25 October (at $t = 1$) after the Swiss Franc has appreciated from $S_0 = 0.67$ (\$/SFr) to $S_1 = 0.72$ (\$/SFr), that is by 500 ticks. The increase in cost to the US importer in the spot market is $V_s(S_1 - S_0) = \$25{,}000$. However, as the Swiss franc spot-rate appreciates, so does the futures price from $F_0 = 0.6738$ (\$/SFr) to $F_1 = 0.7204$ (\$/SFr) that is, an increase of 466 ticks. The gain on the long futures position is \$23,300 [= 466 ticks × \$12.50 × 4 contracts] which nearly offsets the increased dollar cost in the spot market. A key feature of the success of the hedge is that $\Delta S \approx \Delta F$ so the change in the basis $b_1 - b_0 = +34$ ticks, is small.

Consider the final net dollar cost to the US importer in October:

$$\begin{aligned}\text{Net \$-cost of imports} &= \text{Cost in spot market} - \text{Gain on futures} \\ &= \$360{,}000 - \$23{,}300 = \mathbf{\$336{,}700} \\ &= V_s S_1 - N_F z(F_1 - F_0) = V_s(b_1 + F_0) \end{aligned} \tag{7.12}$$

We have used $N_F z = V_s$ and (7.12) makes clear that the hedge 'locks in' the futures price F_0, as long as the final basis $b_1 = S_1 - F_1$ is 'small'. Effectively, UncleSam pays out \$336,700 to

TABLE 7.3 Long hedge (Swiss franc futures)

1 April	$S_0 = 0.6700$ (\$/SFr)
	$N_f = 4$ contracts
	Initial basis $= b_0 = (S_0 - F_0) = -0.0038$ (\$/SFr) (38 ticks)
25 October	$S_1 = 0.7200$ (\$/SFr) (Swiss franc has appreciated, dollar has depreciated)
	$F_1 = 0.7204$ (\$/SFr)
	US importer pays spot:
	$\$V_1 = V_S S_1 = \text{SFr } 500{,}000(0.7200) = \$360{,}000$
	Increase in \$-payments $= \$V_1 - \$V_0 = V_S(S_1 - S_0) = \$25{,}000$
	Gain on futures $= N_f z\,(F_1 - F_0) = 4(\text{SFr } 125{,}000)(0.7204–0.6738) =$ 4(\$12.50)(466 ticks) = \$23,000
	Final basis: $b_1 = S_1 - F_1 = -0.0004$ (4 ticks) Change in basis: $\Delta S - \Delta F = b_1 - b_0 = -4 - (-38) = +34$ ticks

receive SFr 500,000 which implies an effective rate at $t = 1$ of:

$$\frac{Net\ USD\ cost}{V_s} = \frac{\$336{,}700}{\text{SFr } 500{,}000} = (b_1 + F_0) = 0.6734\ (\$/\text{SFr}) \tag{7.13}$$

which is very close to the initial futures price of $F_0 = 0.6738$ (\$/SFr) – the difference being the final basis $b_1 = -4$ ticks.

7.5.4 Short Hedge (US Exporter)

Suppose on 1 April a US multinational, ExpUS, expects *to receive sterling* payments on 25 October, either from exports of goods to the UK or from sterling investments (i.e. dividends, capital gains or interest). ExpUS is 'long sterling' in the cash market and fears a fall in sterling. Hence today it would hedge by shorting sterling futures contracts:

ExpUS will receive sterling in 6 *months* ⇒ *Sell Sterling FX Futures today*

If spot sterling depreciates between April and October then ExpUS obtains less USD in the spot market. However, it will gain by closing out the short-sterling futures position at a profit. Of course, these hedging strategies involve basis risk: if there are sharp changes in either US interest rates or foreign interest rates then F and S will not move closely together and the hedge may perform much worse than expected.

7.6 SUMMARY

- Currency forwards are OTC contracts which can be designed to exactly fit clients' requirements as to amount, delivery dates and currencies. Forward FX contracts have credit (default) risk.
- The cash flows from an actual forward contract can be 'replicated' by using the spot FX-rate and two money market interest rates – this is the 'synthetic forward'. Risk-free arbitrage then ensures that the *quoted* forward rate equals the synthetic ('correct', 'fair', 'no-arbitrage') forward rate. This is the *covered interest parity , CIP,* condition:

$$F = S(1 + r_d T)/(1 + r_f T)$$

- Currency forwards can be used for hedging and speculation. The payoff at maturity for a long position in a forward contract on a foreign currency is, $S_T - F_0$.
- Currency futures are similar to currency forwards – the main differences being that futures contracts can be easily closed out prior to the maturity (delivery) date and futures contracts require margin payments.

- Currency futures provide a low cost method of hedging known future foreign currency receipts or payments (e.g. transactions exposure from imports and exports and repatriating profits or, receipts or payments on foreign assets, such as dividends, capital gains, and interest). There is virtually zero credit risk when using futures, because of margin requirements.
- Basis risk in a hedge arises with futures contracts which are closed out prior to maturity – this is caused by changes in domestic or foreign interest rates over the hedge period. Also there are risks due to the fact that the interest rate applicable to margin calls is unknown at the outset of the hedge.
- Futures prices can be determined using the cost of carry approach which is equivalent to the covered interest parity (CIP) condition. In practice there is little difference between quoted FX forward and futures prices.
- Speculation using currency futures often allows greater leverage than speculation in the spot FX market as only the initial margin payment is required to take an open position in an FX futures contract.

APPENDIX 7: HEDGING USING FX-FUTURES

This appendix puts the hedging examples in the text into algebraic form. Consider a US importer UncleSam, with Swiss franc *payments* of SFr 500,000 in the future. We use the following notation:

$$V_s = \text{value in foreign currency of cash market position } (= \text{SFr } 500{,}000)$$

$$S_0 = \text{spot rate, \$/SFr}$$

$$\$V_0 = V_s S_0, \text{initial USD value of cash market position}$$

$$z = \text{Contract size } (= \text{SFr } 125{,}000)$$

$$\$V_1 = V_s S_1, \text{the amount of US dollars paid at the end of the hedge}$$

Case A: Using 'Foreign Currency' to Calculate N_F

$$N_F = V_s/z \tag{7.A.1}$$

$$\textit{Effective Dollar Cost} = \textit{\$Cost in cash market} - \textit{\$Gain on short futures position}$$

$$\$V_1 = V_s S_1 - N_F z(F_1 - F_0) \tag{7.A.2}$$

Using $V_s = N_F z$, we have:

$$\$V_1 = V_s[S_1 - F_1 + F_0] = b_1 + F_0 \tag{7.A.3}$$

Hence, the hedge locks in the futures price F_0 if the final basis $b_1 = (S_1 - F_1)$ is small. This is an unambiguous way of interpreting the hedge. Using $S_1 = S_0 + dS$ in Equation (7.A.3) and subtracting the initial USD cost, $\$V_0 = V_s S_0$, the increase in USD cost is:

$$\$V_1 - \$V_0 = V_s(dS - dF) \tag{7.A.4}$$

If dS equals dF (on average), then the 'beta' of the regression of dS on dF would be unity. Under these circumstances, the hedge scenario would involve no change in the USD cost of the imports. But this scenario would be unlikely to occur exactly, in practice.

Case B: Using Domestic Currency (USD) to Calculate N_F^*

$$N_F^* = \frac{\$V_0}{\$V_F} = \frac{V_s S_0}{z F_0} \tag{7.A.5}$$

Here the effective US dollar cost of the imports is:

$$\$V_1 = \$\text{-Cost of imports} - \$\text{-gain on futures} = V_s S_1 - N_F^*\, dF\, z \tag{7.A.6}$$

Substituting for N_F^* and using $S_1 = S_0 + dS$:

$$\$V_1 = \$V_0 + \$V_0(dS/S_0 - dF/F_0) \tag{7.A.7}$$

Hence, the final USD cost when using this hedge ratio, will be close to the initial USD cost if S and F move together *proportionately*. Turning now to the *increase* in cost we have:

$$\$V_1 - \$V_0 = \$V_0(dS/S_0 - dF/F_0) \tag{7.A.8}$$

There is no increase in the cost of the hedge position (in US dollars) if the *proportionate change* in F and S are equal. Clearly, if $S_0 \approx F_0$ then the two hedge ratios give very similar results. Given covered interest parity, $F = Se^{(r_d^c - r_f^c)T}$ (using continuously compounded rates) then for small changes we have:

$$dF/F_0 = dS/S_0 + (dr_d^c - dr_f^c)T + (r_d^c - r_f^c)dT. \tag{7.A.9}$$

Hence given the fact that the volatility in S is much greater than the volatility of interest rates, then the proportionate change in F and S will be almost equal if the hedge is over a relatively short time period. Also, if $F_0 \approx S_0$, which will be the case if the initial domestic and foreign interest rates are not too different (or the time to maturity is short) and remain constant, then from (7.A.9) $dF \approx dS$. Hence, under these circumstances it may make little difference in practice whether we use N_F or N_F^*.

EXERCISES

Question 1

You are a US investor and you forecast that the euro will appreciate against the US dollar over the next week. How can the spot FX market be used for speculation?

Question 2

How can you use the spot FX market and the 1-year forward FX market for speculation? Assume the quoted one-year forward rate is $F = 1.0020$ USD per euro and you forecast the spot rate in 1 year's time is $S_T = 1.00$ USD per euro. You speculate using €100 today.

Question 3

What does covered interest parity (CIP) mean? What determines whether CIP holds at any point in time?

Question 4

The spot rate for US dollar (USD) against the pound sterling (GBP) is 1.65 (USD/GBP).
Three-month UK interest rates are at 7.5% p.a. (simple interest, actual/365 day-count basis).
Three-month US interest rates are 6% p.a. (simple interest, actual/365 day-count basis).
Assume there are 30 days in each month. Calculate the 30-day forward rate and the forward margin.

Question 5

You are a US investor. Current interest rates are $r_{UK} = 2\%$ p.a., $r_{US} = 1\%$ p.a. (continuously compounded) and the spot FX rate is $S = 1.3$ (USD/GBP).

(a) Calculate the no-arbitrage (fair) forward FX rate, for a maturity of 6 months ($T = 1/2$ year).

(b) If the current quoted forward rate is $F_q = 1.25$ (USD/GBP), show how a £1,000 investment by a US arbitrageur can result in risk-free arbitrage profit.

Question 6

The spot price of the Swiss franc (SFr) or 'Swissy' is $S = 0.65$ (USD per SFr) and the quoted futures price on a 2-month forward contract is $F_q = 0.66$ (USD/SFr). Interest rates in Switzerland are 2% p.a. and in the USA 8% p.a. (continuously compounded).

(a) What is the no-arbitrage (fair) futures price?

(b) What arbitrage opportunities exist for a US resident?

Question 7

A UK firm knows it will receive \$10m in 1 year's time (from the sale of goods in the USA). Current interest rates are $r_{UK} = 10\%$ p.a., $r_{US} = 12\%$ p.a. (simple rates) and the spot rate $S = 1.6$ (USD/GBP).

Calculate the forward rate (USD per GBP).

Explain how the UK could hedge this inflow of \$10m in 1 year's time by *using the money markets and the spot FX market* (i.e. not using the forward market directly).

Briefly comment on these results.

Question 8

How would a US firm importing goods from Switzerland in 6 months' time hedge its position by using the (FX) futures market?

Question 9

The UK treasurer of Suits plc expects to receive payment for exports of tailored men's suits to a customer in Munich in 3 months' time ($T = 1/4$ year). Her marketing department has sold 1,000 suits at a delivery price of €250 each. In the *Financial Times* on 10 June, she observes the following:

Spot FX rate	0.850 (€/£)
3-month forward FX rate	0.853 (€/£)

Sterling 3-month (simple) interest rate (annualised) $r = 5.5625\%$

Euro 3-month (simple) interest rate (annualised) $r^* = 7.6250\%$

(a) Explain using the above data, how the treasurer can hedge her receipts in euros by:
 (i) taking forward cover
 (ii) taking money-market cover.

(b) What would be the amount of sterling received if the treasurer took an uncovered (open) position and the spot rate S_T in 3 months' time is:
 (i) 0.653 (€/£)
 (ii) 0.658 (€/£)
 (iii) 0.640 (€/£)

 In each case, compare the hedged outcome with the uncovered outcome.

(c) Are the set of interest and exchange rates prevailing on 15 June consistent with covered interest parity? If not, explain how equilibrium will be established in the relevant markets.

PART II

FIXED INCOME: CASH MARKETS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 8

Interest Rates

Aims

- To examine different conventions when returns or interest rates (yields) are 'annualised' – namely 'simple interest', 'compound interest' and 'continuously compounded interest'.
- To calculate present value and terminal value using different interest rate conventions.
- Show how to switch between simple interest rates, compound rates and continuously compounded rates.
- To calculate forward rates from spot rates, using different interest rate conventions.
- To show how a Forward Rate Agreement (FRA) is priced.

This chapter provides an overview of the key interest rates used in the market and different day-count and interest rate conventions used when calculating 'annual yields'. We also discuss how forward rates are derived from spot interest rates and how FRAs are priced.

8.1 LIBOR, REPOS, FED FUNDS, AND OIS RATES

8.1.1 LIBOR

LIBOR (London Interbank Offer Rate) is the interest rate at which an AA-rated bank agrees to lend out funds to another AA-rated bank. LIBOR is an unsecured loan and borrowing is for maturities of one day to one year. The rate *quoted* will often be an average of LIBOR rates taken from a survey of 'reference banks', usually at 11 a.m. London time, under the auspices of the British Bankers Association (BBA). This is known as the 'LIBOR-fixing' and provides a reference rate for use in valuing many derivatives transactions (e.g. interest rate swaps).

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

In recent years there has been an investigation into manipulation of LIBOR rates submitted by the reference banks, which has resulted in lengthy jail terms of up to 14 years for some of those involved. As a result there is a proposal that LIBOR fixing rates will not be determined by a 'survey' of hypothetical (or guesses) of the rate at which you will lend money for various currencies and maturity dates but instead will be based on actual transactions at maturities where there is an active liquid market.

LIBOR rates are applicable to any 'offshore' currency that banks borrow or lend in London. So, for example, there are US dollar, yen, etc. LIBOR rates quoted in London. The equivalent rates elsewhere are Euribor (countries that use the Euro), NYBOR (New York), PIBOR (Paris), FIBOR (Frankfurt), ADIBOR (Abu Dhabi), SIBOR (Saudi or Singapore) and HKIBOR (Hong Kong).

The Fed Funds rate is the rate at which US banks (who hold reserves at the US central bank, the Federal Reserve) lend to each other, overnight. In the UK, overnight leading rates between banks is called the sterling overnight index average (SONIA) and for the Eurozone it is the euro overnight index average (EONIA).

8.1.2 Repurchase Agreement (Repo)

A repo or more accurately a 'sale and repurchase agreement' is a form of collateralised borrowing. Suppose a market maker wishes to borrow \$10m *overnight*. He can agree to 'sell' securities (e.g. T-bills or T-bonds) which he holds, to a counterparty (e.g. broker) for their current market price of \$10m, in exchange for \$10m cash. He also *simultaneously* agrees to 'buy back' the securities the next day at a price higher than \$10m, say (\$10m + \$1,800). This is equivalent to an overnight interest rate of 0.018% or 6.48% p.a. (using an actual/360 day count convention).

If the securities in the repo are less liquid than government bonds (or have substantial credit risk) the amount of cash loaned will be *less than* the market value of the securities – this is the haircut demanded by the lender of the funds. If the haircut is 1% then the lender will only supply \$9.9m cash when the \$10m of bonds are handed over and the overnight interest rate charged on the \$9.9m will be at 6.48% p.a., that is an interest payment of \$1,782 (= 0.0648 × \$9.9m/360). Hence the lender of the funds holds securities which are worth more than the \$9.9m. If the borrower defaults, the lender can subsequently sell the securities at their market price one day later and is virtually assured that he will receive from the sales at least \$9.9m plus the \$1,782 interest.

Note that in principle any security can be used in a repo, such as a bill, bond or even equities but the greater the risk of the collateral losing value over the term of the repo loan, the larger will be the haircut. If the borrower of funds is a highly rated creditworthy institution and provides highly liquid government securities as collateral then the haircut may be between zero and 0.25% (i.e. ¼ of 1%). When viewed from the perspective of the borrower the transaction is a repo. From the point of view of the counterparty, the supplier of funds, it is a *reverse repo*. The most common type of repo is an *overnight repo*, which can be rolled over each day. Longer maturity repos are known as *term repos*. A repo is a secured loan (i.e. collateral is provided) and hence the repo rate is generally slightly below the Fed Funds rate.

8.1.3 Risk-free Rate

Traditionally LIBOR was taken as a measure of the 'risk-free' rate when pricing derivatives but after the 2007–8 crash this is not always the case. A weighted average of the rates in brokered transactions (with the weights proportional to amounts loaned) in the Fed Funds overnight (unsecured lending) market, is known as the *effective federal funds rate*. If a bank borrows overnight in the Fed Funds market for 3 months (say) by rolling over its borrowing each day, then the interest rate it pays over the 3 months is the geometric average of the overnight rates. An *overnight index swap* (OIS) is an arrangement where a *fixed rate* for 3 months (say) can be swapped for the geometric average of the overnight rates (over the 3 months). This swap is a way of fixing the borrowing cost equal to the OIS rate. Hence the OIS rate is also very close to being a risk-free rate. There is a term structure of OIS rates, so a zero-curve can be used to determine a set of 'risk-free' rates that can be used in derivatives pricing. This causes some technical issues when LIBOR rates determine payoffs at maturity of a derivative (e.g. FRA, swaps, caps) but we use OIS as risk-free discount rates. We avoid these problems by assuming LIBOR is an acceptable risk-free rate.

8.2 DAY-COUNT CONVENTIONS

Interest rates (yields) are usually presented as 'annual percentage rates' but it is important to know whether these figures are based on 'simple interest' r_s, 'compound interest' r, or 'continuously compounded' rates, r_c. For example, an interest rate of 2% payable over 3 months may be quoted as a simple interest rate of $r_s = 8\%$ p.a. However, if an investor rolled over four, 3-month investments (*and the interest rate remained constant)*, she could actually earn an annual 'compound rate' of $(1.02)^4 = 1.0824$ or $r = 8.24\%$ p.a. The annual compound rate exceeds the simple rate because the compound rate assumes the investor earns 'interest-on-interest'. We use the following notation:

a = number of days in the year (i.e. the convention used to calculate an 'annual rate')

m = *actual* number of days to maturity of the 'asset'

$T = m/a$ = number of years (or fraction of a year) the investment is held (i.e. maturity)

Initially, for ease of calculation we assume $a = 360$ (even though there are actually 365 days in a year) and if the investment is for 3 months we assume the actual number of days to maturity is $m = 90$, hence $T = 90/360 = 0.25$ years.[1] For an investment which lasts for 2 years we take $m = 720$, hence $T = 720/360 = 2$ years.

8.2.1 Simple Interest

Suppose you place $P_0 = \$100$ in a deposit account at $t = 0$ which pays out $M = \$102$ after 90 days. Similarly, you could purchase an asset (e.g. T-bill or zero-coupon bond or stock) for $P_0 = \$100$ at $t = 0$ and sell it 3 months later for $M = \$102$. You have made 2% over 3 months which is sometimes referred to as the 'periodic' or 'per period' interest rate (yield). Because the interest rate refers to a transaction which begins today and has a single cash flow at one point in the future (with no interim payments), the interest rate is also known as the 3-month *spot interest rate* (or spot yield).

Over the 90 days you have actually made \$2 or 2% on your outlay of \$100. Everything which follows is 'fictitious' since your investment ends after 90 days. But it is useful to have some rules for 'annualising' the 'periodic rate' of 2%, earned over 90 days which we do by using either simple interest, compound interest or continuously compounded interest. As long as everyone knows which 'annualising convention' has been used there will be no confusion and everyone will agree on the final amount received.

If you make 2% over 3 months then the simple annual return is defined as $r_s = 8\%$ p.a. The simple interest rate of 8% p.a. assumes that you do not earn any 'interest-on-interest' on the deal. Conceptually, it is as if you took the \$2 interest at the end of the first 3 months and spent it but you continued to reinvest \$100 (not \$102) over the next 3 months – repeating this procedure at the end of 6 months and 9 months. Hence you have received four interest payments of \$2 every 90 days on an investment of \$100 (at the beginning of each 90-day period).[2] The 'simple' annual interest rate (yield) is *defined as*:

$$r_s = \left(\frac{M}{P_0} - 1\right) \frac{1}{T} = \left(\frac{102}{100} - 1\right) \frac{360}{90} = 0.08 \tag{8.1}$$

[1]In practice when converting to annual rates of interest, the actual number of days in any 3-month period (e.g. 15 January to 15 March) will not usually be exactly $m = 90$ days. Also, the 'convention' when measuring yields is to 'gross up' to an annual rate using $a = 365$. The above calculations are easily repeated using the actual number of days to maturity and $a = 365$.

[2]In annualising the '2% over 3 months', we assume you can earn 2% over the subsequent 3 periods – but we don't actually know what interest rate you will get over the second, third and final periods, until you actually 'get there' in real time. So, the annualised rate (for a 90-day investment) is always 'fictitious' whatever method we use to 'annualise' the 2%. But the *annualised figure (% p.a.)* whether expressed either as a simple rate, or compound rate or continuously compounded rate, is still a useful way of comparing the *relative* return on two different investments A and B (over the same investment horizon), provided we compare them using either annualised simple rates or the annualised compound rates, or the annual continuously compounded rates for A and B, respectively – that is, we compare 'like-with-like' annualised returns.

Hence the market price of the bond is given by:

$$P_0 = \frac{M}{(1 + r_s\, T)} \tag{8.2}$$

This equation says that the value today (i.e. the present value) of an amount $M = \$102$ which accrues in $n = 90/360$ years, when a simple interest rate of $r_s = 8\%$ p.a. is used, is $P_0 = \$100$. Rearranging (8.2) tells us that if you invest $P_0 = \$100$ at simple annual rate of $r_s = 8\%$ p.a. then the terminal value after $T = 90/360$ years is:

$$M = P_0\,(1 + r_s\, T) \tag{8.3}$$

8.2.2 Compound Interest

Suppose you again place $P_0 = \$100$ in a deposit account at $t = 0$ which pays out $M = \$102$ after 90 days. Or you purchase an asset for $P_0 = \$100$ at $t = 0$ and sell it 3 months ($T = 1/4$ of a year) later for $M = \$102$. Your return over 3 months is 2% which implies a compound annual return (yield) of $r = (1.02)^4 - 1 = 0.0824$ or 8.24% p.a. This calculation assumes that the initial \$100 investment is 'rolled over' and any interest earned is also reinvested in each successive 3-month period (at the same rate of return as in the first 3 months). Expressed algebraically:

$$r = \left[\frac{M}{P_0}\right]^{\frac{1}{T}} - 1 = \left[\frac{102}{100}\right]^4 - 1 = 0.08243 \tag{8.4}$$

Again a simple algebraic rearrangement gives:

$$P_0 = \frac{M}{(1 + r)^T} \tag{8.5}$$

It also follows from (8.5) that if you invest P_0 at a compound annual rate $r = 0.0824$ for a period of T years (or fraction of a year), it will eventually accrue to:

$$M = P_0(1 + r)^T \tag{8.6}$$

For example, \$100 invested at a compound rate of 8.243% p.a. over 3 months accrues to:

$$M = 100\,(1 + 0.08243)^{1/4} = 102 \tag{8.7}$$

Alternatively, suppose you invest \$100, but this time you receive $M = \$108$ after $T = 2$ years (i.e. $a = 360$ days and $m = 720$ days). The annual compound return is:

$$r = \left[\frac{108}{100}\right]^{\frac{1}{2}} - 1 = \left[\frac{M}{P_0}\right]^{\frac{1}{T}} - 1 = 0.0392 \tag{8.8}$$

This return is slightly less than 4% p.a. because the compound return assumes that the 'interest' earned at the end of year-1, is reinvested in the second year. Note that the above formulas work regardless of whether T is in years or fractions of a year.

8.2.3 Continuously Compounded Rate

For $P_0 = \$100$, $M = \$102$ and an investment over 90 days, the continuously compounded annual rate (yield) is *defined as*:

$$r_c = (360/90)\ \ln(102/100) = (1/T)\ \ln(M/P_0) = 0.07921\ (7.921\%) \tag{8.9}$$

Once we have defined r_c in this way then it must be the case that:

$$P = Me^{-r_cT}\ and\ M = Pe^{r_cT} \tag{8.10}$$

where $T = m/a$ is the number of years (or fraction of a year) to maturity. The continuously compounded rate assumes you earn interest-on-interest over 'very short periods of time'. In practice, it is almost equivalent to assuming you earn interest daily and you reinvest this daily interest payment, each day. The continuously compounded rate is defined so if you did invest $P = \$100$ today at $r_c = 0.079218$ (7.9218%), and you earned interest 'every second' then you would end up with $M = 100e^{0.0792(90/360)} = \$102$ after 90 days. So the continuously compounded rate is like an ordinary compound rate, but where the compounding takes place 'every second'.

The continuously compounded rate (7.9218% p.a.) is less than the compound rate (8.2432% p.a.). The reason for this is a little involved. If you start off with $P_0 = 100$ then whatever interest rate convention you use, you must always end up with $M = \$102$ after 90 days. But the continuously compounded rate of 7.9218% p.a. can be rationalised by assuming that interest-on-interest accrues 'every second'. To end up with $M = \$102$ at the end of 90 days, therefore requires a lower continuously compounded annual rate (= 7.9218% p.a.) if interest accrues every second, than if interest is 'rolled over' only every 90 days – as is the case when using the compound rate of 8.243% p.a.

8.2.4 Daily Compounding

We now show that a continuously compounded rate is nearly equivalent to using *daily compounding*. Suppose the simple annual rate is $r_s = 0.10$ (10% p.a.). The terminal value after 1 year, of an investment of P_0 today, when compounding is q times per year is:

$$TV_n = P_0\ (1 + r_s/q)^q \tag{8.11}$$

TABLE 8.1 Compounding frequency and terminal value

Compounding frequency	Terminal value of \$100 at end of year (r_S=10% p.a.)
Annually (q = 1)	\$110
Quarterly (q = 4)	\$110.38
Weekly (q = 52)	\$110.51
Daily (q = 365)	\$110.5155
Continuously compounded	TV = \$100e$^{(0.1)}$ = \$110.5171

Suppose the frequency of compounding increases so that q increases from 1, 2, 3, … to say $q = 365$ times per year. The terminal value in 1 year, of P_0 = \$100 invested today is given in Table 8.1, *for different values of* q. If $r_s = 0.10$ (10% p.a.) and interest is compounded each quarter ($q = 4$) then the \$100 accrues to TV = \$110.38 at the end of the year. But if interest is compounded daily ($q = 365$) then TV = \$110.5155. What is the terminal value if we assume the continuously compounded rate is $r_c = 0.10$? The terminal value after 1 year is $M = 100e^{0.10(1)}$ = \$110.5171, which is very close to the result for daily compounding.

8.2.5 Switching Between Interest Rates

For an investment of P_0 = \$100, our conventions when defining simple, compound and continuously compounded rates must all give the same terminal value M = \$102 after $T = 1/4$ year, hence:

$$M = P_0\,(1 + r_s\,T\,) = P_0(1 + r)^T = P_0 e^{r_c T} \tag{8.12}$$

It immediately follows that:

$$r_c = \ln(1 + r) \quad \text{and} \quad (1 + r) = (1 + r_s\,T\,)^{1/T} \tag{8.13}$$

These equations allow us to find values of r_c, r and r_s that *are equivalent to each other*. By this we mean that if we use either r_c or r or r_s in the appropriate formula they will all give *the same value* for M (e.g. after investing P_0 = \$100 for T = ¼ of a year).

For example, if you are given the compound annual rate, $r = 0.082432$ p.a. then the equivalent continuously compounded rate is $r_c = \ln(1 + r) = 0.07921$ p.a. (and vice versa). Similarly, if you are given the simple annual rate $r_s = 0.08$ p.a. (for $T = 90/360$) then the equivalent compound rate is $(1 + r) = (1 + r_s\,T\,)^{1/T} = [1 + (0.08/4)]^4 = 1.02^4 = 1.082432$ (and vice versa). If you substitute these values either for r_c, r or r_s in (8.12) you obtain M = \$102 in all cases.

8.2.6 Present Values

The different interest rate conventions must give the same present value (when used in the appropriate formulas) and these are summarised below.

$$\textbf{Simple rate}: P_0 = \frac{M}{(1 + r_s T)} \tag{8.14a}$$

$$\textbf{Compound rate}: P_0 = \frac{M}{(1 + r)^T} \tag{8.14b}$$

$$\textbf{Continuously compounded rate}: P_0 = Me^{-r_c \mathrm{T}} \tag{8.14c}$$

In the analysis above we invested $P_0 = \$100$ in an asset (e.g. a zero coupon bond) with a known interest rate (yield) and calculated $M = \$102$ after 3 months. The above 'present value' equations allow you to calculate the market price for a (zero coupon) bond given the appropriate (simple, compound or continuously compounded) yield and the time to maturity.

8.3 FORWARD RATES

Spot interest rates (yields) refer to borrowing (or lending) money *starting today,* with a single final payment at a known time in the future. Suppose instead you go to a bank today, but offer to lend $100, starting in 1 year's time, with repayment 1 year later – the agreed interest rate is known as a forward rate. For example, if today you offered to lend $100 starting in 1 year's time and the bank quotes a forward rate $f_{12} = 11\%$ then you would receive $111 at the end of year-2. How does the bank know what forward rate to quote? The bank calculates forward rates from today's observed spot rates of interest.

Figure 8.1 may help here, in terms of a heuristic argument. The 'correct' forward rate is based on the fact that investing $1 over 2 years at r_2 should give the same dollar amount at

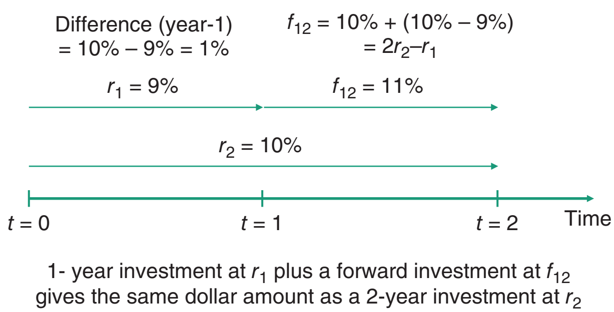

FIGURE 8.1 Forward rate, f_{12}

the end of year-2 as investing at $r_1 = 9\%$ in the first year, followed by an investment earning f_{12} between years one and two.

In Figure 8.1 note that the 2-year investment earns 10% in *each* of the two years. Also, *in the first year*, the 2-year spot rate is 1%(= 10% − 9%) higher than the 1-year spot rate ($r_1 = 9\%$). Hence between $t = 1$ and $t = 2$ the *forward rate* must equal 10% *plus* the amount 'lost' in the first year. Hence the forward rate is:

$$\textit{Forward Rate}:\ f_{12} = 10\% + (10\% - 9\%) = 11\% = 2r_2 - r_1$$

8.3.1 Compound Rates

The above calculation ignores interest-on-interest which makes the correct forward rate a little higher at 11.009% (see below). More formally, let f_{12} denote the (compounded) forward rate applicable between years 1 and 2. Consider the following alternative investment strategies:

Strategy-1: $A = \$100$ invested over 2 years gives $\$A(1 + r_2)^2 = \$121$ at $t = 2$.
Strategy-2: $A = \$100$ invested at r_1 gives $\$A(1 + r_1) = \$109$ at the end of year-1.

At $t = 0$ (today) obtain a forward quote to lend \$109 at the end of year-1 for a further year at a forward interest rate of f_{12}. At $t = 2$ you know that you will receive from strategy-2:

$$\$109\ (1 + f_{12}) = \$A(1 + r_1)(1 + f_{12}) \tag{8.15}$$

Since the two investment strategies are risk-free, then arbitrage will ensure that both strategies will result in the same payout at $t = 2$, hence:

$$A(1 + r_2)^2 = A(1 + r_1)(1 + f_{12}) \tag{8.16}$$

Rearranging (8.16) we can solve for the forward rate:

$$(1 + f_{12}) = \frac{(1 + r_2)^2}{(1 + r_1)} = \frac{(1.10)^2}{(1.09)} = 1.11009 \tag{8.17}$$

If we use the approximation $\ln(1 + r)^T \approx rT$ (for $-1 < r < 1$) then (8.17) can be expressed as a linear relationship:

$$2r_2 \approx r_1 + f_{12} \tag{8.18}$$

For the moment assume the approximations in Equation (8.18) are accurate. We can repeat the above argument for a 3-period horizon to obtain:

$$(1 + r_3)^3 = (1 + r_1)(1 + f_{13})^2 \tag{8.19}$$

$$3r_3 \approx r_1 + 2f_{13} \tag{8.20}$$

where f_{13} is the (annual) forward rate between year-1 and year-3 and hence applies to an investment horizon of 3 – 1 = 2 years. Note that we can also compare our 3-year investment at r_3 with a known 2-year investment at r_2 and the forward rate f_{23} between years 2 and 3 is:

$$(1 + r_3)^3 = (1 + r_2)^2(1 + f_{23}) \tag{8.21}$$

$$3r_3 \approx 2r_2 + f_{23} \tag{8.22}$$

Equations (8.20) and (8.22) can be used to calculate the (compound) forward rates f_{13} and f_{23} from the known spot rates.

8.3.2 Continuously Compounded Rates

If we use 'continuously compounded' spot rates r_2, r_3 then the no-arbitrage equation (equivalent to Equation 8.16) is:

$$Ae^{3r_3} = Ae^{2r_2}e^{f_{23}} \tag{8.23}$$

and (taking logs and rearranging), the continuously compounded forward rate f_{23} is[3]:

$$f_{23} = 3r_3 - 2r_2 \tag{8.24}$$

which is an exact relationship (and not an approximation).

8.3.3 Simple Rates: Day-count Conventions

Some interest rates such as LIBOR are quoted as 'simple' rates and the day-count convention for calculating the interest cash flow is 'actual/360' – so by convention '1 year' = 360 days. For ease of exposition, we assume that the 'actual' number of days in any 3-month period is 90.

Consider investing \$A for 6 months ($n = 180/360$ years) at the (simple) 6-month spot rate $r_n = 0.03$ (3% p.a.). The investment is worth $A(1 + r_n(180/360))$ after 6 months. An alternative is to invest for 3 months ($m = 90/360$ years) at the 3-month spot rate $r_m = 0.02$ (2%) followed by a forward rate agreement at $f_{m,n}$ for a further 3 months ($n - m = 90/360$ years). This alternative strategy is worth $A[1 + r_m(90/360)][1 + f_{m,n}(90/360)]$ after 6 months. Equating these two amounts and rearranging gives the 'simple' forward rate, which applies to an investment which begins in 3 months' time and terminates after a further 3 months:

$$f_{m,n} = \frac{360}{90}\left[\frac{[1 + r_n(180/360)]}{[1 + r_m(90/360)]} - 1\right] = \frac{1}{n - m}\left[\frac{(1 + r_n n)}{(1 + r_m m)} - 1\right] = 0.0398\ (3.98\%) \tag{8.25}$$

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[3]This can also be directly obtained as an *approximate* relationship from Equation (8.21) which uses compound rates: $(1 + r_3)^3 = (1 + r_2)^2(1 + f_{23})$. Take logs and use $r_c \approx \ln(1 + r)$ and we obtain (8.22).

In a forward rate agreement (FRA), the convention is that interest rates are quoted as 'simple' rates on an 'actual/360 basis'. Hence at $t = 0$, the *quoted rate for a* 3×6 FRA is 3.98% p.a. This is known as 'pricing the 3×6 FRA'. Alternatively, if $m = 180/360$ and $n = 270/360$ then the above equation using 6 and 9 month spot rates would 'price' a 6×9 FRA.

Forward rates can be calculated for any horizon by using the appropriate formula, together with data on interest rates (yields) from the (spot) yield curve. From the above analysis, the following equations can be used to calculate forward rates for 'simple', discrete compounded and continuously compounded spot rates, respectively:

$$\text{'Simple' rates } f_{m,n} = \frac{1}{n-m}\left[\frac{r_n n - r_m m}{(1 + r_m m)}\right] \tag{8.26a}$$

$$\text{Discrete compounding } f_{m,n} = \left[\frac{(1 + r_n)^n}{(1 + r_m)^m}\right]^{1/(n-m)} - 1 \tag{8.26b}$$

$$\text{Continuously compounded } f_{m,n} = \left[\frac{n}{n-m}\right] r_n - \left[\frac{m}{n-m}\right] r_m = \left[\frac{n\, r_n - m\, r_m}{n-m}\right] \tag{8.26c}$$

where $n > m$ and n and m are expressed in years or fractions of a year (using 'actual/360' convention for pricing FRAs). For example, if a 3×6 FRA has 91 days to maturity and the interest payment is after a further 91 days then $m = 91/360$ and $n = 182/360$). Alternatively, if simple interest rates have been 'converted' to equivalent continuously compounded spot rates (see Equation 8.13) and $n = 3$ years, $m = 2$ years, then (8.26c) gives $f_{23} = 3r_3 - 2r_2$.

8.3.4 Spot and Forward (Yield) Curves

It is a matter of arithmetic, that if the yield curve for spot rates[4] between times m and n is upward sloping then the forward rate $f_{m,n}$ will be above the spot rate r_n (and vice versa). Algebraically, this can be easily seen by a simple rearrangement of the continuously compounded forward rate equation:

$$f_{m,n} = \left[\frac{n}{n-m}\right] r_n - \left[\frac{m}{n-m}\right] r_m = r_n + \left[\frac{m}{n-m}\right](r_n - r_m)$$

If $r_n > r_m$ then $f_{m,n} > r_n$ (i.e. the forward rate ending at time n is greater than the n-period spot rate).

[4]The spot rate curve is also known as the 'zero coupon yield curve'.

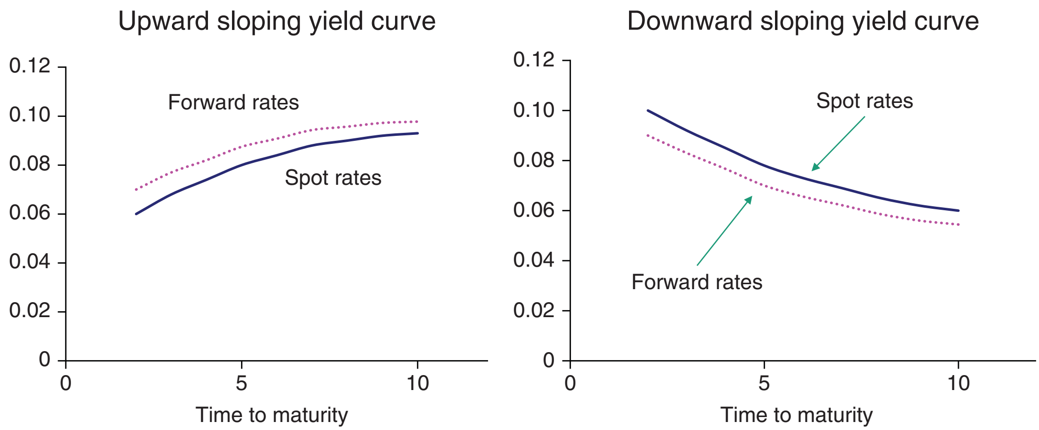

FIGURE 8.2 Yield curve

> *On the website an Excel file takes data on 10 (continuously compounded) spot rates and calculates all the one-period forward rates* $f_{t,t+k}$ $(k = 1, 2, \ldots, 10)$. *In one example we assume an upward sloping (spot) yield curve and in the other, we assume a downward sloping curve and we then plot the spot and forward yield curves. When spot rates increase with time to maturity (i.e.* $r_1 < r_2 < \ldots < r_{10}$*) then the forward rate curve lies above the spot rate curve (and vice versa) – see Figure* 8.2.

Besides the obvious fact that forward rates directly tell you at what rates you can borrow (or lend) money over different horizons, *starting sometime in the future,* they are also widely used in pricing many different types of fixed-income securities – particularly derivative contracts based on interest rate products such as FRAs, T-bill and T-bond futures contracts, options on T-bonds, interest rate swap contracts, caps, floors, forward swaps and swaptions.

8.4 FORWARD RATE AGREEMENTS (FRAs)

In a forward rate agreement (FRA), one party may agree to receive a floating interest rate (e.g. LIBOR) in exchange for paying a known fixed rate of interest (of say 3% p.a.), starting at a specific date in the future. A forward rate agreement (FRA) is a form of forward contract that allows you to 'fix' or hedge interest-rate risk over a specific time period, starting in the future.

The FRA rate agreed in the contract at time $t = 0$ can apply to any two dates in the future. For example, a 3×6 FRA has a fixed FRA rate equal to today's quoted forward rate $f_{3,6}$ for the period beginning in 3 months time and ending in 6 months' time. The quoted forward rate is calculated using 'simple interest' and an 'actual/360' day-count convention. A 6×9 FRA would most likely have a different quoted forward rate, $f_{6,9}$ since this forward rate applies to the period beginning in 6 months' time and ending in 9 months' time.

8.4.1 Bank Loan

If you have a floating rate bank loan with Citibank (at LIBOR), with an interest rate reset in 3 months' time, and you fear a rise in interest rates, then your future borrowing costs will be higher. Conversely, if you have a bank deposit linked to a floating rate which is reset in 3 months' time, then you will lose out if interest rates fall, since your deposit will earn less interest. We show that:

To hedge a rise in future loan rates ⇒ buy ('go long') an FRA.

To hedge a fall in future deposit rates ⇒ sell ('short') an FRA.

Suppose on 16 December Ms Forward has an outstanding loan of $Q = \$10\text{m}$ from Citibank. The previous interest rate reset-date was 15 December, when LIBOR was 2.5% p.a. (say). Hence, over the period 15 December to 15 March the interest rate on the loan is determined by the LIBOR rate on 15 December of 2.5% ($t = 0$), and Ms Forward can do nothing about this. The next loan interest rate *reset date* is 15 March ($t = 3$ months), when the interest rate charged on the loan will be the prevailing 3-month LIBOR rate on 15 March. The *actual payment* of the loan interest will take place on 15 June ($t = 6$ months).

Assume that on 16 December Ms Forward thinks LIBOR rates will rise between 16 December and 15 March, increasing the interest cost of the loan at the next loan reset date. She continues to hold her bank loan from Citibank but she can hedge the interest rate risk by buying a 3×6 FRA today (from Morgan Stanley), at an agreed 'fixed' FRA rate $f_{3,6} = 3\%$ p.a. (simple interest) to begin on 15 March and lasting for a further 92 days until 15 June. The payoff to Ms Forward from the FRA at $T = 15$ March:

'Payoff' from long FRA at $T = Q\ (LIBOR_T - \textit{Agreed FRA rate}, f_{3,6}) \times (92/360)$.

Hence, if the out-turn LIBOR rate on 15 March is 3.5% which is greater than the agreed FRA rate $f_{3,6}$, the seller of the FRA (e.g. Morgan Stanley) will pay Ms Forward:

Payoff Ms Forward, long FRA at $T = \$10\text{m}\ (0.035 - 0.03) \times (92/360) = \$12{,}778$[5]

[5]As we see below, there is a further practical nuance. Although the 'payoff' to the FRA is calculated on 15 March, any cash payout does not accrue until 15 June.

Hence, if Ms Forward has a bank loan but also holds the (long) FRA, then the effective interest cost of the loan repayments on 15 March is:

$$\begin{aligned} \textit{Effective \% interest cost of loan} &= \textit{Actual Interest on Loan} - \textit{Payoff to FRA} \\ &= LIBOR_T\ (92/360) - (LIBOR_T - f_{3,6})(92/360) \\ &= f_{3,6} \times (92/360) \end{aligned}$$

Hence, by taking out the FRA, Ms Forward has effectively swapped the floating-rate payments on her bank loan, for fixed-rate payments at the FRA rate $f_{3,6} = 3\%$ p.a., negotiated on 16 December (with JPMorgan). The effective *dollar* cost of the loan plus the long FRA is:

$$\begin{aligned} \textit{Effective \$-cost of loan} &= \textit{Actual interest on loan} - \textit{Payoff to FRA} \\ &= Q\ [LIBOR_T\ (92/360) - (LIBOR_T - f_{3,6})\ (92/360)] \\ &= Q\, f_{3,6} \times (92/360) = \$89{,}444 - \$12{,}778 = \$76{,}666 \end{aligned}$$

The loan interest of \$89,444 is reduced by the payoff from the long FRA, so the net payment is \$76,666, which is 0.7666% of \$10m over 92 days – that is, 3.0% p.a. (using simple interest and scaling up by 360/92).

Of course, if the 3-month LIBOR rate on 15 March turns out to be 2.5% (i.e. lower than $f_{3,6} = 3\%$) then Ms Forward *will pay \$12,778* to Morgan Stanley (the seller of the FRA). However, in this case, the interest payments on her original bank loan will be at the lower LIBOR rate of 2.5% p.a. so again she effectively ends up with net payments (i.e. the bank loan plus the FRA) equal to the fixed FRA rate $f_{3,6} = 3\%$ p.a.

Banks are the main participants in the FRA market, which are OTC instruments. By taking a long position in the FRA, Ms Forward has effectively 'swapped' the payment on her bank loan at an unknown floating rate (i.e. future 3-month LIBOR), for a known fixed-rate payment at the FRA rate, $f_{3,6} = 3\%$ p.a. As we shall see in later chapters an interest-rate swap is nothing more than a series of FRAs over several reset periods. For example, a 5-year swap with future interest payment reset dates every 6 months, is equivalent to 9 (= 10 − 1) separate FRAs, each with the *same* FRA rate applying to each of the 9 future reset dates. This *constant* FRA rate (which applies at each of the reset dates), is then renamed as 'the swap rate'. Note that the 5-year swap is analytically equivalent to 9 FRAs because the first payment in the swap is at the current known LIBOR rate at $t = 0$ (see Chapter 33 for further details on swaps).

8.4.2 Settlement Procedure

Consider the settlement procedure for the FRA at $T = 15$ March. An FRA is a contract on a notional principal amount \$$Q$. However, there is no exchange of principal at the beginning or end of the contract – only the difference in interest payments on the notional principal is paid.

The ‘settlement rate’ used in the contract is usually an average of the (appropriate) LIBOR rates on the settlement date T (15 March).

The ‘payoff’ from the 3 × 6 FRA can be calculated at $T = 15$ March, but interest is paid in arrears so the actual interest *payment is due* on the 15 June (92 days later). However, if you wish to receive your funds from the FRA on 15 March (i.e. ‘cash settlement’), the amount you will be paid is the *present value* of the amount owed on 15 June. Consider the payment at settlement on the following 3 × 6 FRA:

(i) $f_{3,6} = 3\%$ (fixed on 16 December) on a notional principal of $Q = \$10\text{m}$
(ii) Actual (92-day) LIBOR on 15 March, $LIBOR_T = 3.5\%$

To calculate the ‘cash settlement’ (for Ms Forward) which is to be *paid* on 15 March, note that the amount due on 15 June (92 days later) is ‘Payoff’ $= Q\,(LIBOR_T - FRA\text{-}rate)\,92/360$. So, if the cash payment actually takes place on 15 June, the amount received needs to be discounted at the 92-day LIBOR rate on 15 March hence:

$$\begin{aligned}
\textit{Actual cash payment } 15\ \textit{March} &= \frac{1}{1 + 0.035(92/360)}\,[\$10\text{m}\ (0.035 - 0.03) \times (92/360)] \\
&= \$12{,}778/1.008944 = \$12{,}665
\end{aligned}$$

With an FRA, only the ‘difference in value’ is paid. The FRA is a ‘contract for differences’, based on the difference between the out-turn LIBOR rate and the (fixed) FRA-rate, $f_{3,6}$ (and the notional principal amount, Q). If Ms Forward already has a variable-rate bank loan at 3-month LIBOR then she can use the long-FRA to ‘lock in’ the effective cost of the next loan interest payment at today’s quoted forward rate, $f_{3,6}$. If the bank loan has several future interest rate reset dates then she needs a long-FRA for each reset date. This will lock in the effective cost of all the future loan interest payments at the current quoted forward rates $f_{3,6}, f_{6,9}, f_{9,12}$, etc.

8.4.3 Bank Deposit

Suppose on 16 December Ms Forward has a bank deposit of $Q = \$10\text{m}$ from Citibank, which pays 3-month LIBOR. Ms Forward is worried that interest rates will fall at the next reset date and she will earn less interest on her deposit account. To hedge her future deposit interest she should today, short (sell) an FRA. The payoff to a short-FRA is:

$$\text{‘}\textit{Payoff}\text{’ to short} - FRA = Q\ (\textit{Agreed FRA rate}, f_{3,6} - LIBOR_T) \times (92/360)$$

It follows that the bank deposit plus the short-FRA will provide an effective interest receipts of:

$$\begin{aligned}
\textit{Effective interest received} &= \textit{Actual Interest on Deposit} - \textit{Payoff to FRA} \\
&= LIBOR_T(92/360) + [f_{3,6} - LIBOR_T](92/360) = f_{3,6} \times (92/360)
\end{aligned}$$

and therefore Ms Forward 'locks in' an effective deposit interest rate of $f_{3,6}$ p.a., regardless of the out-turn value for LIBOR at the next reset date. Ms Forward may agree that any payments/receipts from the FRA are actually paid on 15 March, when the amount received needs to be discounted at the 92-day LIBOR rate on 15 March.

8.5 SUMMARY

- Calculating present value or terminal value can be undertaken using simple rates, compound rates, or continuously compounded interest rates. As long as we use the appropriate compounding or present value formulas for each type of interest rate, we obtain the same present value or terminal value.
- The relationship between simple rates, compound rates, and continuously compounded interest rates, enables us to switch between the different rates.
- A *forward rate* is an interest rate which applies to an investment which occurs between two future time periods. Forward rates can be calculated from spot rates. There is a spot yield curve and a forward yield curve.
- Forward rates can be calculated using simple, compound, and continuously compounded spot rates of interest.
- When pricing a forward rate agreement (FRA) we use forward rates that are calculated from spot-rates (using 'simple interest') and an 'actual/360' day-count convention.
- An FRA is a contract agreed today, to exchange a cash flow based on the future LIBOR rate, in exchange for a fixed cash flow at the (pre-agreed) FRA rate. Only the *difference* in interest cash flows is paid (or received).

  A 'long position' in an FRA on a notional principle of $Q = \$100\text{m}$ has:

$$\begin{aligned} \textit{Payoff, long-FRA} &= \$Q\ (\textit{LIBOR}_T - \textit{fixed rate in FRA}) \\ &\quad \times (\textit{actual days to maturity}/360). \end{aligned}$$

- To hedge a future payment on a *bank loan* linked to LIBOR, you should go *long* an FRA. To hedge a future payment on a *bank deposit* linked to LIBOR, you should *short* an FRA. In both cases you then 'lock-in' the current quoted forward rate, regardless of what happens to LIBOR rates in the future.

EXERCISES

Question 1

Suppose a sum of \$2,000 is invested for a period of 7 months. The value of the investment at maturity is \$2,150. Assuming a day-count convention of 360 days p.a. (a) Calculate the simple interest rate. (b) Calculate the equivalent compound interest rate and continuously compounded rate.

Question 2

Suppose you make an investment of \$50,000 today that will attract an annual compound rate of 10% p.a. How long will it take for your investment to amount to €1m?

Question 3

A 90-day T-bill has a face (par or maturity) value $M = \$100$. Its market price is $P = \$97.5$. Calculate the percent return you earn over 90 days if you buy this T-bill today and hold it to maturity – this is the 'periodic return' or 'periodic rate of interest' (yield) over 90 days.

(This example could also apply to an investment in a bank deposit of $P = \$97.5$ and after 90 days you receive $M = \$100$.)

Assume we use 'actual/365' convention to scale up to an annual rate. The 'actual number of days' you hold the T-bill (or bank deposit) is 90, so $T = 90/365$ years.

Show that:

(a) simple interest rate (yield) is 10.399% p.a.
(b) compound interest rate (yield) is 10.8134% p.a.
(c) continuously compounded interest rate (yield) is 10.268% p.a.
(d) What is the intuitive interpretation of these different interest rate conventions? Why is the simple annual rate (10.399% p.a.) less than the compound rate (10.8134% p.a.)?
(e) Why is the continuously compounded rate 10.268% p.a. less than the compound rate of 10.8134% p.a.? Explain.

Note also that the T-bill has a quoted *discount rate* of 10% (actual/360 day count basis) and its market price has been calculated from the discount rate:

$$P = (M - \text{Dollar discount}) = 100 - (90/360)(0.10)100 = 97.5$$

Note the use of 'actual days/360' to get the price – this is the (US) market convention when using the discount rate to calculate the price of a T-bill.

Question 4

Very briefly, mention one practical use for spot interest rates and one practical use of forward interest rates.

Question 5

The 1-year spot yield is 9% p.a., the 2-year spot yield is 9.5% p.a. and the 3-year spot yield is 10% p.a. (compound rates).

(a) Calculate the implied 1-year ahead, 1-year forward rate, f_{12}. Explain.

(b) Explain why a 1-year forward rate of 9.6% would not be expected to prevail in the market.

Question 6

(c) You can lend \$100 to a bank at a quoted forward interest rate $f^q_{1,2} = 13\%$ p.a.. The 1-year and 2-year spot rates (yields) are currently 10% p.a. and 12% p.a., respectively (compound rates). Explain whether you would take the bank's forward rate offer.

Question 7

Suppose a firm plans to borrow \$5 million in 180 days from bank-A at 12% p.a. with tenor of 90 days (actual/360 day-count convention). The loan will be taken out at whatever 90-day LIBOR is on the day the loan begins and will be repaid in one lump sum, 90 days later. The firm would like to lock in the loan rate it pays so it enters into an FRA with bank-B. Determine the annualized cost of the 'loan+FRA' (compounded using 365-day year) for each of the following outcomes:

(a) LIBOR in 180 days is 14%.

(b) LIBOR in 180 days is 8%.

Question 8

A company has a \$20m variable rate loan and faces an interest payment at the 6-month variable interest rate, 6 months from now. There is a concern that interest rates will rise over the next 6 months but this is not certain. The company is struggling to decide whether to hedge or not. After assessing the interest rate outlook, the company decides to *hedge 50% of its exposure* and enters a 6×12 FRA on notional principal of \$10m at a FRA rate of 4% p.a.
Suppose the 6-month spot interest rate at $t = 6$ months is 4.5% p.a. Assume 6 months is ½ year. Explain the company's dollar net position on the FRA settlement date.

CHAPTER 9

Bond Markets

Aims

- To explain spot rates (yields) and the 'yield to maturity' on a bond.
- To show that the 'fair' price of a coupon paying bond is determined by spot yields – otherwise risk-free arbitrage profits can be made by a strategy known as coupon stripping.
- To show how the yield to maturity is 'derived' from the market price of the bond.

There are a wide variety of different types of bonds. A conventional government bond usually pays to the holder a fixed amount of cash ('the coupon') every 6 months plus the maturity (redemption, par, or principal) value at the end of the bond's life. However, not all bonds have fixed coupons or fixed maturity dates. For example, some corporate bonds can be redeemed (or 'called') prior to their maturity date while other corporate bonds (i.e. convertible bonds) can be exchanged for common stock (equity) of the issuer, at some future date. So convertibles have a 'mixture' of payments in terms of coupons and then dividends if converted into equity.

Bonds are usually issued to obtain long term finance – the initial maturities are from 1 year to 30 years (with some being non-redeemable and known as perpetuities). They are issued by governments, and their agencies (e.g. Municipal Securities in the US and Local Authority Bonds in the UK) and by the corporate sector. They may be denominated in the home currency of the issuer or in a foreign currency (e.g. a UK corporate issuing dollar denominated bonds). A key difference between government issued bonds and those issued by corporates, is default

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

risk. Generally speaking government bonds denominated in the home currency are described as 'risk-free', meaning there is no default risk. This is because governments usually have the legal right to raise taxes or to print their own currency, in order to pay the interest and principal on the bonds. But even Government bonds issued by the US government are not entirely free of default risk – but we assume such risk is very low. However, Greek bonds issued in 2010 and denominated in euros were not perceived as being risk-free. This is because holders of Greek bonds might not believe that the Greek government has the (de facto) democratic legitimacy and ability to raise enough euros from taxes to pay the interest and principal – and the Greek government cannot 'print' euros (only the European Central Bank can). Hence Greek government bonds issued in the years after 2010 are subject to a high probability of default. Also, zero default risk would not necessarily be true for government bonds denominated in a foreign currency – as was the case for countries like Mexico and Argentina, who effectively defaulted on dollar interest payments on their dollar denominated bonds.

In this chapter we discuss 'risk-free' government bonds, which provide the main analytic concepts for the reader to study other types of bonds mentioned above and how bond portfolios are hedged using futures and options.

9.1 PRICES, YIELDS, AND RETURN

We begin with an analysis of the relationship between bond prices, spot yields and the yield to maturity. After discussing 'zero-coupon bonds', we demonstrate how we can use risk-free arbitrage to price a coupon paying bond by considering it as a series of zero-coupon bonds. This allows a consistent methodology for the pricing of bonds.

9.1.1 Pure Discount/Zero-coupon Bonds

Bonds which have a single payout, but have a maturity greater than 1 year, are usually classified as pure discount bonds or zero-coupon bonds. Suppose a pure discount bond has a single payout M(dollar) in T years' time. If we know M, T and the observed market price P_0 then we can calculate the current (compound) spot rate (yield), r, using:

$$r = \left[\frac{M}{P_0}\right]^{1/T} - 1 \tag{9.1a}$$

Hence:

$$P_0 = \frac{M}{(1+r)^T} \tag{9.1b}$$

Example 9.1 calculates the spot rate (yield) for a zero-coupon bond with maturity of 6 years.

EXAMPLE 9.1

Spot Rates on a Zero-coupon Bond

Data: The current price, P of a 6-year, zero-coupon paying bond with maturity value \$100,000 is \$81,350.06.

Question: Calculate the compound spot rate (yield), r.

Answer: $r = (\$100,000/\$81,350.06)^{1/6}-1 = 0.035$ (or 3.5% p.a.)

Taking logarithms of (9.1b) and using the approximation $\ln(1+x) \approx x$ we obtain $\ln P = \ln M - rT$ and differentiating (with respect to r) we obtain:

$$dP/P = -T\,dr = -D\,dr$$

$$\textit{Proportionate change in bond price} = -\text{`}\textit{Duration}\text{'} \times \textit{Absolute change in yield} \qquad (9.2)$$

For a zero-coupon bond, the time to maturity (in years) is also the 'duration' of the bond. The proportionate (percentage) change in the bond price is usually referred to as the (holding period) 'return on the bond'.[1]

For example, for a two-year zero-coupon bond ($T = D = 2$), Equation (9.2) implies that when the spot yield *increases* from 2% to 2.1% p.a. over 1 week (say), then the percentage change in the bond price (over 1 week) will be a *fall* of around 0.2%. Hence, if the initial price of the bond is $P = \$99$, the price in 1 week's time would be (close to) \$98.8.

If we have lots of 'zeros' of different maturities then we can observe their current market prices and calculate spot rates for 1 year, 2 year, 3 years, etc. For example, (compound) spot rates at 10 a.m. today might be 5, 5.12, 5.23, … 5.51, 5.52, 5.52, 5.52, … for years 1, 2, 3, …, 15, 16, 17, 18, …, respectively. If we plot a simple graph of the spot rates against time to maturity, then this is known as the (spot) yield curve (at 10 a.m. today) – here the yield curve is upward sloping and then flattens out for maturities longer than 16 years.

What these spot rates imply is that if you want to borrow, say, \$1,000 today and pay back all the interest and principal in exactly 16 years' time, then you will end up owing $\$2,362.42 = \$1,000\,(1.0552)^{16}$. Alternatively, if the maturity value of a 16-year zero-coupon bond is $M = \$2,362.42$, and the 16-year spot yield is 5.52%, then the market price of the bond today is $P = \$1,000$. So the 16-year spot rate is the cost of borrowing money today, with one single repayment in exactly 16 years' time. As our yield curve is currently upward sloping, it costs

[1] If we use continuously compounded yields $\ln(P) = \ln(M) - r_c T$ is exact but (9.2) is still an approximation, but it provides a good approximation for the proportionate change in the bond price, for small changes in yields.

more to borrow money the longer the time horizon over which you wish to borrow. Apart from the bid–ask spread (which we ignore), then the cost of lending money today, with one repayment in 1 year's time is 5% p.a., and with one repayment in 2 years it is 5.12% p.a., etc.

The price of zeros is determined by the supply of bonds (i.e. borrowers of money) and the demand for bonds (i.e. lenders of money). Usually, governments are borrowers of money and issue bonds while pension funds, insurance companies, and (older) individuals are lenders of money and they purchase bonds. The price of a 'zero' will change today, when the balance of supply and demand in the market for funds (for a particular maturity date), changes. But note that it is always true of fixed-income assets like bonds that the bond price and (spot) yields always move in opposite directions.

Generally speaking interest rates of all maturities tend to move up and down together. For example, if at 3 p.m. *all* spot rates are higher by 0.1%, say, this would be called a 'parallel shift' in the yield curve. Analytically, spot yields are the 'building blocks' for analysing fixed-income assets such as coupon-paying bonds, and other fixed income derivatives such as swaps, caps, and floors.

9.1.2 Coupon Paying Bonds

Coupon paying bonds provide a known stream of cash flows called coupons, C, payable each year (say) over the remaining life of the bond. Most bonds have a known final payment at maturity known as the 'par value', the 'principal value' or 'maturity value', M (Figure 9.1). There are some bonds which although they pay coupons, are never redeemed and these are known as perpetuities (e.g. '3% Consols' issued by the UK government).

Bond prices quoted in the financial press and on electronic trading screens are known as 'clean prices'. The pricing formulas below all refer to the clean price. However, the actual price paid by an investor is known as the 'invoice price' and includes accrued or 'rebate' interest (see Cuthbertson and Nitzsche 2008).

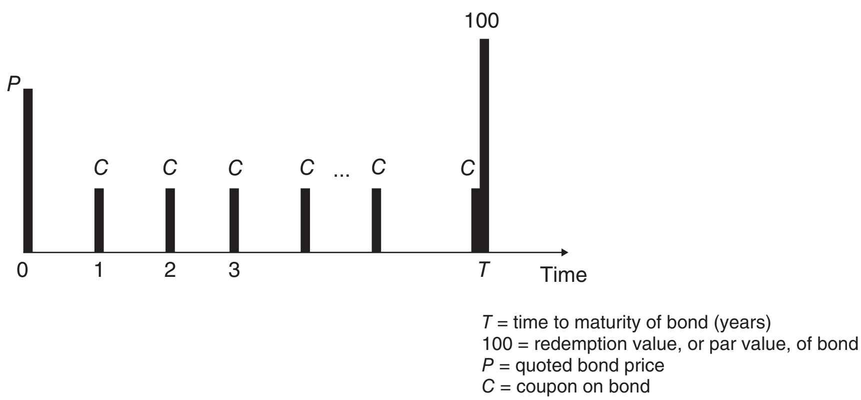

FIGURE 9.1 Coupon paying bond

9.1.3 Coupon-yield (Interest-yield, Flat-yield, 'Current-yield' in USA)

The 'coupon-yield' on a bond is usually quoted in the financial press and is calculated as:

$$\textit{Interest-yield} = \frac{\textit{Annual Coupon}}{\textit{Current Clean Price}} \times 100\% \tag{9.3}$$

The 'coupon-yield' is a very poor measure of the return on a bond since:

- it ignores any capital gains which occur if the bond is purchased at a price below its maturity value, M;
- it ignores 'interest-on-interest' from coupon payments.

9.1.4 Yield to Maturity (Redemption Yield)

Knowing the market price P, the coupon C (here paid annually) the maturity value M, and the number of years to maturity, T then the yield to maturity (YTM), y, is the solution to:

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_T}{(1+y)^T} + \frac{M}{(1+y)^T} \tag{9.4}$$

It can be shown (using the annuity formula) that this simplifies to:

$$P = C\left[\frac{1}{y} - \frac{1}{y(1+y)^T}\right] + \frac{M}{(1+y)^T} = C \times A(y, T) + \frac{M}{(1+y)^T} \tag{9.5}$$

where the 'annuity value' is defined as: $A(\mathrm{y}, \mathrm{T}) = \left[\frac{1}{y} - \frac{1}{y(1+y)^T}\right]$

The 'annuity value' is the present value of \$1 paid at the end of each year, for T years using the YTM as the discount rate. The YTM is the rate y which equates the present value of future cash flows C_i and the maturity value M, with the current market price. Hence, the YTM is the 'internal rate of return' (IRR) of the bond. Here P, C_i M and T are known and we (or the *Financial Times/Wall Street Journal*) calculate y using Excel's 'IRR' inbuilt command.

If coupon payments are annual then $i = 1, 2, 3, \ldots$ refers to years and y is an annual (compound) rate. However, for government bonds the coupons are usually paid semi-annually and in this case we replace y in the above formula by $y/2$ and C_i is replaced by the semi-annual coupon payments $C_i/2$ (and $i = 1$ for 6 months, $i = 2$ for 1 year, etc.). It follows that $y/2$ is the 'semi-annual rate', and y is the quoted 'simple' annual rate (which in the US is known as 'the bond equivalent yield'). If $y =$ the (unknown) 5-year YTM (p.a.), then a 5-year bond with semi-annual coupon payments ($C/2$ every 6 months), has a market price given by:

$$P = \frac{C/2}{(1+y/2)} + \frac{C/2}{(1+y/2)^2} + \frac{C/2}{(1+y/2)^3} + \ldots \ldots + \frac{C/2}{(1+y/2)^9} + \frac{C/2}{(1+y/2)^{10}} + \frac{M}{(1+y/2)^{10}}$$

Knowing the market price of the bond and all the variables on the right-hand side of the above equation, we can calculate the YTM and a simple example is given in Example 9.2, where the YTM is 6% p.a.

EXAMPLE 9.2

Calculation of YTM

Data: The market price, P of a 3-year, 4.5% coupon paying bond with a face value of \$1,000, is \$959.37. Coupon payments are semi-annual (6×6 months periods).

Question: Calculate the YTM.

Answer: $P = \$22.50/(1 + y/2) + \$22.50/(1 + y/2)^2 + \dots + \$22.50/(1 + y/2)^5 + \$1,022.50/(1 + y/2)^6 = \$959.37$

The solution 'y' to the above equation is usually found using a simple algorithm (e.g. the IRR function in Excel. Excel gives $y = 0.06$ (or 6%). This can be verified by working out the PV of the RHS using $y = 6\%$, which is found to be \$959.37.

The YTM is made up of three elements:

YTM = coupon rate + interest on the coupons + capital gain (or loss) from the difference between the purchase price P and the maturity value M.

The YTM measures the (annual compound) rate of return on the bond if (i) it is held to maturity and (ii) all the future coupon payments can be reinvested (when received) at a rate of interest equal to the (current) YTM. If these conditions do not hold, then the YTM will not correctly measure the 'return' on the bond even if it is held to maturity.

It is the assumption that the future coupon payments can be reinvested at the current YTM which creates problems, since we do not know today at what rate of interest we can reinvest these future payments. However, the YTM is the best single figure we can use to (approximately) represent the average annual percentage return on the bond if it is held to maturity. The YTM is widely used when discussing strategies amongst bond market traders and traders of derivatives whose prices depend on interest rates (yields).

The YTM can be calculated using the 'clean price' or the invoice price. For the UK when P is the invoice price then y is known as the gross redemption yield (and is published daily in the *Financial Times*). Note that the YTM does not determine (in an economic sense) the price of the bond – the YTM *is derived from* the observed market price of the bond. However, if someone has already calculated the YTM then Equation (9.4) can of course be used to calculate the bond price (see Example 9.3).

EXAMPLE 9.3

Calculation of Bond Price

Data: 20-year, 4% coupon paying bond, maturity value $M = \$1{,}000$. The current YTM is $y = 5\%$ p.a. Coupon payments are being made semi-annually ($40 \times$ 6-month periods).

Question: Calculate the price of the bond.

Answer: $C = (0.04/2)\$1{,}000 = \$20,$
$T = 2(20 \text{ years}) = 20 \times 6 \text{ month periods},$
$y/2 = 0.05/2 = 0.025 \text{ (semi – annual rate)}$

The PV of the coupon payments is given by the annuity formula:

PV (coupons) $= C[1 - 1/(1 + y/2)^T]/(y/2) = \$20[1 - 1/(1.025)^{40}]/0.025 = \$502.06$

PV (of maturity payment) $= \$1{,}000/(1.025)^{40} = \$372.43$

Market price of bond $= \$502.06 + \$372.43 = \$874.49$

It is clear from Equation (9.4) that the YTM and the market price of a bond are negatively related and the relationship is non-linear ('convex' – see Figure 9.2).

Excel: Bond Price and Yields

The Excel file on the website calculates the prices of two different bonds for different values of the YTM and then graphs the 'convex' (non-linear) bond price-yield relationship for each of the two bonds. Duration and Modified duration are also calculated using the Excel functions '=DURATION' and '=MDURATION'.

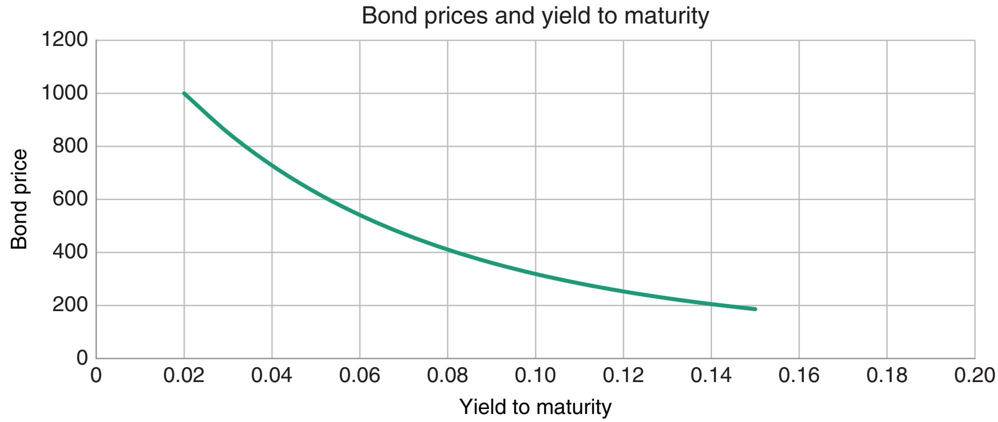

FIGURE 9.2 Bond price and yield

9.1.5 YTM and Coupon Rate

The coupon rate (or coupon yield) is defined as C/M. There are some 'rules of thumb' used by traders when discussing the relationship between the YTM and the coupon rate of a bond. It is easy to show that if the coupon rate equals the YTM, then the bond is currently trading at a market price equal to its maturity (par) value. For example, consider a bond with a 10% (annual) coupon rate, which also has a YTM of 10% and the bond has 2 years to maturity and a par (maturity) value of $M = £100$. We can easily demonstrate that the market price of this bond equals its par value of £100. Using Equation (9.4):

$$P = £10/(1.1) + £110/(1.1)^2 = £100 \tag{9.6}$$

If the coupon rate (C/M) is below the YTM, then the market price P will be below the par value (of $M = £100$) and the bond currently sells 'at a discount'. Conversely, if the coupon rate C/M is above the YTM the bond will today have a market price above its par value of $M = £100$ – it will be trading at a 'premium'. The qualitative relationship between the market price, coupon rate (C/M) and YTM is given in Table 9.1 and they can be deduced from the following rearrangement of Equation (9.4):

$$\frac{P}{M} = \frac{C}{M}\left[\frac{1-(1+y)^T}{y}\right] + \frac{1}{(1+y)^T} \tag{9.7}$$

If the current YTM, $y = C/M$, the coupon rate, then Equation (9.7) gives $P/M = 1$, that is the market price of the bond P, equals its maturity value M. Also, it follows that if (C/M) is less than y, then $(P/M) < 1$ and the current market price will be less than the bond's maturity value.

TABLE 9.1 Yield and price relationship

Bond sells at	Relationship
Par ($P = M$)	Coupon rate = YTM
Discount ($P < M$)	Coupon rate < YTM
Premium ($P > M$)	Coupon rate > YTM

9.2 PRICING COUPON BONDS

This section discusses why bonds are priced using spot rates. Bond prices that do not reflect prevailing spot rates can be subject to coupon stripping and risk-free arbitrage opportunities – which is the reason market participants use spot rates to price bonds. The spot rate r_T is the (annual compound) rate of return on an investment that has a one-off payout in T years' time. It is the return (yield) on a zero-coupon bond.

$$\mathrm{P} = \frac{M}{(1 + r_T)^T} \tag{9.8}$$

If there were zero-coupon bonds for all maturities then we could observe their prices P_1, P_2, etc. and use Equation (9.8) to calculate r_T (for $T = 1, 2, 3, \ldots$, etc.). These spot rates would be the outcome of the demand and supply of zero-coupon bonds in the market. Any (non-callable) coupon bond can be viewed as a 'bundle' of zero-coupon bonds. Hence, the price of a coupon bond should equal the PV of the future coupon payments, where each coupon payment is discounted at the *appropriate spot rate*:

$$P = \frac{C_1}{(1 + r_1)} + \frac{C_2}{(1 + r_2)^2} + \ldots + \frac{C + M}{(1 + r_T)^T} \tag{9.9}$$

Each C_i may be viewed as a one-off payment at times $t = 1, 2, 3, \ldots, T$ (and for most bonds $C_i = C$, a constant). The maturity (redemption) value M is a one-off payment at time, $t = T$. Each separate coupon (and M) may be considered as a strip of zero-coupon bonds, discounted at the spot rate applicable to payments at $t = 1, 2, 3, \ldots T$. Spot rates are the correct way to price government bonds, since all coupons at time t on different bonds are discounted at the same spot rate r_t. Once we have the spot rates, the price of a coupon paying bond is easy to calculate as shown in Example 9.4.

EXAMPLE 9.4

Pricing Coupon Paying Bonds Using Spot Rates

Data: Bond-A : Exchequer Stock 8.75%, annual coupon, 2 years to maturity, $M = £100$

Bond-B : Exchequer Stock 12%, annual coupon, 2 years to maturity, $M = £100$

Current spot rates, $r_1 = 0.05$ (or 5%) and $r_2 = 0.06$ (or 6%)

Question: Calculate the market price of the two bonds.

Answer: $\text{Price (Bond} - \text{A)} = £8.75/(1.05)^1 + £108.75/(1.06)^2 = £105.12$

$\text{Price (Bond} - \text{B)} = £12/(1.05)^1 + £112/(1.06)^2 = £111.11$

9.2.1 Calculation of Spot Rates

We now turn to the problem of estimating spot rates when they are not directly observed in the market. In the US and the UK, T-bills which are risk-free pure discount bonds are only issued with maturities up to 1 year. In general, government bonds with maturities greater than 1 year are issued as coupon paying bonds. However, spot yields can be calculated from coupon-paying bonds in several ways and here we demonstrate the method known as bootstrapping. Example 9.5 considers the case where we can directly observe the 6-month and 12-month spot yields (r_1 and r_2) on zero-coupon bonds but not the 18-month spot yield r_3, since we assume there are no zero-coupon bonds with maturities longer than 1 year. However, the 18-month spot yield r_3 can be derived using the observed market price of a *coupon paying* bond with 18 months to maturity. As we see in later chapters, spot rates can also be calculated from observed Eurodollar futures prices and their implied forward rates and from observed swap rates.

EXAMPLE 9.5

Calculation of Spot Rates by Bootstrapping

Data: Bond-A: 6-month zero coupon bond, spot rate $r_1 = 4\%$ p.a.

Bond-B: 1-year zero-coupon bond, spot rate $r_2 = 4.2\%$ p.a.

Bond-C: Coupon bond with maturity 1.5 years, face value, $M = \$100$, coupon rate = 5%, coupon paid every 6 month. Market price, $P = \$99.448$

Question: Calculate the 18-month spot rate (r_3)

Answer: $\$99.448 = 2.5/(1 + r_1/2) + 2.5/(1 + r_2/2)^2 + 102.5/(1 + r_3/2)^3$

Observed spot rates on 6-month and 12-month bills are $r_1 = 0.04$ and $r_2 = 0.042$. Hence $99.448 = 2.4510 + 2.3982 + 102.5/(1 + r_3/2)^3$

The only 'unknown' in the above equation is r_3 and $r_3 = 0.0542$ (or 5.42% p.a.)

Similarly, if we have coupon paying bonds for 2 years, 2.5 years, etc. then the bootstrapping procedure can be repeated to calculate spot rates for these years. In this way we can obtain the complete spot yield curve. Where a coupon paying bond does not exist for a particular maturity (e.g. for maturity of 7.5 years) then the spot rate can be derived as an interpolation of the (derived) 7-year and 8-year spot yields. In fact, it is a little more subtle than this since after deriving the spot yields for those years for which we have coupon paying bonds, a smooth curve is fitted to be 'close to' all these observed data points – using rather sophisticated techniques (e.g. cubic splines).

9.2.2 Coupon Stripping

Zero-coupon T-bonds can be created by dealer firms, by selling off the 'ownership' of the coupon payments from coupon paying bonds. However, observed interest rates on 'stripped' Treasuries can give a misleading measure of risk-free spot rates because (i) strips are less liquid/marketable, hence their yields reflect a liquidity premium and (ii) there is preferential tax treatment for some holders of strips (e.g. some overseas purchasers) and this 'tax effect' is reflected in observed yields. Even though spot rates are observable in the strips market, it may still be necessary to use 'bootstrap procedures' or 'curve fitting' to obtain good estimates of the true risk-free spot rates.

We now demonstrate why coupon paying bonds should be priced using spot rates. This occurs because if coupon bonds are not priced using spot rates, then there is a potential (risk-free) arbitrage profit to be made from 'coupon stripping' the bond.

Example 9.6 shows that the 'fair price' of a 2-year coupon paying bond, calculated using current spot rates is $\widehat{P} = \$972.4858$. Suppose a dealer has not used spot rates to price the bond and is quoting a price of PV from selling 2nd coupon + Redemption value = $\$1{,}040/(1.055)^2 = \$934.3905$ – the bond is underpriced. You can make an arbitrage profit by 'buying low and selling high'. You purchase the bond for \$970 from the dealer and simultaneously offer to sell the first coupon payment of \$40 to a pension fund and the second payment of \$1,040 to an insurance company. Given current spot-rates, the pension fund will be willing to pay you \$38.0952 today to secure the receipt of \$40 in 1 year's time. Similarly the insurance company will be willing to pay you \$934.3905 today, to secure the receipt of \$1,040 in 2 years' time. So today you receive a total of \$972.4858 from the pension fund and insurance company and you use this to purchase the 2-year coupon bond at $P = \$970$, making a risk-free profit of \$2.4858 today. We assume that your trades do not affect current prices (or current spot yields).

EXAMPLE 9.6

Price of Coupon Paying Bonds

Data: Coupon paying Treasury Bond, 2 years to maturity, 4% (annual) coupon. Face value, $M = \$1{,}000$. Quoted market price is $P = \$970$.

Observed 1-year spot rate = 5%

Estimated 2-year spot rate = 5.5%

Question: Calculate the 'fair' or 'equilibrium' price of the bond and hence show that there are profits to be made from coupon stripping.

Answer: PV from selling 1st coupon = $\$40/(1.05) = \$38.0952$

PV from selling 2nd coupon + Redemption value = $\$1{,}040/(1.055)^2 = \$934.3905$

Using spot rates, 'fair price' $\widehat{P} = \$38.0952 + \$934.3905 = \$972.4858$

Buy the bond for $P = \$970$ today and simultaneously sell the two 'coupons' and redemption value for \$38.0952 and \$934.3905 – making an arbitrage profit of \$2.4858 today.

However, there are lots of arbitrageurs around to take advantage of this mispricing. So, many arbitrageurs buy the 2-year coupon bond, which will tend to increase its price. Also, by selling one-year and two-year strips, their prices fall (or equivalently their spot yields rise). Arbitrage will continue until the market price of the 2-year coupon bond equals the PV of the sum of the receipts from coupon stripping – that is, until the market price of the coupon bond equals its fair value. Since arbitrage here involves a near risk-free transaction – the market price of the bond will quickly reflect its fair value. Our key result is therefore:

The quoted price of a coupon bond is determined by the current term structure of spot rates. The YTM is then derived from the quoted price.

9.3 SUMMARY

- Spot rates (yields) refer to the cost of lending (or borrowing) money today, with the principal and interest payable at one specific time in the future.
- The YTM is an approximate measure of the annual return on a T-bond if it is held to maturity and all the future coupons are reinvested (at the current YTM). Hence, the YTM depends on (i) the coupon payments C, (ii) any interest-on-interest on the reinvested coupons, (iii) the maturity value M, (iv) the time to maturity of the bond,

and (v) any capital gain (or loss) from the difference between the purchase price P and the maturity value M.

- The 'fair' ('no-arbitrage') price of a coupon paying bond is determined by current spot yields – otherwise risk-free arbitrage profits can be made by a strategy known as coupon stripping. Once we have calculated the 'fair' price using current spot yields we can then infer the current YTM that is consistent with this price.
- The market price of a bond and the yield to maturity YTM are inversely related.

EXERCISES

Question 1
What is a Treasury 'strip'?

Question 2
What are the key features of a coupon paying bond that determines its yield to maturity? What are the strengths and defects of the yield to maturity as a measure of the return on the bond if held to maturity?

Question 3
Assume that you require a 10% (compound) return on a 5-year zero-coupon bond with a par (maturity) value of £1,000. What price would you pay for the bond today?

Question 4
Consider a 7%-coupon US government bond that has a par value of $1,000 and matures 5 years from now. The coupon payments are annual. The current yield to maturity (YTM) for such bonds is 8% p.a. (compound rate). Calculate the market price of the bond and state whether you expect this bond to sell at par, at a premium (over par), or at a discount.

Question 5
Why are coupon paying bonds priced using spot yields? What then is the significance of the yield to maturity (YTM)?

Question 6
Quoted (compound) spot rates (yields) are as follows:

Year, *t*	**Spot rate, % p.a.**
1	$r_1 = 5.00$
2	$r_2 = 5.40$
3	$r_3 = 5.70$
4	$r_4 = 5.90$
5	$r_5 = 6.00$

(a) What are the discount factors for each year – that is, the value today of \$1 received in year t?

(b) Calculate the present value PV and hence the fair price of the following Treasury notes/bonds with annual coupons, all of which have $M = \$1{,}000$ par value.

  (i) 5% coupon, 2-year note

  (ii) 5% coupon, 5-year note

  (iii) 10% coupon, 5-year note

(c) What are the one year (compound) forward rates applicable between (i) year-1 and year-2, (ii) year-2 and year-3?

CHAPTER 10

Bonds: Duration and Convexity

Aims

- To demonstrate how *spot-rates* for different maturities give rise to the (spot) *yield curve*.
- To show how *duration and convexity* can be used to provide an approximation to the change in bond prices, after a change in the yield to maturity (YTM).

10.1 YIELD CURVE

Investors borrow (and lend) money over different periods of time. For example, to borrow money today and pay back the principal and interest in 1 year's time, the cost of borrowing might be $r_1 = 9\%$ p.a. To borrow today and pay back the principal and interest in 2 years' time (i.e. there are no interim payments), then the quoted interest rate might be $r_2 = 10\%$ p.a. Because each of these interest rates are quoted for borrowing from today, over a fixed horizon (with no interim payments), they are known as *spot-rates* (or spot yields).

The (spot) yield curve shows the relationship between (spot) interest rates for different maturity investments. We assume we are dealing with risk-free investments – there is no risk of default. For example, the yield curve at 10 a.m. today might look like that in Figure 10.1. The yield curve in Figure 10.1 is upward sloping, which simply means that if you borrow money at 10 a.m. today then the longer the maturity of your loan, the higher the (spot) interest you will pay. Note that spot-rates apply to a transaction that is conceptually different from a 'standard loan'. In a standard loan, the repayments schedule will include interim payments and therefore the interest rate charged cannot be called a spot rate.

Spot-rates at any one time are determined in the market by the interplay of the supply of funds by lenders and the demand for funds by borrowers. As the supply and demand for funds changes, then spot-rates will change and the yield curve will alter its shape or position.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Usually, if there is a change in demand and supply of funds at a particular horizon (e.g. lending over 2 years) then this will also influence the supply and demand at all other horizons. For example, if r_2 increases then all other spot-rates will also tend to increase – the correlation coefficient between changes in any two spot-rates (with different maturities) over a short horizon (e.g. 1 week) is very high, usually in excess of 0.9. Although spot-rates tend to move up and down together, they do not all move by the same absolute amount. In general 'long-rates' tend to move less than 'short-rates'. For example, if the 1-year rate increases by 1% (e.g. from 3% to 4%) over the next week, the 3-year rate might only increase by 0.95% (see curve BB, in Figure 10.1). However, if all spot-rates do happen to move up or down by the same absolute amount this is called a *parallel shift* in the yield curve.

We have described spot-yields in terms of borrowing and lending money over specific horizons. This can be done by buying a zero-coupon bond (i.e. lending money) or issuing (selling) a 'zero' (i.e. borrowing money), so spot-rates can be inferred by observing the market price of zeros. For example, if the market price on a 2-year zero is $P = \$92$ which pays $M = \$100$ in 2 years' time then the 2-year spot-rate is 4.26% p.a. (Calculated using $(1 + r)^2 = M/P = 100/92$. Pure discount government bonds for long maturities do not exist but spot yields can be derived from data on a set of coupon paying bonds (and from swap rates).

Usually the spot-yield curve is upward sloping and flattens out at long maturities. But sometimes it is downward sloping – which implies it costs less to borrow money over say 5 years than it does over one year.

So far we have merely described what the yield curve represents. It tells you how much it will cost you (p.a.) to borrow (or lend) money over a fixed horizon, starting immediately (and with no interim payments). But what determines the shape of the yield curve at any point in time? This analysis is known as the *term structure of interest rates* and broadly speaking the shape of the yield curve today is determined by the market's expectations about price inflation in future years. If today, inflation is expected to be higher (lower), in each future year then the yield curve will be upward (downward) sloping (see Cuthbertson and Nitzsche 2008).

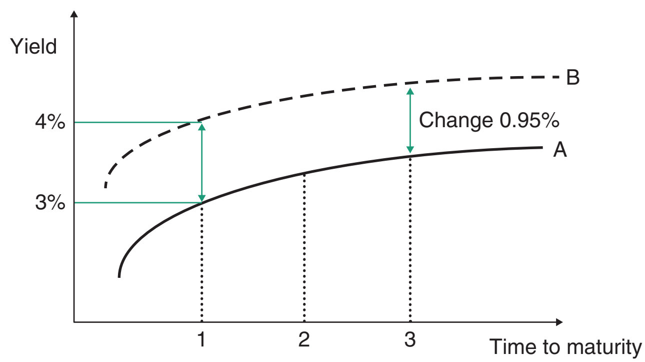

FIGURE 10.1 Yield curve

10.1.1 Estimating Yield Curves

There are many different types of 'yield curve' but all of them are a graph of some measure of 'yield' against maturity (e.g. spot-yield curve, forward-yield curve). There are a wide range of coupon paying bonds with different payment dates, maturities, different tax treatment, etc. Also some maturities may be traded in illiquid markets so that 'posted' prices may not be representative of prices at which you can actually trade these bonds. Hence not all spot yields will lie exactly on a smooth curve and some statistical methods are required to fit a smooth curve to observed data on spot yields. A popular method is the cubic spline technique. Here, separate curves are fitted to various sections of the yield curve (e.g. for rates between 1 and 5-year maturities, then between 5 and 10-year maturities, etc.) and these separate curves are smoothly joined at each of the intersection ('knot') points of the separate curves (e.g. at 5-year maturity, 10-year maturity, etc.), so the whole curve is smooth but has different slopes for each section.

10.2 DURATION AND CONVEXITY

The duration D of a bond is a 'summary statistic' which can be used to tell us (approximately) how much the bond price will change, after a change in the yield to maturity (YTM). For example, consider a speculator who currently holds a coupon paying bond with 7 years to maturity, a current market value of \$1,000 and which has a duration of $D = 5$. Suppose the YTM moves from 6% to 5.5% over the next week, that is the absolute change in the YTM is $dy = -0.5\%$. Then the price of the bond will rise by approximately 2.5% over the next week, since it can be shown that:[1]

$$\%\ \textit{change in bond price} \approx -D \times (\textit{absolute change in } YTM) \tag{10.1}$$

$$\%(dP/P) \approx -D\, dy = -5\ \ (-1/2)\% = +2.5\%$$

The minus sign in the above formula captures the fact that a fall (rise) in the YTM leads to a rise (fall) in the bond price. Note that the duration formula only gives an approximation to the change in price – the actual (or 'true') change in price will differ from that given by the duration formula – but the approximation is quite accurate for small changes in yields (e.g. 25 bps). If we require a more accurate measure of the change in price we need to incorporate a 'convexity' adjustment (see below) or use the present value pricing equation (see Chapter 9).

[1] Equation (10.1) applies only if the yield y is measured as the 'continuously compounded yield' – and the latter is also used to define the duration of the bond – but this subtlety need not detain us here. The formula for D when the yield y is a 'compound yield', and duration is calculated using the compound yield, is given later in the chapter.

The duration formula also assumes that yields at all maturities move by the same (absolute) amount – that is, a parallel shift in the yield curve.

According to the duration formula, the change in value of the bond will be around 2.5% of its current market value of $1,000, that is an increase of $25, so the price of the bond at the end of the week will be close to $1,025. Clearly, duration is useful for fixed-income traders who speculate on changes in interest rates – the larger the duration of the bond, the greater the percentage change in the bond price and hence the greater the ('market') risk of the bond.

The relationship between the 'true' change in bond price and the change in YTM is shown in Figure 10.2 by the curved or 'convex' line. The approximate change in price, given by the duration formula is represented by the 'tangent line' (at the current yield of 5%). Any actual price *rise* will exceed that given by the duration equation – and any actual price *fall* will be less than that calculated using duration. For small changes in yield, the actual price change and the approximated price change – that given using the duration formula – will be very close because the 'curve' and the 'straight line' coincide.

10.2.1 Duration of a Portfolio of Bonds

Suppose you hold N-bonds. The duration of a portfolio of N-bonds is simply a weighted average of the duration of the constituent bonds in the portfolio, where the weights w_i are determined by the market value of the individual bonds:

$$D_p = \sum_{i=1}^{N} w_i D_i \tag{10.2}$$

where (assuming no short-selling) $0 < w_i < 1$ and $\Sigma w_i = 1$. For example, if you hold $200m in bonds, each of which has a duration $D = 4$ and $400m in bonds each with a duration of $D = 12$, then the duration of the bond portfolio is $D_p = (200/600)\,4 + (400/600)\,12 = 9.33$.

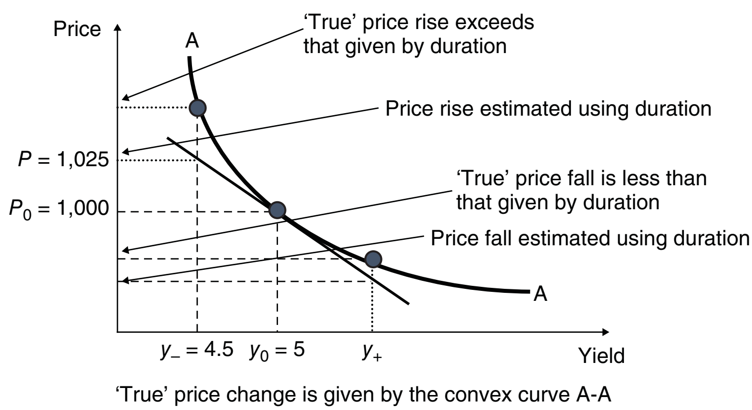

FIGURE 10.2 Duration and price changes

This implies that if bond yields change by 1%, the *value* of your bond portfolio will change by (approximately) 9.33 *per cent,* hence:

$$\text{Proportionate change in value of portfolio}: dV_p/V_p \approx -D_p \times (\textit{absolute change in YTM}) \tag{10.3a}$$

$$\text{\$-change in value of bond portfolio}: dV_p \approx V_p(D_p dy) \tag{10.3b}$$

where dy, the change in the YTM, is expressed as a decimal (e.g. if the current YTM is 3% p.a. and falls to 2% p.a. over 1 week then $dy = -0.01$). For a 1% fall in the YTM over 1 week, the (approximate) change in the (dollar) value of the bond portfolio (over 1 week) is dV_p (= \$600 × 9.3333 × 0.01) = \$56.

It can be shown that the formula for D for an n-period coupon paying bond (annual payments), where y is measured as a 'compound yield' (decimal) is:

$$D = \frac{[PV(C_1)1 + PV(C_2)2 + \ \dots \ + PV(C_n + M)T]}{P} \tag{10.4}$$

where the present value of each coupon payment at time t, is $PV(C_t) = C_t/(1 + y)^t$. In Equation (10.4) the present value of each coupon payment is 'weighted' by the 'time' at which the coupon is received $t = 1, 2, \ \dots, \ T$. Hence, the duration of a bond is sometimes described as a 'time weighted average term to maturity of the bond'. However, what is important is that today, the duration of any bond can be calculated using (10.4) and an approximate change in the bond price is then given by Equation (10.1) (or Equation (10.5) below).

Duration is a useful summary statistic for calculating (approximate) changes in bond prices but what factors determine the duration of a coupon paying bond? Some useful 'rules of thumb' are: (a) duration generally increases with time to maturity (always does so for bonds selling at or above par); (b) duration is higher the lower is the coupon rate (C/M); and (c) duration is (usually) higher when the YTM is low.

Above we have used the 'compound yield' y in the duration formula and when this is the case it can be shown (see Appendix 10) that for small changes in (compound) yields, the *proportionate* change in the bond price is actually given by:

$$\frac{dP}{P} = -D\frac{dy}{(1 + y)} = -(MD)dy \tag{10.5}$$

where 'modified duration' is defined as $MD = D/(1 + y)$. Equation (10.5) is a little more involved than our original formula, Equation (10.1) because of the inclusion of the term $(1 + y)$ in the denominator. Also note that in using this equation we have to express y as a proportion – so a yield of 5% would appear in this formula as $y = 0.05$ and an increase in the yield of 1% (over 1 week, say) would imply $dy = 0.01$. If $D = 5$, then the *proportionate* change in price is $dP/P = -5\ (0.01)/(1.05) = -0.0476$, that is, a price fall of 4.76%. D (when calculated using compound yields as above) is known as *Macaulay duration*. Calculation of duration for a coupon bond is illustrated in Example 10.1, along with the approximate price change using the duration formula. Duration provides a good approximation to the change in price of a bond for parallel shifts in the yield curve and for small changes in yields (of up to 25 bp).

EXAMPLE 10.1

Calculation of Duration

Data: 5 years to maturity, 4.5% coupon (annual), $M = \$100$, YTM = 3.5%

Question: Calculate duration and the approximate change in the price of the bond if the YTM increases to 4% (over 1 week).

Answer: Current price

$$\begin{aligned} P &= \$4.50/(1.035) + \$4.50/(1.035)^2 + \ \dots\ + \$104.50/(1.035)^5 \\ &= \$4.35 + \$4.20 + \ \dots\ + \$87.99 = \$104.50 \end{aligned}$$

$PV(C1) = 4.35$, $PV(C2) = 4.20$, etc.

$$D = [\$4.35(1) + \$4.20(2) + \ \dots\ + \$87.99(5)]/\$104.52 = 4.598 \text{ years (4.6 years)}$$

Approximate change in price over 1 week:

$$dP/P = -[D/(1+y)]dy = -[4.598/(1.035)]\,(+0.005) = -0.0222 \text{ (or 2.22\%)}$$

Approximate new price (at YTM = 4%) = \$104.52(1 − 0.0222) = \$102.19

Sometimes a useful ‘shorthand’ used by bond traders is to refer to the *dollar duration (DD)* of a bond which is defined as:

$$dP = -(DD)dy \quad \text{hence} \quad DD = (MD)P \tag{10.6}$$

Knowing DD one can immediately calculate the (approximate) dollar change in value of the bond. Going one step further, for a 1 bp (0.01%) change in the YTM we have $dy = 0.0001$ $(= 1/10,000)$, hence:

$$dP(\text{for 1 bp}) = PVBP = \frac{MD \times P}{10{,}000} \tag{10.7}$$

The expression in (10.7) is known as *'price value of a basis point' (PVBP)* or the 'duration value of a basis point', usually denoted DV01.

The above equations can be applied to a single bond using that bond's duration D or to a portfolio of bonds using the *portfolio duration*, D_p. For example, to illustrate the use of PVBP, consider a trader who has a *portfolio* of bonds worth $V_p = \$1\text{m}$ and the portfolio 'modified duration' $MD_p = 5$, then PVBP for the portfolio is \$500. Suppose there is now a 1 bp change in the yield – for example a change from YTM = 5% to YTM = 5.01% – then the change in value of the bond portfolio is $dV_p = \$500$.

10.2.2 Convexity

Duration only provides a (first order) approximation to the change in price of a bond and hence is only accurate for small changes in yields (e.g. up to 25 bps). A more accurate approximation is found by incorporating the convexity of the bond. Convexity χ (with annual coupon payments) is defined as:

$$\chi = \frac{\sum_{t=1}^{N} t(t+1)CF_t/(1+y)^t}{P(1+y)^2} \tag{10.8}$$

Convexity measures the curvature of the price-yield relationship.

Unfortunately, there is really no intuition behind the convexity formula. It arises from the mathematics of approximating the non-linear relationship between the bond price and the YTM, using a second order Taylor series expansion for dP (see Appendix 10). However, once we have calculated the bond's convexity (in Excel say), our improved estimate for the change in the bond price is given by:

$$\frac{dP}{P} \approx -MD.(dy) + \frac{1}{2}\chi(dy)^2 \tag{10.9}$$

A high value for convexity is a desirable property in a bond since if you can find two bonds with the same duration, then the bond with the highest convexity (i.e. higher curvature in the price-yield relationship) will exhibit a larger rise in price when yields fall and a smaller fall in price when yields rise – compared with the low convexity bond. However, this 'advantage' will be reflected in the higher price you have to pay for the 'high-convexity' bond.

The price change calculated using duration and 'duration plus convexity' are both approximations to the actual 'true' price change. The actual (true) price change will differ from that given by (10.9) if either the change in yield is large or there is a non-parallel shift in the yield curve, as shown in Example 10.2.

EXAMPLE 10.2

Duration and Convexity

Data: A 5-year pure discount bond, face value \$1,000. The current 5-year YTM is $y = 0.10$ (10%)

Question: Calculate (i) duration, (ii) convexity, (iii) the approximate price change and (iv) the actual price change of the bond for a 2% (200 bp) increase in the yield (over 1 week).

Answer: Duration: $P = \$1{,}000/(1.1)^5 = \$620.92$. Duration of a pure discount bond equals its maturity, $D = 5$ years.

Convexity : $\chi = [5(6)\$1,000/(1.1)]^5/\$690.92(1.1)^2 = 22.28$

Price change, using duration : $dP/P = -5/1.1\ (0.02) = -0.0909$ (or $-$ 9.09%)

Price change, using duration and convexity :

$dP/P = -0.0909 + (1/2)\ 22.28\ (0.02)^2 = -0.0864$ (or 8.64%)

Actual price change (over 1 week):
Initial price (with YTM = 10%) = $\$1{,}000/(1.1)^5 = \$620.92$
Price (with YTM = 12%) = $\$1{,}000/(1.12)^5 = \$567.43$
Actual price change = -8.615%

Using 'duration and convexity' provides a closer approximation (8.64%) to the actual change in price (8.615%) than using only duration (9.09%).

Excel

The calculation of duration and convexity and the 'true' change in the bond price can be easily calculated using Excel and an example can be found on the website.

10.3 SUMMARY

- The *spot rate* is the rate of interest which applies to money borrowed today and paid back at a single point in the future (with no interim payments).
- The *yield curve* is a relationship between interest rates for different maturities (taken at a specific time e.g. 10 a.m. Monday, 20 February).

- Duration D (or 'modified duration, MD) and convexity χ can be used to provide an *approximation t*o the change in price (value) of a bond (portfolio), for a given change in the yield to maturity (here assumed to be a compound rate).

$$\begin{aligned} \%\ \textit{change in bond price} &= -MD \times \textit{absolute change in yield},\ (dy) \\ \%\ \textit{change in bond price} &= -(MD)dy + (\chi/2)(dy)^2 \end{aligned}$$

APPENDIX 10: DURATION AND CONVEXITY

The price of a coupon paying bond, with annual coupon payments is a non-linear (convex) function of the yield to maturity, y (compound yield):

$$P = \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \ldots + \frac{C}{(1+y)^n} + \frac{M}{(1+y)^n} \tag{10.A.1}$$

The change in P for any non-linear function $P = f(y)$ can be approximated by a Taylor series expansion:

$$dP = \frac{\partial P}{\partial y}dy + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(dy)^2 + \text{higher order term in } dy \tag{10.A.2}$$

Differentiating (10.A.1) with respect to y gives:

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{-C}{(1+y)^2} - \frac{2C}{(1+y)^3} - \ldots - \frac{nC}{(1+y)^{n+1}} - \frac{nM}{(1+y)^{n+1}} \\ &= \frac{-1}{(1+y)}\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n}\right] \end{aligned} \tag{10.A.3}$$

Using the first term in the Taylor series $dP = (\partial P/\partial y)dy$ and (10.A.3) we obtain:

$$\frac{dP}{P} \approx -\frac{1}{(1+y)}\left(\frac{1}{P}\right)\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n}\right]dy \tag{10.A.4}$$

Equation (10.A.3) can be written as:

$$\frac{dP}{P} \approx \frac{-D}{(1+y)}\,dy \text{ or } \frac{dP}{P} = -MD.(dy) \tag{10.A.5}$$

where we define duration D and 'modified duration' MD as:

$$D = \frac{1}{P}\left[\frac{(1)C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+M)}{(1+y)^n}\right] \tag{10.A.6a}$$

$$MD = D/(1+y) \tag{10.A.6b}$$

Duration and modified duration for a coupon paying bond are calculated using (10.A.6a) and (10.A.6b) where the inputs are the current YTM (for an n-period bond), current market price of the bond, the coupons and maturity value of the bond. Having obtained the value for D (or MD) Equation (10.A.5) provides a first order approximation for the proportionate change in the bond price (for small changes in the YTM and for parallel shifts in the yield curve).

To calculate the change in the bond price using the second order Taylor series expansion (10.A.2) we differentiate (10.A.3) again with respect to y:

$$\frac{\partial^2 P}{\partial y^2} = \left[\frac{(1)(2)C}{(1+y)} + \frac{(2)(3)C}{(1+y)^2} + \ldots + \frac{n(n+1)C}{(1+y)^n} + \frac{n(n+1)M}{(1+y)^n}\right]\frac{1}{(1+y)^2} \tag{10.A.7}$$

Using (10.A.2), (10.A.4) and (10.A.7) we obtain a second order approximation to the change in the bond price:

$$\frac{dP}{P} \approx -MD.(dy) + \frac{1}{2}\chi.(dy)^2 \tag{10.A.8}$$

where we define the 'convexity' χ of the bond as Equation (10.A.7) divided by P:

$$\chi = \frac{1}{(1+y)^2}\left[\sum_{i=1}^{n} \frac{i(i+1)C}{(1+y)^i} + \frac{n(n+1)M}{(1+y)^n}\right]\frac{1}{P} \tag{10.A.9}$$

The convexity of a coupon paying bond at any point in time can be calculated from (10.A.9). The modified duration and convexity are then used in (10.A.8) to calculate the approximate change in the bond price – over a small interval of time (i.e. dy is small) – and for a parallel shift in the yield curve.

Zero-coupon Bond

It is straightforward to show that the duration of a zero-coupon bond equals its time to maturity, n. We have:

$$P = \frac{M}{(1+y)^n} \tag{10.A.10}$$

$$\frac{dP}{P} = -\frac{1}{P}\left[\frac{nM}{(1+y)^{n+1}}\right]dy \tag{10.A.11}$$

Substituting for P from (10.A.10) in (10.A.11):

$$\frac{dP}{P} = \frac{-n}{(1+y)}dy \tag{10.A.12}$$

Duration is *defined* by the following equation:

$$\frac{dP}{P} = \frac{-D}{(1+y)}dy \tag{10.A.13}$$

Hence, comparing (10.A.12) and (10.A.13) we see that the duration of a zero-coupon bond equals the time to maturity of the zero. Hence a zero-coupon bond with 5 years (left) to maturity has a duration equal to 5.

Continuously Compounded Yields

Repeating the above analysis using the continuously compounded YTM (y) to price a coupon paying bond we have:

$$P = C\,e^{-yt_1} + Ce^{-yt_2} + \ldots + C\,e^{-yt_n} + Me^{-yt_n} \tag{10.A.14}$$

where t_i = time of the cash flow in years (e.g. if the coupons are paid 6 months, 1 year, 1.5 years from today, then $t_1 = 0.5,\ t_2 = 1,\ t_3 = 1.5$ etc. Using (10.A.14) and a first order Taylor series for $dP \approx (\partial P/\partial y)\,dy$, it can be shown that:

$$\frac{dP}{P} \approx -D\,dy \tag{10.A.15}$$

where duration, using the continuously compounded YTM is defined as:

$$D = \frac{1}{P}[t_1C + t_2C + \ldots + t_nC + t_nM] \tag{10.A.16}$$

EXERCISES

Question 1
What is the (spot) yield curve and why is it useful?

Question 2
If all yields are expected to fall by 1% over the next week, would you hold low duration or high duration bonds?

Question 3
What is meant by the convexity of a bond? Why might you be willing to pay more for bond-A which has a greater convexity than bond-B (ceteris paribus)?

Question 4
Consider a 5-year, 10% coupon bond (annual coupons) with par value \$100, yield to maturity $y = 10\%$ p.a. (compound rate). Calculate the current market price, P and the (Macaulay) duration, D.

Question 5

Consider a 5-year, 10% coupon bond (annual coupons) with par value \$100 and yield to maturity, $y = 10\%$ p.a. (compound rate). The current market price, $P = \$100$ and the (Macaulay) duration $D = 4.1679$. Calculate:

(a) The (approximate) price change if the yield to maturity falls to 9.5%.

(b) The 'true' price change if y falls to 9.5%.

Question 6

Portfolio A: 1-year zero-coupon bond, face value = \$2,000 and
10-year zero-coupon bond, face value = \$6,000

Portfolio B: 5.95-year zero-coupon bond, face value = \$5,000

Current yield curve is flat and $y = 10\%$ p.a. (continuously compounded)

(a) Show that the duration of portfolio-A equals that of portfolio-B.

(b) What is the actual percentage change in value of portfolio-A for an increase in yield of 10 bps?

(c) Does the duration formula give approximately the same answer?

(d) Repeat (b) for portfolios A and B for an increase in yield of 5% p.a. Which portfolio has the higher convexity?

PART III

FIXED INCOME FUTURES CONTRACTS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 11

Interest Rate Futures

Aims

- To discuss contract specifications, settlement procedures and price quotes for futures contracts on *3-month Sterling deposits, 3-month Eurodollar deposits and US T-bills*.
- To show how interest rate futures contracts are priced.
- To examine arbitrage strategies using the *implied repo rate* on T-bill futures.
- To examine speculation and spread trades using interest rate futures.

Interest rate futures (IRF) became of increasing importance in the late 1970s and early 1980s when the volatility of interest rates increased dramatically. This was because of high inflation and consequent attempts by Central Banks to control the money supply and exchange rates by altering interest rates. Corporates have bank loans and bank deposits based on floating (LIBOR) interest rates and many financial institutions hold short-term fixed income assets (e.g. T-bills, Commercial bills). Interest rate futures contracts are used to hedge risks due to changes in interest rates (yields) which affect the market value of 'interest sensitive' cash market assets held by corporates, mutual funds, hedge funds and pension funds.

In the US, Eurodollar futures (CME/IMM) are one of the most actively traded contracts – much more so than EuroYen and US T-bill futures. The contract specification for 3-month Sterling futures (on Euronext), Eurodollar futures (CME/IMM) and US T-bill futures are given in Table 11.1.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

TABLE 11.1 Contract specifications

	Sterling 3-month (NYSE-Euronext, London)	90 day Eurodollar Futures (CME-Group, Chicago)	US T-bill futures
Contract size	£500,000	\$1m	\$1m
Delivery months	March/ June/ September/ December	March/ June/ September/ December	March/ June/ September/ December
Quotation	Futures Price F = (100 – forward rate)	IMM index = 100 – futures discount rate	IMM index = 100 – futures discount rate
Tick size (value)	0.01 (£12.50)	1 bp (\$25)	1 bp (\$25)
Settlement	Cash settled	Cash settled	Delivery of 90-92 day T-bill

Notes: Margin requirements are set by the exchange. They are higher for speculators than for hedgers and spread traders.
bp = basis point.

11.1 THREE-MONTH EURODOLLAR FUTURES CONTRACT

The underlying asset in the 3-month Eurodollar futures contract is a deposit which pays the (90-day) LIBOR (Eurodollar) rate.[1] The 'contract size' is for 'notional delivery' of a \$1m, 90-day deposit. Because Eurodollar deposits are non-transferable, the futures contract is settled in cash based on the contract size of \$1m.

Contracts are available which mature (towards the end of) March, June, September, and December of each year. These maturity dates extend out for over 10 years – because Eurodollar futures are widely used to hedge either the floating interest rate cash flows on long-term bank loans or deposits and also to hedge the net positions of interest rate swap dealers.

Suppose on 15 April ($t = 0$) we buy the June-Eurodollar futures contract, which matures on 28 June (Figure 11.1). The quoted 'price' on 15 April is not really a 'dollar price' but is an index known as the *IMM-(June) index* – it is linked to the forward *discount rate* $d_f = 8\%$ p.a., which applies to the 90-day period between 28 June and 26 September. The forward discount rate is measured on an 'actual/360' day count convention. The relation between the IMM-index and the forward discount rate is:

$$IMM_0 = 100 - d_f = 92 \tag{11.1}$$

The next issue to consider is the tick size and tick value. The 'tick size' for the Eurodollar future contract is 1 basis point (bp; i.e. 0.01% p.a.). If the discount rate changes by 1 basis

[1] Note that there are also Eurodollar futures contract on 30-day LIBOR.

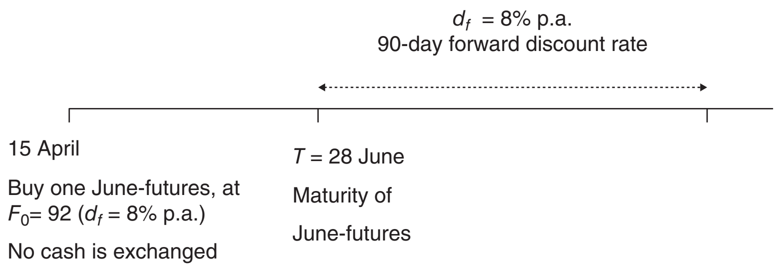

FIGURE 11.1 June Eurodollar futures contract

point, so does the IMM-index (see Equation 11.1), and the dollar value of one (90-day) futures contract with a contract size of \$1m changes by \$25 ('the tick size'):

$$\text{Tick value} = \$1\text{m} \times (0.01/100) \times (90/360) = \$25$$

The reason the IMM-Index appears in the *Wall Street Journal* and on trading screens is that it allows dealers in Eurodollar futures to quickly assess their gains and losses by using the tick size and tick value.

A 1 bp change in the IMM index (or the futures discount rate d_f) corresponds to a \$25 change in the value of one Eurodollar futures contract.

The dollar payoff is linear and hence independent of the initial futures discount rate – a change from 3% to 3.01% or from 6% to 6.01% will both lead to a change in the value of one Eurodollar futures contract of \$25. Similarly, a change in the futures discount rate of 5 bps implies a change in the value of one Eurodollar futures contract of \$125. Also, note that a change of 1 bp (over 1 day say) in the IMM-index for Eurodollar futures contracts of any maturity (e.g. the June-contract, September-contract, etc.) will result in the same \$25 change (over 1 day) in the value of both the June-contract and the September-contract.

Although dealers mainly use the IMM-index quote (e.g. IMM = 92) when analysing Eurodollar futures, there is also (somewhat confusingly) a genuine 'futures price' (in dollars) – which is useful when linking the futures price to the futures yield (see below). Given the IMM-index the Eurodollar futures price is given by:

$$\begin{aligned} F_0 &= 100 - (100 - IMM_0)\,(90/360) \\ &= 100[1 - (d_f/100)(90/360)] = \$98 \text{ (per \$100 face value in the contract)}^2 \end{aligned} \tag{11.2}$$

When discussing hedging using the Eurodollar futures contract we can either calculate the gains and losses on the futures position using the change in the IMM-index, (with each

[2] A similar expression to (11.1) applies for the price P of a cash-market T-bill: $P = 100[1 - (d/100)(90/360)]$, where d is the 90-day *spot* discount rate.

1 bp change being worth $25) or we can calculate the gains and losses using the futures price F (see below) – both will give the same answer.

The contract size for the Eurodollar futures is for a 'notional' delivery of a $1m, 90-day deposit and therefore the 'value of one futures contract' (or 'invoice price' or 'contract price')[3] is:

$$V_F = \$1\text{m}\,(F_0/100) = \$1\text{m}\ (\$98/\$100) = \$980{,}000 \tag{11.3}$$

11.2 STERLING 3-MONTH FUTURES CONTRACT

This futures contract is written on a sterling 3-month deposit with a notional value of £500,000 per contract. The futures price F_0 is:

$$F_0 = (100 - \text{quoted forward interest rate, \% p.a.}) = 100 - f_{12} \tag{11.4}$$

A change in the *annual* forward rate of 1 bp (i.e. 0.01 of 1% p.a.) gives a 1bp change in the futures price (Equation 11.4). But a change of 1 bp over a year is equivalent to a change of 0.25 bps over 3 months, which with a contract size of £500,000 amounts to a change in value of one sterling futures contract of £12.50 (= (0.25/100) × £500,000).

Tick size is 1 bp (0.01% p.a.) which gives a tick value of £12.50 per contract.

11.3 T-BILL FUTURES

Suppose it is 1 July and Ms Long buys the September T-bill futures contract (Figure 11.2). This contract allows delivery of T-bills at maturity of the futures contract, which we assume is 25 September. The contract allows delivery of T-bills each with a face value of $100, which on 25 September have a further 90, 91 or 92 days to maturity. However, the convention is that the *quoted IMM-index* is based on delivery of a 90-day bill. If a 90-day T-bill is delivered on 25 September it can be redeemed (at the Central Bank, 'the Fed') for $100 on 24 December. The contract size is for delivery of $1m of 90-day T-bills.

T-bill futures contracts have maturity dates out to about 2 years. The contract expiration months are March, June, September, and December, with the last trading day being the business day prior to the date of issue of cash market (spot) T-bills, in the 3rd week of the month. So on 15 April there will be June, September, December contracts available, all of which deliver in these expiry months a T-bill with a further 90 days to maturity.

[3]As noted earlier the 'invoice price' is a notional amount, as no cash changes hands at inception of the futures contract (ignoring margin payments). That is why we prefer using 'the value of one futures contract' rather than 'invoice price' or 'contract price' – as the latter terms suggest that money changes hands at inception of the futures contract.

Delivery is allowed on the next business day after the last trading day and any business day thereafter during the expiration month. It is the uncertainty in these variable delivery arrangements that often lead to the contracts being closed out before expiration. Also, the delivery date on the futures is often timed to coincide with the date on which US 365-day T-bills have between 90 and 92 days left to maturity as this facilitates a liquid spot market for delivery of T-bills.

The 90-day T-bill futures contract has an IMM-index quote, discount rate d_f and futures price F, given above, by Equations (11.1), (11.2), and (11.3) for the Eurodollar futures.[4] So the tick value for the T-bill futures contract is \$25. These two futures contracts are therefore very similar, the main differences being that Eurodollar futures are available with very long maturity dates and the T-bill futures allows delivery, whereas the Eurodollar futures is always cash settled at maturity. Of course you can close out either of the contracts at any time (up to the maturity date).

11.4 FUTURES PRICE AND FORWARD RATES

We examine the negative relationship between the forward (interest) rate and the futures price for a T-bill futures contract. Also we see that if a T-bill futures contract is purchased today and held to maturity, then today you 'lock in' a known forward rate (Figure 11.2).

On 1 July you purchase one September T-bill futures for $F_0 = \$98$ (and no cash changes hands). You hold the contract to maturity on 25 September and take delivery of one \$100 face value T-bill, which matures 90 days later on 24 December. On 1 July what is the interest rate you have 'locked in' between 25 September and 24 December?

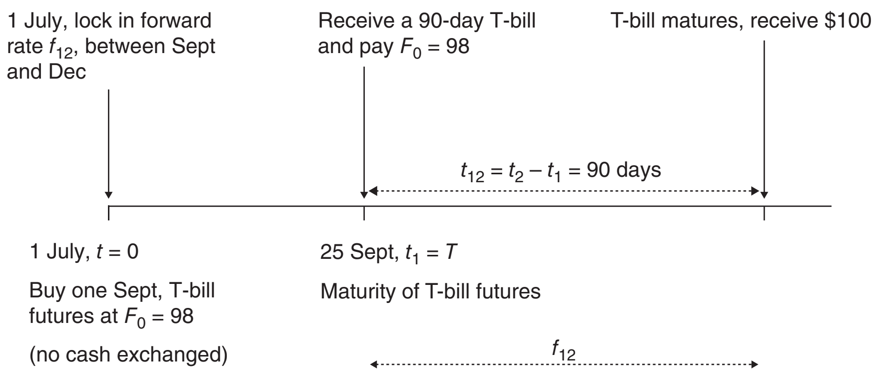

FIGURE 11.2 September T-bill futures contract

[4]Although 90-day T-bills are usually delivered, the contract also allows delivery of 91 or 92-day T-bills and if these are delivered at maturity then 91 or 92 is used in place of '90' in the formula for F, the amount of dollars payable (per \$100 of face value T-bills).

On 24 December your 90 day T-bill matures, you take it to the US Federal Reserve (i.e. the Central Bank) and receive \$100. Hence, on 1 July, you know that the 90-day forward rate you will earn between 25 September and 24 December is:

$$f_{12} = [(100/98) - 1]\,(365/90) = [(M/F_0) - 1]\,(365/90) = 8.28\ \%\ \text{p.a. (simple rate)}^5$$

On 1 July you know that if you hold the futures to maturity and then hold the T-bill delivered in the futures contract for a further 90 days, that you will earn a forward interest rate of 8.28% p.a. (simple rate), between 25 September and 24 December. On 1 July you have therefore 'locked in' this (forward) interest rate. Later we see that you can lock in this forward rate even if you close out the future contract before its maturity date of 25 September – although there will be some residual basis risk. Rearranging the above equation, the September-futures price on 1 July is:

$$F^{Sept} = \frac{\$100}{[1 + f_{12}(90/365)]} \qquad \text{(simple rate)} \tag{11.5a}$$

The equivalent formulas using compound and continuously compounded rates[6] are:

$$F^{Sept} = \frac{\$100}{(1 + f_{12})^{90/365}} \qquad \text{(compound rate)} \tag{11.5b}$$

$$F^{Sept} = \$100e^{-f_{12}t_{12}} \qquad (f_{12} = \text{continuously compounded rate}, t_{12} = 90/365) \tag{11.5c}$$

According to Equations (11.5a)–(11.5c), as the 90-day forward rate rises (or falls) over time (i.e. between 1 July and 25 September), the futures price will fall (or rise). As can be seen from all of the above (alternative) equations for F, all interest rate futures contracts have the feature that the futures price F and the 90-day forward rate f_{12} move in opposite directions.

Note, however, that the forward rate applicable to each contract is different. On 1 July, the September-futures price is determined by the 90-day forward rate that applies from 25 September to 24 December. On 1 July the December-futures price, is determined by the 90-day forward rate that applies from the maturity date of the December-futures (say 24 December) over the next 90 days to 25 March (of the next year). Both are 90-day forward rates but they apply over different time periods.

11.5 PRICING INTEREST RATE FUTURES

Consider an interest rate futures contract which matures at $t = 1$ year and delivers a 1-year cash market T-bill which has a face value at $t = 2$, of \$100. The futures contract 'locks in'

[5]We assume that the day-count convention for scaling up to an annual rate is '365/actual' and we have used 'simple interest' – rather than compound or continuously compounded rates – to gross up to an annual rate.

[6]The three formulas (11.5a)–(11.5c) all give the same value for the futures price (see Appendix 11).

the current quoted forward rate f_{12} (simple interest, which applies between $t = 1$ and $t = 2$). We can show that the no-arbitrage price of the futures contract is:

$$F_0 = \frac{\$100}{(1 + f_{12}t_{12})} \tag{11.6}$$

All Equation (11.6) says is that you would be prepared to enter into a contract today (at $t = 0$) in which you agree to pay F_0 at time $t = 1$, if you have a guaranteed payment of \$100 at $t = 2$. We can show that this is indeed the case by constructing an arbitrage portfolio consisting of a 'synthetic' or 'replication' futures contract – that is, a portfolio that has the same payoff as the futures contract, itself. The idea is that at $t = 0$, a 1-year cash-market T-bill plus a futures contract which matures in one year (and delivers a 1-year T-bill) is equivalent to holding a 2-year cash-market T-bill (at time $t = 0$).

Today, if you invest \$1 over 2 years with Citibank at a (simple) interest rate r_2 this is worth $\$1(1 + r_2t_2)$ in 2 years' time. Alternatively, today you can invest \$1 with Barclays at the 1-year spot rate r_1, which will be worth $\$1(1 + r_1t_1)$ at the end of year-1. Today, you can also agree a *forward* deposit of $\$1(1 + r_1t_1)$ with Bank of America (BoA) to begin in 1 year's time and to last for a further year. The \$1 investment in Barclays together with the forward agreement with Bank of America will accrue to $\$(1 + r_1t_1)(1 + f_{12}t_{12})$ in 2 years' time. The 2-year investment in Citibank and the two 1-year investments in Barclays and BoA must be worth the same in 2 years' time, otherwise arbitrage profits can be made (see Cuthbertson and Nitzsche 2008). Hence, the no-arbitrage forward rate is given by[7]:

$$\$1(1 + r_2t_2) = \$1(1 + r_1t_1)(1 + f_{12}t_{12}) \tag{11.7a}$$

$$f_{12} = \left(\frac{t_2}{t_{12}}r_2 - \frac{t_1}{t_{12}}r_1\right)\frac{1}{(1 + r_1t_1)} \approx \left(\frac{t_2}{t_{12}}r_2 - \frac{t_1}{t_{12}}r_1\right) \tag{11.7b}$$

Figure 11.3 provides the basic steps to determine the (no-arbitrage or 'fair' or 'correct') futures price. Note that we use (simple) yields (not discount rates) throughout – and a day-count convention of 'actual/365'.

m_i = number of days to delivery (maturity) date of futures contract
m_{12} = number of days to maturity of the T-Bill underlying the futures contract
$m_2 = m_1 + m_{12}$
$t_1 = (m_1/365)$, $t_2 = (m_2/365)$ and $t_{12} = (m_{12}/365)$
r_1 = spot rate over period $t = 0$ to $t = 1$, r_2 = spot rate over period $t = 0$ to $t = 2$
f_{12} = forward rate over period $t = 1$ to $t = 2$
\$100 = maturity/face value of (cash market) T-bills delivered in the futures contract

[7] If we use continuously compounded rates then $\$1\, e^{r_2t_2} = \$1\, e^{r_1t_1}\, e^{f_{12}t_{12}}$ and the continuously compounded forward rate is $f_{12} = (1/t_{12})(t_2r_2 - t_1r_1)$, and this relationship is exact.

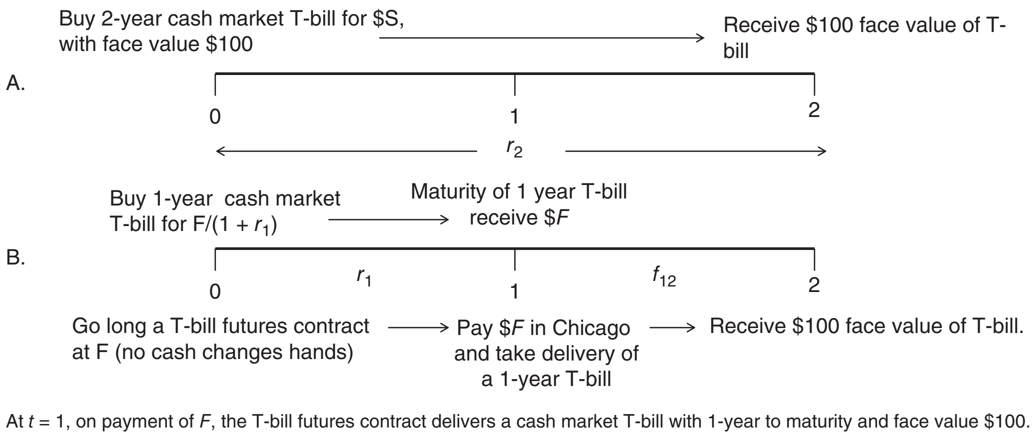

FIGURE 11.3 Pricing T-bill futures

We form two portfolios A and B which have the same payout of $100 after 2 years. Hence for zero arbitrage profits to be made, the two portfolios must be worth the same today.

Portfolio-A ($t = 0$): Purchase a $t_2 = 2$-year cash market T-bill with maturity value $100.
Today, the quoted spot price of the 2-year T-bill is:

$$S = \frac{100}{(1 + r_2 t_2)} = \frac{100}{(1 + r_1 t_1)(1 + f_{12} t_{12})} \tag{11.8}$$

Suppose today a *futures contract* with maturity of 1 year and current quoted futures price of F is available. The futures contract delivers a $100 face value T-Bill in $t_1 = 1$ year's time and the T-bill delivered has a further $t_{12} = 1$ years to maturity. Now consider the replication portfolio-B:

Portfolio-B ($t = 0$)

(i) buy a 1-year cash-market T-bill with maturity value F and

(ii) go long a 1-year T-bill futures contract with quoted price F (payable at $t = 1$), which delivers a cash-market 1-year T-bill at $t = 1$, with face value of $100 at $t = 2$.

Portfolio-B(i) pays out $$F$ at $t = 1$ when the 1-year cash-market T-bill matures. This $$F$ is used at $t = 1$ to pay for delivery of the T-bill in the futures contract. Hence, there is no *net* cash flow at $t = 1$.

At $t = 2$, the T-bill delivered (1 year ago) in the futures contract matures and receipts at $t = 2$ from portfolio-B are \$100. The futures contract cost zero at $t = 0$ (as no money changes hands) and hence the cost of setting up portfolio-B is simply the price of a cash market 1-year T-bill with maturity value $\$F$.

$$\text{Price of cash market 1-year T} - \text{bill at}\, t = 0 = \frac{\$F}{(1 + r_1 t_1)} \tag{11.9}$$

Portfolio-A (the 2-year T-bill), also gives \$100 at $t = 2$. Hence both portfolio-A and portfolio-B have a cash flow of \$100 at $t = 2$, with no intermediate net cash flows – therefore they must have the same value at $t = 0$, otherwise risk-free arbitrage profits can be made:

$$S = \frac{100}{(1 + r_1 t_1)(1 + f_{12} t_{12})} = \frac{F}{(1 + r_1 t_1)} \tag{11.10}$$

Therefore, the no-arbitrage (or fair or correct) futures price (at $t = 0$) is:

$$F = \frac{100}{(1 + f_{12} t_{12})} \tag{11.11}$$

11.6 ARBITRAGE: IMPLIED REPO RATE

As we have seen, cash-and-carry arbitrage is possible between the T-bill futures market and the T-bill cash market – this is the process by which the T-bill futures is priced. Potential arbitrage opportunities can also be analysed in terms of the implied repo rate.

Suppose today (i.e. $t = 0$), you can purchase a T-bill in the spot market with a remaining maturity of 32 days. And if you also go long a T-bill futures contract with maturity $T = 32$ days, at a quoted price F_0, which delivers a 90-day (\$100 face value) T-bill, you have effectively purchased the equivalent of a 122-day cash market T-bill. Put another way you have created a 'synthetic' or 'replication' 122-day T-bill.

If the 32-day and the 122-day T-bills and the futures contract are not correctly priced then a risk-free arbitrage profit can be made by trading these three assets. This arbitrage opportunity can be expressed in terms of the *implied repo rate*, (see Chapter 3) which we now examine.

Suppose at $t = 0$ we purchase the spot asset for S_0 *using our own funds*. At T, we deliver the spot asset when the short futures contract matures and receive a cash amount $F_{0,T}$. The gross (compound) annual return (excluding any interest cost of finance) is known as the *implied repo rate*:

$$\hat{r} = (F_{0,T}/S_0)^{1/T} - 1 \tag{11.12}$$

When the implied repo rate $\hat{r}$ exceeds the cost of finance (i.e. repo rate) r, then arbitrage profits can be made by shorting (selling) the futures and purchasing the spot asset with funds borrowed at the actual repo rate.[8]

An example of a profitable cash-and-carry arbitrage transaction by shorting T-bill futures is given in Example 11.1. We use the (cash market) T-bill day-count convention of 'actual/360' together with the quoted *discount rates* to obtain the cash market price. The underlying asset for delivery in the futures contract is a 90-day T-bill (with face/maturity/par value of \$100). The T-bill futures contract matures in 32 days. Financing the purchase of a cash market T-bill is via a repo transaction (with a bank), where the actual 32-day repo rate (for borrowing cash) is 6% p.a. (compound rate).

EXAMPLE 11.1

Implied Repo Rate and Arbitrage

Today: The 32-day repo rate is 6% p.a. (compound rate)
US T-bill with 122-days to maturity, discount rate $d = 6\%$ (days/360)
Price of T-bill $S_0 = \$97.97$ ($= 100 - 6[122/360]$). Face value = \$100.
T-bill futures, 32 days to maturity, IMM-index = 94.2 (discount rate $d_f = 5.8\%$)
Contract size T-bill futures = \$1m (per \$100 face value T-bills).

Question: (i) Calculate the futures price and the implied repo rate.
(ii) Show how arbitrage profits can be made.

Answer: Price of 122-day T-bill : $S_0 = \$97.97$
Futures price : $F_{0,T} = 100 - 5.8(90/360) = \$98.55$ (delivery of 90-day T-bill)
Implied Repo Rate : $\hat{r} = (F_{0,T}/S_0)^{365/32} - 1 = 0.0696$ (6.96%, compound rate)

Arbitrage: The implied repo rate $\hat{r} = 6.96\%$ exceeds the actual repo rate (cost of borrowing) r = 6%. So cash and carry arbitrage is profitable, namely:

- At $t = 0$: Buy 122-day T-bill for \$97.97, financed by borrowing at the repo rate of 6% p.a. over 32 days and sell one futures contract.
- At $t = T$: Use the 122-day T-bill for delivery against the short futures position in 32 days' time (when it will be a 90-day T-bill) and receive $F_{0,T} = \$98.55$. The amount owed on \$97.97 borrowed at actual repo rate is $\$97.97(1.06)^{32/365} = \$98.47$

[8]The actual repo rate is the rate at which you borrow money from 'the market' (e.g. your prime broker – often a bank). Broadly speaking, this involves providing the broker with collateral (e.g. T-bills) at market value and agreeing to buy back the collateral at a fixed date in the future at an agreed higher price. The (percentage) difference in price is the cost of borrowing or 'repo rate' – for more information see Cuthbertson and Nitzsche (2008).

$$\text{Arbitrage profit} = (\$98.55 - \$98.47) \approx 0.08$$

$$\begin{aligned} \text{Receipt from one futures contract} = (98.55/100) \times \$1\text{m} &= \$985{,}500 \\ \text{Cost repo transaction} = (97.97/100)(\$1\text{m})(1.06)^{32/365} &= \$984{,}717 \\ \text{Arbitrage profit} &= \$782 \end{aligned}$$

The time to maturity of the futures contract is 32 days, at which time a 90-day T-bill is delivered. Thus, for the arbitrage transaction to be risk-free, you need to purchase a 122-day T-bill today, in order to be able to deliver a 90-day T-bill when the futures contract matures. The *implied* repo rate on the cash and carry arbitrage is $\hat{r} = 6.96\%$ which exceeds the actual repo rate $r = 6\%$. Hence selling (one) T-bill futures and financing the purchase of the 122-day cash market T-bills (at the 'low' repo rate of $r = 6\%$) yields positive arbitrage profits of $782.

In practice, financing cash-and-carry arbitrage using actual repos may involve some risk because the repo market is illiquid at 'long' horizons. Indeed, much of the liquidity is in overnight repos. Hence the arbitrageur may have to *roll over* her overnight repo financing until the futures contract reaches maturity (when all positions can then be closed out). But any 'new repos' that are rolled over may be more or less expensive depending on the future course of short-term repo rates – that is, there is 'roll-over risk'.

11.7 SPECULATION

Directional bets on interest rate changes using futures is relatively straightforward. Compared to speculation by purchasing the underlying asset (e.g. spot T-bills), the futures position provides leverage, since you do not pay the futures invoice price at inception. You merely have to provide a relatively small 'good faith deposit' for your initial margin payment (which usually pays a competitive interest rate). Also, you may have to 'top up' your margin account and therefore you need cash or eligible collateral readily available.

For *any fixed income* futures contract, the key feature is the inverse relationship between interest rates and futures *prices*. If you close out a long futures position before maturity then the profit on each futures contract is $F_1 - F_0$. Hence, if you want to speculate on a future fall in interest rates, then today you buy (go long) an interest rate futures contract (e.g. a T-bill futures contract or a Eurodollar futures contract). If interest rates fall, the futures price rises and hence you make a profit from your long position when you close out by selling the futures contract at a higher price.

Today, if you forecast a rise in interest rates, you would sell ('short') interest rate futures contracts – if interest rates subsequently rise, you close out (i.e. buy back) your futures contracts at a lower price, making a profit on the futures position. Of course, a 'naked position' in a futures contract (either long or short) is highly risky, since futures prices can change rapidly in either direction as interest rates change.

11.8 SPREAD TRADES

An *'intracommodity long spread'* position consists of a long position in one *nearby* futures contract (i.e. with a short maturity date t_1) and a *short* position in the far contract (i.e. with a longer maturity date t_2). Both contracts are on the same underlying (e.g. either 90-day T-bill futures or 3-month Eurodollar futures). Clearly, this intracommodity long spread is less risky than an outright (naked) position in either all long or all short contracts (and for this reason the initial margin is usually about one-third of that for a naked (open) position).

As we shall see, if the yield curve undergoes a parallel shift (either up or down) there will be little or no change in value of a spread-futures position. Hence a spread position is hedged against parallel shifts in the yield curve and is a 'bet' placed solely on a change in the *shape* of the yield curve (e.g. a twist in the yield curve).

Spread positions can be used to speculate on changes in the shape of the yield curve.

To see how this works consider the formula for a 3-month sterling-futures contract (where we have substituted from Equation (11.7b) for the forward rate:

$$F = 100 - f_{12} \approx 100 - \left(\frac{t_2}{t_{12}} r_2 - \frac{t_1}{t_{12}} r_1\right) \tag{11.13}$$

Remember that the underlying asset in *all* the sterling contracts (with different maturity dates) is always a 3-month deposit, so t_{12}= 3 months is fixed. Suppose it is 27 December. Consider the June-sterling futures contract which *matures in 6 months* (line b, Table 11.2), and therefore has $t_1 = 6$ months and $t_2 = 9$ months ($t_{12} = 9 - 6 = 3$ months). Here, the 'short-rate' r_1 is the 6-month interest rate and the 'long-rate' r_2 is the 9-month interest rate. From (11.13) we have $\partial F/\partial r_1 = t_1/t_{12} = (6/3) = +2$ and $\partial F/\partial r_2 = -t_2/t_{12} = -9/3 = -3$.

TABLE 11.2 Parallel shift: change in futures prices

Futures contract	3-month rate	6-month rate	9-month rate	Parallel shift $\Delta \mathbf{r}_1 = \Delta \mathbf{r}_2$
a) Long March-futures, ΔF^{March}	+1	+2	0	−1
b) Long June-futures, ΔF^{June}	0	+2	−3	−1

Notes: Today is 27 December. The March-futures matures in 3 months and the June-futures matures in 6 months – the underlying for each contract is a 90-day interest rate. Figures show the change in the March and June futures prices after a 100 bp (1% p.a.) increase in short and long spot interest rates.

The 'payoff' of {0, +2, −3} for the June-futures contract means that (a) the June-futures price will rise by +2 points if the '6-month rate' increases by 1% (over the next week, say) and (b) the June-futures price will fall by 3 points if the '9-month rate' increases by 1% (over the next week). The net result of a simultaneous 1% rise in both short (6-month) and long (9-month) rates (i.e. a parallel shift in the yield curve) is a change in the June-futures price of $\Delta F^{June} = +2 - 3 = -1$, over the next week (line b, last column in Table 11.2).

Now consider being *long* the March-futures which matures in 3 months (line a, Table 11.2). If you repeat the above (with $t_1 = 3$ months and $t_2 = 6$ months and $t_{12} = 6 - 3 = 3$ months) then the effect of a 1% rise in both rates on the change in March-futures price is also $\Delta F^{March} = -1$.

Hence if you are *long* the March-futures and *short* the June-futures (Table 11.2), the net effect of a parallel shift in the yield curve on this 'long futures spread' = −1 + 1 = 0. The long-short spread position is hedged against parallel shifts in the yield curve. We now wish to demonstrate that:

If you hold a 'long-spread' then you gain when the yield curve becomes steeper.

The above conclusion holds regardless of whether the yield curve, as a whole, moves up or down – as long as it becomes steeper. Consider the following 'long spread' on futures on 27 December when you are:

- Long, the March-futures contract (nearby contract)
- Short, the June-futures contract (far contract)

The change in value of this 'long-spread' futures position depends on the change in the 3, 6 and 9 month *spot* rates. The long March-futures contract has a {+1, −2, 0} payoff and the *short* June-futures contract has a {0, −2, +3} payoff. The net exposure in the spread is given in Table 11.3.

TABLE 11.3 Long Eurodollar futures spread

Futures contract	3-month spot rate	6-month spot rate	9-month spot rate
Long March-futures, ΔF^{March}	+1	−2	−0
and Short June-futures, ΔF^{June}	−0	−2	+3
= (Net) Long spread	**+1**	**−4**	**+3**

Notes: Today is 27 December. The March-futures matures in 3 months and the June-futures matures in 6 months – the underlying for each futures contract is a 90-day interest rate. Figures show the change in the March and June futures prices after a 100 bp (1% p.a.) increase in short and long spot rates – which causes changes in the respective forward rates, which then determine the futures prices (see Equation 11.13).

Let us see what happens when there is a steepening in the yield curve – that is, the 9-month spot rate moves up more than the 6-month rate, which in turn increases more than the 3-month rate. For example, $\Delta r_{9m} = +3$, $\Delta r_{6m} = +2$, $\Delta r_{3m} = +1$, so:

$$\Delta r_{9m} > \Delta r_{6m} > \Delta r_{3m}$$

hence $\Delta r_{9m} - \Delta r_{6m} \geq \Delta r_{6m} - \Delta r_{3m}$.

From the long-spread position in Table 11.3, the net effect is:

Change in value of 'long spread':

$$= (+1)\Delta r_{3\text{m}} - 4\Delta r_{6\text{m}} + 3\Delta r_{9\text{m}} = 3(\Delta r_{9\text{m}} - \Delta r_{6\text{m}}) - (\Delta r_{6\text{m}} - \Delta r_{3\text{m}}) > 0$$
$$= 3(+1)–(+1) = +2 \tag{11.14}$$

The above result shows that when the yield curve steepens the value of a long-spread position in Eurodollar futures increases. This also holds even if the Δr are all negative, as long as this involves the yield curve becoming steeper. The converse of the above also holds:

If you hold a 'short spread' (i.e. short March-futures, long June-futures,), you gain when the yield curve becomes less steep (i.e. flatter).

The change in the value of various futures positions, to changes in the position and shape of the yield curve are summarised in Table 11.4. It is clear that naked futures positions (i.e. either all long or all short interest rate futures) can be used to speculate on the general *direction* of changes in all interest rates (i.e. parallel shifts, either up or down). In contrast, spread positions are hedged against parallel shifts in the yield curve but are 'bets' on changes in the *slope* of the yield curve (e.g. 'twists').

TABLE 11.4 Change in futures prices

Scenario	Long futures	Short futures	Long spread	Short spread
All yields rise (upward parallel shift)	–	+	0	0
All yields fall (downward parallel shift)	+	–	0	0
Yield curve steeper	+ or –	+ or –	+	–
Yield curve less steep	+ or –	+ or –	–	+

Notes: The table shows the direction of change of futures prices for long and short positions and for long and short spread positions in interest rate futures contracts, to various movements in the spot yield curve.

11.9 SUMMARY

- Contract details and quotes for short-term interest rate futures on 3-month (90-day) Eurodollars and 3-month (90-day) US T-bills are very similar. The quoted *IMM-index* is linked to the forward *discount rate* d_f (quoted at $t = 0$):

$$IMM_0 = 100 - d_f$$

- A change in the IMM-index of 0.01% (i.e. 1 tick) leads to a change in value of one Eurodollar futures contract or one T-bill futures (both of which have a contract size of \$1m) of \$25 (= tick value).
- Today, the fair price F of a 90-day US T-bill futures contract or 90-day Eurodollar futures contract which matures at $t = 1$ is determined by riskless arbitrage and results in:

$$F = \$100/(1 + f_{12}t_{12})$$

  where f_{12} is today's quoted forward *yield* (simple rate) applicable between the maturity date of the futures t_1 (years) and $t_2 = t_1 + 90/365$, hence $t_{12} = 90/365$.
- If f_{12} is the continuously compounded yield then $F = \$100e^{-f_{12}t_{12}}$. Alternatively if f_{12} is a compound rate then $F = \$100/(1 + f_{12})^{t_{12}}$. Finally, given the IMM index quote then the (futures) discount rate $d_f = 100 - IMM$ and the futures price is $F = 100[1 - (d_f/100)(90/360)]$. All of the formulas give the same value for the futures price, when used with the appropriately measured forward rate.
- A person who speculates on the *direction* of interest rate changes would buy (sell) an interest rate futures contract today, if she forecasts interest rates to fall (rise) in the future.
- The *implied repo rate* (compounded) is the percentage return from buying the underlying asset and selling the futures contract $\hat{r} = (F_{0,T}/S_0)^{1/T} - 1$. The repo rate r is the actual cost of borrowing funds. When the implied repo rate $\hat{r}$ is not equal to the actual repo rate r (i.e. cost of borrowing or lending money), then risk-free arbitrage profits can be made.
- For example, if today $\hat{r} > r$ then arbitrage profits can be made by selling (shorting) an interest rate futures contract (which matures at $T = t_1$) and today borrowing cash at the repo rate to purchase a cash market T-bill with maturity at $t_2 = t_1 + 90/365$). At time $T = t_1$, the cash market T-bill has 90-days to maturity and can be delivered in the futures contract.

- *Spread trades* using interest rate futures, can be used to speculate on changes in the *slope of the yield curve* (e.g. a twist in the yield curve), while being hedged against parallel shifts in the yield curve.

APPENDIX 11.A: FUTURES PRICES AND INTEREST RATES

In this appendix we:

- Derive futures prices from spot yields and forward yields.
- Show how different conventions for calculating spot-yields and forward-yields give different formulas for the futures price, but all formulas result in the same futures price.
- Show how to calculate discount rates and the quoted IMM index from futures prices.

Using Compound Rates

It is 25 May. We want to calculate the T-bill futures prices and IMM-index quotes for the June and September interest rate futures contracts, using only the known term structure of spot yields r at $t = 0$. First, we calculate forward yields from spot yields and use these forward rates to calculate the 'fair' (no-arbitrage) futures prices F for the June and September futures. Finally we derive the IMM-indices from the futures prices. We use compound rates. Our results also apply to the June and September (90-day) Eurodollar futures.

Suppose it is 25 May and we have spot *yields* on cash-market T-bills with 32, 122, and 212 days to maturity (see Figure 11.A.1). Assume the day-count convention is 'actual/365'. Note that $122 - 32 = 90$ days and $212 - 122 = 90$ days. We assume the 32-day spot T-bill matures on exactly the same day that the June-futures contract matures (which delivers a

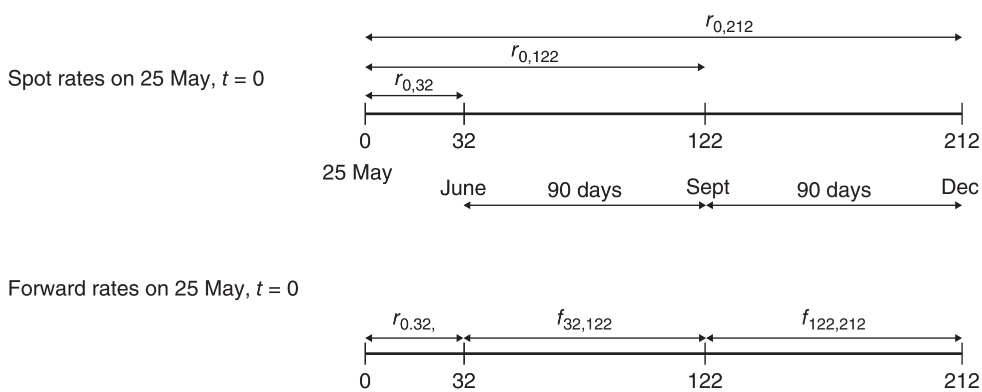

FIGURE 11.A.1 Spot and forward yields (compound rates)

90-day T-bill). Also, the 122-day cash market T-bill matures on exactly the same day that the September-futures contract matures.

Compound spot yields:

$$r_{0,32} = 0.09 \qquad r_{0,122} = 0.10 \qquad r_{0,212} = 0.12 \tag{11.A.1a}$$

$$t_{0,32} = 32/365 \qquad t_{0,122} = 122/365 \qquad t_{0,212} = 212/365 \tag{11.A.1b}$$

Using (compound) spot yields we can calculate the (compound) forward yields:

$$(1 + r_{0,32})^{32/365}(1 + f_{32,122})^{90/365} = (1 + r_{0,122})^{122/365} \tag{11.A.2a}$$

$$(1 + r_{0,122})^{122/365}(1 + f_{122,212})^{90/365} = (1 + r_{0,212})^{212/365} \tag{11.A.2b}$$

$$f_{32,122} = 0.1036 \qquad f_{122,212} = 0.1477$$

Note that because the spot yield curve is upward sloping the forward-rate curve lies above the spot-rate curve (e.g. $f_{32,122} = 0.1036 > r_{0,122} = 0.10$). Each T-bill futures contract delivers a 90-day T-bill. Hence the no-arbitrage futures prices quoted on 25 May are:

$$F_{32,122}(June\ \ Futures) = \frac{\$100}{(1 + f_{32,122})^{90/365}} = \$97.60 \ \ \text{(per \$100 nominal)} \tag{11.A.3a}$$

$$F_{122,212}(Sept\ \ futures) = \frac{\$100}{(1 + f_{122,212})^{90/365}} = \$96.66 \ \ \text{(per \$100 nominal)} \tag{11.A.3b}$$

IMM Futures Index

Now, we take the above futures prices F and calculate the *quoted* (futures) discount rate d_f and IMM-index using:

$$F = 100\left[1 - \left(\frac{d_f}{100}\right)\left(\frac{90}{360}\right)\right] \text{ and } IMM = 100 - d_f \tag{11.A.4}$$

Hence:

$$d_f(32{,}122) = 9.603\%\ \text{(June − futures)}\ \ \text{and}\ \ d_f(122212) = 13.358\%\ \text{(Sept − futures)}$$

$$IMM(32{,}122) = 90.396\ \text{(June − futures)}\ \ \text{and}\ \ IMM(122212) = 86.641\ \text{(Sept − futures)}$$

Note that when calculating IMM and d_f we use the US T-bill futures day-count convention of 'actual/360' and assume a 90-day bill is delivered in the futures contract. By using the no-arbitrage equations (11.A.2a) and (11.A.2b) for the forward rates, we ensure that there are no risk-free profits to be made when the futures prices are calculated in (11.A.3a) and (11.A.3b). For example, it is impossible on 25 May to make a risk-free arbitrage profit by buying (or selling) a combination of the 32-day (June), 212-day (September) cash market T-bills and the June-futures contract, priced at $F_{32,122}$(June-futures) = 97.60.

Above, we started with T-bill *spot yields* and calculated forward rates, futures prices and the IMM-index. Note that we could also reverse this line of reasoning. Given $r_{0,32}$ we could use the two *observed* forward prices $F_{32,122}(June\ \ Futures)$ = \$97.60 and $F_{122,212}(Sept\ \ futures)$ = \$96.66 to calculate $f_{32,122}$ and $f_{122,212}$ and then use these in Equations (11.A.2a) and (11.A.2b) to calculate the term structure of spot yields – namely $r_{0,122}$ and $r_{0,212}$. Here we are constructing the spot yield curve from observed futures prices.

Continuously Compounded and Simple Interest Rates

Assume a day-count convention of t = 'days/365'. \$1 invested over t years (or fraction of a year) must give the same terminal value, independently of what convention we use to measure yields. Hence, the relation between simple yields r^s, compound yields r and continuously compounded yields r^c is:

$$1 + r^s(days/365) = (1 + r)^{days/365} = e^{r^c(days/365)}$$

It follows that:

$$r^c = \ln(1 + r) \quad \text{and} \tag{11.A.5a}$$

$$r^s = (365/\text{days})(e^{r^c(days/365)} - 1) \tag{11.A.5b}$$

Continuously Compounded Rates

Using (11.A.5a), the equivalent continuously compounded spot rates to those in (11.A.1a) are:

$$r^c_{0,32} = 0.08618 \qquad r^c_{0,122} = 0.09531 \qquad r^c_{0,212} = 0.11333$$

Take logarithms of Equations (11.A.2a) and (11.A.2b) and using $r_i^c = \ln(1 + r_i)$ and $f^c = \ln(1 + f)$:

$$t_1 r_1^c + t_{12} f_{12}^c = t_2 r_2^c \tag{11.A.6a}$$

$$t_2 r_2^c + t_{23} f_{23}^c = t_3 r_3^c \tag{11.A.6b}$$

where $t_1 = 32/365$, $t_2 = 122/365$, $t_3 = 212/365$, $t_{12} = t_2 - t_1 = 90/365 = t_{23} = t_3 - t_2$. Hence:

$$f^c_{32,122} = 0.0986 \qquad f^c_{122,212} = 0.1378$$

$$F_{32,122}(June\text{-}Futures) = 100 \ \ e^{-f^c_{32,122}(90/365)} = \$97.60$$

$$F_{122,212}(Sept\text{-}futures) = 100 \ \ e^{-f^c_{122,212}(90/365)} = \$96.66$$

Simple Rates

For the compound spot yields in (11.A.1a), the equivalent simple rates are $r^s = (1/t)(e^{r^c t} - 1)$, hence:

$$r^s_{0,32} = 0.08651 \qquad r^s_{0,122} = 0.09684 \qquad r^s_{0,212} = 0.11714$$

The no-arbitrage (simple) forward rates are determined by:

$$[1 + r^s_{0,32}(32/365)][1 + f^s_{32,122}(90/365)] = [1 + r^s_{0,122}(122/365)]$$

$$[1 + r^s_{0,122}(122/365)][1 + f^s_{122,212}(90/365)] = [1 + r^s_{0,212}(212/365)]$$

Hence:[9] $f^s_{32,122} = 0.09976 \qquad f^s_{122,212} = 0.14012$

$$F_{32,122}(June \ \ Futures) = 100 \ \ /[1 + f^s_{32,122}(90/365)] = \$97.60$$

$$F_{122,212}(Sept \ \ futures) = 100 \ \ /[1 + f^s_{122,212}(90/365)] = \$96.66$$

Hence as expected, the different interest rate conventions all give the same value for the June and September futures prices.

EXERCISES

Question 1
Explain how you can use interest rate futures (e.g. Eurodollar futures) to speculate on (parallel) shifts in the yield curve.

Question 2
Explain why 90-day Eurodollar futures prices for contracts with different maturities, might move by different amounts (over 1 week, say). Assume continuously compounded rates and an actual/360 day-count convention.

[9]The 'simple' forward rates can also be directly calculated using $f^s_t = (1/t)(e^{f^c_t} - 1)$, where $t = 90/365$.

Question 3

Explain how a speculator might use 90-day Eurodollar futures with different maturity dates to speculate on 'twists' (non-parallel shifts) in the yield curve. Assume continuously compounded rates and an actual/360 day-count convention.

Why might a speculator take a long-short position in Eurodollar futures contracts with different maturity dates?

Question 4

The 9-month spot rate is 10% p.a. and the 6-month spot rate is 9% p.a. (continuously compounded). What is the futures price for a contract which delivers a 90-day T-bill (with a face value of $100), in 6 months' time? What is the IMM index quote and the forward discount rate d_f?

Question 5

The quoted IMM index on a US T-bill futures contract is $IMM = 90.00$. The futures contract matures in 30 days.

A 120-day T-bill is also available with a discount rate $d = 10\%$ (day-count convention is actual/360). The price of the cash-market T-bill is obtained from the discount rate using:

$$\text{Price of T-bill: } P = 100\,[1 - (\text{days}/360)(d/100)]$$

What is the implied repo-rate (continuously compounded, 365-day year)?

Explain what arbitrage opportunities exist if the actual repo-rate is $r = 10.3\%$ (continuously compounded).

Question 6

Spot yields for 6–month and 9–month interest rates are $r_6 = 5.30\%$ p.a. and $r_9 = 5.2\%$ p.a. (continuously compounded) and the price of the 6–month and 9–month cash-market T-bills are $S_6 = 100e^{-0.05/2} = 97.531$ and $S_9 = 100e^{-0.05(0.75)} = 96.175$.

The fair (no-arbitrage) price of a T-bill futures contract which matures in 6 months' time and delivers a $100 face value (3-month) T-bill is F = $98.6097. The quoted price for a T-bill futures contract which matures in 6 months is $F_q = \$98$.

Explain how arbitrage profits can be made.

CHAPTER 12

Hedging with Interest Rate Futures

Aims

- To examine how interest rate futures can be used for either hedging the value of fixed income assets (e.g. the dollar value of a portfolio of T-bills), or to 'lock in' future borrowing or lending rates (e.g. on existing floating-rate bank loans or deposits).
- To use the 'price value of a basis point' (PVBP) and the 'duration-based hedge ratio' to determine the optimal number of futures contracts to hedge a cash-market position in a fixed income asset (e.g. T-bills, bank loan, or bank deposit).
- To demonstrate the use of strip, rolling, and stack hedges.

We show how interest rate futures contracts allow investors to hedge spot positions in cash market assets, such as T-bills, bank deposits, and loans. Interest rate futures are a little more complex than futures on, say, stocks or oil. But the key thing to remember is that the *price* of *any* interest bearing asset (e.g. spot T-bill prices, interest rate futures prices) always moves in the opposite direction to the change in *interest rates (yields)*. So, for example, when yields fall, the futures *price* will rise (and vice versa). The futures contracts we consider are the (90-day) T-bill and (90-day) Eurodollar contracts, where the change in futures prices depends on the change in a 90-day (forward) interest rate.

Interest rate futures can be used to lock in a known 'effective' interest rate over a specific future time period. Consider the following situations:

- It is 27 June. You have \$10m in a (dollar) bank deposit which pays 180-day LIBOR. The next interest rate reset date is in about 6 months' time on 21 December. You fear a fall in interest rates over the next 6 months which means you will earn less interest on your deposits, when the (180-day) deposit rate is reset on 21 December.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

You can hedge this risk on 27 June, by going long (i.e. buying) a December-Eurodollar futures contract (which matures a few days after 21 December). If interest rates fall, the December-Eurodollar futures price will rise between June and December. You can then close out (i.e. buy back) your December-futures on 21 December at a higher price, making a cash profit on your futures trades, which can then be used to compensate for the lower interest rate you will receive on your bank deposit (from 21 December over the next 180 days).

- It is 27 July. You have a $10m bank loan at a floating rate (based on 90-day LIBOR), with the next reset date on 21 September. You fear that interest rates will rise over the next 2 months so your borrowing costs will increase on 21 September. Today on 27 July, you hedge by shorting (i.e. selling) a September, T-bill futures contract (which matures towards the end of September). If interest rates rise over the next 2 months, the futures price will fall and you can close out (i.e. buy back) your September-futures contract at a profit on 21 September – which then offsets the higher borrowing costs on your bank loan on 21 September.
- It is 27 July and you know you will receive funds on 15 September which you want to invest in a ('cash market') 90-day T-bill. If spot interest rates fall over the next 2 months the price of your cash market T-bill will increase. So doing nothing is risky. However, today you can hedge against a rise in the price of the cash market T-bill by buying a 'September T-bill futures' on 27 July at an agreed price of F = 98 – no money changes hands in July. If interest rates fall over the next 2 months then you can close out (i.e. sell your futures) on 15 September at a profit – which offsets the higher cost of buying your 'cash market' T-bill.[1]

12.1 NUMBER OF FUTURES CONTRACTS

12.1.1 Three-month Eurodollar Contract

To simplify matters, assume for the moment that yields for all maturities move together and by the same amount (i.e. parallel shift in the yield curve). We also ignore any distinction between discount rates and yields and avoid details of day-count conventions. As interest rates and futures prices move in opposite directions then:

Hedging using interest rate futures:

- You have an existing loan at a floating rate (LIBOR). At the next reset date you fear a *rise* in interest rates. Today, you hedge by *selling/going short* interest rate futures contracts.

[1]Another way of looking at this is to note that when you buy the T-bill futures on 27 June, the forward rate you have 'locked in' between 15 September and December will be close to $f = 8.2\%$ p.a. [$= (100 - 98)/98 \times 4 \times 100\%$].

- You have a bank deposit which pays LIBOR. At the next reset date you fear a *fall* in interest rates. Today, you hedge by *buying/going long* interest rate futures contracts.

The number of futures contracts needed to hedge a cash market position can be found in two broadly equivalent ways:

- Using the 'price value of a basis point' (PVBP).
- Using a 'duration-based hedge ratio'.

12.1.2 Bank Loan at 6-month LIBOR: Single Payment

It is 20 April ($t = 0$) and Ms A has an existing bank loan for \$2m, with interest payable based on *6-month* (180-day) LIBOR and the next interest rate reset date is in 2 months' time on 15 June (Figure 12.1). So the interest on the loan will be determined by the 6-month LIBOR rate on 15 June. The 'exposure period' is 2 months and the loan rate set on the 15 June applies over the 6 months between June and December. (Actual payment of the loan interest is on 15 December, $t = 2$.)

Ms A fears a rise in interest rates over the next 2 months, as her loan will then incur higher interest charges on 15 June. Since futures prices fall when interest rates rise, she can today (20 April) hedge by *selling* June-interest rate futures (with mature on 28 June, say). If interest rates do rise, she makes a gain on the futures when she closes out on 15 June (and buys back at a lower futures price). The profit from closing out the futures position offsets the higher interest rate she has to pay on the loan for the period 15 June to 15 December.

Suppose on 20 April the yield curve is flat so that interest rates at all maturities are equal. On 20 April if the 6-month (spot) interest rate is $y_0 = 7\%$ p.a. then the *3-month* forward rate (beginning in June) is also $d_{f,0} = 7\%$ p.a. On 20 April, the *6-month* forward rate (which applies between 15 June and 15 December) is also 7% p.a. (flat yield curve) and hence:

$$\textit{Expected cost of borrowing between June and December} = \$2\text{m}\,(0.07)(1/2) = \$70{,}000.$$

We hedge using the June-Eurodollar futures contract. The contract size is for 'notional delivery' of a \$1m deposit (with 3 months maturity). The quoted (*June) IMM index* is

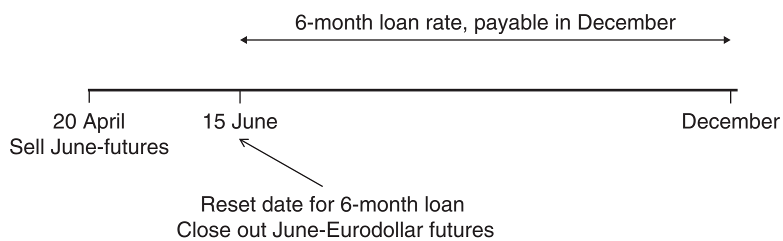

FIGURE 12.1 Hedging 6-month bank loan using Eurodollar futures

determined by the (90-day) forward (discount) rate $d_{f,0} = 7\%$ p.a. (quoted on 20 April, which applies to the 3-month period between 15 June and 15 September)[2]:

$$IMM_0(June) = 100 - d_{f,0} = 93 \tag{12.1}$$

If the forward discount rate changes by 1 bp (= 0.01% p.a.), the change in the dollar value of *one* 3-month Eurodollar futures contract (i.e. 'tick value') is:

$$\text{Tick value : Eurodollar futures} = \$1\text{m} \times (0.01/100) \times (3/12) = \$25$$

Hence, if the IMM index changes from 93.0 to 93.01, (i.e. 1 bp) this implies a change in the value of the futures contract by \$25. We are now in a position to hedge the next interest rate reset (on 15 June) for the \$2m loan, using the June-Eurodollar futures contract. On 20 April we fear a rise in interest rates over the next 2 months so we *go short* (sell) the June-futures contracts. One way of determining the number N_F of futures contracts for the hedge is to use the concept of the 'price value of a basis point' (PVBP). For the bank loan, a change of 1 bp in the LIBOR rate will result in an increase in interest payments over *6 months* of:

$$PVBP(6m\ loan) = \$2\text{m}\ (0.01/100)\ \ (1/2) = \$100 \tag{12.2}$$

So for every 1 bp increase in loan rates, interest payments on the *6-month* loan will increase by \$100.[3] But a 1 bp increase in interest rates results in a change in value of one Eurodollar futures contract by \$25 (the tick value). Hence the number of futures contracts required for the hedge is:

$$N_F = \frac{PVBP(6m\ loan)}{PVBP(3m\ Eurodollar\ \ futures)} = \frac{\$100}{\$25} = 4 \text{ contracts} \tag{12.3}$$

Suppose by 15 June (the 6-month) LIBOR rate increases to $y_1 = 8\%$ p.a. Since the June-futures is now close to maturity then the forward rate will be approximately equal to the June spot-rate so $d_{f,1}(in\,June) = 8\%$ p.a. and therefore $IMM_1(in\,June)$ 100 – 8 = 92. Since $IMM_0 = 93$ the change in the IMM index is 100 ticks. The outcome of the hedge in June is:

Interest cost of bank loan (June-Sept) at LIBOR = \$2m (0.08/2)	=	\$80,000
Gain on futures = 100 ticks × \$25 × 4 contracts	=	\$10,000
Cost of borrowing less gain on futures	=	**\$70,000**

[2]Note that d_f is a discount rate and not a (forward) yield f – but don't worry about this distinction here – as a first approximation you can consider they are equal.
[3]PVBP is also referred as a 'PV01' ('pee-vee-nought-one') or 'DV01' ('dee-vee-nought-one').

LIBOR rates rise from 7% to 8% between April and June which increases borrowing costs on the loan from $70,000 to $80,000. But the cash profit on the 4 futures contracts of $10,000 just offsets the extra LIBOR cost, so the effective interest cost of borrowing is $70,000. The latter is the same interest cost we *expected* when we instituted the hedge in April, when the quoted (6-month) forward rate, which applied to the period June to December was 7% p.a. We can also interpret the hedge in an equivalent way:

$$\begin{array}{lll} \text{Increased cost of loan interest} & = 100\text{bps} \times \text{PVBP } (= \$100) & = \$10{,}000 \\ \text{Profit on futures} & = 100\text{bps} \times \$25 \times 4 \text{ contracts} & = \$10{,}000 \end{array}$$

So even though the 4 futures contracts are closed out, the hedge is effective. The futures hedge has 'locked in' a borrowing rate of $d_{f,0} = 7\%$, the forward rate which 'lies behind' the quoted IMM-index ($IMM_0 = 93$), which is known on 20 April when we initiate the hedge.[4]

Observant readers might have noticed that the loan interest is payable in December but the cash profit on the futures contract is paid out on 15 June. Hence the above figures are not quite comparable. In fact the $10,000 payout for the futures can be invested at $y_1 = 8\%$ between June and December giving $10,400 [$= \$10,000 \times (1 + 0.08/2)$] – so the hedge makes a small profit of $400 in December (that is 0.02% on $2m, over 6 months).

12.1.3 Duration Based Hedge Ratio

An alternative way of determining N_F, which does not require the explicit calculation of the PVBP, is to use a duration-based hedge ratio (see Appendix 12). The contract size for one Eurodollar futures contract is $1m. The duration of a 6-month bank loan is $D(loan) = 1/2$ (year) and the duration of the (notional) 3-month interest rate in the Eurodollar futures contract is $D(futures) = 1/4$ (year). The duration based hedge ratio for N_F is:

$$N_F = \frac{Value\ \ of\ \ loan}{Contract\ \ size}\left[\frac{D(loan)}{D(futures)}\right] = \frac{\$2\text{m}}{\$1\text{m}}\left[\frac{1/2}{1/4}\right] = 4 \text{ contracts} \tag{12.4}$$

Implicit in the above calculation is the assumption that the 3-month forward rate underling the futures contract, moves by the same (absolute) amount as the 6-month (LIBOR) interest rate on the loan. This may not be true. For example, if the 3-month rate (underlying the futures contract) moves by 1%, the 6-month LIBOR rate on the bank loan might change by only 0.9% (on average), so you need slightly less than 4 contracts in the hedge. We can express the relationship between the changes in these two interest rates in a regression equation:

$$\Delta y(loan\ rate) = \alpha + \beta_y(\Delta f) \tag{12.5}$$

[4]Strictly, this result requires a flat yield curve and a parallel shift in the yield curve.

This equation could be estimated using *daily* changes in the *6-month* LIBOR rate regressed on *daily* changes in the *3-month* forward rate – using the last 100 trading days of data, for example – to obtain an estimate of the regression slope, here $\beta_y = 0.9$. The estimate of the intercept α would probably be close to zero but is unimportant here, as it does not figure in the hedge-ratio formula. So our 'complete' formula for the number of Eurodollar futures contracts to hedge a *single future* reset date for the LIBOR interest rate on the bank loan is:

$$N_F = \frac{\textit{Value of Loan}}{\textit{Contract size}}\left[\frac{D(loan)}{D(futures)}\right]\beta_y \tag{12.6}$$

Instead of using the contract size of \$1m for the (Eurodollar or T-bill) futures contract, a small amendment to the above equation is often used:

$$N_F = \frac{\textit{Value of loan}}{V_F}\left[\frac{D(loan)}{D(futures)}\right]\beta_y \quad \text{where } V_F = \$1\text{m}(F_0/100) \tag{12.7}$$

V_F is the 'value of one futures contract' (or 'invoice price' or 'contract price'). For example, if the forward (discount) rate $d_f = 2\%$ p.a. then $F_0 = 99.5$ and $V_F = \$99{,}500$, which is close to the contract size of \$1m. Hence in most cases, using either of the above duration based formulas gives similar results.

Note that using the above formulas will not in general produce a perfect hedge – this is because using 'duration' involves an approximation to price changes and also assumes shifts in the yield curve are always parallel. In addition, 'beta' is a (regression) estimate based on past data and hence may not accurately represent what happens when the hedge is over any future period.

12.2 DIFFERENT TYPES OF HEDGE

12.2.1 Strip Hedge

Suppose it is 15 April and company-XYZ knows that it will borrow at 3-month LIBOR (i.e. the 'tenor' of each bank loan is 3 months), for the following dates and amounts (Figure 12.2):

15 June	\$40m (for 3 months)
10 Sept	\$20m (for 3 months)
20 December	\$30m (for 3 months)
Total future exposure	\$90m

On 15 April, suppose (spot) LIBOR rates for all maturities are 3% p.a. and therefore all forward rates are also 3% p.a. Company-XYZ fears a rise in the LIBOR yield curve in the future.

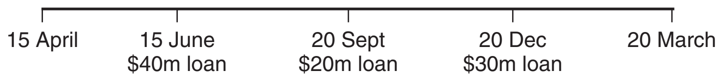

FIGURE 12.2 Hedging 3-month loans at LIBOR using Eurodollar futures

To cover the first loan reset-date on 15 June, firm-XYZ on the 15 April will short (sell) $N_F = 40$ June-contracts:

$$N_F = \frac{PVBP(3m\ loan)}{PVBP(futures)} = \frac{\$40\text{m}(0.0001)(1/4)}{\$25} = \frac{\$1{,}000}{\$25} = 40 \text{ contracts} \tag{12.8a}$$

Alternatively, using Equation (12.6), the tenor of each of the loan interest payments is 3 months, so $D(loan) = 1/4$ year and $D(futures) = 1/4$ for the (3-month) Eurodollar futures contract, hence:

$$N_F = \frac{Value\ of\ loan}{Contract\ size}\left[\frac{D(loan)}{D(futures)}\right]\beta_y = N_F = \frac{\$40\text{m}}{\$1\text{m}}\left[\frac{1/4}{1/4}\right] 1 = 40^5 \tag{12.8b}$$

On 15 April company-XYZ will also sell 20 September-futures contracts and 30 December-futures to cover the future \$20m and \$30m exposures – making a total of 90 contracts in all. This is a strip hedge – we match each loan reset-date with a futures contract which matures just after each reset-date. Because the yield curve is flat, this strategy will 'lock in' an effective interest rate on each of the bank loans of around 3% p.a.

On 15 June company-XYZ will buy back (close out) the 40 June contracts (leaving 20 September and 30 December contracts outstanding). Similarly on 10 September it will close out the 20 September-contracts (leaving 30 December-contracts outstanding) and on 20 December it will close out the remaining 30 December-contracts.

If interest rates rise throughout the period then all the futures contracts will be closed out at a profit, and this profit will offset the higher LIBOR loan rates at each of the 3-month reset dates. The effective cost of borrowing will be an average of the forward interest rates (quoted on 15 April), implicit in the June, September and December, quoted futures prices – that is, about 3% p.a. Conversely, if interest rates fall, then closing out the futures involves a loss, but the interest costs of the loans will be lower – the result of the hedge is again to 'lock in' an effective cost of the loans, equal to the forward rate of about 3% p.a.

The strip hedge locks in an effective borrowing rate on the loan payments equal to the average of the forward rates (known on 15 April) in the contracts used. For example, if on 15 April ($t = 0$) the September-Eurodollar futures is quoted as $IMM_0(Sept) = 97$ then the hedge will lock in an effective borrowing rate between September and December of around 3% p.a. (since $d_f(Sept) = 100 - IMM_0(Sept) = 3\%$). A similar argument applies to the outcomes from

[5] Since the tenor of the loan reset dates and the futures contract are both 3 months and we assume parallel shifts, then for this special case we have $N_F = Value\ of\ loan/Contract\ size = \$40\text{m}/\$1\text{m} = 40$.

the other futures contracts used, so if the yield curve is flat the effective rate on each loan is 3% p.a.[6]

12.2.2 Stack Hedge

If the 'far' December-futures contract is liquid, then you could hedge your interest rate resets on all future loan payments using *only* the December (far-dated) contract – this is often called a stack hedge.

15 April	Sell (short) 90 December-contracts
15 June	Buy back 40 December-contracts
	Retain 50 short December-contracts
	(which cover the remaining reset dates/exposures)
10 Sept	Buy back 20 December-contracts
	Retain 30 December-contracts (covers final exposure)
20 December	Buy back remaining 30 December-contracts

12.2.3 Rolling Hedge

If company-XYZ is worried about the lack of liquidity in the far dated contracts then it might use only the 'nearby contracts'. This is called a *rolling hedge*. Here's what happens in a rolling hedge with Eurodollar futures contracts.

15 April	Sell 90 June-contracts
15 June	Buy back all 90 June-contracts
	Sell 50 September-contracts (covers remaining reset dates/exposures)
10 Sept	Buy back all 50 September-contracts
	Sell 30 December-contracts (covers final exposure)
20 December	Buy back all 30 December-contracts

Note that in the rolling hedge you sell 90 (not 40) June-contracts on 15 April. You close out these 90 June-contracts on 15 June (because the contracts will expire soon after that date). On 15 June you then sell 50 September-contracts (these are now the 'nearby contracts') and

[6]If the spot yield curve is upward sloping on 15 April then so is the forward yield curve and the effective rate paid on all of the bank loans will be an average of these (increasing) forward rates, implicit in the June, September, and December futures prices.

close them out on 10 September, when you also then short 30 December contracts, which you close out on 20 December – when you take out the final bank loan \$40m on which you pay the 90-day LIBOR rate set on 20 December.

On 15 April, interest rates for all maturities are 3%. At first sight it might look as if shorting and then closing out 90 (rather than 40) June-contracts involves too many contracts to hedge the June LIBOR rate on the \$40m bank loan (taken out in June). But consider a 1% increase in *all* interest rates (for all maturities) from 3% to 4%, just before 15 June – *and interest rates then stay at 4% until after 20 December*. Clearly, the profit on 40 of the 90 June-contracts cover the extra loan interest payments on the \$40m loan between June and September. But the profit on the other 50 June-contracts (which are realised on 15 June when you close out) are needed to cover the 1% higher loan interest payments which will occur in September and December, as a result of interest rates *remaining at 4%* between June and December. Hence, there is no *additional* profit when closing out the 50 September-contracts (on 10 September) and the 30 December-futures contracts (on 20 December) as interest rates remain at 4% and hence the June-futures and September-futures prices remain *constant after* 15 June. But this is fine, as we have already covered the 1% higher interest cost of the loan-resets in September and December, with the profit from the 50 (of the 90) June-contracts, that were closed out on 15 June.

In the rolling hedge, the 50 September-contracts you sell on 15 June cover any *further rises* in LIBOR rates between June and September, above the 1% increase that has already occurred in June. Similarly, the 30 December-contracts you sell on the 10 September, cover any *further* rises in interest rates between 15 September and 20 December.

The transactions costs of the rolling hedge compared with the strip hedge might be higher because at each reset date, more contracts are being closed out in the rolling hedge than in the strip hedge. However, if the short dated (nearby) contracts are more liquid than the longer dated contracts (used in the strip hedge) then the transactions costs for the rolling hedge may be less *per contract* and the total transactions cost may also be less. This is a practical matter. However, note that Eurodollar futures contracts are very liquid for maturities out to 10–15 years, so a lower cost strategy using shorter-dated contracts and rolling the hedge really only applies if you are hedging interest rates on a loan with a maturity in excess of 15 years.[7]

The rolling hedge has an additional risks. You close out all of your 90 June-contracts on 15 June when interest rates are 4% p.a. and take a new position on 15 June by selling 50 September-contracts. Similarly, on the 10 September you sell 30 December-contracts. If interest rates increase between June and September from 4% to 5% (say), then the 50 September-futures contracts (sold in June) will cover this additional 1% loan interest in September. But the 50 September futures you sold on 15 June (and that you buy back in September) will lock in an effective interest rate of 4% p.a. (not 3% p.a.) – the implicit forward

[7]The rolling hedge is useful for hedging over long time horizons when liquid long dated futures contracts are not (easily) available on certain underlying assets. For example, future purchases or sales of commodities such as crude oil and natural gas are often hedged using liquid short-dated contracts in a rolling hedge (rather than using a strip hedge with illiquid long-dated contracts).

rate on the September-futures contracts sold in June. A similar argument applies when you close out the December-futures.

Roll-over risk can also work to your advantage. Suppose interest rates fall from 3% to 2% on 15 June and to 1% on 10 September. When you sell your 50 September-futures (on 15 June) and your 30 December-futures (on 10 September), you lock in an effective interest cost on your bank loans of 2% and 1%, respectively.

But a rolling hedge is more risky than a strip hedge. On 15 April, the strip hedge locks in an effective cost on *all the loans* of 3%, no matter what happens to the path of interest rates in the future – this is because the strip hedge locks in the forward rate (of 3%) quoted for all maturities, on 15 April. However, the effective interest cost of the loans when using a rolling hedge depends on the future path of interest rates – if interest rates either rise or fall continuously from April to December then the rolling hedge will result in either a higher or lower effective cost of the loan than 3% p.a. This is because you do not sell your 50 September-futures and 30 December-futures on 15 April but only on their roll-over dates of 15 June and 10 September (respectively), thus locking in the forward rates implicit in the futures prices at *those dates*.

12.2.4 Basis Risk

Of course, all of the above hedges have 'basis risk'. The tenor of the loan interest payments may not exactly match the forward rate that determines changes in the IMM-index (futures) price. For example, a firm's borrowing costs may be determined by the *6-month* LIBOR and this may not change by the same (absolute) amount as the *3-month* Eurodollar rate in the futures contract – this requires a regression to determine the interest rate beta as in Equation (12.5) and this is subject to estimation error.

The rolling hedge is subject to roll-over risk as you do not know what futures prices will be after the first reset date – and these determine the forward rates you lock in, in future periods. The stack hedge locks in the forward interest rate implicit in the December contract – that is, the forward rate which applies between December and the following March. But changes in the loan rate between April and June and June and September may not be the same as the change in the December–March forward rate, if shifts in the yield curve are not parallel.

If we experience parallel shifts in the yield curve then each of the above hedges will produce broadly similar outcomes. But a rolling or stack hedge is highly vulnerable to a change in the *slope* of the yield curve. Hence, our earlier 'strip hedge' using futures with *different* maturity dates would provide a better hedge for non-parallel shifts in the yield curve, assuming long dated futures contracts are fairly liquid so that transactions costs are not too high.

12.3 HEDGING: T-BILL AND EURODOLLAR FUTURES

When hedging with US T-bill or Eurodollar futures we can either use the IMM *index* quote (as appears in the *Wall Street Journal* and on dealers' screens) or the *futures price F*. Our example is a cross-hedge. Suppose it is 15 May and the company CashRich has a \$2m bank deposit paying 6-month LIBOR, with the next interest rate reset date on 15 August (with interest actually paid on 15 February). The maturity (duration) of the 6-month bank deposit is $D_s = 1/2$ (Figure 12.3). CashRich fears a fall in interest rates, which implies that the deposit account will earn less interest beginning on 15 August, for the next 6 months.

A fall in rates implies a rise in futures prices, so on 15 May CashRich buys (goes long) September Eurodollar futures (i.e. with a maturity date closest to, but longer than the hedge period May–August). Any loss of interest on its bank deposit should be offset by the cash gain when closing out the futures contracts. This is a cross-hedge because the *90-day* interest rate underlying the Eurodollar futures contract is not the same tenor as the 180-day interest *rate* on the bank deposit.

On 15 May:

$$\text{September} - \text{IMM index} = 89.2$$

$$F_0 = 100 - (1/4)(100 - IMM) = 97.30 \ \text{(per \$100 nominal)} \tag{12.9}$$

$$\begin{aligned} V_F &= \$1\text{m}\,(F_0/100) &&= \$973{,}000 \\ f_0 &= [(100/97.3 - 1)4] &&= 11.1\% \ \text{(implied forward yield, simple rate)} \end{aligned} \tag{12.10}$$

On 15 May, the forward yield implied by the quoted September-IMM index (or futures price) is $f_0 = 11.1\%$ – this is the rate CashRich hopes to ‘lock in’ for 6 months beginning on 15 August, when the September-futures is closed out.

Number of futures contracts:

The duration-based hedge ratio (assuming a parallel shift in the yield curve) is:

$$N_F = \frac{\$ - Deposit}{Contract\ \ size}\,\frac{D_s}{D_f} = \frac{\$2\text{m}}{\$1\text{m}}\left(\frac{0.5}{0.25}\right) = 4 \tag{12.11}$$

Between 15 May and 15 August assume that all interest rates fall (see Table 12.1) and CashRich only earns a yield of $y_1 = 9.6\%$ on its 6-month deposit. However, lower spot rates

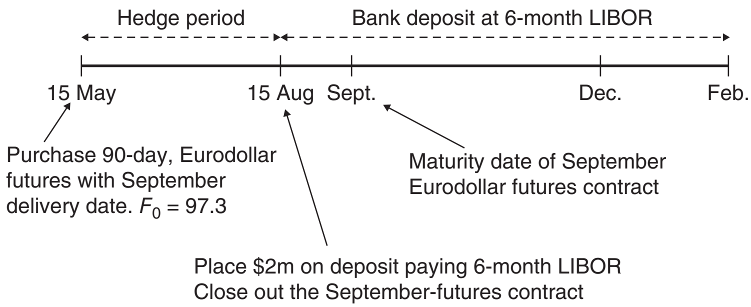

FIGURE 12.3 Hedging using Eurodollar futures

TABLE 12.1 Cross hedge (US T-bill futures)

3-month US T-bill futures (September maturity)				
	Spot rates, 15 May (T-bill yields)	**IMM Index**	**Futures price, F (per \$100)**	**FVF = \$1m(F/100)**
May	y_0 (6 months) = 11%	$IMM_0 = 89.2$	$F_0 = 97.30$	\$973,000
August	y_1 (6 months) = 9.6%	$IMM_1 = 90.3$	$F_1 = 97.575$	\$975,750
Change	−1.4%	1.10 (110 ticks)	0.28	\$2,750 (per contract)

Durations: $D_S = 0.5$, $D_F = 0.25$. Amount to be hedged = \$2m. Number of contracts $N_f = 4$

Notes: We use spot T-bill yields y, whereas in practice discount rates would be quoted and the yields would need to be derived from the discount rates.

imply that the 3-month forward rate also falls and hence the IMM index and the futures *price* rise (see Table 12.1). The profit from closing out the futures offsets the lower interest on the bank deposit, so the company earns an *effective rate* on its deposit of around $f_0 = 11.1\%$.

On 15 August:

$$\begin{aligned} \textit{Gain on 4 futures contracts} &= (\$975,750 - \$973,000)\ 4 &= \$11,000 \\ &= 110 \text{ ticks} \times \$25 \times 4 \text{ contracts} &= \$11,000 \end{aligned}$$

Invest the futures profit of \$11,000 for 6 months (August–February) at $y_1 = 9.6\%$ p.a.

$$\textit{Futures profits invested for 6 months} = \$11,000\ [1 + (0.096/2)] = \$11,528$$

The profit on the futures is equivalent to \$23,056 over 1 year, hence:

$$\begin{aligned} \textit{Effective (simple) annual rate earned on bank deposit} &= y_1 + (\$23,056/\$2\text{m}) \\ &= 0.096 + 0.0115 = 0.1075\ (10.75\%) \end{aligned}$$

Alternatively, we have:

Interest received over 6 months on bank deposit = \$2m (0.096/2)	=	\$96,000
Profit from futures (after 6 months on deposit)	=	\$11,528
Total received over 6 months	=	\$107,528
Total over 1 year (×2)	=	\$215,056
'Effective interest' p.a.(deposit + futures profit) = \$215,056/\$2m	=	10, 75% p.a.

The 10.75% hedged return is substantially above the unhedged rate $y_1 = 9.6\%$ and is reasonably close to the implied (simple) yield on the September-futures contract of $f_0 = 11.1\%$.

The hedge is not perfect because shifts in the yield curve may not be parallel and the contract is closed out on 15 August, well before its September maturity date.

We can look at the hedge in a slightly different way by working out the loss of interest received on the bank deposit on 15 August, relative to the interest rate *we expected* to earn, which is given by the implied forward rate $f_0 = 11.1\%$ on 15 May[8]:

Loss of interest receipts on
6-*month bank deposit* ($f_0 = 11.1\%$, $y_1 = 9.6\%$) $= \$2\text{m}\ [0.111 - 0.096]\ (1/2) = \$15{,}000$
Net loss = *Loss on bank deposit* − *Gain on futures* = \$15,000 − \$11,528 = \$3,472

Had the company remained unhedged then the loss of interest would have been \$15,000 rather than the hedged loss of \$3,472 (which is only 0.17% of the \$2m deposit).

12.4 EURODOLLAR STACK HEDGE

The Eurodollar futures contract with maturities extending out to about 20 years are some of the most actively traded interest rate futures contracts, the reason being the demand by various types of hedger particularly swaps dealers. Short-term contracts are more liquid but there is considerable liquidity in the market out to 10 years maturity. Any financial institution or corporate paying or receiving a floating rate (LIBOR) might wish to hedge using Eurodollar futures. Consider the following situations:

- Bank-A has issued a 5-year 4% *fixed rate* loan of \$100m, financed by \$100m of *floating rate* (LIBOR) deposits (from Citibank), with deposit rates reset every 3 months and currently equal to 3% p.a. (for all maturities).
- Bank-B has issued 10-year *floating rate* loans of \$100m based on 180-day LIBOR and finances these by bidding for *floating rate* deposits of \$110m (from Citibank) based on 90-day LIBOR.
- A corporate (McTrump) with a 5-year, \$100m principal, floating rate (LIBOR) loan with Citibank, with rate resets every 3 months.

Bank-A is currently earning a spread of 1% (=4% fixed on its loans, financed by LIBOR floating-rate deposits currently at 3% p.a.). But if LIBOR deposit rates rise in the future by more than 1% it will make a loss. Now consider Bank-B whose receipts and payments are both at a floating rate (based on LIBOR). But Bank-B has *net* floating rate payments of

[8] Note that the baseline for calculating gains and losses on the 'face value' of \$1m is the quoted forward rate on 15 May ($f_0 = 11.1\%$) and not the spot yield $y_0 = 11\%$. This is because the company cannot hope to lock in the cash market (spot) yield in May of 11% since the deposit reset date is not until 15 August. It can only attempt to lock in the *forward* rate at $t = 0$, which applies to the period starting in August.

\$10m (= \$110m − \$100m). It is therefore vulnerable to a rise in the general level of interest rates and also to a change in the spread between 180-day and 90-day LIBOR rates. McTrump borrowed \$100m at a floating rate and is vulnerable to a future rise in interest rates.

Note that in all cases the participants are *vulnerable over several periods* since the floating rate payments are reset periodically. Today, these positions can be hedged by shorting Eurodollar futures.

12.4.1 Corporate Borrowing: Stack Hedge

Let us consider the case of a company (McTrump) on 1 May ($t = 0$) which has a floating rate loan of \$10m with resets every 30 days, the next is in June, followed by another in July and the principal is repaid in August (see Figure 12.4).[9]

The current (30-day) LIBOR rate in May is $L_0 = 10\%$ p.a. (simple rate). The day-count convention for (USD) LIBOR is actual/360. For simplicity assume that on 1 May there is a flat term structure, so that quoted forward rates beginning in June and August are also 10% (simple interest). The *out-turn values* for future LIBOR rates and futures prices are given in Table 12.2A. We use the September-futures to hedge the next two interest rate resets – this is a *stack hedge*.

McTrump expects to pay \$10m L_i (30/360) each month. Since the (simple) forward rates for June and July are also 10% p.a., the expected *compound annual* rate McTrump is trying to 'lock in' is:

$$\textit{Expected future loan rate (compound)} = \left[1 + 0.10\left(\frac{30}{360}\right)\right]^{\frac{365}{30}} - 1 = 0.1062 \qquad (12.12)$$

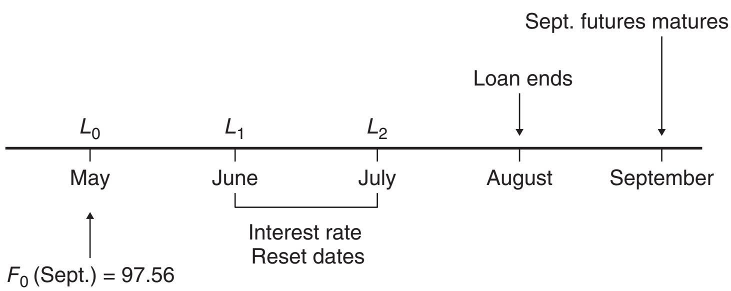

FIGURE 12.4 Eurodollar stack hedge

[9]Aficionados might note that although the interest resets occur every 30 days, the LIBOR rate we use throughout the example is *90-day* LIBOR, to match that in the Eurodollar futures contract – this is to avoid introducing even more interest rates into the calculations.

TABLE 12.2A Eurodollar futures, stack hedge

	t = 0 (1 May)	t = 1 (June)	t = 2 (July)
LIBOR rates (simple)	$L_0 = 10\%$	$L_1 = 10.50\%$	$L_2 = 11\%$
Spot rates (compounded)	$y_0 = 10.62\%$	$y_1 = 11.18\%$	$y_2 = 11.74\%$
Futures	$d_{f,0} = 9.76\%$ $IMM_0 = 90.24$ $F_0 = 97.56$	$d_{f,1} = 10\%$ $IMM_1 = 90.00$ $F_1 = 97.50$	$d_{f,2} = 10.40\%$ $IMM_2 = 89.60$ $F_2 = 97.40$
Implied forward rate (compounded rates)	$f_0 = 10.54\%$	$f_1 = 10.81\%$	$f_2 = 11.28\%$

Notes: The LIBOR day count convention is actual/360. The tick value for *IMM* index is \$25. *F* is the futures price per \$100 and the contract size is \$1m.

The quoted (simple) LIBOR rates L_i correspond to compound rates $y_i = [1 + L_i(30/360)]^{365/30} - 1$. On 1 May the September-futures price $F_0 = 97.56$ and therefore the (compound) forward rate is $f_0 = (100/97.56)^{365/90} - 1 = 10.54\%$ p.a. The loan rate is reset every month so $D_s = 1/12$ and for the 90-day Eurodollar futures contract $D_f = 1/4$, hence:

$$N_F = -\frac{Loan}{Contract\ \ size}\frac{D_s}{D_f} = -\ \frac{\$10\text{m}}{\$1\text{m}}\left(\frac{(1/12)}{(1/4)}\right) = -3.33^{10}$$

To hedge interest payments for both the June and July reset dates using a stack hedge, we require $N_F = -6.67$, that is, on 1 May we short 7 September-futures contracts. On 1 May, 30-day LIBOR is $L_0 = 10\%$ p.a. and the loan outstanding on 1 June is:

$$\$10\text{m}\left[1 + 0.10\frac{30}{360}\right] = 10{,}083,333 \tag{12.13}$$

However, between 1 May and the June reset date, the rise in LIBOR from 10% to 10.5% results in a fall in the *IMM-index* by 24 ticks. In June we need a short position in three

[10] As interest rates are compound rates, the correct expression is

$$N_F = -\frac{Loan}{Contract\ \ Size}\frac{D_s}{D_f}\left(\frac{(1+f_0)}{(1+y_0)}\right) = -\ \left(\frac{(1/12)}{(1/4)}\right)\ \frac{\$10m}{\$1m}\ \ \frac{(1.1054)}{(1.1062)} = -3.33$$

The additional term $(1+f_o)/(1+y_0)$ occurs because we use compound rates rather than continuously compounded rates – omitting this term makes little difference.

September-futures to hedge the second interest rate reset in July.[11] Hence at $t = 1$ (June), we buy back four futures contracts (Table 12.2B):[12]

$$\begin{aligned} \text{Profit on futures (in June)} &= 4\,(IMM_1 - IMM_0)\,\$25 &= \$2{,}400 \\ \text{Reduce loan outstanding to } 10{,}083{,}333 - 2{,}400 & &= \$10{,}080{,}933 \end{aligned}$$

TABLE 12.2B Eurodollar futures, hedge outcome

At $t = 1$ (June)	**Amount owing spot = \$10m(1+0.1(30/360))**	**= \$10,083,333**
	Buy back 4 futures contracts:	
	Profit on futures = $4(\$1\text{m})(F_1 - F_0)/100 = 4 \times 600$	
	Or $= 4 \times (IMM_1 - IMM_0) \times \$25$	= \$2,400
	$= 4 \times 24$ ticks $\times \$25$	= \$2,400
	Reduce loan outstanding to \$10,083,333 – \$2,400	= \$10,080,933
	L_1 = 10.5%, rate reset in June	
At $t = 2$ (July)	Amount owing spot = \$10,080,933(1+0.105(30/360))	= \$10,169,141
	Buy back remaining 3 futures contracts:	
	Profit on futures $= 3(\$1\text{m})(F_0 - F_2)/100 = 3 \times 1{,}600$	= \$4,800
	Or $= 3 \times (IMM_0 - IMM_2) \times \$25$	
	$= 3 \times 64$ ticks $\times \$25$	= \$4,800
	Reduce loan outstanding to \$10,169,141 – \$4,800	= \$10,164,341
	L_2 = 11%, rate reset in July	
At $t = 3$ (August)	Amount owing spot = \$10,164,341(1+0.11(30/360))	= \$10,257,514
	Corporate borrowed \$10m and paid back \$10,257,514 after 3 months:	
	Effective loan rate (compound) = (\$10.257m / \$10m)^4 – 1 = 10.7% p.a.	
	The 10.7% is only 8 bps above the target rate of 10.62% the corporate was hoping to achieve on 1 May. Had the corporate not hedged, the annual compound rate would have been: $[(1 + 0.1/12)(1 + 0.105/12)(1 + 0.11/12)]^4 - 1 = 11.02\%$ which is 40 bps above the target rate of 10.62%.	

[11] This can be ascertained by using the usual formula for N_F with the new value for the loan of \$10,081,533.
[12] In Table 12.2B this is also calculated using the (dollar) change in futures prices F rather than the IMM index.

By July, LIBOR has again increased by a further 0.5% to 11% and the IMM-index has fallen 40 ticks (= 89.6 – 90), so the profit on the 3 remaining futures contracts is:

$$\text{Profit on futures (in July)} = 3\,(IMM_0 - IMM_2)\ \$25 = \$4{,}800$$

which again is used to reduce the principal of the loan. The July LIBOR rate of 11% determines the last loan interest payment, resulting in an outstanding balance in August of \$10,257,514. This gives an effective borrowing rate of 10.7% p.a. which is only 8 bps above the target rate of 10.62% the corporate was trying to lock in at $t = 0$. Without the hedge, the corporate would have paid LIBOR rates of 10%, 10.5%, and 11% in June, July, and August (respectively) resulting in an unhedged cost of 11.02%, which is 40 bps above the target rate of 10.62%.

12.5 SUMMARY

- Hedging fixed income positions in the cash market is based on the inverse relationship between (forward) interest rates and futures prices.
- At $t = 0$, a hedger can 'lock in' the (implied quoted) forward rate f_{12} applicable between *any two future* periods $t = 1$ and $t = 2$, by taking a position in futures contracts today. These futures contracts are (usually) closed out before the maturity date (at $t = 1$).
- Alternatively, a futures hedge can be viewed as trying to match any increase in interest payments on a bank loan or any fall in interest receipts on a bank deposit, with profits/losses on the futures position.
- If you already have or are planning to take out a bank loan at a floating interest rate (LIBOR) in the future, then you can hedge any future rise in LIBOR by today, *shorting (selling)* interest rate futures contracts. If interest rates rise, the futures price falls and you buy back (close out) the futures contract at a lower price – the profit on the futures contract offsets the higher interest cost of the bank loan.
- Conversely, if you have a bank deposit which pays a floating interest rate (LIBID), or are planning to deposit funds in the future, then you should hedge any future fall in deposit rates by today, going *long (buying)* an interest rate futures contract. If interest rates do fall, the futures price rises and you sell (close out) the futures contract at a higher price – the profit on the futures contract offsets the lower interest rate payable on the bank deposit.
- The optimal number of futures contracts is determined by applying either the 'price value of a basis point' (PVBP), or using the duration based hedge ratio. The number

of (Eurodollar) futures contracts required to hedge *each reset date* (using the duration based hedge ratio) is:

$$N_F = \left(\frac{\$Cash\ \ Market}{V_F}\right)\left[\frac{D_s}{D_f}\beta_y\right] \approx \left(\frac{\$Cash\ \ Market}{\$1m}\right)\left[\frac{D_s}{D_f}\beta_y\right]$$

where $V_F = \$1\text{m}(F_0/100)$, F_0 = futures price, D_s = duration of the spot/cash market position, D_f= the duration of the 'underlying' in the futures contract. Beta β_y is obtained from a regression of the change in the spot (cash-market) interest rate (e.g. 6-month LIBOR) on the change in the forward rate (underlying the futures) contract (e.g. 90-day Eurodollar rate). The above formula is often simplified by replacing V_F with the contract size of the Eurodollar futures contract, namely, $1m.

- To hedge a *series* of interest rate reset dates (on a bank loan or bank deposit) you can use a strip hedge, rolling hedge, or stack hedge. The strip hedge locks in the different forward rates implied by the different maturity futures contracts used in the hedge – each of which matures (just) after each reset date. The rolling hedge and stack hedge carry more 'basis risk', particularly if there are non-parallel shifts in the yield curve.

APPENDIX 12: HEDGE RATIOS

The duration based hedge ratio for fixed income assets can be derived in a number of different ways.

$$\begin{aligned}
S &= \text{price of the spot (cash} - \text{market) asset to be hedged} \\
&\quad\ \text{(e.g. price of T} - \text{bills held)} \\
V_s &= \text{total \$} - \text{amount in cash market at } t = 0 \\
&\quad\ \text{(e.g. principal on bank loan)} \\
y_s &= \text{yield on cash market asset (e.g. bank interest rate)} \\
F &= \textit{price} \text{ of a short} - \text{term interest rate futures contract} \\
&\quad\ \text{(per \$100 nominal). (Note, this is not the IMM index)} \\
z &= \text{contract size (= \$1m for T} - \text{bill and Eurodollar futures)} \\
V_F &= \$1\text{m}(F/100) = \text{value of one futures contract ('invoice price')} \\
N_F &= \text{number of futures contracts in the hedge}
\end{aligned}$$

Using Calculus

Suppose you have a bank deposit of ($V_s > 0$) or bank loan ($V_s < 0$) based on (spot) yields (y_s) with resets every 6 months. The change in interest receipts (payments) *over 6-months* is $dV_s = V_s(dy_s/2) = V_sD_sdy_s$ where $D_s = 1/2$ is the duration of a 6-month deposit or loan. The change in interest cash flows on the deposit or loan, plus the gain or loss on N_F (Eurodollar or T-bill) futures contracts is:

$$\begin{aligned} dV &= \text{\$-change in cash market position} + \text{\$-change in futures position} \\ &= V_s D_s dy_s + N_F[(F_1 - F_0)/100]\ z \\ &= V_s D_s dy_s + N_F(zF_0/100)(dF/F_0) \\ &= V_s(D_s dy_s) + N_F V_F(-D_f\ df) \end{aligned} \tag{12.A.1}$$

where $V_F = zF_0/100$ is the face value of one futures contract and $z = \$1\text{m}$ contract size. Using $F = 100/(1 + f(90/360))$ we have $dF/F \approx -df(90/360) = -D_f\ df$. (For *continuously compounded rates* the latter expression is exact, rather than an approximation.) Setting $dV = 0$ gives:

$$N_F = \left(\frac{V_s}{V_F}\right)\frac{D_s}{D_f}\beta_y \quad (V_s > 0 \text{ for bank deposit and } V_s < 0 \text{ for bank loan}) \tag{12.A.2}$$

where β_y is the OLS coefficient in the regression: $\Delta y_s = \alpha + \beta_y \Delta f$. A simplification of (12.A.2), which works well in practice, is to replace V_F with the 'contract size' of \$1m.

If you are hedging a long position $V_s > 0$ in cash-market T-bills (rather than a bank deposit) then the change in the value of the hedge portfolio is:

$$\begin{aligned} dV &= \text{\$-change in cash market T-bills} + \text{\$-change in futures position} \\ &= V_s R_s + N_F V_F(-D_f df) \\ &= V_s(-D_s dy_s) + N_F V_F(-D_f\ df) \end{aligned} \tag{12.A.3}$$

where $R_s \equiv dS/S$ is the proportionate *price change* of the T-bill (i.e. the holding period return on the T-bill), which we assume is adequately approximated by $R_s \equiv dS/S = -D_S dy_s$. Setting $dV = 0$ gives:

$$N_F = -\left(\frac{V_s}{V_F}\right)\frac{D_s}{D_f}\beta_y \tag{12.A.4}$$

Hence, if you currently hold (are long) cash market T-bills ($V_s > 0$) then (12.A.4) implies you hedge by selling ($N_F < 0$) interest rate futures contracts, today.

Compound Rates

Strictly speaking the above formulas for N_F use continuously compounded yields. When using compound yields, the duration approximations are $dS/S = -D_s dy_s/(1 + y_s)$ and

$dF/F = -D_f\, df/(1+f)$. Hence, using compound yields the correct formula is:

$$N_F = \frac{V_p}{V_F}\left[\frac{D_s\ (1+f)}{D_f(1+y_s)}\right]\beta_y \tag{12.A.5}$$

This correction usually makes little difference in practice.

Minimum Variance

Alternatively, we can choose N_F to minimise the variance of dV. From (12.A.1):

$$\sigma^2_{dV} = V_s^2 D_s^2 \sigma^2_{dy_s} + (N_F V_F)^2 D_f^2 \sigma^2_{df} + 2\, V_F N_F\, D_s D_f \sigma_{dy_s,df} \tag{12.A.6}$$

Differentiating (12.A.6) with respect to N_F, setting the resulting expression to zero and rearranging gives Equation (12.A.2) for N_F, where $\beta_y = \sigma_{dy_s,df}/\sigma^2_{df}$ is the OLS coefficient in the regression $\Delta y_s = \alpha + \beta_y \Delta f$.

Hedging Portfolio of Cash-Market T-bonds Using T-bond Futures

It is convenient to include this here even though we do not discuss T-bond futures until Chapter 13. The only additional complication is that the trader who is short a T-bond futures has a restricted choice on which bond to deliver. Hence, the formula for the number of futures contracts in the hedge, includes the 'conversion factor' CF_{CTD} of the 'cheapest to deliver bond':

$$N_F = \frac{V_p}{V_F}\left(\frac{D_s}{D_f}\right)\ \beta_y\ CF_{CTD} \tag{12.A.7}$$

EXERCISES

Question 1

It is 1 January and you expect to receive \$10m in 2 months' time which you wish to place in a bank deposit for a further 5 months, on which you will receive LIBOR.

Explain what Eurodollar futures contract you would use to hedge your cash (spot) market position. What are the risks in the hedge you have chosen?

Question 2

Explain whether you would undertake a long or short Eurodollar futures hedge if you plan to borrow money in 2 months' time (at the then prevailing 180-day LIBOR rate).

Question 3

In 3 months' time you will issue \$10m (market value), 6-month commercial bills. After a further 3 months you will issue \$20m of 6-month commercial bills. You have decided to hedge your positions using Eurodollar futures contracts, with $F = 99$ (per \$100 nominal) with a contract size of \$1m.

Explain how you would hedge your position. What are the risks in the hedge?

Question 4

It is 3 January 2019. On 2 January there was an interest rate reset on your existing \$10m bank loan, based on 180-day LIBOR. The next reset date is on 2 July 2019. The loan ends on 2 July 2020. The current Eurodollar futures price is $F = 97$ (per \$100 nominal) for all maturities.

How many futures contracts will you use to hedge the remaining interest rate payments and will you go short or long the futures contracts? Assume a parallel shift in the yield curve.

Question 5

On 20 January a US Corporate Treasurer realises that she will issue \$10m (market value) of 180-day commercial paper on 25 June. The September-Eurodollar futures $IMM\ index = 92.0$ (contract size \$1m). Today, how can she hedge her exposure to the new issue of commercial bills, on 25 June? How many futures contracts are required? Assume a parallel shift in the yield curve.

Question 6

It is 15 March 2019 and today you take out a 5-year US dollar loan for \$1.5m with payments linked to 180-day LIBOR (+100 bps). (The 'tenor' in the loan repayments is every 6 months).

Carefully explain the steps you might take to hedge this position using a strip of (90-day) Eurodollar futures (contract size, \$1m). What is the optimal number of futures contracts for each reset date?

The current futures price is $F = \$99$ (per \$100 nominal). When the 90-day yield rises by 1% then on average, the 180-day yield rises by 0.9%.

Question 7

On 20 February a US corporate treasurer realises she will have to borrow money on 17 July by issuing \$5m of commercial paper (market value \$4,820,000) with a maturity of 180 days.

On 20 February the September-Eurodollar futures IMM index quote is $IMM = 92.00$. The contract size is \$1m.

How many futures contracts does she need to hedge against a possible rise in 180-day commercial paper rates between today and 17 July? Assume a parallel shift in the yield curve.

CHAPTER 13

T-bond Futures

Aims

- To examine contract details for UK Gilt futures and US T-bond futures including the conversion factor, the cheapest-to-deliver bond and wild card play.
- To determine the optimal number of T-bond futures contracts for hedging.
- To determine the fair price of a T-bond futures contract using cash-and-carry arbitrage.
- To analyse speculative strategies using T-bond futures. This includes spread trades and altering the effective duration of a bond portfolio to take advantage of market timing strategies.

Some of the practical details of T-bond futures are quite intricate. A long T-bond futures position allows the holder to take delivery of a long maturity T-bond at expiration of the futures contract. As with all futures contracts, T-bond futures can be used for speculation, arbitrage, and hedging. Hedging allows the investor to eliminate price risk of her bond portfolio.

For example, suppose Ms Bond holds \$20m in (cash market) 20-year T-bonds and she fears a rise in long-term yields over the next 6 months. Ms Bond should hedge by shorting (selling) T-bond futures. If long rates do subsequently rise, the price of her cash-market T-bonds will fall but so does the futures price. Hence, she can close out her short futures position at a profit by buying back the T-bond futures at a lower price. The profit from the futures position compensates for the loss in value of her cash market position in T-bonds.

If Ms Bond wants to act as a speculator and she forecasts that long-term yields will fall in the future then today she would purchase T-bond futures contracts. If yields fall, then the futures price will rise and she can close out her long position in T-bond futures at a higher price – hence making a speculative profit.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Ms Bond gains leverage by purchasing the futures contract rather than purchasing T-bonds in the cash market (with her own funds) – because she only has to provide a relatively small initial futures margin (and not the full price of the cash market bond). Transactions costs (e.g. bid–ask spreads, clearing, and brokerage fees) in the futures market might also be lower than those in the cash market.

Naked speculative positions in T-bond futures are highly risky, therefore speculators often use *spread trades*. For example, they might purchase one T-bond futures contract with a long maturity date and simultaneously sell another T-bond futures with a short maturity date. This provides possible speculative profits but also reduces risk compared with an outright long or short position in T-bonds.

13.1 CONTRACT SPECIFICATIONS

T-bond futures contracts written on a number of government bonds are traded on several exchanges. The most liquid contracts are on US T-bond futures (CBOT) quoted in US dollars. There are also Euro-Bond futures – for example, on French and German bonds ('Bunds') which are both quoted in euros, and also UK Gilt futures (quoted in sterling) – all traded on NYSE-Euronext. Somewhat less liquid T-bond futures are those on Japanese government bonds, traded on NYSE-Euronext and in Tokyo.

13.1.1 UK Long Gilt Futures Contract

Details of the UK Long Gilt Future (on NYSE-EURONEXT) are given in Table 13.1. The bond deliverable in the contract is a 'notional bond' with a 4% coupon (with a maturity between 8.75 and 13 years). Surprisingly, this notional 4% bond does not actually exist! However, as we shall see, it provides a benchmark from which to calculate the price of possible bonds for delivery (which do exist).

TABLE 13.1 UK Long Gilt Futures (Euronext-LIFFE)

Contract size	£100,000 nominal, notional Gilt with 4% coupon
Delivery months	March / June / September / December
Quotation	Per £100 nominal
Tick size (value)	£0.01 (£10)
Last trading day	11 a.m., 2 business days prior to the last business day in the month
Delivery day	Any business day in the delivery month (seller's choice)
Settlement	List of deliverable Gilts published by the exchange with maturities between 8.75–13 years
Margin requirements	Initial margin £2,000, spread margin £250 (determined by the exchange)

The seller of the futures contract can decide the exact delivery date (within the delivery month) and exactly which bond she will deliver (from a limited set, designated by the exchange). In fact, the seller will select a bond which is known as the *'cheapest-to-deliver'* (CTD). The CTD bond can be shown to depend on the *'conversion factor'* (CF) for each possible deliverable bond. (These concepts are explained below.)

The futures contract size is for delivery of z = £100,000 (face value) T-bonds and futures price quotes are expressed per £100 nominal (of deliverable T-bonds). The tick size is £0.01 per £100 nominal (e.g. 1-tick is a move from £95 to £95.01) – hence the tick value is £10 per contract (= £0.01 × £100,000/£100).

EXAMPLE 13.1

Face Value of Futures Contract

On 27 July the September-futures has a closing (settlement) price quote of F_0 = £103.19 per £100 nominal. Hence the 'face value'[1] of one futures contract is:

$$V_F = (\$100{,}000)(F_0/100) = £103{,}190.$$

13.1.2 US 'Classic' and 'Ultra' T-bond Futures Contracts

The principles underlying these futures contracts (traded on the CBOT) are very similar to those for the UK gilt-futures contract, except price quotes are in 1/32nd of 1% (see Table 13.2). Expiration months are March, June, September, and December. The last trading day is the business day prior to the last 7 days of the expiry month. The *first* delivery day is the 1st business day of the delivery month but delivery can take place on any business day in the delivery month.

Note that the *notional bond* deliverable in the futures contract is assumed to be a *6%-coupon bond*. However, in practice the person with a 'short' futures position can choose from around 30 different eligible bonds to deliver (which have coupons different from 6% and the CF adjusts the delivery price to reflect the type of bonds actually delivered).

For the 'classic' futures contract the T-bonds delivered must have at least 15 years to maturity[2] and less than 25 years to maturity (from the first day of the delivery month).[3] There are also futures on 2, 5 and 10-year T-notes, which only differ from the T-bond futures contract in

[1] Also referred to as the 'invoice price' or 'contract price'. But note that V_F is not paid when the futures contract is initiated, so 'face value' may be a better term to use for V_F.
[2] Or not be callable for at least 15 years.
[3] The 'ultra' T-bond futures contract is exactly the same as the 'classic' contract except that in the case of the 'ultra', the deliverable bond has to have a remaining maturity of not less than 25 years from the first day of the delivery month.

TABLE 13.2 US 'classic' T-bond futures (CBOT)

Contract size	£100,000 nominal, notional US Treasury Bond with 6% coupon
Delivery months	March / June / September / December
Quotation	Per $100 nominal
Tick size (value)	1/32 ($31.25)
Last trading day	7 working days prior to last business day in expiry months
Delivery day	Any business day in the delivery month (seller's choice)
Settlement	Any US Treasury bond maturing at least 15 years from the contract month (or not callable for 15 years) and with less than 25 years to maturity.
Margins	$5,000 initial, $4,000 maintenance (decided by the exchange)
Trading hours	8 a.m. to 2 p.m. – Central Time
Daily price limits	96 points ($3,000)

the maturity of the T-notes which can be delivered in the contract. As with most futures contracts, delivery rarely takes place but it is the possibility of delivery and hence arbitrage profits which keeps the T-bond futures price in line with the spot/cash market price of T-bonds.

EXAMPLE 13.2

Tick Value of Futures Contract

On 7 July, the US T-bond futures settlement price quote (CBOT) for September delivery is '98-14' (= 98 + 14/32) which corresponds to a price of F_0 = \$98.4375 per \$100 nominal. With a contract size of \$100,000, the face value of one contract ('invoice' or 'contract' price) is V_F = \$100,000(F_0/100) = \$98,437.50. The tick size is 1/32 of 1% (= 0.0003125) which on a contract size of \$100,000 nominal, implies a tick value of \$31.25 per contract.

13.2 CONVERSION FACTOR AND CHEAPEST-TO-DELIVER

Before discussing hedging strategies we need to be clear about the use of the conversion factor (CF) and the concept of the cheapest-to-deliver (CTD) bond. We do so with respect to the US T-bond futures contract but similar principles apply to UK Gilt futures. There are a wide variety of bonds with over 15 years to maturity which can be delivered by the investor who is short T-bond futures. These will have different maturities and coupon payments and hence different values for CF and the CTD.

TABLE 13.3 The CTD bond

Deliverable bonds (maturity)	Spot price ($)	Conversion factor (CF_T)	Raw basis ($= S_T - F_T CF_T$)
1. Bond-A (2039)	112-4 (112.125)	1.044	0.156
2. Bond-B (2041)	112-8 (112.25)	1.033	1.461
3. Bond-C (2044)	114-8 (114.25)	1.065	0.029

Notes: *F*(September-futures) = 107-8 (107.25)

13.2.1 Conversion Factor (CF)

This section assumes the reader is largely familiar with the pricing of bonds in the spot market and the conventions used in calculating accrued interest (see Cuthbertson and Nitzsche 2008). The CF adjusts the price of the *actual bond* to be delivered, *by assuming* it has a 6% yield, which makes it equivalent to the notional 6% bond in the futures contract.

When computing the CF the maturity of the underlying bond is defined as the maturity on the *first day* of the delivery month. For example, if *we assume* the actual delivery date is 11 September 2018 and the underlying bond matures on 15 February 2038, then the maturity period used in the calculation of CF is 1 September (i.e. not 11 September) to 15 February 2038.

The CF is best understood using a specific example. To simplify matters we assume the counterparty who is short the futures contract does not yet hold a bond for delivery, so we are going to calculate the CF and CTD bond *at the maturity date of the futures contract (=T).*

Consider an actual 20-year US T-bond, paying semi-annual coupons of 8% (i.e. $4 per 6 months), with $n = 40$ (6-month) periods to maturity (see Example 13.3). If the yield to maturity on this 8%-coupon bond *is assumed to be* 6% p.a. then its ‘fair’ or ‘theoretical’ price would be $123.1 (per $100 nominal) which gives a conversion factor $CF_T = \$1.231$. In essence the deliverable bond (with an 8% coupon) is worth 1.231 times as much as the notional 6% coupon bond (trading at par and hence with $YTM = 6\%$).

EXAMPLE 13.3

Calculation of Conversion Factor

Bond for delivery is an 8%-coupon T-bond with remaining maturity of 20 years (semi-annual coupons). What is the conversion factor of the bond?

(*continued*)

(*continued*)

The theoretical price of an 8%-coupon bond, with \$100 face value, if the YTM is assumed to be 6% (i.e. the same as the notional bond in the futures contract) is:

$$\widehat{P} = \frac{\$4}{(1.03)} + \frac{\$4}{(1.03)^2} + \cdots + \frac{\$4}{(1.03)^{40}} + \frac{\$100}{(1.03)^{40}}$$

Using the annuity formula for the present value of the coupons:

$$\widehat{P} = 23.11(\$4) + 0.3066(\$100) = \$123.1$$

The theoretical price of the actual 8%-coupon bond to be delivered is $\widehat{P} = 123.1$ and hence the conversion factor $CF = 1.231$ (per \$100 nominal).

The conversion factor adjusts the price of the actual bond to be delivered, relative to the notional 6% coupon bond in the futures contract:

$$\textit{If coupon on the bond actually delivered} > 6\% \textit{ then} \Rightarrow CF > 1$$

$$\textit{If coupon on the bond actually delivered} < 6\% \textit{ then} \Rightarrow CF < 1$$

The conversion factor will differ for bonds with different coupon payments and different maturities. The CF for any specific deliverable bond will change over time simply because the maturity date of the deliverable bond gets closer (although it will always exceed 15 years, otherwise it will cease to be an eligible bond). In calculating the CF the CBOT assumes the yield curve is flat at 6%. But in practice this is rarely (if ever) the case. This means that the 'true price' of the deliverable bond, which should be priced using spot rates (see Cuthbertson and Nitzsche 2008) will not equal that used in the calculation of the CF. Hence, there will usually be eligible bonds for delivery that are actually cheaper than the 'cheapest-to-deliver' bond.

Let us now consider the cash amount received by a trader (Ms Short) who is short T-bond futures, when she delivers the underlying bond at maturity T. Let F_T be the futures settlement price (on the 'position day' – see below). When Ms Short delivers the 8%-coupon, 20 year bond she will *receive*:

Ms Short's receipts at settlement

$$= [\textit{Futures Settlement Price} \times CF_T] + \textit{Accrued Interest on Deliverable Bond}$$

$$= F_T CF_T + AI_T$$

Note that F_T is the 'settlement price' at T (and not the price initially agreed at the outset of the futures contract). AI_T is the accrued interest at T and is a fraction of the *next* coupon payment on the bond *delivered* against the futures contract. For example, if the maturity date T of the futures is 11 September 2017 and the deliverable bond matures on 15 February 2038, then the deliverable bond has semi-annual coupons each year on 15 February and 15 August. Assume there are 184 days between 15 August and 15 February. The short therefore delivers a bond at $T = 11$ September 2017, which already has 27 days of accrued interest. Hence the long must pay the short for this loss of accrued interest of $AI_T = \$0.73\,[= (27/184) \times 10/2]$.

13.2.2 Cheapest-to-Deliver

Suppose it is now T (= 11 September 2017) and there are three bonds designated by CBOT for actual delivery in September whose CF are 1.044, 1.033, and 1.065. In practice, the calculation of the CTD bond is quite complex but a rough idea of the CTD bond can be obtained by choosing that bond with the smallest raw basis:

$$Raw\ basis = B_T - F_T CF_T \tag{13.1}$$

where B_T is the spot ('clean') price an eligible bond for delivery, F_T is the settlement futures price and CF_T is the conversion factor of a deliverable bond.

Table 13.3 shows the calculation of the raw basis for three deliverable bonds. The final column indicates that Bond-C is the CTD. Although it is often the case that the bond actually delivered by the short is the CTD, nevertheless this is not always so since the seller may wish to preserve the duration of her own bond portfolio or provide a bond, other than the CTD, in order to minimise her tax bill. We now examine how Equation (13.1) for the raw basis arises. At settlement the short receives:

$$Cash\ received\ by\ short\ (per\ \$100\ nominal) = F_T CF_T + AI_T \tag{13.2}$$

where AI_T is the accrued interest. The conversion factor adjusts the T-bond futures price for the fact that the deliverable bond has a coupon different from the notional 6% bond (stated in the T-bond futures contract). The 8%-coupon bond actually delivered would have a higher price when its coupons are discounted at the notional 6%. To purchase the actual deliverable bond in the cash market, the short will pay:

$$Cash\ paid\ by\ short = \text{`}Clean\ Price\text{'} + Accrued\ Interest = B_T + AI_T \tag{13.3}$$

Hence, the net cost to the short of providing the deliverable 8%-coupon bond is just the raw basis defined above:

$$Net\ cost = (B_T + AI_T) - (F_T CF_T + AI_T) = B_T - F_T CF_T \tag{13.4}$$

13.3 HEDGING USING T-BONDS

Today, if Ms Bond is (net) long in US T-bonds ($V_s > 0$) with a positive portfolio duration ($D_s > 0$) and fears a fall in bond prices, then to hedge she will short T-bond futures. Conversely, if she wishes to purchase bonds in the future and is worried that cash market T-bond prices will rise (i.e. yields will fall) then today she should buy T-bond futures. The optimal number of futures contracts N_F is given by the *duration-based hedge ratio* (see Appendix 13.A):

$$N_F = -\left(\frac{V_s}{V_F}\right)\left(\frac{D_s}{D_f}\right)\beta_y\ CF \tag{13.5}$$

$$dy_s = \alpha + \beta_y dy_f + \varepsilon \tag{13.6}$$

V_s = total *market value* of portfolio of bonds to be hedged

F = price (per \$100 nominal) of the (CTD) bond in the futures contract

z = contract size (= \$100,000 for the 'classic' T-bond futures contract)

$V_F = z(F/100)$ is the value of one bond futures contract ('contract price')

CF = conversion factor of the CTD bond, eligible for delivery in the futures contract

D_s = portfolio duration (*at the expiration of the hedge*) of the (cash-market) bonds

D_f = duration (at maturity of the futures) of the notional bond in the futures contract

y_s = yield to maturity (YTM) of the bonds to be hedged

y_f = YTM of the deliverable bond in the futures contract

Equation (13.5) assumes D_s is defined using *continuously* compounded rates for y_s and y_f. But if y_s and y_f are *quoted* yields to maturity (e.g. using discrete semi-annual compounding) then we use *modified* duration $MD_i = D_i/(1 + y_i)$, in place of D_i in the above equations for $i = s, f$.

13.3.1 Portfolio Duration

For illustrative purposes assume a fund manager has two T-bonds in her portfolio, with market values V_1 = \$20m in Bond-1 and V_2 = \$80m in Bond-2, so that $V_s \equiv V_1 + V_2$ = \$100m. The *portfolio* duration D_S of her cash-market position is:

$$D_S \equiv w_1 D_1 + w_2 D_2 \tag{13.7}$$

where $w_1 = V_1/V_s = 0.20$ (= \$20m/\$100m), $w_2 = V_2/V_s = 0.80$ (= \$80m/\$100m) and D_i is the duration of bond-i, for $i = 1, 2$. Using portfolio duration to calculate N_F assumes a parallel shift in the yield curve and small changes in yields (e.g. 25 bps) – otherwise the hedging error could be large.

13.3.2 Hedging a T-bond Portfolio

Consider a US pension fund manager (Ms Bond) on 1 May who wishes to hedge her portfolio of Corporate bonds (or T-bonds) with market value $V_s = \$1{,}010,000$[4] and portfolio duration $D_s = 6.9$. Ms Bond fears a rise in interest rates over the *next 3 months* (i.e. 1 May to 1 August) and if this occurs the value of her cash-market bond portfolio will fall. The CF of the CTD bond is 1.12 (Table 13.4A). For a portfolio of corporate bonds this would be a cross-hedge, as the corporate bonds are hedged using *T-bond* futures.

Ms Bond, on 1 May, sells September T-bond futures contracts and assuming a parallel shift in the yield curve ($dy_s = dy_f$) and no complications of accrued interest:

$$N_F = -\left(\frac{\$1.1\text{m}}{\$103{,}500}\right)\left(\frac{20}{18}\right)1.12 = -13.2 \ \text{ (short 13 contracts)} \tag{13.8}$$

Suppose interest rates rise between 1 May and 1 August (Table 13.4B). Her cash-market bond portfolio falls by 5% to \$1,045,000 – a capital loss of \$55,000. But the futures price falls by 4 points from $F_0 = 103\text{-}16$ to $F_1 = 99\text{-}16$ and the gain on the short futures position is \$52,000 (= 13 contracts × 128 ticks × \$31.25). The hedge shows a small loss of \$3,000 on an initial cash-market value of about \$1m (i.e. about 0.3%). The unhedged portfolio would have lost \$55,000 (i.e. about 5.5%).

13.4 HEDGING: FURTHER ISSUES

13.4.1 Cross Hedge: Corporate Bond Portfolio

Suppose company-XYZ is going to raise funds by issuing corporate bonds in 6 months' time, after the legal details of the bond issue are completed. The treasurer of company-XYZ may be worried by the possibility of a rise in long-term interest rates over the next 6 months, which will raise the cost of issuing corporate bonds. Company-XYZ therefore hedges by shorting T-bond futures.

TABLE 13.4A Hedging a US bond portfolio

Cash market – 1 May	Futures (September delivery) – 1 May
Market value of bond portfolio, $V_S = \$1.1\text{m}$	CF of CTD bond = 1.12 Size of one contract, $z = \$100{,}000$ Futures Price, $F_0 = 103\text{-}16$ (\$103.5) $V_F = z(F_0/100) = \$103{,}500$
Duration, $D_S = 20$	Duration (futures), $D_f = 18$ Tick value 1/32 equals \$31.25

[4]If all the bonds in the portfolio are 'selling near par' then V_s will be close to the total par value of the bonds.

TABLE 13.4B Hedge outcome

Cash market – 1 August	Futures (September delivery) – 1 August
Loss on bond portfolio = \$1,100,000 – \$1,045,000 = \$55,000	September futures: F_1 = 99-16 (\$99.50 per \$100) Gain on 13 short futures = $N_F\, z(F_0 - F_1)/100$ = 13(\$100,000)(4)/100 = \$52,000 = 13 × 128 ticks × \$31.25 = \$52,000

When hedging corporate bonds, it is important to estimate the relationship between dy_s (the change in the corporate bond yield) and dy_f (the yield on the deliverable bond in the T-bond future contract), since these may change by different amounts over the hedge period.

Regression techniques can be used but results may be subject to error, as changes in corporate 'specific risk' (e.g. IT failures, results of patent applications, environmental issues, default) will affect changes in the corporate bond yield dy_s – whereas changes in dy_f (government T-bond yields) should be largely unaffected by these 'corporate risks'.

13.4.2 PVBP, Convexity, and Perturbation Analysis

Instead of using the duration-based hedge ratio we could obtain the same result using the 'price value of a basis point' (PVBP). This requires the calculation of the PVBP for the cash-market bond portfolio to be hedged and the PVBP for the US T-bond futures contract. This method usually uses the duration approximation for the change in T-bond prices and therefore assumes a parallel shift in the yield curve and small changes in interest rates.

Alternatively, we can calculate the PVBP of the cash-market (spot) bond portfolio using the 'duration + convexity' approximation (for a parallel shift in the spot yield curve of 1 bp):

$$dV_s = -V_s[(D_s dy_s) + (1/2)\chi_s(dy_s)^2]$$

where $\chi_s = \sum_{i=1}^{n} w_i\chi_i$ is the cash-market convexity, $w_i = V_i/V_s$ proportion of the total bond portfolio held in bond-i and χ_i = convexity of bond-i.

Alternatively, we can use the 'full' (present value) pricing formula when considering the effect of a large change in interest rates on the 'total' change in the cash-market value of a T-bond portfolio – this will implicitly incorporate the convexity of the bonds. The 'full valuation' method can be done for either parallel or non-parallel shifts in spot yields – as spot rates might change by different amounts.

13.4.2.1 Non-parallel Shifts

We can also calculate the change in dollar value of *one* futures contract, $\mathrm{dV}_F = \mathrm{V}_F(dF/F)$ using $F_t = S_t e^{r(T-t)} - FVC_T$ where the change is calculated with respect to changes in the spot yields which determine (a) the price of the deliverable bond S_t and (b) the (future) value of any coupon payments FVC_T. The number of futures in the hedge is then:

$$N_F = -\left(\frac{\text{'Total' change in value of cash market, bond portfolio}}{\text{Change in value of one T-bond, futures contract}}\right) \tag{13.9}$$

This is a form of *perturbation analysis* because we choose the size of the change in each spot interest rate along the yield curve and then calculate the impact on the numerator and denominator in Equation (13.9).

13.4.3 Hedging the Market Risk of an Underpriced Corporate Bond

In Chapter 6 we discussed how a hedge fund that buys underpriced stocks can hedge the market risk of the stocks by shorting stock index futures. Consider a similar situation where Ms Bond today, buys what she believes are underpriced corporate bonds (of several companies). She might believe the corporate bonds are underpriced because the rating agencies (and the average bond trader) have given the bond an A-rating but her assessment of their default risk would suggest a higher AA-rating.

She therefore expects a rise in the market price of the bond (even if risk-free government bond yields stay constant), as 'market participants' come to realise that the corporate bond is actually less risky than they first thought. Then she can close out her long corporate bond position at a profit.

However, Ms Bond is exposed to changes in long-term (risk-free) T-bond yields. If long-term risk-free yields increase, this will cause a fall in the cash market price of her corporate bonds, which may more than offset the gains she makes when the 'credit risk underpricing' is corrected. To hedge against future changes in T-bond yields, Ms Bond should *short* N_F, T-bond futures, today:

$$N_F = -\left(\frac{V_p}{V_F}\right)\left(\frac{D_s}{D_f}\right)\beta_y\, CF \tag{13.10}$$

where V_p is the total dollar amount held in her portfolio of underpriced cash-market corporate bonds which have a *portfolio* duration of D_s. If the risk-free T-bond rate increases, then the T-bond futures price will fall and the profit after closing out the futures will just compensate for any loss in value of the cash-market bond position – *solely due to changes in risk-free T-bond yields*.

Ms Bond can then capture any future rise in corporate bond prices due to the correction of the 'credit risk' mispricing. However, her portfolio of underpriced corporate bonds is still subject to any (residual) 'specific risk' (e.g. bankruptcy, regulatory changes, IT failures, reputational damage, etc.) of the bond issuers. The hedge using T-bond futures does not protect the 'mispriced' corporate bond portfolio from specific risks – but overall, specific risk may be small in a well-diversified corporate bond portfolio.

13.4.3.1 Long-Short Bond Portfolio

As in the case of underpriced and overpriced stocks, a bond trader might take a long position in 'underpriced' corporate bonds and short-sell 'overpriced' corporate bonds.[5] The dollar amount of cash-market bonds held long $V_L > 0$ or short ($V_S < 0$), determine the portfolio durations D_L, D_S and hence the number of T-bond futures to go long and short, using Equation (13.10). The net position in futures contracts depends on the sign of $N_L - N_S$, but this ensures the long-short bond speculator is then hedged against parallel shifts in the yield curve, while she waits for the mispricing of the cash-market bonds to be corrected.

13.4.4 Risks in the Hedge

A hedged bond portfolio will not provide a perfect hedge because:

- the hedge period (e.g. 3 months from May to August) may not match the maturity date of the futures contract (e.g. September contract) – this gives rise to basis risk.
- calculating price changes in futures markets are subject to error, in part because it is difficult to ascertain the CTD bond (and hence its duration).
- shifts in the yield curve may not be parallel (there may be twists in the yield curve), so we cannot always assume duration or duration plus convexity provides a good approximation to changes in T-bond cash-market prices.
- If we are hedging a corporate bond portfolio then the change in the value of the corporate bonds due to changes in risk-free T-bond yields can be hedged using T-bond futures but changes in corporate bond prices due to any change in (residual) specific/credit risk will not be hedged.

However, academic studies find that hedging using T-bond futures can reduce price risk, with the duration based hedge ratio performing particularly well. Hedging cash-market positions in *corporate* bonds using *T-bond* futures is also found to be effective, even though this is a cross-hedge and involves possible changes in credit risk.

13.5 MARKET TIMING

Suppose a trader Ms Bond currently holds a bond portfolio and over the next month she expects interest rates to fall sharply. Today, to take advantage of this interest rate forecast she might move out of low duration bonds and into high duration bonds, because the latter will rise in price more than the former. However, this is likely to be costly because it involves 'high'

[5]It may also be possible to find some traders who have price quotes for *cash-market T-bonds* that do not equal the 'fair' or 'correct' T-bond price (calculated using appropriate spot rates of interest) – for example, because they have a poor estimate of the yield curve. In this case a hedge using T-bond futures can also protect the long-short cash market T-bond position from parallel shifts in the yield curve. However, any mispricing of T-bonds is likely to be small, but using highly levered trades may result in substantial gains.

transaction costs of selling low duration bonds and buying high duration bonds – and then reversing these trades in one month's time (when she returns to her 'normal long-run' position in bonds).

A less costly strategy is to continue to hold the original cash market bond portfolio and alter the effective duration of the portfolio using T-bond futures. Suppose, $D_S =$ *portfolio* duration of Ms Bond's cash-market T-bonds, $D_f =$ duration of the T-bond deliverable in the futures contract and $D_d =$ Ms Bond's target (desired) duration. The number of futures contracts required to achieve a desired duration D_d is (see Appendix 13.A):

$$N_F = \left(\frac{D_d - D_s}{D_f} \right) \left(\frac{V_s}{V_F} \right) \beta_y$$

where we usually assume that $dy_s = dy_f$ and hence $\beta_y = 1$.[6] The formula for N_F can be used in a market timing strategy. Suppose you are long a cash-market bond portfolio (so V_s and D_s are positive) then:

- If you expect a *rise in yields* (i.e. fall in bond prices) and therefore require a lower desired duration $D_d < D_s$, then today, *sell* N_F T-bond futures.
- If you expect a *fall in yields* (i.e. rise in bond prices) and therefore require a higher desired duration $D_d > D_s$, then today, *buy* N_F T-bond futures.

13.6 WILD CARD PLAY

There is another practical issue to discuss as regards the US T-bond futures contracts and that is the strategy known as *Wild Card Play*. On the CBOT, trading in T-bond futures ceases at 3 p.m. (EST) while actual T-bonds are traded until 5 p.m. (EST) and the 'short' in the futures contract has until early evening to issue the clearing house with notice of *intention* to deliver. If the notice is issued, the invoice price of the T-bond futures is calculated on the basis of the settlement price (i.e. average of quoted prices just before the close of trading). However, the short can wait and see if she can purchase an eligible bond for delivery, at a lower price in the cash market. To see how this is possible we have to closely examine the delivery process for T-bond futures. Delivery is a 3-day process and involves:

(i) *Position Day:*
The short notifies the clearing house of intention to deliver, two business days later.

(ii) *Notice of Intention Day:*
The clearing house assigns a trader who is long, to accept delivery. The short is now obligated to deliver the next business day.

[6] As expected, this equation reduces to the formula for the minimum-variance hedge ratio when the desired duration is $D_d = 0$ – since the latter implies no change in the value of the 'cash-market bond + T-bond futures' portfolio (for parallel shifts in the yield curve).

(iii) *Delivery Day:*

Bonds are delivered (with the last possible delivery day being the business day prior to the last 7 days in the delivery month).

This delivery process can give rise to a *Wild Card Play* by the short on any 'position day'. If the cash-market price of bonds falls between 3 p.m. and 5 p.m., the short buys the 'low price' CTD bond in the cash market and issues a *notice of intention* to deliver, knowing that on delivery she will receive the 'high' futures settlement price determined as of 3 p.m. that day. However, if the bond price does not decline, she can wait until the next day and repeat this strategy (until the final business day before the final delivery day in the month). In essence, the short has an implicit option that can be exercised during the delivery month, while the long has increased risk because she does not know the exact bond that will be delivered.

There is also a *quality or switching option* for the short, since she can deliver a variety of eligible bonds. Even if she holds a specific bond against delivery on her short futures position, nevertheless if the yield curve shifts she may choose to deliver another yet cheaper bond. The *timing option* applies when the bond held for delivery by the short pays a coupon (rate) that exceeds the cost of financing her cash-market bond position (i.e. the repo rate). Then it is better if the short delays delivery. There is also an *end of the month option* held by the short, since the last trading day for the T-bond futures contract is the eighth-to-last business day. The futures settlement price is fixed in over this period, but the short still has the option to announce her intention to deliver on any day up to the penultimate business day. She can therefore wait to see if cash-market bond prices fall over this period and deliver a 'low' price bond if one becomes available.

All of these embedded options available to the person holding a short T-bond futures position should lead to a lower futures price – in order to compensate the long, for the increased risk. But it is thought that most of the time these options do not distort the cash-and-carry arbitrage relationship. However, these embedded options for the short do make it more difficult to accurately calculate both the optimal hedge ratio and the futures price based on the cash-and-carry approach.

13.7 PRICING T-BOND FUTURES

13.7.1 Futures Price on Deliverable Zero-coupon Bond

Suppose today is time t and the futures contract matures at T, so the time to maturity is $T - t$. The underlying asset to be delivered at maturity T is a single *zero-coupon* bond. Today, we can borrow at r and purchase the underlying bond for a price S in the cash market and later deliver it against the short futures position. The usual risk free arbitrage argument ensures that:

$$F = S[1 + r_s(T - t)] \qquad \text{(simple interest)} \tag{13.11a}$$

$$F = S(1 + r)^{T-t} \qquad \text{(discrete compounded rates)} \tag{13.11b}$$

$$F = Se^{r(T-t)} \qquad \text{(continuously compounded rates)} \tag{13.11c}$$

The implied repo rate ($\hat{r}$) is the return from selling the futures at $t = 0$ for F and simultaneously buying the underlying bond at S, so that $\hat{r} = (F/S)^{1/T} - 1$ (compound rate). In this case, risk-free arbitrage is possible if $\hat{r}$ does not equal r, the cost of financing the arbitrage strategy using the actual repo market.

13.7.2 Futures Price on Deliverable Coupon Paying Bond

In practice there is not a futures contract on a zero-coupon bond. As we saw in Chapter 3 if we ignore some important practical issues, the fair price (at t) of a T-bond futures contract using cash-and-carry arbitrage is:

$$F_t = [S_t - PV(C)]e^{r(T-t)},$$

where S is the invoice price of the cash-market bond (i.e. the 'clean price' plus accrued interest), $PV(C)$ is the *present value* of any coupon payments (on the deliverable bond) between today and the maturity date T of the futures contract and r is the (continuously compounded) spot yield (for maturity $T - t$). Note that the T-bond futures price can also be written $F_t = S_t e^{r(T-t)} - FVC_T$, where the *future* value at T of the coupon payments is $FVC_T = \sum_{i=1}^{N} C_i e^{r_i(T-t_i)}$.

Unfortunately, in practice it is difficult to accurately calculate the fair T-bond futures price. This is because of the flexibility of the short's decision over the delivery date T and the precise choice of the CTD bond and hence its invoice price S.

First, note that the cost of creating a synthetic futures contract is the cost of buying the underlying cash-market bond at S_t using borrowed funds, which accrues to a debt of $S_t e^{r(T-t)}$ at maturity T (of the futures contract). The cost-of-carry is offset by the N coupons C_i received from the cash-market bond at times t_i from today. When these coupons are reinvested at the (expected) risk-free rates r_i (between t_i and T) they have a future value at T of:

$$FVC_T = \sum_{i=1}^{N} C_i e^{r_i(T-t_i)} \tag{13.12}$$

The no-arbitrage futures price is then:

$$F_t = S_t e^{r(T-t)} - FVC_T \tag{13.13}$$

where r is the interest rate over the period (t, T). Considering the additional complexities of the conversion factor, accrued interest etc., the futures price for a contract which matures at T

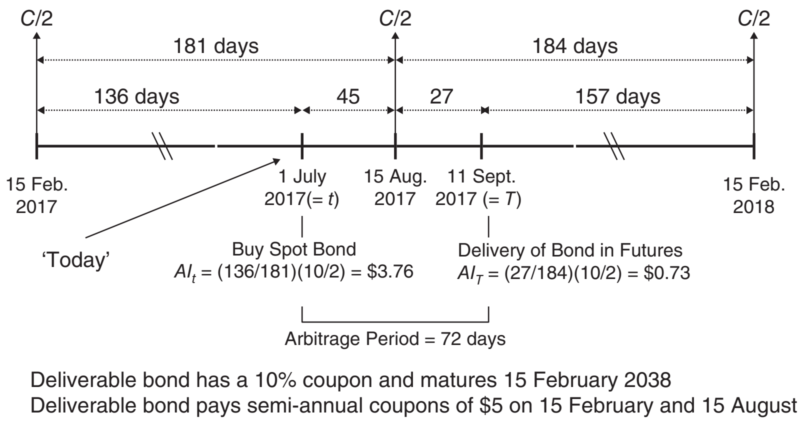

FIGURE 13.1 Pricing a T-bond futures contract

(and has time $T - t$ to maturity) is:

$$F_t = (1/CF_t)(S_t e^{r(T-t)} - FVC_T - AI_T) \tag{13.14}$$

Let's examine this from a more practical point of view, which requires some additional points to be made. Suppose it is 1 July 2017. The yield curve is flat and $r = 0.03$ (3% p.a., continuously compounded). You are deciding whether cash-and-carry arbitrage is possible with the September T-bond futures contract, which has a known maturity date $T = 11$ September (Figure 13.1).

By buying a cash-market T-bond on 1 July and carrying it forward to 11 September, you create a synthetic T- bond future, since the T-bond can be delivered against the futures contract. Assume the CTD bond is a 10%-coupon T-bond which pays semi-annual coupons of $5 each year, on 15 August and 15 February and matures on 15 February 2038. It has a CF of 1.22.

Today at $t = 1$ July to the maturity of the futures, the deliverable bond pays one coupon of $C = \$5$ on 15 August, which when invested over 27 days is expected to accrue to a *future value* at time T of:

$$FVC_T = 5e^{0.03(27/184)} = \$5.022.$$

On 1 July, borrowing at the risk-free (repo rate) r and purchasing the bond in the cash market for an *invoice price* S_t, leads to an amount owing at $T = 11$ September, of $S_t e^{r(T-t)}$ (i.e. purchase cost plus the repo interest cost). Therefore at T, the net cost-of-carry in the cash market is:

$$\text{Net cost-of-carry (cash market) at } T = S_t e^{r(T-t)} - FVC_T \tag{13.15}$$

But the above strategy creates a *synthetic T-bond future* since it ensures that the bond purchased in the cash market on 1 July is available for delivery against the futures contract at

$T = 11$ September. At maturity of the futures T, the *invoice price* the long pays the short for delivery of the underlying T-bond is the quoted price of F_t plus accrued interest[7]:

$$\text{Invoice price of futures at } T\text{:}\ IPF_T = F_t(CF) + AI_T \tag{13.16}$$

$AI_T = \$0.73$ is the accrued interest (between 15 August and $T = 11$ September) on the CTD bond delivered at T (see Figure 13.1). AI_T is a proportion of the *next* coupon which accrues on 15 February 2018.

Since the actual futures contract and the synthetic futures both deliver one bond at T then the cost today must be equal, otherwise risk-free arbitrage profits would be possible. Hence equating (13.15) and (13.16):

$$F_t(CF) + AI_T = S_t e^{r(T-t)} - FVC_T \tag{13.17}$$

The no-arbitrage (fair) futures price is:

$$F_t = (1/CF)(S_t e^{r(T-t)} - FVC_T - AI_T) \tag{13.18}$$

where S_t is the cash-market bond price (i.e. clean price B_t plus accrued interest AI_t payable when initially purchased at time t). If there are no coupon payments over the arbitrage period (i.e. $FVC_T = 0$), no accrued interest ($AI_T = 0$) and the deliverable bond is the one specified in the futures contract ($CF = 1$), then not surprisingly the above formula for F_t reduces to that for the futures price on a zero-coupon bond, namely $F_t = S_t e^{r(T-t)}$.

13.7.3 Arbitrage

Profitable risk-free arbitrage would ensue if the invoice price of the futures IPF_T received at T, exceeds the cost of the synthetic futures ($S_t e^{r(T-t)} - FVC_T$) at T.[8] Today at t, the arbitrageur borrows an amount S_t at a cost of r (the repo rate), purchases the bond and simultaneously sells T-bond futures. The amount *owed* at T, the expiry of the futures contract is:

$$= \textit{Loan plus interest, } S_t e^{r(T-t)} \textit{ less the value at } T \textit{ of (reinvested) coupons } FVC_T.$$

Hence the net (cash-market) cost at T is $S_t e^{r(T-t)} - FVC_T$, but the arbitrageur receives a larger amount IPF_T. This arbitrage creates additional demand for cash-market bonds so S_t increases and sales of bond futures will reduce F_t until the equality in (13.17) is restored and the equilibrium futures price is given by Equation (13.18).

[7] The futures price could also be written $F_{t,T}$ which makes it clear that the quoted price today applies to the futures which matures at T.
[8] Arbitrage is also examined using the implied repo rate in Appendix 13.A.

However, the arbitrage strategy is not completely risk-free because the coupon payments received from the cash market T-bond have to be re-invested at interest rates which are unknown at $t = 0$ (1 July). At t, the best guess of the reinvestment rate for any coupon receipts would be the implied forward rate – and to lock in this rate would require a strip of FRAs. Hence cash-and-carry arbitrage only provides an approximate formula for the T-bond futures price.

EXAMPLE 13.4

Pricing T-Bond Futures

Assume on 1 July, $CF = 1.05$, the yield curve is flat and the risk-free rate $r = 3\%$ p.a. Note that on 1 July the invoice price S_t of the cash market bond will be the 'clean' or quoted price B_t plus any accrued interest on the bond (i.e. from 15 February 2017 to 1 July 2017 – see Figure 13.1). Assume the clean price of the deliverable bond on 1 July is $B_t = 101$. The accrued interest from 15 February 2017 to 1 July 2017 (Figure 13.1) is $AI_t = (136 \text{ days}/181 \text{ days})\$5 = \$3.76$. The cash-market bond price is:

$$S_t = B_t + AI_t = \$101 + \$3.76 = \$134.76 \tag{13.19}$$

Compounding S_t forward to the delivery date of 11 September (i.e. over 72 days) $FVC_T = 5.022$. The net cost-of-carry in the cash market at T is:

$$S_t e^{r(T-t)} - FVC_T = \$104.76\, e^{0.03(72/365)} - \$5.022 = 105.38 - 5.022 = \$100.36 \tag{13.20}$$

Accrued interest payable to the short-futures is $AI_T = (27 \text{ days}/184 \text{ days})\$5 = \$0.73$, hence:

$$F_t = (1/CF_t)(S_t e^{r(T-t)} - FVC_T - AI_T) = (1/1.05)(\$100.36 - \$0.73) = \$94.88 \tag{13.21}$$

Note that this figure, for the no-arbitrage price F_t is an estimate, since we do not know the precise delivery date, the precise bond that will be delivered and the reinvestment rate of any coupons received on the underlying bond (since this may differ from the current repo rate, if the yield curve shifts).

13.8 T-BOND FUTURES SPREADS

A spread is a long position in one asset and a short position in another (similar) asset. A T-bond futures spread can be used to speculate on twists or parallel shifts in the yield curve. For example, a trader Ms Bond could undertake a spread trade by buying a T-bond futures (which delivers a bond with at least 15 years to maturity) and simultaneously selling a 10-year T-bond in the cash market (i.e. which has a different maturity/duration to the CTD bond in the futures). She might then profit after a change in interest rates as the futures price and cash-market prices change by different amounts (because their underlying durations are different). For example, if all interest rates fall by 1% she would gain more on the long futures than she loses on the short bond position, because the duration of the futures is higher than the duration of the cash-market bond.

A spread trade is less risky than holding just a naked futures position – although potential profits are also less. However, such a strategy would have to take account of margin requirements and any 'haircuts' on short-sales of bonds in the cash market. A spread trade may be less costly if *two futures contracts* are used, with different maturities.

13.8.1 Turtle Trade: Arbitrage Profits

13.8.1.1 Buying the Spread

First, let us examine how an arbitrage 'June-September' T-bond futures spread position can be used to exploit any mispricing along the yield curve (Figure 13.2). Suppose at $t = 0$ (1 April 2017) we:

(a) buy a T-bond futures which matures at T_1 (1 June 2017)
(b) sell a T-bond futures which matures at T_2 (1 September 2017) and
(c) you borrow money at the forward repo rate $f = 4.16\%$ (applicable between June and September) by selling the June-Eurodollar futures on 1 April.

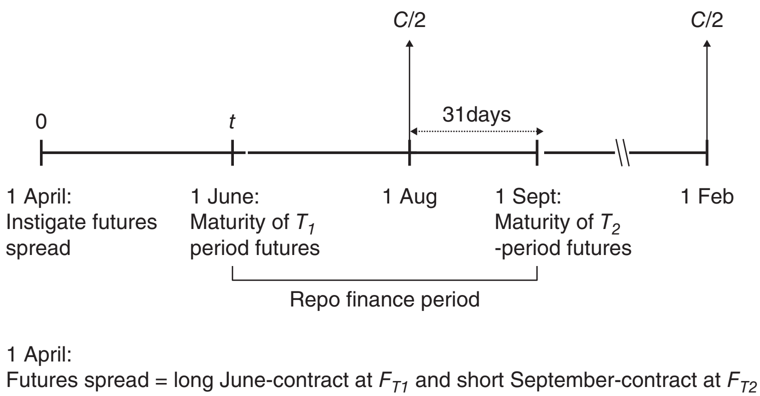

FIGURE 13.2 T-bond futures spreads

We are *buying the spread* (i.e. we buy the nearby T-bond futures contract). When the nearby T-bond futures contract matures we take delivery of the eligible bond (which matures on 1 August 2038, say) and pay F_{T_1} (plus any accrued interest, AI_{T_1}) with finance raised by borrowing at the (forward) repo rate.

We then 'carry' this cash market bond until 1 September when it is delivered against the short September-futures position and we receive F_{T_2} (plus any accrued interest AI_{T_2}, arising from the underlying bond's *next* coupon payment on 1 February 2018). The (compound) return from this long-spread strategy[9] is the implied repo rate $\hat{r}$:

$$(1+\hat{r}) = \left(\frac{Amount\ received\ at\ T_2}{Amount\ paid\ at\ T_1}\right)^{1/(T_2-T_1)}$$
$$= \left(\frac{F_{T_2}CF_{T_2} + AI_{T_2} + FVC_{T_2}}{F_{T_1}CF_{T_1} + AI_{T_1}}\right)^{1/(T_2-T_1)} \tag{13.22}$$

where FVC_{T_2} is the (future) value of any coupons paid on the 'carried' cash-market bond on 1 August 2017, compounded to $T = 1$ September 2017. Note that the implied repo rate is a forward rate. If the implied repo rate $\hat{r} = 4.3\%$ (say) exceeds the cost of borrowing at the actual forward repo rate $f = 4.16\%$ (applicable between 1 June and 1 September), then a long arbitrage spread trade undertaken on 1 April is guaranteed to be profitable.

The actual repo borrowing rate is the forward rate $f = 4.16\%$, when viewed from $t = 0$ (1 April 2017). How can we 'lock-in' this forward rate on 1 April? Today we sell June-Eurodollar futures contracts at $F_{T_1}^{Euro\$} = \$99$. When we close out the Eurodollar futures in June, the effective cost (between June and September) is the implied forward rate $f = 4.16\% (= 100/99)^{365/90} - 1)$.[10] You use the cash inflow on 1 June from closing out several short *Eurodollar* futures contracts, to pay F_{T_1} (plus accrued interest) for delivery of the eligible *T-bond* in the maturing June T-bond futures. Hence if the implied (forward) repo rate from buying the spread is $\hat{r}$ (= 4.5%) and the cost of borrowing is f (= 4.16%, the actual forward repo rate), we undertake the following turtle trade on 1 April:

Turtle trade

Buy the T-bond futures spread and sell June-Eurodollar futures.

When the implied repo rate (i.e. the 'percentage return' on the T-bond spread trade) exceeds the actual (forward) repo cost of borrowing, an overall profit will be made on the turtle trade.

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[9]If we ignore the complications due to coupon payments, the conversion factor and accrued interest then Equation (13.22) is simply $(1+\hat{r}) = (F_{T_2}/F_{T_1})^{1/(T_2-T_1)}$, that is, the return from the long-short T-bond futures positions.

[10]We assume 90 days between 1 June and 1 September, for simplicity.

13.8.1.2 Selling the Spread

Suppose we have the reverse position on 1 April, namely the implied repo rate $\widehat{r} = 4.5\% < f = 4.8\%$, the actual (forward) repo rate. Then you sell the T-Bond spread, so on 1 April:

(a) sell the June T-bond futures
(b) buy the September T-bond futures and
(c) you lend money at the forward repo rate $f = 4.16\%$ (applicable between June and September) by buying the June Eurodollar futures on 1 April.

Selling the June T-bond futures and buying the September T-bond futures is equivalent to borrowing at the forward rate $\widehat{r} = 4.5\%$. By buying the *Eurodollar* futures is equivalent to lending at $f = 4.8\%$. Hence the trade results in an arbitrage profit of 0.3%.

The above strategy is an *intermarket spread* since it involves two T-bond futures contracts which mature at different dates but are on the same underlying asset (i.e. a cash-market T-bond). They are not risk-free trades because of the difficulty of ascertaining the CTD bond in each contract and uncertainty surrounding the reinvestment rate of any coupons paid.

13.9 SUMMARY

- Contract details for T-bond futures are necessarily complex because of the use of the conversion factor (CF), the cheapest-to-deliver bond (CTD), and Wild Card Play.
- Holders of cash-market T-bonds can hedge their position, using the duration based hedge ratio to determine the number of T-bond futures contracts N_F *to short*. Conversely, to hedge a future purchase of cash-market T-bonds, you *buy* N_F T-bond futures, today.
- T-bond futures contracts are useful for hedging a *corporate* bond portfolio, even though corporate bond prices and T-bond futures prices might not be perfectly positively correlated (e.g. due to changes in credit risk of the corporate bond) – hence there will be some hedging error. This is a cross hedge.
- Even when hedging a T-bond portfolio with T-bond futures a perfect hedge is not possible because of basis risk (i.e. the cash-market T-bond prices and futures prices do not move perfectly together). This is exacerbated by non-parallel shifts in the yield curve.
- T-bond futures are used to increase (decrease) the effective duration of an existing cash-market bond portfolio, when the investor forecasts yields to fall (rise). This is a market timing strategy.
- Speculation with T-bond futures often involves *spread trading* where the speculator takes a long position in nearby T-bond futures and a short position in T-bond futures with a longer maturity date (or vice versa). Spread trades can be used to speculate on parallel shifts and twists in the yield curve.

- The fair price of a T-bond futures contract can be determined using cash-and-carry arbitrage.
- Profitable arbitrage opportunities with T-bond futures are usually expressed in terms of the *implied repo rate*. If the implied repo rate differs from the actual repo rate (i.e. the cost of lending/borrowing funds), then profitable arbitrage is possible.

APPENDIX 13.A: HEDGING: DURATION AND MARKET TIMING

First (for illustrative purposes), consider a cash-market (spot) position in two bonds, where S_i = (invoice) price of bond-i and N_i = number of bonds-i held ($N_i > 0$ if held long or $N_i < 0$ if short-sold). The dollar value and change in the cash-market T-bond portfolio is:

Cash Market

$$V_s = N_1 S_1 + N_2 S_2 \tag{13.A.1}$$

$$dV_s = V_1(dS_1/S_1) + V_2(dS_2/S_2) = -[V_1(D_1 dy_1) + V_2(D_2 dy_2)] \tag{13.A.2}$$

where we use the duration approximation, $dS_i/S_i = -D_i\, dy_i$. For a parallel shift in spot yields $dy_1 = dy_2 = dy_s$, then:

$$dV_s = -V_s(D_s dy_s) \tag{13.A.3}$$

where $V_s \equiv V_1 + V_2$, $w_i = V_i/V_s$ for $i = 1, 2$ and $D_S \equiv w_1 D_1 + w_2 D_2$ is the portfolio duration of the cash-market bond portfolio.

T-bond futures

$$\begin{aligned} z &= \text{contract size (\$100,000 for US T-bond futures)} \\ F &= \text{(invoice) price of T-bond futures} \\ V_F &= (z/100)\,F = \text{'face value' of one futures contract} \\ N_F &= \text{number of futures contracts.} \end{aligned}$$

The change in value of a portfolio consisting of 'cash-market T-bonds+T-bond futures' (ignoring accrued interest) is:

$$\begin{aligned} dV &= dV_s + N_F\ z\ (dF/100) = V_s(D_s dy_s) + N_F V_F (dF/F) \\ &= -[V_s(D_s dy_s) + N_F V_F (D_f dy_f)] \end{aligned} \tag{13.A.4}$$

Assume the relationship between the change in the YTM of the cash-market T-bonds and the change in the yield (on the deliverable bond) in the T-bond futures contract is $dy_s = \beta_y dy_f$, then:

$$dV = -[V_s D_s \beta_y + N_F V_F D_f] dy_f \tag{13.A.5}$$

Hedging

To *fully hedge* your cash-market T-bond portfolio we set $dV = 0$, hence:

$$N_F = -\left(\frac{V_s}{V_F}\right)\left(\frac{D_s}{D_f}\right)\beta_y \tag{13.A.6}$$

which is sometimes referred to as the minimum variance hedge ratio.

Effective Duration and Market Timing

Suppose a trader holds a cash-market (spot) T-bond portfolio and she wants to change the effective duration of the portfolio to some desired level D_d, using T-bond futures contracts. The *desired change* dV_d in her cash-market T-bond portfolio is:

$$dV_d = -V_s D_d dy_s = -V_s D_d (\beta_y dy_f) \tag{13.A.7}$$

Equating (13.A.5) and (13.A.7):[11]

$$N_F = \left(\frac{V_s}{V_F}\right)\left(\frac{D_d - D_s}{D_f}\right)\beta_y \tag{13.A.8}$$

The cash-market T-bond position could be either net long or short – depending on the sign of $V_s \equiv V_1 + V_2$ – and the cash-market portfolio duration could be positive or negative depending on the value of D_s. For the moment assume V_s and D_s are both positive. Then (13.A.8) gives the required number of futures contracts to achieve a 'desired duration':

If $D_d > D_s$ $\Rightarrow$ take a long position in T-bond futures,

If $D_d < D_s$ $\Rightarrow$ take a short position in T-bond futures.

Hence, to increase (decrease) the effective duration of an underlying (long) cash-market bond portfolio (with $D_s > 0$), we go long (short) T-bond futures. This makes sense because if interest rates subsequently fall, then you make profits both from the cash market bond

[11] If discrete compound rates are used, the durations D would be 'modified durations'.

portfolio and the long T-bond futures contracts – this is equivalent to increasing the duration of the cash-market bond portfolio and is a form of 'market timing'. If your forecasts are correct then you will make more profit by taking the long futures position rather than just holding your cash-market bond portfolio.

Note that (13.A.6) the *minimum variance hedge ratio* is a special case of (13.A.8), where the desired duration, $D_d = 0$:

$$D_d = 0 \Rightarrow N_F = -\left(\frac{V_s}{V_F}\right)\left(\frac{D_s}{D_f}\right)\beta_y \Rightarrow \text{a short position in T-bond futures}$$

APPENDIX 13.B: IMPLIED REPO RATE AND ARBITRAGE

The implied repo rate is another way of determining whether risk-free arbitrage profits are possible. The *implied* repo rate is the return from buying the underlying cash-market T-bond at $t = 0$ for the invoice price $S_0 = (B_0 + AI_0)$ and simultaneously selling the futures for delivery at T. The return or *implied repo rate* $\hat{r}$ (using discrete compounding) is:

$$\begin{aligned} 1 + \hat{r} &= \left(\frac{\textit{cash received at T from hedge portfolio}}{\textit{cash paid out at } t = 0 \textit{ for the underlying bond}}\right)^{1/T} \\ &= \left(\frac{F_{0,T}\, CF_0 + AI_T + FVC_T}{S_0}\right)^{1/T} \end{aligned} \tag{13.B.1}$$

where $F_{0,T}$ = futures price set at $t = 0$ for payment at T (when the underlying bond is delivered) and the conversion factor is CF_0. The arbitrageur holds the underlying bond and receives coupon payments which can be reinvested over the period $t = 0$ to T to give a future value FVC_T. The accrued interest on the deliverable bond at maturity of the futures contract is AI_T. The cash-market bond has an invoice price of S_0 and this is financed by borrowing at the (actual) repo rate r, at $t = 0$.

Cash-and-carry arbitrage is profitable if the implied repo rate $\hat{r}$ > *actual repo rate* r

It is easy to see using (13.B.1) that if $\hat{r} > r$ then:

$$\frac{F_{0,T}\, CF_0 + AI_T + FVC_T}{S_0} > (1 + r)^T \tag{13.B.2}$$

Part of the reason for using the implied repo rate to determine possible arbitrage profits is that it is straightforward to compare the implied repo rate with the quoted repo rate (cost of borrowing funds). Using (13.B.2), the elimination of any arbitrage profits implies the fair (no-arbitrage) T-bond futures price is:

$$F_{0,T} = (1/CF_0)[S_0(1+r)^T - FVC_T - AI_T] \tag{13.B.3}$$

If we replace the discrete compound rate $(1+r)^T$ by the continuously compounded rate e^{rT} then Equation (13.B.3) is equivalent to the formula for the 'no-arbitrage' or 'fair price' given by Equation (13.14) in the text.

EXERCISES

Question 1
Explain whether you would undertake a long or short futures hedge if you plan to purchase a cash-market T-bond in 30 days' time and you want to hedge against adverse outcomes.

Question 2
On 1 July you hold a bond portfolio of \$10m with duration of 7 (years). The December US T-bond futures price is 95-12 ('95 and 12/32'). Contract size of the T-bond futures is \$100,000.

The cheapest-to-deliver (CTD) bond has a duration of 9 years and a conversion factor of unity. On average, the change in the bond yield equals 90% of the change in the futures yield.

(a) How many T-bond futures contracts are needed to hedge your position over the next 2 months?
(b) How can the duration of your bond portfolio be reduced to 3 (years)?

Question 3
How does the 'wild card play' help the person who holds a short position in a US T-bond futures contract?

Question 4
The current US, T-bond futures price is 101-12 (=101-12/32). Which of the following three bonds is cheapest-to-deliver?

Bond	Bond price (32nds)	Conversion factor (CF)
2	142-20	1.3690
3	120-00	1.1200
4	144-16	1.4100

Question 5

The average duration of your \$1m US bond portfolio, on 15 February is 8 years. The September T-bond futures price is currently 110-16. The cheapest-to-deliver 'Note' against the futures contract has a duration of 7 years. How can you hedge against interest changes over the next 7 months? How many futures contracts do you require? (Ignore the complexities of the conversion factor and cheapest-to-deliver bond).

Question 6

Intuitively, if you are long a bond portfolio with duration $D = 10$ years, how can T-bond futures be used to give a lower effective duration? Why might you want to lower the duration of your cash-market bond portfolio?

PART IV

OPTIONS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 14

Options Markets

Aims

- To discuss the organisation of options markets including the role of the clearing house and interpreting newspaper quotes.
- To consider the payoffs at maturity for long and short positions in calls and puts.
- To show how options can be used for (i) speculation and (ii) providing *insurance* against adverse outcomes, while allowing the investor to also benefit from any 'upside'.
- To explain some terminology applied to calls and puts, such as in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM), as well as intrinsic value and time value.

14.1 MARKET ORGANISATION

Options are traded on individual stocks, stock indices, commodities (e.g. crude oil, gold), foreign currencies, futures contracts and to a much lesser extent on Treasury notes and Treasury bonds. The major exchanges for financial options, are the Chicago Board Options Exchange (CBOE) and the Philadelphia Stock Exchange (PHLX) in the US and Intercontinental Exchange in London. The CBOE was established in 1973, initially trading stock options. It is the largest organised options market, trading standardised contracts and has a deep secondary market.

The over-the-counter (OTC) options market tailors the option contract to the buyer's specifications and is now very large and probably over 10 times larger than the exchange traded options market – although often the *secondary* OTC market is thin. However, some active secondary OTC markets do exist, particularly the market in European options on foreign exchange negotiated for commercial customers. The advantage of OTC markets can

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be illustrated by considering a portfolio manager who wishes to alter the risk profile of her 'specialist' portfolio of stocks by buying an OTC put option. With the OTC contract:

- She can exactly match her own portfolio composition with the underlying assets in the tailor-made OTC put option.
- Expiration date of the put can be tailored to her investment horizon.
- She maintains anonymity, so her belief that the market will fall is not communicated to other traders.
- Drawbacks to using the OTC market are possibly higher transaction cost as well as counterparty risk.

14.1.1 US Stock Options

We use the example of US stock options and stock index options to illustrate some of the administrative procedures which operate when these are traded on an exchange.

14.1.2 Contract Size

Options are traded on CBOT (in Chicago) and options on individual stocks are usually for delivery of 100 stocks. Options on stock *indexes* are also available and are cash settled. The exchange sets the dollar value of an index point (e.g. \$100). For example, the cash payoff to a long call option on a stock index (S&P 500) for $S_T = 2{,}050$ and $K = 2{,}000$ is $\$100(S_T - K) = \$5{,}000$, which is paid by an (assigned) writer (seller) of the option (who has not closed out). The cash payment is usually based on the stock index quote (on the NYSE) at the *end of the day* on which the option is exercised (rather than when the exercise order is made).

14.1.3 Expiration Dates

These are fixed by the exchange and stock options are traded for expiration up to 4.30 p.m. (Central Time) on the third Friday of the expiry month. The holder of a long call on a stock can instruct his broker to exercise the option up until 4.30 p.m. on that Friday. The broker then has until 10.59 p.m. the next day (Saturday) to notify the exchange that exercise will happen.

Expiration dates for options on individual stocks usually extend to about 9 months – with some exceptions. For example, *LEAPS (Long Term Equity Anticipation Securities),* which are primarily options on around 500 individual (different) stocks (but some are also on stock indices), have expiration dates of about 3 years ahead. Similarly *FLEX (short for 'flexible') options* on stocks and stock indices can have any expiration date of up to 5 years ahead and in addition they permit the purchaser to set any exercise price and can be European or American.

14.1.4 Strike/Exercise Prices

These, for example, might be set at \$2.50 intervals when the underlying stock price is less than \$25, at \$5 intervals when the stock price is between \$25 and \$200 and at \$10 intervals for a stock price over \$200. Strike prices are set either side of the *current* stock price and as the stock price moves up or down, options with new strike prices are added by the exchange.

14.1.5 Trading

Over 95% of orders on the CBOE are via an electronic platform. But trading still takes place on the floor of the CBOE. An individual who has purchased (or rents) a seat on the CBOE may be a 'market maker' who stands ready to quote both bid and ask (offer) prices on the option. The 'bid' price is the price the market maker is prepared to buy the option and the 'ask' price is the price at which he is prepared to sell. Prices are quoted by open outcry on the floor of the exchange. Market makers must stand ready to trade with investors. An investor who has purchased an option can close out her position by selling (writing) the same option (i.e. with the same strike and expiration date).

14.1.6 Options Clearing Corporation (OCC)

For US options markets (except those trading futures options) the Options Clearing Corporation (OCC) standardises contracts and acts as an intermediary, effectively creating two separate contracts.

For example, if a trader buys a call option the OCC guarantees that the writer will honour the contract. An option *writer* represents a credit risk to the OCC since she may not have funds to purchase the underlying asset (e.g. stock) in the spot market to facilitate delivery. The OCC therefore requires *the writer* to post a margin payment (usually in cash and equal to at least 30% of the value of the stocks underlying the option contract plus the call premium). There is also a maintenance margin which might be set at a minimum of 15–25% of the value of the stocks underlying the option contract. An option *buyer* has no margin requirement with the OCC since the most she can lose is the option premium, which is paid in full at the outset of the contract.

Initial margins vary depending on whether one has a naked (uncovered) position (i.e. no offsetting holding in the underlying stocks) or a covered/hedged position. The latter is less risky and involves less initial margin. Margin positions on strategies such as straddles, spreads etc. are governed by special rules described in publications on the CBOE website.

Take, for example, a written call on a stock, where the initial margin is 30% of the value of the stock, plus the option premium. However, if the call is out-of-the-money ($S_0 < K$) then the margin is reduced by $K - S_0$. If $S_0 = 110$, $K = 120$ (i.e. out-of-the-money) and $C_0 = \$2$, and each call is written on 100 stocks, then the margin payment would be

$\$2{,}500 = 100[0.3S_0 + C_0 - (K - S_0)]$. If the call had been in-the-money or at-the-money the deduction of $K - S_0$ would not be allowable. A similar rule applies to written puts.

Option prices on a stock *index* are less volatile and hence (uncovered) written index options would have a margin requirement based on, say, 15% of the dollar value of the stock-index, rather than the 30% for individual stock options.

14.1.7 Orders

For example, suppose in January Ms Long wants to buy a December-call option on AT&T stocks with a strike of $K = 90$. Her broker would instruct (electronically) a *market maker* with a seat on the exchange. Trading is conducted in a pit. The buyer shouts out the bid (e.g. \$4) and accepts the lowest offers 'shouted' to him by the market maker (e.g. \$4.1), giving a contract price of \$410 (for delivery of 100 stocks per contract). The OCC keeps track of all trades on the exchange.

Next Ms Long deposits \$410 with her broker, who passes this on to his OCC 'clearing firm-ABC' by the next (business) morning. This money is credited to the option writer's clearing firm-XYZ. The option writer's clearing firm-XYZ also receives a margin payment from the writer of the option. The clearing firms ABC and XYZ then aggregate all the transactions of their customers and deposit funds with the OCC as surety on the net contracts outstanding, according to a prearranged formula. The OCC is therefore the ultimate guarantor for Ms Long, the purchaser of the call option. Because the clearing firms also hold some default risk, the OCC imposes minimum capital requirements on them.

14.1.8 Offsetting Order

If in January Ms Long originally purchased a December-90 call option on AT&T at \$4.10 (total cost = \$410) then she can sell this contract in June (say) by placing an offsetting order. If Ms Long sells at \$4.50 then \$450 will be passed to her clearing firm ABC (and then on to Ms Long) and the OCC will cancel Ms Long's position in this contract. The trader who buys the call option from Ms Long will usually not be the person from whom Ms Long initially purchased the contract. If the purchaser of Ms Long's sale of her option is establishing a new long position in the contract then the 'open interest' would remain the same, with the new 'owner' noted by the OCC. If both traders are closing existing positions, the open interest falls by one contract.

14.1.9 Exercising an Option

If the buyer (Ms Long) of a December-90 AT&T call option holds the option to maturity and exercises the option, her broker notifies the OCC member (ABC) that clears its trades. ABC then places an exercise order with the OCC who '*assigns*' *a* member (XYZ) with an outstanding short position in the same option (i.e. with same strike, same maturity, same underlying asset).

This may be a random assignment or a ‘first-in, first-out’ rule may be used. The OCC member (XYZ) then selects a specific investor (Ms Short) who has an existing short position – using a specific procedure, for example random allocation. Ms Short having then been *assigned,* must fulfil the delivery terms – she *must deliver* 100 stocks of AT&T and she will receive $K = 90$ per stock. A cash amount equal to the strike price is passed from Ms Long’s broker via her clearing firm-ABC to the assigned writer’s clearing firm-XYZ. Hence the ‘assigned writer’ is unlikely to be the original trader who initially sold Ms Long the call option.

Suppose Ms Long holds a December-90 *put option* on a stock to its expiration date. Then a trader holding a short position *must* take delivery of 100 stocks of AT&T from Ms Long and pay her $K = 90$ per stock. When an option is exercised the open interest falls by one. All at-the-money options should be exercised (if the payoff at maturity is greater than any transactions costs) and some brokers will do this automatically for their clients. A very rough estimate is that about 10% of calls and puts on the CBOE are exercised and about 30–40% expire out-of-the money.

14.1.10 Commissions

Market makers trade their own book but are obliged to ‘fill’ all public orders. They make profits (losses) on their own book and also earn the bid–ask spread on a buy-sell ‘round trip’. Market makers buy at the bid price and sell the same contract at a higher offer (ask) price. The exchange sets upper limits for the bid-offer spread – for example, \$0.75 for options prices between \$10 and \$20.

Commissions for retail investors are usually a fixed cost plus a proportion of the value of the trade – and discount brokers charge less than full-service brokers. For example, a dollar trade of between \$3,000 and \$10,000 via a discount broker might involve commission of ‘\$30 +1% of the dollar amount’ in the trade. Larger trades generally involve larger fixed costs but lower variable cost. When offsetting an existing position in an option, the commission must be paid again. When *exercising* a stock option, the commission will probably be the same as for a buy or sell order and may be around 1% of the stock’s dollar value.

14.1.11 Position Limits and Exercise Limits

The exchange (often on the instructions of the regulator) also specifies position limits, namely the maximum number of contracts that an investor can hold on one side of the market. For individual stock options (CBOE) with high market capitalisation and trading volume, the position (and exercise) limit might be set at 250,000 contracts (on the same side of the market). Long calls and short puts are on the same side of the market because each contract’s value increases (decreases) as the underlying price increases (decreases). Put differently, the premia on long calls and short puts are positively correlated. Short calls and long puts are also on one side of the market.

Investors are also limited in the number of contracts that can be *exercised* in any five consecutive business days (i.e. exercise limit). The figure for the exercise limit is usually the same as the position limit. The purpose of position and exercise limits is to prevent single traders or groups of traders acting together and having a significant influence on the market price. However, theses limits may reduce liquidity and may drive some business into the OTC market.

14.1.12 Newspaper Quotes

Illustrative price quotes on 2 November for call and put options on stock-XYZ are shown in Table 14.1. The expiration dates are in the first column. The current price (on the NYSE) of the

TABLE 14.1 Option on Stock-XYZ (2 November)

Expiration	Strike	Call			Put		
		Price	Volume	Open Interest	Price	Volume	Open Interest
Jan	20.00	6.85	216	136,915	0.09	263	146,405
Apr	20.00	7.35	412	2,259	0.34	10	6,422
Nov	22.50	4.29	368	5,888	0.03	100	10,314
Dec	22.50	4.40	14	3,603	0.13	96	3,483
Jan	22.50	4.55	484	165,421	0.26	260	134,777
Apr	22.50	5.04	124	4,788	0.69	190	8,773
Nov	25.00	1.80	6,064	35,262	0.12	1,127	62,772
Dec	25.00	2.25	1,901	8,490	0.47	883	7,619
Jan	25.00	2.61	4,048	204,784	0.74	508	107,419
Apr	25.00	3.45	172	21,996	1.39	448	17,931
Nov	27.50	0.27	13,306	70,256	1.07	5,859	20,912
Dec	27.50	0.81	9,834	20,000	1.47	1,594	6,126
Jan	27.50	1.21	8,063	160,038	1.85	3,016	18,254
Apr	27.50	2.01	1,220	31,842	2.51	293	14,925
Nov	30.00	0.03	168	8,029	3.31	70	1,788
Dec	30.00	0.19	2,865	6,864	...	...	319
Jan	30.00	0.47	2,667	194,423	3.70	310	7,336
Apr	30.00	1.09	496	24,296	4.10	66	2,686

Notes: The cash-market price (NYSE) on stock-XYZ = $26.80.

underlying stock is \$26.80. The strike prices in the second column are set (by the exchange) above and below the current stock price. The columns labelled 'Price' denote the closing price for the call or put option (for the 3 p.m. trade) and these columns are followed by the daily trade volume as well as the open interest (i.e. number of long or short contracts outstanding). Note that the quoted option price for the '3pm trade' in Chicago might not be based on the recorded price for the underlying stock on the NYSE because the two prices might not be taken at exactly the same time, especially if the option is rather illiquid and hence infrequently traded.

From Table 14.1 you can see that the alternative exercise (strike) prices range from 20.00 to 30.00 (there are other strike prices available which are not reported here). Call prices (premiums) for the $K = 22.50$ strike for expiration months November, December, January and April are 4.29, 4.40, 4.55, and 5.04 respectively – so the call premium increases with the maturity date of the option. The quoted option premium assumes delivery of one stock but 100 stocks must be delivered for each contract, so the invoice price for the option = '100 × quoted price'.

By looking at the November contracts with strike prices $K = 20.00, 22.50, 25.00, 27.50$, and 30.00 you can see that the call premia fall as the strike price increases. The converse applies to the put premia which are positively related to the strike price. By looking at the put premia for contracts that have expiration dates of 25 November and 25 April (of the next year), you can see that put premia increase with the time to maturity of the contract.

14.2 CALL OPTIONS

If today you buy a *European call option* and pay the *call premium/price*, then this gives you the right (but not an obligation):

- to purchase the underlying asset at a
- designated delivery point at a
- specified future date (known as the *expiration or maturity date*)
- for a fixed known price (the *exercise or strike price*)
- and in an amount (*contract size*) which is fixed in advance.

A European option can only be exercised on the expiry date itself whereas an *American option* can be exercised any time up to the expiry date. Note, however, that European (and American) options can be sold to another market participant at any time. Most options traded on exchanges are American but as European options are easier to analyse we deal mainly with the latter. Note that we first concentrate on the payoff profiles *at expiration,* in Chapter 16 we discuss what causes the change in options prices second-by-second.

14.2.1 Positions in Options

As we see below there are always two parties to any options trades and these are classified as follows:

$$\begin{aligned} \textit{Long call} &= \text{buy a call option} \\ \textit{Short call} &= \text{sell (or write) a call option} \\ \textit{Long put} &= \text{buy a put option} \\ \textit{Short put} &= \text{sell (or write) a put option} \end{aligned}$$

14.2.1.1 Long Call: Insurance

Consider, for example, the purchase of a (European) call option on stock-XYZ on 15 July, when the current price of stock-XYZ on the NYSE is $S_0 = \$80$. One contract is for delivery of 100 stocks and if the quoted call premium $C = \$3$, the contract will cost \$300. Assume the strike price is $K = \$80$ (i.e. an ATM option) and the expiry date T is in about 3 months' time on 25 October (Figure 14.1).

First consider how the purchases of a call option can provide insurance. Suppose that in July a pension fund knows it will receive an inflow of funds in October and wants to invest in stocks-XYZ. If in July, the pension fund purchases an October-call option (in Chicago), it will have set a *maximum purchase* price of $K = \$80$, if it decides to exercise the contract in October – this is a form of *insurance*.

Clearly, if on 28 October $S_T = \$88 > K = \$80$ then the pension fund will exercise the option in Chicago, take delivery and pay \$80 per stock – which is cheaper than purchasing stocks-XYZ on the NYSE at \$88 (Figure 14.1). A trader who has an outstanding short position in the October-call is legally bound to deliver the stock to the pension fund and receives \$80.

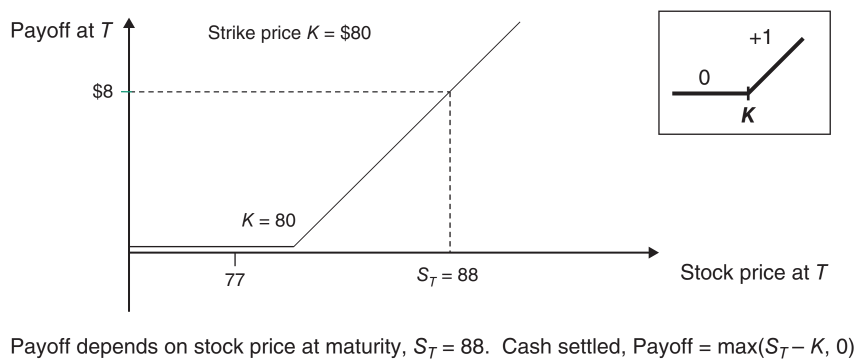


FIGURE 14.1 Long call option (payoff at maturity)

Alternatively, the long call option can be 'cash settled' by the pension fund which receives $S_T - K = \$8$ in Chicago (a trader with an outstanding short position in the October call, provides the \$8). The pension fund can then buy stock-XYZ on the NYSE for $S_T = \$88$, which when offset against the receipt of \$8 from 'cash settled' call, gives an effective cost for the stocks-XYZ of \$80 – the same as the strike price in the call option contract. Hence, (ignoring transactions costs) 'delivery' or 'cash settlement' results in the same value to the holder of the long call.

On the other hand, if the stock price on 25 October is $S_T = \$77$ (i.e. below the strike price of $K = \$80$), then the pension fund will not exercise the call, since it can purchase the stocks-XYZ at lower cost on the NYSE.

Hence, the call option provides insurance in the form of a maximum price payable of $K = \$80$ (if $S_T > K$) but also allows the pension fund to not exercise the call option (if $S_T < K$) and then the pension fund buys stock-XYZ at the lower price on the NYSE. The cost of this insurance is the call premium of $C = \$3$ which is paid in July.

One further thing to note is that the maximum price the pension fund 'locks in' is the strike price in the options contract and not the stock price (on the NYSE) when the pension fund purchases the call option in July. For example, suppose the actual stock price on the NYSE on 15 July, is $S_0 = \$78$. This price of \$78 is *not* the maximum purchase price the pension fund will pay on 25 October when it exercises the call option. The maximum price the pension fund will pay in October when exercising the call option is the strike price $K = \$80$.

14.2.1.2 Long Call: Speculation

Now consider purchasing an October-call option on 15 July for $C = \$3$, in order to speculate on the future price of stock-XYZ over the next 3 months (Table 14.2). If the actual price of stock-XYZ on 25 October turns out to be $S_T = \$88$, then the holder of the call option can take delivery at $K = \$80$ and sell each stock-XYZ on the NYSE at \$88 – which implies a payoff of

TABLE 14.2 Buy (long) call option

Current stock price, $S_0 = \$78$
Traders' desk (today, 15 July)
Contract size = 100 stocks
Strike price, $K = \$80$
Call premium (price), $C = \$3$
Premium paid = 100(\$3) = \$300
Outcome (3 months later on 15 October, time T)
Stock price at expiry, $S_T = \$88$
Cash payoff at expiration: $(S_T - K)100 = (\$88 - \$80)\ 100 = \$800$
Profit (net of call price): $(S_T - K - C)\ 100 = (\$8 - \$3)\ 100 = \$500$

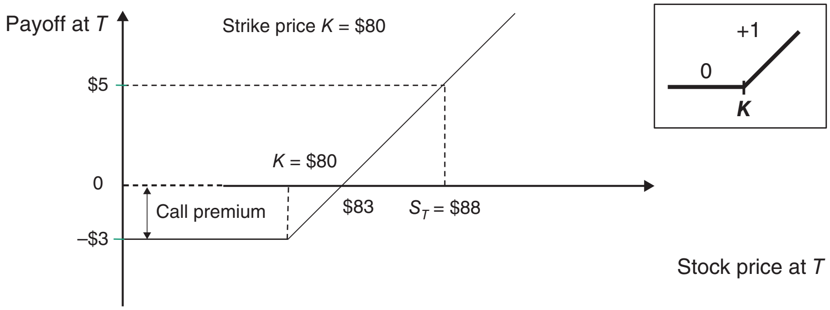


FIGURE 14.2 Long call option (profit at maturity)

$8. Alternatively the speculator can cash settle the long call and receive $8 (from the options clearing house). In either case the 'payoff' from exercising the call on 15 October is $8 per stock:

Payoff: long call $= (+1)[\max(0, S_T - K)]$[1] *if* $S_T > K$ *exercise the option.*

The '+1' implies we are long one call option. In October, the 'profit' Π (after deduction of the call premium $C = \$3$) is $5 per stock (Figure 14.2).[2]

$$\begin{aligned}\Pi &= (+1)[\max(0, S_T - K) - C] \\ &= (S_T - K) - C \qquad \text{for } S_T > K \text{ (exercise the option)} \\ &= -C \qquad \text{for } S_T < K \text{ (do not exercise)}\end{aligned} \tag{14.1}$$

The breakeven stock price occurs when $\Pi = 0$:

$$S_{BE} = K + C = \$80 + \$3 = \$83 \tag{14.2}$$

Each call option contract is for 'delivery' of 100 stocks; hence, if $S_T = \$88$ the speculator makes an overall profit of $500. On the other hand, if at expiration $S_T = \$78$ (< K) the speculator would not exercise the option. The call option expires worthless but the speculator has only lost the call premium of $300, which was paid when she initially purchased the option in July.

Hence a speculator who holds (i.e. is long) a call option can at most lose the call premium. However, the speculator can benefit from an unlimited upside potential, if the stock price in October ends up well above the strike price. The return to the speculator from a long call is asymmetric.

If the stock price at expiration is $82 then the profit to the speculator from the long call is –$1 (= $82 – $80 – $3), that is, a loss of $1. However, in this case it is still worth exercising

[1] $\max(0, S_T - K)$ is also written as $(S_T - K)^+$.
[2] When calculating 'profit' we ignore the fact that the payoff of $8 occurs in October but the cost of buying the call occurs in July – so we ignore any foregone interest that would have been earned on payment of the initial call premium of $300 paid at $t = 0$.

the long call, since if the speculator does not do so, her net loss is the premium of $C = \$3$, which is larger than the \$1 loss from exercising the call. In fact, if the stock price exceeds the strike price ($S_T > K$) at the expiration date, it *always pays* to exercise the call option (since this positive payoff will contribute to reducing the effective cost of the call premium).

It is useful to designate the payoff profile from options in terms of direction vectors – this will be especially useful when we look at options strategies in Chapter 17. The payoff from a long call is represented by the direction vector $\{0, +1\}$. When $S_T < K$, the payoff to the long call is zero – hence the use of '0' in the payoff vector. When $S_T > K$, the long call earns a positive payoff which increases dollar-for-dollar with the stock price – this is a positive relationship, hence the use of '+1' in the direction vector (see inset in Figures 14.1 and 14.2). For a light-hearted view of how call options can be used, see Finance Blog 14.1.

Finance Blog 14.1 Call Options – Ken and Barbie

Assume there are lots of 'Kens' (i.e. desirable but identical males) which can be picked up at the local nightclub. It is January and Barbie decides she likes the Kens she has met but is unsure about marrying one of them. She wants to wait a year to see how things work out for these Kens and only then will she decide whether to marry a Ken in 1 year's time. She will marry a Ken in a year's time if his prospects then look good. But she knows that if Kens behave badly and don't have a decent job at the end of the year, it will probably not be worth marrying any Ken, at that point.

Let's get Barbie to buy a *December-call option* on Ken with a strike price of $K = \$100$ where the delivery point at the maturity date of the option in December, is the local church. She pays a call premium of $C = \$3$ in January (which is paid to the market maker who sells the call option to Barbie). Barbie's long call option sets a maximum price of $K = \$100$, she will pay for 'delivery' of a Ken December (if she then wishes to take delivery).

If over the next year, Kens improve their salaries so that they are worth $S_T = \$110$, then Barbie will exercise her call option. She then turns up at the church in December and marries Ken for a mere payment of $K = \$100$ (to the clergyman officiating), even though a 'spot' or 'cash-market' Ken would cost her $S_T = \$110$ at the local nightclub round the corner. (Note that if Barbie holds the call option to maturity and decides to exercise the option then Ken is contractually bound to turn up at the church, he has no choice.)

Hence, Barbie by purchasing the December \$100-call option in January has a form of 'insurance'. By paying $C = \$3$ in January she knows the *maximum price* she will pay for delivery of Ken next December is $K = \$100$.

On the other hand, if Kens have no pay rises, get demoted or lose their jobs over the next year, their spot (cash market) price in the local club might drop to say $S_T = \$90$ (i.e. fall below $K = \$100$). In this case Barbie *would not exercise her call option* and would simply

(*continued*)

(*continued*)

not turn up at the church. Under the terms of the option contract, she does not even have to tell Ken she will not be turning up – so Ken could suffer abject humiliation at the altar in December.

If the spot-price for Kens in December is $S_T = \$90$ then Barbie could *if she wished*, purchase a Ken in the spot market at her local club for \$90, and take immediate delivery. So Barbie's long call option ensures that in 1 year's time, she pays a maximum price of $K = \$100$ for delivery of a Ken, yet she can take advantage of the lower spot price for Kens, in December should this occur.

If you saw the film *Four Weddings and a Funeral* you will know that Hugh Grant's character 'Charles' did turn up at the church with the intention of marrying 'Duckface' (Henrietta). But he really could have saved himself the trouble and embarrassment by purchasing a long call on Duckface earlier in the year – then he would have had the option of whether to turn up or not (depending on what he thought the value of Duckface was at the maturity of the call option, on the prospective wedding day).

Again, those of you who know the ending of *Four Weddings and a Funeral* will realise that what Hugh Grant (Charles) probably held was an embedded *rainbow call option (or alternative option)*. This type of option has two underlying assets (e.g. two different stocks/people) and at maturity of the option, you can take delivery of the underlying asset which has the best 'payoff' of the two.

In *Four Weddings and a Funeral,* Charles implicitly held a long rainbow call option based on two underlying assets: Duckface and the Andie MacDowell character, Carrie. He took the view that the highest payoff at the expiration date was Carrie rather than Duckface – so he took delivery of Carrie. Incidentally, a rainbow option is more expensive to buy than either a plain vanilla option on Carrie or a plain vanilla option on Duckface. But you already know that 'dating' two people simultaneously is generally more expensive than dating just one of them.

Those of you who have seen the film several times will recall that Duckface actually had an option of her own. This was a *down-and-out knockout call option* on Charles, which she executed with a right hook – before the ceremony was completed (i.e. before maturity of the option) – so for her, Charles was 'knocked out' (literally) and hence Duckface accepted that the payoff to the call would be zero, at any point thereafter.

Duckface realised the low 'cash-market' value she placed on Charles that day because his behaviour had fallen below the barrier of decency even expected of an English 'fop'. Therefore she decided to terminate ('knockout') her call option on Charles. Even if Charles had later redeemed himself in Duckface's eyes, she could not in future exercise the call option and pay K for delivery of Charles – the option she held had been 'knocked out' before maturity (i.e. before the exchange of rings at the wedding ceremony).

Down-and-out options do exist in the real world. For example, a down-and-out call option on a stock is one that 'dies' (i.e. effectively no longer exists) if the stock price falls below a pre-specified level ('the lower barrier'), before the maturity date of the option contract. Then even if the stock price at maturity of the call option contract has risen above the strike price (i.e. $S_T > K$), the 'down-and-out call is worthless at maturity – as it had 'died' before maturity.

Source: Adapted from Cuthbertson and Nitzsche (2008).

14.2.1.3 Write (Sell) a Call

One simple way to find out what happens to the profit at maturity for the writer of a call is to work out the profit to the long call (+\$5) and simply reverse the sign. If the long call makes a cash profit of \$5 then the person who wrote (sold) the call makes a cash loss of \$5. Let us look at this in more detail.

If $S_T = \$88$, on 25 October then the *writer* of the call has to purchase the stock in the cash market (NYSE) at $S_T = \$88$ but receives only $K = \$80$ when she delivers the stock (in Chicago) to the holder of the long call (Figure 14.3). The writer of the call has a payoff of –\$8 (= $-(S_T - K)$.

Alternatively, if the long call is cash settled then the trader with an outstanding short positon (i.e. the writer) makes a cash payment of \$8 to the holder of the call but on 15 July she had received $C = \$3$ when she sold the call option, so her profit is $-\$5 = -(S_T - K) + C$, that is, a loss. The writer makes a total loss on one contract of \$500 which is the mirror image of the \$500 gain, made by the long call.

Alternatively, if the actual stock price on 25 October is (say) $S_T = \$77$ (i.e. below the strike price of $K = \$80$), then the *holder* of the call option ('the long') will not exercise it (in Chicago), since she can purchase the stocks at lower price on the NYSE. Hence, when $S_T < K$, *the writer*

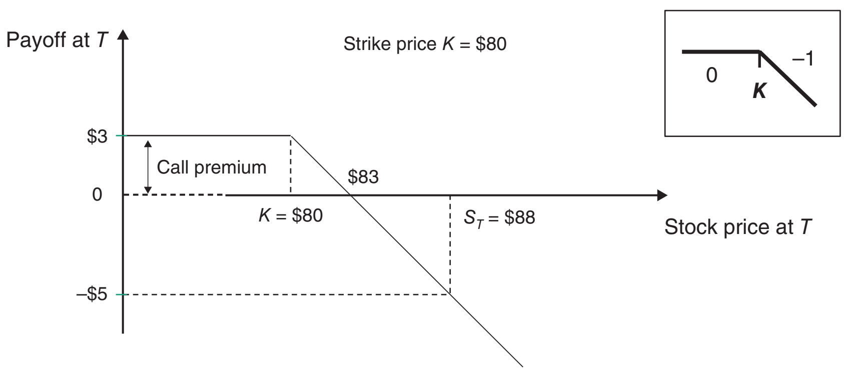


FIGURE 14.3 Short (sell, write) call option

of the call option makes a profit of \$3, which is the option premium she received in July. The payoff at maturity from *one written call* is:

$$\text{Payoff: written call} = (-1)[\max(0, S_T - K)]$$

The '–1' implies you initially wrote (sold) one call option. The profit at maturity for the writer of the call is:

$$\begin{aligned} \Pi &= (-1)\,[\max(0, S_T - K] + \mathrm{C} \\ &= -[S_T - K] + C \qquad\qquad \text{for } S_T > K \\ &= +C \qquad\qquad\qquad\qquad\; \text{for } S_T \leq K \end{aligned} \tag{14.3}$$

14.3 PUT OPTIONS

If you *buy* a European put option (in Chicago) this gives you the right *to sell* the underlying asset (in Chicago) at some time in the future, at a strike price fixed in the option contract.

If today you buy a *European put option* and pay the *put premium/price,* then this gives you the right (but not an obligation):

- to sell the underlying asset at a
- specified future date (*the expiration/maturity date*) at a
- designated delivery point
- for a known fixed price *(strike/exercise price)*
- in an amount *(contract size)* which is fixed in advance.

14.3.1 Long Put + Stock: Insurance

First let us see how the purchase of a put option (i.e. you go long the put) can be used to set a minimum value you will receive in the future, *for stocks that you already own*. The stock + put is like an insurance contract which sets a minimum future selling price for the stock but allows most of the 'upside capture' should stock prices rise in the future.

Suppose on 15 July Ms Prudence, a pension fund manager, holds 100 stocks of XYZ, with a price of $S_0 = \$72$ on the NYSE. Ms Prudence has to pay out someone's lump-sum pension in 3 months' time (on 25 October) and she is worried that the stock price will fall *below* \$70. But she also wants to take advantage of any rise in stock prices, should this occur. Ms Prudence can eliminate most of the downside risk by purchasing a put option, with a strike price $K = \$70$ and expiration date on 25 October. Each put contract is for delivery of 100 stocks. Ms Prudence must pay the put premium $P = \$2$ today.

By purchasing the put, Ms Prudence guarantees that she can, if she wishes, sell her stocks for $K = \$70$ each, by exercising the put contract in Chicago on 25 October – even if the price of the stocks on the NYSE is much lower, say $S_T = \$20$ per stock. The options market in Chicago must honour Ms Prudence's decision to exercise the put, by delivering her stocks to Chicago, for which she receives the strike price $K = \$70$ in the put contract. The $K = \$70$ paid to Ms Prudence, will be provided by a Chicago options trader (Mr Short) who has a short position in the October-70 put, on the 25 October (i.e. Mr Short has previously sold an October-put with a strike of $K = \$70$ and has not closed out his position before 25 October). Mr Short's \$70 payment to Ms Prudence will be paid via the options clearing house.

Alternatively, the long put can be 'cash settled' by Ms Prudence for receipt of $K - S_T = \$70 - \$20 = \$50$ cash (provided by Mr Short via the options clearing house). Ms Prudence then sells the stock (owned by the pension fund) on the NYSE for $S_T = \$20$, giving an 'effective' cash value for the put+stock position of Ms Prudence of \$70 – the same as the strike price in the put contract.

On the other hand, if in October, $S_T = \$75 (> K = \$70)$ then Ms Prudence can 'walk away' from the put contract (i.e. not exercise it in Chicago) but instead sell stocks-XYZ on the NYSE at the high price of \$75 per stock.

This means that whatever happens to the stock price on the NYSE over the next 3 months, Ms Prudence can either exercise the put and sell stocks-XYZ at a minimum price of $K = \$70$ (in Chicago), or if the stock price rises above \$70, she can 'throw away' the put option (i.e. not exercise it in Chicago) and sell her stocks-XYZ at the higher spot price on the NYSE. Hence, *if you already own some stocks*, buying a put option today provides insurance in the form of a guaranteed minimum price when you sell your stocks (on the expiration date of the put), whilst also allowing you to benefit from any 'upside potential', should the stock prices rise on the NYSE.

For a light-hearted analysis of the payoffs when holding a 'spot market asset' and using puts to insure a minimum selling price, see Finance Blog 14.2.

Finance Blog 14.2 Put Options – Ken and Barbie

Let's get Barbie to insure the minimum future value she will obtain for 'Ken' by using a put option. Suppose that Barbie has been married to Ken for some time but she is getting rather tired of him. So Barbie *already holds a Ken* who is currently worth $S_0 = \$100$ but she is worried he may be worth much less in 1 year's time.

Should she have a trial separation and then divorce him at the end of the year? The problem is that if she waits a year to divorce him, Ken may lose his job and his market value may fall over the year, thus reducing Barbie's pay-out in the divorce settlement. On the other hand, if Ken does rather well in the jobs market and is worth more at the end of the year then she may wish to stay married to Ken. How can options help to solve Barbie's dilemma?

In January Barbie can buy (go long) a *December-Ken put contract* with strike price $K = \$100$. Assume the put contract costs Barbie $P = \$2$ (which she pays in January to the

(*continued*)

(*continued*)

options trader who sells Barbie her put contract). The long put contract gives Barbie the right to 'deliver' Ken in Chicago next December and receive $K = \$100$, but only if she finds this advantageous.

Next December, if Ken's earning power and hence his 'cash market' price has deteriorated to say $S_T = \$90$, Barbie will deliver him (along with the put contract) to the attorney's office in Chicago (the designated delivery point) and get divorced. Under the terms of the put contract she will receive the generous divorce settlement of $K = \$100$ on delivery of Ken to the attorney's office, even though Ken is only currently worth $S_T = \$90$ in the 'jobs market'.

Of course, if Ken's earning power and his cash-market price rises over the coming year to $S_T = \$110$, then Barbie will not exercise her put contract to sell Ken for \$90 at the attorney's office, since he is currently worth \$110. So Barbie will 'throw away' her put contract – it is worthless. She may stay with Ken and the marriage will continue. Or, she can sell Ken in the 'cash market' in her local club for $S_T = \$110$ (to another 'Barbie' who is looking for immediate delivery that night of a Ken). Again, Ken has no choice in the matter – Barbie has purchased the put contract on Ken and hence holds 'all the cards', which is sometimes the case in a divorce. This 'insurance contract' on Ken may have cost Barbie a mere \$2 last January, when she purchased the put option on Ken.

Source: Adapted from Cuthbertson and Nitzsche (2008).

14.3.2 Long Put: Speculation

There are also opportunities for speculation with put options. On 15 July, suppose a speculator Ms Doom, expects the price of stock-XYZ to fall substantially between July and October (say). To benefit from a future stock price fall, Ms Doom buys an October-put option contract on 15 July, with $K = \$70$, exercise date 25 October and pays the put premium $P = \$2$ on 15 July. (Note that here Ms Doom only holds a put option, she does not hold any stocks).

Suppose the price of stock-XYZ on the NYSE on 25 October is $S_T = \$65$ ($< K = \$70$) – Figure 14.4. Then on 28 October, Ms Doom could buy 100 stocks-XYZ on the NYSE for $S_T = \$65$ per stock, and then exercise the put option by delivering these to Chicago for which she receives $K = \$70$ per stock from the options clearing house. Ms Doom has a positive payoff of \$5 ($= K - S_T$) per stock[3] and a net profit of \$3 after paying the put premium of $P = \$2$ (Figure 14.5). If one put contract is for delivery of 100 stocks then the outcome is given in Table 14.3.

[3]If the long put is cash settled then Ms Doom receives a cash payment of \$5 from the options clearing house in Chicago (i.e. ultimately from a trader with an outstanding short position in the December put).

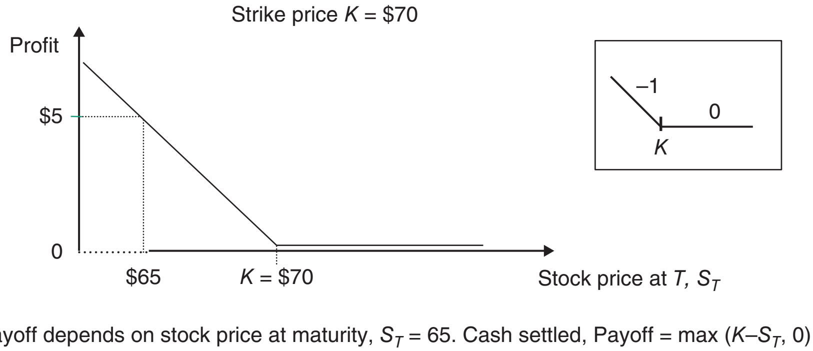


FIGURE 14.4 Long put option (payoff at maturity)

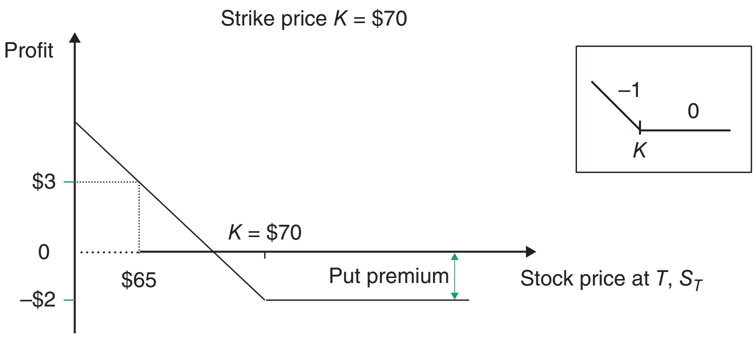


FIGURE 14.5 Long put option (profit at maturity)

TABLE 14.3 Buy (long) put option

Current stock price, S_0 = \$72

Traders' desk (today, 15 July)

Contract size = 100 stocks
Strike price, K = \$70
Put premium (price), P = \$2
Premium paid = 100(\$2) = \$200

Outcome (3 months later: 25 October, time *T*)

Stock price at expiry, S_T = \$65
Cash payoff at expiration: (K – S_T) 100 = (\$70 – \$65) 100 = \$500
Profit (net of call price): (K – S_T – P) 100 = (\$5 – \$2) 100 = \$300

Hence the payoff from holding one long put is:

Payoff: long put $= (+1)[\max(0, K - S_T)]$[4] exercise the put if $S_T < K$

The profit from the long put is:

$$\begin{aligned}\Pi &= (+1)[\max(0, K - S_T) - P] \\ &= (K - S_T) - P \qquad \text{for } S_T < K \\ &= -P \qquad \text{for } S_T > K\end{aligned} \tag{14.4}$$

If $S_T > K$ then Ms Doom does not exercise the put (which expires worthless) but the most she loses is the put premium, $P = \$2$. The breakeven stock price occurs when $\Pi = 0$:

$$S_{BE} = K - P = \$70 - \$2 = \$68 \tag{14.5}$$

The payoff profile for a long put (Figure 14.5) is represented by the direction vector {–1, 0}. When $S_T > K$, the payoff to the long put is zero – hence the use of '0' in the payoff vector. When $S_T < K$, the long put earns a *positive* payoff which increases dollar-for-dollar as the stock price *falls* – this is a negative relationship, hence the use of '–1' in the direction vector.

14.3.3 Write (Sell) a Put

The payoff to the seller of a put is the opposite of that of the buyer of a put – there are two sides to every trade – if one side 'wins' the other 'loses'. Let us look at this in more detail.

Suppose Ms Doom already has a long position in the October-put. If $S_T = \$65 < K = \$70$, then Ms Doom will exercise the put and receive a payoff of \$5. Hence, Ms Writer, who has previously sold (written) a put, will have a negative payoff if the (long) put is exercised (i.e. $S_T < K$). Ms Writer is legally obliged to 'receive' the stock from Ms Doom in Chicago and pay Ms Doom $K = \$70$. But Ms Writer can only sell the stock for $S_T = \$65$ on the NYSE (Figure 14.6) – a loss of \$5 to Ms Writer. But Ms Writer when she initially sold (written) the October-put in July received the put premium $P = \$2$, so her net loss is \$3. Hence the payoff to Ms Writer (at expiration) is:

Payoff: Short put $= (-1)[\max(0, K - S_T)]$ *(–1, implies Ms Writer has sold a put).*

The profit at maturity for Ms Writer is:

$$\begin{aligned}\Pi &= (-1)[\max(0, K - S_T) - P] \\ &= -[K - S_T] + \mathrm{P} \qquad \text{for } S_T < K \\ &= +P \qquad \text{for } S_T > K\end{aligned} \tag{14.6}$$

[4] $\max(0, (K - S_T))$ is also written as $(K - S_T)^+$.

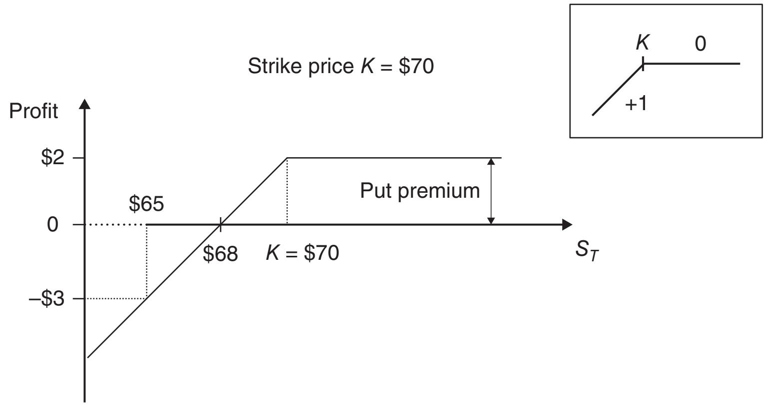


FIGURE 14.6 Short (sell, write) put option

The payoff to the writer of the put option is just the 'mirror image' of the payoff for Ms Doom who is long the put (see Equation 14.4).

14.4 INTRINSIC VALUE AND TIME VALUE

Illustrative prices for call options (on stock-A), on 1 July for strike prices $K = \$360$ and $K = \$390$ are shown in Table 14.4. These are American call options so they can be exercised immediately by the holder, if this is advantageous. The current stock price on the NYSE is $S_0 = \$376$.

In exchange traded options markets there would be many more strike prices and maturity (expiration) dates available and the OTC market (i.e. mainly large banks such as JPMorgan-Chase, Barclays) will in principle provide equity options with almost any strike and maturity date.

TABLE 14.4 Call premia on stock-A, 1 July

Strike price	**Expiry month**		
	October	**January**	**April**
360	36.5 (16 \ 20.5)	50.0 (16 \ 34)	57.5 (16 \ 41.5)
390	21.5 (0 \ 21.5)	35.5 (0 \ 35.5)	44.0 (0 \ 44)
Stock Price (NYSE, 1 July): S_0 = \$376			

Notes: Prices denoted in dollars. (.\.) = (intrinsic \ time value) of call option.

14.4.1 In, Out, and At-the-Money

For a long position in a call option the option is said to be:

In-the-Money (ITM): if Current Spot Price > Strike Price ($S_0 > K$)

At-the-Money (ATM): if Current Spot Price = Strike Price ($S_0 = K$)

Out-of-the-Money (OTM): if Current Spot Price < Strike Price ($S_0 < K$)

Strictly speaking an option is 'in-the-money' if S_0 exceeds the *present value* of the strike price (i.e. $S_0 > Ke^{-rT}$) but the simpler definitions above are usually used.

14.4.2 Newspaper Quotes: Calls

On 1 July, the 360-October calls (Table 14.4) must be worth at least \$16 because the long has the right to exercise the American call option and buy stock-A at $K = \$360$ (in Chicago) and immediately sell the stocks for $S_0 = \$376$ on the NYSE. The cash payout from immediately exercising a call option is the 'intrinsic value' of the option.

$$\text{Intrinsic value} = S_0 - K = 376 - 360 = \$16 \tag{14.7}$$

The October-390 call has a call premium of \$36.5, which is greater than the intrinsic value of \$16. The reason for this is that there is a chance that between 1 July and the October expiration date, the stock price will increase even further (thus increasing the value of the call). Hence, the option has an additional source of value known as the 'time value' of the option.

$$\begin{aligned}\text{Time Value} &= \text{Call Premium} - \text{Intrinsic Value}\\ &= 36.5 - 16 = \$20.5\end{aligned} \tag{14.8}$$

Now consider the October-390 call. This has an intrinsic value of zero since the holder of this call would not wish to take immediate delivery in the option contract and pay $K = \$390$ and then immediately sell the stock on the NYSE for $S_0 = \$376$. However, there is a chance that on or before the maturity date of the option, the stock price will rise above $K = \$390$. For the October-390 call, the long is willing to pay \$21.5 on 1 July, for that chance.

Notice that for either of the calls ($K = \$360$ or $K = \$390$) the long is willing to pay a higher call premium, the longer the time to expiry – hence these options have time value that increases with time to maturity (look along the rows in Table 14.4). This is because there is a longer time over which the stock price might rise above (or well above) the strike price. In summary we have:

$$\textit{For Long Calls:} \quad \begin{array}{lcc} \textit{Intrinsic Value}, IV & \equiv & \max\{S_0 - K, 0\} \\ \textit{Time Value} & \equiv & C_0 - IV \end{array}$$

14.4.3 Newspaper Quotes: Puts

Let us undertake a similar analysis of put options on stock-A (Table 14.5). The October-390 puts have an intrinsic value of \$14 ($= K - S_0$). This is because the holder of the put option could buy the stocks on the NYSE for $S_0 = \$376$ and sell (deliver) them immediately for $K = \$390$ (in Chicago) by immediately exercising the American put option.

The 390-puts have an intrinsic value of $IV = \$14$, but the put premium for the October expiry is $P = \$31$. The time value is therefore \$17 ($= P - IV = 31 - 14$). This reflects the possibility that between 1 July and the October maturity date of the option, the price of stock-A might fall futher, thus increasing the payoff $K - S$ to holding the put option.

$$\textit{For Long Puts:} \quad \begin{array}{lcc} \textit{Intrinsic Value}, IV & = & \max\{K - S_0, 0\} \\ \textit{Time Value} & = & P - IV \end{array}$$

Put premia increase as the time to expiry increases (look along the rows in Table 14.5), since over a long horizon there is an increased chance that the stock price will end up below the strike price and the put payoff will be positive.

14.4.4 In/Out-of-the-Money

Note that the 360-calls (Table 14.4) are in-the-money (ITM) (i.e. $S_0 - K > 0$) while the 390-calls are out-of-the-money (OTM). Hence the 360-calls have relatively high intrinsic values and relatively low time values. The converse applies for the 390-calls which are OTM and hence have zero intrinsic value and relatively large time values. Similar considerations apply to the puts.

TABLE 14.5 Put premia on company-A, 1 July

Strike price	Expiry month		
	October	January	April
360	16 (0 \ 16)	25 (0 \ 25)	27.5 (0 \ 27.5)
390	31 (14 \ 17)	40 (14 \ 26)	43.5 (14 \ 29.5)
Stock price (NYSE, 1 July): $S_0 = \$376$			

Notes: Prices denoted in dollars. (.\.) = (intrinsic \ time value)

14.5 SUMMARY

- For a speculator, a *long call* allows for the possibility of large upside gains to be made if the price of the underlying asset (e.g. stock) ends up well above the strike price, K. Downside losses for the speculator are limited to the call premium paid, C.
- A speculator should buy a (European) call option if she thinks stock prices will rise (above $K + C$) at maturity of the option. For a speculator who holds a long call, it provides payoff $\max(S_T - K, 0)$ when exercised.
- For a speculator, a *long put* allows for the possibility of large gains to be made if the price of the underlying asset ends up well below the strike price K. Downside losses for the speculator who is long a put is limited to the put premium paid, P.
- A speculator who holds a long put, has a payoff of $\max(K - S_T, 0)$ on the exercise date of the put.
- A speculator should buy a (European) put option if she thinks stock prices *will fall* (below $K - P$) at the expiration date of the option.
- A *written (short) call* gives unlimited downside risk should stock prices rise. A written (short) put gives substantial downside risk should the stock price fall. Writers of options therefore have to post margin payments with the clearing house.
- A long call can be used to provide insurance. A long call implies the *maximum price* you will pay for a stock (at maturity of the option) is the strike price, K. But if the stock price falls in the future then you can 'walk away' from the call option contract (i.e. not exercise the call) and purchase stocks on the NYSE at their current low price. Hence a long call provides insurance (i.e. an asymmetric payoff) and the call premium is the price of this insurance.
- If you already hold stocks then a long put can be used to provide insurance. Holding a stock and a put implies you can exercise the put and deliver the stock (in Chicago) at expiration of the put contract and receive a minimum selling price of K for your stocks (in Chicago) – even if the stock price on the NYSE is less than K. But you can also take advantage of high stock prices ($S_T > K$) should they materialise by not exercising the put and selling your stock on the NYSE for $S_T (> K)$. The put premium is the price of this insurance.
- The call premium comprises two components. The *intrinsic-value* of a long call is the payoff to be made on immediate exercise, $IV \equiv \max[S_0 - K, 0]$. But option premia also incorporate *time value*, which reflects the fact that the option may eventually end up in-the-money, at (or before) the maturity date. The time-value of a call option with current price C_0 is $C_0 - IV$.

- For a long put option the intrinsic value is $IV \equiv \max[K - S_0, 0]$ and the time-value is $P_0 - IV$.
- The options clearing house facilitates an active market in traded options by minimising credit (default) risk, as traders holding short positions in either calls or puts have to post margin payments.

EXERCISES

Question 1
If the stock price at maturity of a long call option is $S_T = \$82$, and $K = \$80$ and $C = \$3$ would you exercise the option? What is the payoff and profit for the long call?

Question 2
You think the stock market will rise over the next 2 months. What are the advantages and disadvantages of purchasing a call with either a low or a high strike price?

Question 3
Suppose you have written a call option on a stock-XYZ. What will happen at maturity if the option is either out-of-the-money or in-the-money?

Question 4
You purchase a long call option on Apple stocks (on CBOE) from the writer, 'Writecorp', using clearing firm-XYZ and hold it to maturity. At maturity the assigned short in this option is 'Shortcorp'. What happens at maturity if delivery takes place?

Question 5
When is a long call 'in-the-money' (ITM) and when is a long put 'out-of-money' (OTM)?

Question 6
Why does the buyer of a call option not have to provide margin payments to the options clearing house?

Question 7
Why does the writer of a call option have to make margin payments to the options clearing house? Consider the possible outcomes at maturity of the call.

Question 8
Suppose you purchase a put. In what sense does the strike price act like a 'deductible' in a standard insurance contract?

Question 9

Frank purchased a call option on 100 stocks of Gizmo plc 1 year ago at a call premium of $C = \$10$. (Each call delivers one stock at maturity). The stock price at the time was $S = \$110$ and the strike price $K = \$120$. At expiration, 1 year later, the stock price is $S_T = \$135$.

(a) State whether the option should be exercised.

(b) Calculate the payoff and the profit or loss on the option (including the option premium)

(c) Would Frank have done better by investing the same amount of cash 1 year ago in a bank offering 10% p.a.?

CHAPTER 15

Uses of Options

Aims

- To show how a protective put is used to control downside risk for an individual stock or a portfolio of stocks.
- To show how 'put–call parity' is established by using a no-arbitrage approach.
- To demonstrate how put–call parity may be used to structure a 'guaranteed bond'.
- To show how we can use stock index options to 'insure' a diversified stock portfolio against a general fall in stock prices, whilst also maintaining upside potential, should stock prices increase.
- To outline some 'exotic options'.

15.1 PROTECTIVE PUT

15.1.1 Stock-XYZ

If you already hold stocks-XYZ then you may wish to limit any potential losses on the stocks but also be able to take advantage of any rise in the price of the stock, should this occur. Hence you want to limit the downside, but share in any upside potential. The way to do this is to continue to hold the stocks-XYZ asset but also to purchase a (European) put option on stock-XYZ, with a desired strike price and a time to maturity which matches the period you feel vulnerable to a fall in the price of stocks-XYZ. Finance Blog 15.1 sets the scene by interpreting your car insurance contract in terms of a protective put.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Finance Blog 15.1 Car Insurance as a Protective Put

Car insurance is like a (put) option, with maturity of one year and with the strike price K set equal to the insured value of your car. Suppose your car is currently worth $S_0 = \$40{,}000$ and you decide to insure it for a year for a maximum of \$36,000, so the 'deductible' (or 'excess') in your insurance contract is \$4,000 (i.e. 10% of its market value). You pay an insurance premium upfront which we assume is $P = \$720$ (i.e. 2% of the insured value of \$36,000). The insurance premium of $P = \$720$ is just like paying a put premium. The fact that you choose a deductible of \$4,000 means that the strike price in the put $K = \$36{,}000$ is below the *current* spot (market) price of your car $S_0 = \$40{,}000$ – so your insurance contract with the deductible is like purchasing a '10%-out-of-the-money' put.

Suppose that by the end of the year ($=T$) you have not had an accident and the value of your car remains largely unchanged at $S_T \approx \$40{,}000$, so you do not make an insurance claim. Viewed as a put option, you could deliver your car in Chicago and receive $K = \$36{,}000$ by using ('exercising') the put option. But you would not do this as your car is currently worth $S_T \approx \$40{,}000$ if you sell in the 'cash (auction) market'. Your car insurance contract is like an (implicit) put option that has expired 'out-of-the-money' (since $K = \$36,000 < S_T \approx \$40,000$) and your put option (insurance) is worthless.

However, suppose you have an accident and the market value of your car falls to $S_T = \$5{,}000$.

The insurance company will only pay you a maximum of \$36,000 if you 'deliver' your car to them. This is like delivery of the underlying asset, at maturity of a put option contract with a strike price of $K = \$36{,}000$.

Alternatively, the insurance company will allow you to keep your car worth $S_T = \$5{,}000$ and will give you a cash payout of \$31,000 – giving a total value of \$36,000. This is like 'cash settling' the put contract, which pays a cash amount of $K - S_T = \$36{,}000 - \$5{,}000 = \$31{,}000$. So the market value of your car $S_T = \$5{,}000$ plus the cash settlement from the put contract comes to $K = \$36{,}000$.

On the other hand, if you had initially insured your car for \$40,000 then you would lose nothing after your accident, since the insurance company would pay out \$40,000 and take delivery of your wrecked car. But this insurance contract would cost you more than \$720, since it gives you more protection should you have an accident. Here the actual car insurance policy is like buying an *at-the-money* put (i.e. $K = S_0 = \$40{,}000$) since this put contract would pay you $K = \$40{,}000$ on delivery of your wrecked car (to the delivery point in Chicago).

Source: Adapted from Cuthbertson and Nitzsche (2001).

15.1.2 Stock Index Options

Stock index options (SIO), are frequently used to hedge the market (systematic) risk of a *diversified portfolio* of stocks. The most popular index options in the USA (which trade on the

CBOE) are written on the S&P 500 (SPX) index, the S&P 100 (OEX) index, the Nasdaq 100 index (NDX) and the Dow Jones Industrial Index (DJX). These contracts are European, except for options on the S&P 100, which are American. A fund manager using index options is hedging the market risk of her diversified stock portfolio (but leaving the portfolio exposed to some specific risk, although in a well-diversified portfolio this should be relatively small). To understand the hedging process we need to consider contract specifications for index options and we focus on the S&P 500 contract.

The S&P 500 stock *index* must be translated into a dollar amount. In the options market the value of one index point on the S&P 500 is taken to be $z = \$100$. Hence, if the current S&P 500, $S_0 = 2{,}000$ index points (on the NYSE), then the dollar value of the index is $V_I = zS_0 = \$200{,}000$. Stock index options are cash settled and one option contract is to buy or sell \$100 times the index, at the specified strike price. Hence, the dollar payoff to a call on the S&P 500 index is *\$100* $max(S_T - K, 0)$ and for a put is *\$100* $max(K - S_T, 0)$. The call and put premia are quoted in 'index points'.

15.1.3 Protective Put for a Portfolio of Stocks

Suppose you hold a diversified stock portfolio worth $V_s = \$4\text{m}$, whose composition mirrors the S&P 500 (i.e. your portfolio has $\beta_p = 1$) and you fear a fall in the overall stock market. You can limit the downside risk by purchasing index puts. This is called a *protective put*. If the S&P 500 index currently stands at $S_0 = 2{,}000$ (index points), then to protect your stock portfolio you should purchase N_p index-puts:

$$N_p = \frac{\text{\$-value of stock portfolio}}{\$100 \times S_0(index)} \beta_p = \frac{V_p}{V_I} \beta_p = 20 \tag{15.1}$$

Assume the strike price $K = 2{,}000$ (an ATM-put). If the index falls by 400 points (20%) to $S_T = 1600$, then the value of your stock portfolio falls by \$800,000. But if you cash-settle the put (at maturity) the payoff from your 20 contracts is \$800,000 (= 20 contracts × 400 points × \$100 per index point). Here, the loss on the stock portfolio is exactly offset by the gain on the puts. However, there is also the put premium to consider. If the put premium paid is $P = 50$ index points, the 20 put contracts would have cost \$100,000 (= 20 contracts × \$100 × 50 index points). Without the puts you would have lost \$800,000, with the puts the net value of your stock+put position is unchanged at \$4m, but the cost of this insurance is the put premium of \$100,000.

If instead you had chosen a 10% out-of-the-money put option ($K = 1{,}800$) then the payoff at maturity is \$400,000 (= 20 contracts × 200 points × \$100 per index point) which partly offsets the loss of \$800,000 on your stocks. The net position is a loss of \$400,000, which is 10% of the initial value of your stock portfolio of \$4m. (This put with a lower strike price $K = 1{,}800$ will cost you less than the ATM put with $K = 2{,}000$.)

Of course, if the S&P index at maturity of the put is above K, the puts are worthless and are not exercised. But the value of your stock portfolio would have risen. In this case the 'insurance' provided by the put option is not needed. However, insurance does not come 'free' – you pay for it in the form of the put premium of $100,000 here.

15.2 PUT–CALL PARITY: EUROPEAN OPTIONS

In this section we derive the put–call parity relationship for *European options* (on a non-dividend paying stock-ABC). Put–call parity is an arbitrage relationship between the (European) put premium P, the call premium C, the stock price S, and holding an amount of cash equal to Ke^{-rT} in a risk-free asset (e.g. as a bank deposit or a T-bill or zero-coupon bond which matures at T, with face value K). The call and put options have the same underlying asset (stock-ABC), strike price K and time to maturity, T. The put–call parity relationship can be expressed as:

$$\begin{aligned} (Long)\ Stock + (Long)\ Put &= (Long)\ Call + Cash\ (equal\ to\ Ke^{-rT}) \\ S + P &= C + Ke^{-rT} \end{aligned} \tag{15.2}$$

Note that the term Ke^{-rT} uses continuously compounded interest rates. (The equivalent expression using compound rates is $K/(1+r)^T$.) We can *nearly* produce the above put–call parity result using our direction vectors to mimic the above equation, namely long stock $\{+1, +1\}$ plus long put $\{-1, 0\}$, equals $\{0, +1\}$ – so the net result is a payoff profile which mirrors that of a long call. (However, we have 'lost' the 'cash' so the direction vectors do not provide the full story.) Note that the 'signs' in (15.2) indicate whether you are long or short. They are all '+', indicating long stocks, long puts, long calls and long 'cash' of $\$Ke^{-rT}$, which implies the cash of $\$Ke^{-rT}$ is invested in a bank deposit at the risk-free rate, r.

To demonstrate put–call parity, we form two portfolios and show that they have the same payoff at time T – hence they must be worth the same today, otherwise risk-free arbitrage is possible. Consider the following two portfolios:

Portfolio-A: One put option plus one stock at $t = 0$.

Portfolio-B: One call option plus an amount of cash equal to Ke^{-rT} at $t = 0$.

Consider the value of portfolio-A at expiration (Table 15.1). If the stock price at expiration $S_T > K$ then the put option expires worthless but the stock is worth S_T. Alternatively, if $S_T < K$ then the put payoff is $K - S_T$ and the stock is worth S_T, hence portfolio-A has a payoff $= (K - S_T) + S_T = K$.

Now consider portfolio-B at expiration. If $S_T > K$, the call option's payoff is $(S_T - K)$ while the cash amount Ke^{-rT} held in the risk-free asset earns interest and is worth K at time T.[1]

[1] $\$A$ invested today, earning a continuously compounded rate of r over a horizon T, is worth Ae^{rT} at T. Let the initial investment be $A = Ke^{-rT}$, then this is worth $(Ke^{-rT})e^{rT} = K$ at time T.

TABLE 15.1 Returns from two portfolios: put–call parity

	$S_T > K$	$S_T < K$
Portfolio-A	S_T	K
Portfolio-B	$(S_T - K) + K = S_T$	K

Portfolio-A = One put option plus one stock at $t = 0$

Portfolio-B = One call option plus cash of Ke^{-rT} at $t = 0$

Hence the payoff to portfolio-B is $(S_T - K) + K = S_T$. Alternatively, if $S_T < K$ the call option expires worthless but the amount held in the risk-free asset plus interest earned is worth K at time T.

These payoffs are shown in Table 15.1 and it can be seen that both portfolio-A and portfolio-B give the same outcomes at T and so must have the same value today – hence Equation (15.2) holds, otherwise risk-free arbitrage profits can be made.

15.3 GUARANTEED BOND

We are now in a position to understand 'structured finance', which lies behind the advertisement in the 'Texas Wall Street Journal'[2] (Finance Blog 15.2).

Finance Blog 15.2 'Texas Wall Street Journal'

'Invest in our US "Bush Bonanza Fund."
Steer your way to success in the S&P bull market.
Defy gravity, stay ahead if the bears attack.'

We guarantee you will gain most of any upside in the S&P 500 index – but even if the index should fall to zero over the next 2 years you still get all your money back.*
For a total investment of $106.60 today, we guarantee you a minimum amount of $100 after 2 years, but we also offer you the opportunity to obtain very high returns should the stock market increase – this is too good to miss!

*There is an administration fee of 6.6% of capital invested for this particular structured product which is known as a 'guaranteed bond'.

Source: Adapted from Cuthbertson and Nitzsche (2008).

[2] As far as we are aware there is no such publication and therefore this example has nothing to do with the 'Wall Street Journal'.

TABLE 15.2 Stock, option and bond prices

Current stock price, S = \$100

Interest rate, r = 0.05 (5%) (continuously compounded)

Current price 2-year bond = \$90.50 (pays out \$100 in 2 years)

Investment horizon, T = 2 years

Option prices (maturity in 2 years)

	Call premium	Put premium
K = 100	16.1	6.6
K = 90	22.0	3.5

Notes: Continuously compounded interest rates are used here. \$90.50exp (0.05 × 2) = \$100

Suppose in the cash market and the options market you are today faced with prices in Table 15.2. How can Ms Lego 'structure' a 'guaranteed bond' over the 2-year investment horizon described in the advert in Finance Blog 15.2? For simplicity assume the S&P 500 stock *index* is a single stock with a price of S = \$100 today.

Ms Lego must structure the guaranteed bond by investing a total of \$106.60 today, so that at the end of 2 years the investor gets \$100 back, even if the stock market falls (to zero). But if the stock market does well, the investor will share in (most) of the 'upside'. Here are the questions Ms Lego needs to answer:

- What should she do today?
- Then what happens if the stock price in 2 years ends up at either 50 or 150?
- What might Ms Lego do in order to 'keep' some of the investor's money for the structured products division of her bank (and for her own future bonus) without taking any risks?

First, notice that the call and put premia satisfy put–call parity:

$$C + Ke^{-rT} = P + S$$

$$16.1 + 90.5 = 6.6 + 100$$

15.3.1 Guaranteed Bond Using Stocks and Put Option

Today Ms Lego buys \$100 of the stock on the NYSE. To guarantee a minimum price at which she can sell the stock in 2 years' time, today she also buys a 2-year at-the-money (ATM) put with K = \$100 at a cost of P = \$6.60. The total cost is \$106.60 – which equals the funds received from the investor.

Two Years Later

If $S_T = 50$: Ms Lego delivers her stock in Chicago with the put contract and receives $K = \$100$.

If $S_T = 150$: The stock is worth \$150 (on the NYSE). Ms Lego does not exercise the put – it is worthless.

$$\%\ Return\ to\ investor = 50/106.60 = 47\%$$

Hence the 'stock plus put' guarantees a minimum value of $K = \$100$ if the stock price falls, but allows considerable 'upside' (47%) for the investor if the stock price rises by 50% to \$150.

15.3.2 Guaranteed Bond Using Bond and Call Option

Ms Lego could use the put–call parity equation to structure the 'guaranteed bond' using the bond and a call option. Ms Lego uses the \$106.60 to invest in the bond (@\$90.50) and also buys the (2-year) call with $K = 100$ at a cost of $C = \$16.10$. The total cost is again \$106.60 (because of put–call parity).

Two Years Later

If $S_T = 50$: The bond has a maturity value of \$100 which is paid to the investor. Do not exercise the call, it is worthless.

If $S_T = 150$: The call has a cash payoff $= S_T - K = \$50$, plus the bond with maturity value \$100 gives a total cash payout of \$150.

Hence the payoff to the 'bond+call' is the same as the payoff to the 'stock plus put' – but we know this must be the case if put–call parity holds. So far Ms Lego who works in the structured products division of the bank has made no money. How might she keep some money for the structured products division of the bank (to pay her salary and bonus), without taking any risk?

- Take \$107.60 from the investor for this structured product and keep \$1 for the bank.
- Tell the investor that at the end of the 2 years his actual cash 'payout' will take place 1 month later (in order to sort out administrative details). The investment bank then places the funds in a bank deposit and 'pockets' the deposit interest over the 1-month period.
- If the structured products department uses the 'stock+put' and the guaranteed return is based on a stock *index* which only takes account of changes in prices (and not dividends paid out, as with a 'total return index'), then the bank might also 'pocket' any dividends paid on the stocks over the 2 years.

- Ms Lego could also offer an alternative deal where the investor could lose a maximum of about 10% (over the 2 years) but offer the investor a lower 'administrative fee' of \$5.50. Ms Lego can structure this product by buying an out-of-the-money put with a strike of $K = 90$, at a cost of $P = \$3.50$ (see Table 15.2). The cost of 'stock plus put' $= \$100 + \$3.50 = \$103.50$. Ms Lego therefore earns (for certain), \$2 $(= \$105.5 - \$103.5)$ for the structured products division of the bank, for each 'guaranteed bond' she sells to investors.

15.4 OTHER OPTIONS

The underlying asset in an options contract can be individual stocks, stock indices, currencies, precious metals such as gold and silver, futures contracts and T-bonds. There are also options whose payoff depends on the future outcome for interest rates (e.g. caps and floors). There is a very large OTC market in options and some options are only available OTC (e.g. caps and floors).

15.4.1 Caps, Floors, and Collars

Simplifying a little, a *caplet* is a call option which pays off max $(r_T - K_{cap}, 0)$ where r_T is the interest rate (LIBOR) at the expiration of the option contract and K_{cap} is the strike (interest) rate. Clearly a caplet can be used to speculate on a future rise in (LIBOR) interest rates – if you expect interest rates to rise in the future (above K_{cap}) then today you would purchase a caplet. However, let us consider how it can be used to insure against interest rate rises.

Suppose in January LIBOR interest rates are currently $r_0 = 5\%$. You have a floating rate loan of \$1m on which you pay LIBOR with a reset date in March and you are worried that interest rates will increase between January and March, so the interest cost of your loan will increase. You decide to purchase a caplet with $K_{cap} = 5\%$ which expires in March. In March if interest rates turn out to be $r_T = 7\%$, the caplet payoff is 2%. The cap contract also includes a notional principal amount of (say) \$1m and hence the cash payoff would be \$20,000.

In March, if LIBOR, $r_T = 7\%$ then your loan costs you 2% more as interest rates have risen, but the caplet provides a cash payoff of 2% which will compensate you for your higher borrowing costs. So the effective *maximum* interest rate you will pay, if you have a bank loan and a caplet and if interest rates are high, is $K_{cap} = 5\%$. But things can get even better. If in March, interest rates have fallen to $r_T = 4\%$ then you can just 'walk away' (i.e. not exercise) the caplet and simply pay the low LIBOR interest rate of 4% on your loan. Hence, once again a call option allows you to set the maximum effective interest rate you will pay on your loan but also allows you to benefit from any favourable outcomes (i.e. low loan rates). For this privilege you would have to pay the caplet premium 'up front' (i.e. in January).

If your floating rate loan has a number of futures reset dates (e.g. March, June, September, December) then you can set the maximum interest you will pay on these future loan payments by buying a series of caplets, each with a maturity date which matches the reset dates for your loan. A series of caplets is known as a cap. Financial institutions will 'design' and sell you a cap in the OTC market. (Caps are not traded on an exchange.)

A *floorlet* has a payoff equal to max $(K_{FL} - r_T, 0)$ and is therefore a long put on interest rates. Clearly, if you are a speculator and you think interest rates are going to fall below K_{FL} (in 3 months' time say) then you can make a profit if today you buy a (3-month maturity) floorlet.

Alternatively, if you have a bank deposit with a variable (floating) interest rate, with a reset date in 3 months' time and you are worried that interest rates will fall, then a long floorlet will ensure that the *minimum effective interest rate* you earn on your deposits will be $K_{FL} = 8\%$, say. For example, if interest rates turn out to be $r_T = 7\%$ in 3 months' time, you exercise the floorlet and obtain a payoff of 1% $(= K_{FL} - r_T)$, which when added to the interest on your deposit of $r_T = 7\%$ implies the effective interest rate you receive is 8%. On the other hand, if interest rates turn out to be $r_T = 9\%$, say, then you would not exercise the floorlet (since it is out-of-the-money), but your bank deposit would earn the current high interest rate of 9%.

A floor is a series of floorlets, with different maturity dates and can be used to ensure that the *effective* minimum interest rate you will receive from your deposit account (at floating rates) is $K_{FL} = 8\%$ at each reset date (e.g. every 6 months).

To summarise. A bank deposit on which you receive a variable interest rate is risky as you do not know what interest rates will be in the future. If you buy a floorlet (i.e. a put) with a strike of $K_{FL} = 8\%$ then you are certain of receiving an effective minimum interest rate on your deposit of 8% (even if market interest rates are low) but you can also take advantage of high deposit rates should they occur.

Finally, the combination of a floating rate bank loan plus a long cap with $K_{cap} = 10\%$ and a short floor with $K_{FL} = 8\%$, is known as a *collar*. This is because the collar implies that the *effective* interest rate payable on the bank loan cannot go above 10% nor fall below 8% – so the effective interest payable is constrained at a desired upper and lower level.

15.4.2 Exotic Options

There is no end to the types of option that can be offered in the OTC market and those with complex payoffs are known as *exotic options*. For example, Asian options have a payoff which is based on the average price over the life of the option. An *Asian (average price) put* option has a payoff which depends on max $(K - S_{av}, 0)$ where S_{av} is the average price over the life of the option (e.g. the average of end-of-the-month prices over one year). So, an Asian average price put option on euros would be useful for a US firm that wants to fix the average level of US dollars it will receive from its future export receipts in euros, each month over the next year. The Asian option costs less than using 12 vanilla options for each of the individual monthly euro receipts.

Another type of exotic option are *barrier options*. These either 'die' or 'come alive', before the maturity date in the options contract. For example a (European) *knockout option* may specify that if the stock price rises or falls to a 'barrier level', the option will terminate ('die') on that date, and hence cannot be exercised later. If the option is terminated when the stock price falls to the barrier, then they are referred to as *down-and-out options*, while if they are terminated when the price rises to the barrier, they are *up-and-out-options*. These options, *unless they have already been knocked out,* have a payoff at maturity which is the same as the payoff from 'plain vanilla' calls and puts we have already discussed. But clearly, sometimes knock-out options 'die' before their maturity date and such outcomes are less favourable than payoffs from plain vanilla options. For this reason knock-out options have lower premiums than plain vanilla options (with the same underlying asset, strike price, and time to maturity). In finance, if you 'get less' you pay less.

15.4.3 Other Options

There are also options which are 'embedded' in other securities. Examples of *embedded options* include callable bonds, convertible bonds, warrants and executive stock options, and stock underwriting. The latter involves the underwriter agreeing to purchase any stocks which are not taken up by the public, at an agreed minimum price. The agreed minimum price is equivalent to the strike price in a put contract and the underwriting fee is the put premium. The corporate is long the put and the underwriter has written the put.

15.5 SUMMARY

- Put–call parity for European options links call and put premia for options (on the same underlying asset, with the same strike price and time to maturity). It provides two equivalent methods to structure a 'guaranteed bond' – namely, buying a 'stock and a put' or buying a 'call and a bond'. A guaranteed bond provides the investor with a guaranteed minimum future value for her investment but also allows her to benefit from a rise in stock prices, should this occur.
- An options contract can be written on individual stocks, stock indices, currencies, precious metals, futures contracts, and T-bonds. Some options have payoffs which depend on the future level of interest rates (e.g. caps, floors, and collars). There is a very large OTC market in options (particularly for currencies and interest rates) and some options are only available OTC.
- Options with complex payoffs are known as *exotic options*. Examples include *Asian options and barrier options,* but there are many more.
- Some options may be *embedded* in other securities (e.g. callable bonds, convertible bonds, warrants and executive stock options).

EXERCISES

Question 1
What is a guaranteed bond and its relationship to put–call parity?

Question 2
A long straddle is constructed from a long call with a premium of $C = \$13$ and a long put with $P = \$9$. Both options have the same strike $K = 100$ (and the same maturity date). The current stock price is \$100. By how much must the stock price move so that you make a positive net profit on your long straddle, at maturity?

Question 3
What is put–call parity for European options on a non-dividend paying stock? Why is the put–call parity useful when pricing calls and puts?

Question 4
You currently hold a stock with $S_0 = 100$. You buy an at-the-money put for $P = 6$. What is the maximum loss from holding the put and the stock?

Question 5
Let $P = \$6$, $K = 165$ and the current stock price $S_0 = 164$ (i.e. the put is currently ITM).

What is the profit profile (at expiration) and the breakeven stock price for a portfolio consisting of 100 stocks and 100 long puts? The profits are determined by the change in the price of the stocks, the payoff from the long puts less the cost of the puts.

What are the profits for $S_T = 163$?

Question 6
You are going to take out a 1-year bank loan for \$1,000 in 6 months' time at a floating (LIBOR) interest rate (i.e. you will pay whatever the interest rate happens to be in 6 months). You are worried that interest rates will rise over the next 6 months, so you will pay more interest on your bank loan.

How could a caplet with $K_c = 3\%$ p.a., be used to 'insure' yourself against higher interest rates in 1 year's time? What is the outcome in 1 year's time, if interest rates turn out to be either $r_T = 7\%$ or 1%?

Question 7
You are going to take out a 1-year bank loan for \$1,000 in 6 months' time, at floating (LIBOR) interest rate (i.e. you will pay whatever the interest rate happens to be in 6 months' time). You are worried that interest rates will rise over the next 6 months, so you will pay more interest on your bank loan.

A colleague suggests you insure yourself against high future (LIBOR) loan rates by buying a 6-month floorlet with $K_{FL} = 3\%$ p.a., notional principal of \$1,000 today. Do you agree?

CHAPTER 16

Black–Scholes Model

Aims

- To demonstrate how option prices change with changes in the price of the underlying asset, its volatility, interest rates and time to maturity.
- To establish upper and lower bounds for the price of European calls and puts.
- To show how the Black–Scholes formula is used to price European calls and puts.
- To show how options and the underlying asset (e.g. stock-Z) can be combined into a portfolio which does not change in value when there is a small change in the price of the underlying asset – this is delta hedging.
- To explain implied volatility and its use in options trading.

16.1 DETERMINANTS OF OPTION PRICES

It can be shown that the option premium varies second-by-second as the stock price, the risk-free interest rate and the volatility of the stock change, over time. Let us develop some intuitive arguments which give some insight into the determination of European option prices. We consider each factor in turn, holding all the other factors constant. This intuitive approach will help us understand the mathematical formulas for option prices which we present later. In each case we assume the investor has a long options position (i.e. has purchased a call or a put) and we only consider European stock options (where the stock pays no dividends). The results are summarised in Table 16.1.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

TABLE 16.1 Factors affecting the option premia

	Long European call option	Long European put option
Time to expiration, T	+	+
Current stock price, S_0	+	–
Strike price, K	–	+
Stock return volatility, σ	+	+
Risk-free rate, r	+	–

Notes: We only consider options on stocks which pay no dividends. '+' indicates a positive relationship between the option price and the variable chosen. That is, a rise (fall) in the variable is accompanied by a rise (fall) in the option premium. A '–' indicates a negative relationship between the option price and the variable chosen. That is, a rise (fall) in the variable is accompanied by a fall (rise in the option premium.

16.1.1 Time to Expiration, T

Calls: The price today of a European call option C is higher, the longer the time to maturity of the option. This is because a long-maturity option has more time to end up well in-the-money, that is for S_T to be much higher than K.

Of course, S could also fall over the life of the option, but losses are limited to the call premium no matter how far stock prices fall. Hence, the longer the time to maturity the higher the *expected* payoff at maturity and therefore you are willing to pay more for long-maturity calls.

Puts: The put premium on a European option increases with time to maturity, for the same reason as given for the call. This is because a long-maturity put option has more time to end up well in-the-money (i.e. $S_T < K$), yet the most you can lose is the put premium. (For puts there are some rare cases where the put premium falls as the time to maturity increases – but we ignore these).

16.1.2 Strike Price, K and Stock Price, S

Calls: If stock price movements are random, then the higher the *current* stock price S_0 relative to the strike price K, the more likely it is that the stock price *at expiration* S_T, will end up above the strike price and the call will have a positive payoff. Hence the purchaser of a long European call option, will today be willing to pay a higher call premium C, the higher is S_0/K.

Puts: It should be obvious that the price of a put option varies with S in the opposite way to that for a call. Hence the price of a European put option P depends negatively on S_0/K.

16.1.3 Volatility, σ

Calls: The greater the volatility of the return on the stock σ, the greater the possible range of stock prices that might occur at maturity of the option. But the most the holder of a long call option can lose is the call premium. High volatility increases the chance of a very high stock price and hence a very high payoff for the call option (at maturity), while downside risk (if $S_T < K$) is limited to the call premium.

Therefore the current price of a call option is higher, the higher is the volatility of the underlying stock return, since this implies a higher *expected* payoff to the call.[1] The latter statement does not imply that holders of call options like risk – it is just that higher volatility of stock returns implies a higher expected payoff (at maturity) to the holder of the call and she is therefore willing to pay a higher call premium.

Puts: The owner of a long put has a payoff at maturity of $K - S_T$ (if $S_T < K$). The higher the volatility, the larger is the possible payoff at maturity, if the stock price ends up below K. However, high volatility also implies that the stock prices could rise well above K, in which case the put is worthless at maturity. But no matter how high the stock price at maturity, the most you can lose is the put premium.

Hence, the *expected* payoff at maturity to holding the put is higher, the higher is the volatility of the underlying stock. So you are willing to pay a higher price today for a put which has an underlying stock with high volatility.

16.1.4 Risk-free rate, r

Calls: The European call option premium is higher the higher is the current interest rate, r – although unfortunately we cannot give a full intuitive interpretation of this result. But try this. At maturity, if you exercise the call option you have *to pay* the strike price K. The present value of this payment is Ke^{-rT}. The higher is the interest rate the lower is the *present value* of your payment[2] – hence the more you are prepared to pay for the call option.

[1] Consider a call with $K = \$100$ and suppose the initial stock price is also $S_0 = \$100$. Suppose a 'low volatility' stock can end up at either \$98 or \$102 with equal probability and a 'high volatility' stock can end up at either \$90 or \$110. The expected dollar payoff to both stocks is the same, namely \$100. But the expected payoff for the call on the low volatility stock is $\$1 = \frac{1}{2}(0) + \frac{1}{2}(2)$ and for the call on the high volatility stock is higher, at $\$5 = \frac{1}{2}(0) + \frac{1}{2}(10)$. You can easily adapt the above argument to see why the call premium today is higher, *the longer the time to maturity*. For any given level of volatility, the stock will have a wider range of outcomes the longer the time to maturity and hence the call will have a higher expected payoff – so you are willing to pay more for it today.

[2] The higher is the interest rate today, the less you have to place in a bank account, to be certain of having $\$K$ at maturity of the option. Remember we are also holding the stock price constant in this scenario.

Puts: Higher interest rates reduce the put premium. If you exercise the put at expiration, you deliver a stock and receive K. But the higher is the interest rate, then K-dollars received at T are worth less today, so you are willing to pay less today for the put.

16.1.5 Price Bounds for European Options (Non-Dividend Paying Stocks)

It is possible to establish limits on the range of possible values for the call and put premia and if the quoted price of an option lies outside these bounds then risk-free arbitrage profits can be made. The upper bounds for European options (on a non-dividend paying stock) are:

$$C_e \leq S \quad \text{and} \quad P_e \leq Ke^{-rT}$$

The lower bounds are:

$$C_e \geq \max(S - Ke^{-rT}, 0) \quad \text{and} \quad P_e \geq \max(Ke^{-rT} - S, 0)$$

As shown in Appendix 16, these upper and lower limits on the no-arbitrage values for calls and puts, only depend on the risk-free rate being positive.

It would be extremely useful if all of the above factors could be included in a single equation to determine the call C or put premium P. This 'closed form' solution is the *Black–Scholes equation* for option prices and has the general form:

$$C \ or \ P = f[(S/K), r, \sigma, T] \tag{16.1}$$

To work out the exact functional form for the Black–Scholes equation is rather difficult but results in a 'curved' or 'non-linear' or 'convex' relationship between the call (or put) premium and the price of the underlying stock, S. This positive non-linear relationship for a long call is shown as the curve A-B in Figure 16.1. In fact, as the option approaches its maturity date, the 'curve' moves towards the 'kinked line' which represents the payoff on the day the option matures. If the current stock price is S_t then the current call premium would be C_t (point B, Figure 16.1). However, if the stock price remained at S_t the call premium would fall-towards point D and the option is said to 'lose time value' as it approaches maturity.

16.1.6 Speculation with Calls

Figure 16.1 shows how we can speculate over short horizons, using call options. Suppose a speculator purchases a call option (on a stock) at $C_0 = \$3.1$, and the stock price then increases from $S_1 = \$100$ to $S_2 = \$100.5$. According to the Black–Scholes formula the call price rises to $C_1 = \$3.3$. If the speculator paid the \$3.1 to the clearing house at $t = 0$, then at $t = 1$ when she closes out her initial (long) position by selling the call at \$3.3 (to another trader), the clearing

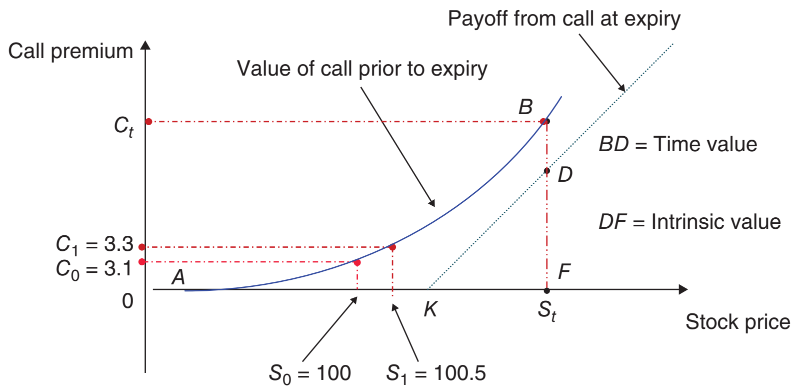


FIGURE 16.1 Black–Scholes

house in Chicago credits the speculator's (margin) account with a profit of \$0.2, which is a percentage return[3] of 6.45% = (\$0.2/\$3.1) × 100%.

The ratio of the dollar change in the call premium (\$0.2) to the dollar change in the stock price (\$0.5) is known as the *call option's delta*, which here is $\Delta_c = 0.4$ (= \$0.2/\$0.5).[4] The delta of a call always lies between 0 and +1. But note that although the *dollar* change in the call premium is usually less than the *dollar* change in the stock price, the option still provides 'leverage' compared to speculating with the stock – because leverage refers to *percentage changes*. To see this, note that in the above example, the option gives a return (per dollar invested) of 6.45%. Had you invested your 'own money' in the stock you would have earned a considerably smaller percentage return on your capital invested – you would have paid \$100 for the stock and sold it for \$100.5 – a return on your 'own capital' of 0.5%.[5]

The change in dollar value and the 'percentage return' are two different concepts. In the above example, the percentage return on the stock is $R_s \equiv (S_1 - S_0)/S_0 = 0.5\%$ but the percentage return from the call is much larger at $R_c \equiv (C_1 - C_0)/C_0 = 6.45\%$. As a speculator, it is the percentage return (on your 'own funds') that you are interested in, when you compare two investment strategies – betting on the call price versus betting on the stock price.

Another way of looking at this is to note that the percentage change in the call premium for each *1% change* in the stock price is 12.9% (= $6.45/0.5 = R_c/R_s$). The figure of 12.9% is known as the *option's elasticity*[6] and it clearly indicates that investing in the option provides leverage (i.e. a larger percentage return), compared to investing (the same amount of money)

[3] Algebraically the *proportionate* return is $R_c \equiv (C_1 - C_0)/C_0$ which is often written in 'calculus form', $R_c \equiv dC/C$ if the change in the call premium is small.

[4] Strictly, the option's delta provides an *approximation* to the change in the call premium when the stock price changes by a *small* amount – see below for further clarification.

[5] Clearly here we are considering a 'cash purchase' of the stock and not a levered purchase of the stock (which requires borrowed funds to finance part of the stock purchase).

[6] The elasticity $E \equiv R_c/R_s \equiv (dC/C)/(dS/S)$.

in the stock. Note that leverage works on the downside too. If the stock price falls by 1% then the call premium will fall by 12.9%. Leverage implies that the percentage change in the option premium will be larger than the percentage change in the underlying stock price.

A speculator would purchase a call option if she expected a bull market, that is, a rise in stock prices. If stock prices rise (above K) she may achieve a higher percentage return from the long call than investing in the stock.

Prior to maturity, the relationship between the put premium and the stock price is also a 'curved line' but for a put, the put premium P and the stock price S are *negatively related*. Hence, a speculator would purchase a put option if she felt that stock prices would fall in the near future. She would pay the put premium $P_0 = \$2$ at $t = 0$. If the stock price subsequently fell then at $t = 1$ the put premium would increase, to say $P_1 = \$2.3$. If she then closes out her initial position by selling the put (to another options trader), she will receive \$2.3 from the clearing house, making a profit of 15% on her initial outlay of \$2 ($= \$0.3/\$2 \times 100\%$).

We also noted above that the Black–Scholes formula (and our intuition) show that both the call and put premia are positively related to the volatility of the stock return. Suppose the *perceived* volatility of a stock increases (e.g. because traders think the future economic outlook has become more uncertain) then both call and put premia would increase. Had the speculator been long either calls or puts she could now close out her position at a profit. Options therefore allow speculators to make a potential profit (or loss) by predicting changes in volatility, σ. This is sometimes referred to as *trading volatility*.

16.2 BLACK–SCHOLES

In the 1960s, options were being traded in the US over-the-counter but rather bizarrely no-one knew how to correctly price them. It was easy to work out that for a call option, say, the premium should be higher, the lower the strike price, the longer the time to maturity, the higher the interest cost and the greater the volatility of the underlying stock. But how could all of these be combined to give an explicit equation that could be used to quickly calculate the correct or fair price for the option? The Binomial Options Pricing Model (BOPM) had not been invented and the first pricing equation for options was the work of Black, Scholes, and Merton – a brief history of the route to the Black–Scholes equation is given in Finance Blog 16.1.

Finance Blog 16.1 Noble Pursuits

It was the combined work of Fischer Black, Myron Scholes, and Robert Merton in Boston that finally solved the option pricing problem in the early 1970s. Black (after his degree in physics) initially worked on the pricing of warrants (i.e. options on stocks, issued by the company itself).

Scholes was also working on options pricing in the late 1960s at the Sloan School of Management (MIT), where he met Black and they began working together. Meanwhile, a young applied mathematician, Robert Merton, joined MIT as a research assistant to Paul Samuelson in the economics faculty. Samuelson, based on his own earlier work, encouraged Merton to explore the theory of warrant pricing.

At this time, Merton developed the use of continuous time finance and Brownian motion. These mathematical ideas were to provide the basis for the Black–Scholes formula. Merton, Black, and Scholes exchanged ideas over several years at MIT. Their path to success might be described as a 'Brownian motion with positive drift' – there were many false starts but ultimately the problem was solved.

In 1970, Black and Scholes completed their options pricing paper. They acknowledged Merton's suggestion of combining the option and the underlying asset, to yield a risk-free portfolio. Black and Scholes' paper was initially rejected by the *Journal of Political Economy* (JPE) – a publication of the economics faculty of the University of Chicago – because it was too specialised.

It was then rejected by Harvard's *Review of Economics and Statistics* but finally, with the support of Eugene Fama and Merton Miller, it was accepted by the JPE and published in May/June 1973 under the title 'The Pricing of Options and Corporate Liabilities'. Merton, who had collaborated with Black and Scholes, also produced a paper on options pricing in the *Bell Journal* of spring 1973.

Coincidentally, the Chicago Board Options Exchange (CBOE) began trading options (initially in the large smoking room of the Chicago Board of Trade) in April 1973 and the 'new' Black–Scholes formula was soon in use by traders. (For more details of this 'story' see the excellent book by Bernstein 2007.) The Ivory Towers of academia produced something of real practical value (as well as aesthetically pleasing). Whether it be 'Black Holes' or 'Black–Scholes', the power of mathematics to solve problems is impressive – not least in modern finance dealing with derivatives.

Source: Adapted from Cuthbertson and Nitzsche (2001).

The Black–Scholes (1973) formula for pricing European options (on a stock which pays no dividends) was derived using continuous time finance and stochastic calculus. The assumptions of the Black–Scholes model are:

- All risk-free arbitrage opportunities are eliminated.
- No transactions costs or taxes.
- Investors can borrow and lend unlimited amounts at the risk-free interest rate which is constant over the life of the option.
- Stock prices are random, like a 'coin flip' – if your first flip gives 'heads' this does not help you to predict whether the next flip will give you a head or a tail. The technical term for the random stock price process (over very small time intervals) is a 'geometric Brownian motion'.

- Stock prices are continuous and do not experience sudden extreme jumps – such as after an announcement of a takeover or other major unexpected firm specific events (e.g. new patents granted).
- The stock pays no dividends.
- The volatility of the stock (return) is known and constant over the life of the option (or is a deterministic function of time).

16.2.1 Call Option

To work out the Black–Scholes equation is rather difficult and at first sight the formula looks rather formidable. For the call premium:

$$
\begin{aligned}
C &= SN(d_1) - N(d_2)Ke^{-rT} \\
d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T} = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}
\end{aligned}
\tag{16.2}
$$

where

C = price of call option (call premium)
r = risk-free rate of interest for horizon T (continuously compounded).
S = current stock price
T = time to expiration/maturity (as proportion of a year)
σ = annual standard deviation of the (continuously compounded) return on the stock.
ln = natural logarithm of a variable[7]

The call premium depends positively on the current stock price relative to the strike price (S/K) and the volatility of the stock return over the life of the option ($\sigma\sqrt{T}$). In practice, the remaining 'life' of the option – that is, the time to maturity T – is usually measured using trading days (rather than calendar days):

$$T = (\text{number of trading days to maturity of the option})/252$$

[7]The term 'e^{-rT}' is the exponential function (where $e^1 = 2.718$, and for example $e^2 = 2.718 \times 2.718 = 7.389$, $e^{-1} = 1/2.718 = 0.3678$, etc.).

where there are (approximately) 252 trading days in a year (i.e. Monday–Fridays and excluding 'national' holidays). The Black–Scholes 'curve' shows the relationship between the call premium and the stock price (Figure 16.1).

Note that the only input to the Black–Scholes formula that is not known precisely is the volatility of the stock, σ which has to be estimated. If two traders have different forecasts for the volatility of the stock (over the life of the option) then even if they both use the Black–Scholes formula they will quote different options prices. Traders can apply statistical models to forecast volatility[8] but they mainly use an estimate of 'implied volatility' from 'similar' options (e.g. traded options with the same underlying stock and time to maturity as the option they wish to price, but with slightly different strike prices) – implied volatility is discussed further below.

What is interesting yet very counterintuitive is that the call premium does not depend on the expected rate of return μ on the stock, in the real world. For example, if one trader thinks the stock price will grow at 5% p.a. and a second trader that it will grow at 20% p.a. – they will still agree on the same price for the option – even though the expected payoff at maturity of the call option is clearly higher given the second trader's 20% growth assumption. This seems very counterintuitive. How can this be?

It is not possible to fully explain this counterintuitive result at this point. But as we shall see, by combining the option and the stock into a risk-free portfolio, such a portfolio must earn the risk-free rate – otherwise traders can make risk-free arbitrage profits. This results in the risk-free rate r 'appearing' in the Black–Scholes formula rather than the expected stock return μ.

The term $N(d_1)$ is the *cumulative* probability distribution function for a standard normal variable. It gives the probability that a variable with a standard normal distribution $\sim N(0, 1)$ will have a value less than d_1. For example, when $d_1 = -1.96$ then $N(d_1) = 0.025$ which means there is 2.5% of the total area of the normal curve to the left of the point –1.96 (Figure 16.2). The values of $N(d_1)$ can be found in standard statistical tables or you can use the Excel function 'NORMSDIST(–1.96)' which will 'return' a value 0.025. So at the end of the day, $N(d_1)$ is just a 'number' (which lies between 0 and +1). Here are some other values of $N(d_1)$ that you might remember from statistics classes:

$$\begin{array}{lllll} N(-\infty) = 0, & N(-2.33) = 0.01, & N(-1.96) = 0.025, & N(-1.65) = 0.05, & \\ N(0) = 0.5, & N(2.33) = 0.99, & N(1.96) = 0.975, & N(1.65) = 0.95, & N(+\infty) = 1 \end{array}$$

[8]The simplest forecast of daily volatility is the sample standard deviation $\hat{\sigma} = \sqrt{\sum_{t=1}^{n} (R_{cc,t} - \overline{R}_{cc})^2/(n-1)}$ where $R_{cc,t} = \ln(P_t/P_{t-1})$ is the continuously compounded daily return on the stock. When pricing options, daily (continuously compounded) returns are usually used to estimate daily volatility and the *annual* volatility is then taken to be $\hat{\sigma}\sqrt{252}$ – based on 252 trading days in the year.

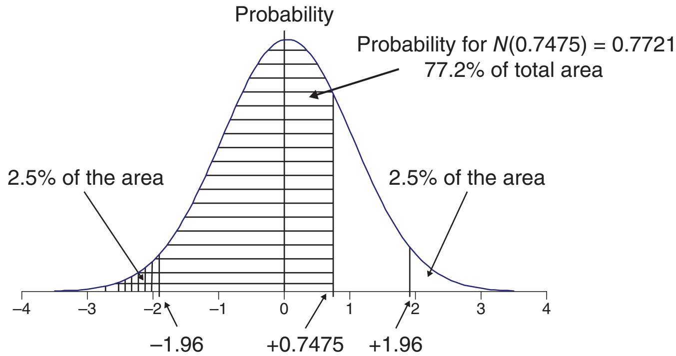


FIGURE 16.2 Standard normal distribution, $N(0,1)$

For negative values of d_1 we proceed as follows. We know, for example, that N(+1.96) = 0.975. Because the normal distribution is symmetric, $N(-d_1) = 1 - N(d_1)$ and therefore $N(-1.96) = 0.025$. In Example 16.1 we calculate the Black–Scholes price for a European call and put option (on a non-dividend paying stock).

EXAMPLE 16.1

Pricing Calls and Puts

$S = \$45$ $\qquad K = \$43$

$r = 0.10$ (= 10% p.a.) $\qquad \sigma = 0.20$ (20% p.a.)

$T = 0.5$ (years, approximately 126 trading days)

Call Premium, C

$$Ke^{-r.T} = 43 \ e^{-(0.1).(0.5)} = 40.9029$$

$$d_1 = \frac{\ln(45/43) + (0.1 + 0.2^2/2)0.5}{0.2\sqrt{0.5}} = 0.7457$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.7457 - 0.2(0.5)^{1/2} = 0.6043$$

Using the Excel function NORMSDIST(d_i), $N(0.7457) = 0.7721$ and $N(0.6042) = 0.7272$

$$C = SN(d_1) - N(d_2)Ke^{-rT} = 45\ (0.7721) - 0.7272\ (40.9029) = \$5.00$$

Put Premium, *P*

The price P, of a European put using the Black–Scholes formula is:

$$P = Ke^{-rT}N(-d_2) - SN(-d_1)$$

$$N(-d_1) = 1 - N(d_1) = 1 - 0.7721 = 0.2279$$

$$N(-d_2) = 1 - N(d_2) = 1 - 0.7271 = 0.2728$$

$$P = 40.9029\ (0.2728) - 45(0.2279) = 0.9069$$

Check the price of the put by using the put–call parity relationship:

$$P = C + Ke^{-rT} - S = 5 + 43e^{-0.10(0.5)} - 45 = 0.9069$$

We have moved a long way from the simple 'payoff' diagrams for calls and puts. We now know what causes option prices to change second-by-second.

An Excel file is available on the website to graph the relationship between the underlying stock price S *and the call premium* C *(or put premium,* P*) given by the Black–Scholes formula.*

In practice, traders have all these calculations programmed using 'in-house' software. For example, an element of code is provided in Example 16.2 in MATLAB to calculate European call and put premia for alternative values of the stock price, S.[9] (Other programs would have a similar structure.)

[9]The formulas for the call and put premia assume the stock pays a continuous dividend at the rate 'd' per annum. For example, if the dividend payout is 4% p.a., then $d = 0.04$ in the option price formulas. If we set $d = 0$, then the formulas match those in the text for options on a non-dividend paying stock.

EXAMPLE 16.2

MATLAB Code, Price Plain Vanilla Call and Put

```
k = 43  ;    % strike price  (Note that in MATLAB '%' indicates a comment)
s0 = 45 ;     % price of the underlying asset (stock)
r = 0.10  ;   % interest rate, contin. comp, decimal
d = 0.0  ;    % dividend yield, contin. comp, decimal
sigma = 0.20  ;   % standard deviation, annual, decimal
tau = 0.5  ;    % time to maturity, years or fraction of a year
c0 = bscall(k,s0,sigma,tau,r,d);       % activate function/procedure for call
premium
p0 = bsput(k,s0,sigma,tau,r,d);        % activate function/procedure for put
premium
pp0 = c0 - s0.*exp(-d.*tau) + k.*exp(-r.*tau) ; % put-call parity: check on BS
formulas
% ---------------------------------------------------------------------------
%                 Setting up Stock Price data for graph
% ---------------------------------------------------------------------------
s = s0-10:1:s0+10;                % S values are s0-10 to s0+10 in
increments of 1
c=bscall(k,s,sigma,tau,r,d);           % Activate function/procedure for call
p=bsput (k,s,sigma,tau,r,d);           % Activate function/procedure for put
% ---------------------------------------------------------------------------
%          Function: Black-Scholes Call Premium - Dividend paying stock
% ---------------------------------------------------------------------------
function c = bscall(k,s,sigma,tau,r,d)
   d1 = ( log( s./k ) + ( r - d + sigma.^2./2).*tau )./ ( sigma.*sqrt(tau) );
   d2 = d1 - sigma.*sqrt(tau);
   c=s.*exp(-d.*tau).*normcdf(d1) -k.*exp(-r.*tau).*normcdf(d2);
% ---------------------------------------------------------------------------
%          Function: Black-Scholes Put Premium - Dividend paying stock
% ---------------------------------------------------------------------------
function p=bsput(k,s,sigma,tau,r,d)
   d1 = ( log( s./k ) + ( r - d + sigma.^2./2).*tau )./ ( sigma.*sqrt(tau) );
   d2 = d1 - sigma.*sqrt(tau);
   p=k.*exp(-r.*tau).*normcdf(-d2) - s.*exp(-d1.*tau).*normcdf(-d1) ;
```

After setting the (scalar) inputs K, r, tau (time to maturity), σ (sigma) and the current stock price $s0$, the program uses the functions 'bscall' and 'bsput' (shown at the bottom of the program) – to calculate the option premia, c and p (using alternative values for the stock price).

Put–call parity is used to calculate the put premium *pp*0 – this provides a check that we have programmed the formulas for *c*0 and *p*0 correctly. The functions 'bscall' and 'bsput' calculate 21 values for the call and put premia corresponding to $s = s0–10:1:s0+10$ where $s0=45$ and s runs from 35 to 55 in increments of '1'. We can use the output from the MATLAB program to graph the call or put premium against the stock price.

16.3 ARE STOCKS LESS RISKY IN THE LONG RUN?

We can use the Black–Scholes equation to throw some light on whether stocks are less risky if you hold them for a long period rather than for a short period of time. It is often asserted that stocks are safer in the long run than in the short run – so a young person should hold lots of her wealth in stocks and only a little in a risk-free asset (e.g. bank deposit or government bond) and vice versa for a person nearing retirement. Is this true?

It is true that if you invest \$1m in the stock market today then the *probability of a loss* (i.e. ending up with less than \$1m) becomes smaller, the further ahead is the date at which you want to sell your stocks. But a more realistic approach is to ask what is the probability of ending up with a return on your stock market portfolio which is less than the return on the risk-free asset (e.g. bank deposit or AAA-government bond) – in Example 16.3 we will refer to this as the 'shortfall probability'.

EXAMPLE 16.3

Shortfall Probability

Suppose the risk-free rate is $r = 5\%$ p.a. and the mean return on the stock market is $\mu_R = 15\%$ p.a. with a standard deviation of $\sigma = 20\%$ p.a. Then the mean excess return is $\mu_X = \mu_R - r = 10\%$ p.a. The standard deviation of the excess return is also $\sigma = 20\%$ p.a. The Sharpe ratio of the stock market portfolio is 0.5 (= 10/20) – so these are reasonable 'ball-park' figures to use.[10]

(*continued*)

[10]We assume that continuously compounded returns are identically and independently distributed, which allows us to use the result that if the annual mean return and standard deviation are μ and σ then the mean return over T years is $\mu.T$ and the standard deviation is $\sigma\sqrt{T}$. We also assume that stock returns are normally distributed, the standard deviation of the risk free rate is zero and is (therefore) uncorrelated with the stock return – these are not unreasonable assumptions for this illustrative example.

(*continued*)

The shortfall probability over 1 year is the probability of observing an annual stock return R which is less than r or equivalently, the probability that the annual excess stock return $X = R - r$ is less than zero. The excess return is assumed to be normally distributed: $X \sim N(\mu_X, \sigma)$. The mean value of X is $\mu_X = 10\%$ p.a. Over a 1-year horizon if the out-turn value for $X = 0$, then this is a fall of 10% (below the mean), which is half a standard deviation (= 10/20). In 'statistical language' if X turns out to be less than zero then the 'standard normal variable' $z = (X - \mu)/\sigma < -0.5$. From the standard normal distribution, the probability of $z < -0.5$ is 46%, which is therefore the shortfall probability over a 1-year horizon.

Over 20 years, the mean excess return is $\mu_X T = 200\%$ and the standard deviation is $\sigma\sqrt{T} = 89.44\%$. So over 20 years, the probability of an out-turn value for X that is less than zero, is equivalent to the probability that $z < -200/89.44 = -2.236$, which is 1.27%. Hence the shortfall probability is much lower over a 20-year horizon than over a 1-year horizon. This is because the mean excess return of the stock market is positive at 10% p.a. However, the return on your stock portfolio could be less than the risk-free return of 5% even over a 20-year horizon – although there is only a 1.27% chance of that happening.

The result from calculating the shortfall probability over 20 years might seem 'comforting', since it is much lower than the shortfall probability over 1 year, but it should not really be as comforting as it first seems. This is because the 'low probability' of 1.27% tells you nothing about *how much* you might lose if you did end up with a loss at the end of your 20-year investment horizon. As an investor you are not just interested in the probabilities attached to certain outcomes but in the potential *dollar outcomes* at the end of your investment horizon. It is dollars that buy things, not 'probabilities'.

For example, if the stock price is initially $W_0 = \$100$ and can have only two outcomes +20% and –20%, then at the end of one year the *maximum* you can lose is 20% ($20) of your initial investment. After 2 years the maximum you can lose is 36% of your initial investment (i.e. wealth after two consecutive negative returns, $S_2 = S_0 \times 0.8 \times 0.8 = 64\%$ of initial wealth). But after 20 years, one possible outcome is that your final wealth could be as low as $S_{20} = S_0(0.8)^{20} = S_0(0.0115)$, so over 99% of your initial wealth could disappear. This outcome has a very low probability (about 1 in 100,000) but if it occurred you would be devastated.

Most people are presumably concerned about both the likelihood of a loss *and* the severity of the loss, should it occur. How can we get a handle on both? Well, buying a European put option on a stock (index) protects your portfolio of stocks from any 'downside' and therefore deals with the severity of a loss. The put also reduces the probability of a loss to zero (assuming your counterparty does not default). It insures the value of your stocks at a level equal to the strike price in the put.

The strike price you might choose could be set equal to your initial $1m investment, compounded at the risk-free rate until the end of your investment horizon T, that is a put with a strike price $K(T) = S_0 e^{rT} = (\$1\text{m})e^{rT}$. Then you are guaranteed that your wealth at retirement will end up at least as high as if you had invested your money in the risk-free asset.

The put premium you pay today ensures that the minimum future value of your stock portfolio in T years will at least equal what you would have got by investing in the risk-free asset. But you also have the additional benefit that if the stock market rises substantially then you will not exercise your put but simply take advantage of a high value for your portfolio of stocks.

The price of a put option increases with the time-to-maturity and with the volatility of the stock return. For example, suppose $S_0 = K = \$100$, $\sigma = 0.20$ and the strike price in the put increases with the risk-free rate, $K(T) = S_0 e^{rT} = (\$100)e^{rT}$. Then a put option with maturities $T = \{1, 5, 10, 20, 30, 70, 100\}$ years will today cost $P = \{\$8, \$17.7, \$24.8, \$34.5, \$41.43, \$59.7, \$68.3\}$ – this is shown in the lower graph of Figure 16.3.[11]

On the website you will find Excel and MAT-LAB files to reproduce figure 16.3.

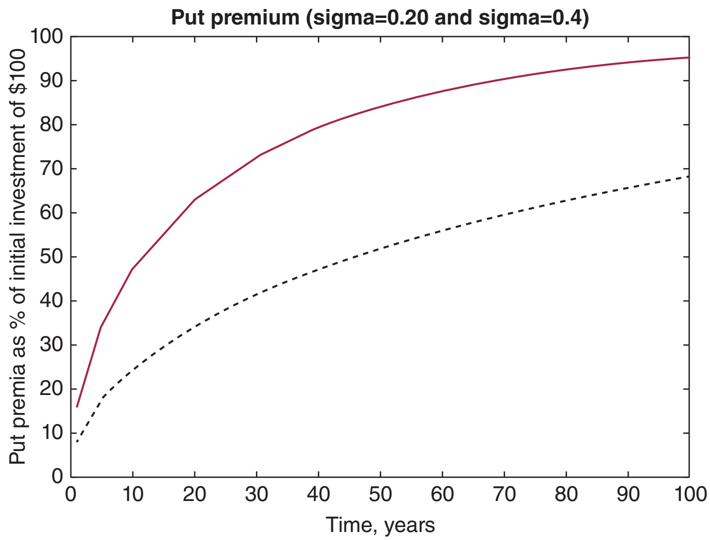


FIGURE 16.3 Cost of put versus maturity, T: $K(T) = S_0\, e^{rT}$

[11]If you set the strike price $K(T) = S_0 e^{rT}$ in the Black–Scholes formula for a put then $d_1 = \sigma\sqrt{T}/2$, $d_2 = -\sigma\sqrt{T}/2$ and $P/S = N(d_1) - N(d_2)$. We take $\sigma = 0.2$. It can then be shown that P/S increases as T increases, as noted in the text. Also, P/S is independent of the risk-free rate. It is also the case that using put–call parity with $K(T) = S_0 e^{rT}$, gives $P = C$ so the put costs the same as a call. So buying the put, given that you already have the stocks, is equivalent to having the payoff for a European call.

The put option may cost you quite a proportion of the value of your initial investment – depending on the horizon over which you want to insure your stock portfolio. For example, for a horizon of 30 years the cost of the put is \$41.43. For a horizon of 100 years the put premium is \$68.3. If current stock return volatility is high $\sigma = 0.40$ (i.e. 40% p.a. say) then the put premium (for each horizon, T) is substantially higher than for $\sigma = 0.20$ (upper curve in Figure 16.3) – this is because the higher stock market risk, implies a higher insurance premium.

So now you have your answer. Market participants (who trade options) think stocks are more risky the longer the investment horizon, because today they set a higher put premium the longer the maturity date of the put. You pay a bigger insurance premium, the longer the period over which you want to insure the value of your stocks at some minimum future level.

An even broader question is how your holdings of stocks relative to holding risk-free assets (e.g. AAA rated government bonds) *should vary* over your life. It turns out that there are no clear-cut answers other than the usual intuitive ones. At any point in your life cycle, diversify as much as possible (i.e. hold a ‘passive’ international portfolio of stocks, bonds, and ‘alternative assets’). If you want to buy insurance at some point, say a few years before retirement, then do so. By buying a put you guarantee a minimum value for your level of final wealth, but you can also benefit from any upside in the stock market, should this occur.

It might be advisable to buy a put when you are nearing retirement, so at a ‘reasonable cost’ you secure a floor price for your stocks but can also benefit from any remaining upside in the stock market. For example, with 5 years to retirement (and $\sigma = 0.20$) then you can guarantee that the return on your \$100 initial investment will at least equal the risk-free rate, if today you pay the put premium of \$17.7 – a not unreasonable insurance premium. Of course, after paying your insurance premium, if the return on the stock market over the next 5 years is higher than the risk-free interest rate, then you will not exercise the put, but you can sell your stocks at the current high price on the NYSE.

16.4 DELTA HEDGING

Black, Scholes, and Merton derive their option price formulas by assuming ‘delta hedging’ takes place. They set up a risk-free portfolio consisting of a position in the option and a position in the underlying stock. This portfolio is risk-free because any gain or loss from a small change in the stock price is exactly offset by changes in the value of the option’s position – this is *delta hedging*. As we shall see, the ratio of the number of stocks to the number of options required to set up the hedge is given by the *option’s delta* and is denoted Δ_c (for a call).

$$\Delta_c = \frac{Change\ \ in\ \ options\ \ price}{Small\ \ change\ \ in\ \ stock\ \ price} = \frac{C_1 - C_0}{S_1 - S_0} = N(d_1) = +0.4 \tag{16.3}$$

It can be shown that the delta of a (long) call option (on a non-dividend paying stock) is $\Delta_c = N(d_1)$, so the delta can be calculated directly from the Black–Scholes formula.[12] Pictorially, Δ_c is the slope of the line which is tangent to the Black–Scholes curve (at the current stock price). This tangent line exactly coincides with the Black–Scholes curve for small changes in the stock price (e.g. Figure 16.4, when $S_0 = \$100$). But for large changes in the stock price the 'straight line' and the 'curve' do not coincide. The true change in the call premium is given by the Black–Scholes curve and the *approximate change* in the call premium is given by the 'tangent line' – that is, the option's delta.

$$\text{Approximate \$-change in call premium} = \Delta_c \times (\text{Small \$-change in stock price})$$

For simplicity, suppose each call is written on 1-stock. Assume the current stock price $S_0 = 100$, the call premium is $C_0 = \$10$, and option's delta is $\Delta_c = +0.4$. The option's delta of $+0.4$ simply means that when the stock price increases by \$1 the call premium increases to \$10.4 – that is, the call premium changes by (approximately) \$0.4 (in the same direction as the stock price).

Suppose Ms Short works for a large investment bank on the options desk and she has just sold 100 call options to an investor (Mr Long). Ms Short has 'written' 100 calls. If Ms Short does nothing she is subject to market risk – if the stock price rises, so will the call premium and if Ms Short wants to close out her position by buying back the options, she will make a loss. To hedge her options position Ms Short should today buy 40 stocks for every 100 calls she has sold (written).

If the stock price rises by \$1, the call premium rises by (approximately) \$0.4 so that $C_1 = \$10.4$. Mr Long who holds one long call gains \$0.4 since he bought each call for \$10 and could now sell it for \$10.4. The options trader Ms Short has a (mark-to-market) loss of \$0.4 – since she sold each call at \$10 and to close out her position she would have to buy back the call at \$10.4.

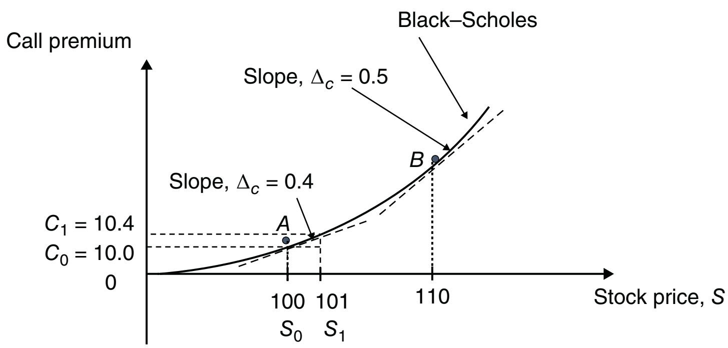


FIGURE 16.4 Delta of a call

[12]As we see later, the delta of a put option (on a non-dividend paying stock) is $\Delta_p = N(d_1) - 1$.

Ms Short has a loss of \$40 (= \$0.4 × 100 calls) on her 100 *written* calls, but this is offset by the \$40 gain on her long position in stocks (= 40 stocks × \$1) – she is delta-hedged. There is no loss on Ms Short's portfolio consisting of '0.4 stocks for each call she has sold' or equivalently 'holding 40 stocks and selling 100 calls'. Ms Short's portfolio is *delta neutral* – as long as the change in stock prices is small.

To maintain a delta neutral position requires continuous *rebalancing* since Δ_c changes as the stock price changes. At a stock price of \$100, $\Delta_c = 0.4$ (say). But given the shape of the Black–Scholes curve, if the stock price on day-2 is \$110 then the slope is $\Delta_c = 0.5$ (Figure 16.4). Hence, for a delta-neutral position to be maintained an extra 0.1 stocks (= 0.5 – 0.4) would have to be purchased by Ms Short for each call she has sold. Since Ms Short initially sold 100 calls then she will have to purchase an additional $0.1 \times 100 = 10$ stocks (making a total of 50 stocks held on day-2), in order to maintain her delta-hedged position over day-3. This is known as *dynamic delta hedging*.

Delta hedging only provides a hedge against small changes in the stock price and not against large changes. Also the option's delta does not take account of changes in the call premium due to changes in the volatility of the stock return, the risk-free rate or the time to expiration of the option – which all affect the price of the call via the Black–Scholes formula. To hedge your written calls against such changes requires additional hedging strategies involving 'the Greeks'.

16.5 IMPLIED VOLATILITY

How can you determine 'the options market's' average forecast of volatility for a particular stock (or stock index)? This is obtained from observed options prices by 'inverting the 'theoretical Black–Scholes equation'. Let's see how this is done.

Today, data on all the variables $z = \{S, K, r, T\}$ are available (Table 16.2) and the *quoted* call premium, $C_q = \$5.748$ is observable on the Chicago options exchange. The Black–Scholes (B–S) call premium is given by Equation (16.2), here represented as $C_{BS} = f(z, \sigma)$. Implied volatility is that value for volatility σ_i which makes the theoretical B–S call premium C_{BS} equal to the quoted call premium, $C_q = \$5.748$, that is, σ_i is the solution to:

$$C_q = \$5.748 = C_{BS} = f(S, K, r, T, \sigma_i) \tag{16.4}$$

σ_i is the option traders' current forecast of stock return volatility, over the life of the option, assuming the B–S pricing equation is the correct model for determining the quoted call price.

Conceptually, to solve for implied volatility, we choose alternative 'trial values' σ_i and calculate the 'theoretical' B–S price, $\widehat{C}_{BS,i} = f(z, \sigma_i)$ and see if the B–S price equals the quoted price $C_q = \$5.748$. For example, for $\sigma_i = 0.282$ we find $\widehat{C}_{BS,i} = 5.6265$ (Table 16.2), which is below the current quoted price $C_q = \$5.748$. We know that the B–S call premium increases with volatility so we now assume a new trial value $\sigma_i = 0.288$ which gives $\widehat{C}_{BS,i} = 5.748$ and equals the quoted

TABLE 16.2 Implied volatility

$S = 164$
$K = 165$
$r = 0.0521$
T(years) = 0.0959
Quoted call premium = 5.748

Implied volatility $\sigma_i = 0.288$ (28.8% p.a.)

Trial values for sigma, σ_i	d_1	d_2	$N(d_1)$	**Theoretical B–S call premium $\hat{C}_{BS,i} = f(z, \sigma_i)$**
0.282	0.031267	−0.05606	0.398747	5.626545
0.283	0.031466	−0.05617	0.398745	5.646796
0.284	0.031664	−0.05628	0.398742	5.667047
0.285	0.031862	−0.0564	0.39874	5.687298
0.286	0.03206	−0.05651	0.398737	5.707549
0.287	0.032257	−0.05662	0.398735	5.727799
0.288	0.032454	−0.05673	0.398732	**5.74805**
0.289	0.032651	−0.05685	0.39873	5.7683

price $C_q = \$5.748$. Hence, $\sigma_i = 0.288$ (28.8% p.a.) is the current value of implied volatility. This is done in Table 16.2 by trial and error.

> *Clearly, a more efficient way of proceeding would be to have a computer program choose alternative trial values for implied volatility σ_i to minimise $(\hat{C}_{BS,i} - C_q)^2$. An example of an iterative search procedure to calculate implied volatility can be found in the Excel spreadsheet on the website – which uses the Excel optimiser SOLVER.*

There are many iterative search algorithms which can be used to calculate implied volatility (e.g. Newton–Raphson procedure). An initial guess for implied volatility is $\sigma_1 = [|(\ln(S/K) + rT)(2/T)|]^{1/2}$ followed by the iterative process:

$$\sigma_{i+1} = \sigma_i - \frac{[\hat{C}_i - C]e^{d_1^2/2}\sqrt{2\pi}}{S\sqrt{T}} \tag{16.5}$$

where $d_1 = \frac{\ln(S/K)+(r+\sigma_i^2/2)T}{\sigma\sqrt{T}}$ is from the B–S formula and is computed using σ_i. The term $\widehat{C}_{BS,i} = f(z, \sigma_i)$ is the B–S call premium and $C_q = \$5.748$ is the *quoted* call premium. The iteration proceeds until we obtain a value for σ_i which satisfies $C_q = \widehat{C}_{BS,i} = f(z, \sigma_i)$ to a high degree of accuracy.[13]

One of the problems in interpreting implied volatilities is that it is possible to calculate different implied volatilities from options with different strike prices (but on the same underlying stock and with the same maturity date). For example, an at-the-money option might give an implied volatility of 20% p.a. while a deep out-of-the-money option (on the same stock) might yield a figure of 24%. One way out of this dilemma is to take a weighted average of these two volatilities as a measure of the 'true' volatility. The weights could be based on the sensitivity of the option premium to changes in volatility (i.e. the option's 'vega' – see Chapter 28). As the premium for an at-the-money option is more sensitive to changes in volatility (than for an OTM option) then the weights might be 0.8 and 0.2, for example, with the best estimate of implied volatility being 0.8(20%) + 0.2(24%) = 20.8%.

You can see how the market's view of volatility has changed over time using the implied volatility of the S&P 500 (Figure 16.5) – available on the CBOE website.

16.5.1 Trading Volatility: Mispriced Options and Delta Hedging

Delta hedging can also be used to create a risk-free portfolio so that you can profit from any mispriced options. Suppose Ms Short thinks a European call option is currently overpriced at C

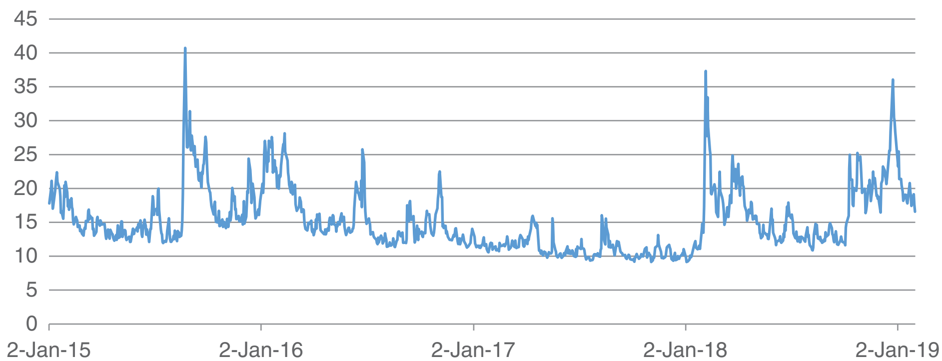


FIGURE 16.5 Implied volatility, S&P 500

[13]Black–Scholes implied volatilities vary depending on whether the options are ATM, OTM, or ITM. For example, the convention for options on FX is that you might report the implied volatility for an at-the-money (ATM) option, or for a 25 Delta (out-of-the-money, OTM) or for a 10-Delta (far OTM option). The convention in the FX market is that a 25-Delta put, for example, has a delta of –0.25 and a 25-Delta call has a delta of +0.25.

= \$10.1 because she thinks other traders are using a forecast of volatility in the Black–Scholes formula that is too high. Today, writing (selling) 100 options for \$10.1 will result in a profit for Ms Short if the option price falls towards its true (fair) value – as other option traders recognise they have used too high a figure for volatility. Ms Short will then be able to close out her options positions, by buying the options at their new lower market price of say \$9.8. Ms Short has been using options to *'trade volatility'*.[14]

However, this is a highly risky strategy by Ms Short since the mispriced option could increase in price if stock prices increased, before market participants correct their forecast of volatility. Can Ms Short profit from mispricing in the options (due to a miscalculation of volatility by other traders) but hedge any price risk on the options, due to changes in stock prices? Yes, by delta hedging.

Suppose the options delta is 0.4. Then Ms Short can today hedge her options position *against changes in stock prices* by buying 0.4 stocks for every option she has sold – so she buys 40 stocks ($= 0.4 \times 100$). If the stock price increases by \$1 she gains \$40 on the 40 stocks but the call premium increases by \$0.4 and she makes a (mark-to-market) loss of 40 (= \$0.4 × 100 options) on her 100 written options. Her short options position is now hedged against changes in stock prices. She can therefore wait until the call premium falls, when 'other traders' revise downwards their forecasts of volatility. Ms Short can then close out by buying back the calls at \$9.8 (say).

Notice that Ms Short only makes money if *her own* volatility forecast (which is different from the 'average market view' of volatility) is correct, and the other options traders later recognise this fact. She does not make any money from changes in the stock price, because she is delta-hedged. In practice there are some risks and costs attached to the above strategy (and to delta hedging):

- when rebalancing the hedged portfolio, you incur transactions costs (e.g. bid–ask spreads).
- delta hedging only gives a good hedge if changes in the stock price are small.

16.6 SUMMARY

- The call premium (on a European option) is positively related to the stock price, while the put premium is negatively related to the stock price. Both call and put premia increase when the volatility of the stock return σ, increases. Call and put premia both

[14]You can also speculate on changes in volatility using futures contracts. In March 2004 the CBOE futures exchange introduced a futures contract on the 30-day implied volatility of the S&P 500 stock index – the ticker symbol of this futures contract is VIX ('the vix'). The payoff to the futures contract depends on the change in implied volatility between today and when you close out the futures contract. If you thought the volatility of the S&P 500 was going to increase in the future, then today you would go long 'VIX futures contracts'.

increase with the maturity date of the option T and they also depend on the risk-free rate r. These relationships are a consequence of the Black–Scholes formulas for calls and puts.

- We can establish upper and lower limits for call and put premia. If quoted options prices lie outside these bounds then risk-free arbitrage profits can be made.
- A speculator who thinks that stock prices will rise (fall) in the future will buy calls (buy puts). A speculator who forecasts that stock return volatility will increase (decrease) in the near future will buy (sell) either calls or puts or both. If these volatility forecasts are correct then the speculator can close out her options positions at a profit.
- A (naked/open) position in options is risky but this risk can be removed (over small intervals of time) by delta hedging. For example, if you have sold 100 call options each with a delta of 0.4, then you can delta-hedge your position (over the next day) by buying 40 stocks, today. Your hedge-portfolio of '100 written calls and 40 long stocks' will not change in value over the next day, for small changes in stock prices. This is a delta-neutral portfolio.
- Because the delta of an option changes over time (as the stock price changes), then to maintain a delta-neutral (hedged) position, you need to frequently alter the number of stocks you hold – this is *dynamic delta hedging*.
- Using the Black–Scholes equation and the quoted option price, we can solve for the market's current forecast of volatility (of the underlying stock return), over the life of the option – this is the option's *implied volatility*.
- If your own forecast of a stock's return volatility is lower (higher) than 'implied volatility', then you believe the 'correct price' of the option is below (above) its quoted price. You can take advantage of this mispricing by today selling (buying) the overpriced (underpriced) option. You also need to delta hedge your options position against changes in the options price *due to changes in the stock price*. You are '*trading volatility*' but hedging the value of your options portfolio from any change in stock prices.

APPENDIX 16: PRICE BOUNDS ON EUROPEAN OPTIONS

Upper Bounds: Calls and Puts (on Non-Dividend Paying Stock)

If a quoted option price is below the lower bound or above the upper bound (as set out below) then there are arbitrage profits to be made. Establishing these bounds only requires the assumption of a positive risk-free (nominal) rate of interest.

A European call option (on a non-dividend paying stock) gives the holder the right to take delivery of a stock, and the option can never be worth more than the current price of the stock. Hence the upper bound for European calls is $C_e \leq S$.

For a European put the minimum value is K at maturity T, so today the European put cannot be worth more than $P_e \leq Ke^{-rT}$.

In the above cases if the upper bound does not hold then arbitrage profits can be made. For example, if $P_e > Ke^{-rT}$ an arbitrageur Ms A sells a European put for P_e, invests the proceeds at the risk-free rate giving a cash amount to Ms A of $P_e e^{rT}$ at T. If $S_T > K$, the long put is not exercised (as it is worthless) and Ms A is worth $P_e e^{rT}$ at T. If $S_T < K$, the long put is exercised, with a cash payout from Ms A of $K - S_T$. At T, Ms A is worth $P_e e^{rT} - (K - S_T) = (P_e e^{rT} - K) + S_T > 0$.

Lower Bounds: Calls and Puts (on Non-Dividend Paying Stock)

The lower bounds for European calls and puts are:

$$C_e \geq \max(S - Ke^{-rT}, 0) \tag{16.A.1a}$$

$$P_e \geq \max(Ke^{-rT} - S, 0) \tag{16.A.1b}$$

To show that (16.A.1a) must hold for a European call option (on a non-dividend paying stock) assume that $C_e < S - Ke^{-rT}$. Then we can show that an arbitrageur Ms A by 'buying low – selling high' can earn risk-free profits. Today, Ms A buys the call for C_e, borrows a stock from her broker and short-sells the stock for S. Today the net receipts of $S - C_e$, accrue to $(S - C_e)e^{rT}$ at $t = T$. At T, if $S_T > K$, Ms A exercises the call option by taking delivery of the stock and paying K and then returns the stock to her broker. Ms A's net profit at T is $\Pi_T = (S - C_e)e^{rT} - K$, which has a present value (today) of $PV = (S - C_e) - Ke^{-rT} > 0$ (given our initial assumption that $C_e < S - Ke^{-rT}$). On the other hand, if $S_T \leq K$ then Ms A does not exercise the call but she can buy the stock in the cash-market for $S_T \leq K$ (and then returns it to her broker). Hence, she makes an even greater arbitrage profit at T of $\Pi_T = (S - C_e)e^{rT} - S_T$. The worst outcome from a long call is that it expires out-of-the-money and hence is worth zero at T – this implies a lower bound today for a European call is zero, $C_e \geq 0$.

To show that the lower bound for a put is given by (16.A.1b) we take the counter example where $P_e < Ke^{-rT} - S$. To make an arbitrage profit Ms A would buy the put and the stock today by borrowing $P_e + S$. At T, this portfolio is worth the larger of S_T (when $S_T > K$) or K (when $S_T < K$). At T, Ms A owes the bank $(P_e + S)e^{rT}$. But given our initial starting assumption that $P_e + S < Ke^{-rT}$, then Ms A has a positive arbitrage profit at T. As the worst that can happen is that the put option expires worthless, its value today is greater than zero.

EXERCISES

Question 1

Intuitively, would you pay more today for a European put option on a stock-A, with strike price $K_A = 100$ or a put option-B with strike $K_B = 98$ (on the same underlying stock-A)? Assume the current stock price is $S = 96$. Briefly explain.

Question 2
Intuitively, why would you pay more for a (European) call option-A on a (non-dividend paying) stock-A which has an annual return volatility of 20% p.a. or for a call option-B on stock-B which had an annual volatility of 10% p.a.? Assume all other variables which influence the call premium remain constant. Briefly explain.

Question 3
If the delta of a long call option is +0.6 what does this mean?
Assume the current stock price is \$100.

Question 4
Why is the delta of a long call option positive and the delta of a long put option negative?

Question 5
You have just purchased 100 puts (on stock-A) with a delta of –0.4. How can you hedge your risk over a small interval of time? Assume the current stock price is \$100.
Explain, what happens if the stock price subsequently falls by \$2.

Question 6
Assume, $S = 100$, $r = 10\%$ p.a. (continuously compounded), $K = 100$, $\sigma = 20\%$ p.a. and a call has 6 months to maturity.
Use the Black–Scholes formula to price the European call option on a (non-dividend paying) stock.
What happens to the call premium if next day, volatility increases to $\sigma = 30\%$ p.a.?

Question 7
Explain the concept of 'implied volatility' σ_{imp} for a put option (on stock-A).

Question 8
Explain how 'implied volatility' σ_{imp} for a put option (on stock-A) is calculated.

CHAPTER 17

Option Strategies

Aims

- To show how options can be combined to produce different payoff profiles – this is financial engineering.
- To demonstrate how bull and bear spreads are speculative strategies based on forecasting the *direction of change* in stock prices.
- To show how options can be combined to benefit from strategies based on forecasting *the range* (whether up or down) of future movements in stock prices. These are *volatility* strategies and include straddles, strangles, butterfly spreads, and condors.

A combination of calls, puts, and stocks can produce a wide variety of payoffs, the generic term for which is financial engineering or structured products. One example of financial engineering we have already met is the put–call parity relationship, which resulted in a structured product often referred to as a 'guaranteed bond'. In this chapter we examine how options can be combined in order to speculate on the future value (or range of values) for the underlying asset (or underlying futures contracts). In analysing these trades the main thing to remember is that the call premium is positively related to the price of the underlying asset and the put premium is negatively related to the price of the underlying asset. A second key fact is that both the call and put premia increase (decrease) when the volatility of the underlying asset return increases (decreases).

The underlying asset in the option's contract could be the stock of a particular company (AT&T), an industry sector (e.g. Internet Index Options on CBOE), a market index (e.g. S&P 500), a commodity (e.g. oil, silver, wheat), a futures contract, an exchange rate, bond price or interest rates (the latter are dealt with in Chapters 40 and 41). Our examples assume the options are written on a stock.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

17.1 SYNTHETIC SECURITIES

To illustrate possible outcomes from combining several options we first consider payoffs at maturity, T. A long call, if held to maturity (Figure 17.1) gives a zero payoff if the stock price at maturity is less than the strike price and a \$1 payout for every \$1 the spot price at expiry exceeds the strike price: this is designated a $\{0, +1\}$ payoff. A short (written) call if held to maturity, has a payoff profile which is the 'mirror image' of the long call.

Prior to expiration, the payoff profile of a long call is the dashed line in Figure 17.1, as given by the Black–Scholes formula. This curve moves towards the 'kinked' $\{0, +1\}$ payoff as the option approaches maturity (i.e. as it loses time value). Similarly, prior to maturity a written call has the 'curved' payoff profile in Figure 17.1 and its time value increases as the call approaches maturity (for any given stock price).

A long put (Figure 17.2) involves a zero payoff if the stock price at maturity is above the strike price $S_T > K$. A long put has a positive payoff at maturity if $S_T < K$. The *positive* payoff increases the *lower* is the stock price – this is a negative relationship, so the payoff is designated $\{-1,\ \ 0\}$.[1]

Prior to expiration, the payoff profile of a long put is the dashed line in Figure 17.2. At 'reasonably high' stock prices the long put loses time value represented by the curved line moving towards the 'kinked' line. However, at 'very low' stock prices a long (European) put may increase in value as the time to maturity approaches.[2] The payoff for a short put is the mirror image of the long put (see Figure 17.2).

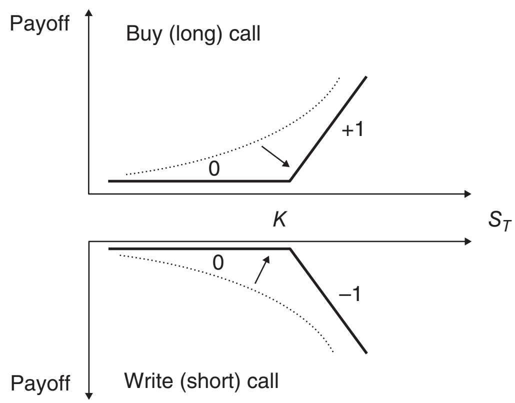


FIGURE 17.1 Payoff for calls

[1]Hence a direction vector '+1' indicates a positive relationship between the option payoff and the stock price: a rise (fall) in the stock price results in a rise (fall) in the payoff. A direction vector '–1' indicates a negative correlation, so that a rise (fall) in the stock price indicates a fall (rise) in the option payoff.

[2]The reason for this is a little esoteric but arises from the fact that at low stock prices the holder of a long put will almost certainly exercise the put and has to deliver a stock to the writer of the put option, and hence the long put must 'tie up' some of her 'money' in the stock, in order to hedge her position. The long put therefore foregoes interest receipts – the increase in time value on the long put is 'compensation' for this loss.

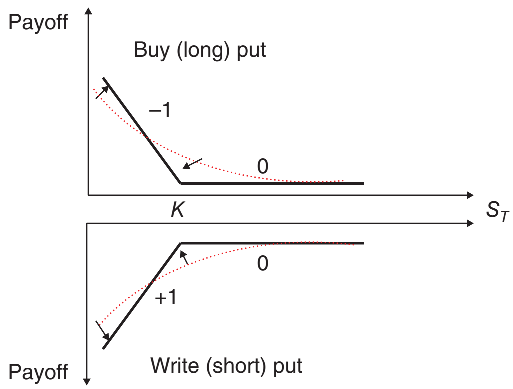


FIGURE 17.2 Payoff for puts

The payoff profile from being long stocks is $\{+1, +1\}$. If you are long stocks and the stock price increases then there is a positive payoff, as your stocks are worth more (Figure 17.3). The payoff profile if you short-sell stocks is $\{-1, -1\}$. If you short-sell stocks and the stock price increases then your position has fallen in value – since you would have to close out at a higher price and you make a loss – this is a negative relationship between the stock price and the payoff (Figure 17.3). If we assume the futures price moves dollar-for-dollar with the stock price, then the payoff profile for a futures contract is the same as for the stock. We could therefore use the futures price in place of the stock price.

Suppose we have three assets: stocks, calls, and puts (with the same underlying, maturity dates and strike price, K). Taking any two of these three assets we can replicate the payoff profile of the remaining one. Put another way we can create a 'synthetic' or 'replication portfolio' to mimic the outcome for a call, put, or position in the stock.

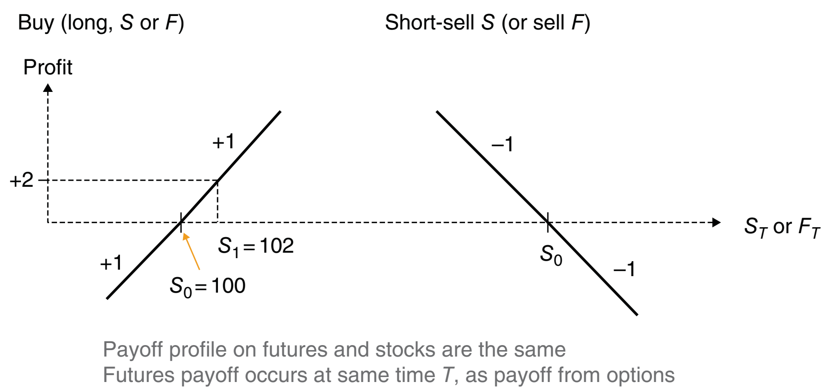


FIGURE 17.3 Payoff for futures/stocks

17.1.1 Synthetic Long Call

Figure 17.4 demonstrates that the payoff profile for a synthetic long call can be obtained by combining a long stock and a long put (with the same strike K, and time to maturity, T)[3]:

$$\begin{aligned}&\text{Long stock} + \text{Long Put} = \text{Long Call}\\&\{+1,+1\} + \{-1,\ \ 0\} \quad\ \ = \{0,+1\}\end{aligned}$$

17.1.2 Synthetic Short Put

The payoff profile of a synthetic short put (Figure 17.5) can be obtained from a long stock and short call (with same strike K, and time to maturity T):

$$\begin{aligned}&\text{Short put} &&= \{+1,0\}\\&= \text{Long stock} &&= \{+1,+1\}\\&+ \text{Short call} &&= \{0,-1\}\end{aligned}$$

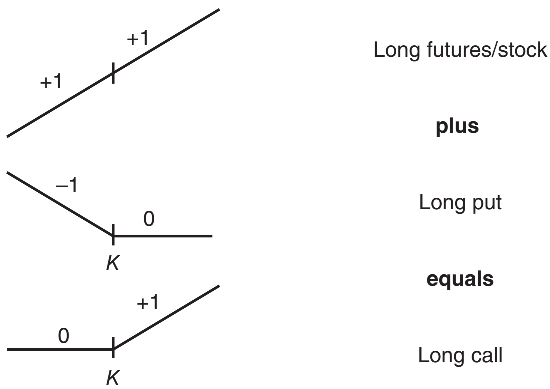


FIGURE 17.4 Synthetic long call

[3]If we include an amount of cash equal to Ke^{-rT} as well as holding the call, then this is 'put–call parity': $C + Ke^{-rT} = S + P$ (for European options on a non-dividend paying stock). But cash amounts cannot be included in the payoff diagrams – and do not influence the 'shape' of the payoff profiles.

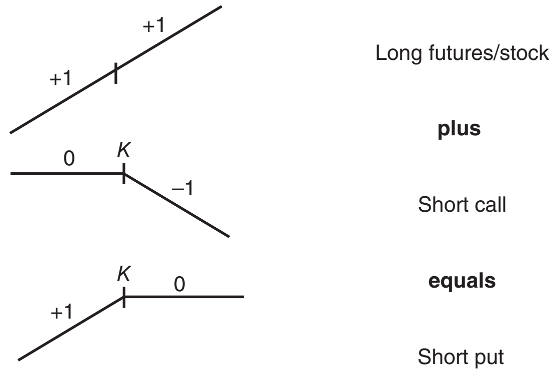


FIGURE 17.5 Synthetic short put

17.1.3 Synthetic Long Forward

A synthetic long stock (or futures) can be generated by combining a short put and a long call (with the same strike, K and time to maturity, T) – Figure 17.6.

$$\begin{aligned}\text{Long stock (futures)} &= \{+1, +1\}\\ = \text{Short put} &= \{+1,\ \ 0\}\\ + \text{Long call} &= \{0, +1\}\end{aligned}$$

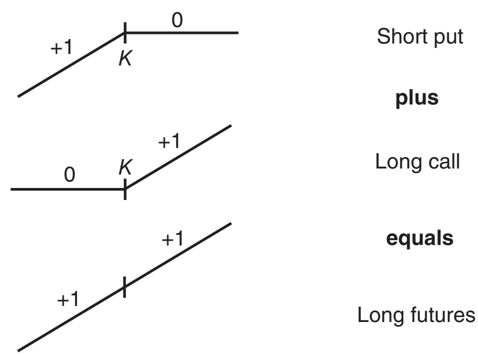


FIGURE 17.6 Synthetic long futures

17.1.4 Spreads and Straddles

Investors use *option spreads* because they involve less risk than an open or naked option's position but this risk reduction comes at a cost in terms of lower potential profits. Spreads may involve just calls or just puts or a combination of calls and puts. There are several different types of spread:

- *Vertical (or money) spread:* Use either calls or puts with the same expiration date but with different strike prices K – these are directional bets – that is, a speculative position which depends either on a rise or fall in the underlying asset price.
- *Straddle, Strangle, Butterfly, and Condor:* These use calls and puts and are speculative bets which depend on the volatility of the underlying asset price – large movements in either direction can cause profits or losses.
- *Horizontal (or time or calendar) spreads:* Use options with the same strike price K but with different maturity dates.
- *Diagonal spread:* Use options with different maturities and with different strike prices.

The term 'vertical spread' arises from the fact that newspaper quotes list the different strike prices in a (vertical) column and the term 'horizontal spread' because different expiry months are printed across the row. We use the convention that '*buying the spread'* (i.e. a long spread position) involves a positive net payments (at time $t = 0$) to set up the options trade. For example, buying an April-call at C_1 with a low strike price $K_1 = 102$ and selling an April-call at C_2 with a high strike price $K_2 = 105$ (i.e. both with the same expiry date in April) will involve an 'up front' net cash payment since $C_1 > C_2$. This would usually be referred to as buying the April-102/105 call spread, where it is understood that the first figure '102' refers to the option which is purchased. The April-102/105 put spread would be '*selling the spread'* since the cost of purchasing the 102-put would be less than the receipts from selling the 105-put.

If we are dealing with a time-spread with only one strike price, then the position will be long if we buy the nearby contract and sell the far contract. Hence, buying the April-102 call and selling the July-102 call would be a net long position (i.e. buy the spread). The investor would receive a net cash inflow from a long time-spread, as the price of the July contract is higher than the price of the April contract.

17.2 BULL AND BEAR SPREADS

In the first set of money spreads we consider, the investor takes a bet on the *direction* the stock price will move but she wishes to insure against large downside losses should the bet subsequently prove to be incorrect. These strategies are known as bull and bear spreads.

17.2.1 Bull Spread with Calls

If an investor expects a rise in stock prices then she could profit either by:

- purchasing the stock (or forward contract) today or
- buying a call option or
- selling a put option.

Buying stocks could be expensive (e.g. bid–ask spreads, commissions, price impact) and involves downside risk and selling a put option has a large downside risk, if the stock price subsequently falls.

An alternative to the above strategies, which involves a loss of some upside potential, but also reduces the downside risk, is to undertake a *bull spread* – the payoff is given in Figure 17.7 and is of the form $\{0, +1,\ \ 0\}$. A bull spread can be constructed using two calls with the same maturity T but different strikes K:

$$\begin{aligned}
&\text{Bull spread payoff} && = \{0, +1,\ \ 0\} \\
&= \textit{Buy} \text{ a call with a low strike price } K_1 && = \{0, +1, +1\} \\
&\text{and } \textit{Sell} \text{ a call at a high strike price } K_2 > K_1 && = \{0,\ \ 0, -1\}
\end{aligned}$$

There is a 'trick' to producing these results, which may be useful. Start with the bull spread payoff {0, +1, 0} – this payoff profile is what we are aiming for (line-1 above). Start producing this payoff by working from the left. First you 'fix' the '0' and '+1' but then you must continue

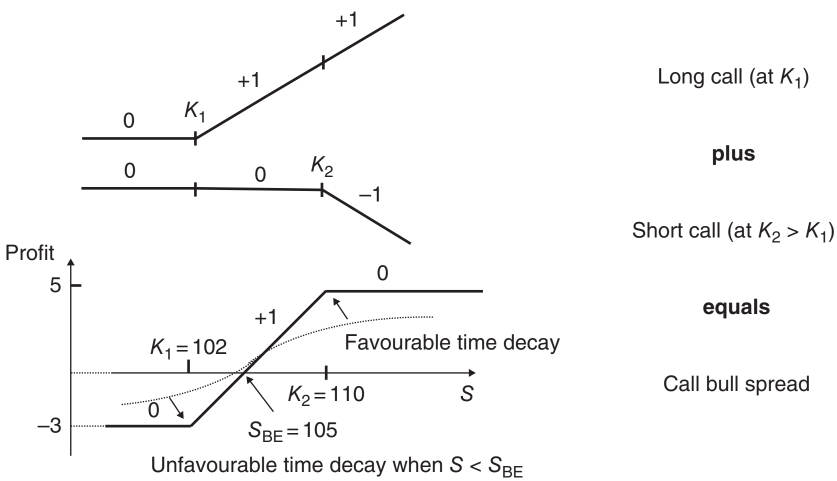


FIGURE 17.7 Bull spread with calls

with another '+1' – as an option only has one 'kink' in its payoff profile (line-2 above). When deciding on the entries in line-3 you make sure you do not alter your original '0, +1' (in line-2), by putting '0, 0' as the first two elements on the left (line-3) and then include '–1' so that the sum of the third column for lines 2 and 3 is 0 – as required to reproduce line-1. Practise this, it can be useful.

A bull spread is a *money spread* since the options have the same expiration date but different strike prices. So, for example, setting up this position in January, *both* calls could be for expiration in October (say). Compared with holding a naked long-call, there is a lower breakeven and a reduction in cost because of the premium received from the sold (written) call. However, there is also the loss of some upside potential, compared with holding just the long call.

Let us examine this strategy in a little more detail. Consider the following two calls (which have the same expiration date):

$$C_1 = 5 \;\; K_1 = 102 \;\; C_2 = 2 \;\; K_2 = 110$$

Note that the low strike price call ($K_1 = 102$) has the higher call premium and hence it costs $C_1 - C_2 = 3$ to go 'long K_1' and 'short K_2', to set up the spread trade. Since you pay out cash at $t = 0$, you are *buying the spread*. The profit (net payoff) at expiration is[4]:

$$\begin{aligned} &\Pi = \text{Max}(0, \;\; -K_1) - \text{Max}(0, \;\; S_T - K_2) - (C_1 - C_2) && (17.1)\\ &S_T \le K_1 \le K_2: \quad \Pi = -(C_1 - C_2) \\ &K_1 < S_T \le K_2: \quad \Pi = S_T - K_1 - (C_1 - C_2) \\ &S_T > K_1 > K_2: \quad \Pi = S_T - K_1 - (S_T - K_2) - (C_1 - C_2) \\ &\qquad\qquad\qquad\quad = K_2 - K_1 - (C_1 - C_2) \end{aligned}$$

When the stock price S_T is less than both K_1 and K_2 then both options expire out-of-the-money and you pay the premium $C_1 = 5$ and receive the premium $C_2 = 2$ on the written call, making an overall loss of 3 (which is independent of how far S_T is below K_1). When S_T exceeds both K_1 and K_2 then both calls are exercised. The stock delivered in the long call is used to deliver against the short call with net payoff $K_2 - K_1 > 0$. Alternatively, both calls could be cash settled (i.e. no delivery) and again the payoff is $K_2 - K_1 > 0$. The cash profit is:

$$\Pi = K_2 - K_1 - (C_1 - C_2) = 8 - 3 = 5 \tag{17.2}$$

which is independent of S_T. (This payoff will always be positive since the maximum payoff is $K_2 - K_1$ and this must always exceed the cost of setting up the position, namely

[4]Strictly speaking, since the net cost of the two calls when setting up the bull spread are incurred at $t = 0$ but the payoffs to the bull spread are at T, we should compound the net cost of $(C_1 - C_2) = 3$ to time T (using the risk-free rate).

$(C_1 - C_2)$, given by the Black–Scholes formula.) For the intermediate case, the profit is $\Pi = S_T - K_1 - (C_1 - C_2)$ and the breakeven stock price (i.e. $\Pi = 0$) is:

$$S_{BE} = K_1 + (C_1 - C_2) = 102 + 3 = 105 \tag{17.3}$$

The breakeven stock price S_{BE} (at expiry) must exceed the strike price K_1 by at least the up-front cost of setting up the bull spread $C_1 - C_2$.

What about the payoff profile when the bull spread has been in existence for some time? Suppose the original bull spread was purchased in January with a maturity date in October. In March the payoff profile is given by the curved line in Figure 17.7 and notice that the arrows indicate possible time decay characteristics. If the stock price is above the break-even price, then the bull spread has positive time value and benefits the holder. This is because the positive time value of the written option benefits the holder of the bull spread and this more than counteracts the negative time decay of the long call option. The converse applies if the stock price is 'low' – then the time decay works against the holder of the bull spread – and its value falls over time (at any given stock price).

17.2.2 Bull Spread with Puts

The bull spread profit profile {0, +1, 0} can also be constructed from puts with different strike prices:

$$\begin{array}{ll} \text{Bull spread with payoff} & \underline{\{\ \ 0, +1, 0\}} \\ = \text{Sell put with } K_2 (> K_1) & \{+1, +1, 0\} \\ \text{and buy a put with strike } K_1 & \{-1, \ \ \ 0, 0\} \end{array}$$

Our above 'trick' works with puts by starting at the *right-hand side,* in order to end up with {0, +1, 0} bull spread profile. In the second line starting with the first two entries on the right we have 0 followed by +1 but we must then continue with the +1 to give the full second line as '{+1 +1, 0}'. Then we complete the third line (starting at the right-hand side). We 'insert' {0, 0} in the third line to keep the 0 followed by +1 in line-2, unchanged. Then in the third line we include –1, so the first column in rows 2 and 3, sum to 0 – which gives the payoff for the bull spread, as required.

17.2.3 Bear Spread with Calls

The payoff here (Figure 17.8) is the mirror image of the bull spread and is undertaken when the investor believes stock prices will fall. The strategy requires a payoff {0, –1, 0} – the opposite of that for a bull spread.

$$\begin{array}{ll} \text{Bull spread with payoff} & \underline{\{0, -1, \ \ \ 0\}} \\ = \textit{Sell} \text{ a put with } K_1 & \{0, -1, -1\} \\ \textit{and buy} \text{ a put with } K_2 \ \ (> K_1) & \{0, \ \ \ 0, +1\} \end{array}$$

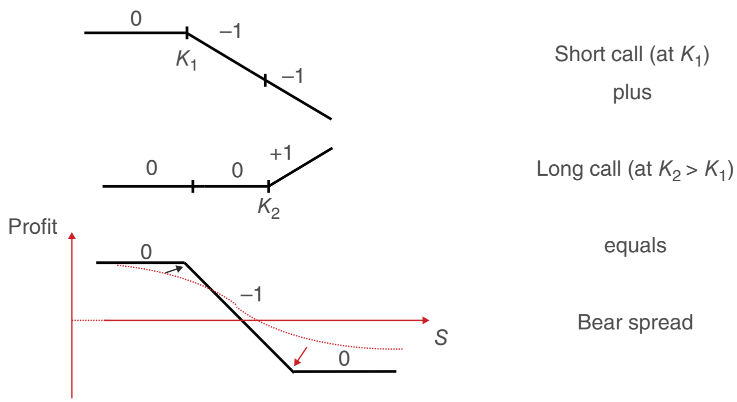


FIGURE 17.8 Bear spread with calls

17.2.4 Bear Spread with Puts

A bear spread {0, −1, 0} can also be engineered using puts with different strike prices, as follows:

Bull spread with payoff	$\{\ 0, -1,\ \ 0\}$
= *Buy* a put with $K_2(> K_1)$	$\{-1, -1,\ \ 0\}$
and sell a put with K_1	$\{+1,\ \ \ 0,\ \ 0\}$

Not surprisingly, selling a bull spread would also result in a bear spread.

17.3 STRADDLE, STRANGLE, BUTTERFLY, AND CONDOR

Another type of money spread can be used when the investor thinks the stock price is likely to change by a substantial amount but she is uncertain about the *direction* of movement. Here, the investor can make a profit if stock prices move outside a particular range (either up or down) or remain within a particular range. These are often referred to as *volatility strategies* – we consider straddles, strangles, butterflies, and condors. Before considering how to create these synthetic securities let us examine each of their payoffs (see Figure 17.9).

There is favourable payoff to the long straddle if the stock price moves up or down from its initial value of $S_0 = K = 102$ by a relatively large amount (i.e. to below 94 or above 110: Figure 17.9, panel A). There is also a limited loss of \$8 should price changes turn out to be small and lie between 94 and 110.

For example, a long straddle might be a sensible investment strategy if there is going to be a 'yes/no' decision on a contract bid by a particular firm, or when awaiting a decision on whether a merger with another firm will be allowed to go ahead, or awaiting the outcome

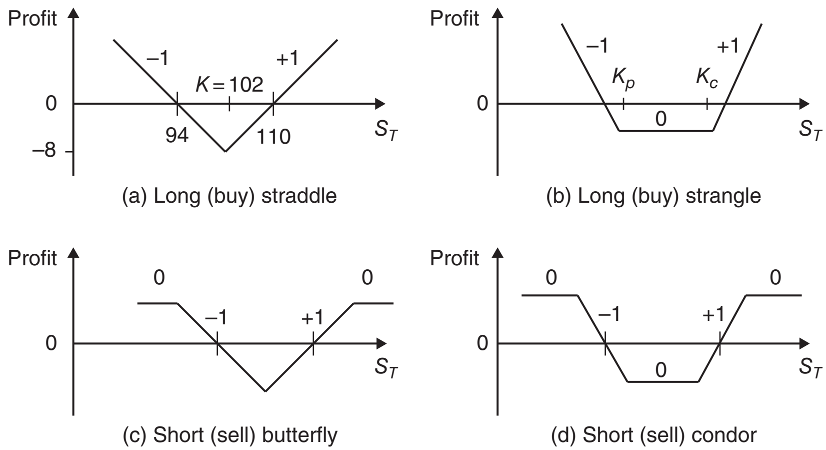


FIGURE 17.9 Payoff, volatility strategies

of a pharmaceutical company's licence application to manufacture a new drug. All of these straddles have a positive payoff if there is a subsequent large rise or fall in the stock price, depending on the outcome of these 'yes/no' decisions.

The strangle (Figure 17.9, panel B) has a large payoff if the underlying price moves in either direction by an even larger amount than for the strangle, and the downside risk is limited as indicated by the 'flat bottom' – it is a volatility strategy. The short butterfly (Figure 17.9, panel C) and the short condor (Figure 17.9, panel D) limit the upside potential compared with a long straddle or strangle. The short butterfly and short condor each have a positive outcome if stock prices move up or down by a 'large' amount, but after a certain point the profit does not increase with further movements in the stock price. Because the payoff to a short butterfly and short condor limit the upside compared with the straddle, they will cost less to set up (in terms of the option premia paid) than the straddle. The above strategies are often undertaken by the trading desks of large institutional investors or by floor traders on the exchange.

17.3.1 Long Straddle

On 15 April (say) a *long straddle* involves buying a call {0, 1} and put {–1, 0} with the same strike price K (and time to maturity) to give the {−1, 1} payout profile in Figure 17.9. Assume the options purchased are at-the-money ($S_0 = K = 102$) and both expire in October:

$$K = 102 \qquad P = 3 \qquad C = 5$$

The profit outcomes in October from the long straddle are (see Figure 17.10):

$$\Pi = \max(0, S_T - K) - C + \max(0, K - S_T) - P \tag{17.4}$$

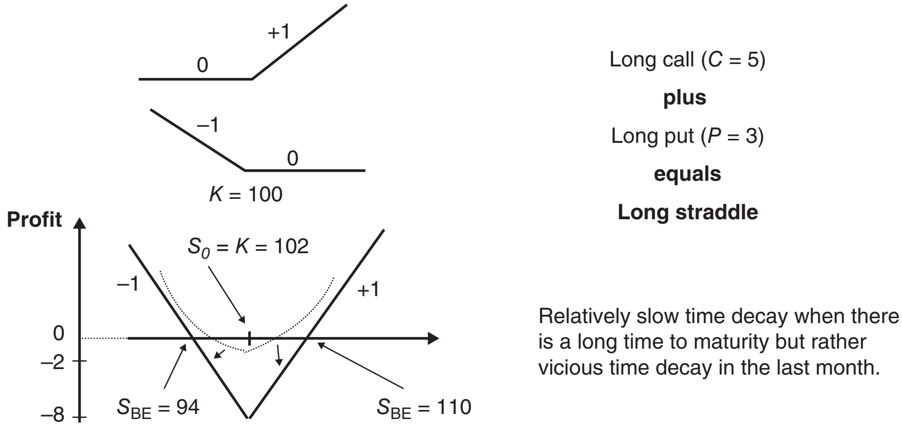


FIGURE 17.10 Long straddle

$$S_T > K \qquad \Pi = S_T - K - (C + P)$$
$$\Rightarrow S_{BE}^{+} = K + (C + P) = 102 + 8 = 110$$

$$S_T < K \qquad \Pi = K - S_T - (C + P)$$
$$\Rightarrow S_{BE}^{-} = K - (C + P) = 102 - 8 = 94$$

The breakeven stock price is that value of S_T which makes $\Pi = 0$. Notice that the breakeven prices are symmetric around the strike price $K = 102$, with $S_{BE}^{+} = 110$ and $S_{BE}^{-} = 94$. The 'gap' between the breakeven stock prices is $S_{BE}^{+} - S_{BE}^{-} = 2(C + P) = 16$, so for at-the-money options the stock price will have to move by plus or minus 8 before the straddle moves into profit. The maximum loss occurs when $S_T = K$ since then both the long call and put will not be exercised and the loss equals the sum of the call and put premia $C + P = 8$.

If S_T is above (below) K then the call (put) will be exercised. Although Π increases symmetrically as S_T increases or decreases (relative to K), the maximum gain is infinite for $S_T > K$ but is limited for $S_T < K$ by $S_T = 0$ (e.g. for $S_T = 0$, $\Pi = K - (C + P) = 94$).

Note that for a straddle to be profitable the investor must have a view about possible price movements which is *more extreme* than that held by the 'average' of all market participants. This is because if 'the market' as a whole expects extreme price movements, this will be reflected in a higher value for the volatility of stock returns σ and hence higher call and put premia. This implies that the 'market determined' breakeven stock prices for the long straddle will be 'wider'. Hence a speculator, using a long straddle must expect even wider movements in stock prices than does the average market participant.

The profit from holding a long straddle prior to maturity is given by the curved line in Figure 17.10. If stock prices remain unchanged, then as we get closer to maturity the 'curve' moves down towards the 'V', but an interesting feature of the straddle is that this movement is slow, until about 1 month before expiry. Hence, if the anticipated large shift in the stock price (in either direction) does not occur until the end of September, the holder of the October long straddle can sell the options a month before the expiration date and the speculator will not have lost much of the initial 'call plus put premia' of \$8, she paid in April. In other words, the straddle has relatively slow time decay when there is a long time to maturity but unfortunately it has a rather vicious time decay in the last month of its life. (Maybe this is a metaphor for life in general.)

Above, we have interpreted 'volatility' as an actual movement up or down in stock *prices*. Now consider what happens to the value of the long straddle if market participants' perception of stock price volatility σ increases (while the stock price remains constant). For example, if market participants think the forecast for future growth in output and profits is unchanged at 3% then there will be no change in stock prices. But increased uncertainty about the outcome of trade negotiations (say) might imply an increase in the forecast volatility of stock returns from 20% p.a. to 23% p.a. (say).

Since the long straddle consists of a long call and a long put, the value of the long straddle will increase as σ increases. In fact, the dashed line in Figure 17.10 will move up (down) as σ increases (decreases). To show the change in value of the straddle when both the stock price and implied volatility change requires a three dimensional diagram (see Finance Blog 17.1).

17.3.2 Short Straddle

It is fairly obvious that a short straddle which involves selling a call and put will have a {+1, −1} payoff at expiry that is, an inverted 'V' shape. Providing the stock price *at expiry* is 'close to' the strike price ($S_T \approx K$) an at-the-money short straddle has a positive profit at maturity, equal to the initial receipt of the call and put premia. However, the short straddle is subject to potentially large losses if there is a large rise or fall in stock prices. Perhaps the most famous loss on short straddles is that of Nick Leeson whose total losses on options (and futures) positions caused Barings Bank to go bankrupt with derivatives losses of \$1.4bn (see Finance Blog 17.1).

By the end of 1994 Nick Leeson, working for Barings Bank (London) but trading from Singapore on the Tokyo exchange, had sold about 35,000 *Nikkei 225 straddles* – that is, he sold 35,000 puts and 35,000 calls (with the same maturity and strike price). The calls and puts were written on the Nikkei 225 stock market index. Leeson therefore received the call and put premia. These positions were built up over several months and the strike prices were in the range of 18,500–20,000 (on the Nikkei 225 index). He did not report these trades to Barings Bank (as he was officially not allowed to sell options) but hid the trades in his secret 'error account 88888'. The value of the Nikkei 225 index, underlying these options positions was around \$7bn.

The Nikkei at the end of 1994 had been trading in the range 19,000–20,000. If the Nikkei 225 had remained in its 'historic range' then Leeson's short straddles would earn a profit at

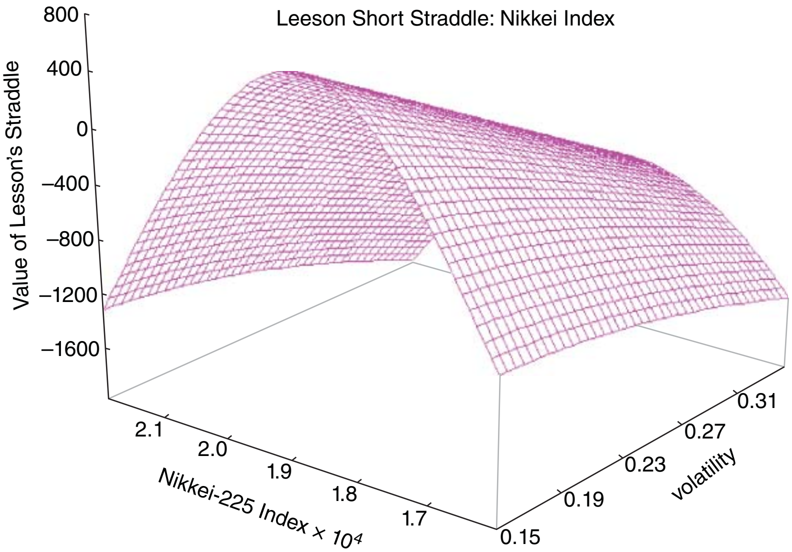


FIGURE 17.11 Leeson's short straddle

maturity. Leeson's receipts from selling the calls and puts would more than cover any 'small' payoffs at maturity to investors holding either the long calls or long puts. This is obvious from the inverted-V payoff for Leeson who held the short straddle. For example, if Figure 17.10 represents a long straddle, then Leeson's *short straddle* would be profitable at expiration, providing the stock index does not move more than 8 points in either direction, but the maximum gain to Leeson at expiration is the sum of the call and put premia $C + P$ (if $S_T \approx K$). However, if the options are held to maturity, and the Nikkei moves in either direction by a large amount, the potential losses to Leeson could be very high since either the calls ($S_T > K$) or the puts ($S_T < K$) could be exercised by the 'long'.

Leeson's short straddle positions change in value day-by-day as the Nikkei 225 stock market index and its volatility change over time, as shown in Figure 17.11.

Lesson's short straddle position became very problematic as it resulted in large losses prior to maturity of the options – see Finance Blog 17.1.

Finance Blog 17.1 Leeson's Straddle

Prior to maturity, as the Nikkei moved outside the 19,000–20,000 range Leeson's short straddle fell in value (Figure 17.11) and Leeson as the 'short' would have to provide increased margin payments. The payoff before maturity is represented by the 'arch shape' at the front in Figure 17.11 (the Nikkei 225 index is on the 'x-axis'). The value of Leeson's position is on the vertical axis and as the Nikkei 225 index moves up or down (while implied volatility remains constant), the value of Leeson's position follows this 'arch shape'.

On 17 January 1995 the Nikkei was at 19,350 but then the Kobe earthquake struck and by the end of the week the Nikkei was around 18,950 so Leeson's written puts fell in value and these losses triggered additional margin calls (from the Tokyo exchange). Had Leeson been forced to close out his short straddle he would have crystallised these losses. Prior to expiry, the value of Leeson's short straddle position is:

$$V = N(P + C)$$

where $N = -35{,}000$ = number of written puts/calls. The change in value of the straddle position for small changes in the Nikkei 225 is given by the deltas of the calls and puts (see Chapter 16):

$$\partial V / \partial S = N(\Delta_p + \Delta_c)$$

From put–call parity (for a European option, on a non-dividend paying stock), $(1 + \Delta_p) = \Delta_c$. For example, if the calls and puts are (close to) ATM then we might have $\Delta_c = 0.51$, $\Delta_p = -0.49$ then $dV = +(0.02)NdS = 0.02(-35,000)dS = -700dS$. Hence Leeson's position would deteriorate as the Nikkei 225 *fell*, but the deterioration at first sight does not look too drastic as the delta of the straddle is small so that a 1 point fall in dS would lead to a 700 fall in value of Leeson's straddle.

The short-straddle position falls in value by '700 × change in the Nikkei 225 index'. But note that delta only gives a first order approximation for dV which is only accurate for *small changes* in the Nikkei – and pictorially is represented by the nearly 'flat top' of the arch in the above figure, which has a slope that is very small. It is clear from the above figure that the actual losses would be much higher if the Nikkei moved substantially – which it did.

This illustrates the danger in using the 'delta approximation' to calculate the change in value of an option's position – using a delta-gamma-vega approximation to the change in value of the straddle would be more accurate and we deal with this in Chapter 27.

After the Kobe earthquake the increased uncertainty about the future course of the Japanese economy, increased traders' perceptions of future volatility σ (of the Nikkei 225) and this *immediately* caused an increase in both call and put premia (see Chapter 16). Hence Leeson's *written* calls and puts experienced additional (mark-to-market) falls in value (see the volatility axis in Figure 17.11). This is known as vega risk.

As Leeson's mark-to-market losses on his short straddle became larger, the clearing house (in Tokyo) asked for additional margin payments which Barings did not provide and Leeson's straddle positions were closed out at a loss (by the clearing house). This was part of the reason Barings Bank went bust.

Source: Adapted from Cuthbertson and Nitzsche (2001).

17.3.3 Long (Buy) Strangle

This strategy is very similar to the long straddle except the long call and put have different strike prices, which gives a V-shaped profit profile but with a 'flat bottom' between the two strike prices. If K_0 is the strike in a *straddle*, then a long *strangle* involves buying a call with $K_c > K_0$ and buying a put with $K_p < K_0$. Because the calls and puts in a strangle are more out-of-the-money than in the straddle, the strangle costs less than the straddle. The payoff for a long strangle is {–1, 0, 1} which gives a flat-bottomed 'bucket shape' (see Figure 17.12).

$$\begin{array}{ll} \text{Long strangle with payoff} & \underline{\{-1, 0, +1\}} \\ = \text{Buy put with } K_p & \{-1, 0, \quad 0\} \\ + \text{Buy at call with } K_c (> K_p) & \{\quad 0, 0, +1\} \end{array}$$

The breakeven stock prices are:

$$\text{for } S_T > K_c : \quad S_T - K_c - (C + P) = 0 \;\Rightarrow\; S^{+}_{BE} = K_c + (C + P)$$
$$\text{for } S_T < K_p : \quad K_p - S_T - (C + P) = 0 \;\Rightarrow\; S^{-}_{BE} = K_p - (C + P)$$

The 'gap' between the breakeven points for the strangle is $S^{+}_{BE} - S^{-}_{BE} = K_c - K_p + 2(C + P)$ and this is greater than that for the straddle, $S^{+}_{BE} - S^{-}_{BE} = 2(C + P)$. Thus although the strangle costs less than the straddle, the stock price has further to move before the strangle is in-the-money.

The strangle has one positive feature relative to the straddle. It can be shown that the strangle increases in value by the same amount as the straddle, as *implied volatility* σ increases,

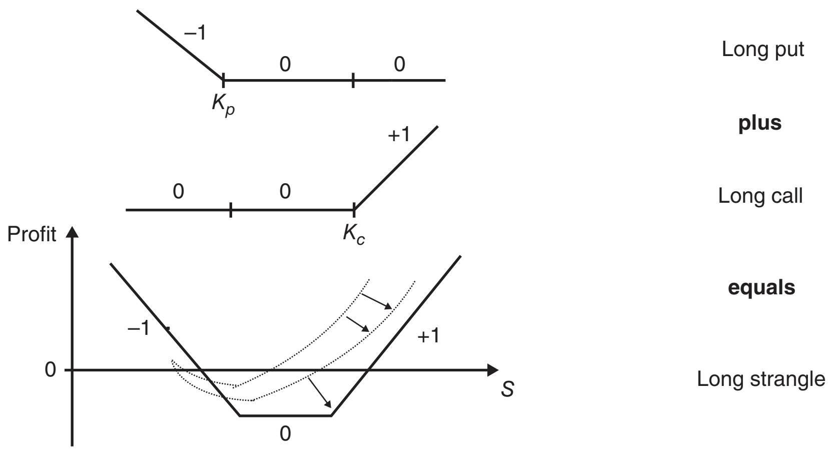


FIGURE 17.12 Long strangle

but the strangle costs less at the outset. The strangle is therefore 'better value' than the straddle if you are speculating on an increase in volatility.

17.3.4 Short Butterfly

A butterfly spread is another variant on the straddle – it is a V-shape with 'wings'. A *short butterfly* can be 'engineered' entirely from calls: selling one call at a low strike, buying two calls at the middle strike and buying one call at the highest strike price. This gives a payoff profile of {0, –1, 1, 0}, see Figure 17.13.

A profit is made if the stock price either increases or decreases by a substantial amount – it is a volatility strategy but has a limited upside (whichever direction stock prices move). The short butterfly requires:

Short butterfly with payoff	$\{0, -1, +1, \quad 0\}$
$=$ Sell a call with K_1	$\{0, -1, -1, -1\}$
$+$Buy a call with $K_2(> K_1)$	$\{0, \quad 0, +1, +1\}$
$+$Buy a call with $K_2(> K_1)$	$\{0, \quad 0, +1, +1\}$
$+$Sell a call with $K_3(> K_1)$	$\{0, \quad 0, \quad 0, -1\}$

That is, sell two 'outer-strike price' call options (K_1, K_3) and purchase two 'inner-strike price' call options (with strike price, K_2).

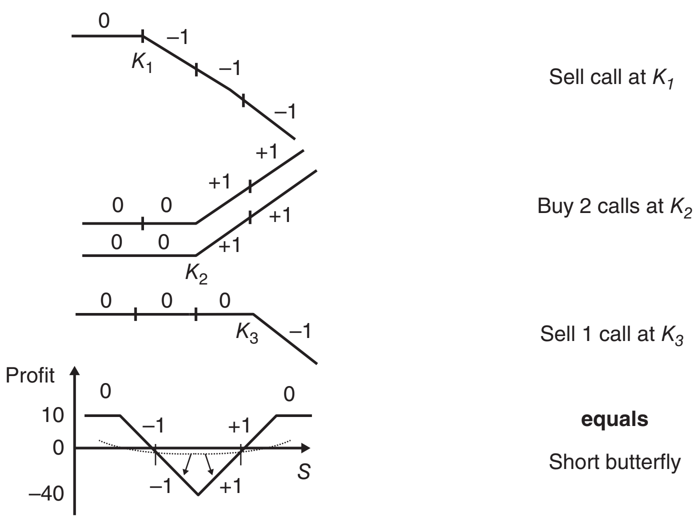


FIGURE 17.13 Short butterfly

The payoff profile when the options are not close to expiration is the almost flat dashed line at the bottom of Figure 17.13 – hence the butterfly spread does not change in value by much, when the stock price changes (that is, the delta of the butterfly is close to zero). When the options are close to the expiration date, the short butterfly loses time value rather quickly.

Also, the payoff profile before maturity is almost invariant to changes in implied volatility. This is because for long-dated options, vega does not change very much with the stock price and hence the positive vega of the two purchased options will almost exactly offset the negative vega of the options sold (see Chapter 28 for a detailed discussion of vega). Hence if you wish to speculate on changes in volatility σ, you would use straddles and strangles rather than butterfly spreads.

A short butterfly can also be replicated with *puts*. Again, selling two 'outer strike' puts and purchasing two 'inner strike' puts we obtain:

Short butterfly with payoff	$\{\ 0, -1, +1,\ 0\}$
= Sell put with K_3	$\{+1, +1, +1,\ 0\}$
+Buy put with K_2	$\{-1, -1,\ \ 0,\ 0\}$
+Buy put with K_2	$\{-1, -1,\ \ 0,\ 0\}$
+Sell put with K_1	$\{+1,\ \ 0,\ \ 0,\ 0\}$

17.3.5 Long Butterfly

A long butterfly spread has the shape given in Figure 17.14 and can be engineered from either calls or puts by 'reversing the trades' for the short butterfly, given above.

A long butterfly spread can also be constructed from bull and bear spreads:

long call bull spread = *buy* K_1*-call and sell* K_2*-call* = $\{0, +1, 0, 0\}$

plus

long call bear spread = *buy* K_3*-call and sell* K_2*-call* = $\{0, 0, -1, 0\}$

equals

long butterfly $\{0, +1, -1, 0\}$

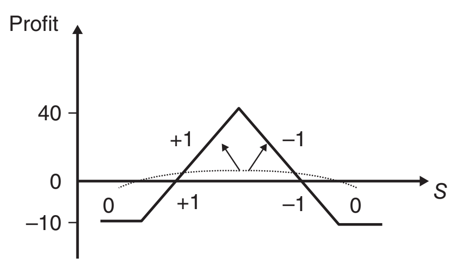


FIGURE 17.14 Long butterfly

17.3.6 Short Condor

This is similar to the short butterfly (Figure 17.13) but with a 'flat bottom' and a profit profile of $\{0, -1, 0, +1, 0\}$. The 'flat bottom' is obtained by replacing the two 'central options' with the *same* strike price K_2 in the butterfly, with two options with different strike prices – this flattens the 'V-shape' of the short butterfly – see Figure 17.9d. (This is similar to the relationship between the straddle and the strangle.) A short condor with calls is constructed as follows:

Short condor with payoff	$\{0, -1, \ 0, +1, \ 0\}$
$=$ Sell a call at K_1	$\{0, -1, -1, -1, -1\}$
$+$Buy a call at $K_2(> K_1)$	$\{0, \ 0, +1, +1, +1\}$
$+$Buy a call at $K_3(> K_2)$	$\{0, \ 0, \ 0, +1, +1\}$
$+$Sell a call at $K_4(> K_2)$	$\{0, \ 0, \ 0, \ 0, -1\}$

Specifically, a short condor with calls involves selling a call at the lowest strike K_1, buying a call at a higher strike K_2, buying another call at the next strike K_3 and selling a call at the highest strike K_4. To see how you engineer a short condor with puts use our above 'trick' by starting at the *right-hand side,* of the short condor payoff $\{0, -1, 0, +1, 0\}$. You will find that a short condor with puts requires the same position in puts as given for calls above – try it.

A condor costs about the same as the equivalent butterfly and the gap between their respective breakeven points and the maximum gain possible are similar. The *maximum loss* for the short condor occurs over the range of stock prices between the middle strikes (see Figure 17.9d), rather than at a single point as with the butterfly. The value of the condor prior to maturity behaves similarly to that of the butterfly, noted above.

17.4 HORIZONTAL (TIME, CALENDAR) SPREADS

All of the above strategies are 'vertical spreads' which are constructed from options with different strike prices but all have the same maturity date. Horizontal or time spreads have options with different maturity dates but the same strike price. A horizontal spread cannot be held until *both* options expire (because the short maturity option will have expired some time before the longer maturity option!). Because the long-maturity option will have time value when the 'nearby option' reaches maturity, the expected payoff from horizontal (time) spreads depends on the specific options used. There are no general rules we can apply here.

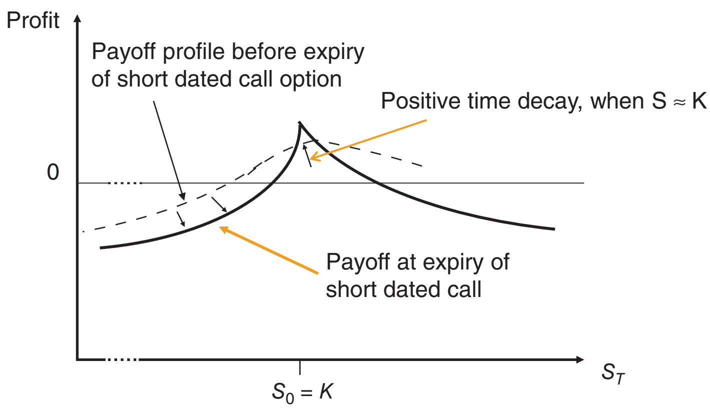
However, consider a simple case of buying a long-dated (T_L = 360-days) call option for C_L and selling a short-dated option (T_S = 180-days) at C_S where both options are at-the-money (i.e. $K = S_0$). The cost of setting up the calendar spread will be positive, since $C_L > C_S$. The time decay will be positive for the short-dated option and negative for the long-dated option.

But the positive time decay of the short-dated option is faster than for the long-dated option, so the spread will make money as time passes (Figure 17.15, dashed line) – as long as the stock price does not change over time $S_t \approx S_0 \approx K$.

The profit profile of a calendar spread is usually shown when the short-maturity option matures and it is assumed that the long maturity option is then closed out. The profit at expiration of the calendar spread is the solid curved line (Figure 17.5) and shows that if $S_T \approx S_0 \approx K$ the (calendar) spread-investor makes a profit but if the stock price is well above or below the strike price the investor makes a loss.

At maturity of the short-dated option, the long-dated option still has 180 days to maturity. First consider what happens at expiration on the short-maturity option if $S_T \approx S_0 \approx K$. The short call is worth virtually nothing but the long maturity call is quite valuable because of its positive time value – so the calendar spread makes a profit. If $S_T \gg K$ then the short call option (which is exercised by the long) costs the investor $S_T - K$. But the long call is also close to being worth $S_T - K$ (plus some positive time value). Overall when the investor in the calendar spread closes out she makes a payoff of zero but there is a net loss which is about equal to the initial cost of setting up the spread (i.e. $C_L - C_S$). If $S_T \ll K$ when the short maturity option expires, then the short-maturity call option is worthless and the value of the long maturity call option (which is also well out-of-the-money) is also close to zero. In this case, the investor in the calendar spread makes a net loss which is close to the cost of setting up the spread. This explains the payoff profile for the outcomes at maturity of the short-dated call which are represented by the solid line in Figure 17.15.

To summarise. In a relatively static market (i.e. stock price does not change $S_t \approx S_0 \approx K$) the horizontal call-spread will make money from time decay. However, it will lose money if the stock price moves substantially either side of its initial value, S_0 (Figure 17.15). Calendar



Calendar Spread: A long position in a 360-day option and a short position in a 180-day option, both at-the-money.

FIGURE 17.15 Calendar spread (profit profile) spreads can be created with put options and buying a long-maturity put and selling a short-maturity put gives a profit profile similar to that when using calls.

17.5 SUMMARY

- Combinations of different options can be used to set up positions which have a wide variety of alternative payoff structures.
- Bull and bear spreads using (stock) options are directional bets – the investor is gambling on the stock price either rising (bull spread) or falling (bear spread). For both strategies losses and gains are capped.
- Straddles, strangles, butterfly spreads, and condors (on stock options) are bets on the movement of stock prices in either direction – they are volatility bets.
- A long straddle or strangle may have a substantial positive profit if there is either a large rise or fall in the stock price – but losses are limited to the cost of buying the options.
- A short straddle or strangle has a positive profit if the stock price does not change much in either direction but can result in large losses if the stock price either rises or falls by a large amount.
- Short butterfly spreads and short condors have positive profits if the stock price moves in either direction by a large amount, but results in losses if the stock price changes by a small amount, in either direction. For both the short butterfly and condor, losses and gains are capped. Long butterfly spreads and long condors have the opposite ('mirror image') profit profiles to their respective short positions.
- All options strategies can lead to gains or losses 'second-by-second' if either the stock price changes or volatility changes.

EXERCISES

Question 1
What is a long straddle and why might you hold this position to maturity?

Question 2
You construct a long straddle by buying an at-the-money (ATM) call and an ATM put (both with the same underlying asset, strike price and time to maturity). Is the effective delta of this straddle large or small? Does the delta imply the straddle is not very risky?

Question 3
A long strangle has a payoff profile of $\{-1, 0, +1\}$ – a 'bucket shape'. Show how you can construct a long strangle and algebraically derive the break-even stock price.
Draw the payoff diagram for $C = 5$, $P = 3$, $S_0 = 100$, $K_p = 90$ (put) and $K_c = 105$ (call), when $S_T > K_c$ and $S_T < K_p$.

Question 4

You expect the stock market to rise over the next 3 months. What are the advantages and disadvantages of buying (i) a call (with a low strike price, K_1) versus buying (ii) a (call) bull spread (with strikes $K_1 < K_2$)?

Question 5

How do you set up a long straddle and a short butterfly spread? Qualitatively, how are the outcomes different at maturity and is this intuitively reasonable? More specifically, explain how the butterfly costs you less, but potentially it gives you less at maturity.

Question 6

A put option with a strike $K_1 = 35$, costs $P_1 = \$4$ and a put with a strike of $K_2 = 40$ costs, $P_2 = \$8$. How can you construct a bull spread using these puts?
What are the payoffs and profits from the strategy?
For different values of the stock price at maturity: $S_T > K_2$, $K_1 > S_T > K_2$ and $S_T < K_1$ show the payoffs from the bull spread (with puts) and also show the profits from the strategy and the breakeven stock price. Put your results in the following table.

$S_T < K_1 = 35$	$K_2 > S_T > K_1$	$S_T > K_2 = 40$
Payoff long K_1 put		
Payoff short K_2 put		
Payoff bull spread		
Option premia		
Profit		

Question 7

You purchased a (call) bull spread some time ago. Later you note that the bull spread is currently well 'in-the-money' and has positive value. Would you gain or lose by holding on a bit longer before closing out, if you felt the stock price (as well as volatility, interest rates etc.) would remain unchanged?

CHAPTER 18

Stock Options and Stock Index Options

Aims

- To examine stock options and stock index options.
- To show how you can provide a minimum (floor) value for a *portfolio* of stocks but also be able to capture most of the 'upside' if stock prices rise – this is a protective put.
- To show how you can hedge a portfolio of stocks using dynamic delta hedging.
- To show how several options (on the same underlying asset) can be combined to give a risk-free portfolio – this is a ratio spread.
- To demonstrate how you can make a profit from mispriced options, while hedging any changes in market risk.

We have seen in Chapter 17 that investors can use stock options to speculate on the direction of stock price changes and on changes in the volatility of stock returns. Investors can also *insure or hedge* a cash market position consisting of stocks held in a specific firm (e.g. 50,000 stocks in AT&T) or in certain specific industries (e.g. oil industry) or in a 'market portfolio' (e.g. S&P 500) by using various types of stock index options. First we discuss hedging using options on individual stocks and then using stock index options.

18.1 OPTIONS ON STOCKS

If you hold a number of stocks in *one* particular company (e.g. AT&T) you can alter or eliminate the (market and specific) risk, using options on this stock.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

18.1.1 Static Hedge: Covered Call

A static hedge assumes the initial options positions are held to maturity. Suppose an investor holds AT&T stocks. She can offset *some* of the downside risk by writing (selling) a call – this is a covered call strategy. Downside risk is reduced slightly because the investor receives the call premium:

$$\begin{array}{ll} \text{Long stock} & \{+1,+1\} \\ +\text{Short call} & \{\ 0,-1\} \\ = \text{Covered call payoff} & \{+1,0\} \end{array}$$

Figure 18.1 shows the payoff to a covered call with $C = \$3$, $K = \$25$ and an initial stock price $S_0 = \$24$. The profit at maturity, is given by:

$$\begin{aligned} \Pi &= S_T - S_0 - max(0, S_T - K) + C \\ &= S_T - S_0 + C \quad \text{for } S_T < K \end{aligned} \tag{18.1}$$

$$\begin{aligned} \text{and} \quad S_{BE} &= S_0 - C = \$24 - \$3 = \$21 \\ \Pi &= K - S_0 + C = \$4 \quad \text{for } S_T > K \end{aligned}$$

For $S_0 = \$24$ the downside risk is reduced by the amount of the call premium of \$3 to a breakeven of \$21. However, the upside potential is considerably reduced – the maximum profit is $K - S_0 + C = \$4$.[1]

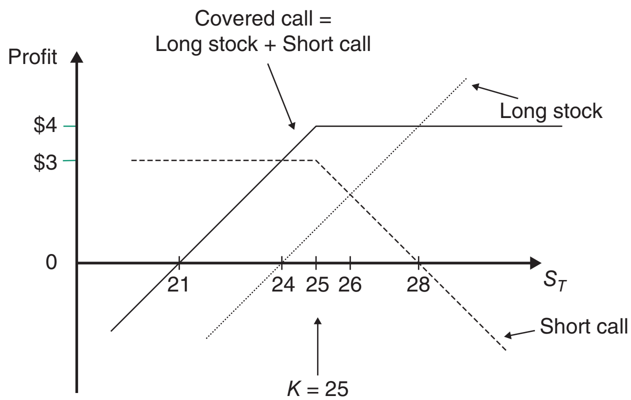


FIGURE 18.1 Covered call

[1]The payoff profile of the covered call is equivalent to a written put (with the same strike and time to maturity as the call) plus cash of Ke^{-rT} – this is a consequence of put–call parity, $S - C = Ke^{-rT} - P$.

18.1.2 Static Hedge: Protective Put

An investor can protect her long stock position but without sacrificing all the upside potential (of holding stocks) by buying a put – this is a *protective put* (Figure 18.2) also sometimes called a ‘guaranteed bond’ – see Chapter 15.

$$\begin{array}{ll} \text{Long stock} & \{+1, +1\} \\ +\text{Long put} & \{-1, \ 0\} \\ = \text{Protective put} & \{0, +1\} \end{array}$$

The payoff profile for the protective put is the same as for a long call. (This is put–call parity again.) Suppose $S_0 = \$24$ and the put has $K = \$25$ and $P = \$5$. The profit from the protective put is:

$$\Pi = S_T - S_0 - P \qquad \text{for } S_T \geq K \tag{18.2}$$

$$\text{and} \quad S_{BE} = S_0 + P = \$29$$

$$\begin{aligned} \Pi &= S_T - S_0 + (K - S_T) - P \quad \text{for } S_T < K \\ &= K - S_0 - P = -4 \end{aligned}$$

As can be seen in Figure 18.2 the protective put has a lower limit (a ‘floor’) on losses but allows most of the upside capture. The protective put is an insurance contract – in return for the put-premium, a minimum value for the stock is guaranteed at maturity. (Note that for an at-the-money put $K = S_0$, so the ‘floor value’ would be the current value of the stock.)

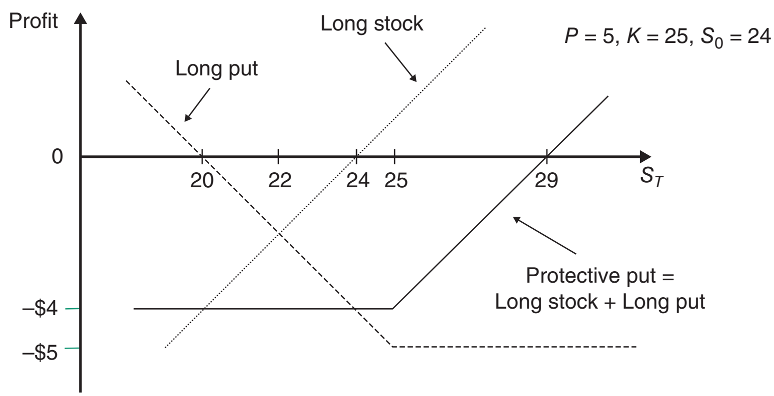


FIGURE 18.2 Protective put

18.1.3 Delta Hedging a Stock Portfolio with Puts

The static protective put does not ensure that the value of the portfolio remains unchanged *at all times*. Instead, it ensures a minimum value (= K) for the stock+put portfolio, at maturity of the option contract. However, it is possible to *continuously hedge* a stock+put portfolio. We require any change in the cash-market value of the stocks to be offset by changes in the value of the puts. Suppose you hold N_s stocks (of AT&T) with price S, with \$-value $V_s = N_s S$. A portfolio of N_s stocks plus N_p puts is worth:

$$V = N_s S + N_p \phi P \tag{18.3}$$

where $\phi = 100$ is the *number* of stocks underlying each put contract. A dynamically hedged portfolio has to satisfy:

$$\partial V / \partial S = N_s + N_p \phi \, \Delta_p = 0 \tag{18.4}$$

$$\Rightarrow N_p = -\frac{N_s}{\phi \, \Delta_p} = -\frac{V_s}{(100\ S)\Delta_p} \tag{18.5}$$

If you are long stocks $V_s > 0$ then as $\Delta_p < 0$, the hedge portfolio consists of *buying* N_p puts. The term '100 S' is the dollar value of the 100 AT&T stocks deliverable in each put contract. Suppose the investor is long $N_s = 800$ stocks at $S = \$10$, so that $V_s = \$8{,}000$ and assume $\Delta_p = -0.4$. Then $N_p = 20$ put contracts should be purchased to hedge against (small) changes in the stock price. As the option's delta changes (because of changes in S, r, σ, and T), the number of put option contracts to maintain the delta hedge needs rebalancing over time.

A dynamic hedge for a portfolio of stocks on AT&T using puts, can eliminate the market and specific risk of the AT&T stocks. As the put-delta is only valid for small changes in the stock price, the value of the stock+put portfolio is not hedged against large changes in stock prices (gamma risk) or changes in volatility (vega risk) – these issues are elaborated on in Chapter 28.

18.1.4 Ratio Spread

If you already hold a position in options then you can create a 'delta neutral' (i.e. risk-free) options portfolio by combining two (or more) options (on the same underlying asset). The latter possibility also gives rise to a *ratio spread* whereby traders attempt to make profits from *mispriced* options (on the same underlying asset).

The analysis below applies to both calls and puts. Two stock options (both on the same underlying) might have different deltas because they have different strike prices or maturity dates. Each option contract is for delivery of $\phi = 100$ stocks. The value of a portfolio consisting of N_A holdings of option-A and N_B holdings in option-B (on the same underlying) if the option prices are f_A, f_B is:

$$V = (N_A f_A + N_B f_B)\phi \tag{18.6}$$

For a zero change in value of the options portfolio:

$$\begin{aligned} \partial V/\partial S = 0 \Rightarrow \quad & N_A\Delta_A + N_B\Delta_B = 0 \\ N_A &= -(\Delta_B/\Delta_A)N_B \end{aligned} \tag{18.7}$$

Suppose we already hold a naked position in $N_B = 10$ (long) call options, with $\Delta_B = +0.5$ and we are worried about the risk to our options position from small changes in the stock price. If another *call* option (on the same underlying) has $\Delta_A = +0.25$, then a 'delta neutral' portfolio requires $N_A = -20$ that is, short 20 calls of option-A. (But if option-A is a put option with $\Delta_A = -0.25$, the delta-neutral position would involve *buying* $N_A = 20$ put options-A).

18.1.5 Underpriced Options

Consider a trader Ms Long, who thinks that call option-B (on Boeing stocks) with $\Delta_B = 0.5$ is under-priced by 1% because of stale price quotes. To take advantage of the under-pricing Ms Long could today buy $N_B = 10$ of options-B, hoping to close out later at a 1% profit – after the mispricing is corrected.

But if the stock market as a whole falls (i.e. S&P 500) then the price S_B of Boeing stock will also fall, which would produce a fall in the price of call option-B (written on Boeing), and this might eliminate any profit due to Ms Long's position in the 'underpriced' option-B.

In Chapter 16 on delta hedging we noted that Ms Long could create a delta-neutral portfolio by short-selling $N_s = 0.5(10) = 5$ stocks of Boeing. If option-B is correctly priced then the hedge position does not change in value (for small rises or falls in S_B). So the delta-hedge protects the value of the (option-B+Boeing-stocks) portfolio, while the investor waits for the option-B's mispricing to be corrected.

But if, as Ms Long believes, option-B is under-priced then when the mispricing is corrected, option-B's price will rise *by 1% more than implied by the hedge* ($dC_B = \Delta_B dS_B$), Hence, when Ms Long closes her positions in option-B and stocks-B she earns a profit of 1%, even if S_B has fallen due to a fall in the market (S&P 500) index.

To undertake the above arbitrage transaction requires short-selling stock-B.[2] This may not be possible because of the 'uptick rule' (i.e. in the US you are only allowed to short-sell if the last quote was a price rise), and may be expensive to implement because of high transactions costs (e.g. 50% margin on short-selling in the US plus any 'haircuts' demanded by the broker from whom you borrow the stocks and also the risk of a 'short-squeeze'). However, all is not lost.

The trader can exploit $N_B = 10$ under-priced calls (on Boeing) by simultaneously selling $N_A = (\Delta_B/\Delta_A)N_B = 20$ call options-A (also on Boeing) but with different strikes or time to maturity and hence a different delta, $\Delta_A = +0.25$). This is a *ratio spread*.

[2] Instead of using stocks, the options can also be delta hedged using futures contracts on Boeing stocks, which may involve lower transactions costs and there are no problems with shorting futures contracts.

If option-A and option-B (both on Boeing) are correctly priced, the ratio spread provides a hedge (for small changes in the stock price of Boeing). However, given that option-B is actually under-priced by 1%, the ratio spread makes a riskless profit of 1%, when option-B eventually rises to its correct (fair) value given by Black–Scholes.[3]

18.2 STOCK INDEX OPTIONS (SIO)

Stock index options (SIO) are frequently used to hedge the market (systematic) risk of a *diversified portfolio* of stocks. A fund manager using index options can either obtain insurance or can dynamically hedge the market risk of her portfolio of stocks.

18.2.1 Contract Specification

We focus on the S&P 100 (American style) and the FTSE 100 (European-style) contracts. The S&P 100 (American) index option is often referred to by its ticker symbol (OEX) and is the most actively traded option on CBOE. SIO are settled in cash. If z is the dollar-value of one index point and the current stock index is S, then the dollar-value of the S&P index is $V_I = zS$. For S&P 100 index options, $z = \$100$ (Table 18.1), hence if the S&P 100 index is $S = 1{,}000$, then $V_1 = \$100{,}000$. Put slightly differently, if the S&P 100 index changes by one point (e.g. from 1,000 to 1,001) then this implies a change in the value of the S&P index of $100.

TABLE 18.1 S&P 100 (American-OEX) Index Option (CBOE)

Unit of trading:	$100 × S&P 100 Index
Expiry month:	Four, near term months (plus one additional month from the March quarterly cycle)
Min. price movements:	For an option trading below 5, the minimum tick is 0.05 ($5) and for all other series 0.10 ($10)
Tick size (& value):	Either 0.05 ($5) or 0.10 ($10)
Exercise:	American. May be exercised on any business day before the maturity date. European also available.
Settlement day:	Cash settled based on closing index value on the business day of exercise
Last trading day:	Business day preceding the expiration date
Settlement:	In cash, based on $(S_T - K)^+ \times \$100$ for a long call and $(K - S_T)^+ \times \$100$ for a long put.

Source: CBOE website.

[3]We are ignoring other (gamma and vega) risks in this strategy, which we deal with when discussing the other 'Greeks' in Chapter 28.

The 'settlement price' S_T is the index value at the market close. If, at maturity $S_T = 1{,}000$ (index points) and the strike price in a put contract is $K = 1{,}100$ (index points), then the holder of a long put receives:

$$\text{\$-Payoff for one put} = z\,(K - S_T) = \$100\,(1,100 - 1,000) = \$10{,}000 \tag{18.8}$$

Call and put premia are quoted in terms of index points. Suppose a quote for the S&P 100, March-950 call is $C = 38$ (index points). This implies that one call contract costs \$3,800 (= \$100 × 38), hence:

$$\text{Invoice price of one S\&P 100 Call} = Cz \tag{18.9}$$

Index options are also available on the S&P 500 (a European-style option traded on CBOE), the Major Market Index (traded on AMEX) and the Value Line Index (traded on PHLX) and the NYSE-Composite (traded on NYSE). In the US there are also index options available on industry indices (e.g. oil, utilities).

For the FTSE 100 (European Style) index option (Table 18.2) the value of an index point is set at $z = £10$. So if the FTSE 100 index is at 6,500 then the value of the index is $V_1 = £65{,}000$. Quotes for option premia are in index points. For example, the April-6500 put on the FTSE 100 (European Style) index option, quoted at $P = 63$ would have an invoice price of £630 (= £10 × 63).

18.2.2 Static Hedge Using Stock Index Options: Protective Put

Suppose you hold a stock portfolio-A with current value $V_A = \$1\text{m}$, whose composition mirrors the S&P 100 ($\beta_A = 1$) and want to protect its value in 1 year's time. You fear a price fall so you buy index puts with maturity $T = 1$ year. This is a *protective put*. If the S&P 100

TABLE 18.2 FTSE 100 (European) Index Option

Unit of trading:	£10 per index point
Expiry month:	March, June, September, December (plus additional months so that the nearest 3 calendar months are always available for trading)
Min. price movements:	0.5 (index point)
Tick size (& value):	0.5 (£5) (Min. block trade is 500 contracts)
Exercise:	By 18.30 on last trading day only
Settlement day:	1st business day after maturity date
Last trading day:	10.15 (London time) on 3rd Friday of expiry month

Source: ICE-Intercontinental Exchange website.

index currently stands at $S_A = 1{,}000$ index points, then to insure a diversified portfolio of stocks (with $\beta_A = 1$), the number of puts required is (see Appendix 18.A):

$$N_p = \frac{\$ - \textit{value of stock portfolio}}{\$ - \textit{value of stock index}} \beta_A = \frac{V_A}{z\, S_0} \ \beta_A = 10 \tag{18.10}$$

To insure your stock portfolio at T from *all* downside risk (given $\beta_A = 1$) you need to choose ATM-puts with a strike price $K = 1{,}000$. At T, if you lose on your stock portfolio you want to be fully compensated by the payoff from the puts. If the S&P 100 index falls 20% to $S_T = 800$, then the value of your stock portfolio falls by \$200,000. But if you exercise the puts you make a profit of 200 index points ($= K - S_T$) per contract and with 10 contracts the dollar payoff is \$200,000 ($= 10 \times 200 \times \$100$ per point). The loss on the stock portfolio over the year is exactly offset by the payoff to the puts. If the put premium paid was $P = 30$ index points, the 10 put contracts cost \$30,000 ($= 10 \times \$100 \times 30$ points) which is the cost of the insurance.

Let's take a slightly more complex case, where $\beta_A = 1.2$ so that $N_p = 12$. The key factor is the choice of strike price in the put. If the S&P 100 index currently stands at $S_0 = 1{,}000$ and you choose a strike of $K = 900$, this implies you are willing to accept a fall in the market index of 10%. Hence, the maximum acceptable fall in the value of your stock portfolio is 12% ($= 1.2 \times 10\%$) that is \$120,000.

Suppose the S&P 100 index falls by 20% to $S_T = 800$ index points, so your stock portfolio falls by 24%, ($= \$240,000$). The payoff to the puts is \$120,000 ($= 12$ puts $\times (K - S_T) \times \$100$) hence, the net outcome from the protective put strategy is a loss of \$120,000. This is exactly the loss you were willing to incur when choosing a strike of $K = 900$ at the outset.

If the S&P index rises (above K) you do not exercise the puts but the value of your stock portfolio has increased. In this case the 'insurance' provided by the put was not needed but of course insurance does not come 'free' – as you pay the put premium. This is a static stock-put hedge because we have assumed the option contract is held to maturity. It provides a 'floor' for the value of the stock portfolio but also allows most of the upside potential.

18.2.3 Dynamic Delta Hedge Using Stock Index Options

Assume you hold V_A (dollars) in a diversified stock portfolio-A with beta β_A (with respect to the S&P 100 '*market index*'). The change in value of portfolio-A is $dV_A = V_A R_A$. To preserve the value of our stock portfolio over a small interval of time, we use a dynamic hedging strategy of stocks+puts. It can be shown (see Appendix 18.B) that the number of index puts to delta-hedge stock portfolio-A is:

$$N_p = -\frac{V_A}{zS_0}\left(\frac{\beta_A}{\Delta_p}\right) = \frac{V_A}{zS_0}\left(\frac{\beta_A}{|\Delta_p|}\right) \tag{18.11}$$

Equation (18.11) implies that if you are long a portfolio of stocks, then to delta-hedge you go long (buy)N_p index puts. Note that the cost of setting up the protective put is $N_p(zP)$ and there will be transactions costs of rebalancing the portfolio as the delta of the option changes over time. We discuss dynamic hedging in Chapter 27.

18.3 SUMMARY

- Options are available on individual stocks (e.g. AT&T, Coca-Cola), broad groups of stocks (e.g. index of oil stocks) and on broad market indexes (e.g. S&P 100, S&P 500, FTSE 100, Russell 2000).
- A *static* stock-put hedge provides a minimum (floor) value for a portfolio of stocks and also allows upside gains if stock prices are high, at maturity of the option.
- A *dynamic* stock-put delta hedge ensures that any gains (losses) on the stock portfolio over a small interval of time are offset by losses (gains) on the puts. Hence, over a small interval of time there is no change in the value of the 'stock+put' portfolio. A dynamic delta hedge requires frequent rebalancing.
- *Ratio spreads* allow traders to delta hedge an existing option position using 'other' options with different strikes or time to maturity (but on the same underlying asset). If an option is mispriced, then a ratio spread can be used to hedge the position, while waiting for the mispricing to be corrected.

APPENDIX 18.A: STATIC HEDGE: INDEX PUTS

You have $V_A = \$1\text{m}$ in portfolio-A of stocks, with $\beta_A = 1.2$. If the S&P 100 stock index is currently $S_0 = 1{,}000$ and you choose an index put with a strike $K = 900$, this implies the acceptable maximum dollar-loss on your portfolio of stocks is $V_A\beta_A(K - S_0)/S_0 = \$120{,}000$ (and hence the minimum (floor) value required is \$880,000).

To calculate N_p consider the outcome at maturity of the put. If the S&P 100 stock index falls to $S_T = 800\ \ (< K)$ (at expiration of the put option) then the actual loss on your stock portfolio is $V_A\beta_A(S_T - S_0)/S_0 = \$240{,}000$. The payoff from N_p index puts is $N_p(K - S_T)z$ where $z = \$100$ per index point. Hence, for $S_T < K$ we choose the number of puts so that:

Actual loss on stock portfolio + *payoff from puts* = *Acceptable* max *loss on stock portfolio*

$$V_A\beta_A(S_T - S_0)/S_0 + N_p(K - S_T)\$100 = V_A\beta_A(K - S_0)/S_0 \tag{18.A.1}$$

$$\Rightarrow N_p = \frac{\$-\textit{value of stock portfolio}}{\$-\textit{value of stock index}}\beta_A = \frac{V_A}{z\,S_0}\ \beta_A = 12 \tag{18.A.2}$$

The payoff from the 12 long puts is $N_p(K - S_T)z = \$120{,}000$ so that Equation (18.A.1) is satisfied. The actual loss on the stock portfolio is \$240,000, the payoff from the puts is \$120,000 which gives a net loss of \$120,000, which equals your maximum acceptable loss set by your choice of strike price, $K = 900$.

Equation (18.A.2) is often described as follows. The number of *index units* held in the stock portfolio is V_A/S_0. If $\beta_A = 1$ then the number of index units held in puts should therefore also equal V_A/S_0. But as each index point is worth $z = \$100$, and the stock portfolio beta β_A may not be equal to one, then the required N_p is given by (18.A.2).

APPENDIX 18.B: DYNAMIC DELTA HEDGE

A diversified stock portfolio consists of N_i different stocks-i with prices X_i. The value of portfolio-A consisting of m-stocks is $V_A = \sum_{i=1}^{m} N_i X_i = \sum_{i=1}^{m} V_i$ hence:

$$dV_A = V_A R_A \tag{18.B.1}$$

where $R_A \equiv \sum_{i=1}^{m} w_i R_i$ is the return on portfolio-A, $R_i = dX_i/X_i$ is the return on stock-i and $w_i \equiv V_i/V_A$ is proportion held in each stock-i. Assume, portfolio-A has a market beta β_A with respect to the S&P 100 market index, R_m. To preserve the value of the stock portfolio over a small interval of time, we use a dynamic hedging strategy with stocks+puts. The value of the hedge portfolio is $V = V_A + N_p z\ P$, hence:

$$dV = V_A R_A + N_p z\ \Delta_p\ dS = V_A R_A + N_p \Delta_p (zS_0)\ (dS/S_0) = 0 \tag{18.B.2}$$

where $\Delta_p \equiv \partial P/\partial S < 0$. Substituting $R_m = dS/S_0$, the return on the S&P 100 '*market index*' (which is the underlying in the put contract) and $R_A = \beta_A R_m$ then from (18.B.2):

$$N_p = -\frac{V_A}{zS_0}\left(\frac{\beta_A}{\Delta_p}\right) = \frac{V_A}{zS_0}\left(\frac{\beta_A}{|\Delta_p|}\right) \tag{18.B.3}$$

EXERCISES

Question 1

If the initial stock price is S_0 and the call premium is C_0 show the payoff and profits at maturity for a covered call. Is a covered call strategy risk free?

Question 2

A pension fund has to pay out a 'lump sum' to its pensioners in 6 months' time.

Why might the pension fund (which holds a diversified portfolio of stocks), purchase index puts, with 6 months to maturity, that are currently 3% out of-the-money (OTM)?

Question 3

What is a protective put? Why is the payoff (profile) to a protective put, qualitatively like the payoff to a long call?

Question 4

What is the payoff profile (at expiration) and the breakeven strike price for a portfolio consisting of an equal number of long stocks and long puts? The puts have $K = 164$ and $P = \$6$. What is the profit at expiration of the puts, if $S_T = 163$? Assume the initial stock price is $S_0 = 162$.

Question 5

The current stock price is $S_0 = 100$. A put with a strike of $K = 98$ (with 6 months to maturity) is available at a price of $P = \$4$. In a table, show the payoff and profit from a protective put for outcomes at maturity of $S_T \geq K$ *and* $S_T < K$. What is the breakeven stock price (which gives zero profit)? Who might use a protective put?

Stock price (Note: $K = 98$)	**Payoff Long stock and long put**	**Profit**
$S_T \geq K$		
$S_T < K$		

Question 6

You hold a portfolio of $N_s = 1{,}000$ stocks of Coca-Cola with current price $S = \$80$. A put option on Coca-Cola is available with $K = \$75$, put premium $P = \$10$ and a delta of $\Delta_p = -0.2$. (Each put is written on $\phi = 100$ stocks). How would you delta-hedge your stock position and what would happen to the value of your stock-put portfolio if stock prices rise by \$2 over the next day?

Question 7

You hold a position in $N_p = 40$ put options (on stock-A), with market price $P = \$3$ and $\Delta_p = -0.40$. Assume each put option is written on $\phi = 100$ stocks. You believe these puts are underpriced by 1% because you think volatility will increase in the future – although all other options traders believe volatility will not change).

(a) How can you take advantage of the underpricing of the put options, while protecting yourself against the change in the put premium, due to unexpected changes in the stock price?

(b) If the stock price falls by \$2 over the next day and the underpricing of the put is *not* corrected, what is the outcome of your strategy? Explain.

(c) If the stock price falls by \$2 and the underpricing of the put option of 1% *is* corrected, what is the outcome of your strategy? Explain.

(d) What are the risks in your strategy?

CHAPTER 19

Foreign Currency Options

Aims

- To examine contract specifications for foreign currency options and the payoffs from calls and puts.
- To analyse the advantages and disadvantages of hedging future foreign currency receipts/payments with either futures or options
- To briefly outline some non-standard ('exotic') currency options.

19.1 CONTRACT SPECIFICATIONS

Most FX-options contracts are traded in the OTC market as they can be tailored to meet the needs of corporate treasurers in terms of size of the deal, currency used, and maturity dates. The exchange-traded FX options markets are much smaller in volume although exchange traded options on spot FX rates are available on NASDAQ (which was the Philadelphia Stock Exchange, now called NASDAQ-OMX-PHLX).

Most contracts traded are against the US dollar and the option premium is quoted in cents per unit of foreign currency, with the only exception being contracts on the Japanese yen which are quoted in *hundredths* of a cent. Delivery is the foreign currency (in exchange for USD). There are also options contracts which deliver an FX-*futures contract* – futures options are dealt with in Chapter 20.

For example (Table 19.1), the contract size for the (British) pound sterling (GBP) is for £31,250 and entries are quoted in cents per GBP. If the current spot rate on 26 July is $S_0 =$ 1.5000(\$/£) then October-options would be available with strike prices around this value; for example, for strikes of 1.490, 1.495, 1.510, and 1.520 (although these are not the only strike prices available). If the quoted call premium for one October-1.5200 (American style) contract

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

TABLE 19.1 Foreign currency options (NASDAQ-OMX-PHLX)

Currency against dollar	Contract size	Strike price increments	Min. price change
Pound sterling	£31,250	\$0.025	\$0.0001 (1 pip or point) = \$3.125
Euro	€62,500		
Japanese yen	¥6,250,000	\$0.050	\$0.000001 = \$6.25
Canadian dollar	C\$50,000	\$0.050	\$0.0001 = \$5.00

on GBP is 2.05 cents (per GBP) then the invoice price for one contract is \$640.63 [$= £31,250 \times 0.0205$ (\$/£)].

In addition to exchange traded options, there is also a large OTC market in currency options where clients are provided with 'tailor-made' options with different strikes, contract sizes, and time to maturity.

If a US company has to pay for imports from the UK at a known time in the future then it may be worried about a rise in sterling, as the imports will then cost more in USD. It can set a maximum amount of USD it will pay for the imports in the future by buying a call option on sterling today. The call option gives the US company the right *to purchase (receive) GBP* at a strike price, K (USD per GBP) even if the spot FX-rate for sterling at expiration is higher than K. The call also allows the US company the right not to exercise the call option if the spot-FX rate is below K. The US company will then purchase sterling at the low spot FX-rate, $S_T < K$ (USD per GBP) thus reducing the USD cost of the imports.

On the other hand if a US company has to convert sterling into USD in the future (e.g. from export sales in the UK), it may be worried about a fall in sterling, as it will receive less USD for each GBP. It can set a minimum amount of USD it will receive in the future by buying a put option on sterling today. The put option gives the US company the right *to sell GBP* at a strike price, K (USD per GBP) even if the spot FX-rate for sterling at expiration is lower than K. The put also allows the US company not to exercise the option if the spot-FX rate is above K. The US company will then sell sterling at the high spot FX-rate, $S_T > K$ (USD per GBP,) thus increasing its USD receipts.

19.2 SPECULATION

19.2.1 Profit from a Long Call

Foreign currency options can be used for speculation. Long positions have limited downside risk (the option premium) yet provide the possibility of large speculative profits by exploiting

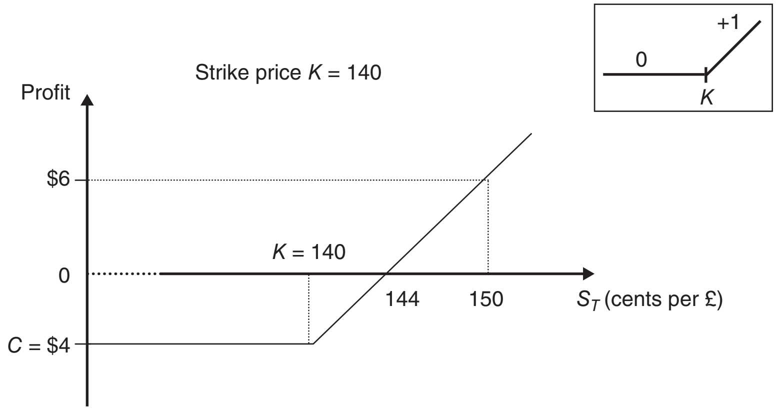


FIGURE 19.1 Foreign currency call option

leverage. We demonstrate the payoffs from a long position in either a (European) call or put. If held to maturity, a long call gives a profit (Figure 19.1):

$$\begin{aligned} \Pi &= \max(0, S_T - K) - C \\ &= -C \qquad \text{if } S_T \leq K \\ &= S_T - K - C \qquad \text{if } S_T > K \end{aligned} \tag{19.1}$$

The breakeven spot rate at T is:

$$S_{BE} = K + C \tag{19.2}$$

A long call on sterling gives the investor the right to receive z = £31,250 at expiration, at an exchange rate of $K = 140$ cents (per GBP). Suppose in January, the March-140 call on sterling has a premium of $C = 4$ cents (per GBP) then:

$$\text{Invoice price per contract} = zC = £31{,}250\ (0.04) = \$1{,}250 \tag{19.3}$$

If the spot price in March at expiration turns out to be $S_T = 1.50(\$/£)$ (see Figure 19.1), then:

$$\text{Payoff} = (S_T - K)z = (1.50 - 1.40)\ £31{,}250 = \$3{,}125 \tag{19.4}$$

The *profit* on one contract is:

$$\Pi = (S_T - K - C)\ £31{,}250 = \$1{,}875 \tag{19.5}$$

If $S_T < K$ then the option will not be exercised and the loss will be limited to the call premium, or more precisely the invoice price of one contract \$1,250. The breakeven spot rate at expiration is:

$$S_{BE} = K + C = 1.40 + 0.04 = 1.44(\$/£) = 144(\text{cents}/£) \tag{19.6}$$

It should be obvious that the appropriate speculative strategy is:

Buy (go long) a call on sterling if you expect sterling to appreciate (above $S_{BE} = K + C$).

19.2.2 Profit from a Long Put

A long put on sterling gives the investor the right to sell £31,250 at a strike price of K(\$/£). Hence the investor makes a profit if spot sterling depreciates below K, at expiration of the option (Figure 19.2). The profit is:

$$\begin{aligned}\Pi &= \max(0, K - S_T) - P \\ &= K - S_T - P \qquad \text{if } S_T < K \\ &= -P \qquad\qquad\quad\ \ \text{if } S_T \geq K\end{aligned} \tag{19.7}$$

The breakeven spot rate (at T) is:

$$S_{BE} = K - P = 1.44 - 0.025 = 1.44 - 0.025 = 1.415 \tag{19.8}$$

The profit profile is given in Figure 19.2 for $K = 1.44$ (\$/£) and $S_T = 1.40$ (\$/£) where the put premium $P = \$0.025$ (2.5 cents).

$$\text{Invoice price per contract} = Pz = (0.025(\$/£))\ £31{,}250 = \$781.25 \tag{19.9}$$

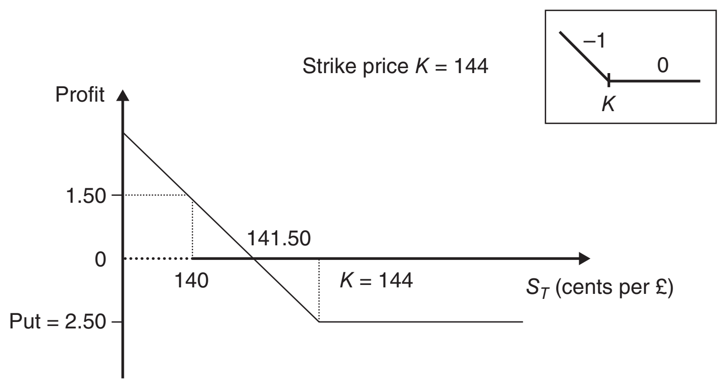


FIGURE 19.2 Foreign currency put option

$$\text{Gross profit} = (K - S_T)z = (1.44 - 1.40)\ £31{,}250 = \$1{,}250 \tag{19.10}$$

The *profit* per contract is:

$$\Pi = (K - S_T - P)£31{,}250 = \$468.75 \tag{19.11}$$

The maximum profit occurs if the spot FX-rate S_T falls to zero:

$$\Pi_{\max} = (K - P)£31{,}250 = (1.44 - 0.025)\ £31{,}500 = \$44{,}218.75 \tag{19.12}$$

The appropriate speculative investment strategy is:

Buy (go long) a put on sterling if you expect sterling to depreciate (below $K - P$).

Foreign currency calls and puts can be combined to give more complex payoffs (at maturity) such as straddles, straps, butterfly spreads, condors etc. (as discussed in Chapter 17).

19.3 HEDGING FOREIGN CURRENCY EXPOSURE

Foreign currency options are not as widely used in hedging as foreign currency futures, nevertheless options provide a useful alternative particularly when the receipt or payment of foreign currency *is not known with certainty*. We have seen how foreign *currency futures (or forwards)* can be extremely useful in hedging when the investor knows *for certain* that she will either receive or have to pay out foreign currency at some time in the future. Using options can be useful when the investor *might* receive or *might* have to make foreign currency payments in the future *but she doesn't know her future cash flow position for certain*.

For example, suppose a US firm is making a bid on an investment project to supply IT hardware to the UK but it doesn't know in advance whether the bid will be accepted and hence whether it will receive any sterling in the future. Next, consider a US investor who has *prospective* foreign currency receipts in the form of dividends on foreign stocks or she feels she may have to sell some stocks in 1 year's time to obtain additional USDs, but she does not know for certain if this is what she will actually do when the time arrives.

Next consider a US multinational that may have to buy foreign currency in 1 year's time when it starts to build a new plant in a foreign country (i.e. direct investment abroad) but it raises USD finance from its US 'correspondent bank'. Similarly, a US investor who is contemplating the purchase of a foreign security (e.g. stocks or bonds) financed from USD cash inflow in the future, may wish to remove exchange rate risk. If the USD finance becomes available, all is well, but she might not want to go ahead with the FX deal if her (dollar) cash inflow does not materialise.

In all these cases a foreign currency option is appropriate since it gives the holder the right to exchange currencies in the future at an exchange rate fixed today, but crucially the option also allows the holder to 'walk away' from the deal if it is not to her advantage.

For our specific example we consider a US firm UncleSam, that in January is making a bid on an investment contract, to supply IT hardware to the UK. But UncleSam doesn't know in advance whether the bid will be accepted and hence whether it will receive any sterling (GBP) in the future. Suppose the outcome of the bid will be announced in 1 year's time (December) and if successful involves an immediate receipt by UncleSam of $V = £12.5$m. The current (1-year) forward rate $F_0 = 1.61(\$/£)$ giving a dollar equivalent of \$20.125m. Now consider the alternatives of having no hedge in place versus using either forwards, futures or options to hedge its sterling exposure.

19.3.1 Numerical Example

The initial data is given in Table 19.2. In January, the initial spot and forward rates are $S_0 = 1.60(\$/£)$ and $F_0 = 1.61$ $(\$/£)$ and the two possible outcomes for the spot rate in December are $S_T = 1.65$ and $S_T = 1.50$.

19.3.2 No Hedge

Clearly under the 'no hedge' scenario (Table 19.2A, column 2) *if the bid is successful*, UncleSam receives sterling and will gain (lose) if sterling appreciates (depreciates) – the payoffs being $V S_T$ which are \$20.625m for an appreciation of sterling (to $S_T = 1.65$) and \$18.75m for a depreciation (to $S_T = 1.50$). Hence, if the bid is successful, UncleSam has exchange rate risk. However, if *the bid is unsuccessful* there are no cash flows and no risk.

TABLE 19.2 Foreign currency hedge (January)

Value of bid	$V = £12.5$m
Spot rate	$S_0 = 1.60(\$/£)$
Forward rate	$F_0 = 1.61(\$/£)$
Strike price	$K = 1.60(\$/£)$
Put premium	$0.025(\$/£)$
Size of option contract	$z = £31{,}250$
Invoice price of put contract	$IP = z.P = \$781.25$
Number of put contracts	$N_P = V/z = £12.5\text{m}/£31{,}250 = 400$ contracts
Total costs of put contracts in USD	$N_P(z.P) = Np.IP = V.P = \$312{,}500$
Possible outcomes for spot FX	$S_T = 1.65(\$/£)$ or $S_T = 1.50(\$/£)$

TABLE 19.2A Bid successful

(1) Scenario	(2) No hedge	(3) Sell £ in forward market at $F_0 = 1.61(\$/£)$	(4) Put option
A : $S_T = 1.65(\$/£)$ (appreciation GBP)	$\Pi = VS_T$ $= (£12.5\text{m})1.65(\$/£)$ $= \$20.625\text{m}$	$\Pi = VF_0$ $= (£12.5\text{m})1.61(\$/£)$ $= \$20.125\text{m}$ Lock in F_0	$S_T > K$, hence puts not exercised Convert £s at 1.65($/£) Cash FX market $= (£12.5\text{m})1.65(\$/£) = \$20.625\text{m}$ Cost of puts = $312,500 Net profit $\Pi = \$20{,}312{,}500$
B : $S_T = 1.50(\$/£)$ (depreciation GBP)	$\Pi = VS_T$ $= (£12.5\text{m})1.50(\$/£)$ $= \$18.75\text{m}$	$\Pi = VF_0$ $= (£12.5\text{m})1.61(\$/£)$ $= \$20.125\text{m}$ Lock in F_0	Exercise puts Payoff $= VK = (£12.5\text{m})1.60(\$/£)$ Less cost of puts $VP = \$312{,}500$ Net profit $\Pi = \$19{,}687{,}500$

19.3.3 Using the Forward Rate

UncleSam selling £12.5m sterling in the forward market (Table 19.2A, column 3) at $F_0 = 1.61$ implies certain USD receipts of $VF_0 = £12.5\,(1.62) = \$20.125\text{m}$ – *but only if the bid is successful* (Table 19.2A, column 3).

If in January, UncleSam hedges by selling £12.5m sterling in the forward market at F_0 then a problem may arise if the firm *fails to win* the contract, next December. It will then have a naked short futures position in sterling. The US firm has agreed to sell £12.5m in 1 year's time at F_0, in exchange for receipt of $20.25m.

If the bid is *unsuccessful* and sterling appreciates $S_T > F_0$, then UncleSam must purchase sterling at $S_T = 1.65$ and sell sterling at the agreed forward rate $F_0 = 1.61$, with a resulting loss of $500,000 ($= V(F_0 - S_T) = -\$500,000$) – Table 19.2B, column 3.

However, if the bid is *unsuccessful* but spot sterling depreciates to $S_T = 1.50$ the cash inflow to Uncle Sam is $V(F_0 - S_T) = £12.5\,(1.61 - 1.50) = \$1.375\text{m}$. Hence:

> *When the bid is unsuccessful, hedging using forwards (or futures) does not reduce risk since UncleSam has an open short position in forward sterling and will either 'win' (if sterling depreciates) or lose (if sterling appreciates).*

19.3.4 Put Options (Bid Successful)

Now consider the risk if UncleSam buys put options on sterling and the bid is successful. If sterling *appreciates* the put expires worthless but UncleSam converts the $V = £12.5\text{m}$ sterling

TABLE 19.2B Bid unsuccessful

(1) Scenario	(2) No hedge	(3) Sell £ in forward market at F_0 = 1.61($/£)	(4) Put option
C : S_T = 1.65($/£) (appreciation GBP)	No cash flows	Purchase £12.5m at S_T = 1.65($/£) and receive at F_0 = 1.61($/£) $\Pi = (F_0 – S_T)V$ = -$500,000	Put not exercised Lose put premium = VP = $312,500
D : S_T = 1.50($/£) (depreciation GBP)	No cash flows	Purchase £12.5m at S_T = 1.50($/£) and receive at F_0 = 1.61($/£) $\Pi = (F_0 - S_T)V$ = -$1,375m	Purchase at S_T = 1.50($/£) and exercise puts at K = 1.60($/£) $\Pi = (K - S_T - P)V$ = $937,500

at a 'high' spot rate of 1.65, giving VS_T = $20.625m less the cost of the puts (Table 19.2A, column 4). This profit is the 'upside capture' of the protective put.

$$\begin{array}{lll} \text{Profit} & \Pi = V(S_T - P) & = \$20{,}312{,}500 \\ \text{Cost of puts} & VP & = \$312{,}500 \end{array}$$

When the bid is *successful* but sterling *depreciates,* the sterling receipts are worth 'less' in USD (= $\$V.S_T$) but the puts end up in-the-money with a payoff = $V(K - S_T)$, hence:

$$\text{Gross payoff} = VS_T + V(K - S_T) = \text{VK} = 1.60\ (\pounds 12.5\text{m}) = \$20\text{m}$$

$$\text{Profit} = V(K - P) = \$20\text{m} - \$312{,}500 = \$19{,}687{,}500$$

The profit is independent of the level of S_T. Hence, when the bid is *successful* and sterling depreciates, UncleSam has minimum cash inflow of $19,687,500. Also if the bid is *successful* but sterling appreciates this allows Uncle Sam to take advantage of the high USD-GBP spot FX-rate (and the put expires worthless). In summary we have:

Bid successful:

$$\begin{array}{lll} \textit{If } S_T > K, \textit{ puts are worthless:} & \Pi = V(S_T - P) & (\textit{upside capture}) \\ \textit{If } S_T < K, \textit{ exercise puts:} & \Pi = VS_T + V(K - S_T - P) = V(K - P) & (\textit{insurance/floor}) \end{array}$$

19.3.5 Put Options (Bid Unsuccessful)

If the bid is *not successful* UncleSam has a naked position in the long put but the most Uncle-Sam can lose is the put premium. Hence, if sterling *appreciates*, the puts are not exercised but

UncleSam's losses are limited to the put premium: $\Pi = VP = \$312{,}500$ (Table 19.2B, column 4). If sterling *depreciates*, the put ends up in-the-money giving a net profit of:

$$\Pi = (K - S_T - P)£12.5\text{m} = (1.60 - 1.50 - 0.025)\ £12.5\text{m} = \$937{,}500.$$

Bid unsuccessful:

If $S_T > K$,	*puts are worthless:*	$\Pi = -VP$	(*Downside limited and known*)
If $S_T < K$,	*exercise puts:*	$\Pi = (K - S_T - P)$	(*Profit for* $S_T < K - P$)

The key difference between hedging with the put and the forward contract is that even in the worst-case scenario – when the bid is unsuccessful and sterling appreciates – the maximum loss is limited to the cost of the puts. Overall, our conclusion is that whether or not the bid is successful, the put limits downside risk but allows upside capture. But with the forward contract, if the bid is not successful, UncleSam could experience substantial losses if sterling appreciates.

19.3.6 Using Futures

Finally, for completeness, we consider the case where the firm hedges in January by selling (shorting) futures contracts. *If the futures are held to maturity,* the outcomes are (analytically) the same as when using the forward market.

At maturity, the profit from the *short* futures position is $V(F_0 - F_T)z_f N_f$ where $z_f = £62{,}500$ (for sterling futures on the CME). However, to hedge the position, the number of futures contracts to short is $N_f = V/z_f$ so the profit from the short futures position is simply $V(F_0 - F_T)$. Furthermore, at maturity of the futures $F_T = S_T$, so the cash profit from the futures position at maturity is $V(F_0 - S_T)$.

If the bid is *successful*, then net receipts at maturity in USD are $VS_T + V(F_0 - S_T) = VF_0$. If the bid is *unsuccessful,* net receipts are $V(F_0 - S_T)$. These results are the same as those obtained when using *forward contracts*. If the short futures position is closed out at $t = 1$ (before maturity), then the story is a little more complex. UncleSam's USD receipts are:

Bid successful	$\Pi = VS_1 + V(F_0 - F_1) = V[F_0 + (S_1 - F_1)]$
Bid unsuccessful	$\Pi = V(F_0 - F_1)$

If basis risk ($= S_1 - F_1$) is small, the cash flow if the bid is successful will be close to VF_0, which is the same as we obtained for the *forward* contract. If the bid is unsuccessful, the profit from the short futures position $=V(F_0 - F_1)$. This will be close to the outcome for the *forward contract* if the futures contract is closed out near its maturity date (i.e. $F_1 = F_T = S_T$). In practice, the latter is usually the case.

19.4 OTHER CURRENCY OPTIONS

Some of the more complex currency options come under the heading of 'exotic options'. Consider, for example, a type of barrier option known as a 'down-and-in' (at-the-money, $S_0 = K$) call option, with a lower barrier $L < K$. This 'down-and-in' call option only 'comes alive' when the spot FX rate at any time t ($< T$) S_t falls below the lower barrier L, set in the option contract. Hence, there is a positive payoff at maturity to the long call if $S_T > K$, as long as the exchange rate has fallen below the lower barrier (i.e. $S_t < L$), before the maturity date of the option. If $S_T > K$, but the exchange rate has never fallen below the lower barrier, then the payoff at maturity to the long call is zero. Hence the payoffs to the down-and-in call are less favourable than for a plain vanilla call and this is reflected in the lower call premium for the down-and-in call (see Chapter 31).

There are also FX options whose payoff is based on the *average value* of the spot-FX rate (e.g. USD-GBP) over the life of the option. These are known as *Asian FX options* and can be used by a corporate to hedge *a series* of future foreign cash inflows or outflows each month, over a period of a year (say). The Asian option is cheaper than buying a series of plain vanilla options, each of which has an expiration date at the end of each month.

Also, a useful option for a US multinational corporation, UncleSam, which has net cash inflows or outflows in a number of different foreign countries is a *basket option*, sold in the OTC market. Here the payoff is denominated in dollars using the FX-rate on a basket of currencies. For example, if UncleSam has net cash inflows in many different currencies then it can guarantee minimum receipts of USDs by buying put options on each *separate* currency – but this could be expensive. However, if FX-rates on different currencies are not perfectly correlated then the volatility of a 'basket of currencies' will be less than the sum of the volatilities of the individual currencies. From Black–Scholes we know that option premia (on calls and puts) are positively related to the volatility of the 'underlying asset' in the option contract. Hence, a 'basket' FX put option will have a premium which is less than the sum of the premia for the individual plain vanilla put options, on each separate currency.

19.5 SUMMARY

- Traded foreign currency options, both European and American style, are available (OTC) for all major currencies against the US dollar. There are also a (smaller) number of contracts on cross-rates (e.g. Euro-GBP and Euro-Yen). The OTC market is very active since contracts can be 'structured' to suit clients' needs as regards contract size, maturity date, currencies etc.
- Foreign currency options provide leverage when used for speculation and the usual types of options strategies (e.g. straddles, straps, spreads) on a spot FX rate are possible by combining currency options with different strikes (but the same expiration dates and currency).

- A US firm with *known future receipts* in sterling (e.g. from sales in the UK) can 'lock in' known USD receipts, by today shorting (selling) forwards or futures on sterling. However, if the future sterling receipts do not materialise (e.g. are not paid), then the 'naked' short forward/futures position will result in a gain if sterling appreciates and a loss if sterling depreciates. The forward/futures hedge is therefore 'risky', if future sterling cash flows are uncertain.
- Foreign currency *options* are useful for when future foreign currency receipts or payments are uncertain. If a US firm has uncertain future sterling receipts then buying a foreign currency put option on sterling today, provides a known minimum receipt in USD (of $K - P$), while allowing upside potential (should sterling appreciate), regardless of whether the future sterling receipts are received or not. If the sterling receipts do not materialise the most that can be lost is the put premium (times the number of puts purchased).

EXERCISES

Question 1

Explain the key items in the following quotes for calls and puts on sterling (GBP). The contract size is for delivery of GBP31,250 and the options expire on 28 September. Assume the options are European.

£31,250 British Pound. Cents per unit (19 August)			
British pound Spot FX rate	Strike	Calls (28 Sept)	Puts (28 Sept)
160.60	164	0.48	4.14

Question 2

A US exporter expects to receive £1m (GBP) in 1 year's time. The current spot FX rate is S_0 = 1.1 $/£ (USD/GBP). What type and how many options will the US exporter require to cover her position and set a minimum value for the FX rate of 1.1 USD/GBP in 1 year's time? Contract size is for delivery of GBP 31,250.

Question 3

It is January. You are a US portfolio manager, holding €20m in stocks in 25 large European companies and these stocks are likely to be sold in 6 months' time (in July). You predict that the Euro spot-FX rate is likely to fall from its current level of $S_0 = 1.20$ against the USD over the next 6 months (because of the interest rate policy of the European Central Bank). You need a currency strategy that will be beneficial if your prediction is correct but will not lead to large losses if you are incorrect. Discuss the relative merits of using either a European FX option (with $K = 1.10$ USD/Euro) or an FX futures contract to achieve your aims. Assume the spot FX rate in 6 months' time turns out to be either 1.15 or 1.0. Note any risks in your strategy.

Question 4

It is 12 December and a US exporter expects to receive €0.5m from the sale of Barbie dolls in Germany, and payment will arrive in March. The March-102 put contract on euros has a premium of 6.0 (cents per euro) and the current spot rate is $S_0 = 100$ (cents per euro). The contract size is €62,500. What is the hedged/insured position using puts with a strike of $K = 102$ cents/euro? What is the outcome if the spot FX rate in 3 months is $S_T > K$ or $S_T < K$.

Question 5

On 26 September you are faced with the following quotes for December calls and puts on the Philadelphia Stock Exchange for options on sterling (GBP).

B. Pound GBP	Strike price	Calls (December)	Puts (December)
144	140	5.10	–
144	145	–	2.5

All quotes are in cents per GBP.
The current spot rate is $S = 144$ US cents/GBP. Contract size = £31,250.

Draw the payoff/profit profile for a long call and a long put taken individually, indicating the breakeven spot FX rate at expiry.

Question 6

On 26 September you are faced with the following quotes for December calls and puts on the Philadelphia Stock Exchange for options on sterling (GBP).

B. Pound GBP	Strike price	Calls (December)	Puts (December)
144	140	5.10	–
144	145	–	2.5

All quotes are in cents per GBP.
The current spot rate is $S = 144$ US cents/GBP. Contract size = £31,250.

You purchase one call and one put on GBP, with the above strikes. Calculate the payoff and profit outcomes at expiration, when the spot FX rate ends up (i) below 145, (ii) between 140 and 145, and (iii) above 140.

What is the general shape of the payoff to the long call and put (with the above strikes)? What strategy have you implemented and under what circumstances does it give a positive profit?

Question 7

In 6 months' time a US firm has to pay for imports from Germany of either €2m or €3m (depending on the amount of imports delivered). What are the advantages and disadvantages of hedging with FX-futures? Consider the outcomes if the US firm decides to cover €3m but only €2m are required to pay for imports and consider the outcome if either the euro appreciates or depreciates over the next 6 months.

CHAPTER 20

Options on Futures

Aims

- To analyse futures options, that is, options contracts where the underlying asset is a futures contract.
- To explain price quotes, delivery and settlement procedures.
- To establish upper and lower bounds for the price of European calls and puts on futures.
- To examine payoffs and trading strategies using futures options.
- To examine the relative merits of options on spot assets and options on futures contracts.

A futures option is an option where the underlying asset to be delivered is a futures contract. The futures contract itself could, for example, be written on T-bonds, Eurodollar interest rates, spot-FX rates or commodities such as corn or oil. Most futures options are American style options. The main advantage they have over options on the spot asset is that they deliver a futures contract, which is highly liquid and easy to close out at low cost. The latter can sometimes make them more attractive than using options on the underlying spot asset itself.

20.1 MARKET CONVENTIONS

A long American call (put) option on a futures contract gives the holder the right, but not the obligation, to buy (or sell) a futures contract at the strike price, up to and including the maturity date of the option.

Some of the most popular futures options are on T-bond and T-note futures (CBOT), on Eurodollar futures (CME), on EURIBOR futures (Euronext-LIFFE) and on the S&P 500 futures contract. Other futures options which are also actively traded, include those written on futures

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contracts on the Japanese yen (CME), the euro (CME), the Canadian dollar (CME), crude oil and other oil products (NYMEX), gold (CMX), corn, soybeans, cotton, and sugar (CBOT).

When you buy a call or put futures option you must immediately pay the full option premium (invoice price). At maturity, a futures option delivers a futures contract which itself has a maturity date. Futures options are referred to by the month in which the underlying *futures contract matures* (and not by the expiration date of the option itself). However, the expiration date of a futures option is usually on, or a few days before, the maturity date of the underlying futures contract.

For example, a futures option on the S&P 500 futures index expires on the same day as the futures contract. However, the T-bond futures option expires in the month prior to the month that 'names' the option contract. For example, the October T-bond futures option expires in September and delivers a T-bond *futures contract* which expires in October. CME currency futures options expire two business days prior to the expiration of the underlying futures contract. Where the contracts on the option and the deliverable futures contract expire simultaneously $F_T = S_T$, so the futures option is analytically equivalent to an option on the spot asset, but with cash settlement.

The closing prices of futures options are available for different strike prices K and expiration months. For example, suppose on 1 January, the US T-bond *October-call* futures option (CBOT) for expiration in September, has a strike price of $K = 96$ per \$100 nominal of T-bonds) and a quoted call premium of 2-39. The '39' represents 39/64th, so the call premium is $C = 2(39/64)\% = 2.609375\%$, with invoice price:

$$\textit{Invoice price of T-bond futures option} = (0.02609375)\ \$100{,}000 = \$2,609.38$$

At expiration (in September), the October-call futures option delivers one T-bond futures contract with a contract size of \$100,000.

20.1.1 Expiration and Delivery

Let us now consider what happens on the expiration date $t \equiv T$ of a European futures option or when an (American) futures option is exercised ($t \leq T$). If a *long call* option is exercised, the holder acquires a *long position* in the underlying futures contract (with a delivery price of K) *plus* a cash amount equal to $F_{settle} - K$, where F_{settle} is the futures settlement price at the end of the *previous* day. However, it turns out (see Example 20.1) that exercising the call option at t, (where $t \equiv T$ for a European option) results in a payoff equivalent to:

$$\text{Effective payoff from call} = \max(0, F_t - K) \tag{20.1}$$

where F_t is the futures price at the time of exercise of the option. Since the options are American, exercise can take place at any time $t > 0$ and not just at T.

EXAMPLE 20.1

Call Payoff

Suppose at 11 a.m. on 15 August the T-bond futures price is $F_t = 103$ (per \$100 nominal of deliverable bonds). The price of the call futures option is 1-08 (1 8/64 = 1.125% of the principal). The payoff to a call option on T-bond futures with $K = 100$ is:

$$\textit{Payoff from call} = (103 - 100)(\$100,000/100) = \$3{,}000$$

This payoff arises as follows. On 15 August it is actually the settlement price for the futures the night before (14 August) $F_{settle} = 102.9$ which is used. So the cash payoff from the option is actually based on $F_{settle} - K$. But on 15 August you also take delivery of one futures contract which can immediately be closed out (in the futures market) at $F_t - F_{settle}$, hence:

$$\textit{Total payoff} = \textit{Option payoff} + \textit{Profit on futures} = (F_{settle} - K) + (F_t - F_{settle}) = F_t - K$$

You do not have to close out your long futures contract if you do not wish to do so, but the value of your cash payoff of $F_{settle} - K$ plus the mark-to-market value of your long futures position $F_t - F_{settle}$, is equal to $F_t - K = \$3{,}000$. The invoice price of the call futures option is 11.25 (=0.01125 × \$100,000/100).

If the investor decides not to close out the futures contract she has to pay the initial margin on the long futures position. When the call is exercised, *the writer of the call* has $F_{settle} - K > 0$ deducted from her margin account and also has to provide any additional margin requirements on the futures – she 'inherits' a short position in T-bond futures, when the option is exercised. If on the expiration date $F_T < K$, then the call is not exercised, and the writer retains the call premium.

If a *long put* futures option is exercised, the holder acquires a short position in the underlying futures contract (with a delivery price of K) *plus* a cash amount equal to $K - F_{settle}$. For example, exercise of the September-100, T-bond put-futures option results in a short position in T-bond futures at a 'delivery price' of K = 100 together with a cash amount of $K - F_{settle}$. Again, the position is immediately marked-to-market[1] which for a short futures position is $-(F_t - F_{settle})$. The payoff to the put *if it is exercised* is $K - F_{settle} - (F_t - F_{settle})$, hence:

$$\text{Payoff from Put} = \max(0, K - F_t)$$

[1]If the required margin is not forthcoming, the futures position is closed out.

At expiration, the *writer* of a put 'inherits' a long position in the futures at a delivery price of K and the position is immediately marked-to-market. Writers of calls or puts, must post margin payments on the options contracts (as with options on other cash-market assets) and the option writer's margin account is marked-to-market.

Nearly all futures options are American and hence early exercise is possible. For American style futures options it is often *not* possible to derive an analytical (closed form) solution, (e.g. Black–Scholes equation as for certain European style options) – hence, to price futures options may require numerical techniques such as the binomial option pricing model (BOPM – see Chapters 21–23) and Monte Carlo simulation (MCS – see Chapter 26).

20.2 PRICE BOUNDS ON EUROPEAN FUTURES OPTIONS

If a quoted futures option price is below the lower bound or above the upper bound (as set out below) then there are arbitrage profits to be made. Establishing these bounds only requires the assumption of a positive risk-free (nominal) rate of interest. The easiest way to establish the no-arbitrage bounds for European futures options is to use the put–call parity relationship (see Chapter 25)[2]:

$$P = C + (K - F)\, e^{-rT} \tag{20.2}$$

where we assume the option and the futures mature at the same time (and the call and put, both have the same strike and time to maturity). Because the price of the put cannot be negative, then Equation (20.2) gives a *lower bound* for the call futures price:

$$C + Ke^{-rT} \geq Fe^{-rT} \tag{20.3}$$

$$C \geq (F - K)e^{-rT} \tag{20.4}$$

Similarly, the price of the call cannot be negative, then Equation (20.2) gives a *lower bound* for the put futures price:

$$Ke^{-rT} \leq Fe^{-rT} + P \tag{20.5}$$

$$P \geq (K - F)e^{-rT} \tag{20.6}$$

Since the call and put cannot be worth less than zero the lower bounds for European futures-options are:

$$C \geq \max[(F - K)e^{-rT}, \quad 0] \tag{20.7a}$$

$$P \geq \max[(K - F)e^{-rT}, \quad 0] \tag{20.7b}$$

[2]As shown in Chapter 25, this is most easily obtained by taking the put–call parity relationship for an option on a non-dividend paying stock $P + S = C + Ke^{-rT}$ and substituting $S = Fe^{-rT}$

A European call futures option gives the holder the right to take delivery of a futures contract and the call can never be worth more than the current futures price. Hence the upper bound for European call futures options is $C \leq F$. For a European put futures option the minimum value is K at maturity T, so today the put cannot be worth more than $P \leq Ke^{-rT}$.

20.3 TRADING STRATEGIES

The popularity of futures options is due to:

- Greater liquidity in the futures market than in the spot (cash) market.
- Transactions costs using options on stock *index futures* are lower than for options on the underlying stock indices. This is also true for futures on agricultural products and metals, since these options deliver a futures contract which can be closed out at low cost. In contrast, for options on the spot asset, delivery of the underlying asset may be costly or inconvenient.

Since futures and spot prices move closely together, a long call on a futures contract would constitute a speculative (open) position, where the investor expects a rise in futures prices. If the speculator expects a fall in futures prices, she would profit by purchasing put futures contract. We consider each of these in turn.

20.3.1 Long Call

The profit at expiration is:

$$\begin{aligned} \Pi &= \max(F_T - K, 0) - C \\ &= -C \quad \text{if } F_T \leq K \\ &= F_T - K - C \quad \text{if } F_T > K \end{aligned} \tag{20.8}$$

with a breakeven futures price of $F_{BE} = K + C$. For a call option on the futures stock index with $K = 475$, and the value of an index point $z = \$250$, Figure 20.1 gives the possible outcomes:

The call premium is $C = 7$ and the invoice price of one contract is \$1,750 (= \$250 C). The breakeven point is $F_{BE} = K + C = 475 + 7 = 482$. If, at expiry, $\mathrm{F_T} = 485$ ($> K = 475$), then the net profit per contract is :

$$\Pi = (F_T - K - C)\ \$250 = (485 - 475 - 7)\ \$250 = \$750 \tag{20.9}$$

If $F_T \leq K$, then the call expires worthless and with $C = 7$, the invoice cost per contract is \$1,750.

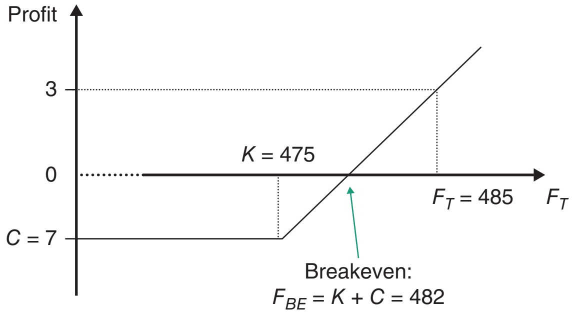

FIGURE 20.1 Call on stock index futures contract

20.3.2 Long Put

The payoff at maturity is:

$$\begin{aligned}\Pi &= \max(K - F_T, 0) - P \\ &= K - F_T - P \quad \text{if } F_T < K \\ &= -P \quad \text{if } F_T \geq K \end{aligned} \tag{20.10}$$

and the breakeven futures price is $F_{BE} = K - P$. If $P = 5$ then the invoice price is (\$250) $P =$ \$1,250. Figure 20.2 gives the profit profile at expiry. If $F_T = 462$ and $K = 475$ then the profit is $(K - F_T - P)$ \$250 = 8 (\$250) = \$2,000.

Note that if the underlying futures contract matures at the same time as the (European) option then $F_T = S_T$ and the European futures option and the European option on the spot asset (with the same strike) are equivalent – and they both have the same (call and put) premia.

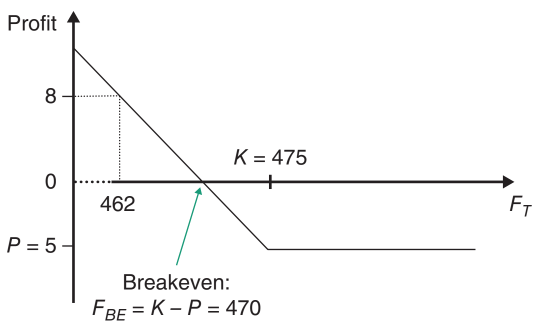

FIGURE 20.2 Put on stock index futures contract

20.3.3 Covered Call

Holding a long futures position is highly risky – if the futures price falls you lose money. You can *mitigate* some of this potential loss if you write a call futures option, since you then receive the call premium. However, this does not eliminate much of the potential downside loss if the futures price falls. Consider a covered call using futures options, where the underlying futures contract is on a stock index (e.g. the S&P 100 or S&P 500).

$$\textit{Covered Call} = \textit{Long Futures} + \textit{Written Call on Futures Option}.$$

The payoff profile for long-futures {+1, +1} plus written call {0, –1} gives the {+1, 0} payoff for a covered call (see Figure 20.3). If the futures price falls (below $K = 475$), the *long* call expires worthless, so the *writer* of the call does not have to pay anything at T. The losses on the long futures position $F_T - F_0$ are (partially) offset by receipt of the call premium, $C = 8$. Hence, downside losses from a covered call are slightly less than when holding only the long futures.

At T, if the futures price rises (above K), the long futures is closed out at a profit $= F_T - F_0$ but the long call is in-the-money, so the *written* call makes a loss $-(F_T - K)$. Overall, payoff from the long futures plus the short call cancel each other out, so the holder of the covered call merely retains the call premium.

The profit from the covered call is:

$$\begin{aligned}\Pi &= F_T - F_0 - \max(F_T - K, 0) + C \\ &= K - F_0 + C \quad \text{if } F_T > K \text{ (independent of } F_T) \\ &= F_T - F_0 + C \quad \text{if } F_T \le K \text{ (depends on } F_T) \end{aligned} \tag{20.11}$$

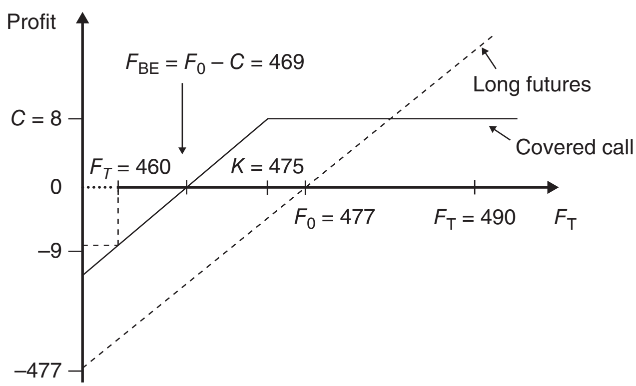

FIGURE 20.3 Covered call

Note that the outcome for the futures price F_T only affects the covered call payoff when $F_T \leq K$. The breakeven futures price (for $F_T \leq K$) is $F_{BE} = F_0 - C$. Figure 20.3 gives the profit profile on the covered call for Example 20.2.

EXAMPLE 20.2

Covered Call Futures Option

Suppose $F_0 = 477, K = 475$ and the call futures option has $C = 8$. What is the payoff to a covered call, if the futures price at expiry of the option is either $F_T = 490$ or $F_T = 460$ and the value of an index point is \$250? (see Figure 20.3).

For $F_T = 490 > K = 475$, the long call is exercised and the call writer pays $F_T - K = 15$. But the long futures can be closed out at a profit of $F_T - F_0 = 13$. The net profit is –2 plus the call premium of 8, that is:

$$\Pi = K - F_0 + C = (475 - 477) + 8 = 6 \text{ (per contract)}$$

giving an overall profit of \$1,500 (= \$250 × 6). If $F_T = 460 < K = 475$, the breakeven futures price is $F_{BE} = F_0 - C = 477 - 8 = 469$. The long call is not exercised and the call writer receives the call premium of $C = 8$. However, there is a loss on the long futures position of $F_T - F_0 = 460 - 477 = -17$ and hence a net profit of $8 - 17 = -9$, that is:

$$\Pi = F_T - F_0 + C = 460 - 477 + 8 = -9 \text{ (i.e.a loss).}$$

All the options strategies we discussed in Chapter 17 with respect to options on the *underlying spot* asset (e.g. spreads, straddles, condors etc.) can be undertaken with *futures options*.

20.4 SUMMARY

- For a futures option the underlying deliverable asset is a futures contract. The futures contract that is delivered usually has a maturity date slightly later than the expiration date of the futures option.
- A call futures option, is an option to take delivery of a futures contract (i.e. go long the futures) at a delivery price K, at expiration of the option. The payoff to a long call option on a futures contract is max $(F_T - K, 0)$.
- A put futures option, is an option to deliver a futures contract (i.e. go short the futures) at a delivery price K, at expiration of the option. The payoff to a long put option on a futures contract is max $(K - F_T, 0)$.

- We can establish upper and lower limits for call and put premia on European futures options. If quoted options prices lie outside these bounds then risk-free arbitrage profits can be made.
- The most popular futures options are on US T-bond and T-note futures (CBOT), on Eurodollar futures (CME), and the Euro-LIBOR contract (Euronext-LIFFE), as well as on futures contracts on the major currencies (Euro, Japanese yen, Canadian dollar, on CME). Options on futures contracts on commodities, for example, crude oil and other oil products (NYMEX), gold (CMX) and corn (CBOT) are also actively traded.
- Futures options are often used rather than options on the underlying asset itself. This is because at maturity of the option the underlying futures contract can be easily closed out at relatively low transactions costs.

EXERCISES

Question 1
What is a futures option?

Question 2
In January, a June-futures option is purchased. Does the option expire in June?

Question 3
When does the October-futures option on the S&P 500 index expire?

Question 4
A long call futures option on T-bonds is exercised but the futures contract is not closed out. What happens to the trader holding the long call?

Question 5
If you exercise a call futures option by cash-settling the option contract, what is the cash payoff?

Question 6
A T-bond *call* futures option with expiration date in September has a strike price of $K = 96$ and a quoted call premium of 2-60. The contract size is \$100,000. What is the invoice price of the call futures option?

Question 7
Why might you hedge your future heating oil purchases using a call *futures* option on heating oil rather than a call option on (spot) heating oil (itself)?

PART V

OPTIONS PRICING

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CHAPTER 21

BOPM: Introduction

Aims

- To demonstrate how the binomial option pricing model (BOPM), is used to determine option premia by establishing a risk-free arbitrage portfolio consisting of a position in stocks and the option – this is delta hedging.
- To show how delta hedging and the no-arbitrage approach can be interpreted in terms of risk-neutral valuation (RNV), which allows us to price options using a simple backward recursion.
- To demonstrate how RNV leads to other useful approaches to pricing options such as Monte Carlo simulation.

We present a detailed account of the BOPM for pricing options using the no-arbitrage principle – the option is priced so that traders faced with this 'BOPM price' cannot undertake trades which result in risk-free profits. We construct a risk-free portfolio from two risky assets, namely calls and stocks. Using the principle of delta hedging, whereby the proportions held in stocks and the option gives a risk-free portfolio (over small intervals of time) – we obtain the BOPM formula for the price of the option. We then interpret the BOPM formula in terms of risk-neutral valuation (RNV). Using insights from RNV we can then price an option without going through all the details involved in delta hedging and forming a 'risk-free arbitrage portfolio' – instead we price the option directly by using the BOPM formula and 'backward recursion'.

21.1 ONE-PERIOD BOPM

To understand delta hedging and risk-neutral valuation we first price a European call option (on a non-dividend paying stock), which has one period to expiration. The basic idea is to

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construct a portfolio consisting of the call option and some stocks, so that the portfolio is risk-free (over a small interval of time).

The current stock price is $S = 100$ and the stock pays no dividends. Consider a one-period problem where there are only two possible outcomes for the stock price at $t = 1$, namely an 'up' move with $U = 1.1$ and a 'down' move with $D = 0.9$, hence:

$$S = 100 = \text{stock price today}$$

$$S_u = SU = 110 = \text{stock price after an 'up' move,}$$

$$S_d = SD = 90 = \text{stock price after a 'down' move.}$$

$$K = 100 = \text{strike price of a call option,}$$

$$r = 0.05 = \text{one-period risk-free rate of interest}$$

$$C = \text{the } \textit{unknown} \text{ call premium (i.e. price of the call)}$$

Suppose the 'real world' probability of the stock price moving up or down is $p = 1/2$, so the 'real world' expected stock price at $t = 1$ is $ES_1 = 100 = 1/2(110) + 1/2(90)$ and since $S = 100$ the expected stock *return* $\mu = (ES_1/S) - 1 = 0$. As we see below, somewhat surprisingly, neither the 'real world' probability p of a rise in the stock price nor the 'real world' expected return on the stock μ determine the call premium.

The payoffs when holding *either* one stock or *one long* call are given in Figure 21.1. Note that if the payoff to a long call is +10 (at $t = 1$), then the payoff to a short (sold/written) call is –10. The *difference* between the two possible outcomes for the stock is $S_u - S_d = 20$ and for the call is $C_u - C_d = 10$. We define the hedge ratio as $h = (C_u - C_d)/(S_u - S_d) = 1/2$.

Below, we see that if today ($t = 0$) Ms Arb sells one call and buys 1/2 stock (or alternatively if Ms Arb sells 100 calls and buys 50 stocks) then she has set up a risk-free portfolio (over a small interval of time). To be 'risk-free' this portfolio must have a known value at $t = 1$ no matter what the values of the stock price and call premium are at $t = 1$. To set up this 'hedge

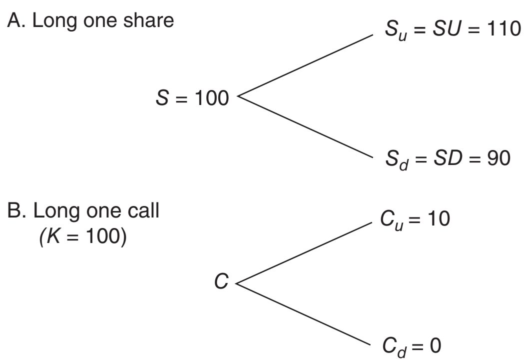

FIGURE 21.1 One-period BOPM – call

portfolio' involves a cash outlay at time $t = 0$ but because the payoff at $t = 1$ is known, Ms Arb's hedge portfolio must have a return equal to the risk-free rate – or profitable arbitrage opportunities are possible. Consider the payoff to portfolio-A where Ms Arb is long 1/2-stock and short 1-call.

Payoff to Portfolio-A: long 1/2 stock, short 1-call

$$\text{Payoff for price rise} = (1/2)\,(110) - 10 = 45$$

$$\text{Payoff for price fall} = (1/2)\,(90) - 0 \quad = 45$$

The payoff to portfolio-A at $t = 1$ is known with certainty, no matter what the outcome for the stock price (i.e. either 110 or 90). Ms Arb has created a risk-free portfolio using a hedge ratio of $h = 1/2$. The cost of constructing portfolio-A at $t = 0$ is the cost of buying the stocks less the receipt from the sale of the call, that is, $(1/2)100 - C$. Portfolio-A is risk-free and if there are to be no arbitrage profits to be made, portfolio-A must earn the risk-free rate:

$$\begin{aligned} 1 + r &= \frac{\textit{Certain payout} \;\; \textit{at} \; t = 1}{\textit{Cost of initial investment} \;\; \textit{at} \; t = 0} \\ 1.05 &= \frac{45}{(1/2)100 - C} \end{aligned} \tag{21.1}$$

Therefore the 'fair', 'correct' or 'no-arbitrage' price of the call is $C = 7.1428571$. Alternatively, if you borrow $(1/2)100 - C$ to set up portfolio-A then you will owe the bank $[(1/2)100 - C]1.05$ at $t = 1$ and for no arbitrage profits to be made, this must equal the (known) payoff of 45 at $t = 1$:

$$\begin{aligned} &\text{Bank debt (at } t = 1) = \text{payoff from portfolio-A at } t = 1 \\ &[(1/2)100 - C]1.05 = 45 \end{aligned} \tag{21.2}$$

Again the solution to Equation (21.2) is $C = 7.1428571$.

21.1.1 Arbitrage: Overpriced Call

The fair price of the option is $C = 7.143$ – otherwise *risk-free* arbitrage profits can be made. To see this, suppose the actual quoted option premium (by all option traders) is $C_q = \$10$ which exceeds the BOPM 'fair price' $C = 7.143$ – the call is overpriced. Ms Arb spots this pricing anomaly. As usual, Ms Arb 'sells high and buys low'. At $t = 0$, she sells (writes) the overpriced call and receives $C_q = \$10$ and hedges by purchasing ½ a stock at S = 100, at a net cost of \$40, which she borrows from the bank. The guaranteed cash value of this hedged 'portfolio-A' at $t = 1$ is \$45 (no matter whether the stock price goes up or down) and as she owes the bank \$42 (= \$40 × 1.05) she makes an arbitrage profit of \$3.

Viewed slightly differently, the net cost of setting up portfolio-A at $t = 0$ is \$40 (= (1/2)100-10) and if Ms Arb uses her 'own funds' of \$40 today then her return is

7.5%(= \$3/\$40), which is in excess of the risk-free (borrowing) rate of 5% – again indicating risk-free arbitrage profits.

Note that there will be lots of traders who try to implement this risk-free, yet profitable strategy. Selling calls would tend to lead to a fall in the quoted price of $C_q = 10$ and move it towards the 'fair' no-arbitrage price $C = 7.143$. Hence, we expect the quoted call premium to deviate from its (BOPM) fair price only by a small amount and for any discrepancies to be quickly eliminated by the actions of arbitrageurs.

21.1.2 Arbitrage: Underpriced Call

Consider how to make arbitrage profits if the call is temporarily underpriced at a quoted price $C_q = \$7$. Ms Arb buys low, sells high. So today, she *buys the call at* $C_q = 7$ and hedges this position by short-selling ½ stock, for which she receives \$50 (= ½ of%100) at $t = 0$. Her net cash inflow at $t = 0$ is \$43 (= 50 − 7). This can be invested at the risk-free rate to give receipts at $t = 1$ of \$45.15 (= 43 × 1.05). Ms Arb's hedged portfolio will be worth *minus* \$45 at $t = 1$ – that is, when she closes out all her short stock and long call positions she will have lost \$45 (regardless of whether the stock price rises or falls). Hence she makes an overall profit at $t = 1$ of \$0.15 (= 45.15 − 45).

21.1.2.1 Formal Derivation

We now derive the price of the call option algebraically. The two outcomes for the stock price are $S_u = SU$ and $S_d = SD$ (where in our example $U = 1.1$ and $D = 0.9$ as shown in Figure 21.1).[1] Let:

$$C_u = \text{payoff to a long call if the stock price is } S_u$$

$$C_d = \text{payoff to a long call if the stock price is } S_d$$

Hence:

$$C_u = S_u - K = SU - K \qquad (\text{for } S_u > K \text{ otherwise } C_u = 0)$$

$$C_d = S_d - K = SD - K \qquad (\text{for } S_d > K \text{ otherwise } C_d = 0)$$

Portfolio-A: long h-stocks, short 1-call

$$\text{Payoff for price rise at } t = 1 \qquad V_u = hS_u - C_u$$

$$\text{Payoff for price fall at } t = 1 \qquad V_d = hS_d - C_d$$

[1] In fact, the risk-free rate must lie between the rate of return if the stock goes up and its rate of return if the stock goes down so that $U > R > D$ (with $D > 0$). This ensures that no risk-free arbitrage profits can be made (see Appendix 21). Note that in this example $U \neq (1/D)$ and there is no reason that it should. But later we show that it is often useful to set $U = (1/D)$ and with suitable changes, we can still obtain the correct (no-arbitrage) price for the call.

The payoff to a long call at $t = 1$ is C_u or C_d so the payoff to a short call is $-C_u$ or $-C_d$. For the two payoffs to be equal:

$$hS_u - C_u = hS_d - C_d \tag{21.3}$$

$$h = \frac{(C_u - C_d)}{S_u - S_d} = \frac{(C_u - C_d)}{S(U - D)} = \frac{(10 - 0)}{100\,(0.2)} = \frac{1}{2} \tag{21.4}$$

In the above analysis we are *short one* call and (21.4) indicates that the hedge will then involve going *long h*-stocks. The formula for the hedge ratio h in (21.4) is the change in value of the option divided by the change in value of the stock and this is the option's 'delta' – hence this approach is called *delta hedging*. Given the risk-free portfolio-A, we determine the call premium C by equating the amount of bank debt owed at $t = 1$ (due to the cost of setting up portfolio-A at $t = 0$), with the known payoff from portfolio-A at $t = 1$:

$$\text{Bank debt (at } t = 1) = \text{payoff from portfolio-A (at } t = 1)$$

$$(hS - C)(1 + r) = hSU - C_u \tag{21.5}$$

Substituting from Equation (21.4) for h and (carefully) rearranging, the BOPM equation for the call premium is:

$$C = \frac{1}{R}\left[qC_u + (1 - q)C_d\right] \tag{21.6a}$$

$$R = 1 + r \tag{21.6b}$$

and

$$q = (R - D)/(U - D) \tag{21.6c}$$

21.2 RISK-NEUTRAL VALUATION

Note that the formula for the call premium does not depend on the 'real world' probability p of an 'up' move for the stock price (and hence is independent of the 'real world' expected return μ on the stock). In the formula for the call premium the 'weights' applied to the two option payoffs C_u and C_d are q and $1 - q$ (which sum to unity), but there is little intuitive insight at present to be gleaned from Equation (21.6c) for q.

The 'weight' q is known as the *risk-neutral probability* of a rise in the stock price, which must not be confused with the actual 'real world' probability of a rise in the stock price p. The risk-neutral probability is simply a number which lies between 0 and 1 and is derived under the assumption that portfolio-A is risk-free.[2] The expected payoff to the option, using risk-neutral probabilities is:

$$\textit{Expected payoff to long call option} = \left[qC_u + (1 - q)C_d\right]$$

[2]Note that for q to be interpreted as a probability lying between zero and one, we must have $U > R > D$ (and $D > 0$) and these inequalities also ensure no arbitrage opportunities are possible (see Appendix 21).

In some ways the term 'risk-neutral probability' could be misleading since it appears to imply that we are assuming investors are 'risk neutral' and hence do not care about risk. Investors in options do care about the level of risk they hold but in the BOPM they can set up a risk-free hedge portfolio and by definition this has zero risk. An alternative is to call q, *an equivalent martingale probability* and the latter is used frequently in the continuous time literature. However, we will stick with the more commonly used term, 'risk-neutral probability'. Using Equation (21.6a) we can price the option using the risk-neutral probability q and this method of pricing is known as risk-neutral valuation (RNV) and it plays a major role in the pricing of all types of options. Hence we can now state:

> *The call premium C is the expected payoff at maturity, where the expectation uses risk-neutral probabilities $(q, 1-q)$ and the expected payoff is discounted using the risk-free rate.*

q is not a 'real probability' it is a 'pseudo-probability': q lies between 0 and 1 and the sum of q and $1-q$ for the two possible outcomes is 1. But why is q known as a risk-neutral probability? We use a 'what if' argument here, to help explain the term, 'risk-neutral probability'. If the probability of an 'up' move equals q, then what would we expect the stock price to be at $t = 1$? This is given by:

$$r = 5\% \tag{21.7}$$

where E^* is the expected value, using risk-neutral probabilities.

But the initial stock price is $S = 100$, so if we *interpret* q as the probability of an 'up' move then this implies that the stock price (in the BOPM) is expected to grow at 5% (= (105/100) − 1). But this is also the value of the risk-free rate, $r = 5\%$. Hence, the no-arbitrage BOPM price for the option given by Equation (21.6a) *is consistent with the assumption* that the stock price grows at a rate (or has an expected return) equal to the risk-free interest rate. Confused? Yes, RNV is a tricky concept. However, it turns out to be a brilliant insight since it allows us to price very complex options by:

- using the 'pseudo-probabilities' q, to 'weight' the option payoffs, to give an 'expected payoff' to the option (in a risk-neutral world).
- Discount these expected payoffs using the risk-free rate, to give the correct no-arbitrage price of the option.

Rather miraculously we obtain the correct no-arbitrage 'real world' option price using the above two assumptions, even though in the real world the stock price does not grow at the risk-free rate.[3] As long as we use our two 'tricks', they compensate for each other and enable

[3] Valuing an option using the BOPM with q as the probability of an 'up' move is consistent with the assumption that the growth in the stock price is equal to the risk-free rate r. Thus, in moving from the real world to a

us to obtain the 'fair' or 'correct' (i.e. no-arbitrage) option price, in the real world. What do we mean by 'correct'? We mean a price for the option that does not allow any risk-free arbitrage profits to be made in the 'real world' – now that does sound sensible and 'real'.

Finally note that in Equation (21.6a) neither the *actual* probability p of a rise in the stock price nor the real world growth rate of the stock μ, nor the risk preferences of investors, enter into the calculation of the price of the call. Hence all investors, regardless of their differing degrees of risk aversion or their different guesses about the 'real world' probability of a rise or fall in future stock prices can agree on the 'fair' or 'correct' price for a call option.

21.2.1 RNV and No-arbitrage

We can use some more tricks. If q really is a risk-neutral probability, then in a risk-neutral world, the expected return on a stock must equal the risk free rate and hence q must satisfy:

$$E^*(S_1) = (SU)q + (1-q)SD = S\ R \tag{21.8}$$

From Equation (21.8) we can immediately deduce that $q = (R-D)/(U-D) = 0.75$ which we know to be true from our 'no-arbitrage' approach. Similarly, in *a risk-neutral world* the call option has a $q = 0.75$ chance of being worth $C_u = 10$ and a 0.25 chance of being worth $C_d = 0$. Hence its expected value (at $t = 1$) in this risk-neutral world is:

$$E^*(C_1) = 0.75\ (10) + 0.25\ (0) = 7.5 \tag{21.9}$$

Discounting at the risk-free rate, the call premium at $t = 0$ in a risk-neutral world is:

$$\mathrm{C} = (1/R)\ E^*(C_1) = (1.05)^{-1} 7.5 = 7.143 \tag{21.10}$$

Thus, using RNV the BOPM Equation (21.6a) gives *the same value* for C as when we use the full 'no-arbitrage' approach. This establishes the equivalence of the two approaches. When pricing an option, if you interpret q as the probability of an 'up' move, this is equivalent to assuming the expected stock return is equal to the risk-free rate r – that is, you are in a risk-neutral world. However, the resulting value for the option premium using Equation (21.6a) is valid in the real world, since Equation (21.6a) is consistent with no risk-free arbitrage profits. This is the principle of RNV.

Using RNV we could say that the call premium at any time $t-1$ is given by:

$$C_{t-1} = (1/R)(E^* C_t) \tag{21.11}$$

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risk-neutral world the expected return on the stock changes from its real world expected return of μ to the risk-neutral return r. This general result is known as *Girsanov's Theorem* and the move from the 'risky' real world (of p and μ) to the risk-neutral world of q and r is known as a *change of measure*.

where $E^*(C_t)$ is the expected payoff to the option at time t, and the expectation $E^*(C_t)$ assumes we are in a risk-neutral world – that is, the stock price grows at the risk-free rate. This is also the basis for pricing options using Monte-Carlo simulation (MCS), as we see later. MCS involves simulating the stock price assuming the expected stock return equals the risk-free rate ($\mu = r$), calculating the expected payoff from the option at maturity T and discounting this expected payoff using the risk-free rate.

21.3 DETERMINANTS OF CALL PREMIUM

21.3.1 Call Premium and Stock Returns

Somewhat counter-intuitively the binomial pricing formula implies that if we have two otherwise identical call options (i.e. same strike price, expiration date, and same stock return volatility) but the underlying stock-A for one of the options has a 'real world' *expected return* of $\mu = 0$, while the other stock-B, has an expected return of $\mu = 100\%$ p.a. (say), then the two options will have exactly the same call price. This is because in each case we can create a risk-free hedge portfolio, which by arbitrage arguments can only earn the risk-free rate. Put another way, the call premium is independent of the 'real world' *expected return* of the underlying stock, μ.

21.3.2 Call Premium and Volatility

Note, however, that we are *not saying* that the call premium is independent of the *volatility* of the underlying stock (represented in the BOPM by $S_u - S_d$ or equivalently '$U - D$', which enters the definition of q).[4] Expected growth and volatility are very different concepts. After all, we can have a stock price with an *expected* growth (return) of zero but we may feel that the range of possible outcomes (around its expected value of zero) is very large. In our simple one-period model the call premium *does* depend on the *range of possible values* for S – that is on U and D – which measure the volatility of the stock price in the BOPM (and as we have seen, volatility also plays a key role in the Black–Scholes option price formulas).

To show that the call premium depends positively on the volatility of the stock, go back to our original example where $S_u = 110$ and $S_d = 90$ and the 'volatility' $S_u - S_d = 20$. Now change the volatility of the stock price, by assuming $S_u = 120$ ($U = 1.2$) and $S_d = 80$ ($D = 0.8$), so the volatility increases to $S_u - S_d = 40$. The 'real world' expected stock price (at $t = 1$) with $p = 1/2$ is $S_u - S_d = 20$ and the 'real world' expected return remains unchanged at $\mu = (ES_1/S_0) - 1 = 0\%$.

[4]As we see in Chapter 22, volatility actually depends on '$\ln U - \ln D$', rather than '$U - D$'.

The payoff to the call is either $C_u = 20$ or $C_d = 0$. Also $q = (1.05 - 0.8)/(1.2 - 0.8) = 0.625$ and hence using Equation (21.6a) the call premium $C = (0.625)20/1.05 = 11.905$,[5] which is higher than the call premium of $C = 7.1428$ (when we assumed stock price volatility was lower (i.e. $S_u - S_d = 20$)).

Of course, the above examples have two key simplifying assumptions, namely that there are only two possible outcomes for the stock price and that the option expiration is after one period. However, by extending the number of branches in the binomial 'tree' we can obtain a large number of possible outcomes for the stock price. If we consider each 'branch' as representing a short time period, then conceptually we can see how the BOPM 'approaches' a continuous time formulation, which forms the basis of the famous Black–Scholes approach.

21.4 PRICING A EUROPEAN PUT OPTION

We can price a one-period put option by constructing a similar risk-free portfolio of stocks and the put. But this time the risk-free portfolio-B is obtained by going *long* some stocks and simultaneously going *long* one put. So, if S falls the loss on the stock will be offset by the gain on the put, making the portfolio of 'stock+put', risk-free. The payoff to the put is max $(0, K - S_T)$. If $K = 100$ (as before, Figure 21.2) then for $S_u = 110$ the put payoff is $P_u = 0$ and for $S_d = 90$ we have $P_d = 10$.

The hedge ratio is $h_p = -(P_u - P_d)/(S_u - S_d) = 1/2$. For each long put option a delta hedge requires the *purchase of ½ stock*. The cost of setting up the delta hedge at $t = 0$ is $(1/2)S + P$ and the payoff at $t = 1$ is:

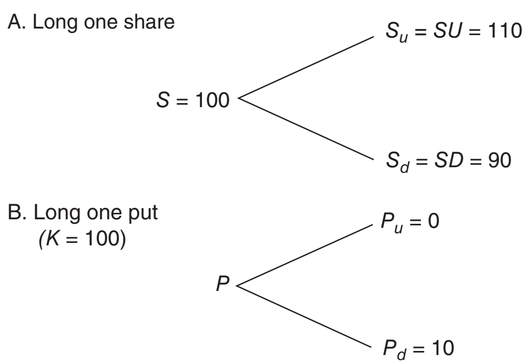

FIGURE 21.2 One-period BOPM – put

[5] Again we can establish the call premium by forming a risk-free arbitrage portfolio. The hedge ratio $h = (C_u - C_d)/(S_u - S_d) = 20/40 = 1/2$. The cost of the hedge portfolio at $t = 0$ is $(1/2)S - C = 1/2\,(100) - C$. The payoff at $t = 1$ is certain and is either $(1/2)S_u - C_u = 10$ or $(1/2)S_d - C_d = 40$. Since the outcome at $t = 1$ is the same whether the stock price goes 'up' or 'down', then for no risk-free arbitrage opportunities (see Equation (21.5)) we require $(1/2\,(100) - C)\ 1.05 = 40$, which we can solve to give $C = 11.905$.

Payoff to Portfolio-B: long 1/2-stock + long 1-put

$$\text{Payoff for price rise at } t = 1: \ 1/2(110) + 0 = 55$$
$$\text{Payoff for price fall at } t = 1: \ 1/2(90) + 10 = 55$$

Equating the return on the risk-free hedge portfolio-B to the risk-free rate gives:

$$\frac{\textit{Certain payout at } t = 1}{\textit{Cost of initial investment at } t = 0} = (1 + \text{risk-free rate})$$
$$\frac{55}{(1/2)100 + P} = 1.05 \tag{21.12}$$

Hence $P = 2.38$. Alternatively using the BOPM risk-neutral valuation formula, we directly obtain the put premium:

$$P = \frac{1}{R}\left[qP_u + (1-q)P_d\right] = \frac{(0.75)0 + (0.25)10}{1.05} = 2.381 \tag{21.13}$$

where (again) $q = (R-D)/(U-D) = 0.75$. Equation (21.13) has the same form as that for the call premium except that we use the put payoffs P_u and P_d. The BOPM put premium $P = 2.381$ can be checked by using put–call parity.

$$P = C - S + K/(1+r) = 7.143 - 100 + 100/(1.05) = 2.381$$

21.5 SUMMARY

- By constructing a *risk-free portfolio* consisting of the option and the underlying stock and *delta hedging,* the BOPM can be used to determine the 'no-arbitrage' call and put premia.
- The hedge ratio is given by the delta of the option.
- The call and put premia are determined by (i) the current stock price S, (ii) the risk-free rate $R = 1 + r$, (iii) U and D (which can be shown to determine the stock return volatility), (iv) $q = (R-D)/(U-D)$, the risk-neutral probability, and (v) the time to maturity, T. The option premia do *not* depend on the real-world expected stock return, μ.
- The parameter q can be *interpreted* as a risk-neutral probability of an 'up' move and two key results follow. First, the option premium (C or P) given by the BOPM is the *expected* payoff to the option at T, using risk-neutral probabilities q and the payoff is discounted at the risk-free rate. Second, the BOPM formula Equation (21.6) derived via arbitrage *is consistent with* the assumption that the expected growth rate of the stock price equals the risk-free rate. This is risk-neutral valuation (RNV).

- RNV provides a way of obtaining the correct value for option premia using the BOPM Equation (21.6), which involves backward recursion – this considerably simplifies the calculations. But behind this approach is the assumption that options traders have eliminated any risk-free arbitrage opportunities.
- RNV also leads to other useful approaches to pricing options such as Monte Carlo simulation.

APPENDIX 21: NO-ARBITRAGE CONDITIONS

We show that there are arbitrage opportunities if the inequalities $U > R > D$ do not hold for the underlying asset (stock), with current price S. First, note that it is always the case that $U > D$ (by construction). Consider the arbitrage opportunity when $D > R$ (and hence $U > R$). Here the outcome for either U or D is greater than the cost of borrowing. Hence at $t = 0$, borrow S (e.g. bank loan) and buy the stock for S – the net cash flow is zero (Table 21.A.1). The outcome for either U or D at $t = 1$ is a debt to the bank of RS. But at $t = 1$ the value of the long position in the stock is either SU or SD and the net position is either $S(U - R) > 0$ or $S(D - R) > 0$. This is an arbitrage opportunity as the net cash flow at $t = 0$ is zero but there is a positive cash flow at $t = 1$, for both the U and D outcomes.

For $R > U > D$, the risk-free rate exceeds the return on the stock for both the U and D outcomes. The arbitrage strategy at $t = 0$ involves short-selling the stock at S and investing the proceeds (in a bank deposit) which accrues to RS (Table 21.A.2). The cost of buying back the stock at $t = 1$ is either SU or SD. Hence there is a zero cash flow at $t = 0$ and a positive cash flow of either $S(R - U) > 0$ or $S(R - D) > 0$ at $t = 1$.

TABLE 21.A.1 Case A: $U > D > R$

Action	Cash Flow, $t = 0$	Cash Flow 'Up-Move', $t = 1$	Cash Flow 'Down Move', $t = 1$
Borrow \$$S$ @ R	$-S$	$-SR$	$-SR$
Buy stock @ S	$+S$	SU	SD
Net cash flow	0	$S(U - R) > 0$	$S(D - R) > 0$

TABLE 21.A.2 Case B: $D < U < R$

Action	Cash Flow, $t = 0$	Cash Flow 'Up-Move', $t = 1$	Cash Flow 'Down Move', $t = 1$
Short-sell stock @ S	$+S$	$-SU$	$-SD$
Invest \$$S$ @ R	$-S$	SR	SR
Net cash flow	0	$S(R - U) > 0$	$S(R - D) > 0$

EXERCISES

Question 1

Show that the risk-neutral probability $q = (R - D)/(U - D)$ in the (one-period) BOPM is consistent with the assumption that the underlying stock price S (which pays no dividends) grows at the risk-free rate. $R = 1 + r$, where r = risk-free rate (per period, simple interest).

Question 2

In the context of the equation to determine the call premium in the *one-period* BOPM, explain and interpret the equation for the call premium, using the concept of a risk-neutral probability.

Question 3

In the one-period BOPM the current stock price $S = 90$, $U = 1.1$ and $D = 0.9$. A one-period European call option (on a non-dividend paying stock) has $K = 90$. The risk-free interest rate is $r = 2\%$ per period (simple interest).

(a) Form a risk-free (hedge) portfolio, assuming you sell one call and hedge using stocks. Price the call option by showing that there are zero profits from your hedge portfolio at $t = 1$.

(b) What is the value of the hedge portfolio at $t = 1$ (for the two stock price outcomes)?

(c) What is the return on the hedge portfolio between $t = 0$ and $t = 1$?

Question 4

In the one-period BOPM the current stock price $S = 90$, $U = 1.1$ and $D = 0.9$. A one-period European put option (on a non-dividend paying stock) has $K = 90$. The risk-free interest rate is $r = 2\%$ per period (simple interest).

(a) Form a risk-free (hedge) portfolio, assuming you buy one put and hedge using stocks. Price the put option by showing that there are zero profits from your hedge portfolio at $t = 1$.

(b) What is the value of the hedge portfolio at $t = 1$ (for the two stock price outcomes)?

(c) What is the return on the hedge portfolio between $t = 0$ and $t = 1$?

(d) If the price of a call option (same strike, underlying stock and time to maturity) is $C = 5.2941$, show that the put premium satisfies put–call parity.

Question 5

In the one-period BOPM the current stock price $S = 90$ and $U = 1.1$ and $D = 0.9$. A one-period European put option (on a non-dividend paying stock) has $S = 90$. The risk-free interest rate is $r = 2\%$ per period (simple interest).

The return on a (non-dividend paying) stock in a risk-neutral world must equal the risk-free interest rate – use this condition to calculate the risk-neutral probability, q. Use risk-neutral valuation (RNV) to calculate the price of the put.

Question 6

In the one-period BOPM the current stock price $S = 90$, $U = 1.1$ and $D = 0.9$. A one-period European put option has $K = 90$. The risk-free interest rate is $r = 2\%$ per period (simple interest). The no-arbitrage price of the put is $P = 3.5294$.
If a market maker is quoting a price for the put of $P_q = 3.0$, show how you can make a risk-free arbitrage profit.

CHAPTER 22

BOPM: Implementation

Aims

- To show how *dynamic* delta hedging can be used to price a (two-period) call option, using a portfolio comprising stocks and calls, which is risk-free over small intervals of time.
- To show how dynamic delta hedging is consistent with the no-arbitrage binomial pricing equation – the latter is a backward recursion that can be interpreted using risk-neutral valuation (RNV).
- To *replicate* the payoff to an option, using stocks and (risk-free) borrowing or lending (e.g. using a bank deposit/loan). This provides an alternative derivation of the binomial formula for options.
- To show that as each time-step in the binomial tree becomes smaller, the tree more closely approximates a continuous time process (Brownian motion) for the stock price – as used in the Black–Scholes approach. Hence, as we increase the number of time periods in the binomial tree, the option price calculated from the BOPM formula converges to the Black–Scholes price.

Using insights from RNV, we price a two-period call option using the BOPM without going through all the details involved in delta hedging and forming a 'risk-free arbitrage portfolio' – instead we price the option assuming RNV, using a 'backward recursion'. This allows us to generalise the BOPM to many periods and to price many different types of option. We show that RNV is consistent with there being no arbitrage opportunities at any node in the binomial tree.

We demonstrate another method of pricing an option using a 'replication portfolio'. We construct a portfolio consisting of stocks and risk-free borrowing or lending, which replicates

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

the payoffs to the option. The price of the option must then equal the cost of setting up this replication portfolio, otherwise risk-free arbitrage profits can be made.

22.1 GENERALISING THE BOPM

We extend the BOPM (analysed in Chapter 21) and price a two-period call option with strike $K = 100$. The current stock price is $S = 100$, the one-period risk-free rate $r = 0.05$ and $C =$ the *unknown* call premium. As previously, we take $U = 1.1$ and $D = 0.9$ which gives the stock price tree and values for a long call at expiration as indicated in Figure 22.1.

Because we create a risk-free hedge portfolio at each node of the binomial tree, we can invoke RNV and use 'backward recursion' to calculate C from the *known* values of C_{uu}, C_{ud} and C_{dd}. For example, from the two upper branches in Figure 22.1 we have:

$$C_u = \frac{1}{R}[qC_{uu} + (1-q)C_{ud}] = \frac{0.75(21) + 0.25(0)}{1.05} = 15 \tag{22.1}$$

where $q = (R - D)/(U - D) = 0.75$. From the two lower branches:

$$C_d = \frac{1}{R}[qC_{ud} + (1-q)C_{dd}] = 0 \tag{22.2}$$

We can now solve for C, the call premium for this two-period case:

$$C = \frac{1}{R}[qC_u + (1-q)C_d] = \frac{0.75(15) + 0}{1.05} = 10.7142 \tag{22.3}$$

Note that the call premium for the option with two periods to maturity has a higher value than our 'identical' option with one period to maturity, where we found $C = 7.143$

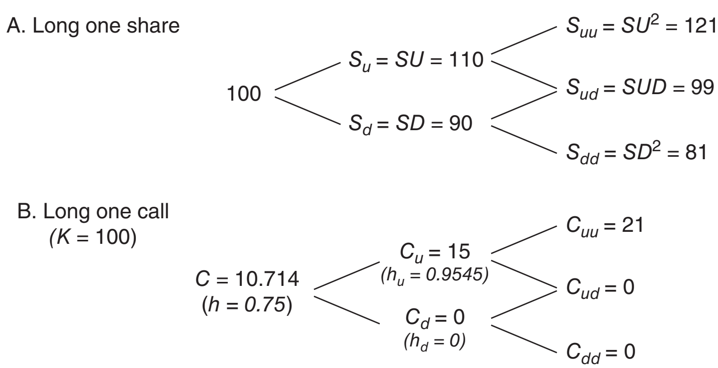

FIGURE 22.1 Two-period BOPM

(see Chapter 21). Backward recursion (under RNV) is the easiest way of obtaining the option price. If we just consider *European* options, (where only the payoff at maturity determines the value of the option), then RNV provides a general formula for pricing calls and puts.

22.1.1 Many Periods

Equations (22.1) and (22.2) give the values of C_u and C_d in terms of the final payoffs C_{uu}, C_{ud} and C_{dd} and if we substitute (22.1) and (22.2) in (22.3) we obtain:

$$C = \frac{1}{R^2}[(1)q^2C_{uu} + 2q(1-q)C_{ud} + (1)(1-q)^2C_{dd}] \tag{22.4}$$

The European option price is equal to the expected value (using risk-neutral probabilities) of the option payoffs at maturity, discounted at the risk-free rate of interest.

The '2' in the middle of Equation (22.4) represents the two *possible paths* to achieve the stock price SUD (that is paths UD and DU) and the '1's' represent the single path to achieve either SU^2 or SD^2. We *interpret* q as the risk-neutral probability of an 'up move' for S and $(1-q)$ the probability of a 'down move'. The (risk-neutral) probabilities of achieving the outcomes UU, UD or DU and DD are $q^2 = 0.5625$, $2q(1-q) = 0.375$ and $(1-q)^2 = 0.0625$, respectively. In general the number of possible *paths* to any final stock price are given by the binomial coefficients.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{22.5}$$

where n is the number of periods in the binomial tree, k is the number of upward price movements and $n! = n(n-1)(n-2)\ldots 1$ and $0! = 1$. Let's try out Equation (22.5) for $n = 2$:

Number of paths with $k = 2$ 'ups'	=	$(2!)/(0!\ 2!) = 1$	(i.e. UU)
Number of paths with $k = 1$ 'ups'	=	$(2!)/(1!\ 1!) = 2$	(i.e. UD or DU)
Number of paths with $k = 0$ 'ups'	=	$(2!)/(2!\ 0!) = 1$	(i.e. DD)

The reader might like to draw a tree with $n = 3$ periods (with 8 possible final outcomes *UUU, UUD, UDU, UDD, DUU, DUD, DDU, DDD*), and verify that the number of possible paths to achieve $k = 1$ 'up' moves is $\binom{n}{k} = (3!)/(2!\ 1!) = 3$ and these are *UDD, DUD*, and *DDU*. This can also be repeated using Equation (22.5) for $k = 2$ or 3 'up' moves. In general, over n-periods in the tree, the BOPM formula for a European call option is:

$$C = \frac{1}{R^n}\sum_{k-0}^{n}\binom{n}{k} q^k(1-q)^{n-k}\max[SU^kD^{n-k} - K, 0] \tag{22.6}$$

The probability of the stock price reaching the value $SU^k D^{n-k}$ after n-periods is $\binom{n}{k} q^k(1-q)^{n-k}$. Note that the term in square brackets in (22.6) is just another way of writing the payoffs at the final nodes. For example, for $n = 2$, these are:

$$C_{uu} = \max[0, SU^2 - K], \quad C_{ud} = \max[0, SUD - K], \quad C_{dd} = \max[0, SD^2 - K] \tag{22.7}$$

> *The hedge ratios in the BOPM can be calculated at each node and this is done for $n = 10$ time periods in the Excel file provided. The call option premium using RNV and backward recursion is found to be $C = 39.087$.*

The price of a put option is also given by Equation (22.6) but with the term *max[...]* replaced by the sequence of put-option payoffs, namely *max[*$0, K - SU^k D^{n\text{-}k}$*]*. Equation (22.6) indicates that the price of an option depends on the strike price K, the underlying asset price S, the risk-free rate r and the asset's volatility (which is determined by U and D), but it does not depend on the risk preferences of individuals or the 'real-world' probability of a price increase/decrease or the real world expected return on the stock. Below we show that as the number of nodes n in the tree increases, we obtain a more accurate value for the option price and $n > 30$ generally gives reasonably accurate results for plain vanilla European options.

22.1.2 Where Do *U* and *D* Come From?

At $t = 0$, r, K, and S are known. Above we have shown that if we know U and D (and hence $q = (R - D)/(U - D)$), then we can price an option by invoking RNV. It can be shown that the size of U and D are determined by the *actual real-world* volatility of the stock return, and one method of achieving this is known as the Cox-Ross-Rubinstein (CRR) parameterisation:

$$U = e^{\sigma\sqrt{dt}} \quad \text{and} \quad D = e^{-\sigma\sqrt{dt}} \tag{22.8}$$

where σ = the observed *annual* standard deviation of the (continuously compounded) stock return (decimal), T is the time to expiration of the option in years (or fraction of a year), n is the number of steps chosen for the binomial tree so $dt = T/n$ is a small interval of time. For example, if the expiration date of the option is at $T = 1/4$ year (3 months) and we choose a binomial tree with $n = 30$ steps, then $dt = 0.008333$ years (i.e. dt represents about 3 days out of a total of 365 calendar days per year).

Given Equation (22.8), note that the 'spread' of the binomial lattice/tree (in percentage terms) at any two adjacent points (in a vertical direction) is $\ln(SU) - \ln(SD) = 2\sigma\sqrt{dt}$, so the proportionate gap between U and D (i.e. $\ln(U/D)$) does depend directly on the 'real world' value of σ. Our particular choice for U and D imposes symmetry that is $U = 1/D$ but it can be shown that this is not restrictive if our aim is to construct a 'risk-neutral' lattice. Note also that when $U = 1/D$, the nodes SUD and SDU both have a value equal to S and the lattice recombines. For example, if S (at $t = 0$) is 100 it will also be 100 in the middle node at $t = 2$. Finally, note that U and D do *not* depend on the real world expected return on the stock and hence neither does the option premium – this is RNV again.

We now have a very useful method of pricing a European call option (say), using RNV and backward recursion through the binomial tree:

- Choose $n > 30$ and divide the time to maturity T (in years) of the call option into small time intervals each representing $dt = T/n$ (years).
- If r (decimal) is the (annual) continuously compounded interest rate then $R = e^{-r\,dt}$.
- Construct the tree for the stock price using $U = e^{\sigma\sqrt{dt}}$ and $D = e^{-\sigma\sqrt{dt}}$ – this ensures the volatility of the stock return mimics its real world volatility.[1]
- Calculate the possible final payoffs, $\max[0, S_T - K]$ at T for the call option.
- Use $q = (R - D)/(U - D)$ to calculate the *expected* payoffs for the call – this is RNV.
- Undertake backward recursion through the tree, discounting the option values at the risk-free rate each period (this is RNV again), to finally obtain the option price at $t = 0$.

Although the above recursive method is very useful, it is worth remembering that the reason it works is because 'behind the scenes', at each node of the tree, we are implicitly assuming options traders are forming a risk-free delta-hedged portfolio so that no arbitrage profits are possible at any node – this is examined further in Appendix 22.

The option price will change as the stock price, stock return volatility, interest rate or the time to maturity change over time. We can calculate the (approximate) change in the price of the option using the option's 'Greeks' which include not only delta but the option's gamma, vega, theta, and rho. Calculation of the Greeks for the BOPM is explained in Chapter 28.

22.2 REPLICATION PORTFOLIO

22.2.1 Replicating a Long Call: One-period BOPM

In our original example, we priced the (one-period) call option by establishing a risk-free portfolio consisting of a written call and a long position in 'delta' stocks. We can also price the

[1]The observant reader will have noted that we have not used this condition so far, yet we still obtained the correct (no-arbitrage) option prices using the binomial equation and the RNV approach. We get the correct option price because RNV does not allow any arbitrage profits to be made. What we have not done so far is to match the volatility of the stock price in the tree to its real world volatility σ, as measured by statisticians – this is because up to this point, for expositional purposes, we wanted to keep the numbers in the tree simple whole numbers, hence our choice of $U = 1.1$ and $D = 0.9$ in our initial examples.

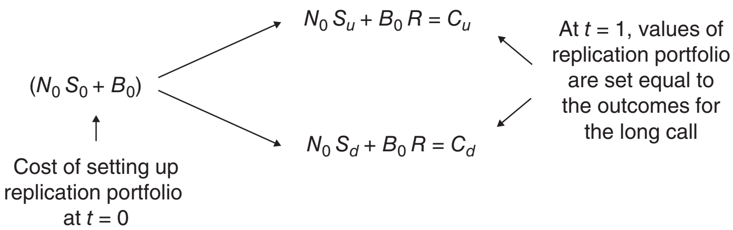

FIGURE 22.2 Replication portfolio

call by establishing a *synthetic call* or a *replication portfolio* for the call, using stocks and the risk-free asset. We combine stocks and the risk-free asset at $t = 0$ into a 'replication portfolio' which gives exactly the same payoffs as the call C_u, C_d at $t = 1$. Because our 'replication portfolio' has the same payoff as the call (at $t = 1$), then the price of the call must equal the cost of setting up the 'replication portfolio' (at $t = 0$) – otherwise risk-free arbitrage profits are possible.

Consider purchasing N_0 stocks at a price S_0 and buying \$$B_0$ of risk-free (zero-coupon) bonds with a return $R = (1 + r)$ – see Figure 22.2. When $B_0 > 0$ this implies a bond purchase (lending money) and $B_0 < 0$ implies issuing bonds (borrowing money). Hence, $B_0 < 0$ could just as easily be the amount borrowed in the form of a bank loan and $B_0 > 0$, represents the amount placed in a bank deposit.

Replication Portfolio-A: Stocks plus Bonds

$$\text{At } t = 0: \text{ Cost of Replication Portfolio-A} = (N_0\, S_0 + B_0) \tag{22.9}$$

At $t = 1$, portfolio-A is worth either $(N_0 S_u + B_0 R)$ or $(N_0 S_d + B_0 R)$ and to replicate the payoff from a long call, we set these two values equal to C_u and C_d, respectively:

$$(N_0 S_u + B_0 R) = C_u \tag{22.10a}$$

$$(N_0 S_d + B_0 R) = C_d \tag{22.10b}$$

Subtracting Equation (22.10b) from (22.10a) gives:

$$N_0 = (C_u - C_d)/(S_u - S_d) \tag{22.11}$$

From (22.10a) and (22.10b),

$$B_0 = \frac{C_u - (N_0 S_u)}{R} = \frac{C_d - (N_0 S_d)}{R} \tag{22.12}$$

Substituting N_0 from Equation (22.11) into Equation (22.12) and using $S_u = SU$, $S_d = SD$:

$$B_0 = \frac{S_u\, C_d - S_d\, C_u}{R(S_u - S_d)} = \frac{U\, C_d - D\, C_u}{R(U - D)} \tag{22.13}$$

It is easy to see that the two expressions for B_0 in Equation (22.12) are equal by noting that they imply $(C_u - C_d) = N_0(S_u - S_d)$ and given the definition of N_0 in (22.11) these two expressions must be equal. As the portfolio of N_0 stocks and B_0 (dollars) bonds is constructed to replicate the payoff of the call option at $t = 1$, then the call premium C_0 (at time $t = 0$) must equal the cost of the replication portfolio at $t = 0$ (otherwise arbitrage profits could be made):

$$\textit{Cost of Replication Portfolio}: (N_0 S_0 + B_0) = C_0 \tag{22.14}$$

Substituting in (22.14) for N_0 from (22.11) and for B_0 from (22.13), then after some manipulation we obtain:

$$C_0 = N_0 S_0 + B_0 = \frac{1}{R}[qC_u + (1-q)C_d] \tag{22.15}$$

where $q = (R - D)/(U - D)$. The number of stocks N_0 in Equation (22.11) required to replicate the payoffs of the call, is the hedge ratio h in our earlier derivation.

22.2.2 Replicating a Long Call: Two-period BOPM

Now we use this approach to replicate the option values in a two-period lattice using stocks and the risk-free asset. Consider what is happening at $t = 0$ (Figure 22.1). From (22.11) and (22.12) we have:

$$N_0 = \frac{C_u - C_d}{S_u - S_d} = \frac{15 - 0}{100(0.2)} = 0.75 \qquad \text{(as before)}$$

$$B_0 = \frac{C_u - (N_0.S_u)}{R} = \frac{15 - 0.75(110)}{1.05} = -64.286$$

Note that here we are *replicating* the payoff of the *long* call (at $t = 1$) with a *long* position in 0.75 of stocks and a *short* position in the bond (i.e. borrowing cash). At $t = 0$ the replication portfolio consists of borrowing \$64.286 and purchasing \$75 of stocks ($N_0 S_0 = 0.75 \times \$100$). This is a net investment of \$10.714 which, not surprisingly, we have earlier found is the value of the option premium C (at $t = 0$) for a two-period call. The outcome in the 'up' and 'down' nodes for our 'replication portfolio' are:

$$\begin{aligned} \textit{Node-U:}\quad N_0 S_u + B_0 R &= 0.75(110)–64.286(1.05) = 82.5–67.5 = 15 \\ \textit{Node-D:}\quad N_0 S_d + B_0 R &= 0.75(90)\ –64.286(1.05) = 67.5–67.5 = 0 \end{aligned}$$

which, of course, are the outcomes for the value of the call, $C_u = 15$, and $C_d = 0$ at the first two nodes. We now rebalance our replication portfolio so at the U-node:

$$N_u = \frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}} = 0.9545$$

$$B_u = \frac{C_{uu} - (N_u S_{uu})}{R} = \frac{21 - 0.9545(121)}{1.05} = -90$$

The reason for borrowing \$90 at node U is that you must increase the number of stocks by (0.9545 – 0.75) = 0.2045 at a price of \$110 per stock, giving a total cost of \$22.5, which when added to your existing debt of 67.5 brings your debt to \$90. The outcomes for the replication portfolio when moving from the U-node to the nodes UU and UD are:

$$\textit{Node-UU: } N_u S_{uu} + B_u R = 0.9545(121) - 90(1.05) = 115.5 - 94.5 = 21$$

$$\textit{Node-UD: } N_u S_{ud} + B_u R = 0.9545(99) - 90(1.05) = 94.5 - 94.5 = 0$$

Again we have replicated the value of the call at these two nodes (see Figure 22.2). Finally, consider the D-node. Here $N_d = (C_{ud} - C_{dd})/(S_{ud} - S_{dd}) = 0$ and the replication portfolio consists of zero stocks and is entirely composed of bonds $B_d = C_{ud}/R$ but because $C_{ud} = 0$ (Figure 22.1), we hold no bonds at the D-node. The replication portfolio at node-D is therefore worth zero – but this exactly replicates the value of the call at the nodes UD and DD (Figure 22.1).

Naturally, we obtain the same BOPM formula for the price of the call using either the 'replication portfolio' of stocks plus bonds or by using our earlier 'delta hedge' risk-free portfolio.

22.3 BOPM TO BLACK–SCHOLES

By increasing the number of steps n, in the binomial tree and seeing what happens to the option price in the BOPM, we obtain some insight into the Black–Scholes pricing formula for European options. As we increase the number of steps we are also shortening the time interval between each node in the binomial tree $dt = T/n$, so the BOPM becomes 'more like' the Black–Scholes approach, which uses continuous time mathematics, and the option price given by the BOPM formula converges towards the Black–Scholes price. Suppose we have:

$$S = 90 \qquad K = 100 \qquad r = 0.10 \qquad \sigma = 0.20 \qquad T = 0.3 \text{ (years)}$$

then the Black–Scholes formula gives a call premium $C^{BS} = 5.33$. To translate these inputs into the BOPM we use $dt = T/n$ where n is the number of steps in the binomial tree. We then calculate U, D and R as follows:

$$U = e^{\sigma\sqrt{dt}} \qquad D = e^{-\sigma\sqrt{dt}} \qquad R = e^{r\,dt}$$

For example, for $n = 1$ we have:

$$dt = T/n = 0.3/1 = 0.3 \qquad R = e^{r\,dt} = 1.0304$$
$$U = e^{0.20\sqrt{0.3}} = 1.0618 \qquad D = e^{-0.20\sqrt{0.3}} = 0.9418$$

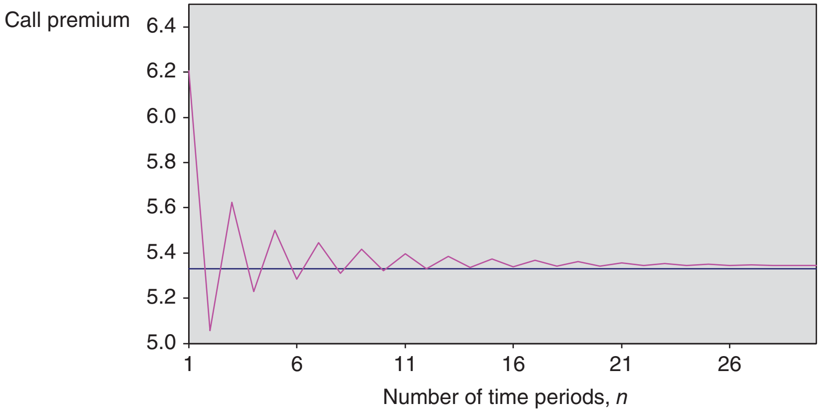

FIGURE 22.3 Call premium – BOPM and Black–Scholes

The call premium given by the BOPM using only one time-step is:

$$C^{(1)} = \frac{qC_u + (1-q)C_d}{R} = 6.21 \tag{22.16}$$

where $q = (R-D)/(U-D)$, $C_u = \max(SU-K, 0)$, $C_d = \max(SD-K, 0)$. For $n = 1$ then we have $C^{(1)} = 6.21$ which is not particularly close to the Black–Scholes value $C^{BS} = 5.33$.

However, as we apply the recursive binomial Equation (22.6) for $n = 2, 3, \ldots$, etc. and $dt = T/n$, the binomial call price for $n = 30$ is $C^{(30)} = 5.345$, which is close to the Black–Scholes value of $C^{BS} = 5.33$ – see Figure 22.3. In general, for *plain vanilla* European options (but not necessarily for complex exotic options) choosing $n = 30$ in the BOPM gives reasonably accurate results for the option price. The CRR parameterisation oscillates between over- and under-approximations (which are approximately symmetric) and which gradually dampen as the number of steps in the tree increases. The average of these over- and under-approximations converges rapidly towards the Black–Scholes price – for example, using only $n = 20$ steps in the binomial tree we have $C^{(20)} = 5.3615$ and $C^{(21)} = 5.3422$ and the average of the two is 5.3518, which is very close to $C^{BS} = 5.33$.

The Excel file which demonstrates that the BOPM solution for a European option does approach the Black–Scholes price as n becomes large, can be found on the website.

Of course, one problem with a numerical method like the BOPM is that it may not converge quickly and the solution can 'bounce around' the 'correct' option price given by Black–Scholes.

This is the price you pay for the flexibility of the binomial approach. The option premium from the BOPM approaches that given by the Black–Scholes formula, as the number of steps increases (i.e. $n \to \infty$ and hence $dt = T/n \to 0$). The 'up–down' lattice of the BOPM then has many nodes and there are many possible paths the stock price could take (e.g. for just three nodes you can have eight possible paths (UUU, UUD, UDD, DDD, etc. – see below)). Hence as $dt = T/n \to 0$ the BOPM lattice approximates the geometric Brownian motion used by Black, Scholes and Merton in deriving the pricing formulas for European options.

Also notice that when the number of nodes in the binomial tree increases, the possible outcomes for stock prices in the *final period* (T) begin to look more like a 'normal curve'. For example, with a probability of ½ for an 'up' move, $U = 1.1$, $D = 0.9$ and for $n = 3$ nodes the outcomes and probabilities are:

Path	**Probability**	**Final stock prices**
UUU	1/8	$SUUU = 133.1$
UUD,DUU,UDU	3/8	$SUUD = 108.9$
UDD,DDU,DUD	3/8	$SUDD = 89.1$
DDD	1/8	$SDDD = 72.9$

If the final stock prices are plotted in a histogram it looks (slightly) more like a 'normal curve' than if we just had $n = 1$ with two outcomes 110 and 90 (each with probability of ½). This is because for $n = 3$ the 'extreme' UUU and DDD outcomes each only occur 1/8th of the time but the central portion of the histogram for the paths with a one-up move or a one-down move, each occur 3/8ths of the time. In fact, as the number of nodes n increases (i.e. the time period between each node gets smaller) the 'histogram' for the final stock prices in the BOPM does approach a 'normal curve' – which is the assumption used in deriving the Black–Scholes formula.[2]

22.4 SUMMARY

- RNV provides a way of obtaining the BOPM formula for option premia using backward recursion, which considerably simplifies the calculations. But behind this approach is the assumption that options traders are able to undertake *dynamic* delta hedging to eliminate any risk-free arbitrage profits.
- For European options, the BOPM is a backward recursion starting with the option payoffs at maturity T, then calculating the expected value of the option at each node in the tree using risk-neutral probabilities and discounting these payoffs using the risk-free rate. Repeating this procedure as you move backwards through the tree, gives the 'correct' or 'no-arbitrage' option price.

[2]To be more accurate (continuously compounded, log) stock *returns* are normally distributed but the distribution of the final stock *price* is actually lognormally distributed. This distinction need not concern us here.

- The BOPM formula for the option premium can also be derived by *replicating* the payoffs to the option, using stocks and risk-free borrowing or lending (i.e. using either a risk-free bond or bank deposit/loan).
- In the BOPM the 'size' of the 'up' $U = e^{\sigma\sqrt{dt}}$ and 'down' $D = e^{-\sigma\sqrt{dt}}$ movements in the stock price depend on the 'real world' volatility of the stock return – and via q in the BOPM, the option price depends on the volatility of the stock return.
- The European option premium given by the BOPM, converges towards the Black–Scholes price, as the number of steps n in the binomial tree increases (so each time-step in the tree represents a smaller interval of time).
- The BOPM is a numerical technique, so it may suffer from convergence problems and only gives an approximation to the 'true price' – but it is a very flexible method which can be used to price exotic options.

APPENDIX 22: DELTA HEDGING AND ARBITRAGE

Given values for the call option determined by RNV in the two-period BOPM, we show that dynamic delta hedging ensures that no risk-free arbitrage profits can be made at each node in the tree. The hedge ratios at each node are easily calculated (see Figure 22.A.2).

$$h_u = \frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}} = \frac{21}{(121 - 99)} = 0.9545 \tag{22.A.1}$$

$$h_d = \frac{C_{ud} - C_{dd}}{S_{ud} - S_{dd}} = 0 \tag{22.A.2}$$

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{15 - 0}{110 - 90} = 0.75 \tag{22.A.3}$$

The hedge ratio at $t = 0$ is 0.75, then if the upper branch ensues, it rises to 0.9545 whereas on the lower branch it is zero. We show how we can maintain a delta-neutral position at each node of the tree and this implies our risk-free portfolio earns the risk free rate, $r = 5\%$ (per period). We assume a trader has written 1,000 calls (at $t = 0$) and she needs to delta-hedge this position with stocks.

$$\textit{At } t = 0\text{:} \quad h = 0.75, \quad C = 10.714, \quad S = 100$$

Write 1,000 calls and buy 750 stocks

$$\begin{aligned} \text{Buy 750 stocks @ \$100} &= \$75{,}000 \\ \text{Write 1,000 calls @ \$10.714} &= \$10{,}714 \\ \text{Net investment} &= \$64{,}286 \text{ ('own funds')} \end{aligned}$$

The outcomes at the *U*-node and *D*-node are:

U-Node: $C_u = 15, S_u = 110$

$$\text{Value of portfolio } V_u = 750\ (\$110)\text{–}1{,}000(\$15) = \$67{,}500$$

$$\text{Return over period-1} = \$67{,}500/\$64{,}286 = 1.05\ (5\%)$$

D-Node: $C_d = 0, S_d = 90$

$$\text{Value of portfolio } V_d = 750\ (\$90)\text{–}1{,}000(0) = \$67{,}500$$

$$\text{Return over period-1} = \$67{,}500/\$64{,}286 = 1.05\ (5\%)$$

The outcomes at the *D*-node and *U*-node are equal since the hedge is designed so that $V_u = V_d$. At the *U*-node, the new hedge ratio $h_u = 0.9545$. As we have 1,000 written options then we need to hold 954.5 stocks. Hence we buy an additional (954.5 – 750) stocks @ \$110 = \$22, 4995 using borrowed funds at an interest cost $r = 5\%$. The outcomes at the *UU*-node and *UD*-node are:

UU-node: $C_{uu} = 21, S_{uu} = 121$

$$\text{Value of stocks: } 954.5 \text{ @ } \$121 = \$115{,}494.50$$

$$\text{Less (written) call payoff: } 1{,}000 \text{ @ } \$21 = \$21{,}000$$

$$\text{Less loan outstanding: } \$22{,}495\ (1.05) = \$23{,}619.75$$

$$\text{Value of portfolio: } V_{uu} = \$70{,}874.75 \approx \$70{,}875$$

$$\text{Return over period-2}: = \$70{,}875/\$67{,}500 = 1.05 \text{ (or 5\%)}$$

UD-node: $C_{ud} = 0, S_{ud} = 99$

$$\text{Value of stocks: } 954.5 \text{ @ } \$99 = \$94{,}495.50$$

$$\text{Less (written) call payoff: } 1{,}000 \text{ @ } \$0 = \$0$$

$$\text{Less loan outstanding: } \$22{,}495\ (1.05) = \$23{,}619.75$$

$$\text{Value of portfolio: } V_{uu} = \$70{,}874.75 \approx \$70{,}875$$

$$\text{Return over period-2}: = \$70{,}875/\$67{,}500 = 1.05 \text{ (or 5\%)}$$

To reach the *D*-node from $t = 0$ we move from holding 750 stocks ($h = 0.75$) to zero stocks, since at node-D, $h_d = 0$. Selling 750 stocks at $S_d = 90$ results in a cash inflow of \$67,000. The 1,000 written calls sold at $t = 0$ are worth zero, at node-*DD*. (Notionally, we could buy back 1,000 calls at a cost of $C_d = 0$.) The cash inflow of \$67,000 is the same as V_d calculated above.

Explaining the move from node-D, to either node-DD or node-UD is trivial. We have $h_d = 0$ stocks which are worth zero at nodes UD and DD and the calls are also worth zero ($C_{ud} = C_{dd} = 0$).

Changing the Number of Calls in the Hedge

What would the hedge look like at node-U if we decided to change the number of calls (rather than the number of stocks) in order to maintain the delta hedge? At node-U the hedge ratio is $h_u = 0.9545$ and hence a hedged portfolio also consists of:

Hold the 'original' 750 stocks and write 785.7 calls (= 750 / 0.9545)

At the outset we sold 1,000 calls and at node-U the delta hedge requires 785.7 written calls, hence we must buy back 214.3 calls:

At node-U buy back 214.3 calls @ $15 = $3,214 (= borrowed funds)

We can show that delta hedging by changing the number of calls produces the same outcomes as our above analysis (i.e. with a fixed 1,000 written calls). For example, the outcome at the UU-node of our 'new' hedge is:

UU-node: $C_{uu} = 21, S_{uu} = 121$

$$\begin{aligned}
\text{Value of stocks: } 750 \text{ @ } \$121 &= \ \$90{,}750 \\
\text{Less call payoff: } 785.7 \text{ @ } \$21 &= \ \$16{,}500 \\
\text{Less loan outstanding: } 3{,}214\,(1.05) &= \ \$3{,}375 \\
\text{Value of portfolio: } V_{uu} &= \ \$70{,}875 \\
\text{Return over period-2: } \ \$70{,}875 \,/\, \$67{,}500 &= \ 1.05\,(5\%)
\end{aligned}$$

This is exactly the same payoff as when we hedged at node-U using a fixed 1,000 written calls and delta hedging with 954.5 stocks. In fact the latter is a more realistic outcome as options traders in banks tend to be net sellers of calls to their retail and corporate customers and they dynamically delta-hedge this position by changing their stock holdings, day-by-day until the maturity date of the option (or until they close out their options position prior to maturity).

Making arbitrage profits from a *mispriced* call with two periods to maturity is similar to that for the one-period case, except the 'arbitrage profit' may accrue in either or both of the two periods depending on when the mispricing is corrected. Clearly if the mispricing is not corrected in the first period, the 'delta-hedged' calls and stocks earn the risk-free rate. But in the second period the mispricing must be corrected, since the option reaches maturity. Then a return in excess of the risk-free rate is earned between $t = 1$ and $t = 2$ – hence over the two periods, the arbitrageur earns more than the risk-free rate.

For example, suppose a call is initially overpriced. At $t = 0$ you sell 1 call and buy h stocks. If the mispricing is not corrected in period-1 then you earn the risk-free rate. But if the mispricing is corrected, the call becomes correctly priced at the end of the first period, and you earn more than the risk-free rate over period-1 – in this case you earn the risk-free rate in period-2 since the mispricing has already been corrected.

EXERCISES

Question 1

How are the size of the 'up' moves (U) and 'down' moves (D) determined in the BOPM and how is the stock price volatility represented in the tree for the BOPM?

Question 2

For a binomial tree with $n = 3$ periods there are $2^3 = 8$ possible paths to arrive at the final values for the stock price.

(a) List these 8 different paths (to reach the stock prices at $n = 3$).
(b) How many *distinct* values for the stock price are there at $n = 3$?
(c) How many alternative ways (paths) are there to reach a node at $n = 3$ which has (i) two up moves, or (ii) two down moves, (iii) 3 up moves, (iv) 3 down moves? List these alternative paths.

Question 3

In the BOPM, when delta hedging an option position, why does the hedge position have to change as you move through the lattice?

Question 4

In the BOPM, give an intuitive interpretation using risk neutral valuation, of the formula for the put premium on a stock, with two periods to expiration:

$$P_0 = \frac{1}{R^2}[q^2 P_{uu} + 2q(1-q)P_{ud} + (1-q)^2 P_{dd}]$$

where $q = (R - D)/(U - D)$, $R = 1 + r$ and $r =$ risk-free rate.

Question 5

How can you use $\$B_0$ held in a risk-free asset (e.g. a zero-coupon bond or bank deposit/loan) together with N_0 stocks with current price S_0, to *replicate* the payoff to a one-period (long)

European put? What does this imply for the number of stocks to buy or sell to replicate the put payoff?

Briefly explain what happens in your replication strategy if the stock price increases by \$2 (over a short time horizon).

Question 6

Consider the two-period BOPM. The current stock price $S = \$105$ and the risk-free rate $r = 3\%$ per period (simple rate). Each period, the stock price can go either up by 10% or down by 10%. A European call option (on a non-dividend paying stock) with expiration at the end of two periods ($n = 2$), has a strike price $K = \$100$.

(a) Draw the stock price tree (lattice).
(b) Show that the (no-arbitrage) price of the call is 12.47.
(c) Calculate the hedge ratio at $t = 0$.
(d) Show how you can hedge 100 *written* calls at $t = 0$, and how the hedge portfolio earns the risk-free rate over the first period (i.e. along the path from node $t = 0$, either to node-D or node-U).
(e) What would an investor do at $t = 0$ if the call is overpriced at $C_g = 13$? What is the outcome at $t = 1$ (given that the stock price and the call premium are the same as in the lattice in parts (a) and (b))?

Question 7

Consider the two-period BOPM. The current stock price $S = \$105$ and the risk-free rate $r = 3\%$ per period (simple rate). Each period, the stock price can go either up by 10% or down by 10%. A European put option (on a non-dividend paying stock) expiring at the end of the second period has an exercise price of $K = \$100$.

(a) Sketch the stock price tree (lattice).
(b) Calculate the fair (no-arbitrage) price of the put, P.
(c) Calculate the hedge ratio h, at time zero.
(d) Show how you can hedge 100 long puts at $t = 0$, and how the hedge-portfolio earns the risk-free rate over the first period (i.e. along the path from node $t = 0$, either to node-D or node-U).
(e) What would an investor do at $t = 0$ if the put is overpriced at $P_g = 2$? What is the outcome at $t = 1$ (if the stock price and the put premium are the same as in the lattice in parts (a) and (b))?

CHAPTER 23

BOPM: Extensions

Aims

- To show how the BOPM is used to price American options – these are path-dependent options and subject to early exercise.
- To adapt the BOPM to price options on stocks that pay a continuous dividend, options on foreign exchange, and options on futures contracts.
- To use the BOPM to price options on stocks that pay dividends at discrete intervals.
- To demonstrate how the binomial approach can be speeded up using control variate techniques and trinomial trees.
- To show how stock price movements in the binomial tree are determined by the 'real world' volatility of stock returns.

23.1 AMERICAN OPTIONS

So far we have used the BOPM to price European options (which can only be exercised at maturity). American options can be exercised at any time, over the life of the option. The question arises as to when it is optimal to exercise an American option and how this affects the price of American options. The following results hold for American options:

- For a *call* option on a non-dividend paying stock, early exercise is never optimal.
- For a *call* option on a dividend paying stock, early exercise is sometimes optimal.
- For a *put* option on a stock (with or without dividends), early exercise is sometimes optimal.

As we shall see American options on stock indices, commodities, currencies and futures contracts can be priced using results for options on stocks that pay continuous dividends.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

23.1.1 European Put

We price a two-period *European* put with $K = 100$ using RNV. The tree for the stock price has $S = 100$, $U = 1.1$, $D = 0.9$, $R = 1 + r = 1.05$ and the risk-neutral probability $q = (R - D)/(U - D) = 0.75$. First we calculate the payoffs at maturity (Figure 23.1) $P_{uu} = \max(K - S_{uu}, 0) = 0$, $P_{ud} = \max(K - S_{ud}, 0) = 1$ and $P_{dd} = \max(K - S_{dd}, 0) = 19$. We then move backwards through the tree:

$$P_u = \frac{1}{R}[qP_{uu} + (1-q)P_{ud}] = \frac{0.75(0) + 0.25(1)}{1.05} = 0.238 \tag{23.1a}$$

$$P_d = \frac{1}{R}[qP_{ud} + (1-q)P_{dd}] = \frac{0.75(1) + 0.25(19)}{1.05} = 2.857 \tag{23.1b}$$

The (European) put premium is:

$$P = \frac{1}{R}[qP_u + (1-q)P_d] = \frac{0.75(0.238) + 0.25(2.857)}{1.05} = 0.850 \tag{23.2}$$

23.1.2 American Put

For an American put option the payoffs at maturity are the same as for the European put. But early exercise may be profitable at nodes U, D and at $t = 0$. To price the American put we see if the intrinsic value $IV = K - S$ (which is the cash received for early exercise) at any of the nodes is greater than the 'recursive values'[1] for the put, P_u or P_d (that is, the value of the put if you do

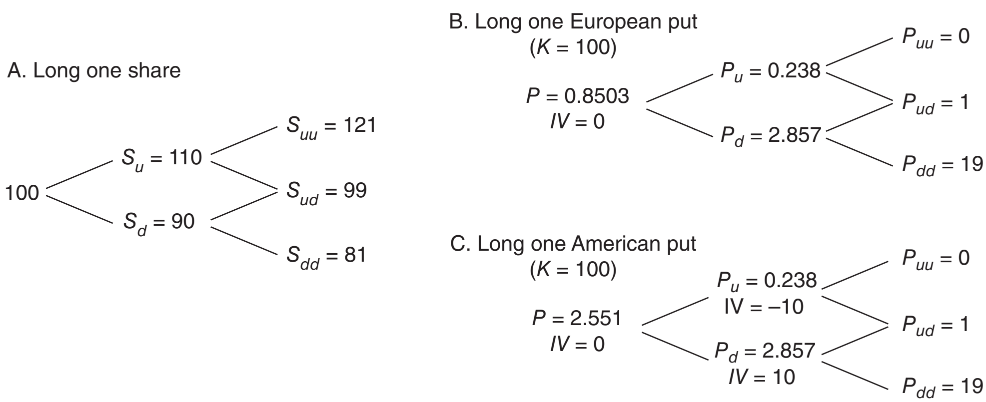

FIGURE 23.1 European and American put

[1]The 'recursive value' is also referred to as the 'continuation value', since option values at each node of the lattice/tree depend on option values in future time periods.

not exercise it at that node). If $IV >$ 'recursive value' then early exercise is profitable and we replace the recursive value, 'P_u or P_d' with the IV at that node. Expressed mathematically the payoff to an American put at node-U is:

$$P_u^* = \max(IV_u, P_u)$$

At node-U, $P_u = 0.238$ and $IV_u = K - S_u = -10$ so early exercise is not profitable and we retain $P_u = 0.238$ in the tree. At node-D, $P_d = 2.857$ and $IV_d = K - S_d = 10$ so early exercise is profitable and we replace $P_d = 2.857$ in the tree with $P_d^* = 10$. The 'recursive value' at $t = 0$ now becomes:

$$P^* = \frac{1}{R}[qP_u + (1-q)P_d^*] = \frac{0.75(0.238) + 0.25(10)}{1.05} = 2.551$$

Finally, we compare $P^* = 2.551$ and $IV_0 = \max(K - S_0, 0) = 0$, which indicates that early exercise is not profitable at $t = 0$ and hence the American put premium is $P^* = 2.551$. Notice that the American put is worth more than the equivalent European put (which has $P = 0.8503$) – the American put has extra 'optionality' as it allows for the possibility that early exercise may be profitable.

This general principle of working backwards through the tree and seeing if early exercise is worthwhile applies when pricing all types of American options: on stock indices, currencies, futures contracts and commodities (e.g. corn, oil, gas etc.).

23.2 OPTIONS ON OTHER UNDERLYING ASSETS

It is convenient here to use the annual *continuously compounded* interest rate, denoted r (decimal). One dollar today is worth $\$1e^{r\,dt}$ after a small time interval dt $(= T/n)$. The equation for the risk-neutral probability q when pricing an option on a stock which pays no dividends is:

$$q = (e^{r\,dt} - D)/(U - D) \tag{23.3}$$

23.2.1 Continuous Dividend Yield

If a stock has a *continuous annual* dividend yield δ (decimal), then the (total) return on the stock consists of the capital gain plus the dividend yield $R_s = d(\ln S) + \delta$, where $d(\ln S) \approx (dS/S)$ is the growth (or proportionate change) in the stock price. In a risk-neutral world, the (total) return on a stock must equal the risk-free rate $R_s = r$ and hence, the expected growth in the stock *price* $d(\ln S)$ equals $r - \delta$ and $S_1 = S_0 e^{(r-\delta)dt}$. In a risk-neutral world, the expected stock price at time t_1 is given by:

$$E^* S_1 = qS_0U + (1-q)S_0D = S_0e^{(r-\delta)dt}$$

Hence:

$$q = (a - D)/(U - D) \tag{23.4}$$

where $a = e^{(r-\delta)dt}$.

When using the binomial recursion formula to price an option on a stock which pays continuous dividends at a (constant) rate δ, the only change required is that '$Se^{r\,dt}$' in the definition of q is replaced by $Se^{(r-\delta).dt}$. The values $U = e^{\sigma\sqrt{dt}}$ and $D = 1/U = e^{-\sigma\sqrt{dt}}$ remain unchanged.

23.2.2 Options on Foreign Currency and Futures

To price options on a foreign currency, the equivalent to δ is the foreign risk-free rate r_f. For options on a futures contract the underlying futures price grows at the rate $\delta = r$, hence the definitions for a become:

- **Option on foreign currency:** $a = e^{(r_d - r_f)dt}$ underlying S = spot exchange rate
- **Option on futures:** $a = 1$ underlying is the futures price, F

When setting up the tree, $U = e^{\sigma\sqrt{dt}}$ and $D = 1/U$ remain unchanged and σ is the real world historical volatility of the underlying asset (i.e. volatility of the stock return, volatility of the return dS/S on the exchange rate or the volatility of the return dF/F of the futures price, depending on the types of option being considered). Hence, the binomial tree for the underlying asset is constructed in the usual way. Hence, the only difference in the BOPM is the definition of 'a' (and hence q in Equation 23.4).

The maturity of the option T is divided into $n = 3$ or more time steps, with each step $dt = T/n$ – this gives a reasonably accurate value for the option's price. For a European option on a stock we only require the payoffs at maturity T to price the option. For $n = 30$ time steps there are $n + 1 = 31$ *terminal* stock prices – which is manageable. But for $n = 30$ there are 2^{30} or about a billion alternative stock price *paths* (e.g. even for $n = 3$ there are 8 possible paths). Since many exotic options are path dependent, the BOPM with $n = 30$ takes considerable computing power and it may take quite a while for the computer to 'grind out' a value for the option price. Hence various methods to speed up the calculations are used, such as trinomial trees and control variate techniques.

23.2.3 Control Variate Techniques

Control variate techniques can be used in the BOPM framework to obtain more accurate values for option premia (for any fixed number of nodes, in the tree). To illustrate this approach suppose we wish to price an American option – which is path dependent. If we value the American option using the standard BOPM with $n = 100$, this should give a good estimate of its true price. Assume this 'true BOPM price' is $f_{Am}^{n=100} = 5.27$ – but computational time will be considerable.

To save on computational time, suppose we decide to price an American option using only $n = 5$ steps in the BOPM and this gives $f_{Am}^{n=5} = 5.48$. In order to get closer to the true price $f_{Am}^{n=100} = 5.27$, the control variate technique adjusts $f_{Am}^{n=5} = 5.48$ in the following way. First, we use the standard BOPM with $n = 5$ to calculate the price of a *European* option $f_{Eur}^{n=5} = 5.3$ (on the same underlying, with the same strike and time to maturity). Both the American and European binomial prices with $n = 5$ are subject to error. However, we know the 'exact' *Black–Scholes price for a European option* $f_{Eur}^{BS} = 5.1$ (say). Using the control variate technique, the price of the American option f_{Am}^{CV} is:

$$f_{Am}^{CV,n=5} = f_{Am}^{n=5} + (f_{Eur}^{BS} - f_{Eur}^{n=5}) = 5.48 + (5.1 - 5.3) = 5.28 \tag{23.5}$$

The control variate technique adjusts the American binomial price $f_{Am}^{n=5}$ (obtained using $n = 5$), by the error in the BOPM estimate when pricing the (equivalent) *European* option, $(f_{Eur}^{BS} - f_{Eur}^{n=5}) = -0.2$. If the BOPM overpriced the *European* option by 0.2 (with $n = 5$) we assume it will overprice the American option by the same amount. The control variate price $f_{Am}^{CV,n=5} = 5.28$ is much closer to the 'true' price of the American option (using $n = 100$), $f_{Am}^{n=100} = 5.27$ than the 'unadjusted' binomial estimate $f_{Am}^{n=5} = 5.48$ with $n = 5$, and the control variate approach takes much less computing time.

23.3 OPTIONS ON FUTURES CONTRACTS

Below, we show that RNV and backward recursion using the BOPM equation produces the same price for an option on a futures contract, as the 'full no-arbitrage' approach. A one-period call option on a futures contract delivers a long position in a futures contract. If the futures price at expiry of the option contract is F_T and the strike is K then a long call option at maturity can be cash settled for:

$$\text{Payoff from call} = \max(0, F_T - K)$$

Suppose the risk-free rate $r = 10\%$ (per period, continuously compounded), $F_0 = 100$, $U = 1.15$ and $D = 0.9$, so $F_u = F_0U = 115$ and $F_d = F_0D = 90$. The payoffs for a one-period ($T = 1$) call option on a futures contract with strike price $K = 100$ are:

$$C_u = \max(0, F_u - K) = 15 \qquad C_d = \max(0, F_d - K) = 0.$$

We noted above that an option on a futures contract can be priced under RNV using:

$$q = (1 - D)/(U - D) = 0.4 \tag{23.6a}$$

which gives:

$$C = e^{-rT}[qC_u + (1 - q)C_d] = e^{-0.10(1)}[0.4(15) + 0.6(0)] = 5.429. \tag{23.6b}$$

We now show that we obtain the same call premium using no-arbitrage and delta hedging. Suppose you are long one call at $t = 0$ with (an unknown) call premium C. To create a risk-free portfolio you would *short* futures contracts, since if F increases you would lose on the short futures but gain on the long call. If you *short* h-futures for each *long* call then the payoffs at $T = 1$ are:

$$V_u = C_u - h(F_u - F_0) \tag{23.7a}$$

$$V_d = C_d - h(F_d - F_0) \tag{23.7b}$$

To delta hedge we choose h so these payoffs are equal, which implies:

$$h = \frac{C_u - C_d}{F_u - F_d} = \frac{15 - 0}{115 - 90} = 0.6 \tag{23.8}$$

The cost of setting up the hedge portfolio at $t = 0$ is simply the cost of the long call C, since it cost nothing to enter the futures contact. (We ignore margin payments.) Our portfolio of one long call and h short futures is risk-free and must therefore earn the risk-free rate $r = 10\%$ (continuously compounded), otherwise arbitrage profits can be made. The cost of setting up the hedge portfolio at $t = 0$ is simply the cost of the call C, financed using borrowed funds. At T the bank loan outstanding is Ce^{rT} and for no arbitrage profits this must equal the known payoff on the hedge position V_u (or V_d):

$$\begin{aligned} Ce^{rT} &= V_u = C_u - h(F_u - F_0) \\ Ce^{0.10(1)} &= 15 - 0.6\,(115 - 100) \\ C &= 5.429 \end{aligned} \tag{23.9}$$

We can obtain the solution for C algebraically by substituting for h from Equation (23.8) in (23.9), using $F_u = F_0U$ and $F_d = F_0D$ and rearranging to give:

$$C = e^{-rT}[qC_u + (1 - q)C_d] \quad q = (1 - D)/(U - D) \tag{23.10}$$

Hence the 'full no-arbitrage' approach produces the same equation for the call premium as directly invoking RNV via (23.6b). Extending the above approach to the n-period case, for *European* call or put futures options, is straightforward. We simply work backwards through the tree from $t = n$ to $t = 0$ using $q = (1 - D)/(U - D)$.

23.3.1 American Futures Option

What about pricing an American futures option where we have the possibility of early exercise? To keep things simple, suppose we hold a *two-period* American call option on a futures contract. Early exercise at node-U is worthwhile if the intrinsic value $IV_u = \max(F_u - K, 0) > C_u$, where the recursive formula gives $C_u = e^{-r\Delta t}[qC_{uu} + (1 - q)C_{ud}]$.

If this is the case, we replace C_u in the tree by $F_u - K$. This calculation is repeated for node-D (and at $t = 0$) – that is comparing IV_d, and C_d and then IV_0 and C_0 – and taking the maximum value in each case.

23.3.2 Numerical Example

Suppose an American call futures option has a time to maturity $T = 1/3$ year, strike price $K = 100$ and the current futures price is $F_0 = 100$, with the volatility of the futures (return) $\sigma = 30\%$ p.a. We divide the time period T into $k =$ number of 'ups' periods so each step in the tree is $\Delta t = T/n = 1/12$ (which represents 1 month). We set $U = e^{\sigma\sqrt{dt}} = e^{0.3/\sqrt{12}} = 1.0905$ and $D = e^{-\sigma\sqrt{dt}} = e^{-0.3/\sqrt{12}} = 0.9170$.

For a futures option $\delta = r$ so $q = (1 - D)/(U - D) = 0.4784$ (and hence the option price is independent of the risk-free rate). The intrinsic value of the option at each node is $IV = \max(F - K, 0)$. Let each node be denoted (t, k) where $t =$ time and $k =$ number of 'ups'.

In Figure 23.2 the upper cells show the stock price and the lower cells the option value (price). At $n = 4$, the option is exercised at the two upper nodes, only. For $(t, k) = (3, 2), (3, 3)$ and (2,2), the intrinsic value of the option exceeds its binomial recursive value – hence here the option is assumed to be exercised and the lower cells at these nodes show the option's intrinsic value $IV = \max(F - K, 0)$, rather than its binomial recursive value. The IVs at these nodes are then used in calculating the next recursive values as we move backwards through the tree and the American call premium is 6.387. (As we increase the number of nodes in the tree, so that $n > 30$, we obtain a more accurate estimate of the call premium.)

# Ups \ Time	0	1	2	3	4
4					141.3982 **41.3982**
3				129.6681 **29.6681**	
2			118.9110 **18.9110**		118.9110 **18.9110**
1		109.0463 11.2137		109.0463 **9.0463**	
0	100.0000 6.3870		100.0000 4.2987		100.0000 0.0000
		91.7042 2.0427		91.7042 0.0000	
			84.0965 0.0000		84.0965 0.0000
				77.1200 0.0000	
					70.7222 0.0000
Time	0	1	2	3	4

FIGURE 23.2 Binomial tree for American call on index futures
Note: Shaded areas indicate intrinsic value, used in the calculations.

For American options on stocks paying continuous dividends at a rate δ and options on FX, the procedure is the same as above. The tree is constructed using $U = e^{\sigma\sqrt{dt}}$ and $D = 1/U$ where σ is the real world (estimated annual) standard deviation of the stock return or FX return. Also $a = e^{(r-\delta)dt}$ for the option on the stock and $a = e^{(r-r_f)dt}$ for the FX-option (where r = domestic interest rate and r_f = foreign interest rate).

The BOPM can also be used to price options where the underlying asset is a stochastic *interest rate* (e.g. options on T-bonds, on T-bond futures or options on interest rates, known as caps and floors) but this requires a tree where interest rates are allowed to vary at each node. This is dealt with in Chapter 41.

23.4 OPTIONS ON DIVIDEND-PAYING STOCKS

23.4.1 Dividends and the BOPM

We have already seen that for a stock (or stock index) that pays a *constant continuous* dividend yield δ, we simply use $a = e^{(r-\delta)dt}$ as the risk-neutral probability and proceed in the usual fashion to price European or American options on the dividend paying stock (index). In practice, the continuously compounded dividend rate δ has to be estimated and clearly while the assumption of a *constant dividend rate* is not unreasonable for a *stock index (which contains many stocks)*, it is not plausible for individual stocks, where dividend payments tend to be bunched in certain months of the year. The BOPM gets a little tricky when dividends are discrete.

23.4.2 Single Known Dividend Yield

Assume the option matures in 30 days, $T = 30/252$, $\Delta t = 1/252$ so that $n = T/\Delta t = 30$. We now apply the BOPM to a European call option where the underlying stock (index) pays *a single dividend* at time $t = i(\Delta t)$. If the time $t = i(\Delta t)$ is *prior* to the stock going ex-dividend, the nodes on the tree correspond to stock prices:

$$SU^j D^{i-j} \qquad (j = 0, 1, 2, \dots i)$$

If time $t = i(\Delta t)$ is *after* the stock goes ex-dividend the nodes would have values

$$S(1-\delta)U^j D^{i-j} \qquad (j = 0, 1, 2, \dots i)$$

where δ is the known single dividend yield. For example, given a single dividend payment prior to the 2nd node, the binomial tree is shown in Figure 23.3. If there are *several* known dividend yields δ_i over the life of the option, then the nodes after the ex-dividend dates would be $S(1-\delta_i)U^j D^{i-j}$.

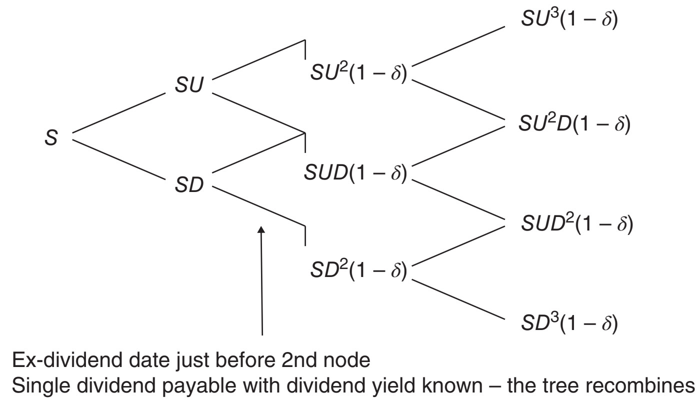

FIGURE 23.3 Single dividend, known dividend yield

23.4.3 Known Dollar Dividend

First note that when a dividend is paid, the stock price falls by the amount of the dividend payment D. (We ignore any tax issues here.)[2] If we let $U = 1/D$ then unfortunately with discrete dividends, the binomial tree for S does not recombine and there are a very large number of nodes to evaluate. To avoid this problem and obtain a recombining tree we proceed as follows. We let U and D apply to the stock price minus the *present value* of all known future dividends (over the life of the option), which we denote S^*. Suppose a *single* ex-dividend date is at τ and the dividend paid is Div. Then the values for S^* at times $i\,\Delta t$ are:

$$\begin{array}{lll} \text{Before ex-dividend date:} & S^* = S - (Div)\, e^{-r(\tau - i\Delta t)} & \text{for } i\,\Delta t < \tau \\ \text{After ex-dividend date:} & S^* = S & \text{for } i\,\Delta t > \tau \end{array}$$

The tree for S^* is constructed using $U = e^{\sigma^* \sqrt{\Delta t}}$, $D = 1/U$ where σ^* = volatility of S^*.[3] This gives us a recombining tree for S^*. To obtain a 'new' tree, we now *add back* the PV of future dividends, *at each node*. Suppose we have calculated S_0^* at $t = 0$. Then a 'new' tree for S at times $i\,\Delta t$ is:

$$\begin{array}{lll} \text{Before ex-dividend date:} & S_{new} = S_0^* U^j D^{i-j} + Div\, e^{-r(\tau - i\Delta t)} & \text{for } i\,\Delta t < \tau \\ \text{After ex-dividend date:} & S_{new} = S_0^* U^j D^{i-j} & \text{for } i\,\Delta t > \tau \end{array}$$

[2] This occurs because if the stock price were to fall by less than D (e.g. $Z < D$) then you could buy the stock for S immediately prior to the ex-dividend date, capture the dividend D and immediately sell the stock for $S - Z$. Your profit $= (D + S - Z) - S = D - Z > 0$ (ignoring any problems due to discounting, when dividends are paid with a lag).

[3] σ^* is slightly larger than σ, the volatility of S. In practice the input for σ^* is usually an implied volatility.

The option is then priced off this 'new' tree S_{new} using $q = (e^{r\,\Delta t} - D)/(U - D)$ as the risk-neutral probability.[4]

For example, suppose $r = 5\%$, there is one dividend of \$10 with an ex-dividend date at the end the second month (0.1667 years), then $PV(Div) = \$10\, e^{-0.05(0.1667)} = \$9.917$. If $S_0 = \$100$ then $S_0^* = \$90.083$. We calculate the tree for S_{new} using the above equations and then work backwards through the tree (from the maturity date of the option) to give the *European* call (or put) premium.

To value an *American call* with strike $K = 110$ on a dividend paying stock we would calculate the intrinsic value at each node and the early exercise decision is based on $IV = S_{new} - K$ (not $S^* - K$). For example, if the stock has just gone ex-dividend and at the next 'upper node' $S^* = \$110$ and the dividend at τ is $Div = \$3$ then $S_{new} = \$113$ (since the present value of the dividend at τ is the dividend itself of \$3). For an instant, $S_{new} = \$113$ and then 'immediately' it falls to \$110 as it goes ex-dividend. But just before the stock goes ex-dividend the call has an intrinsic value of $IV = 3$. If $IV = 3$ at this node and this is greater than the recursive value C_τ in the tree, then we replace C_τ with $IV = 3$. We proceed in this way at each node, to see if the intrinsic value exceeds the recursive value. So, apart from the construction of the stock price tree, an American option on a stock which pays discrete dividends is priced in the usual way.

23.5 SUMMARY

- For options on a stock paying dividends at the *continuous rate* δ (decimal), the stock price tree is constructed using $U = e^{\sigma\sqrt{dt}}$ and $D = 1/U$ but the risk-neutral probability is now $q = (a - D)/(U - D)$ where $a = e^{(r-\delta)\Delta t}$. The option is then priced in the usual way by backward recursion (under RNV).
- For options on a *foreign currency,* $\delta = r_f$ the foreign interest rate and hence $a = e^{(r-r_f)\Delta t}$ (where $a = 1$), S = spot-FX rate and σ = volatility of the return (proportionate change) in the spot-FX rate.
- For options on a *futures contract,* $\delta = r$ hence $a = 1$ and $q = (1 - D)/(U - D)$. In the tree S is replaced by F the forward rate, and σ is the volatility of dF/F.
- American, European and many path-dependent 'exotic options' can be priced using the BOPM under RNV – so the method is very flexible.
- American options are valued using backward recursion but at each node we test to see if early exercise is profitable by comparing the intrinsic value IV (when exercised) with the binomial recursive value (no exercise), and we take the maximum of these two values. For example at node-U, the value of the put can be written $P_u^* = \max(IV_u, P_u)$, where P_u is the BOPM recursive value.

[4]Note that (perhaps surprisingly) the formula for q is for an option on a stock that does *not* pay dividends. This is because we have adjusted the values of S in the tree to reflect dividend payments, so to use $q = (e^{(r-\delta)dt} - D)/(U - D)$ would be a form of 'double counting'.

- To price an option on a stock that pays discrete dividends we construct a tree where we let U and D apply to S^*= 'the stock price minus the present value of all known future dividends over the life of the option'. This allows the tree to recombine, which substantially improves computational efficiency. The option is then priced off a 'new' tree for S_{new} where we *add back* the PV of future dividends, at each node. Expected payoffs are calculated using the *(usual)* risk-neutral probability, $q = (e^{r\,dt} - D)/(U - D)$.
- Computational time in the BOPM can be reduced by using control variate techniques or a trinomial tree (see Appendix 23).

APPENDIX 23: BOPM AND RISK-NEUTRAL VALUATION

As we see in Chapters 47 and 48, continuous time models of stock prices S can be represented in terms of continuously compounded ('log') returns $d(\ln S)$ or proportionate changes dS/S over a short period of time, $dt \to 0$. The 'up' and 'down' movements in the binomial tree for stock prices are an approximation to these continuous time processes and are designed to produce an outcome for the stock price S_T at T, which is (approximately) lognormal. This requires movements of the stock price in the tree to replicate the 'real world' volatility of the stock price. In addition, when pricing options (on a non-dividend paying stock) using the BOPM under risk-neutral valuation (RNV), we must set the growth rate of the stock price equal to the risk-free rate.

In the BOPM we divide the time to maturity of the option T(years) into n-periods of equal length, $dt = T/n$. Over a small interval of time dt, the expected return of the stock is measured as $\mu.dt$ (where μ = continuously compounded annual growth rate, decimal). Over a small time interval, the variance of the stock return is $\sigma^2 dt$ (where σ = *annual* standard deviation (decimal) of the continuously compounded stock return and is calculated from historical data). Hence:

$$E(S_1)/S_0 = e^{\mu\,dt} \qquad \text{(expected return in the 'real world')}$$
$$\text{var}(S_1/S_0) = \sigma^2 dt \qquad \text{(variance in the 'real world')}$$

We price an option (on a non-dividend paying stock) using the BOPM under RNV. Therefore we calibrate U, D and q, so the stock price in the tree satisfies two conditions (over the time period dt):

1. Expected return equals the *risk-free rate* $E(S_1)/S_0 = e^{r\,dt}$ (RNV)
2. Variance of the stock price, $var(S_1) = S_0^2\sigma^2 dt$ ('real world' volatility)

Hence, RNV and replicating the 'real world' volatility gives two equations and three unknowns (U, D, q):

$$E(S_1) = qS_0U + (1 - q)S_0D = S_0e^{r\,dt} \tag{23.A.1}$$

$$\begin{aligned} \text{var}(S_1) &\equiv E(S_1^2) - [E(S_1)]^2 \\ &= q(S_0U)^2 + (1-q)(S_0D)^2 - [qS_0U + (1-q)S_0D]^2 = S_0^2\sigma^2 dt \end{aligned} \quad (23.A.2)^5$$

From (23.A.1):

$$[qU + (1-q)D] = e^{r.dt} \quad (23.A.3)$$

Multiplying (23.A.3) by $(U + D)$ and simplifying:

$$(U + D)[qU + (1-q)D] = (U + D)e^{r.dt} \quad (23.A.4)$$

$$qU^2 + (1-q)D^2 + UD = (U + D)e^{r.dt} \quad (23.A.5)$$

Simplifying (23.A.2):

$$[qU^2 + (1-q)D^2] - [qU + (1-q)D]^2 = \sigma^2 dt \quad (23.A.6)$$

Substituting in (23.A.6) from (23.A.5) and (23.A.3):

$$[(U + D)e^{r.dt} - UD] - e^{2r.dt} = \sigma^2 dt \quad (23.A.7)$$

We have three unknowns q, U and D and only two equations – the RNV equation (23.A.1) and the (simplified) volatility equation (23.A.7). We arbitrarily use our one 'degree of freedom' by setting $U = 1/D$. Equation (23.A.1) or equivalently (23.A.3) gives directly:

$$q = \frac{a - D}{U - D} \qquad a = e^{rdt} \quad (23.A.8)$$

If higher order terms than dt are ignored, a solution to the volatility equation (23.A.7) (with $U = 1/D$) is:

$$U = e^{\sigma\sqrt{dt}} \quad \text{and} \quad D = e^{-\sigma\sqrt{dt}} \quad (23.A.9)$$

In a risk-neutral world U and D are independent of the expected growth rate of the stock (i.e. the expected 'real world' stock return μ), and therefore so is the option price. From (23.A.9) we have $U/D = e^{2\sigma\sqrt{dt}}$ so U/D is determined by the 'real world' volatility of the stock return and hence so are the option premia.

As we move from the 'real world' to our equations in a 'risk-neutral' world, the expected return on the stock changes from μ to r (see 23.A.1) but the volatility of the stock return is the same as in the real world – this is a manifestation *of Girsanov's theorem*. It is easy to see that

[5] Here we use the standard result, $\text{var}(X) = E(X^2) - [E(X)]^2$

(23.A.9) satisfies the volatility equation (23.A.7) by substituting (the Taylor series approximations up to order dt):

$$\begin{aligned} e^{\sigma\sqrt{dt}} &= 1 + \sigma\sqrt{dt} + (1/2)\sigma^2 dt \qquad & e^{-\sigma\sqrt{dt}} &= 1 - \sigma\sqrt{dt} + (1/2)\sigma^2 dt \\ e^{rdt} &= 1 + rdt & e^{2rdt} &= 1 + 2rdt \end{aligned}$$

in the left-hand side of (23.A.7) (and ignoring terms in dt^2 or higher).

The above analysis can be repeated for an option on an asset that pays a continuous yield (e.g. dividend yield) at a rate δ. The return on a stock equals the capital gain dS/S plus the (continuously compounded, dividend) yield δ. In a risk-neutral world the asset (stock) return equals the risk-free rate r and hence the expected value of the asset price $ES_1 = S_0 e^{(r-\delta)\,dt}$. Therefore, to price an option on an asset that pays a continuous yield, the only change in the above analysis is in (23.A.1) where we replace $S_0 e^{r\,dt}$ with $S_0 e^{(r-\delta)\,dt}$ which results in $a = e^{(r-\delta)dt}$ in (23.A.8).

Negative Risk-neutral Probabilities

Sometimes when σ is very small, the above formulas can give negative probabilities for q – which are meaningless. One 'trick' to avoid this problem is to assume the option is written on a futures contract with futures price F (even though in reality it is not!), then $a = 1$ and we never get negative risk-neutral probabilities. The tree for F is constructed at each node and the underlying cash market price at each node is obtained using $S_i = F_i\, e^{-(r-\delta)i\,dt}$, where $\delta =$ constant dividend yield (or the foreign interest rate for a foreign currency option).

Other Risk-neutral Probabilities

In the above derivation we found we had 'one degree of freedom' and imposed $U = 1/D$ (Cox, Ross and Rubinstein 1979). This gives unique values for U, D and q with which to construct the binomial tree, which is then used to price the option. But we could have used another 'trick' in the derivation of U, D and q, which results in q being *the same* for options with *different* underlying assets, S (e.g. options on stocks that pay no dividends, on stocks that pay continuous dividends, options on FX-spot rates or commodities or futures contracts).

This seems a little counter-intuitive but it is to do with how we 'allocate' our one degree of freedom. Given our two equations to determine the three 'unknowns' U, D and q we can *arbitrarily set* $q = 0.5$. Then solving our two equations (23.A.3) and (23.A.7) for U and D we obtain (when terms of higher order than dt are ignored):

$$U = e^{(r-\sigma^2/2)dt\; +\; \sigma\sqrt{dt}} \tag{23.A.10a}$$

$$D = e^{(r-\sigma^2/2)dt\; -\; \sigma\sqrt{dt}} \tag{23.A.10b}$$

Clearly, using these values of U and D would give a different tree for the stock price than if we use the Cox, Ross and Rubinstein formulas but the value of the option premium from backward recursion using the BOPM under RNV is the same using either approach. The different values for q in the two trees would exactly offset the different values for S, and the price of the option using backward recursion turns out to be the same. (After all we can only have one 'correct' or 'no-arbitrage' price for the option.)

For a stock (index) paying a continuous dividend at a rate δ, we replace r by $r-\delta$ in Equations (23.A.10a) and (23.A.10b):

$$U = e^{(r-\delta-\sigma^2/2)dt+\sigma\sqrt{dt}} \tag{23.A.11a}$$

$$D = e^{(r-\delta-\sigma^2/2)dt-\sigma\sqrt{dt}} \tag{23.A.11b}$$

In addition, for currency options $\delta = r_f$ the foreign interest rate and for options on futures contracts $\delta = r$, so r is omitted from the above equations. Hence, Equations (23.A.11a) and (23.A.11b) enable construction of a tree for the underlying asset and hence price options on dividend paying stocks, currencies and futures contracts when using $q = 0.5$.

Note that the size of U and D (and the value of q) have all been derived assuming a risk-neutral world. So, when *pricing options,* the tree for the stock price does *not* represent actual movements in the stock price but it still correctly prices the option because of the equivalence of backward recursion using RNV and the 'no-arbitrage' approach.

Trinomial Tree

When pricing an option, the use of a *trinomial tree* rather than a binomial tree can reduce computational time. The tree is set up so that at each node there is an up, middle, and down step. For example, for a non-dividend paying stock, the tree mimics the 'real world' volatility and has the stock price growing at the risk-free rate if:

$$\begin{aligned}
U &= e^{\sigma\sqrt{3.dt}} \qquad\qquad D = 1/U \\
q_d &= -\sqrt{\frac{dt}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6} \\
q_u &= \sqrt{\frac{dt}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6} \\
q_m &= 2/3
\end{aligned}$$

where q_d, q_u, q_m are the risk-neutral probabilities for the down move, up move and for the 'middle' path. We then use backward recursion on the trinomial tree to calculate the option premium. For assets paying a continuous dividend yield at a rate δ, we replace r by $r - \delta$ in the above equations. Also, for currency options $\delta = r_f$ the foreign interest rate and for options on futures $\delta = r$, so r is omitted from the above equations. The trinomial tree is equivalent to the explicit finite difference method, discussed in Chapter 48.

EXERCISES

Question 1
Why is the BOPM (and other 'tree methods') often seen to be more flexible than closed-form solutions for the options price, such as the Black–Scholes formula for calls and puts?

Question 2
What are the drawbacks of using the BOPM to price options?

Question 3
You want to price an American put option (on a non-dividend paying stock) using the BOPM with $n = 20$ steps. How does the control variate technique improve the accuracy of the price of the American put? Explain.

Question 4
You hold a long (European) put option on a futures contract. The current futures price is $F_0 = 100$ and the futures price can move to either $F_u = 115$ or $F_d = 90$. The futures option has a strike price $K = 100$, $T = 1$ period to maturity and the risk-free rate $r = 10\%$ p.a. (continuously compounded).

(a) Create a risk-free portfolio consisting of one long put and futures contracts.
(b) Using the no-arbitrage condition, calculate the European put premium.
(c) Check that the value of your hedge portfolio is the same at the up and the down nodes.
(d) Check your answer in (b) by using the BOPM formula for the price of the put option.

Question 5
The index futures price is $F_0 = 100$. An American put option on the futures index has $K = 100$, $r = 8\%$ p.a. (continuously compounded), $\sigma = 30\%$ p.a., $T = 1/3$ year (4 months).
Use a tree with $n = 4$ steps to calculate the 'up' and 'down' moves for the futures price and show that the price of the American put is $P = 6.3870$.

Question 6

The spot FX-rate is $S_0 = 1.52$ (\$/£, USD per GBP). An American put option on the USD has $K = 1.5$ (USD/GBP), the interest rate in the US is $r = 4\%$ p.a. (continuously compounded), the volatility of the USD-GBP spot exchange rate $\sigma = 12\%$ p.a., the option has $T = 1$ year to maturity and the interest rate in the UK is $r_f = 5\%$ p.a. (continuously compounded).

Use a tree with $n = 4$ steps to calculate the 'up' and 'down' moves for the spot FX-rate and show that the price of the American put is $P = 0.0658$ (USD/GBP).

CHAPTER 24

Analysis of Black–Scholes

Aims

- To examine alternative ways of measuring and forecasting volatility.
- To test the validity of the Black–Scholes equation.
- To assess the limitations of the Black–Scholes equation for pricing European options.

24.1 VOLATILITY

In the Black–Scholes formula all variables are directly observable, except for the volatility of stock returns (over the life of the option). To price an option we therefore require a *forecast* of volatility. Typical values of the annual standard deviation might be in the range of $\sigma = 0.2 - 0.6$ (20% to 60% p.a.) for individual stocks and around 20% p.a. for stock indices (e.g. S&P 500). If stock returns are assumed to be independent (and identically distributed), the standard deviation over a horizon T (measured in years or fractions of a year), is given by $\sigma_T = \sigma\sqrt{T}$. For example, if $\sigma = 30\%$ p.a. the standard deviation over 3 months ($T = 1/4$) is $\sigma_{0.25} = 30\sqrt{0.25} = 15\%$. Hence if we use the $\sqrt{T}$-rule ('root-tee' rule) we only require a single estimate of σ using returns over some fixed horizon.

24.1.1 Estimating Volatility

There are a variety of methods used to forecast volatility, including an equally weighted average, an exponentially weighted moving average (EWMA), more complex measures based on high/low ranges for returns, time series models such as ARCH (Autoregressive Conditional Heteroscedasticity) and GARCH (Generalised Autoregressive Conditional Heteroscedasticity), and stochastic volatility models.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

24.1.1.1 Equally Weighted

If we have daily observations on the stock price then the daily (continuously compounded) return is:

$$R_i = \ln(S_i/S_{i-1}) \qquad (i = 1, 2, 3, \ldots, n \text{ days}) \tag{24.1}$$

The mean (daily) return is:

$$\overline{R} = (1/n)\sum_{i=1}^{n} R_i \tag{24.2}$$

The usual formula for the sample standard deviation is:

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(R_i - \overline{R})^2}{n-1}} \tag{24.3}$$

The above formulas can be applied at any chosen frequency for the returns data (e.g. daily, weekly, monthly) to obtain the mean return and standard deviation for any particular horizon.[1] For example, using *daily returns* in (24.3) gives $\hat{\sigma} = 0.01$ (i.e. 1% per day) then assuming $T = 252$ trading days per year, the annual volatility used in the Black–Scholes formula is:

$$\hat{\sigma}_{252} = 0.01\sqrt{252} = 0.1587 \;\; (15.87\% \text{ p.a.}) \tag{24.4}$$

A measure of the accuracy of our estimate of σ_{252} is given by its standard error (stde). If returns are *niid* (i.e. normally, independent and identically distributed) then it can be shown that $stde(\hat{\sigma}_{252}) = \sigma/\sqrt{2n}$ and if we use $n = 25$ daily observations to estimate the standard deviation of $\sigma = 0.01$, then:

$$stde(\hat{\sigma}_{252}) = \frac{0.1587}{\sqrt{50}} = 0.0224 \;\; (2.24\% \text{ p.a.}) \tag{24.5}$$

Using only 25 observations means that our estimates of σ_{252} is not terribly precise. In practice between 90 and 180 days of data are used, thus reducing the standard error to 1.167% or 1.183% p.a. respectively. If the stock pays dividends D then for those periods which include an ex-dividend day, the continuously compounded stock return is[2]:

$$R_i = \ln[(S_i + D_i)/S_{i-1}] \tag{24.6}$$

[1]Usually non-overlapping data is used. Using overlapping data creates additional problems. For example, if you use daily returns data to measure the *return over 5 days*, and then move forward one day at a time, each 5-day return has 4 days in common, so this introduces serial correlation into the 5-day returns series. Non-overlapping data would measure the 5-day return, by moving forward 5 days at a time.

[2]We have ignored any complications due to the taxation of dividends. If the tax implications are complex then sometimes data that includes an ex-dividend date are excluded from the calculation of R_i and hence of $\hat{\sigma}$.

Above, we calculated the annual volatility based on the number of trading days (252) rather than the number of calendar days (365). Which one is 'correct' depends on what are the causes of volatility. According to the efficient markets hypothesis (EMH), volatility is caused solely by the random arrival of new information, therefore the volatility over 2 days should be twice that for 1 day. An alternative view is that volatility is caused by the activity of trading itself. French (1980) tested these alternatives. He calculated:

(i) the variance of *daily* stock price changes (closing prices) over consecutive trading days (i.e. all closing prices on working days Monday–Friday, if there are no weekday holidays).
(ii) the variance of stock price changes between close of trading on Friday and close of trading on Monday (i.e. 3 calendar days).

If trading and non-trading days are equivalent then the *variance* in (ii) should equal 3 times the variance in (i). However, French found that results in (ii) are only about 20% higher – hence volatility is far higher when the exchange is open than when it is closed. Proponents of the EMH can of course argue that more information is likely to arise when the exchange is open. But 'news' about the weather, which affects agricultural futures prices is equally likely to arise Monday–Friday as on Saturday or Sunday. Hence agricultural futures prices which depend crucially on the weather should obey the '3 times' rule, but they do not (see French and Roll 1986), suggesting it is trading *per se* that causes volatility. The above arguments suggest that when scaling up daily volatility to an annual volatility we should use $T = 252$, the number of *trading days* in a year.

Another question when measuring σ is how many past data points to use. If the volatility of stock returns changes dramatically over short time horizons then this suggests using only recent observations to calculate $\hat{\sigma}$ (e.g. the past 90 trading days), since recent data probably more accurately represent the *expected* volatility over the life of the option contract. On the other hand, using more data points might give a more accurate measure of the 'true' volatility, if the latter doesn't vary too much. Some compromise must be reached depending on how time varying the 'true variance' is thought to be and there are 'backtesting' techniques which can be used to assess the forecasting accuracy of alternative estimates of volatility. A reasonable compromise, for input to the Black–Scholes option pricing equations, might be to estimate $\hat{\sigma}$ using data over the most recent 90 to 180 trading days.

There is an inconsistency in using our estimate of $\hat{\sigma}$ in the Black–Scholes formula since the latter assumes volatility is constant, whereas our historical *measure* of volatility uses a fixed 'window' of n-data points, implying that volatility varies over time (which in practice, it does). However, assuming volatility is constant *over the life* of the option does not in general lead to large pricing errors.

24.1.1.2 Exponentially Weighted Moving Average (EWMA)

In the EWMA method, more recent data is given greater weight in forecasting tomorrow's volatility than more distant data. The weights used to forecast volatility are geometrically declining $w_k = (1-\lambda)\lambda^k$ so if $\lambda = 0.9$ the values of λ^k are 0.9, 0.81, 0.73, etc. The EWMA volatility forecast is then:

$$\sigma_{t+1} = (1-\lambda)\sum_{m=0}^{n} \lambda^m (R_{t-m} - \overline{R})^2 \tag{24.7}$$

If the sum in Equation (24.7) goes to infinity ($n \to \infty$), then $\sum_{m=0}^{n} \lambda^m = 1/(1-\lambda)$ and the weights on past returns sum to unity and the EWMA formula can be rewritten in recursive form:

$$\sigma^2_{t+1} = \lambda\sigma^2_t + (1-\lambda)(R_t - \overline{R})^2 \tag{24.8}$$

This simplifies the forecast of volatility. Given an initial value for σ^2_0, we can 'update' each value of σ^2_{t+1}, namely $\sigma^2_1, \sigma^2_2, \sigma^2_3, \ldots$, etc. and the only new piece of information required is $(R_t - \overline{R})^2$ for each time period.[3] At time t, the forecast one period ahead σ^2_{t+1}, then only requires our previously calculated σ^2_t and the current value for $(R_t - \overline{R})^2$– that is, two data points (and not the infinite sum of data points in (24.7)). For forecasting daily volatility we usually assume a zero mean return, hence:

$$\sigma^2_{t+1} = \lambda\sigma^2_t + (1-\lambda)R^2_t \tag{24.9}$$

A 'best estimate' of λ can be obtained by choosing that value for λ that minimises the ex-post forecast errors of volatility over some past 'backtesting' data period. For daily returns λ is usually around 0.95.

24.1.1.3 ARCH and GARCH

More sophisticated methods of forecasting σ include ARCH and GARCH regressions. These models assume that the forecast of tomorrow's volatility is a weighted average of today's volatility, plus a 'surprise' (or unexpected) contribution due to today's observed stock return. The simple GARCH model is rather similar to the EWMA model and both are autoregressive (i.e. tomorrow's forecast of volatility depends on a weighted average of past volatilities – Cuthbertson and Nitzsche 2004). The GARCH approach is a type of stochastic volatility model, since volatility is partly influenced by past random events.

Consider a simple GARCH model. Daily returns R_t are assumed to be (conditionally) normally distributed with a mean μ and a time-varying variance, σ^2_t:

$$R_t = \mu + \varepsilon_t \qquad \varepsilon_t \sim niid(0, \sigma_t) \tag{24.10}$$

[3] If we are using daily data then the initial value for σ^2_0 is usually taken to be the (standard) sample variance using about 25 daily returns, at the beginning of the data set. Using the recursion in Equation (24.8), the forecast value of σ^2_{t+1} for about 80 days later is then independent of our (arbitrarily chosen) starting value for σ^2_0.

The (conditional) variance of R_t is time varying, $\mathrm{var}(R_t) = E(R_t - \mu)^2 = \sigma_t^2$. The GARCH(1,1) model assumes that volatility is autocorrelated (i.e. tomorrow's variance depends on today's variance).

$$\sigma_{t+1}^2 = \alpha_o + \alpha_1\sigma_t^2 + \alpha_2\varepsilon_t^2 \qquad \alpha_o > 0 \text{ and } \alpha_1 + \alpha_2 < 1 \tag{24.11}$$

If $\alpha_1 + \alpha_2$ is close to unity then the variance is *persistent* – periods of high (low) variance or 'turbulence' are generally followed by further periods of high (low) variance. GARCH models can utilise any appropriate distribution for the error term and hence are very flexible (e.g. Student's *t*-distribution, which has fatter tails than the normal distribution). An estimated GARCH equation for daily returns (e.g. on a stock index) might be:

$$\sigma_{t+1}^2 = 0.0000004 + 0.88\sigma_t^2 + 0.08\varepsilon_t^2 \tag{24.12}$$

Persistence in variance is indicated by $\alpha_1 + \alpha_2 = 0.96$, which is very close to unity. This means that if volatility ε_t^2 is high (low) today it will tend to remain high (low) for a considerable time in the future. The long-run steady state variance is $\sigma^2 = (4x10^{-6})/(1 - 0.96) = 1 \times 10^{-4}$, which implies $\sigma = 0.01$ (1% per day). So the GARCH models implies that day-by-day the standard deviation of stock returns moves in long swings around its long-run mean value of 1% per day, but takes some considerable time to return to its long-run level.

Forecasts of volatility (over the life of the option) from the above statistical models can be used as an input for σ^2 in the Black–Scholes equation to price option-A (say). But in most cases implied volatilities can be calculated from 'similar' options-B (e.g. same or similar underlying asset, same strike but slightly different times to maturity) and these implied volatilities are often used as inputs in the Black–Scholes equation to price option-A, rather than statistical forecasts from ARCH and GARCH models. However, the above statistical forecasting models for volatility are used in value at risk (VaR) calculations, which are discussed in Chapters 44–46.

On a slightly different point, Chiras and Manaster (1978) have examined whether volatility estimates based on historical data (e.g. sample standard deviation using the last 90-day returns), predict future volatility better than do (weighted) *implied volatilities*. They find that implied volatilities forecast future volatility better than 'historical volatility' estimates, implying that option traders are using more information than contained in simple averages of historical data.

24.2 TESTING BLACK–SCHOLES

Tests of the validity of the Black–Scholes equation are joint tests of:

(a) the assumptions behind the Black–Scholes European option pricing formula are correct

(b) the market is efficient so that quoted options prices move quickly to equal their no-arbitrage ‘fair value’ given by the Black–Scholes equation.

The Black–Scholes model does not price American options (where early exercise is possible). Hence we only deal with European options. In principle, testing Black–Scholes is very simple. Using current data on the known variables S, T, K, r and a forecast for σ, one can calculate the Black–Scholes theoretical or fair price C_t^{BS} of a European call (on a non-dividend paying stock). If the actual quoted price of the call option is C_t then we expect $C_t^{BS} = C_t$. In practice, the following difficulties arise in testing the Black–Scholes model:

- Quoted option price C_t and the inputs to Black–Scholes, the stock price S_t and interest rate r_t must be observed synchronously.
- The estimate of σ^2 used to calculate the Black–Scholes value C_t^{BS} can be calculated in various ways (as discussed above) and each will give a different value for C_t^{BS}.
- For options on stocks that pay dividends some assumptions have to be made about the timing of future dividend payments.

Market prices for S_t and C_t might be misleading if they refer to trades separated by even one minute. Since we are dealing with potentially risk-free arbitrage profits, we need ‘high quality’ synchronous data. Similar arguments apply to the choice of the risk-free rate (e.g. should you use the rate on T-bills or the repo rate).

It is generally found in empirical studies that for at-the-money options C_t^{BS} is close to the quoted price C_t, and the Black–Scholes formula holds. However, for in-the-money (ITM) or out-of-the-money (OTM) options, $C_t \neq C_t^{BS}$ and the Black–Scholes formula is ‘biased’. But how inaccurate is the Black–Scholes equation? What is important is whether it is possible to make risk-free profits, after taking account of transactions costs (bid–ask spreads) when delta hedging an options portfolio, while waiting for any mispricing to be corrected. The vast amount of evidence on this question suggests that there are very few occasions where even small arbitrage profits can be made (Black and Scholes 1972; Galai 1977) – hence for plain vanilla European options on stocks, stock indices and currencies, the Black–Scholes formula works relatively well.

24.2.1 Implied Volatility

An indirect method of showing that the Black–Scholes formula yields biases is to look at *implied volatilities* for European call and put options on the same underlying asset (e.g. stocks, foreign currency). If the Black–Scholes equation correctly prices all these options, then their implied volatilities should not vary with the strike price (or time to maturity). For foreign currency options (e.g. on USD/GBP) if we take quoted prices of calls and puts with different strikes (but the same maturity) and back out the implied volatilities (see Chapter 16) we find that a plot of σ^{imp} against K produces a ‘smile’ (Figure 24.1).

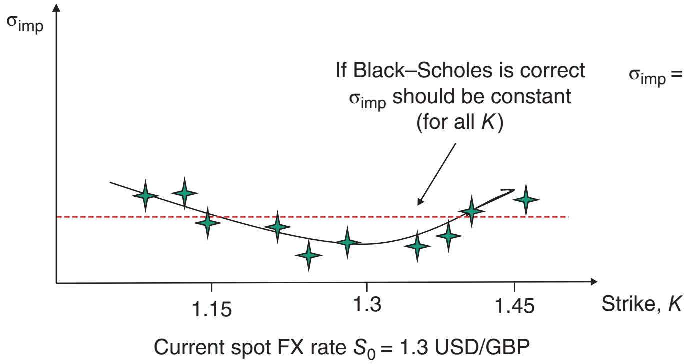

FIGURE 24.1 Volatility smile: USD/GBP

For options on equity indices (e.g. S&P 500) implied volatilities plotted against different strikes produce a *volatility skew*; implied volatilities for very 'low' strikes (e.g. OTM puts) have much higher values of σ^{imp}, than for ATM or ITM puts (Figure 24.2). The above results, qualitatively, also apply to American calls and puts.

The positions of the graphs in Figures 24.1 and 24.2 depend on the current price of the underlying asset. For example, the lowest point of the volatility smile (Figure 24.1) is usually close to the current spot FX rate and if the spot rate increases (say), the 'smile curve' moves to the right. Hence the volatility smile (or skew) is often a plot of σ^{imp} against either K/S_0 or K/F_0 where F_0 is the forward/futures price with maturity T, the same as the maturity of the options.

Implied volatilities can also be calculated for calls and puts with different maturities (but on the same underlying and same strike) – this is the term structure of (implied) volatility. Historical implied volatilities for different strikes and maturities are shown in Table 24.1 – this represents the *volatility surface*. In Table 24.1 for short-dated options (1-month), there is a

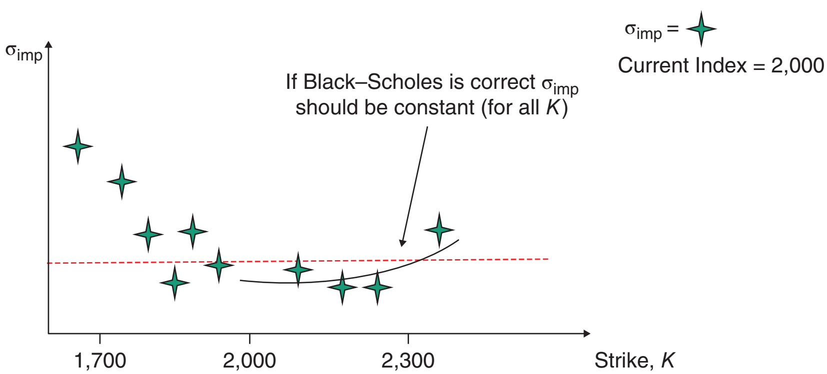

FIGURE 24.2 Volatility skew: equity index

TABLE 24.1 Volatility surface (% p.a.)

Time to maturity	$K/S_0 = 0.90$	$K/S_0 = 1$	$K/S_0 = 1.10$
1 month	17.2	15.0	17.5
6 months	17.1	15.1	17.3
1 year	17.6	16.5	17.8
5 years	17.7	17.8	18.0

pronounced volatility smile with respect to the strike price but this becomes less pronounced at longer maturities (e.g. for 5-year maturities).

Some of the values for implied volatilities are obtained directly from prices of liquid options and some values in Table 24.1 will be interpolated. When using implied volatilities to price a 'new' call (or put) option-X (on FX say), a financial engineer will look at option-X's current value for $K/S_0 = 1.10$ (say) and maturity $T = 1$ year (say) and then read off the appropriate value of implied volatility $\sigma^{imp} = 17.8\%$ p.a. (Table 24.1). This volatility would be used to price option-X using the Black–Scholes formula or in the BOPM (to calculate U and D) or in a MCS to generate the underlying asset price (see Chapter 26). If option-X has values of K/S_0 or T that do not exactly match those in Table 24.1 then interpolation is used.

Volatility is found to be mean reverting, so if short-term volatility is below (above) its long-run level then it tends to increase (decrease) over longer horizons. Hence, when short-term volatility is historically low, implied volatility tends to be an increasing function of maturity – as in the columns of Table 24.1. (The opposite is true when historical volatility is high.)

Some practitioners define the volatility smile as the relationship between σ^{imp} and $(1/\sqrt{T})\ln(K/F_0)$ and then 'the smile' is less dependent on T. With enough data points in Table 24.1 it may be possible to run a regression of σ^{imp} against $\ln(K/F_0)$ and (some function of) T to give $\sigma^{imp} = \mathrm{b}_0 + \mathrm{b}_1 \ln(K/F_0) + \mathrm{b}_2 f(T)$. Given the estimated values of b_i you can then directly calculate the value for σ^{imp} for option-X.

24.3 LIMITATIONS OF BLACK–SCHOLES

Using the Black–Scholes formula only gives the 'correct' (no-arbitrage) price if the assumptions underlying the mathematical model are true and if the market eliminates all (risk-free) arbitrage opportunities. Empirical studies show that the Black–Scholes formula does accurately price ATM options but not ITM or OTM options. A key assumption of the Black–Scholes model is that the continuously compounded stock return is *niid*, which implies that the stock *price* (at maturity of the option contract) is lognormal. Not surprisingly, if the 'true' (empirical) distribution of the stock price is not lognormal, then the quoted price will differ from the Black–Scholes price.

For example, a significantly OTM call option has a positive value only if there is a large increase in the stock price (so that at expiration $S_T > K$), where the right tail of the stock price distribution matters. The fatter the right tail of the terminal stock price distribution, the more valuable the call option. Hence, if the true distribution for the stock price has a right tail which is 'fatter' than the lognormal distribution, the Black–Scholes formula will tend to underprice this option (relative to the market's correct valuation based on the true fat tailed distribution). A similar argument can be made if the right tail is thinner than the lognormal. Here the traded price of the call will be below that given by the Black–Scholes formula.

For an out-of-the-money put, a positive payoff at maturity requires a fall in stock prices, hence if the true terminal distribution for the stock price has a fatter left tail than the lognormal, then the Black–Scholes formula will underprice the OTM put.

The Black–Scholes model assumes a constant volatility, which must be independent of the stock price. However, suppose the stock price is positively correlated with volatility. This implies that when S_T is high, the right tail of the distribution will be 'fat' and the left tail 'thin' (relative to the constant volatility case). Hence the Black–Scholes formula will misprice ITM and OTM options. These distortions in the Black–Scholes price are due to the assumption of a constant volatility and will increase with the time to maturity of the option (because $\sigma_T = \sigma\sqrt{T}$ and hence any mismeasurement of σ is magnified as T increases).

24.3.1 Jump Diffusion Process

The Black–Scholes formula assumes $\ln(S_t/S_{t-1})$ is *niid*. However, if the true distribution is one where from time-to-time relatively large positive or negative 'jumps' occur then the true distribution has 'fat' right and left tails and the Black–Scholes formula will misprice the options. We explore some implications of the stock price being subject to 'jumps'. A geometric Brownian motion for the stock price:

$$S_t = S_{t-1} \exp[(\mu - \sigma^2/2)\, dt + \sigma\sqrt{dt}\, \varepsilon_t] \qquad \varepsilon_t \sim niid(0, 1) \tag{24.13}$$

gives a lognormal distribution for the terminal stock price. The stock price is equally likely to increase or decrease because $\varepsilon_t \sim niid(0, 1)$. Suppose we want to generate a series for the stock price that exhibits both larger and more frequent 'sharp' price *falls* than given by the lognormal distribution. This can be represented by a jump diffusion process. When ε_t is negative, Equation (24.13) implies that this contributes to a fall in the stock price. We want 'occasionally' to make the stock price fall by more than given by Equation (24.13). Let the *size of the jump* be $J > 0$ and the frequency or *intensity of the jumps*, $I > 0$. We simply add the following term to Equation (24.13) (in Excel):

$$-J.IF(RAND(.) < [I \times J], 1, 0)S_{t-1} \tag{24.14}$$

The function *RAND*(.) draws any number between 0 and 1 with equal probability. If *RAND*(.) is less than $I \times J$ then the 'IF' statement takes a value of 1 and the whole expression equals $-JS_{t-1}$, and hence the stock price undergoes an additional fall. For example, if $J = 0.8$ then the stock price falls by an additional 20%. On the other hand, if *RAND*(.) returns a 'large' positive number, then 'IF' returns a '0' and the stock price follows the usual *niid* process of Equation (24.13). The frequency with which the jumps occur is usually taken to follow a Poisson process. Hence, if the jump is denoted dq then:

$$dq = 1 \text{ with probability } I\,dt$$

$$dq = 0 \text{ with probability } (1 - I\,dt)$$

There is a probability $I\,dt$ of a jump dq in time period dt. Thus the jump diffusion process introduces a fat left tail (and a 'left' skewed distribution). However, in practice, if there are jumps, then traders cannot perfectly delta hedge. Traders can choose to either hedge only the small 'Brownian motion changes' given by (24.13) or they can choose the hedge ratio to minimise the variance in the value of their hedge portfolio, which is now subject to the jump process. Neither approach is particularly satisfactory. In the former case traders are not completely hedged when a large fall in the stock price occurs and in the latter case they are only hedged for large changes 'on average' – and this may be of little comfort immediately after a substantial crash.

Turning back to option pricing. Another alternative is to assume an upper and lower range σ^+ and σ^- for the possible values of the volatility (estimated from the empirical jump process) and price the option based on the 'worse case' outcome. Here one makes no assumptions about the probability distribution of the timing (frequency/intensity) of the crash and we do not need to estimate the average size of the fall J or the probability distribution of J. The only assumption needed is to place an upper bound on the size of the fall and to limit the number of crashes per unit of time (e.g. no more than three crashes per year and that immediately after a crash, there cannot be a crash within the next 6 months). This is a considerable simplification and avoids potential estimation error (see Wilmott 1998, who calls this 'crash modelling' and also proposes a 'Platinum Hedge' to minimise risk).

Where the option price depends on the price of more than one underlying asset (e.g. on two different stock prices) then we need to make some additional assumptions about the correlation of these prices in crash periods. For example, we could assume independence (not particularly realistic) or we could assume all assets fall by the same proportionate amount at the same time (slightly more realistic).

With certain types of jump processes it is possible to obtain a partial differential equation (PDE) for the option price, solve this numerically and hence price the option. For example, if the jump component of the asset return represents non-systematic risk (i.e. the risk is not priced) then we can still apply risk-neutral valuation to obtain a (complex) closed-form solution for a European call option (Merton 1976; Naik and Lee 1990). However, this 'theoretical price' depends on the estimated parameters I and J of the jump process, which are subject to error. Perhaps this is why jump diffusion models are not widely used to price options. The potential advantages are just not worth the effort and traders prefer to stick with the Brownian motion assumption for $\ln S$ and 'trade with their eyes wide shut' as far as 'jumps' are concerned.

In Chapter 26 we determine option prices using Monte Carlo simulation and this technique requires a 'representative' stock price process. There is nothing to stop us using a jump diffusion process for stock prices and if so, we will obtain a different value for the option premium to that given by the Black–Scholes formula (which assumes a geometric Brownian motion). 'Jumps' tend to have a greater effect on option premia when the option is close to maturity. Over longer horizons the jumps tend to 'average out'. However, Ball and Torous (1985) for standard 'plain vanilla' options find no significant differences between the Black–Scholes formula and option premia based on 'jump processes'. However, this conclusion would not hold for certain exotic options (e.g. barrier options).

Intuitively, to see how a very specific 'jump process' can influence the call premium, take a simple yet extreme example. It may be that a special event, such as the threat of a takeover, is imminent. In this case traders may anticipate a very large jump depending on the outcome of the takeover – a price rise if the takeover is successful and a price fall if it is not. This probability distribution is bimodal around the expected 'high' or 'low' price. Therefore the Black–Scholes formula does not apply here and observed option premia on such stocks *should differ* from those given by Black–Scholes.

24.4 SUMMARY

- In the Black–Scholes formula, option premia are very sensitive to the input for volatility. Forecasts of volatility can be obtained from the sample variance, from exponentially weighted forecasts and from more sophisticated statistical models such as ARCH/GARCH.
- Most options are priced by 'backing out' a value for implied volatility from an option-B which is 'similar' to 'option-A' you wish to price (e.g. option-B would be on the same underlying asset as option-A but would have a different strike price or time to maturity). Several values for implied volatility can be obtained from the prices of traded options-B (e.g. options-B which are ATM, ITM, and OTM). These different implied volatilities from options-B are then used to obtain a 'best estimate' for implied volatility (usually a weighted average), which is then used as an input into the Black–Scholes formula to price option-A.
- The Black–Scholes formula exhibits some systematic mispricing patterns, nevertheless it is accurate for pricing a wide range of plain vanilla European options on stocks (which are close to being ATM).

- Stock prices may be subject to large infrequent 'jumps'. Jump diffusion processes can be incorporated into (some) options pricing models and these models will generally result in different option prices to those given by the Black–Scholes equation.

EXERCISES

Question 1
The volatility of the change in the stock price is 40% p.a. What is the standard deviation of the stock price change over 1 trading day? Assume 252 trading days in a year. What assumptions are you making about the stochastic behaviour of the stock price?

Question 2
The EWMA model can be represented as $\sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)R_t^2$ with $0 < \lambda \leq 1$ and we assume the mean value for R is zero. Express this equation in terms of the change in the variance (over time) $\sigma_{t+1}^2 - \sigma_t^2$ and briefly comment on your interpretation of this equation as 'mean reverting'.

Question 3
Why do we scale up our estimate of daily volatility to annual values by using trading days rather than calendar days?

Question 4
What, if any, is the intuition behind using the EWMA forecasting scheme for volatility?

The EWMA can be written as either:

$$(1)\ \ \sigma_{t+1} = (1-\lambda)\sum_{m=0}^{n} \lambda^m (R_{t-m} - \overline{R})^2 \quad \textit{with}\ 0 < \lambda < 1$$

Which for large n can be shown to be equivalent to:

$$(2)\ \ \sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)(R_t - \overline{R})^2$$

Question 5
A European call option on a stock (with 3 months to maturity) has a quoted price which is 1% below the 'correct' (no-arbitrage) Black–Scholes price. How might you take advantage of this mispricing over the next 4 weeks, without incurring a high level of risk?

Question 6
Suppose you calculate implied volatilities from the *quoted prices* of European calls and puts on the same stock, with the same maturity date, but the options have different strike prices.

If the Black–Scholes equation correctly prices these options, what would you expect to observe in a graph of implied volatility (on the y-axis) against the different strike prices (on the x-axis)?

Question 7
The GARCH(1,1) model can be represented as:

$$(1)\ R_t = \mu + \varepsilon_t \qquad \varepsilon_t \sim niid(0, \sigma_t)$$
$$(2)\ \sigma_{t+1}^2 = \alpha_o + \alpha_1 \sigma_t^2 + \alpha_2 \varepsilon_t^2 \qquad \alpha_o > 0 \ and \ \alpha_1 + \alpha_2 < 1$$

The EWMA model (with an assumption of zero mean return) can be represented as:

$$(3)\ \sigma_{t+1}^2 = \lambda \sigma_t^2 + (1 - \lambda) R_t^2$$

What assumptions are required, so that the GARCH(1,1) model is equivalent to the EWMA model?

CHAPTER 25

Pricing European Options

Aims

- To adapt the standard Black–Scholes formula to price European options on stocks that pay dividends.
- To price European foreign currency options and futures options.
- To demonstrate the links between pricing formulas for European options on dividend paying stocks, foreign currency options and options on futures contracts.
- To examine the links between the put–call parity relationship for European options on dividend paying stocks, on foreign currency and on futures contracts.

The standard Black–Scholes formulas for call and put options on a non-dividend paying stock are:

$$C = SN(d_1) - Ke^{-rT}N(d_2) \quad \text{and} \quad P = Ke^{-rT}N(-d_2) - SN(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

25.1 WHAT DO $N(d_1)$ AND $N(d_2)$ REPRESENT?

Even without detailed proofs, we can get some insight into the interpretation of these two concepts. The term $N(d_2)$ is the probability that the call option will be exercised (i.e. $S_T > K$) in a risk-neutral world. If the call option is exercised then the holder of the call pays K, so the expected cash outflow at T is $KN(d_2)$, with expected present value $e^{-rT}KN(d_2)$. The delta of the call is equal to $N(d_1)$. In the delta-hedged portfolio you hold $N(d_1)$ stocks and their current value (conditional on the call being exercised) is $SN(d_1)$. Hence today the expected payoff to the call in a risk-neutral world is $C = SN(d_1) - Ke^{-rT}N(d_2)$.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

25.2 EUROPEAN OPTIONS: DIVIDEND PAYING STOCKS

We show how to adapt the Black–Scholes equations to price options where the underlying asset generates a cash flow. To incorporate dividends in the Black–Scholes model for *European* options, two alternative methods are available depending on whether dividends are paid continuously or are discrete payments. Discrete dividend payments are applicable to options on individual stocks as dividends are often paid twice a year, while the assumption of continuous dividends is more applicable to options on stock indices, where the assumption that dividends are paid in a continuous stream (from many different stocks) is not too unrealistic.

25.2.1 Discrete Dividends

Assume there are n known dollar dividend payments D_i over the life of the option, which are paid at *discrete times* t_i *from today* (fraction of a year) on the ex-dividend dates. To price an option on a stock which pays discrete dividends, we merely replace the stock price S in the Black–Scholes equation with S^* where:

$$S^* = S - PV(dividends) = S - \sum_{i=1}^{n} D_i e^{-r_i t_i} \tag{25.1}$$

25.2.2 Continuous Dividend Payments

Suppose dividends are paid continuously and the (annual) dividend yield δ (decimal) is constant. There is a 'trick' which enables us to obtain the correct option pricing formula for an option on a stock that pays continuous dividends:

In the Black–Scholes formula use $S^* = Se^{-\delta T}$ in place of S.

The standard Black–Scholes formulas then become:

$$C = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \tag{25.2}$$

$$P = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1) \tag{25.3}$$

Also $\ln(Se^{-\delta T}/K) = \ln(S/K) - \delta T$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{25.4}$$

$$d_2 = \frac{\ln(S/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{25.5}$$

where

C = call premium	P = put premium
S = stock price	δ = dividend yield (continuously compounded)
σ = standard deviation	r = risk-free rate (continuously compounded)
T = time to maturity	

The intuition behind the above 'trick' is as follows. Payment of dividends causes a fall in the stock price by the amount of the dividend payment. Hence the *growth rate* for a stock which pays continuous dividends is reduced by δ.

25.3 FOREIGN CURRENCY AND FUTURES OPTIONS

The above formulas for pricing European calls and puts *on dividend paying stocks* can be easily adapted to price European options on foreign currencies and European options on futures contracts. Before we get into the detailed formulas note that the changes required are:

Option on foreign currency:

Replace δ with the foreign interest rate r_f.
S is now the spot exchange rate.
σ is the volatility of the (log) return on the spot-FX rate.

Options on futures:

Replace δ by r (this implies r 'disappears' from the definition of d_1 and d_2).
Replace S by F (the futures price).
σ is the volatility of the (log) return of the futures price.

When the cost of carry and the convenience yield are functions only of time it can be shown that the volatility of the futures price equals the volatility of the underlying asset, and σ is usually taken to be the volatility of the futures price. Also, note that the option pricing formula for futures options (known as Black's model) does not require that the futures option and the (underlying) futures contract have the same maturity date.

25.3.1 European Option on Foreign Currencies

Holding a foreign currency is analogous to holding a stock paying a known dividend yield (Garman-Kohlhagen 1983). The foreign currency pays a 'dividend' equal to the foreign risk-free rate r_f. Since we assume the same stochastic process for the exchange rate S as for a stock (paying dividends), our earlier formulas for pricing apply with:

δ (dividend yield) is replaced by r_f, the foreign interest rate

$$C = Se^{-r_f T}N(d_1) - Ke^{-rT}N(d_2) \tag{25.6}$$

$$P = Ke^{-rT}N(-d_2) - Se^{-r_f T}N(-d_1) \tag{25.7}$$

$$d_1 = \frac{\ln(S/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{25.8}$$

$$d_2 = \frac{\ln(S/K) + (r - r_f - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{25.9}$$

The spot rate S is measured as domestic per unit of foreign currency. For example, for a US resident if S is measured as dollars per euro, then r = the domestic (US) interest rate and r_f = interest rate in the Euro area. σ is the volatility of $\ln(S_t/S_{t-1})$ – the change in the (log) exchange rate. Also, r and r_f are T-period interest rates (continuously compounded).

The above equations can be expressed in terms of the forward FX-rate (for the same maturity date as the option) since:

$$F = Se^{(r-r_f)T} \tag{25.10}$$

which makes the FX-option price formula the same as Black's generic formula for pricing options on any type of futures contract (see Chapters 38 and 39). Substituting $S = Fe^{-(r-r_f)T}$ in the above Equations (25.6)–(25.8), gives the *'forward price'* versions of European call and put premia on a foreign currency:

$$C = e^{-rT}[FN(d_1) - KN(d_2)] \tag{25.11}$$

$$P = e^{-rT}[KN(-d_2) - FN(-d_1)] \tag{25.12}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} \tag{25.13}$$

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{25.14}$$

The above formulas for C and P are easy to use and an example is given in Example 25.1.

EXAMPLE 25.1

Pricing a Call Option on Sterling (GBP)

Inputs:

$$\begin{aligned} S &= 142 \text{ (cents/GBP)} & K &= 145 \text{ (cents/GBP)} \\ r &= 0.05 \text{ (US interest rate)} & r_f &= 0.09 \text{ (UK interest rate)} \\ \sigma &= 0.15\ (\sigma^2 = 0.0225) & T &= 0.1370\ (\approx 50 \text{ days}) \end{aligned}$$

$$d_1 = \frac{\ln(142/145) + (0.05 - 0.09 + 0.0225/2)0.1370}{0.15\sqrt{0.1370}} = -0.4475$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.4475 - 0.0555 = -0.5030$$

$$N(d_1) = 0.3264, \qquad N(d_2) = 0.3074$$

$$C = 142\ e^{-0.09(0.1370)}(0.3264) - 145\ e^{-0.05(0.1370)}(0.3074)$$

$$= 45.7808 - 44.2687 = 1.5121 \text{ (cents/GBP)}$$

25.3.2 European Option on a Futures Contract

At maturity, a futures option delivers a futures contract. The payoff at maturity for a call futures option is $\max(0, F_T - K)$ and for a put futures option is $\max(0, K - F_T)$. It can be shown that the futures price can be treated like a security paying a continuous dividend at a rate $\delta = r$. Hence, we adapt the Black–Scholes formula (for dividend paying stocks) by:

δ(dividend yield) is replaced by r and S is replaced by F.

This gives Black's (1976) formula for pricing options on futures contracts:

$$C = e^{-rT}[FN(d_1) - KN(d_2)] \tag{25.15}$$

$$P = e^{-rT}[KN(-d_2) - FN(-d_1)] \tag{25.16}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} \tag{25.17}$$

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{25.18}$$

Not surprisingly the above formulas are the same as those for the 'forward price' version of an option on a foreign currency (Equations 25.11–25.14).

The assumptions, when using Black's (1976) formula, are that the interest rate is constant and the underlying futures price has a lognormal distribution at maturity of the option. The latter is not unreasonable in the case of options on stock index futures and currency futures. However, for options on an interest rate dependent security (e.g. options on T-bonds, on T-bond futures, and options on interest rates), the assumption of a *constant* interest rate as used in Black's model is less tenable – but even here Black's model is often used to price European options on these fixed income securities (see Chapters 38 and 39).

25.4 PUT–CALL PARITY

25.4.1 European Options on Dividend Paying Stocks

Similar to the above analysis, the put–call parity condition for an option on a stock that pays continuous dividends is obtained by replacing S by $S^* = Se^{-\delta T}$ in the put–call parity condition for a non-dividend paying stock:

$$\text{Non-dividend paying stock: } S + P = C + Ke^{-rT} \tag{25.19}$$

$$\text{Stock paying continuous dividends: } Se^{-\delta T} + P = C + Ke^{-rT} \tag{25.20}$$

To show that Equation (25.20) holds, consider the following two portfolios:

Portfolio-A: One European put option plus $N = e^{-\delta T}$ stocks, with dividends being reinvested in *additional stocks* at the rate δ.

Portfolio-B: One European call plus cash of Ke^{-rT}.

Both options have the same strike and time to maturity. For $S_T > K$, portfolio-A consists of *one*-stock worth S_T, since the put is not exercised. For $S_T > K$, portfolio-B has a payoff from the call of $S_T - K$ plus cash of K, making a total payoff of S_T – the same payoff as portfolio-A.

For $S_T < K$, portfolio-A has a put payoff of $K - S_T$ plus one stock worth S_T, giving a total payoff of K. For $S_T < K$, portfolio-B has a payoff of K in cash and the call is out-of-the-money. Hence portfolios A and B have the same payoffs at T and therefore must be worth the same today, otherwise risk-free arbitrage profits are possible. This gives rise to the put–call parity relationship in Equation (25.20).

25.4.2 European Options on Currencies

For European options on currencies we replace δ with r_f so put–call parity is:

$$Se^{-r_f T} + P = C + Ke^{-rT} \tag{25.21}$$

Put–call parity for foreign currency options can be established in a similar fashion to that for an option on a dividend paying stock. We take the US dollar as the domestic currency and the (pound) sterling (GBP) as the foreign currency. The spot exchange rate is S (\$/£). Consider the following two portfolios:

Portfolio-A: One put with price \$$P$ (strike K and maturity T)

plus cash of USD $= Se^{-r_f T}$ invested in a foreign risk-free asset.

Portfolio-B: One long call at \$$C$ (strike K and maturity T)

plus cash of USD $= Ke^{-rT}$ invested in a domestic (US) risk-free asset.

The risk-free asset could be T-bills or a bank deposit. We show in Table 25.1 that the payoffs at maturity are the same for both portfolios and hence they must be worth the same today, which gives the put–call parity condition of Equation (25.21).

For portfolio-A, if cash of USD, $Se^{-r_f T}$ is switched into sterling (at the current exchange rate S \$/£), you receive GBP of $e^{-r_f T}$ today. If this sterling amount is invested in a sterling bank deposit paying r_f it will accrue to £1 at T which can then be switched back into S_T dollars (Table 25.1, top row).

First consider the outcome for the two portfolios when $S_T > K$. Portfolio-A pays S_T dollars, since the put is out-of-the money. At T, portfolio-B pays out $S_T - K$ dollars from the call and \$$K$ on the domestic (US) asset, giving a total payoff S_T – the same as portfolio-A.

Consider the outcomes for $S_T < K$. The put in portfolio-A pays out $K - S_T$ dollars and the sterling asset pays £1, equivalent to S_T dollars at T – that is, a net payoff of K dollars. For $S_T < K$ the call in portfolio-B is not exercised and the US bank deposit pays out \$$K$ at T – which is the same payoff as portfolio-A.

TABLE 25.1 Put–call parity, currency options

	Today	$S_T > K$	$S_T < K$
Portfolio-A:	Cash of USD Se^{-rT} invested in foreign risk free asset *plus*	S_T	S_T
	Long put	0	$K - S_T$
	Outcome Portfolio A	S_T	K
Portfolio-B:	Long call *plus*	$S_T - K$	0
	Cash of USD Ke^{-rT} invested in US risk free asset	K	K
	Outcome Portfolio B	S_T	K

25.4.3 European Futures Options

For European options on futures we take the put–call parity condition for a dividend paying stock, Equation (25.20) and

replace δ by r and replace S by F

which gives put–call parity for futures options:

$$P = C + (K - F)e^{-rT}. \tag{25.22}$$

An alternative derivation is to note that $F = Se^{(r-\delta)T}$, hence $S = Fe^{-(r-\delta)T}$ and if this is substituted in (25.20) we again obtain (25.22). To demonstrate put–call parity, we construct two portfolios which have the same payoff at expiration and hence must have the same value today (Table 25.2). The two portfolios are:

Portfolio-A: Long futures at cost of zero + Long put option on futures.

Portfolio-B: Long call option on futures + Cash of $(K - F)e^{-rT}$.

The futures and the option both expire at T. The cost of the long futures at $t = 0$ is zero, since no money is required to initiate a futures position (we ignore margin payments). Note that if $K > F$ then today we hold cash of $(K - F)e^{-rT}$ and if $K < F$ we borrow an amount today of $|(K - F)e^{-rT}|$. The amount $(K - F)e^{-rT}$ will accrue to $(K - F)$ at time T.[1] The long futures position at T has a cash payout of $F_T - F$, while the put payoff is $\max(K - F_T, 0)$.

TABLE 25.2 Put–call parity, futures options

Portfolio	Current value	$S_T > K$	$S_T < K$
Long futures	0	$F_T - F$	$F_T - F$
Long put	P	$K - F_T$	0
Portfolio A: Long futures + Long put		$K - F$	$F_T - F$
Long call	C	0	$F_T - K$
Cash	$(K - F)e^{-rT}$	$K - F$	$K - F$
Portfolio B: Long call + Bonds/cash		$K - F$	$F_T - F$

[1]If $K > F$ then this is a long position in a bond (or 'lending'). If $K < F$, this is a short position in the bond (i.e. we *issue* bonds with *face value* $F - K$ and hence borrow funds). In either case the cash flow is $K - F$ on maturity.

First consider $F_T < K$. For portfolio-A the long futures payoff is $F_T - F$ and the put payoff is $K - F_T$, giving a net payoff of $K - F$. For portfolio-B when $F_T < K$, the long call has a payoff of zero and cash of $(K - F)e^{-rT}$ accrues to $K - F$, so portfolio-B has the same payoff as portfolio-A.

Next consider $F_T > K$. Portfolio-A has the long futures payoff $F_T - F$ and the put payoff is zero. For portfolio-B when $F_T > K$, the call has a payoff $F_T - K$ and the cash accrues to $K - F$ giving a net payoff of $F_T - F$ – the same as for portfolio-A.

Portfolios A and B have the same payoff at T and therefore must be worth the same today – otherwise risk-free arbitrage profits can be made. This gives rise to the put–call parity condition of Equation (20.22).

25.5 SUMMARY

- The standard Black–Scholes formula (on a non-dividend paying stock) can be adapted to price European options on stocks which pay discrete dividends by replacing S in the Black–Scholes formula with:

$$S^* = S - PV(dividents) = S - \sum_{i=1}^{n} D_i e^{-r_i t_i}$$

- To price European calls and puts on stock indices which pay *continuous* dividends at the rate δ, we take the standard Black–Scholes formula (without dividends) and replace S with $S^* = Se^{-\delta T}$, hence:

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \tag{25.23}$$

$$P = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \tag{25.24}$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{25.25}$$

$$d_2 = \frac{\ln(S/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{25.26}$$

- European options on foreign currencies can be priced using Equations (25.23)–(25.26) by substituting the foreign interest rate r_f in place of δ, with r = domestic interest rate, S = spot exchange rate (domestic per unit of foreign currency) and σ is the volatility of the (log) return of the spot-FX rate.
- European options on futures contracts can be priced using Equations (25.23)–(25.26) by 'replacing' δ with r and replacing S by F. This gives Black's (1976) formula for pricing options on futures contracts.

$$C = e^{-rT}[FN(d_1) - KN(d_2)] \qquad P = e^{-eT}[KN(-d_2) - FN(-d_1)]$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

- Put–call parity condition for European options on a stock that pays continuous dividends is obtained by replacing S with $S^* = Se^{-\delta T}$ in the put–call parity condition for a non-dividend paying stock:

$$\text{Non-dividend paying stock:} \qquad S + P = C + Ke^{-rT}$$

$$\text{Stock paying continuous dividends:} \qquad Se^{-\delta T} + P = C + Ke^{-rT} \tag{25.27}$$

- Put–call parity for European options on a foreign currency can be obtained using Equation (25.27) by replacing δ with the foreign interest rate r_f.
- Put–call parity for European options on futures contracts can be obtained from Equation (25.27) by replacing δ with the (domestic) interest rate r and replacing S by F:

$$P = C + (K - F)e^{-rT}$$

- The put–call parity relationships for European options all assume that risk-free arbitrage opportunities have been eliminated.

EXERCISES

Question 1

The stock index is $S = 1{,}100$ with volatility $\sigma = 20\%$ p.a. and the risk-free rate of interest is $r = 5\%$ p.a. (continuously compounded). European calls and puts are available with a strike of $K = 1{,}000$ and time to maturity $T = 6/12$ years. Over the life of the option, the stocks in the index pay a dividend at a (continuously compounded) rate of $\delta = 0.03$ p.a.

Use the Black–Scholes equation to calculate d_1 and d_2 *and* the price of the call and put. Check your calculations using put–call parity.

Question 2

Swiss franc (SFr) January-60 European call and put options have 6 months to maturity. The current spot-FX rate is 64, the current US risk-free rate is 6% p.a. and the Swiss interest rate is 3% p.a. (continuously compounded). The volatility σ of the USD-SFr exchange rate is 20% p.a. (All currency quotes are in US cents per Swiss franc.)

(a) Calculate the call and put premia (for these European options) using the Black–Scholes (Garman–Kohlhagen) formulas.
(b) Check your calculations for call and put premia using put–call parity.
(c) What is the intrinsic and time value of the call and put?

Question 3
What is the link between the original Black–Scholes (European) option pricing formula on an underlying asset with price, S paying a continuous ‘dividend yield’ δ, and Black’s formula for the price of a (European) futures option?

Question 4
The price of a European call on a foreign currency is given by

$$C = Se^{-r_f T}N(d_1) - Ke^{-rT}N(d_2)$$
$$d_1 = \frac{\ln(S/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Using the equation for the forward rate for foreign currency show that this can be expressed as:

$$C = e^{-rT}[FN(d_1) - KN(d_2)]$$
$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Question 5
A European futures call option on oil with 6 months to expiration has a strike price $K = \$100$, the current 6-month futures price $F = \$100$, the risk free rate $r = 4\%$ (continuously compounded) and the volatility of the (log) change in the oil price is $\sigma = 20\%$ p.a.. The call premium using Black’s formula is $C = 5.5256$. A European put option (with the same underlying, strike and time to maturity as the call) has a quoted price of $P_q = 5.0$.

Using put–call parity, show how an arbitrage profit can be made.

CHAPTER 26

Pricing Options: Monte Carlo Simulation

Aims

- To analyse how plain vanilla European options can be priced under risk-neutral valuation (RNV), using Monte Carlo simulation (MCS).
- To show how we can reduce computational time in MCS.
- To show how 'the Greeks' are calculated using MCS.
- To apply MCS in pricing complex options, such as options whose payoff depends on several underlying variables (e.g. spread options, barrier options, options with stochastic volatility).

The concept of RNV can be used to determine option premia using MCS. This method is especially useful for some exotic options where the option payoffs depend on the complete path taken by the underlying asset S_t (and not just on the final value S_T), or where the option payoff depends on more than one underlying asset. The method can be adapted to accommodate any stochastic path for the underlying asset. However, the method has some drawbacks. It is computationally time consuming and not particularly suited in handling situations where early exercise is possible (i.e. American options).

26.1 BROWNIAN MOTION: PARALLEL UNIVERSE

Pricing options using MCS requires a stochastic process for the underlying asset. Suppose we assume that the stock return has an annual mean return of $\mu = 0.15$ (15% p.a.)

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

with standard deviation, $\sigma = 0.20$ (20% p.a.) and is subject to random events:

$$R_t = \mu + \sigma\,\varepsilon_t \qquad \varepsilon_t \sim niid(0,1) \tag{26.1a}$$

where

$$R_t \equiv (S_t/S_{t-1}) - 1 \tag{26.1b}$$

We assume the random error ε_t is normal, independent, and identically distributed with mean zero and a standard deviation of one, $niid(0,1)$. The standard deviation of the stock return R_t is σ. The random variable ε_t represents 'firm specific' events (such as strikes, legal disputes, regulatory and environmental issues, unexpected cost increases, breakdown of equipment, reputational damage, etc.). Substituting (26.1b) in (26.1a) and rearranging, gives the stochastic process for the stock *price*:

$$S_t = (1 + \mu + \sigma\,\varepsilon_t)S_{t-1} \tag{26.2}$$

This equation represents the behaviour of the stock price over annual intervals. Over a small interval of time, for example, $dt = 0.01$ years (i.e. approximately 2.5 trading days) the mean return is $\mu\ dt$ and the standard deviation is $\sigma\sqrt{dt}$. Hence, the path for stock prices over a small interval of time is:

$$S_t = S_{t-1}(1 + \mu.dt + \sigma\sqrt{dt}\ \varepsilon_t) \tag{26.3}$$

which is known as a (discrete) *Brownian motion*. Setting $S_0 = 100$, we draw successive values for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$ from $\varepsilon_t{\sim}niid(0,1)$ and using Equation (26.3) we generate a random series for S_t. Note that (26.3) is a *recursion,* so if $\varepsilon_1 = 0.49$:

$$S_1 = 100(1 + 0.15(0.01) + 0.20\sqrt{0.01}\ 0.49) = 101.13 \tag{26.4}$$

S_1 is then used to calculate S_2. If the next random draw from the normal distribution is $\varepsilon_2 = 0.765$ then:

$$S_2 = 101.13(1 + 0.15(0.01) + 0.20\sqrt{0.01}\ 0.765) = 103.08 \tag{26.5}$$

We can repeat this say $n = 200$ times, to obtain 200 data points for the stock price, which represents 2 years in total ($n = T/dt = 2/0.01 = 200$). This gives us just *one* possible path that the stock price might have taken over this 2-year horizon. Starting again with $S_0 = 100$, we can draw another 200 random values for ε_t and get a 'new' path for the stock price over 2 years – one in which the 'true' mean return is still 15% p.a. and the standard deviation 20% p.a. but the series of random shocks ε_t are different.

We have created a 'parallel universe' where the initial value S_0, the mean return and standard deviation are fixed but we introduce random 'firm specific' events ε_t into our stock

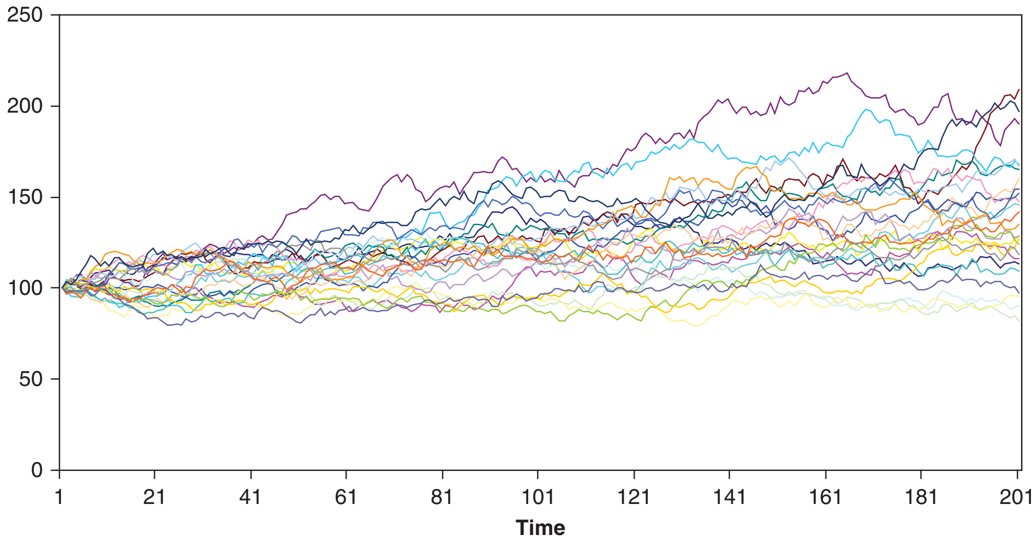

FIGURE 26.1 Brownian motion

price series. We are replaying history but at each point pricing a European call in time we 'roll the dice' to simulate the positive or negative random variable ε_t, which represents random events that might affect the firm's stock price. The simulated possible paths for the stock price over the 2-year horizon are shown in Figure 26.1.

Along any *single realised path* the stock price will not grow at exactly 15% p.a. because of the 'randomness' in the price series. For example, if the first few random shocks ε_t are relatively large negative numbers, then the stock price after the first few periods will be very low. Hence even after 2 years the final stock price S_{200} may sometimes end up below its initial value of $S_0 = 100$ (Figure 26.1). On average, across all the simulated stock price series, the stock price grows at 15% p.a. but there is substantial variation around this mean value, because the standard deviation of stock returns, $\sigma = 20\%$ p.a. is quite large.

> *Excel and MATLAB files to generate a Brownian motion for stock prices, given the mean return and standard deviation of returns, can be found on the website.*

26.2 PRICING A EUROPEAN CALL

Consider pricing a European call on a stock using Monte Carlo simulation. First we need to set up a stochastic equation for the stock price in a risk-neutral world – that is assuming the stock price grows at the risk-free rate. Let T (= 1 year) be the life of the option, the time step $dt = 0.01$ (about 2.5 trading days) and therefore we generate $n = T/dt = 100$ values for the stock price, for each run of the MCS. Assume the following values apply:

$S_0 = 100$, $K = 100$, $\sigma = 0.20$ (20%), $r = 0.05$ (5%), $T = 1$ year, $n = T/dt = 100$

In continuous time finance (see later chapters) the return on a stock over a small interval of time (measured in the conventional way $R_t \equiv (S_t/S_{t-1}) - 1 \approx dS/S$), is assumed to follow a *geometric Brownian motion*:

$$dS/S = \mu\, dt + \sigma\sqrt{dt}\, \varepsilon_t \tag{26.6}$$

where μ = actual ('real world') growth rate of the stock price. (Equation 26.6 is the continuous time version of the discrete geometric Brownian motion, Equation (26.3.) It is shown in Chapter 47 (using Ito's lemma) that Equation 26.6 implies the following equation for the continuously compounded (log) return, $d[\ln(S)]$:

$$d[\ln(S)] = (\mu - \sigma^2/2)dt + \sigma\sqrt{dt}\, \varepsilon_t \tag{26.7}$$

Discrete time versions of Equations (26.6) and (26.7) are:

$$S_t = (1 + \mu.dt + \sigma\sqrt{dt}\, \varepsilon_t)S_{t-1} \tag{26.8a}$$

$$S_t = S_{t-1} \exp[(\mu - \sigma^2/2)dt + \sigma\sqrt{dt}\, \varepsilon_t] \tag{26.8b}$$

Equation (26.8a) has a random term which is normally distributed but it only produces an approximation to a *lognormal* distribution for S_T (and approaches the lognormal as $dt \to 0$). However, (26.8b) gives an exact lognormal distribution for S_T for all values of dt. Hence, if the payoff to an option depends only on S_T then (26.8b) can be used to give the value of S_T in 'one step' – thus saving on computational time. The reason we use (26.8b) is that the Black–Scholes model assumes the terminal stock price has a lognormal distribution – although both recursions give paths for S_t which are similar, when $dt < 0.01$. An example of MATLAB code to generate 'nsim' columns of values for S_t, each column of which contains 'nobs' = $n = T/dt$ observations is given in Example 26.1 (where $\mu = r$, the risk-free rate, as we are pricing the option in a risk-neutral world).

EXAMPLE 26.1

MATLAB file for Geometric Brownian Motion

```
r =0.02 ; sigma = 0.20 ; T = 1; dt = 1/100 ;  nobs  = T/dt ;  nsim  = 5000;

      j=1;                                                % j is the column index
          while j<=nsim ;
```

```
        e = randn(nobs,1);      % generate nobs x 1 series of N(0,1) errors
        t=1;
          while t < nobs;       % t is the row index
        t = t+1;
          Stock(t, j) = Stock(t-1, j).* ( exp( ( r-sigma.^2./2 ).*dt +
               sigma.*sqrt(dt).*e(t) )  );
      end
    j=j+1 ;
  end

% 'Stock' has dimension (nobs x nsim)
```

We can use MCS under RNV to obtain the price of a *European* option (on a non-dividend paying stock). For any run-i ($i = 1, 2, \ldots, m$) in the MCS:

- Use the Brownian motion in Equations (26.8a) or (26.8b) with $\mu = r$ (under RNV)[1] to generate the final stock price S^i_{100}. The payoff to the call is:

$$\text{Call payoff} = \max(S^i_{100} - K, 0)$$

- Discount the call payoff at the risk-free rate (this is RNV again) to obtain our first estimate of the option premium:

$$\widehat{C}^i = e^{-rT}\max(S^i_{100} - K, 0)$$

- Repeat the above $m = 10{,}000$ times to obtain 10,000 values for $\widehat{C}^i$
- The best estimate of the option premium from the MCS is:

$$\widehat{C} = (1/m)\sum_{i=1}^{m} \widehat{C}^i \tag{26.9}$$

For a standard European option where the payoff depends only on S_T we can use Equation (26.8b) to obtain S_T in 'one step' (i.e. set $dt = T$) and the distribution for S_T will be lognormal. (But for *path-dependent* options we have to recursively generate *all 100* values of S_t, for each run of the MCS – see Section 26.6.)

In the Excel spreadsheet in Table 26.1 we find that $\widehat{C} = 8.29$ but as we have only used $m = 200$ runs of the MCS (for pedagogic purposes) this will be a long way from the 'correct'

[1]When pricing a European option on a stock *paying continuous dividends* at the rate δ we replace μ with $r - \delta$. For foreign currency options we replace μ with $r - r_f$, that is, the domestic minus the foreign interest rate and for an option on a futures contract we set $\mu = 0$.

TABLE 26.1 Pricing a Vanilla Call Using MCS

Time	Sim 1	Sim 2	Sim 3	Sim 4	Sim 5	Sim 6	Sim 7	Sim 8	Sim 9	Sim 10
0	100	100	100	100	100	100	100	100	100	100
0.01	102.0683	98.24254	98.87855	101.5431	103.5184	95.2749	100.6461	99.78637	99.00934	100.2764
0.02	101.4991	98.61617	97.02375	101.1719	102.4285	94.31592	104.9234	97.97129	97.66684	100.5254
...	...	...	...	...	...	...	...	...	...	...
0.95	132.726	115.302	121.9	93.21048	74.42675	104.8382	137.0511	83.4953	78.38809	106.5505
0.96	132.0004	115.3534	121.933	95.04815	75.91156	106.4852	138.3202	81.60178	77.16147	104.5654
0.97	133.2309	113.4452	125.1421	99.45439	74.88303	108.0976	138.6531	81.6501	77.80683	107.2737
0.98	131.967	110.7992	122.8945	98.97497	74.8229	106.5214	135.3777	84.09324	77.13906	108.8608
0.99	132.0259	112.8501	120.0849	97.57811	75.52392	106.7636	137.2358	84.23462	77.61826	109.955
1	130.4202	115.9259	122.0662	97.48344	75.87743	105.2443	135.0986	85.72814	78.28576	108.3915
Call Payoff: Max(S_T-K, 0)										
Call Payoff for each "run"	25.42017	10.92586	17.06618	0	0	0.24429	30.09859	0	0	3.391519
Mean payoff for Call	8.71466									
PV of "mean payoff"	8.289641	Equals the MCS call premium								
Put Payoff: Max(K-S_T, 0)										
Put Payoff for each "run"	0	0	0	7.516557	29.12257	0	0	19.27186	26.71424	0
Mean payoff for Put	8.262523									
PV of "mean payoff"	7.859555	Equals the MCS put premium								

value given by the Black–Scholes equation. MCS requires a large number of simulations before $\widehat{C}$ settles down to a near constant value. Each run of the MCS gives a value for the discounted payoff $\widehat{C}^i$ which could be plotted in a histogram – which (because of the asymmetric payoffs to the option) will have a lower bound of zero and a long right tail. The standard deviation of $\widehat{C}^i$ (i.e. the 'width' of this non-symmetric histogram) is:

$$stdv(\widehat{C}^i) = [1/(m-1)]\sum_{i=1}^{m}(\widehat{C}^i - \widehat{C})^2 \tag{26.10}$$

Although the outcomes for each $\widehat{C}^i$ are not normally distributed we know from the central limit theorem, that the distribution of the mean value approaches the normal distribution as m increases. Also the 'standard error'[2] of our estimate of $\widehat{C}$ is given by $stde(\widehat{C}) = stdv(\widehat{C}^i)/\sqrt{m}$. A 95% confidence interval for the 'true' option price C from the MCS is:

$$\widehat{C} - 1.96\frac{stdv(\widehat{C}^i)}{\sqrt{m}} < C < \widehat{C} + 1.96\frac{stdv(\widehat{C}^i)}{\sqrt{m}}.$$

To increase the accuracy by a factor of 10 requires an increase of the number of simulations by '100m'. The number of runs in the MCS has to be rather large before $\widehat{C}$ settles down and we find that for $m = 5{,}000$ trials (using MATLAB), the final MCS value is $\widehat{C} = 10.541$, which is 0.86% above the Black–Scholes value, $C^{BS} = 10.451$.

Figure 26.2 shows how the MCS values for the call and put premia change as we increase the number of simulations ('nsim'). Because MCS is a numerical technique there is always some error in the calculation of the option premium. The 95% standard error bands for the MCS call premium gives a range of 10.13 to 10.94, which includes the Black–Scholes price $C^{BS} = 10.451$.

Excel and MATLAB files to calculate European call and put premia can be found on the website.

[2]Another way of looking at this, is we should increase the number of simulations 'm' in the MCS, until the histogram of estimated call values looks like the 'bell shape' of the normal distribution. As we increase the number of simulations 'm', then the 'width' of the bell shape, narrows (as $stde(\widehat{C}) = stdv(\widehat{C}^i)/\sqrt{m}$), and our estimate of the option premium becomes more accurate.

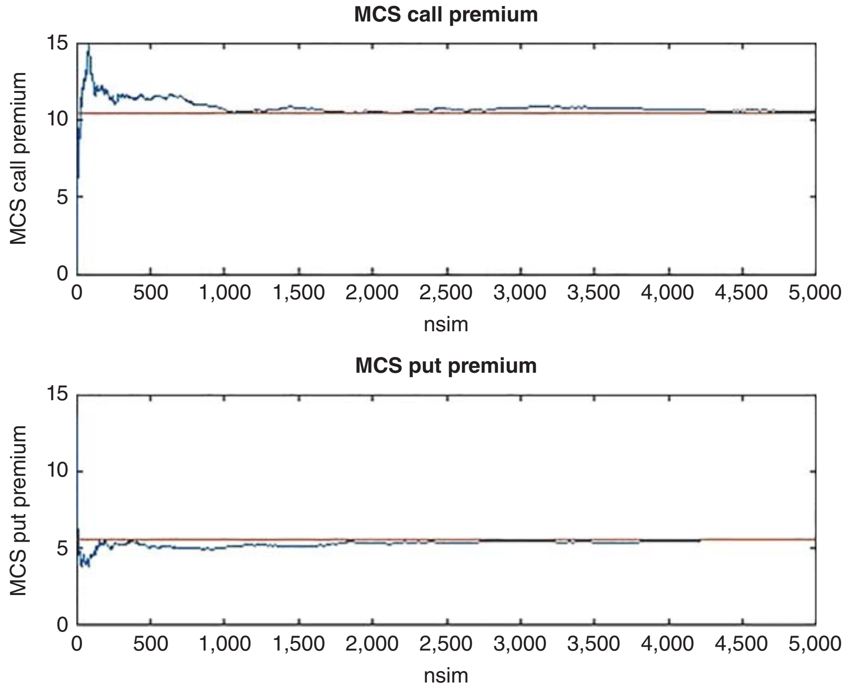

FIGURE 26.2 Call and put premia using MCS

26.3 VARIANCE REDUCTION METHODS

The computation time of the MCS can be reduced by using a number of ‘variance reduction methods’, such as the control variate method and antithetic variables.

26.3.1 Antithetics

In the antithetic method, each time we draw a value for $+\varepsilon_t$ we also use $-\varepsilon_t$ and both are used to generate two new values for the stock price. We therefore get ‘two stock prices from one random draw’. This technique can be applied to any symmetric distribution. Using antithetics we have two *discounted* payoffs for the call for each simulation: C^i_+ using $+\varepsilon$ and C^i_- using $-\varepsilon$. We then take the average of these two (discounted) payoffs, as the payoff for simulation i. The estimate of the option premium for run i of the simulation and the final option premium are:

Option premium for run i: $$C^{i*} = (C^i_+ + C^i_-)/2$$

Final option premium: $$\widehat{C}^* = (1/m)\sum_{i=1}^{m} C^{i*}$$

If the standard deviation of C^{i*} using (26.10) is $stdv(C^{i*})$ then the 'standard error' in estimating the true option premium is $stdv(C^{(i)*})/\sqrt{m}$. We obtain a much more accurate estimate from the m-pairs (C^i_+, C^i_-) than if we used $2m$ values of C^i. This is because in the antithetic variates case, the two payoffs C^i_+ and C^i_- are equivalent to holding two calls, one which depends on $+\varepsilon_t$ and the other that depends on $-\varepsilon_t$ – hence these are perfectly negatively correlated and therefore the variance in outcomes from our 'pseudo-portfolio' of two options is less than that when using $2m$ values C^i for the 'single option'.

26.3.2 Control Variates

This approach is exactly the same as that used when trying to speed up the calculations in the BOPM (see Chapter 23). Suppose we want to calculate the price f_A^{MCS} of a 'complex' option-A using MCS – this might be an Asian call option, for example, where the option payoff depends on the *average value* of the underlying S, over the life of the option.[3]

Suppose we consider a plain vanilla European call option-B[4] whose price f_B^{BS} is directly given by the Black–Scholes equation. We could also price this European option-B using MCS, giving a price f_B^{MCS}. The value of f_B^{MCS} would be close to, but not equal to f_B^{BS} (because of MCS sampling error). The control variate technique adjusts the MCS price of the Asian call option-A (f_A^{MCS}) depending on how big the error is in the MCS valuation of (the plain vanilla European) call option-B. The new 'improved' control variate MCS estimate for the Asian call option-A is therefore:

$$f_A^{new} = f_A^{MCS} + (f_B^{BS} - f_B^{MCS}) \tag{26.11}$$

There are a number of other numerical techniques available to 'speed up' MCS and the interested reader should consult Hull (2018), Wilmott (1998) and the excellent book by Clewlow and Strickland (1998).

26.4 THE GREEKS

The delta of an option can be calculated using MCS by repeating the whole Monte Carlo procedure with a different starting value for the underlying asset price (and with all other factors held constant) – this is the 'perturbation approach' to calculating the option's delta (which we have used with the BOPM).

26.4.1 Perturbation Approach

For example, suppose we consider the call option of Section 26.2, which we have already priced using MCS. Denote the price of the call option $f(S)$, where $S = 100$ is the initial stock price. To

[3]The payoff to this Asian call option is $\max(0, S^i_{av} - K)$ where $S_{av} = (1/n)\sum_{k=1}^{n} S_k$ and there are $n = T/dt$ periods in each run of the MCS.
[4]This option has the same underlying asset, strike price, and time to maturity as the Asian option-A.

calculate the call option's delta, we repeat the MCS with a new starting value $S^+ = S + h$, where h is small (e.g. $h = 0.1$). Keep all other factors (K, r, σ, T) unchanged, including $n = T/dt$, the original random number series, and number of runs m of the MCS. Using the new starting value for the stock price S^+, we obtain a new MCS value for the call option premium $f(S^+)$. The option's delta is then calculated as:

$$\Delta^+ = \frac{f(S^+) - f(S)}{S^+ - S} = \frac{f(S^+) - f(S)}{h} \tag{26.12a}$$

A more accurate estimate of Δ can be obtained using a *central difference* approximation. The initial stock price is S. Rerun the full MCS using $S^- = S - h$ and calculate:

$$\Delta^- = \frac{f(S^-) - f(S)}{S^- - S} = \frac{f(S^-) - f(S)}{-h} \tag{26.12b}$$

The average (central difference) estimate of delta is:

$$\Delta_{av} = \frac{\Delta^+ + \Delta^-}{2} = \frac{f(S^+) - f(S^-)}{2h} \tag{26.13}$$

The 'Greeks' are more fully dealt with in Chapter 28 but it is worth briefly outlining how some of these are calculated using MCS. The above calculation for delta could be repeated for any of the variables that influence the option price (e.g. r, σ, t) to obtain an estimate of rho, vega, and theta, respectively. For example, the vega Λ of an option is the change in the option premia for a small change in volatility $\Lambda = \partial f/\partial \sigma$. Using a central difference, vega is calculated (following Equation 26.13) from a MCS as:

$$\Lambda_{av} = \frac{\Lambda^+ + \Lambda^-}{2} = \frac{f(\sigma^+) - f(\sigma^-)}{2h} \tag{26.14}$$

where $\sigma^+ = \sigma + h$ and $\sigma^- = \sigma - h$. An option's gamma is the change in delta as the stock price changes $\Gamma = \partial^2 f/\partial S^2 = \partial\Delta/\partial S$ and can be approximated using:

$$\Gamma = (\Delta^+ - \Delta^-)/\mathrm{h} = [f(\mathrm{S}^+) - 2f(\mathrm{S}) + f(\mathrm{S}^-)]/h^2 \tag{26.15}$$

26.5 MULTIPLE STOCHASTIC FACTORS

Using MCS to price a European option that depends on more than one underlying asset is straightforward. We simply assume a stochastic process for each underlying asset, for example for two different stocks (or stock indices paying continuous dividends δ) we might use the two geometric Brownian motions:

$$dS_A = (r - \delta_A)S_A dt + \sigma_A S_A dz_A \tag{26.16a}$$

$$dS_B = (r - \delta_B)S_B dt + \sigma_B S_B dz_B \tag{26.16b}$$

where $dz_i = \varepsilon_i\sqrt{dt}$ $(i = A, B)$. Note that the 'real world' growth rate of the stock price μ has been replaced by the risk-neutral term $r - \delta$ – this ensures that (for a dividend paying stock) the expected stock return (capital gain plus dividend yield) equals the risk-free rate r – this is RNV.

We can approximate these two continuous time stochastic differential equations (SDE) using discrete time versions. In addition, we must ensure that the two random variables dz_A and dz_B have the same correlation coefficient as the 'real world' returns on the two stocks A and B (see the Choleski decomposition in Appendix 26).

26.5.1 Pricing a Spread Option

The payoff to a European *call spread option* is $\max(S_{A,T} - S_{B.T} - K, 0)$. Each run of the MCS gives two values $S_{A,T}$, $S_{B,T}$ at the maturity date of the call option, which we assume occurs after $n = 100\ (= T/\Delta t)$ time steps. The call premium for any run-i of the MCS is $\widehat{C}^i = e^{-rT}\max(S^i_{A,100} - S^i_{B,100} - K, 0)$ and the MCS estimate of the call premium is:

$$\widehat{C} = (1/m)\sum_{i=1}^{m}\widehat{C}^i \tag{26.17}$$

26.5.2 Stochastic Interest Rates

Suppose we want to value a European call option on stock-A (which pays continuous dividends) under the assumption of *stochastic interest rates*. The Brownian motion for stock-A under risk-neutrality is:

$$dS_A = (r - \delta_A)S_A dt + \sigma_A S_A dz_A \tag{26.18}$$

Our second equation has to mimic the behaviour of interest rates and because interest rates are usually assumed to be mean reverting we cannot use a Brownian motion. There are many mean reverting equations for interest rates (see Chapters 41 and 49) and a fairly straightforward one is:

$$dr = \beta(\overline{r} - r)dt + \sigma_r dz_r \qquad \beta > 0 \tag{26.19}$$

For example, suppose the current interest rate r is above its long-run value $\overline{r} = 3\%$ (say, given by our sample of data). Then according to Equation (26.19) the change in the interest rate next period will be negative – thus moving the interest rate closer to its long run level of 3%. The speed at which the interest rate approaches its long-run value is given by the parameter β (which has to be estimated).

The random errors in Equation (26.18) and (26.19) will have a correlation coefficient equal to the correlation between returns on stock-A and the change in the interest rate, in our sample of 'real world' data. (The correlation between stock returns and changes in interest rates is likely to be small.) We then simulate the stock price and interest rate using (discrete versions of) *both* Equations (26.18) and (26.19). The payoff to the European call option on stock-A is

$\max(S^i_{A,T} - K, 0)$. Because interest rates are stochastic, we take the *average value* of r in each Monte Carlo simulation as our discount rate:

$$r^i_{av} = (1/n)\sum_{k=1}^{n} r^i_k \quad \text{where} \quad n = T/dt$$

Hence the call premium for run-i of the MCS and for the final option premium is:

$$\widehat{C}^i = e^{-r^i_{av}T}\max(S^i_{A,T} - K, 0) \quad \text{and} \quad \widehat{C} = (1/m)\sum_{i=1}^{m}\widehat{C}^i \tag{26.20}$$

26.5.3 Stochastic Volatility

Suppose we want to value a European call option on a stock (paying a continuous dividend) under the assumption of stochastic volatility – unlike the Black–Scholes approach, which assumes a constant volatility. We require a stochastic differential equation to model the random behaviour of volatility $V \equiv \sigma^2$. Volatility is usually modelled as a mean reverting process. The stochastic equations for the stock price and its volatility (variance) are[5]:

$$dS = (r - \delta)S\,dt + S\sqrt{V}dz_s \tag{26.21a}$$

$$dV = \beta(\overline{V} - V)dt + \xi V^{\alpha}dz_v \tag{26.21b}$$

where $dz_i = \varepsilon_i\sqrt{dt}$, $\overline{V}$ is the long-run average level of the variance and ξ is the *volatility of the variance* V. The terms $\overline{V}$, ξ, α, and β are constants which have to be estimated (often α is set equal to ½). In generating the simulated values for V and S, the correlation between the stock return and its volatility is reflected in the Monte Carlo 'draws' of dz_s and dz_v which mimic this correlation.[6] It is clear from the first equation that to obtain a value for S requires a value for V (and vice versa) but given starting values for S_0 and V_0, future values for both are given by using these two recursive equations.

For reasonable values of the parameters it can be shown that the generated series for stock returns (from the above two equations) has 'fat tails' and some skewness – as found in real world stock return data. Hence our MCS gives the price of the European call under an assumption of 'non-normality', unlike the Black–Scholes model. After generating values for the terminal stock price (using the above two equations), for $n = 100$ (time steps, say) then the option payoff and call premium are calculated in the usual way from the m-simulations:

$$\widehat{C}^i = e^{-rT}\max(S^i_{100} - K, 0) \quad \text{and} \quad \widehat{C} = (1/m)\sum_{i=1}^{m}\widehat{C}^i \tag{26.22}$$

[5]A GARCH model could also be used to model stochastic volatility.

[6]This can be done using a Choleski decomposition and in continuous time notation the correlation is expressed $E(dz_s dz_v) = \rho dt$.

If we really believe that volatility is stochastic we could compare the MCS option price under stochastic volatility with the (incorrect) value given by the Black–Scholes formula to see how misleading the latter might be. For standard European options with less than one year to maturity, the impact of stochastic volatility on the option price is not particularly large (except for deep out-of-the-money options) but the effect is larger for options with longer maturity dates.

MCS is useful when pricing options under stochastic volatility but sometimes a closed-form solution is possible if the stochastic process for V is relatively simple. In fact, a closed-form solution is possible given the above equation for V, as long as V is *uncorrelated with S* (Hull and White 1987). Also for $\alpha = 0.5$, Heston (1973) provides an *approximate* closed-form solution for the option price P_H, of the form:

$$P_H = f[(S, K, r, \delta, T-t); \quad (\overline{V}, \xi, \rho, \alpha, b)] \tag{26.23}$$

The first set of variables in (26.23) are the usual Black–Scholes inputs but the second set arise from the stochastic volatility model and these parameters must be estimated – which implies the resulting option price using Equation (26.23) will be subject to estimation error.

26.6 PATH-DEPENDENT OPTIONS

Consider using MCS to price a 1-year European ATM up-and-out call option (with $K = 100$ and $S_0 = 100$). This is a 'knock-out barrier option' with the upper barrier set at $H = 110$ say.[7] The option payoffs are as follows. If over the life of the option, the stock price hits (or crosses) the upper level $H = 110$ (from below) then the call is 'knocked out' – that is, at this point the option ceases to exist and it must therefore expire worthless, even if at maturity $S_T > K$. If the stock price does not cross H (at any time over the life of the option), the option payoff at maturity T is the same as for a plain vanilla call, $\max(S_T - K, 0)$. Assume the stock price is (only) checked at 4 p.m. each day to ascertain whether it has crossed the barrier or not.

To price this option under RNV we simulate a discrete geometric Brownian motion (GBM) for S (using Equation 26.8b), setting $\mu = r$, (RNV) and using $dt = 1/252$ and $n = T/dt = 252$ time steps. At each of the 252 time steps in the MCS, we check to see if S has crossed the barrier $H = 110$. If in run-i of the MCS the stock price crosses the barrier, we assign a payoff of zero for the option at T for that 'run' of the MCS, even if $S_T^i > K$ at maturity of the option. If S does not cross the barrier $H = 110$ at any of the 252 time steps then the option payoff is the usual $\max(S_T^i - K, 0)$. Hence:

$$\begin{aligned} payoff^i &= \max(S_{100}^i - K, 0) \qquad && \text{if } S_k^i < H \text{ for } k = 1, 2 \ldots n \\ payoff^i &= 0 && \text{if } S_k^i \geq H \text{ for } k = 1, 2 \ldots n \end{aligned}$$

[7] Chapter 31 discusses path-dependent 'exotic' options in more depth.

The MCS option premium $\widehat{C}$ is the average over all $m = 10{,}000$ simulations of the option payoffs (discounted at the risk-free rate)[8]:

$$\widehat{C} = e^{-rT}(1/m)\sum_{i=1}^{m}(payoff^{i}) \tag{26.24}$$

> *MATLAB files to calculate call and put premia for barrier options can be found on the website.*

A limitation of the basic MCS method is that it can only usefully be used for European style options. When pricing American options we have to know the value of the option for *all* values of *S at each point in time in the MCS*, so that we can compare the 'MCS price' with the option's intrinsic value, at each time period. In its basic form, the 'mathematical problem' with MCS arises because the method solves in a forward direction (i.e. start at S_0 and simulate values to S_T), rather than in a backward direction (e.g. as with the BOPM and the backward finite difference method discussed later). However, there are more complex simulation methods that can be used to price American and other non-standard options (e.g. see Longstaff and Schwartz 2001).

26.7 SUMMARY

- The stochastic behaviour of stock prices can be represented by a geometric Brownian motion (GBM), in continuous time. Using a discrete time approximation to a GBM we can generate a series for the stock price once we have an estimate of the mean return on the stock and its standard deviation.
- When pricing an option using MCS *under risk-neutral valuation*, we (a) set the growth rate of the stock price equal to the risk-free rate ($\mu = r$) in the GBM, and (b) we discount the option payoffs using the risk-free rate. The only estimated input in the GBM is therefore the volatility of the stock return.
- MCS is a numerical technique and therefore the resulting option price is subject to error.
- For options whose price is determined by two or more 'variables' (e.g. two underlying stochastic variables S_A and S_B), MCS tends to be a relatively efficient numerical method of obtaining option premia, and it can easily accommodate correlations between the underlying stochastic variables (e.g. using the Choleski decomposition).

[8]To reduce computational time we can use the known closed-form solution for a continuously monitored barrier option as a control variate.

- MCS is very flexible and can handle any parametric stochastic processes for the underlying assets (e.g. normal, mixture-normal, Student's-*t*, etc.) and for any other variables that may influence option prices, such as stochastic interest rates and stochastic volatility. This makes MCS very useful in pricing complex European options.
- Antithetic variables or the control variate technique can be used in a MCS to speed up calculations and to improve the accuracy of the estimated option price (for any given number of simulations).

APPENDIX 26: MCS, SEVERAL STOCHASTIC VARIABLES

To price options using MCS we often assume that (continuously compounded) returns on the underlying assets (i.e. proportionate change in prices over a small interval of time) are multivariate normal and therefore price *levels* are multivariate lognormal. We will demonstrate how correlated variables can be generated in a MCS and we concentrate on the two-asset case, but this can easily be generalised to N-assets.

Standard computer software can be used to generate two normally distributed, *identical* and *independent* random variables ε_1 and ε_2, with zero mean ($E\,\varepsilon_i = 0$) and a standard deviation of 1, $\text{var}(\varepsilon_i) \equiv E(\varepsilon_i^2) = 1$. Hence $\varepsilon_i\ (i = 1, 2) \sim niid(0, 1)$. These random variables are independent and therefore have zero correlation (and covariance) by construction, that is $E(\varepsilon_1\varepsilon_2) = 0$. We wish to create two normally distributed variables z_1 and z_2 both with mean zero and standard deviation of unity but with a correlation coefficient equal to ρ. This is accomplished using the following two equations:

$$z_1 = \varepsilon_1 \tag{26.A.1a}$$

$$z_2 = \rho\varepsilon_1 + \sqrt{1-\rho^2}\,\varepsilon_2 \tag{26.A.1b}$$

The variables z_1 and z_2 have ε_1 in common and this is the source of the correlation between z_1 and z_2. It is easy to see that:

$$\begin{aligned}
Ez_1 = Ez_2 &= 0 \qquad\qquad \text{since } E(\varepsilon_1) = E(\varepsilon_2) = 0\\
\text{var}(z_1) &= E(\varepsilon_1^2) \equiv \text{var}(\varepsilon_1) = 1\\
\text{var}(z_2) &= \rho^2\text{var}(\varepsilon_1) + (1-\rho^2)\text{var}(\varepsilon_2) = 1\\
\text{cov}(z_1, z_2) &= E(z_1 z_2) = \rho E(\varepsilon_1^2) + \sqrt{1-\rho^2}E(\varepsilon_1\varepsilon_2) = \rho\\
\text{corr}(z_1, z_2) &= \text{cov}(z_1, z_2)/(\sigma(z_1)\sigma(z_2)) = \rho
\end{aligned}$$

Let μ = annual mean return (growth rate of the stock price), σ= annual standard deviation of stock returns, T = time to maturity of the option (years), n = number of time-steps chosen in the MCS and the time interval is $dt = T/n$. We now simulate (multivariate lognormal) stochastic variables S_{1t}, S_{2t} using z_{1t} and z_{2t}:

$$S_{1t} = S_{1t-1} \exp\left[(\mu_1 - \sigma_1^2/2)dt + \sigma_1\sqrt{dt}\, z_{1t}\right]$$
$$S_{2t} = S_{2t-1} \exp\left[(\mu_2 - \sigma_2^2/2)dt + \sigma_2\sqrt{dt}\, z_{2t}\right]$$

The correlation coefficient between z_1 and z_2 is ρ so the two return series $\ln(S_{i,t}/S_{i,t-1})$ also have a correlation coefficient of ρ. The variables S_{1t}, S_{2t} could be any two underlying 'assets' depending on the option pricing problem at hand (e.g. two stock prices or one stock price and one interest rate or one stock price and the volatility of the stock return).[9] If we are simulating 'real world' values S_1 and S_2 then μ is the 'real world' mean growth rate of the variable. If we are pricing an option using MCS under RNV then we replace μ by its 'risk-neutral equivalent' – which would be $\mu = r - \delta$ for an option on an asset (e.g. stock) with a continuous payment (e.g. dividend yield) of δ p.a.

Choleski Decomposition to Obtain Correlated Variables

We now present the two-variable case in matrix form. Assume continuously compounded asset returns $d(\ln S_i)$ are multivariate normal, $N(0, \Sigma)$, and we have statistical estimates of the variance-covariance matrix of returns Σ, from which we can construct the correlation matrix, C. For the (2×2) case we have:

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{26.A.2a}$$

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{26.A.2b}$$

where $\sigma_{12} = \sigma_{21}$ and the correlation coefficient $\rho = \sigma_{12}/\sigma_1\sigma_2$. A simple piece of matrix algebra allows us to map the correlation matrix into two matrices A such that:

$$C = AA' \tag{26.A.3}$$

where A is the lower triangular matrix:

$$A = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \tag{26.A.4}$$

[9] The stochastic interest rate and volatility equations would probably not be represented by a GBM but would be mean reverting stochastic processes.

The A-matrix is the Choleski factorisation of the correlation matrix C. It is easy to check that $C = AA'$. The reason for the Choleski factorisation is that if $\varepsilon = (\varepsilon_1, \varepsilon_2)'$ (2×1 vector) consists of two *independent* $N(0,1)$ variables then we can generate our two correlated standard normal variates $Z = [z_1, z_2]'$ using:

$$Z = A\varepsilon \tag{26.A.5}$$

The (two) variables in Z will have zero mean, unit variance but have a correlation of ρ. Most computer programs can generate N correlated normally distributed random variables $Z = [z_1, z_2, \ldots..z_n]'$ using the Choleski decomposition (or some other numerical algorithm), given the $N \times N$ covariance matrix $\Sigma_{N\times N}$ as an input.

> *There is a MATLAB file on the website which uses the Choleski decomposition to produce correlated multivariate normal random variables.*

Practical Issues

Most software packages produce uniform $U(0,1)$ pseudo-random numbers which lie between 0 and 1 and occur with equal probability. These can be 'transformed' into standard normal variables in a variety of ways. For example, if u_i is distributed $U(0,1)$ then $\varepsilon = \sum_{i=1}^{12} u_i - 6$ gives an (approximate) standard normal variable $\varepsilon \sim niid(0,1)$. In Excel the uniform $U(0,1)$ distribution is given by the command: RAND( ), so $\varepsilon = \sum_{i=1}^{12} RAND()_i - 6$ is *niid(0,1)*. A more succinct way of generating ε in Excel is to use the inverse cumulative normal command, so ε=NORMSINV(RAND( )). However, these numerical methods give too many values for ε_i that are close to the mean, than for a 'true' normal distribution. This problem can be mitigated by other 'transformations' (e.g. Box–Muller transformation).

Two possible numerical problems in a MCS are either that our simulated 'pseudo' random numbers are not in fact random at all or, that the Choleski decomposition fails numerically (when the number of assets N is large). Random number generators actually use a deterministic algorithm. They take a particular 'seed number' as a starting point and generate numbers that appear random (and pass tests for statistical independence, etc.). For a given 'seed number', the same set of random numbers will be repeated. The problem is that as you increase the number of 'runs' in the MCS, the random number generator may 'choose' a seed it has already used. This leads to repetitions or cycles in your 'pseudo' random numbers and a spurious increase in accuracy of your Monte Carlo results.

Another problem is that standard MCS tends to produce numbers which 'cluster', so additional observations do not provide new information – and this tends to bias the simulation results. However, so called quasi-Monte Carlo (QMC) techniques or 'low discrepancy sequences' avoid producing 'clusters', so MCS becomes much more efficient and can produce accurate results with substantially fewer 'runs'. The idea is that the 'quasi-random' sample, 'remembers' the previous sample and tries to position itself away from any previous samples, thus 'filling' the sample space without any clustering.

Finally, in order to undertake the Cholesky decomposition the *estimated* variance-covariance matrix Σ must be positive semi-definite. This requires (i) the number of data observations to be greater than the number of assets N in the covariance matrix Σ, and (ii) there is no perfect collinearity amongst the asset returns. However, if the dimension of Σ is large, then near perfect collinearity may become a practical problem.

EXERCISES

Question 1

The return on a stock can be represented by $R_t = \mu + \sigma \varepsilon_t$ where $\varepsilon_t \sim niid(0, 1)$ and $R_t \equiv (S_t/S_{t-1}) - 1$. What does this imply for the stochastic equation for the stock price S_t?

Question 2

When using MCS, assume the path for the stock price S, over a small interval of time is:

$$S_t = (1 + \mu.dt + \sigma\sqrt{dt}\ \varepsilon_t)S_{t-1}$$

$\mu = 10\%$ p.a. (expected return on the stock), $\sigma = 20\%$ p.a. (volatility of stock return) and the initial stock price is $S_0 = 100$. Explain how you would simulate the stock price for each trading day over a horizon of 10 days.

Question 3

Suppose stock prices follow a GBM with mean $\mu = 0$ and standard deviation $\sigma > 0$. After many periods (e.g. $T = 100$ periods) would you expect most of the simulated paths for the stock price to end up close to the initial stock price?

Question 4

Discrete time versions of a Brownian motion (over small time intervals dt) can be represented as:

(1) $S_t = (1 + \mu.dt + \sigma\sqrt{dt}\ \varepsilon_t)S_{t-1}$

or

(2) $S_t = S_{t-1}\exp[(\mu - \sigma^2/2)dt + \sigma\sqrt{dt}\ \varepsilon_t]$

What, if any, are the different outcomes for the stock price produced by these equations after many time periods?

Question 5
How can you calculate the delta of a European option (on a non-dividend paying stock) when using MCS to price the option?

Question 6
Suppose you price a European option using MCS under the assumption that the underlying stock price S follows a geometric Brownian motion and assuming that volatility (variance) V of the underlying stock price is stochastic and mean reverting. As the number of runs in the MCS is increased, would you expect the option price given by the MCS to approach the Black–Scholes price?

Question 7
Suppose you price a European option using MCS under the assumption that the underlying stock price S follows a geometric Brownian motion and assuming that volatility (variance) V of the underlying stock price is stochastic and mean reverting.
Explain what stochastic equation you might use to simulate the behaviour of volatility (variance) V and provide any intuitive interpretation of this equation.

PART VI

THE GREEKS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 27

Delta Hedging

Aims

- To demonstrate simple delta hedging of calls and puts and the need for rebalancing.
- To show how dynamic delta hedging is used to protect the value of an options portfolio from (small) changes in the price of the underlying asset.
- To show that after dynamic delta hedging, you can close out your positions at any time and you will have neither gained nor lost money.

27.1 DELTA

A portfolio can be constructed so that any gain or loss from a small change in the option price (over a short interval of time) is offset by changes in the price of the underlying asset – this is delta hedging. Delta hedging applies to all kinds of options, where the underlying S can be a stock price, stock index, commodity price, exchange rate, futures price, or an interest rate. The delta of a long call (which is written on one stock) is defined as:

$$\Delta_c = \frac{\textit{Approximate change in option's price}}{\textit{Small change in stock price}} = \frac{\partial C}{\partial S}$$

Hence:

$$\textit{Approximate change in call premium}, dC = \Delta_c \times (\textit{Small change in stock price}, dS)$$

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

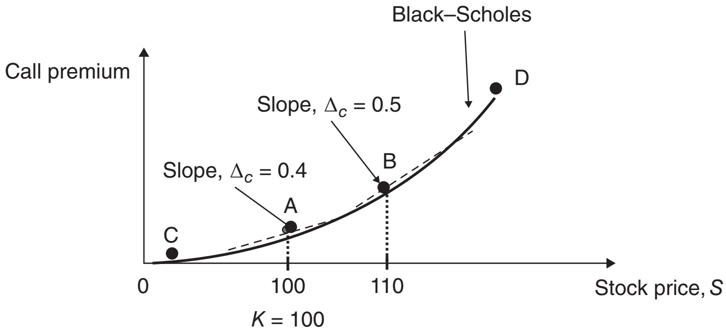

FIGURE 27.1 Delta of a long call

The delta of a long call $\Delta_c \equiv \partial C/\partial S$ is positive because a rise in the stock price leads to a rise in the call premium (Figure 27.1). Consider delta hedging a position in European call options. The value V of a portfolio consisting of N_s stocks and N_c calls is:

$$V = N_s S + N_c C = N_c(hS + C) \tag{27.1}$$

where $h \equiv N_s/N_c$ and is known as the *hedge ratio*. The convention is that N_s or $N_c > 0$ if the stock or option is held long (i.e. 'buy') and N_s or $N_c < 0$ if either is held short (i.e. the stock is short-sold or the option is sold). To hedge we set $\partial V/\partial S = 0$:

$$\frac{\partial V}{\partial S} = h + \frac{\partial C}{\partial S} = 0$$

$$h \equiv N_s/N_c = -\partial C/\partial S \equiv -\Delta_c \tag{27.2a}$$

$$N_s = -N_c \Delta_c \tag{27.2b}$$

The 'minus sign' indicates that if you are long *one* call option ($N_c = +1$), then to hedge you must short-sell Δ_c stocks ($N_s = -N_c\Delta_c = -\Delta_c$). If you have sold (written) one call option ($N_c = -1$), then to delta hedge you must go long (buy) Δ_c stocks ($N_s = -N_c\Delta_c = \Delta_c$).

27.1.1 Delta of a Call

To maintain a delta-neutral position requires continuous *rebalancing* of the stock-call portfolio since Δ_c changes as the stock price changes. At a stock price of \$100, the call has a delta of $\Delta_c = 0.4$ (see Figure 27.1, point A). Given the curvature of the relationship between C and S, if the stock price on day-2 is \$110 then the value of $\Delta_c = 0.5$ (Figure 27.1, point B). Assume for each call the underlying asset is for delivery of 1 stock.

If, for example, the initial position on day-1 consists of 100 written calls then to delta hedge over one day requires buying (go long) 40 stocks (= 100×0.4) at a cost of \$4,000 ($40 \times \$100$). On day-2, $\Delta_c = 0.5$ so the 100 written calls can be hedged over the next day by holding 50 stocks (100×0.5) – that is on day-2 you need to purchase an extra 10 stocks at an additional cost of \$1,100 (= $10 \times \$110$) to maintain a delta hedge position (between day-2 and day-3). This is *dynamic delta hedging*.

When hedging our 100 written calls, if the stock price *falls* from \$110 on day-2 to \$100 say on day-3 then Δ_c falls, from 0.5 to 0.4. Hence, on day-3 to delta hedge over the next day we need to hold 40 stocks and at the beginning of day-3 we sell 10 (= $50 - 40$) stocks (@ \$100) to maintain our delta-hedged position (and we receive \$1,000).

Delta hedging, therefore involves selling stocks just after a price fall and buying stocks just after a price rise. It could be termed (somewhat paradoxically) a 'buy high, sell low' strategy, and this rebalancing is part of the cost of dynamic hedging.

Delta hedging only provides a hedge against small changes in the stock price. To hedge an options position against large changes in the stock price or changes in the volatility of stock returns or changes in the risk free rate, requires additional hedging strategies, which comes under the heading of 'the Greeks' (see Chapter 28).

When the current stock price is very low (i.e. well below the strike price K), the call is well out-of-the-money (OTM) and $\Delta_c = 0$ (see Figure 27.1, point C). Hence to delta hedge (over the next day) requires no stocks to be held, since over a short interval of time, the change in value of 'zero holdings' of stocks is zero and the change in the call premium is also (approximately) zero ($dC = \Delta_c \times dS$). When the stock price is very high (i.e. well above the strike price K), the call is well in-the-money, ITM and $\Delta_c = +1$ (see Figure 27.1, point D). Hence to delta hedge 100 written calls you must hold 100 stocks. If the stock price changes by \$1 (over the next day) the value of your stocks changes by \$100, but the value of the written calls also changes by \$100 (= $100\ calls \times dC = 100 \times \Delta_c \times dS$), and you are hedged.

27.1.2 Delta of a Put

The delta of a long put $\Delta_p \equiv \partial P/\partial S$ is negative because a rise in the stock price leads to a fall in the put premium (Figure 27.2). Using the Black–Scholes formula it can be shown that $\Delta_p = N(d_1) - 1$. Alternatively the put-delta can be obtained from the call delta (for a call with the same underlying asset, time to maturity, and strike price) by differentiating the put–call parity relationship:

$$P + S = C + Ke^{-rT} \Rightarrow \Delta_p = \Delta_c - 1 \tag{27.3}$$

When S is very low (much lower than K), the put is 'well ITM' and a dollar fall in the stock price gives a dollar rise in the put premium, hence $\Delta_p = -1$ (Figure 27.2, point D). When S is

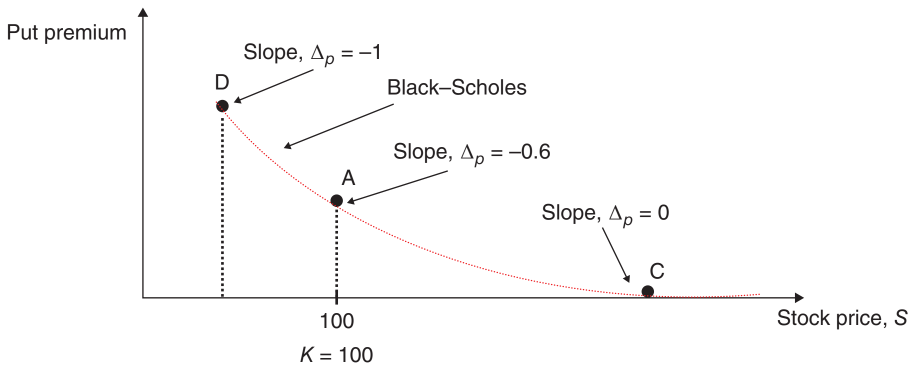

FIGURE 27.2 Delta of a long put

very high (much higher than K), the put is 'well OTM' and a dollar fall in the stock price has virtually zero effect on the put premium and hence $\Delta_p = 0$ (Figure 27.2, point C).

27.1.3 Summary

From the Black–Scholes formula for European calls and puts (on stocks that pay no dividends) it can be shown that:

$$\begin{aligned} \Delta_c &= \partial C/\partial S = N(d_1) && \text{and } 0 \leq \Delta_c \leq 1 \\ \Delta_p &= \partial P/\partial S = N(d_1) - 1 && \text{and } 0 - 1 \leq \Delta_p \leq 0 \end{aligned}$$

Since d_1 depends on S then delta changes as S changes. The hedge ratio for a position in calls is $h \equiv N_s/N_c = -\Delta_c$. Hence, a delta neutral position for European calls involves a 'long-short' position:

Delta-neutral positions with calls (hedge portfolio = 'long-short position')
Portfolio-A: Short 1 call and long Δ_c stocks
Portfolio-B: Long 1 call and short-sell Δ_c stocks

Delta hedging with puts gives:

$$V = N_s S + N_p P = N_p(h_p S + P)$$
$$h_p = N_s/N_p = -\Delta_p = |\Delta_p| \tag{27.4}$$

For a long put Δ_p is negative, hence if you are long puts ($N_p > 0$) then to hedge you go long stocks ($N_s > 0$). Similarly, if you are short (i.e. you have sold/written) a put ($N_p < 0$), a delta-neutral hedge involves short-selling stocks $N_s < 0$.

Delta-neutral positions with puts
Portfolio-A: Long 1 put option and long $|\Delta_p|$ stocks
Portfolio-B: Short 1 put option and short-sell $|\Delta_p|$ stocks

In practice there are some risks and costs attached to dynamic delta hedging since:

- when rebalancing you incur transaction costs (bid–ask spreads, commissions, price impact).
- delta hedging is only effective for small changes in stock prices.
- the volatility of the stock or the risk-free rate may change and hence the actual change in the option's value will not equal that given by the option's delta.
- different European options have different deltas – so you must choose the correct 'delta'.[1]

27.2 DYNAMIC DELTA HEDGING

The Black–Scholes formula (on a non-dividend paying stock) shows that the option premium varies with S, r, σ, T, all of which may change minute-by-minute, over the life of the option. The purpose of this section is to show how delta hedging an options position implies that you neither gain nor lose money, regardless of what point in time you choose to close out your hedged position (either before maturity or on the maturity date itself) and regardless of what has happened to the stock price over time. The only requirement is that you rebalance your hedged portfolio frequently (or, in the limit 'continuously').

The current stock price $S_0 = 100$. Suppose you are a derivatives dealer for Morgan Stanley (Ms Short) and you have just sold (i.e. *written*) one at-the-money (ATM) call option ($S_0 = K = 100$) to a hedge fund client ('*GoldFinger*').[2] The call premium is $C_0 = 10.45$ and $\Delta_0 = 0.6368$ (with $K = 100, \sigma = 20\%, r = 5\%, T = 1$ year). Assume each call option delivers

[1] For example, an option on dividend paying stock has $\Delta_c = e^{-\delta T} N(d_1)$, where δ is the dividend yield.
[2] We are not directly concerned about what happens to GoldFinger (only what happens to Ms Short), but note that GoldFinger is long the call option.

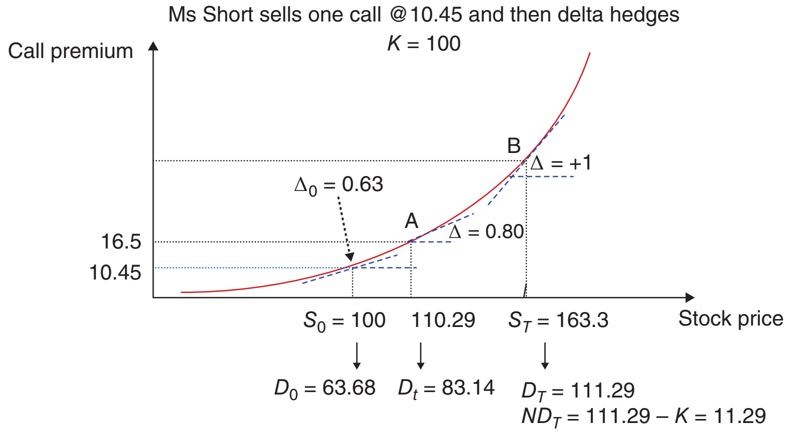

FIGURE 27.3 Delta hedging: rebalancing

one stock at maturity. At $t = 0$ to delta hedge the call, Ms Short borrows money to buy Δ_0 stocks (Figure 27.3)[3]:

Buy $\Delta_0 = 0.6368$ stocks @ $S_0 = 100$ using borrowed funds (e.g. bank loan) of \$63.68

Initial debt: $D_0 = \Delta_0 S_0 = \$63.68$

When the stock price changes from day-to-day over the life of the option, Ms Short has to continually rebalance her hedge portfolio by buying and selling stocks to preserve delta neutrality. However, as we shall see, the average (percentage) cost of the hedge is equal to the risk-free rate (if we ignore transactions costs of buying and selling stocks). We can view the outcome of the delta hedge at two different times – at maturity and before maturity:

1. If Ms Short continues to delta hedge *to the maturity date* of the option contract then when the option matures she should have neither lost nor gained. More precisely, since Ms Short's hedge is risk free, the cost of the hedge should equal the risk-free rate.
2. Any *change* in the value of Ms Short's hedge portfolio over a small interval of time should be approximately zero. Also, if Ms Short closes out all her hedge positions at any time *before maturity* then she should have neither lost nor gained. After closing out, the *value* of her hedge portfolio should be close to zero.

[3]To highlight the use of the delta of the call, we assume you can buy fractions of a stock. All the figures used can be scaled up by 100 (say), if for example, Ms Short is assumed to sell 100 calls (each of which delivers one stock) or if Ms Short sells one call which delivers 100 stocks. Then, for example, if Ms Short has sold 100 calls (each delivering one stock) at $C_0 = \$10.45$, she hedges by buying 64 stocks ($\Delta_0 = 0.6368$) @ $S_0 = \$100$ using borrowed funds (e.g. bank loan) of \$6,400, so her initial debt is $D_0 = \Delta_0 S_0 = \$6{,}400$. The receipts from the 100 written calls is \$1,045 which she places in a deposit account. Here we have rounded the number of stocks to equal a whole number.

Consider two possible cases. First, where the stock price rises and the call eventually ends up well in-the-money ($S_T \gg K$) and second, where the stock price falls and the call ends up well out-of-the-money ($S_T \ll K$).

27.2.1 Option Ends in-the-Money (ITM)

27.2.1.1 Ms Short's Position at Maturity

Our simulations show that the outcome of the hedge at maturity of the option (after 1 year) is a net debt of $ND_T = \$11.29$ for Ms Short (Morgan Stanley) who initially sold the option at $C_0 = \$10.45$ (Figure 27.3).

We can interpret this result as follows:

- The receipt of \$10.45 from the sale of the option could be placed in a (risk-free) bank deposit at 5% interest and would be worth \$10.98 [= 10.45 exp(rT)] after one year. Hence, in 1 year's time the *net* debt of Ms Short is small \$0.31 (= \$11.29 − \$10.98). This is about 3% (= \$0.31/\$10.45) of the initial receipt of the call premium of \$10.45 and approximately equal to the risk-free rate of 5%.

27.2.1.2 Ms Short's Net Position at t

Suppose instead, Ms Short closes out her hedged position after $t = 0.2$ years (chosen arbitrarily). At this point, our simulations show that the stock price, call premium, number of stocks held and the debt level have all increased to (see Table 27.1):

$$C = 16.5317, \qquad N_s = \Delta_c = 0.8052 \text{ stocks}, \qquad S = 110.2871, \qquad D = 83.14$$

Hence, when Ms Short closes out all her positions in the hedge at $t = 0.2$ years, we find:

- Value of stocks sold = 88.8032 (= 0.8052 × \$110.2871)
- Less debt at $t = 0.2$ = –83.1478 (buying on rising market and interest paid)
- Ms Short's loss on the call = –6.0817 (= 10.45 – 16.5317, closes out the option)
- Net position at $t = 0.2$ = –0.4265 (small loss on the hedge)

Hence, Ms Short again neither gains nor loses (much) and the value of her hedge position is close to zero.

27.2.1.3 Ms Short's Dynamic Delta Hedge

Let us now see how the above results come about by examining Ms Short's dynamic delta hedging strategy in more detail. If S increases then Δ_c increases and Ms Short has to borrow more money to purchase more stocks to maintain the delta hedge – this is the 'hidden' cost of hedging, since Ms Short buys stocks on a rising market. The additional borrowing increases the level of debt, as does the interest cost on the existing debt.

Suppose at $t = 1$ the stock price rises to $S_1 = \$100.1$ with $\Delta_1 = 0.6381$. Then to delta hedge over the next day, Ms Short buys $\Delta_1 - \Delta_0 = 0.001226$ additional stocks at S_1 costing an additional \$0.1227. Ms Short's new debt position is:

$$New\ debt = (Initial\ debt + interest\ cost) + cost\ of\ buying\ stocks$$

$$\$63.84 = \$63.68\ e^{0.05(0.01)} + \$0.1301$$

$$D_1 = D_o e^{r\Delta t} + (\Delta_1 - \Delta_0)S_1$$

At $t = 0.2$, the stock price has increased, so Ms Short buys extra stocks for the delta hedge, giving a debt level of \$83.1478 at $t = 0.2$. S increases over time and just before expiration of the option, $S \gg K$ so the delta of the (long) call has increased to $\Delta_c = 1$ (see Table 27.1). Hence at maturity T, the dynamic delta hedge results in Ms Short holding 1-stock to deliver against the written call – so there is no uncertainty about having a stock to deliver. At T, Ms Short delivers the stock to GoldFinger (assuming the hedge fund has held the long call to maturity)

TABLE 27.1 Delta hedge a written call

Time, t	Time left, $T - t$	Asset price, S	d_1	d_2	Delta	Cash inflow (minus = outflow)	Interest on debt	Stock of debt	Change in value of hedge	Call option premium
0	1	100	0.3500	0.1500	0.6368			63.6831	0	10.4506
0.01	0.99	100.1	0.3533	0.1543	0.6381	−0.1227	0.0318	63.8376	0.0313	10.4500
0.02	0.98	104.364	0.5622	0.3642	0.7130	−7.8232	0.0319	71.6928	5.5070	13.2683
...										
0.17	0.83	108.2762	0.7553	0.5731	0.7750	−6.0212	0.0368	79.6779	4.8706	15.1530
0.18	0.82	107.3599	0.7091	0.5280	0.7609	1.5135	0.0398	78.2042	−1.5239	14.3790
0.19	0.81	109.3247	0.8103	0.6303	0.7911	−3.3077	0.0391	81.5510	2.9105	15.8337
0.2	0.8	110.2871	0.8604	0.6815	0.8052	−1.5560	0.0408	83.1478	1.4187	16.5317
...										
0.95	0.05	154.3975	9.7909	9.7461	1.0000	0	0.0555	111.0204	−5.5528	54.6472
0.96	0.04	157.2825	11.3918	11.3518	1.0000	0	0.0555	111.0759	5.6645	57.4823
0.97	0.03	154.1949	12.5616	12.5270	1.0000	0	0.0556	111.1314	−6.2806	54.3448
0.98	0.02	157.3344	16.0727	16.0444	1.0000	0	0.0556	111.1870	6.1735	57.4344
0.99	0.01	160.6512	23.7383	23.7183	1.0000	0	0.0556	111.2426	6.5280	60.7012
1	0.0001	163.3499	245.3658	245.3638	1.0000	0	0.0556	111.2983	5.2923	63.3504

and receives K. Ms Short has outstanding debt of $D_T = 111.29$ from the funds borrowed to purchase the additional stocks in the delta hedge and from interest on her bank loan. Hence:

$$\text{\$-net debt at } T : ND_T = D_T - K = \$111.29 - \$100 = \$11.29 \tag{27.5}$$

The cost of the hedge as a proportion of Ms Short's initial receipts from the sale of the call is:

$$\% \text{ Cost of hedge} = \frac{ND_T - C_0}{C_0} = (\$11.29 - \$10.45)/\$10.45 = 8\%$$

For our simulation (Table 27.1), the cost of the hedge is 'in the ball-park' of the risk-free rate of 5%. If the hedge is repeated a large number of times (which in practice will happen to Ms Short as she is an options trader for Morgan Stanley), then *on average,* the percentage cost of all her hedges will be close to the risk-free rate of 5%, if she rebalances frequently (and for the moment we ignore transactions costs). The hedging error in our *single* simulation does not produce this 'average result' because our simulation contains some large changes in the stock price so the delta hedge is not perfect.

27.2.2 Option Ends Out-of-the-Money

Let us now consider delta hedging Ms Short's written call when the stock price falls over time and the option ends up well OTM at maturity, T. Her initial level of debt is $D_0 = \$63.68$. As the stock price falls the delta of the option falls. To maintain the hedge, Ms Short (at Morgan Stanley) therefore sells some of her original $\Delta_0 = 0.6368$ stocks and uses this cash inflow to reduce her bank debt. If the stock price falls monotonically then Ms Short will always be selling off some of her stocks but at lower prices – hence she would end up with positive debt at maturity of the call – some of which is also due to interest payments.

In our simulation where the stock price falls, the final debt level at maturity of the option is $D_T = \$10.19$. Since $S_T \ll K$, the call delta has fallen to $\Delta_T = 0$ and hence at T, Ms Short holds no stocks and the call expires out-of-the-money (and is not exercised by GoldFinger). The percentage net cost of the hedge portfolio turns out to be:

$$\% \text{ Cost of hedge} = |D_T - C_0|/C_0 = 2.49\%$$

Now let us examine the *change in value* of Ms Short's hedge portfolio over a small interval of time, using some algebra:

$$\begin{aligned} dV_{t+1} &= N_{st} dS_{t+1} + D_t(r\,dt) - dC_{t+1} \\ &\approx \Delta_t\, dS_{t+1} + D_t(r\,dt) - (\Delta_t\, dS_{t+1}) = D_t(r\,dt) \end{aligned} \tag{27.6}$$

where $dS_{t+1} = S_{t+1} - S_t$ etc., $N_t = \Delta_t$ is the number of stocks held at time t, $dC_{t+1} \approx \Delta_t dS_{t+1}$ is the (approximate) change in the call premium, r is the annual interest rate, dt is a small interval of time. From Equation (27.6) we see that $dC_{t+1} \approx \Delta_t dS_{t+1}$ is a good approximation for the change in the call premium, then $dV_{t+1} = D_t(r\,dt)$ and the cost of the hedge over a small interval of time, equals the risk-free rate.

Perhaps the most intuitive way of looking at the hedge is to assume that Ms Short who initially sold the call for $C_0 = \$10.45$ at $t = 0$, places this cash in a risk-free deposit account and she ends up at T, with cash of $C_0 e^{rT}$. But she also has an outstanding bank loan of ND_T. If she repeats the hedge a large number of times, the difference $ND_T - C_0 e^{rT}$ should be small and on average should be close to zero.[4] A measure of the (percentage) efficiency of the hedge (over a large number of delta hedges) is:

$$\text{\% Hedge error:} \qquad \% HE = \frac{stdv[ND_T - C_0 e^{rT}]}{C_0} \tag{27.7}$$

The more frequently Ms Short rebalances the hedge, the closer to zero we expect the hedging error to be. If we hold the volatility σ of the underlying stock constant and generate repeated 'runs' of the delta-hedge scenario we can obtain measures of *%HE* for different rebalancing periods. As we rebalance over {25, 5, 1, 1/2} days the *%HE* improves (ignoring transactions costs) from an average of about 0.5% to 0.25% to 0.15% to 0.1%, respectively (Kurpiel and Roncalli 2009). However, in practice there is a trade-off here because the hedging error may be larger when we rebalance more frequently because of higher trading costs.

The option's delta only provides a first order approximation to the change in the price of the call, which in practice could change due to changes in other factors (e.g. interest rates or volatility). In practice, the change in value of the delta-hedged portfolio will not exactly match the *actual change* in value of the written call. Also, when the volatility of the underlying asset σ, is allowed to vary (i.e. in a stochastic volatility world) then the *%HE* will be worse than the figures given above. Kurpiel and Roncalli (2009) report a worsening of *%HE* for delta hedging, with 1-day rebalancing from 0.15% when σ is constant (i.e. a Black–Scholes environment) to around 0.2–0.25% in a stochastic volatility world.

Note that in our examples of delta hedging we have ignored rebalancing costs (i.e. bid–ask spreads on stocks, commissions, brokerage fees, stamp duty etc.) but in a 'real world' hedge these must be factored in to the cost of the hedge. Normally this would be done by Ms Short, the options trader at Morgan Stanley, initially selling the option above its Black–Scholes price of $C_0 = \$10.45$. How much she can sell the option at a price in excess of \$10.45, depends on the degree of competition between options dealers, the market awareness of the hedge fund Goldfinger buying the option, and the transparency of prices quotes (and other fees) in the market place.

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[4]Notice that the net positon of Ms Short at $t = 0$ was also zero. She has debt (bank loan) of \$63.84 matched by holding of 0.6368 stocks each worth $S_0 = \$100$. She has sold a call option (to Goldfinger) for \$10.45 – this is a liability to Ms Short as immediately closing out the position would cost her \$10.45 to buy back the call. Ms Short's 'options liability' of \$10.45 is matched by the \$10.45 cash she placed in her bank deposit account.

27.2.3 Using Futures

In practice, delta hedging an options position is usually undertaken with futures contracts on the underlying asset, rather than using the underlying asset itself, because transactions costs are lower in futures markets than in the cash markets. The futures price on a stock (index) paying a continuous dividend at a rate δ or a currency futures (where $\delta = r_f$) we have:

$$F = Se^{(r-\delta)T^*} \qquad \text{hence } \partial F/\partial S = e^{(r-\delta)T^*} \tag{27.8}$$

where T^* is the maturity of the futures contract (which need not equal the maturity of the options being hedged). The delta of a futures contract is $e^{(r-\delta)T^*}$. If the stock price S changes by 1 unit then (27.8) shows that F changes by $e^{(r-\delta)T^*}$ units. Hence $e^{-(r-\delta)T^*}$ futures contracts have the same sensitivity to changes in the stock prices as does one stock. Hence, if N_S is the required position in the underlying asset (e.g. stocks) for delta hedging the options position, then the number of *futures contracts* needed to delta hedge the options position is $N_F = N_S e^{-(r-\delta)T^*}$ contracts.[5]

Note that the delta of a *forward* contract may differ from that for a futures contract. The value of a forward contract on a dividend paying stock at time t is $V_{fwd} = (F_t - F_0)e^{-rT} = S_t e^{-\delta T} - F_0 e^{-rT}$ where $F_0 = S_0 e^{(r-\delta)T}$ is the price agreed when the forward contract was initiated (at $t = 0$). Hence $\partial V_{fwd}/\partial S = e^{-\delta T}$ is the delta of a forward contract (on a dividend paying stock) and, the number of forward contracts to delta hedge the options position is $N_F = N_S\, e^{-\delta T}$. If the *forward contract* is on a non-dividend paying stock then $N_F = N_S$.

EXAMPLE 27.1

Delta Hedging Using Futures Contracts

A portfolio of options on stock-A has $\Delta_0 = -2{,}000$. To simplify, assume dividends on stock-A are paid continuously (over the life of the option). Domestic interest rates are 4% p.a. A futures contract on stock-A with a maturity of 9 months, which pays dividends at $\delta = 2\%$ p.a. is available.

The portfolio of options on stock-A can be delta hedged with a long position in 2,000 stocks-A. Alternatively, the portfolio of stock options can be hedged by buying $H_F = 2{,}000\, e^{-(0.04-0.02)(9/12)} = 1{,}970$ futures contracts on stock-A.

[5] Alternatively, the value of a hedge portfolio of futures and calls (say) is $V = N_F F + N_c C$, setting $dV = N_F dF + N_c dC = 0$ and using $dF = dSe^{(r-\delta)T^*}$ gives $N_F = (-N_c\Delta_c)e^{-(r-\delta)T^*}$ where $N_S = (-N_c\Delta_c)$ is the number of stocks to hedge the portfolio of calls.

27.2.4 Delta Hedging Under Stochastic Volatility

Above we have assessed the success of a delta-hedging strategy assuming a GBM for the stock price and a value for delta given by the Black–Scholes equation, which assumes a constant volatility. But what if in the real world volatility is stochastic, how might we undertake the delta hedge?

In this case the 'correct' delta of the call is not given by the Black–Scholes equation $\Delta_c = N(d_1)$. In the presence of stochastic volatility, Heston (1973) provides an (approximate) closed-form solution for the option premium and a 'correct' formula for the option's delta, which we denote Δ_H. Assume Ms Short sells a call option at the price C_H given by Heston's approximate closed-form solution (see Chapter 26):

$$C_H = f[(S, K, r, \delta, T - t); \quad (\overline{V}, \xi, \rho, \alpha, b)] \tag{27.9}$$

and she dynamically delta hedges by initially buying Δ_H stocks. To simulate changes in the stock price (under stochastic volatility) we need two equations:

$$dS = \mu S\,dt + S\sqrt{V}dz_s \quad and \quad dV = \beta(\overline{V} - V)dt + \xi V^{\alpha} dz_v \tag{27.10}$$

where $dz_i = \varepsilon_i\sqrt{dt}$ and $\mu =$ the real world growth in the stock price.[6] Given simulated 'real world' values of S, Ms Short then delta hedges each day (say) using the changing value of Heston's Δ_H and calculates the daily standard error of hedging error $\%HE$, over repeated simulations.

We can repeat this experiment for different rebalancing periods (e.g. 2 days, 5 days, 25 days, etc.) in order to find the smallest $\%HE$ after incorporating realistic rebalancing costs of buying and selling stocks and hence taking account of transactions costs. The $\%HE$ will tend to be smaller the more frequently Ms Short rebalances but this may be offset by higher transactions costs.

We could also see whether simulating S in a stochastic volatility world using (27.10) but hedging using the 'incorrect' delta Δ_{BS} from the Black–Scholes model (which assumes a constant volatility) is substantially larger than the hedging error when using the 'correct' Δ_H. Ignoring transactions costs, we expect the hedging error using Heston's Δ_H to be smaller than the hedging error using Δ_{BS}, but with transactions costs the outcome is not obvious. It is possible that with transactions costs, the hedging error using the 'incorrect' Black–Scholes delta Δ_{BS} may not be substantially different from using the more complex Δ_H from the Heston model.

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[6] We do not set $\mu = r$, since we are not *pricing* the option – the latter is done using Heston's formula. We are simply simulating the stock price in the 'real world'.

27.3 SUMMARY

- Dynamic delta hedging a written (sold) call requires a long position in Δ_c stocks (i.e. delta hedging calls, requires a long-short position). Delta hedging a long position in puts requires a long position in $|\Delta_p|$ stocks (i.e. delta hedging puts requires a long-long position).
- Ignoring transactions costs, a trader who sells a call option (say) and continuously rebalances her hedge portfolio by buying and selling stocks, will end up neither gaining nor losing whenever she closes out all her positions (in stocks and calls) and pays off any loans (debts) incurred. This is true if she closes out at any time up to and including the expiration date of the options.
- Hedging using the Black–Scholes delta does not provide a hedge against *large* changes in S, or against changes in volatility or interest rates, unless we explicitly include stochastic models for these variables (e.g. Heston's stochastic volatility model).
- The transactions costs of delta hedging an options position are recouped by initially selling the call or put, above the 'theoretical' price given by the Black–Scholes equation. The quoted price will be influenced by the forces of competition (e.g. wage costs of traders, costs of IT, etc.) amongst market makers in options (either traders on exchanges such as the CBOT or by the trading desks of large universal banks such as JPMorgan-Chase, Morgan Stanley, Citibank).

EXERCISES

Question 1
Suppose you currently hold several (European) put options on the same underlying asset (Apple-stocks). Will these call options all have the same value for delta? Explain.

Question 2
Explain how you delta hedge a long call option (on a stock) with $\Delta_0 = 0.6$. After delta hedging explain what happens if the stock price rises by \$2.

Question 3
Explain how you would delta hedge a short put (on a stock) with $\Delta_0 = -0.4$. After delta hedging explain what happens if the stock price rises by \$2.

Question 4
How would you replicate the price movements of a long put, when the put has a delta of minus 0.4?

Question 5

Today you own 2,000 stocks with price $S = \$100$. A call option with $\Delta_0 = 0.4$ is available. Assume each call delivers one stock. Explain how to set up a delta-neutral portfolio. Explain what happens when the stock price falls by \$2.

Question 6

You have just *purchased* (at $t = 0$) one long call option on a stock, with strike $K = 98$ and $\Delta = 0.4$. The call premium, $C_o = \$5$, the time to maturity is $T = 1/4$ (year, 3 months). The current stock price is $S_0 = \$100$ and the risk-free rate is $r = 3\%$ p.a..

Using some or all of the above illustrative figures carefully explain how you can delta hedge your options position over the next 3 months (when the call option expires).

(a) Consider the outcome after 1 day if the stock price falls by \$2.

(b) Consider the outcome at expiration, T, if the stock price falls monotonically and the call option ends out-of-the-money (OTM, $S_T \ll K$). Assume at expiration, T you have \$5.14 in a deposit account.

Question 7

At $t = 0$ you buy an at-the-money put option for $P_0 = \$5_,$ with $K = 100$, and $\Delta = -0.5$. The time to maturity is $T = 1/4$ (year, 3 months). The current stock price is $S_0 = \$100$ and the risk-free rate is $r = 3\%$ p.a..
You delta hedge the long put to maturity. Assume the stock price falls monotonically and ends up well below the strike price at $S_T = \$60$. At T you have debt (bank loan) of $D_T = \$100.2$.
Using some or all of the above illustrative figures carefully explain what happens in the hedge, particularly at $t = 0$ and at $t = T$.

Question 8

Suppose you believe that the volatility of the stock return over the next 3 months will be $\hat{\sigma} = 25\%$ p.a., which implies a Black–Scholes price of $\widehat{C} = \$2$. Given the current quoted price of a 3-month European call option of $C_{imp} = \$3$, you calculate that implied volatility is $\sigma_{imp} = 30\%$ p.a.. What should you now do to speculate on your view of volatility while protecting your position against changes in the stock price?

CHAPTER 28

The Greeks

Aims

- To show how 'the Greeks' (e.g. delta, gamma, rho, vega, and theta) provide useful 'summary statistics' for options, which can be used to provide an approximation to the change in the option price.
- To show how the Greeks are used to protect the value of an options portfolio from small changes (delta hedging) and large changes (delta-gamma hedging) in the price of the underlying asset.
- To demonstrate how the Greeks are used to protect the value of an options portfolio from large and small changes in the underlying asset's price, when the latter is also accompanied by (small) changes in volatility – this is '*gamma-vega-delta'* hedging.
- To demonstrate how to calculate 'the Greeks' for the BOPM.

28.1 DIFFERENT GREEKS

The Black–Scholes formula (on a non-dividend paying stock) shows that the option premium varies with S, r, σ, T, all of which may change minute by minute. In this chapter we show how the change in price of the option can be represented in terms of a number of 'summary statistics' which are generally referred to as 'the Greeks'. Knowing the numerical values for the various 'Greeks' also allows an options trader to set up effective hedge positions for her portfolio of options. As we shall see, the more sources of uncertainty the options trader wants to hedge (stocks, interest rates, volatility) and the smaller the desired hedging error, the more complex the trader's hedging strategy needs to be.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

28.1.1 Portfolio Delta

Suppose you have a portfolio of n options consisting of N_i calls and puts (with different strikes and time to maturity) but on the same underlying stock (e.g. Microsoft). The portfolio delta of the options is defined as:

$$\Delta_{port} = \sum_{i=1}^{n} N_i \Delta_i \tag{28.1}$$

We take $N_i > 0$ if you are long (= buy) a call or put and $N_i < 0$ if you are short (=sell) a call or put.

EXAMPLE 28.1

Delta of Portfolio

Ms Smart is long (i.e. holds) 1,000 puts (on AT&T) with strike of $K = 100$ and time to maturity of 3 months. Each (long) put has a delta of –0.4. Ms Smart has also written (sold) 2,000 calls (on AT&T) with a strike of $K = 90$ and time to maturity of 6 months. Each (long) call has a delta of +0.3.

$$\Delta_{port} = (1,000)\,(-0.4) + (-2,000)\,(+0.3) = -400 - 600 = -1{,}000$$

If the stock price of AT&T goes up by \$1, then the value of Ms Smart's portfolio of options will fall by \$1,000 (approximately). The 1,000 long puts will fall in value by \$400. Also, 2,000 long calls will rise in value by \$600 and therefore Ms Smart's 2,000 written calls will fall by \$600. Ms Smart is effectively 'short options' ($\Delta_{port} = -1,000$) – if the stock price rises the value of her options portfolio will fall.

To delta hedge her portfolio of AT&T options Ms Smart will go long (i.e. buy) 1,000 stocks, today. If the stock price goes up by \$1 over the next day then her stocks increase in value by \$1,000 (= $1,000$ stocks $\times$ \$1), which (approximately) offsets the loss of around \$1,000 on her portfolio of options. She is delta hedged.

28.1.2 Gamma

In the following sections we denote the derivatives price as f so our remarks will apply to either call or put options. The gamma of an option is defined as:

$$\Gamma \equiv \partial^2 f/\partial S^2 = \partial\Delta/\partial S \tag{28.2}$$

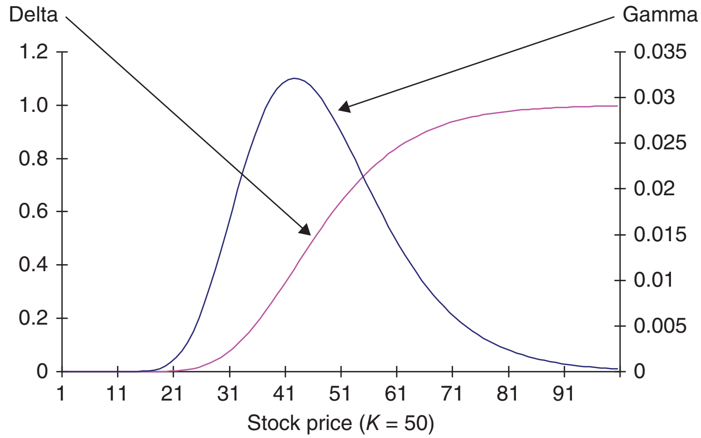

FIGURE 28.1 Delta and gamma – long call

Gamma is the second derivative of the option price with respect to the stock price.[1] Gamma is also the change in the option's delta as S changes by a small amount $\Gamma \equiv \partial\Delta/\partial S$, hence:

$$\textit{Approximate change in delta} = \Gamma \times (\textit{Small change in stock price}) \tag{28.3}$$

Gamma (like delta) takes on different values for different values of S, r, σ, T, K. The relationship between gamma and the stock price (other variables held constant) is roughly 'bell shaped' and the maximum value for gamma occurs when the current stock price is close to the strike price – that is, for a 'near-ATM' option (Figure 28.1).

> ### EXAMPLE 28.2
>
> #### Delta and Gamma
>
> Suppose today an option has a $\Delta = 0.5$ and $\Gamma = 0.05$. Then if $dS = +\$2$ over the next day, the (approximate) change in delta will be 0.1 ($= 0.05 \times \$2$) and the new delta will be 0.6. Hence, if you are delta hedging, you will have to increase your holdings of stocks by approximately +0.1 to maintain a delta-hedged position.

If $S = K = 50$ and the stock price changes by a small amount (e.g. \$1), the slope of the Black–Scholes curve (i.e. the option's delta) will not change by very much, say from 0.5 to 0.53, giving a gamma of 0.03. (However, note that the value of gamma for an ATM option which is also 'close to maturity' can be much larger than this – see below.)

[1]The gamma of an option is a similar concept to the convexity of a bond.

Clearly Figure 28.1 shows that when a call is either very *ITM* ($S \gg K$) or very *OTM* ($S \ll K$) then Γ is small and fairly constant (if the stock price changes by a small amount). This implies that for OTM and ITM options, a delta hedge will work quite well even if you do not rebalance very frequently (e.g. every 5 days), since a small gamma means that delta does not change very much as S changes, from day to day.

In contrast, for an option that is currently ATM (i.e. $S \approx K$) then gamma can be relatively large and vary a great deal as S changes (Figure 28.1). A large gamma alerts an options trader who is delta hedging her position, that to remain delta hedged she will probably have to rebalance her portfolio frequently (e.g. twice per day). Alternatively she can 'eliminate' the current large value of gamma by 'gamma hedging' – see below.

The value for gamma is obtained from the Black–Scholes equation for European options (on a stock paying continuous dividends):

$$\Gamma\,(call\ or\ put) = \frac{N'(d_1)^{e-\delta T}}{S\sigma\sqrt{\mathrm{T}}} \geq 0 \quad \text{where} \quad N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \geq 0 \tag{28.4}$$

Gamma is positive for either a long call or a long put. Also the gamma for a call is equal to the gamma for a put option (with the same underlying asset, strike price, and time to maturity). To obtain a 'negative gamma' you must sell (write) calls or puts.

Gamma is useful when calculating the change in the option price for a large change in the stock price – using gamma is sometimes referred to as a 'curvature adjustment'. A delta-hedged position does not give protection against *large changes* in the stock price. This can be seen by noting that the change in the option premium can be approximated by a second-order Taylor series expansion of the option price (with respect to S):

$$df \approx \left(\frac{\partial f}{\partial S}\right) dS + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}\right)(dS)^2 \approx \Delta\, dS + \frac{1}{2}\Gamma(dS)^2 \tag{28.5}$$

The change in the option price depends on both delta and gamma – as can be seen in Example 28.3.

EXAMPLE 28.3

Delta-Gamma Approximation

If a call option has $\Delta = 0.5$, $\Gamma = 0.05$ and $dS = +\$3$ then the change in the call premium calculated using only the delta is +\$1.5. But to obtain a more accurate value for df, we use both delta and gamma:

$$df \approx 0.5\,(3) + (1/2)\,(0.05)\,(+3)^2 = 1.5 + 0.225 = \$1.725.$$

The change in the call premium df (for one call) using the delta-gamma approximation is \$1.725 and using only the delta approximation is \$1.5 – a difference of \$0.225 (=15% of the delta change of \$1.5). For 10,000 long calls this difference amounts to \$2,250. The figure of \$1.725 is a more accurate measure of the 'true' change in the option price than using only delta. (But remember that the 'true' or 'correct' change in the option price is given by the Black–Scholes formula.)

For small changes in S (e.g. \$0.1), the change in call premium using just delta is \$0.05 and using the gamma term $\Gamma(dS)^2/2 = 0.00025$ makes little difference. An option's 'gamma risk' only becomes quantitatively important when the change in the stock price is relatively large (or the gamma for a portfolio of options is large – see below).

28.1.3 Portfolio Gamma

The gamma for a portfolio of options (with different strikes and time to maturity but on the same underlying stock) is defined as:

$$\Gamma_{port} = \sum_{i=1}^{n} N_i \Gamma_i \tag{28.6}$$

EXAMPLE 28.4

Portfolio Gamma

Ms Smart has sold (written) 10,000 puts (on a stock) which each have a gamma of 0.10 and she holds 1,000 calls each with a gamma of 0.05. The portfolio gamma is:

$$\Gamma_{port} = (-10,000)\ (0.10) + 1{,}000\ (0.05) = -1{,}000 + 50 = -950.$$

Suppose Ms Smart has taken a position in stocks such that the delta of her options+stocks portfolio is now zero. Then if the stock price moves up or down by \$3, the value of her portfolio will fall by about \$4,275 ($= (1/2)\ (-950)\ 3^2$). Hence, even if Ms Smart is delta hedged, but her portfolio has a large negative gamma, she loses money if the stock price moves either up or down by a substantial amount. Traders who have delta hedged their options portfolio should 'beware' of any large 'gamma exposure'.

If Ms Smart's options portfolio is delta hedged (i.e. $\Delta_{port} = 0$) but the gamma of the portfolio of options is large, then the portfolio will not be hedged against 'large' changes in the stock price. But if Ms Smart can form a 'new portfolio' of stocks and options which makes both the *portfolio* delta and the *portfolio* gamma equal to zero, then she is protected from both small and large changes in the stock price – this is 'delta-gamma' hedging and is discussed below.

Note that the 'gamma of a stock' is zero – this is because the value of a stock is linear in the stock price. If $V_s = N_s S$ then $\partial V_s/\partial S = N_s$ and $\partial^2 V_s/\partial S^2 = 0$.

28.1.4 Theta

The change in the option premium due to the passage of time is non-stochastic and is measured by the option's theta (see Appendix 28):

$$\theta = \partial f/\partial t \quad \text{hence} \quad df = \theta dt \tag{28.7}$$

In the Black–Scholes formula 'time' is measured in years or fractions of a year and when discussing theta we can take dt = '1 trading day', so $dt = 1/252 = 0.004$ (years) or we can measure theta per calendar day ($dt = 1/365$). The option's theta depends on the current stock price, time to maturity, stock volatility, and interest rate.

Theta is usually negative for most long positions in options – both long calls and long puts lose value as they get closer to maturity (all other variables held constant). Theta is most negative for options that are nearly ATM, so ATM options lose time value rather quickly, day by day. Alternatively, if the current stock price is very low then theta is close to zero (for calls and puts). When the time to maturity is short (e.g. few weeks to maturity) the value of theta for ITM, OTM and especially for ATM options can become relatively large – so these options all lose time value relatively quickly (all other variables held constant).

EXAMPLE 28.5

Theta

Suppose a long call option has $\theta = -8$ which implies that each day it loses 'time value'. The approximate fall in the price of the call (all other variables held constant) over 1 trading day (i.e. $dt = 1/252 = 0.004$) is $dC = \theta dt = -8\ (0.004) = -0.032$ (3.2 cents per trading day) or 2.19 cents per calendar day. As the option gets closer to maturity, the time decay (per day) for the call becomes more severe.

For example, for the same option with 1 week to maturity (and all other variables held constant) we might have $\theta = -20$, so that the call premium would fall by 0.08 (8 cents) per trading day (ceteris paribus). If we wish to measure the change in the option premium per calendar day, then we repeat the above with $dt = 1/365$.

Note that as 'time' is deterministic, the theta of the option is not used in any kind of hedging as there is no uncertainty. Theta is useful as an estimate of the change in value of the option over time. There is another more esoteric use for theta. It can be shown (using the Black–Scholes partial differential equation [PDE], see Chapter 48) that for $\$V$ in a *portfolio* of options on a single underlying asset with current price S the following holds, $\theta + rS\Delta + (1/2)\sigma^2 S^2\Gamma = rV$. For a delta-neutral portfolio we have:

$$\theta + (1/2)\sigma^2 S^2\Gamma = rV$$

This shows that for a delta-neutral portfolio, when theta is large and positive the gamma tends to be large and negative – so here theta is also an indication of the gamma-risk in the options portfolio.

28.1.5 Rho

The change in the option price due to a change in r is known as 'rho', $\rho = \partial f/\partial r$. For call and put options (for both a dividend paying and non-dividend paying stock):

$$\text{Call: } \rho_c = TKe^{-rT}N(d_2) > 0 \qquad (28.8a)$$

$$\text{Put: } \rho_p = -TKe^{-rT}N(-d_2) > 0 \qquad (28.8b)$$

In the Black–Scholes pricing formula, r is entered as a decimal. Hence, a change in interest rates of 1% is entered as $dr = 0.01$ in the formula $df = \rho dr$.

EXAMPLE 28.6

Rho

A long call has $\rho_c = 27$. If interest rates increase over 1 day from 3% to 3.5%, that is, by ½% (+0.005), then the call premium increases by $dC = \rho_c\, dr = 27\,(0.005) =$ \$0.135 (13.5 cents).

Theta and rho are the 'lesser Greek Gods' since usually the option premium is not particularly sensitive to either changes in r or changes in the time to maturity.

28.1.6 Vega

The change in the option's premium for a small change in volatility σ, is measured by the option's vega Λ:

$$\Lambda = \frac{\partial f}{\partial \sigma} \qquad (28.9)$$

'Vega' is not a Greek letter and 'vega' is also referred to as lamda, kappa, or sigma, which *are* Greek letters. The numerical value of vega is the same for a long call or long put (with same

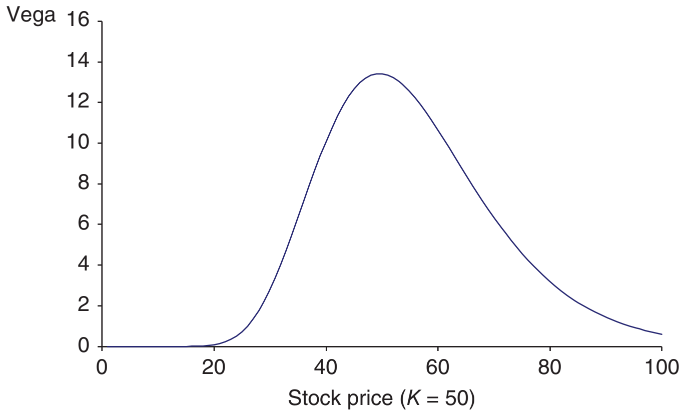

FIGURE 28.2 Vega (long call or put)

strike, time to maturity, and underlying stock) and has a bell shape with respect to the stock price (Figure 28.2). To obtain a 'negative vega' you must sell (write) calls or puts.

If the vega of an option is large this implies the option price is highly sensitive to small changes in volatility σ. For a European option on a stock, the vega is large when the option is close to being at-the-money (i.e. $S \approx K$) and is zero for options that are well OTM or well ITM (Figure 28.2). The analytic expression for the vega of a European (call or put) option on a stock (or stock index) paying a continuous dividend yield δ is:

$$\text{Call (Put) } Vega, \Lambda = S\sqrt{T}\, N'(d_1)\, e^{-\delta T} \geq 0 \qquad \text{where } N'(x) = \frac{e^{-x^2}}{\sqrt{2\pi}} \geq 0$$

Strictly speaking, using the Black–Scholes vega to hedge against *changes* in volatility σ of the underlying asset is inconsistent – as the Black–Scholes model assumes volatility is constant – but this approach seems to work reasonably well in practice (i.e. even when volatility is stochastic). It should be obvious that the vega of a *stock* is zero – this is because the value of a stock portfolio does not depend on the stock's volatility (if $V_s = N_s S$ then $\partial V_s / \partial \sigma = 0$).

EXAMPLE 28.7

Vega

The vega of a long call is +20 and you have just sold a call for \$3. The market's view of volatility increases over the next day by 1%, from $\sigma = 25\%$ p.a. to $\sigma = 26\%$ p.a. The change in the value of the written call over one-day is:

$$dC = (-1)\Lambda d\sigma = -20\,(+0.01) = -0.2 \text{ (20 cents).}$$

The (approximate) price of the call, the next day is \$2.8 (assuming S, r are constant).

Explicit expressions for ‘the Greeks’ can be obtained if we have a closed-form solution for the option premium. These are given in Appendix 28 for European calls and puts (on a stock that pays a continuous dividend yield), based on the Black–Scholes equation. Using the Greeks enables an options trader to hedge her existing portfolio of options either from small or large changes in the price of the underlying asset, or from changes in volatility and changes in interest rates.

28.1.7 Approximating Option Price Changes

The Greeks also provide a convenient way of finding the *approximate* change in the option premium df, as all the variables S, r, t, σ change. A Taylor series expansion f the equation for the (Black–Scholes) option price $f = f(S, r, t, \sigma)$ is:

$$\begin{aligned} df &\approx \left(\frac{\partial f}{\partial S}\right) dS + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}\right)(dS)^2 + \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial t} dt + \textit{higher order terms} \\ &\simeq \Delta\, dS + \frac{1}{2}\Gamma (dS)^2 + \Lambda.d\sigma + \rho.dr + \theta.dt \end{aligned} \tag{28.10}$$

where dS = change in the stock price, etc. This is also a useful expression for calculating the approximate change in the option price df, once we have any reasonable way of obtaining estimates of the Greeks (e.g. from an *approximate* closed-form option pricing equation, from MCS or from the BOPM).

Note also, that the values for *all* the Greeks change over time as the values of S, σ, r and time to maturity t, change. For example, a portfolio of options which currently has $\Gamma = 0$ will not represent a gamma-neutral position some time later because as S, σ, t and r change over time, so does the option’s gamma, Γ.

28.2 HEDGING WITH THE GREEKS

28.2.1 Portfolio of Options

Suppose you have a portfolio of n options consisting of N_i calls and puts (with different strikes and time to maturity) but on the same underlying stock (e.g. Microsoft). The delta, gamma, and vega of the portfolio of options are:

$$\Delta_{port} = \sum_{i=1}^{n} N_i \Delta_i \qquad \Gamma_{port} = \sum_{i=1}^{n} N_i \Gamma_i \qquad \Lambda_{port} = \sum_{i=1}^{n} N_i \Lambda_i \tag{28.11}$$

- $N_i > 0$ if long (buy) a call or put
- $N_i < 0$ if short (sell) a call or put

- $\Delta_i > 0$ for long call (from Black–Scholes)
- $\Delta_i < 0$ for long put (from Black–Scholes)

Also, each long stock has a delta of $+1$ ($\partial S/\partial S = +1$) and a gamma and vega of zero. If you hold (buy) stocks then $N_s > 0$ and if you short-sell stocks $N_s < 0$.

28.2.2 Gamma Neutral

The gamma for Microsoft stock (or futures on the stock) are both zero. So, the only way you can change the gamma of an existing options portfolio on several Microsoft calls and puts is to take new positions in some other *options* on Microsoft. Suppose you already have a portfolio-A, of several calls and puts on Microsoft which have a *portfolio* delta of Δ_A and gamma of Γ_A. If you now add a further N_Z traded options on Microsoft (with different strike prices and maturity dates) into your existing options portfolio-A, then the gamma of this new portfolio is:

$$\Gamma_{new} = N_Z\Gamma_Z + \Gamma_A \tag{28.12a}$$

Your new portfolio is gamma-neutral $\Gamma_{new} = 0$ when:

$$N_Z = -\Gamma_A/\Gamma_Z \tag{28.12b}$$

But including N_Z additional options will also alter the overall portfolio's delta. Hence we then buy or sell a number of Microsoft stocks (or futures contracts on Microsoft stock) to make the overall options portfolio delta neutral. An example is given in Example 28.8.

EXAMPLE 28.8

Delta-Gamma-Neutral Portfolio

Portfolio-A: Call and put options on Microsoft stock, with different strikes and maturity dates:

Portfolio-delta, $\Delta_A = 500$. Portfolio-gamma $\Gamma_A = -300$

Other call options (Z) on Microsoft are available, which for simplicity we assume all have the same strike prices K_{Micro} and time to maturity, T_{Micro}. Each of these Microsoft options-Z has:

$$\Delta_Z = 0.6 \quad \text{and} \quad \Gamma_Z = 0.03.$$

To make our existing portfolio-A, gamma neutral:

$$N_Z\,\Gamma_Z + \Gamma_A = 0$$
$$N_Z = -\Gamma_A/\Gamma_Z = -(-300)/0.03 = +10.000$$

We buy 10,000 of options-Z on Microsoft (each with K_{Micro} and T_{Micro}).

The delta of our 'new' portfolio (i.e. options-A and options-Z, all on Microsoft stock) is now:

$$\Delta_{new} = \Delta_A + N_Z\Delta_Z = 500 + 10{,}000(0.6) = 6{,}500$$

Hence to establish delta neutrality we short-sell 6,500 Microsoft stocks.

Note that as time passes S, σ, r and T change, hence so do gamma and delta – therefore periodic rebalancing is required to maintain delta-gamma neutrality.

Kurpiel and Roncalli (1998) find that (in a Black–Scholes constant volatility world) when you just delta hedge a call option with rebalancing over either {5, 1, 1/2} days, the percentage hedging error is $\%HE = \{0.25, 0.15, 0.1\}$[2] whereas with a delta-gamma hedge these figures improve to $\%HE = \{0.08, 0.03, 0.02\}$, which are 'reasonable' even when rebalancing takes place only every 5 days.

28.2.3 Vega Neutral

The vega of an existing options portfolio on Microsoft can only be altered by adding another traded option on Microsoft to the existing portfolio. Suppose we initially hold a portfolio of Microsoft stocks with a portfolio vega of Λ_A and another option-X on Microsoft is available with a vega Λ_X. To obtain a vega-neutral position we require N_X options which satisfy:

$$\Lambda_{new} = N_X\Lambda_X + \Lambda_A = 0 \tag{28.13a}$$

hence

$$N_X = -\Lambda_A/\Lambda_X \tag{28.13b}$$

Our new portfolio will not be delta neutral but we know that if we now take a position in Microsoft stocks, we can also achieve delta neutrality. For daily rebalancing in a *stochastic volatility world*, Kurpiel and Roncalli (2009) report an improvement in hedge performance as we move from purely delta hedging with $\%HE = 0.20 - 0.25\%$, to delta-vega hedging where $\%HE = 0.135\%$. However, we still have a potential problem because our delta-vega neutral portfolio may have a non-zero gamma and therefore may not be protected against *large* changes in the stock price.

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[2] From an earlier chapter, the percentage hedging error is the bank debt (loan outstanding) at T less the invested call premium, expressed as a percentage of the initial call premium, that is $\%HE = stdv[ND_T - C_0e^{rT}]/C_0$

28.2.4 Gamma-Vega-Delta Neutral

A *gamma-vega-delta neutral* position implies that the value of the portfolio is (nearly fully) protected against small or large changes in the stock price and against small changes in the stock's volatility.

Suppose we have a portfolio-A of options on Microsoft which is already vega neutral but has a gamma of $\Gamma_p \neq 0$. How can we *simultaneously* achieve gamma and vega neutrality? At first sight it may seem as if we just include $N_X = -\Gamma_p/\Gamma_X$ new options-X (on Microsoft) to achieve gamma neutrality. However, this is incorrect, as adding the new options-X destroys our initial vega neutrality, because the new N_X options themselves have non-zero vegas. To achieve a delta-gamma-vega neutral portfolio we proceed in two steps:

- First, we include 'new' options on Microsoft (with different strike prices and maturity dates and hence different gammas and vegas) to our original portfolio-A (of Microsoft options), so that we *simultaneously* achieve gamma-vega neutrality.
- Second, we go long or short Microsoft *stocks* to make our 'new' options portfolio delta neutral – this will not alter our 'new' gamma-vega neutrality because stocks have a gamma and a vega of zero.

An example of setting up a 'delta-gamma-vega' neutral position is given in Example 28.9.

EXAMPLE 28.9

Delta-Gamma-Vega Neutral

You currently hold a Portfolio-A of options on Microsoft:

Portfolio-A $\Delta_a = -500 \quad \Gamma_a = -5{,}000 \quad \Lambda_a = -4{,}000$
Also:

Option-Y (on Microsoft) is available with $\Delta_y = 0.6, \quad \Gamma_y = 0.05, \quad \Lambda_y = 15$

Option-Z (on Microsoft) is available with $\Delta_z = 0.5, \quad \Gamma_z = 0.10, \quad \Lambda_z = 10$

First, use options Y and Z (on Microsoft) to give a portfolio which is simultaneously 'gamma-vega' neutral.

Gamma-vega neutrality

$$\Gamma_a + N_y\Gamma_y + N_z\Gamma_z = 0 \qquad -5{,}000 + N_y\,(0.05) + N_z(0.10) = 0$$
$$\Lambda_a + N_y\Lambda_y + N_z\Lambda_z = 0 \qquad -4{,}000 + N_y(15) + N_z(10) = 0$$

Hence: $N_z = +74{,}800$ (i.e. buy options-Z) and $N_y = -49{,}600$ (i.e. sell options-Y)

New portfolio delta:

$$\Delta_a - 49{,}600\ \Delta_y + 74{,}800\ \Delta_z = -500 – 49{,}600(0.6) + 74{,}800(0.5) = +7{,}140$$

Second, use Microsoft stocks to achieve delta neutrality. This requires short-selling 7,140 Microsoft stocks.

The value of your ‘final options and stocks’ position consists of your initial portfolio-A of several options on Microsoft (with different strikes and time to maturity) plus our new positions in options-Y and options-Z (on Microsoft) which give gamma-vega neutrality, and finally short-selling Microsoft stocks to achieve delta neutrality. Your new positions in options and stocks means that your ‘final portfolio’ is now hedged against small (delta) and large (gamma) changes in S and also against (small) changes in volatility (vega).

28.2.5 Frequency of Rebalancing

A long call or a long put on a stock (with the same strike price and maturity) have the same *positive* values for either gamma or vega. Short calls and puts have *negative* gamma and vega. Although options traders might frequently rebalance their portfolios to maintain delta neutrality, they generally do not rebalance often to maintain gamma and vega neutrality, since it may be difficult to find appropriate ‘offsetting options’ at competitive prices. They therefore adjust their options position to give gamma and vega neutrality only after these ‘Greeks’ become unacceptably large. However, their portfolio of options positions will usually be *monitored daily* and their *potential vulnerability* to possible changes in either stock prices or volatility (of stock returns) will be calculated – this is the subject of Chapter 46 on risk measurement.

The options trading desks of many banks mainly *write* calls and puts for their clients and hence build up negative gamma and negative vega positions, in the normal course of trading. They therefore often step in as buyers of options when these are available at competitive prices, in order to reduce their gamma-vega exposures. Because an option’s vega and gamma are large when the options are close to ATM, an options dealer may gamma-vega (and delta) hedge a portfolio of ATM options under these circumstances, even when the options have a considerable time to maturity.

It is also the case that the gamma and vega for ATM options *which are close to expiration* are exceptionally large and hence the options trader may consider a gamma-vega (and delta) hedge, if she holds options that are both ATM and close to expiration. A trader who initially sells ATM calls and puts and delta hedges her position hopes that the options will become well OTM or ITM, since then the vega and gamma become very small (Figure 28.2) and there may be no need to undertake costly gamma-vega hedging.

A portfolio of *futures options* (e.g. options on AT&T futures contracts) can be used together with futures contracts (on AT&T stocks) in implementing dynamic delta hedging. An existing portfolio of futures options-A (e.g. on AT&T futures contracts) can be combined with other futures options-B (on AT&T futures contracts, but with different strikes and time to maturity) to *simultaneously* produce a gamma-neutral and vega-neutral portfolio. Finally, one can use futures contracts (on AT&T) to achieve a portfolio which is gamma-vega-delta neutral.

As we have seen, if you hold a portfolio of options *on AT&T stock*s then you can gamma-vega-delta hedge using other options (written on AT&T stock) together with the AT&T stocks themselves. However, because the stock price and the futures price move closely together you can also gamma-vega-delta hedge a portfolio of options (on AT&T *stocks)*, using options *on futures contracts of AT&T* – together with futures contracts on AT&T stocks (to ensure delta neutrality).

28.3 GREEKS AND THE BOPM

Suppose we do not have a closed-form solution like Black-Scholes for option premia but we wish to calculate 'the Greeks' for a specific option. We have already seen this in the context of MCS. Now we examine the calculation of the Greeks in the BOPM.

Suppose we construct a lattice for the stock price and a $T = 1$ year option, with $n = 52$ steps, so that each time period corresponds to 1 week ($dt = T/n = 1/52$ years) and $U = e^{\sigma\sqrt{dt}}$ and $D = e^{-\sigma\sqrt{dt}}$. How can we determine the Greeks so that if we currently hold an option, we can hedge against delta, gamma and vega risk?

There are two ways of doing this. The first is to apply the perturbation approach used in the MCS in Chapter 26. Suppose the current stock price is S_0 $(= 100)$ and we calculate the call premium C_0 (under RNV) using backward recursion in the binomial tree. To find the option's delta we perturb the initial stock price by a small amount $h = 1$, say, so that $S_0^+ = S_0 + h$ is the new *initial* stock price. We keep everything else unchanged and produce a new tree using $S_u^+ = S_0^+ U$ etc., and calculate the new price for the option C_0^+. The delta of the option is then calculated as $\Delta^+ = (C_0^+ - C_0)/(S_0^+ - S_0) = (C_0^+ - C_0)/h$. The option's vega and rho can be calculated in a similar way.

If we want to find the option's gamma (which is the change in the delta as S changes by a small amount) we repeat our calculation for delta but this time using a lattice which starts off with $S_0^- = S_0 - h$. Suppose for this lattice we obtain a call premium equal to C_0^- so that our new $\Delta^- = (C_0^- - C_0)/(S_0^- - S_0) = (C_0^- - C_0)/(-h)$. Then gamma is given by the change in delta:

$$\Gamma = (\Delta^+ - \Delta^-)/h = (\mathrm{C}^+ - 2C + C^-)/h^2 \tag{28.14}$$

The second way of proceeding is to directly calculate (some of) the Greeks using only the initial tree for S and the tree for the option premium f. Define the stock price at each node of the tree as $S_{t,i}$. We designate i as the *number* of 'up' moves in the lattice (e.g. for UU, $i = 2$). A 'down'

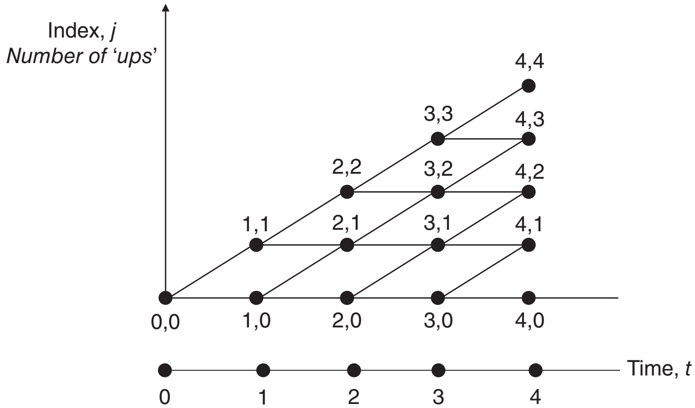

FIGURE 28.3 BOPM lattice

move is denoted '0' (which geometrically is represented as a horizontal move – Figure 28.3). Thus at $t = 2$, S_{ud} becomes S_{21} where the '1' represents the number of 'up' moves. We also have a similar lattice for the values of the option price $f_{t,i}$ obtained via backward recursion (under RNV). Use of this notation allows the lattice to be represented in computer programs.

At $t = 0$ we calculate the option's delta at node (0, 0) using:

$$\Delta_{00} = \frac{f_{11} - f_{10}}{S_{11} - S_{10}} \tag{28.15}$$

To estimate gamma, note that we have two estimates of Δ at $t = 1$:

$$\Delta_{11} = \frac{f_{22} - f_{21}}{S_{22} - S_{21}} \quad \text{and} \quad \Delta_{10} = \frac{f_{21} - f_{20}}{S_{21} - S_{20}}$$

The values of S that correspond to these values of Δ are calculated as follows. The average value of S in the upper and lower parts of the lattice (at $t = 2$) are $S^* = (S_{22} + S_{21})/2$ and $S^{**} = (S_{21} + S_{20})/2$. Hence their difference is:

$$\lambda = [(S_{22} + S_{21}) - (S_{21} + S_{20})]/2 = (S_{22} - S_{20})/2 \tag{28.16}$$

We can now calculate gamma at $t = 0$, as the change in delta divided by the 'distance' between our two measures of delta:

$$\Gamma_{0,0} = \frac{\Delta_{11} - \Delta_{10}}{\lambda} \tag{28.17}$$

Even though our estimates of Δ and Γ require the use of nodes at $t = 1$ and $t = 2$, it is assumed that these represent Δ and Γ at $t = 0$, because the time interval dt in the lattice is small. We can make dt as small as we like so we can also obtain estimates of Δ at $t = 0$ by approximating the derivatives at other nodes (time periods). For example, using values at $t = 2$ an estimate of Δ is:

$$\Delta = \frac{f_{22} - f_{20}}{S_{22} - S_{20}}$$

An estimate of the option's theta $\theta = \partial f/\partial t$ (i.e. $t = 1$ held constant) is easily obtained by interpolating between the first two nodes at $t = 1$. The average value of the option at $t = 1$ is $(f_{11} + f_{10})/2$ and hence an estimate of θ is[3]:

$$\theta = \frac{(f_{11} + f_{10})/2 - f_{00}}{dt} \tag{28.18}$$

An estimate of vega requires two trees as in any single tree, σ is constant and $U = e^{\sigma\sqrt{dt}}$ and $D = 1/U$ are constant. Hence we must use the perturbation approach. We choose a new value for sigma $\sigma^* = \sigma + h$ (where h is small) and calculate new values for U and D and hence a 'new tree' for the stock price. We keep $dt = T/n, r, K, S_0$ and T unchanged and calculate the 'new' option price using backward recursion on the new tree. If the new option price is f^*, the option's vega is:

$$\Lambda = \frac{f^* - f}{\sigma^* - \sigma} \tag{28.19}$$

The option's rho can be calculated in a similar way to vega by changing the risk-free rate by a small amount (and keeping *dt, n, U, D* the same as in the original tree) which gives $\rho = (f^* - f)/(r^* - r)$.

28.4 SUMMARY

- Delta hedging one written (sold) call, requires a long position in Δ_c stocks.
  Delta hedging one long (buy) call, requires short-selling Δ_c stocks.
  Hence, delta hedging calls requires a long-short position.
- Delta hedging one long (buy) put, requires a long position in $|\Delta_p|$ stocks.
  Delta hedging one written (short) put, requires short (selling) $|\Delta_p|$ stocks.
  Delta hedging puts requires a long-long position or short-short position.

[3] An estimate of theta at $t = 0$ can also be obtained (for $dt \to 0$) using other elements in the tree, for example: $\theta = (f_{21} - f_{00})/2dt$ or $\theta = (f_{42} - f_{00})/4dt$. There are many different ways of estimating the Greeks using, for example, forward, backward and central differences (see Chapter 47).

- Delta is a measure of the sensitivity of option premia to small changes in the price of the underlying stock. The Greek, 'rho' measures the sensitivity of option premia to small changes in interest rates, the option's vega to small changes in volatility and theta to small changes in the time to maturity. The gamma of an option measures the sensitivity of option premia to large changes in the price of the underlying stock.
- The Greeks can be calculated from any (approximate) closed-form solution for the option price (e.g. Black–Scholes) or alternatively by numerical methods using the BOPM or MCS.
- 'The Greeks' allow an options trader to construct a hedge portfolio whose value is (largely) unaffected by small changes in S, σ, and r. If the options trader also gamma hedges then the options portfolio is also hedged against large changes in S. In practice, rebalancing takes place at discrete intervals so there is always some risk in any dynamic hedge.
- If an options trader wishes to hedge against both small and large changes in the stock price S and against (small) changes in volatility then she must first buy and sell stock options to *simultaneously* make the options portfolio 'gamma and vega neutral' and second, buy or sell the underlying stocks to make the portfolio delta neutral.

APPENDIX 28: BLACK–SCHOLES AND THE GREEKS

The Black–Scholes formula for the price of a European call option on a stock which *pays a continuous dividend yield* at a rate δ (per annum) can be obtained by taking the Black–Scholes equation for an option on a non-dividend paying stock and replacing S by $Se^{-\delta T}$.

Call and Put Premia: European Options, Dividend Paying Stock

For calls and puts on a stock (index) paying continuous dividends:

$$C = Se^{-\delta T}N(d_1) - K\,e^{-rT}N(d_2) \tag{28.A.1}$$

$$P = K\,e^{-rT}N(-d_2) - S\,e^{-\delta T}N(-d_1) \tag{28.A.2}$$

Since $\ln(Se^{-\delta T}/K) = \ln(S/K) - \delta T$ then:

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{28.A.3a}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{28.A.3b}$$

$C, (P)$ = call (put) premium
T = time to maturity (years, fractions of a year)
S = stock price
δ = dividend yield (continuously compounded, proportionate, 4% = 0.04)
σ = standard deviation of stock return, (proportionate)
r = risk-free rate (continuously compounded, proportion)

Delta: $\Delta = \partial f/\partial S$

Call Delta: $\Delta_c = \partial C/\partial S = e^{-\delta T} N(d_1)$

Put Delta: $\Delta_p = \partial P/\partial S = e^{-\delta T}(N(d_1) - 1)$

Gamma: $\Gamma = \partial^2 \mathrm{f}/\partial \mathrm{S}^2$

$$\textbf{Call Gamma} = \textbf{Put Gamma:} \quad \boldsymbol{\Gamma} = \frac{N'(d_1)e^{-\delta T}}{S\sigma\sqrt{\mathrm{T}}} \geq 0 \quad \textbf{where } N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \geq 0$$

Vega: $\Lambda = \partial f/\partial \sigma$

Call Vega = Put Vega: $\Lambda = S\sqrt{T}\, N'(d_1)e^{-\delta T} \geq 0$

Rho: $\rho = \partial f/\partial r$ (for both a dividend paying and non-dividend paying stock)

Call: $\rho_c = TKe^{-rT}N(d_2) > 0$

Put: $\rho_p = -TKe^{-rT}N(-d_2) > 0$

Theta $\theta = \partial f/\partial T$:

a. Non-dividend paying stock

$$\theta_c = \frac{-S\,N'(d_1)\sigma}{2\sqrt{T}} - r\,K\,e^{-rT}N(d_2) > 0$$

$$\theta_p = \frac{-S\,N'(d_1)\sigma}{2\sqrt{T}} + r\,K\,e^{-rT}N(-d_2) > or < 0$$

Note that T = 'time to maturity' and therefore as the call gets closer to maturity, T falls. It can be shown that as the option approaches the maturity date, theta increases and the call premium becomes more sensitive to changes in 'time' – the call loses time-value more quickly.

b. Dividend paying stock

$$\theta_c(with\ dividends) = \frac{-SN'(d_1)\sigma e^{-\delta T}}{2\sqrt{T}} - rKe^{-rT}N(d_2) + \delta SN(d_1)e^{-\delta T}$$

$$\theta_p(with\ dividends) = \frac{-SN'(d_1)\sigma e^{-\delta T}}{2\sqrt{T}} + rKe^{-rT}N(-d_2) + \delta SN(-d_1)e^{-\delta T} > or < 0$$

In the above formulas time is measured in years. For example, if $\theta = -20$, then the change in price of the option *over 2 trading days* is –20(2/252) = −0.16 (16 cents).

Theta is usually negative for a long position in an option – as time passes the option becomes less valuable – this is 'time decay'. However, there are some (pathological) cases where theta can be positive – for example, for an ITM European put on a non-dividend paying stock or an ITM European call on a currency that pays a high interest rate.

If we set $\delta = 0$ in any of the above formulas then we will obtain the Black–Scholes 'Greeks' for an option on a *non-dividend paying stock*.

Options on Foreign Currency and Options on Futures

The above formulas for options on dividend paying stocks can be easily adapted to apply to European options on foreign currencies and European options on futures. The changes required are:

Foreign currency options:

Replace δ with the foreign interest rate r_f, S = spot exchange rate and σ = volatility of the spot FX-rate. There are two rhos, one with respect to the domestic interest rate which is given above and one with respect to the foreign interest rate $\delta = r_f$ which is given by:

$\rho = \partial f / \partial r_f$

Call: $\rho_c = -Te^{-r_f T}SN(d_1) > 0$

Put: $\rho_p = -Te^{-r_f T}N(-d_1) > 0$

Futures option:

Replace δ by r (so that r 'disappears' from the definition of d_1 and d_2), replace S by F (the futures price, with the same expiration date as the option) and σ is the volatility of the futures price.

EXERCISES

Question 1
Briefly explain the concept of the delta of a *portfolio* of options (on the same underlying asset).

Question 2
What is the vega of an option and why is it useful?

Question 3
How does the gamma of a long call vary as the price of the underlying (stock) changes around the strike price? What are the risk management implications after selling an ATM call with a high gamma?

Question 4
Suppose portfolio-A, which consists of several options, is delta neutral but has a gamma of minus 300. A call option-Z on the same underlying stock is available which has a delta of 0.62 and a gamma of 1.5.
How can you use option-Z and any stocks you might buy or sell to make your overall portfolio gamma and delta neutral?

Question 5
You are a trader (market maker) who supplies clients with whatever calls or puts they want to buy or sell. At the end of the trading day you have a *portfolio* of (long and short) calls and put options on Microsoft stock and on AT&T stock. Carefully explain to your risk manager what additional trades you are immediately going to do, to hedge your current options positions over the next 3 days. What are the costs in setting up your hedged postion?

Question 6
You hold a Portfolio-A of options with the following Greeks:

$$\Delta_a = -450, \qquad \Gamma_a = -6{,}000, \qquad \Lambda_a = -4{,}000$$

(Λ_a is the vega of the option). Option ‘Z’ is available with $\Delta_z = 0.6$, $\Gamma_z = 1.5$, $\Lambda_z = 0.8$ and option ‘Y’ is available with $\Delta_y = 0.1$, $\Gamma_y = 0.5$ and $\Lambda_y = 0.6$.
Explain how you can combine options A, Y and Z to give a portfolio which is ‘gamma, vega and delta neutral’.

Question 7
In the BOPM how do you calculate the gamma of an option at $t = 0$?

CHAPTER 29

Portfolio Insurance

Aims

- To show how *static* stock+put insurance can achieve a lower bound for the value of a diversified stock portfolio, while maintaining most of the upside potential.
- To demonstrate how static stock+put insurance is equivalent to a replication portfolio of calls and T-bills. This is an example of put–call parity using stock index options.
- To analyse how day-to-day price changes of a 'stock+put' portfolio can be replicated using either a 'stock+futures' portfolio or a 'stock+T-Bill' portfolio. Replication portfolios are used because they are often less costly than directly using the 'stock+put' combination.

Portfolio insurance is a general term which refers to a strategy of hedging an equity portfolio to ensure that it does not fall below some prescribed minimum value, while also retaining most of the upside potential, should stock prices increase. We have already outlined the static hedging strategy of stock+put insurance which at maturity (of the put) ensures a lower bound equal to the strike price of the put. Upside potential is also maintained because if the stock price (at maturity of the put) is greater than the strike price, the put expires worthless but the investor benefits from high stock prices (less the put premium). However, in practice, stock+put insurance has a number of drawbacks, namely:

- the hedging horizon of pension funds and mutual fund managers may be long and stock index options then have to be rolled over and this may be costly.
- most traded stock index options are American and their prices reflect the early-exercise premium. Portfolio managers with fixed hedging horizons are reluctant to bear this cost.
- position limits (set by the clearing house) may prevent portfolio managers setting up their optimal hedged position in stock index puts.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Because of the above practical problems, methods are available to set up a replication portfolio which mimics the price movements of stock+put insurance. The originators of these alternative forms of portfolio insurance are Leland and Rubinstein (Leland 1980; Rubinstein 1985; Rubinstein and Leland 1981).

A key distinction in undertaking portfolio insurance is whether the position is held to maturity (static hedge) or is continuously rebalanced (dynamic hedge). From put–call parity we know that a portfolio of 'stocks+puts' is equivalent to a portfolio of 'calls+T-Bills' (i.e. zero-coupon bond or cash held in a deposit account). Hence, we can undertake stock+put insurance with a replication portfolio of calls and T-bills.

We also show that the change in value of a stock+put portfolio can be replicated using 'stocks+futures' and this is generally referred to as dynamic portfolio insurance. Prior to the 1987 stock market crash, dynamic portfolio insurance was very popular as transactions costs are lower for rebalancing a 'stock+futures' portfolio, than for the actual stock+put position. However, during the 1987 crash investors using 'stock+futures' portfolio insurance were unable to undertake trades quick enough to replicate the stock+put position and hence the method became less popular immediately after the 1987 crash.

29.1 STATIC HEDGE

29.1.1 Stock+Put (Protective Put)

Here we are concerned only with the value of the 'stock+put' portfolio at maturity T of the option contract. Suppose it is September and you have $V_0 = \$560,000$ in a diversified stock portfolio which replicates the S&P 500 ($\beta_p = 1$). Assume the S&P 500 is at $S_0 = 280$ index points and the December-280, put option on the S&P 500 is priced at $P_0 = 2.97$ (index units). Assume each index unit for the put has a dollar value of $z_p = \$500$, hence:

Invoice price, put option, $z_p P = \$500\ (2.97) = \$1,485$
Value of stocks underlying the put option, $z_p S = \$500\ (280) = \$140,000$.

For a cost of \$1,485 the put is a claim on \$140,000 of the index. The put premium is 1.06% of the value of stocks underlying the contract ($= P_0/S_0$), the fact that this is relatively small can be useful in simplifying some of the mathematical expressions below. The number of index units held in your portfolio of stocks is:

$$N_0^* = V_0/S_0 = \$560,000/280 = 2,000 \quad \text{(index units)} \tag{29.1}$$

Holding 2,000 index units implies that a change in the S&P 500 index of 1 point would lead to a \$2,000 change in the value of your stock portfolio (with $\beta_p = 1$). To implement stock+put insurance we hold N_0 index units in both stocks and puts:

$$V_0 = N_0(S_0 + P_0) \tag{29.2}$$

Hence our initial holding of stocks and puts in index units is:

$$N_0 = \frac{V_0}{S_0 + P_0} = \frac{\$560{,}000}{280 + 2.97} = 1{,}979 \;\text{(index units)} \tag{29.3}$$

$$N_0 = \frac{V_0}{S_0 + P_0} = \frac{V_0}{S_0}\frac{1}{(1 + P_0/S_0)} = N_0^* \frac{1}{(1 + P_0/S_0)} \tag{29.4}$$

so if P_0/S_0 is small then N_0 is very close to N_0^*. Since the index multiple is $z_p = \$500$ the actual number of put contracts purchased is:

$$N_{p,0} = \frac{N_0}{z_p} = \frac{1{,}979}{500} = 3.96 \qquad (= 4 \text{ contracts}) \tag{29.5}$$

At expiration, the stock+put portfolio is worth:

$$V_T = N_0 S_T \qquad \text{if } S_T > K \qquad \text{(Upside potential)} \tag{29.6a}$$

$$V_T = N_0 S_T + N_0(K - S_T) = N_0 K \qquad \text{if } S_T < K \qquad \text{(Insured floor level)} \tag{29.6b}$$

In each case the *net profit* is $V_T - N_0 P_0$. We have chosen an ATM put ($S_0 = K = 280$), although in practice we can choose any strike price. The stock+put portfolio has an 'insured' lower value at T of $N_0 K$, so that:

$$V_{\min} = N_0 K = 1{,}979\,(280) = \$554{,}120 \tag{29.7}$$

The static stock+put strategy ensures a minimum value for the portfolio of \$554,120 (less the cost of the puts of $4 \times \$1{,}485 = \$5{,}940$), but also allows any upside capture if $S_T > K$. For example, if $S_T = 310$ ($> K = 280$) then the puts are not exercised and:

Value of insured portfolio,	$V_T = N_0 S_T$	$= 1{,}979\,(310)$	$= \$614{,}490$
Value of *uninsured* portfolio,	$N_0^* S_T$	$= 2{,}000\,(310)$	$= \$620{,}000$

Hence, for $S_T = 310$ the upside capture is 98.95% ($= \$614{,}490/\$620{,}000 = 1{,}979/2{,}000$).

29.1.2 Call+T-bill: Fiduciary Call

From put–call parity we know that 'stock+put = call+T-bill', hence the payoff from the stock+put portfolio can be replicated using 'calls+T-bills'. Our task, therefore, is to choose the number of calls and T-bills which replicates the *minimum* payoff at maturity $V_{\min} = N_0 K$

of the stock+put portfolio. This strategy is sometimes referred to as a fiduciary call. The value of a portfolio of N_C calls and N_B T-bills (i.e. zero-coupon bond) is:

$$V_{C,B} = N_C C + N_B B \tag{29.8}$$

If B is the market price of a T-bill with face value $M = \$100$ and maturity date T then[1]:

$$B = Me^{-rT} \tag{29.9}$$

In a static hedge, the value of the 'call+T-bill' at expiration is:

$$V_T = N_B M \qquad \text{if } S_T < K \tag{29.10a}$$

$$V_T = N_C(S_T - K) + N_B M \qquad \text{if } S_T \geq K \tag{29.10b}$$

The worst outcome is when the call expires out-of-the-money and hence the minimum value is:

$$V^{C,B}_{\min} = N_B M \tag{29.11}$$

We now set $V^{C,B}_{\min}$ equal to the minimum value of the stock+put portfolio $V^{S,P}_{\min}$ since this is what we are trying to replicate (at expiration). Hence, the number of T-bills required is

$$N_B = \frac{V^{S,P}_{\min}}{M} = \frac{N_0 K}{M} = \frac{\$554{,}120}{\$100} = 5{,}541.2 \quad \text{(T-bills)} \tag{29.12}$$

Having fixed N_B, the number of calls is then determined using (29.8):

$$N_C = (V_0 - N_B B)/C \tag{29.13}$$

Using (29.13), substituting N_B from equation (29.12), and for M from equation (29.10) and using put–call parity ($C = P + S - Ke^{-rT}$) it can be shown that the number of index units held in calls is:

$$N_C = \left(\frac{V_0}{S_0 + P_0}\right) = 1{,}979 \text{ index units } (= 1{,}979/500 = 3.96 \text{ contracts}) \tag{29.14}$$

which is the same as the number of puts in our original 'stock+put' portfolio (Equation (29.3)). In practice, the problem with static 'stock+put' and 'call+T-bill' portfolio insurance is that the required exchange traded calls and puts may not be available with a maturity date which matches the stock investor's desired holding period, with the desired strike price. The investor

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[1]This is equivalent to holding a cash amount B, which is invested in a risk-free bank deposit.

could try and obtain the required options from the OTC market but this could be expensive. An alternative is to replicate the period-by-period payoffs from 'stock+put' insurance using *dynamic hedging* with 'stocks+index futures' or 'stocks+T-bills'. By construction, we make sure the change in value of these two replication portfolios exactly mirrors that of the 'stock+put' portfolio, over any small interval of time. Hence, as we approach the maturity date of the options, either of these two replication portfolios will also ensure a minimum value N_0K – the same as that from the 'stock+put' portfolio.

29.2 DYNAMIC PORTFOLIO INSURANCE

Our aim is to continuously mimic the changes in value of a 'stock+put' portfolio first by using 'stocks+index-futures' and second by using 'stocks+T-bills'. The initial values of the options and futures contracts are given in Example 29.1, which also summarises the replication portfolios required.

EXAMPLE 29.1

Portfolio Insurance

Value of stock portfolio: $V_0 = \$560{,}000$,	S&P 500 index: $S_0 = 280$
Maturity of derivatives: $T = 0.10$,	Risk-free rate: $r = 0.10$ (10%)
σ(S&P500): $\sigma = 0.12$,	Put premium: $P_0 = 2.97$ (index units)
Strike price: $K = 280$,	Put delta $\Delta_p = -0.3888$
Call delta: $1 + \Delta_p = \Delta_c = 0.6112$,	Futures ($t = 0$): $F_0 = S_0e^{rT} = 282.814$
Price of T-bill: $B = Me^{-rT} = 99.00$,	Value of index point, $z_F = \$500$

Hedge positions:

Number of units held in stocks $N_0^* = V_0/S_0 = 2{,}000$ index units

Stock+put insurance: $N_0 = V_0/(S_0 + P_0) = 1{,}979$ index units

Stock+futures insurance:

$$N_F = [N_0\Delta_c - N_0^*]\,\frac{e^{-rT}}{z_F}$$

$$= [(1{,}979)(0.6112) - 2{,}000](0.99/500) = -1.565 \text{ (short)}$$

(continued)

(*continued*)

Stock+T-bill insurance:

$$N_S = \frac{V_0}{(S_0 + P_0)}\Delta_c = 1{,}979\ (0.6112) = 1{,}209.6 \text{ (index units)}$$

$$N_B = \frac{V_{s,p} - (N_0\Delta_c)S}{B} = 2{,}235.3 \text{ (long T-bills)}$$

For simplicity we take the time to maturity for both the futures and options contracts to be the same and the options are ATM. These assumptions are easily changed and do not affect the principles involved. Note that the hedge period will often be less than the time to maturity of the derivatives contracts. We assume that rebalancing takes place over small intervals of time, so the actual change in option premia are $\Delta\, dS$.

29.2.1 Stock+Put Portfolio

We have V_0 invested in a stock+put portfolio consisting of N_0 index units held in both stocks and puts:

$$N_0 = \left(\frac{V_0}{S_0 + P_0}\right) = 1{,}979 \text{ index units} \tag{29.15}$$

Note that N_0 is fixed throughout the hedge. At $t > 0$ the 'stock+put' portfolio has:

$$V_{s,p} = N_0(S + P) \tag{29.16}$$

$$\partial V_{s,p}/\partial S = N_0(1 + \Delta_p) \tag{29.17}$$

29.2.2 Stock+Futures Portfolio

We now replicate changes in value of the 'stock+put' portfolio using N_0^* stocks and N_f futures contracts. The futures contract requires no up-front payment (ignore margin calls) so the number of index units held in stocks is:

$$N_0^* = V_0/S_0 = 2{,}000 \qquad \text{(index units)} \tag{29.18}$$

which is fixed throughout the hedge. The value of the 'stock+futures' portfolio at $t > 0$ is:

$$V_{S,F} = N_0^* S + N_F z_F F \tag{29.19}$$

where z_F is the value of an index point on the S&P 500 futures contract. Also, $F = Se^{rT}$ (on a non-dividend paying stock). The change in value of the 'stock+futures' portfolio is:

$$\frac{\partial V_{S,F}}{\partial S} = N_0^* z_F + N_F \left(\frac{\partial F}{\partial S}\right) \tag{29.20}$$

Our task is to find that value for N_F which replicates changes in the value of the 'stock+put' portfolio. Equating (29.17) and (29.20) gives:

$$N_0(1 + \Delta_p) = N_0^* + z_F N_F(\partial F/\partial S) \tag{29.21}$$

Solving for N_F:

$$N_F = [N_0(1 + \Delta_p) - N_0^*] \quad \frac{e^{-rT}}{z_F} = -1.565 \text{ (Short futures)}^2 \tag{29.22}$$

From (29.22), N_F is negative, hence the replication portfolio requires a short position in index futures. The number of futures contracts to replicate the 'stock+put' portfolio must be continually rebalanced (by either buying or selling futures) as the put-delta changes with the stock price, time to maturity, volatility etc. Note that if we assume $N_0 \approx N_0^*$ then (29.22) simplifies to $N_F = N_0\Delta_p[e^{-r(T-t)}/z_F]$.

29.2.3 Stock+T-bill Portfolio

We know from put–call parity that a replication portfolio for 'stock+put' is a 'call+T-bill'. Suppose we have a portfolio of N_S index units in stocks and N_B T-bills, then:

$$V_{S,B} = N_S S + N_B B \quad \text{and} \tag{29.23a}$$

$$\partial V_{S,B}/\partial S = N_S \tag{29.23b}$$

Equating (29.23b) with (29.17) gives the number of stocks required to replicate the 'stock+put' portfolio:

$$N_s = N_0(1 + \Delta_p) = N_0\Delta_c = 1{,}209.6 \text{ (index units)} \tag{29.24}$$

The number of T-bills in the replication portfolio is then derived from (29.23a), using $V_{S,B} = V_{s,p}$:

$$N_B = \frac{V_{S,B} - N_S S}{B} = \frac{V_{s,p} - N_S S}{B} \tag{29.25}$$

$$N_B = \frac{V_{s,p} - (N_0\Delta_c)S}{B} = 2{,}235.3 \quad \text{(long T-bills)} \tag{29.26}$$

The \$-value of T-bills (cash) held is $N_B B$, which will vary through time as the replication portfolio must be continually rebalanced.

29.2.4 Numerical Example

The hedge outcomes are given in Example 29.2 for a (small) fall in the stock index $dS = -1$ point (which for simplicity we assume takes place over a small interval of time, so the time to maturity T of the derivatives remains unchanged).

> **EXAMPLE 29.2**
>
> **Outcomes Replication Portfolios**
>
> Fall of 1 unit in the S&P 500 index, $dS = -1$. Hence $S_1 = 280 - 1 = 279$
>
> **New derivatives prices (at t = 1)**
>
> *Put premium*, P $= 3.377$ (using Black-Scholes)
>
> *Futures* $(t = 1)$ $F_1 = S_1 e^{rT}$ $= 281.804$
>
> *Change in put premium* $dP = \Delta_p dS$ $= -0.3888$ (approximated by the put-delta)
>
> *Change in futures price* $dF = 281.804 - 282.814 = -1.01$
>
> **1. Stock+put portfolio**
>
> Gain on stocks $= N_0 dS = 1{,}979\ (-1)$ $= -1{,}979$
>
> Gain on puts $= N_0 \Delta_p dS = 1{,}979\ (-0.3888)(-1)$ $= 769.4$
>
> Net gain $= \mathbf{-1,209.6}$ (0.2% of portfolio value)
>
> **2. Stock+futures**
>
> Gain on stocks $= N_0^* dS = 2{,}000(-1)$ $= -2{,}000$
>
> Gain on futures $= N_F\, dF\, z_F = (-1.565)(-1.01)500$ $= 790.3$
>
> Net gain $= \mathbf{-1,209.6}$
>
> **3. Stock+T-bill**
>
> Gain on stocks $= N_S dS = 1,209.6(-1) = -1{,}209.6$
>
> Gain on T-bills $= 0$ (No change in T-bill price)
>
> Net gain $= \mathbf{-1,209.6}$

Example 29.2 shows that the change in value of the two replication portfolios equals that for the 'stock+put' portfolio (of –$1,209.6). However, this is because we have assumed that the change in the put premium is *exactly* given by $dP = \Delta_p dS$. The latter is an approximation and the 'new' (correct) put-premium is given by the full Black–Scholes formula $P = P(S, K, \sigma, r, T - t)$, so the 'true change' in the 'stock+put' portfolio is actually –1,173.58, rather than –1,209.6. In practice our two replication portfolios will not produce exactly the same change in value as the 'stock+put' portfolio because:

- option deltas only give an approximation to changes in options premia
- we ignore transactions costs of rebalancing
- the option deltas do not take into account changes in the option price due to large changes in the stock price, due to the loss of time value and due to changes in interest rates and volatility – hence the replication portfolios will not exactly mimic changes in value of the stock+put portfolio.
- in practical situations there would also be some rounding errors due to the requirement of purchasing whole numbers of calls or futures contracts.

Replicating the change in value of a 'stock+put' portfolio over many time periods using 'stocks+futures' and 'calls+T-bills' is provided in Excel. We assume that rebalancing is over a 'small' time horizon and (to simplify) we hold constant interest rates, volatility and the time to maturity of the option.

The above practical difficulties of replicating price movements of a 'stock+put' portfolio were highlighted in the 1987 stock market crash when the 'floor' provided by dynamic portfolio insurance (rather than directly holding 'stocks+puts') turned out to be rather more expensive and less effective than the model outlined above would indicate. These issues are discussed in Finance Blog 29.1.

Finance Blog 29.1 Replicating Stock+Put Insurance

After the 1974 stock market crash, the idea of portfolio insurance was to dominate the thinking of a UCLA Berkley academic Hayne Leland and his colleague Mark Rubinstein. In 1974 put options on stock indices were not available but Leland realised that you could replicate the payoff to a stock+put portfolio by using 'stocks+T-bills'. Given a desired floor ($= K < S_0$) for your stock portfolio, you could achieve this minimum portfolio value by selling stocks

(*continued*)

(*continued*)

as their prices fell and investing these funds in T-bills. If stock prices continue to fall you would continually sell stocks so that you would eventually be 100% in T-bills (and hence insured against any further fall in stock prices). On the other hand, as stock prices rise and you move further above the floor K, then in your replication portfolio you buy more stock and sell off some of your T-bills – you therefore have more upside potential.

The portfolio insurance marketed by Leland-Rubinstein Associates promised protection against five moves of 5% in the market index and remained operative until these 'five moves' had taken place. Because of the complexity of portfolio insurance, Leland and Rubinstein's first 'pitch' in 1979 at the investment banks yielded few takers. So they hired a marketing manager, John O'Brien, whose name was added to their consultancy firm. Interest grew, but active fund managers (i.e. 'stock pickers') did not like being told to sell individual stocks in a falling market. The advent of stock index futures in 1983 solved this problem, since the replication portfolio could now be engineered from 'stocks plus futures', so existing stock portfolios could be held unchanged and portfolio insurance implemented by selling index futures, as the S&P 500 index fell.

Business grew throughout the early 1980s but then the 1987 crash came along. With portfolio insurance using 'stocks+futures', you have *to try* and sell stocks or stock index futures in a falling market. Now in principle, market participants should be aware that 'portfolio insurers' are selling stocks and index futures, not because they have negative information about the *future course* of stock prices but merely to protect their existing stock portfolio. Hence, there should be some willing buyers at current market prices.

On Friday, 16 October 1987, the Dow Jones fell by 10% and about 40% of all selling came from portfolio insurers. There was an overhang of sell orders on Monday, 19 October 1987. Many stocks were failing to trade on the Monday and the Dow fell by a record 23% in one day. So you couldn't execute 'sell orders' from your replication portfolio. But things became worse. The arbitrage link between futures prices and spot prices also broke down. The 'fair' futures index value was around 280 but the futures contract opened at around 260, a discount of about 7.5%.

Arbitrageurs should have stepped in and bought futures (at a 'low' price) while simultaneously short-selling stocks. But they couldn't execute the latter trades, so futures prices were 'uncoupled' from spot prices – thus invalidating the 'mathematics' of the number of futures to short in the stock+futures replication strategy. The Brady Commission (1988) placed much of the blame for the 1987 crash on the large amount of sales by portfolio insurers. However, some of the fall was undoubtedly also due to simple 'mechanical' stop-loss orders by investors, whereby sales are triggered when the stock price falls below a pre-specified price.

By and large, portfolio insurance did result in portfolios staying at or above their declared minimum values. But a lack of market liquidity meant this was more costly than predicted from the theoretical model. The moral of the story here might be that if you expect a really severe crash then buy the 'real thing' – the put – since the writer has to pay you K per stock (and there are margin requirements to ensure 'writers' can meet their obligations). Otherwise, if you do decide to use portfolio insurance, it would be prudent to allow for the possibility of 'illiquid' markets in crisis periods. A lack of liquidity in derivatives markets (e.g. for credit default swaps [CDSs] and collateralised debt obligations [CDOs]) was also a contributing factor to the crash of 2008–9 – see Chapters 42 and 43.

Source: Adapted from Cuthbertson and Nitzsche (2001).

29.3 SUMMARY

- Portfolio insurance is a technique which allows fund managers to secure a minimum insured value for their diversified stock portfolio, while still allowing considerable upside capture should the stock market increase.
- Holding a portfolio of stocks and then purchasing N_0 puts and *holding them to maturity* ensures a minimum value for the stock portfolio of $V_{\min} = N_0 K$, at maturity of the option. This is static portfolio insurance. Stock+put insurance can be costly and the puts with the required strike prices and maturity dates may not be available.
- Although a 'stock+put' portfolio has a minimum insured value at maturity, its value changes prior to maturity, as the stock price and hence the put premium change over time. The dynamic price behaviour of the 'stock+put' portfolio can be 'replicated' using a number of alternative portfolios which may be more liquid and less costly. This is replication using dynamic trading.
- A long position in a portfolio of stocks plus a short position in stock index futures provides a low cost method of replicating the change in value of a 'stock+put' portfolio. To replicate changes in a 'stock+put' portfolio, the number of index futures contracts to short must be frequently rebalanced, as the stock index, volatility, and time to maturity of the option change.
- A 'stock+T-bill' portfolio will also replicate the change in value of a 'stock+put' portfolio – again continuous rebalancing is required.
- The delta of an option only provides an approximation to the change in value of the option. Therefore, replicating changes in the value of a 'stock+put' portfolio via dynamic trading will only be successful if changes in stock (index) prices are small (or equivalently, frequent rebalancing takes place) and there are no changes in the volatility of the stock index or in the risk-free rate (both of which influence the options price).

EXERCISES

Question 1
Briefly explain the idea of portfolio insurance, if you currently hold a stock portfolio.

Question 2
Why undertake dynamic portfolio insurance using stocks+futures, instead of insuring your stocks by buying put options (on the stock) and holding the put option to maturity, T?

Question 3
Why does dynamic portfolio insurance using stocks+futures to mimic the price changes of a stock+put portfolio, cause rapid buying or selling of futures contracts when the stock market moves up or down by a large amount?

Question 4
You hold \$675,000 in a diversified portfolio that tracks the S&P 500 index which currently stands at $S_0 = 1{,}500$. The value of an index point on the S&P 500 is \$250. You would like to buy puts to insure your portfolio so that the minimum value is \$600,000 at maturity of the puts.

(a) What strike price will the puts have?
(b) What is the total cost of the puts if $P = 16$ (index units)?

Question 5
You currently have V_0 to invest. Consider a portfolio consisting of an equal number of index units N_0 in stocks with price S_0 and puts with price P_0, so that $V_0^{s,p} = N_0(S_0 + P_0)$.

(a) Derive the formula for the number of futures contracts N_f needed to replicate the price dynamics of the stock+put portfolio.
(b) Why might the stock+futures portfolio not accurately mimic the change in value of the stock+put portfolio?

Question 6
On 1 June, you hold a stock portfolio with a current value of $V_0 = \$9{,}750{,}000$, that mimics the S&P 500 index. On 1 June the S&P 500 index, $S_0 = 1{,}500$. (Value of an index point is \$250). On 1 June index-puts with maturity $T = ½$ year strike $K = 1{,}400$ (points) and $\Delta_p = -0.23$, have a premium of $P = 31$ (points).
The S&P 500 futures index is $F_0 = 1{,}538$ and the risk-free rate is $r = 5\%$ p.a. (continuously compounded). The time to maturity of futures and the put contract is $T = ½$ year (6 months, maturity 1 December).

Use the information to set up a dynamic replication portfolio using stocks+index futures which 'tracks' the price movements of a portfolio of an equal number N_0 of stocks and puts.
Show that the change in value of the stock+put portfolio equals that of the stock+futures portfolio, for $dS = +1$. Assume you can use fractional numbers of futures contracts.

Question 7

You have V_0 to invest. You hold N_0 index units of the S&P 500 in a portfolio of stocks and N_0 index units in puts. How can you ensure that the change in value of this stock+put portfolio (for small changes in the S&P 500 index, S) will mimic the change in value of a call+T-bill portfolio? (Hint: use the put–call parity condition, for a non-dividend paying stock index).

PART VII

ADVANCED OPTIONS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 30

Other Options

Aims

- To show how a corporate's debt and equity can be valued using options theory.
- To examine different types of equity warrants, which are long-term options on a company's stock.
- To examine quantos which are long-term equity options written on foreign stocks but with the payout in the home currency.
- To show how an equity collar enables a portfolio manager to set an upper and lower limit on the performance of her existing equity portfolio. If this is achieved at zero 'up-front' cost then it is known as a *zero cost collar* or a *risk reversal*.

In this chapter we show how the debt and equity of a firm can be valued using options theory. Then we examine equity warrants, which are stock options 'attached to' bonds. Finally, we discuss how an equity collar places a floor price and a ceiling price on a stock (or stock portfolio).

30.1 CORPORATE EQUITY AND DEBT

Merton (1974) and Black–Scholes (1973) noted that the debt and equity of a firm can be valued using options theory. Suppose a corporation has two sources of finance, debt D and equity E. If the value of the firm's assets V_t at time t, exceeds the face value of debt (bonds) outstanding K_B then the equity holders have a positive stake in the firm. On the other hand, if $V_t < K_B$ the bondholders may put the firm into liquidation. In this case the equity holders receive nothing but they can 'walk away', as they have limited liability and any other assets they own cannot

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

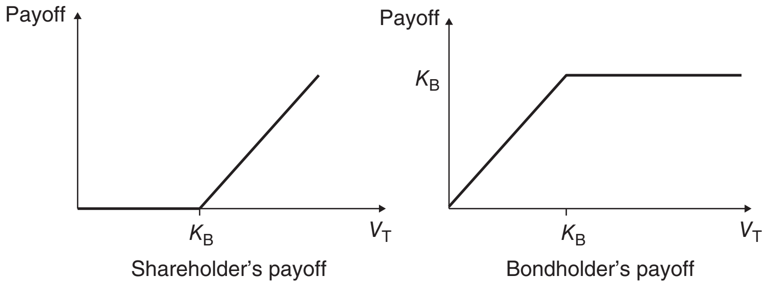

FIGURE 30.1 Payoffs

be taken by the liquidator. The payoff to equity holders is therefore like a call option with a payoff, max $[V_t - K_B, 0]$. We have:

K_B = face value of (zero coupon) bonds (debt) issued, which mature at time T

D_t = value of bonds (debt) at times t ($\leq T$)

V_t = market value of the firm's assets at t ($\leq T$)

E_t = value of shareholder's equity at time t ($\leq T$)

Case A: Solvency at Time T

If the value of the firm V_T exceeds the face value K_B of the bonds ($V_T \geq K_B$), the bonds are worth $D_T = K_B$ and the equity is worth $E_T = V_T - K_B$.

Case B: Insolvency at Time T

If $V_T < K_B$ then the bonds are worth $D_T = V_T$ and $E_T = 0$.

For the above two states, the payoffs to equity and debt holders are (see Figure 30.1):

$$E_T = \max[0, V_T - K_B] \tag{30.1}$$

$$D_T = K_B - \max[0, K_B - V_T] \tag{30.2}$$

The value of the equity is therefore like a European call on the value of the firm's assets with a strike price of K_B. If we assume that V_t follows a geometric Brownian motion (GBM), then value of the firm's equity E_t (at $t \leq T$) is given by the Black–Scholes formula for a European call option:

$$E_t = C_E(V_t, K_B, \sigma, r, T - t) \tag{30.3a}$$

Now consider the market value of the debt at time T. The equity holders can 'hand over' the firm to the bondholders if $V_T < K_B$. The bondholders have written a put option on the assets of the firm. The payoff to the bondholders consists of a short put plus an amount K_B (at time T). The value of the debt today D_t equals the PV of the face value of the bonds, $D_t = K_B e^{-r(T-t)}$ *less* the value of the put held by the equity shareholders:

$$D_t = K_B e^{-r(T-t)} - P_E(V_t, K_B, \sigma, r, T-t) \tag{30.3b}$$

where $P_E(V_t, K_B, \sigma, r, T-t)$ is the current value of a European put on the firm's assets with a strike price of K_B. The call and put premia $C_E(.)$ and $P_E(.)$ are given by the Black–Scholes formulas. Note that the current value of the debt D_t is less than the value of a risk-free bond $(= K_B e^{-r(T-t)})$ because of the risk of default represented by the written put $P_E(.)$ because if default occurs, the bondholders will not be paid the full face value of their bonds.

30.1.1 Pricing

It was Merton (1973) who provided a closed-form solution for pricing the corporate bond D_t. Since the value of the firm is assumed to follow a Brownian motion, the equation for D_t has similar features to the Black–Scholes formula. The market price of the risky corporate debt is:

$$D_t = Be^{-r\tau}[(1/L)N(d_1) + N(d_2)] \tag{30.4}$$

$\tau = T - t =$ time to maturity of the loan/debt/corporate bond

$r =$ risk-free rate

$L =$ leverage ratio $= Be^{-r\tau}/V$

$B =$ market value of outstanding debt, at time t

$d_1 = -[(1/2)\sigma_V^2\tau - \ln L]/\sigma_V\sqrt{T}$ and $d_2 = -[(1/2)\sigma_V^2\tau + \ln L]/\sigma_V\sqrt{T}$

$\sigma_V^2 =$ volatility of firm value (variance of proportionate change in V)

The Merton model also provides an equation for the yield spread (over risk-free T-bonds) that should be charged to a corporate borrower:

$$Spread = r_{corp} - r = (-1/\tau)\ln[N(d_2) + (1/L)N(d_1)] \tag{30.5}$$

The predictions of the model are quite intuitive. In particular, the spread should increase the higher is either the leverage ratio or the volatility of the firm's assets. An extended version of the above approach can be used in measuring credit default risk in the following way.[1] If D_t and V_t could be accurately measured then Equation (30.4) can be inverted to solve for the implied volatility σ_V^2 of the firm's total assets. Suppose $\sigma_V = \$12.12$m. Then assuming normality, there is only a 5% chance that the value of the firm V will fall below \$20m ($= 1.65\sigma$). Suppose the current value of the firm is \$100m and its outstanding debt is \$80m. Then our option's model implies that there is a 5% chance of the firm going into 'financial distress', over the life of the debt (i.e. the period τ). Unfortunately V and D are not easily measured for a levered firm (e.g. which has non-marketable bank loans) so further analysis is needed to make the model operational.

It is worth noting that we have taken a fairly simple example, where all debt matures on the same date (i.e. a zero-coupon bond). Valuing *coupon paying* corporate bonds is obviously more difficult since each coupon payment represents a put option and if an 'early' put option is exercised, the 'later' put options are worthless. Also, corporate bonds often contain convertible or call provisions (i.e. the payoffs are path dependent). This means that it may not be technically possible to derive closed-form solutions and other methods such as binomial trees, Monte Carlo simulation and numerical solution of PDEs are required – these techniques are discussed in other parts of the book.

30.2 WARRANTS

Equity warrants are one of the oldest manifestations of options. They are call options written by a firm on its *own* stock. Initially they arose because a firm issuing long-term bonds felt the bonds would be more attractive to investors (and therefore could be issued at a lower yield) if warrants were 'attached' to the bond. The warrants give the bondholder the opportunity to purchase the firm's stock at some time in the future, at a price fixed today.[2] If the firm does well in the future and its stock price increases, then the warrants would be 'in-the-money' and could be exercised at a profit by the warrant holder.

Equity warrants are sometimes referred to as 'equity kickers' since they give the holder the opportunity to participate in the 'upside' if the firm is successful in the future but also allow the investor the relative security of a corporate bondholder (who receive payments ahead of stockholders). These warrants are often 'stripped' from the bonds and traded separately on stock exchanges. Warrants are also sometimes given out by companies, either in payment for underwriters' fees or as part of a company's executive remuneration package – they are then referred to as *executive stock options*. The maturity of a warrant could be anything from about 2 to 12 years (or more) and hence warrants are 'long-term' options carrying the credit risk of the issuing firm.

[1]For example, as used by the KMV corporation of San Francisco.

[2]The firm must issue additional stocks if the warrants are exercised in the future. This dilutes the holdings of existing stockholders, as the profits of the firm are distributed across more stockholders.

30.2.1 Valuing European Warrants

European warrants can only be exercised on a certain day. They can be valued much like ordinary European stock options. However, if a warrant is exercised, then the company has to issue more stocks and hence increase the number of stocks outstanding. This 'dilution' does not occur when an exchange traded option is exercised, as the option writer has to purchase stocks on the NYSE.

Suppose a company has N stocks outstanding with current price S_0 so the value of the company to equity holders is $V_0 = NS_0$. Today the company issues M (European) warrants and each warrant allows the holder to purchase one stock from the company at time T, at a strike price of K. The value of the company does not change on announcement of the warrant issue (assuming any future cash flow from the warrants does not change the underlying profitability of the company by improving incentives or lowering costs of production). After the announcement of the warrant issue, the value of the company remains at $V_0 = NS_0$ as the future exercise of the options (and consequent 'dilution') is already reflected in the current stock market price, S_0 – the market is said to be 'efficient'. If the stock price at maturity of the warrant is S_T, the value of the equity in the company will be $V_T = NS_T$ (with or without the warrants).

If the warrants are exercised at T, there is a cash inflow to the company of $M.K$ and the market value of the company's equity increases from V_T to $V_T + MK$, while the number of stocks outstanding rises to $N + M$. The stock price immediately after exercise is therefore:

$$S_T^* = \frac{\textit{Value of firm}}{\textit{Number of stocks}} = \frac{NS_T + MK}{N + M} \tag{30.6}$$

If exercised, the payoff to the warrant holder is $S_T^* - K$. Substituting for S_T^* from (30.6) and rearranging:

$$\text{Warrant payoff at } T = \frac{N}{N + M}\max[S_T - K, 0] \tag{30.7}$$

The warrant payoff is equivalent to holding $N/(N + M)$ regular call options. The current price of the stock is S_0 which using Black–Scholes gives a call premium C_0, so the value of each warrant is:

$$\text{Value of one warrant at } t = 0 : V_w = \frac{N}{N + M}\,\mathrm{C}_0 \tag{30.8}$$

So the total cost of the warrant issue is MV_w. The total value of the company's equity will decline by MV_w as soon as the decision to issue the warrants is announced and therefore the stock price will fall by MV_w/N at $t = 0$.

A bond with a warrant attached will sell at a lower yield (higher price) than a conventional bond. This makes the 'bond-plus-warrant' an attractive source of finance for small firms with growth potential. The bond-plus-warrant will have lower coupon payments than a conventional bond (with the same maturity and tenor) and therefore involves less cash outflow for the firm but also gives the warrant holder a share in high profits should these occur in the future.

'Warrant' is often used as a generic term for an option with a long maturity date. As well as warrants on individual stocks there are also warrants (often on stock indices) written by third parties (e.g. Morgan Stanley, Citigroup) which are sold to investors and then traded on an exchange (e.g. American Stock Exchange), rather than OTC.

30.2.2 Quanto

A quanto is a long maturity option based on a *foreign stock index* such as the Nikkei 225 and traded (say) on the American Stock Exchange (AMEX). For example, if $S_T > K$ a long call (quanto) on the Nikkei 225 gives a US holder a payoff at expiration of $S_T - K$. However, the special feature of a quanto is that at the time the option is purchased, the contract fixes the rate of exchange between the yen and the US dollar which will apply at maturity of the quanto. Hence there is no exchange risk.

For example, if the payoff $S_T - K$ on the Nikkei 225 is equivalent to ¥20,000 and the exchange rate *agreed at* $t = 0$ is 100 Yen/USD, then the USD payoff is \$200. Thus a quanto allows a US investor to speculate on future value of the Nikkei 225 index without incurring any exchange rate risk.

30.3 EQUITY COLLAR

Suppose you hold stocks (with price S_0) but are worried about a fall in prices and want to secure a minimum value for your portfolio, then today you could buy a put, with a low strike price $K_1 (< S_0)$. The 'stock+put' allows unlimited upside potential, should stock prices rise. However, if you are willing to forego some of the upside potential, you could sell a call with a high strike price, $K_2 (> S_0)$. The cash received from selling the call can be used to offset the cost of the put. The payoff to this strategy is an equity collar – it establishes a minimum and maximum value for your existing stocks (Figure 30.2).

An equity collar has the same payoff profile as a bull spread which we discussed in Chapter 17. The key difference is that the bull spread is constructed using only options but an equity collar starts with a stock and then uses options to provide a floor and a ceiling on the future value of the stocks.

The payoff from the equity collar is given in detail in Table 30.1 for three possible outcomes for the stock price at maturity T, of the options. The lower bound for the payoff on the collar is K_1 and the upper bound is K_2. The net cost of establishing the collar is:

$$\text{Net cost} = P(S_0, K_1, \sigma, r, T) - C(S_0, K_2, \sigma, r, T) \quad \text{where } K_2 > K_1 \tag{30.9}$$

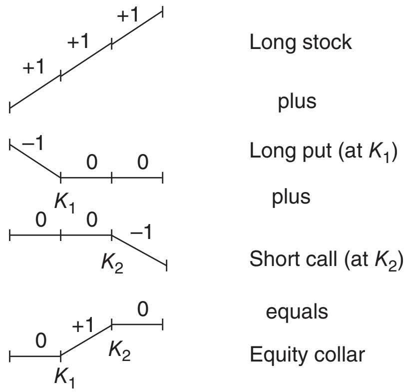

FIGURE 30.2 Equity collar

TABLE 30.1 Equity collar payoffs

	$S_T < K_1$	$K_1 \leq S_T \leq K_2$	$S_T > K_2$
Long stocks	S_T	S_T	S_T
Long put (K_1)	$K_1 - S_T$	0	0
Short call (K_2)	0	0	$-(S_T - K_2)$
Payoff	K_1	S_T	K_2
Profit	$K_1 - (P - C)$	$S_T - (P - C)$	$K_2 - (P - C)$

If the stock price at maturity lies between the two strike prices, then the profit and the breakeven stock price are:

$$\Pi = (S_T - S_0) - P + C = 0 \tag{30.10}$$

$$S_{BE} = S_0 + (P - C) \tag{30.11}$$

30.3.1 Zero-cost Collar (Risk Reversal, Range Forward)

If the strike prices are chosen so that the put and call premia exactly offset each other (i.e. $P = C$) then the collar can be set up at zero cost – this is called a *risk reversal, range forward* or simply, *a zero-cost collar*. By definition a ‘risk reversal’ has $P = C$ and hence $S_{BE} = S_0$

(Equation 30.11). Suppose we chose a floor level K_1 and hence via Black–Scholes P is fixed, then a zero-cost collar requires:

$$C(S_0, K_2, \sigma, r, T) = P \tag{30.12}$$

We 'invert' the Black–Scholes formula for C and solve (30.12) for K_2 (e.g. by using Excel's 'Solver'). Once you have chosen K_1, you must accept whatever value that arises from Equation (30.12) for K_2, which ensures $C = P$.

The beauty of the zero-cost collar is that an investor holding stocks can fix a maximum and minimum value for her stocks at no 'up-front' cost. There is a guaranteed minimum value for the stocks and yet the investor can still share in some of the upside should stock prices rise. However, this is not a 'something for nothing' outcome – financial markets never give you that! True, the (zero cost) collar gives you a floor value – which you like. But the 'hidden cost' is the fact that it eliminates the possibility of very large upside gains (which you might get if you only held the stocks plus the put, but this would involve a cost equal to the put premium).

In a zero-cost collar $P = C$ implies that you can only choose one of the strikes, either for the call or the put. Also the zero-cost collar will probably involve at least one strike price for which there may not be exchange traded options available. For example, suppose the portfolio manager chooses $K_1 = 159$, for the put which using Black–Scholes gives $P = 2.2$. For a zero-cost collar the portfolio manager now requires $C = 2.2$. But suppose inverting the Black–Scholes equation for the call premium (e.g. using Excel's 'Solver') gives $K_2 = 170.80$. Since there is no traded call option with $K_2 = 160.80$ then either the collar will not quite be 'zero cost' or OTC options must be used.

30.4 SUMMARY

- Options theory can be used to value a corporate's equity and debt (i.e. bank loans and bonds issued). The payoff to equity holders is like a call option. The value of the equity in the firm is therefore equal to the value of a European call on the firm's assets, with a strike price equal to the face value of the bonds (debt).
- Bondholders have written a put option on the firm's assets. Hence the value of the firm's debt is equal to the market value of the bonds issued, less the value of the put held by the equity holders (with a strike price equal to the face value of the bonds).
- Warrants are 'long maturity' options on a stock with maturities ranging from 2 to 12 years. Warrants are often initially attached to bonds, which have a lower yield than plain vanilla bonds. For investors, the warrant combines the relative safety of a conventional bond but allows upside potential if the stock price rises. The warrant is like a call option on the value of the firm and can be priced using a variant of the Black–Scholes formula.
- A quanto is an option with a payoff at maturity which depends on the level of a *foreign* stock index but with payments made in the *home* currency, based on an exchange

rate which is fixed when the quanto is purchased. Quantos therefore provide a low cost method for portfolio managers to take a position in foreign stocks (or stock indices) without incurring FX risk.

- An equity collar establishes a minimum and maximum value for stocks (or stock portfolio) already held by an investor. An equity collar consists of the initial stock holdings, together with a long put with a low strike price K_1 and a short call with a high strike price K_2. If the collar has zero 'up-front' cost (that is, $P = C$) it is called a zero-cost collar (risk reversal).

EXERCISES

Question 1

An investment manager holds a portfolio worth \$4,351,700, which can be thought of as 10,000 shares of a single stock worth $S_0 = 435.17$ (which pays no dividends). Her performance will be evaluated in 78 days ($T = 78/365$) and she would like to establish a maximum and minimum profit over the remaining period until the evaluation is made.

She finds that a *zero-cost* equity collar can be constructed by buying a put with $K_p = 430$ and selling a call with $K_c = 448.75$, both with maturity $T = 78/365$ *years*. The continuously compounded risk-free rate is 4.14% p.a. and the volatility of the stock is 15% p.a.

Use Black–Scholes (with Excel) to show that $P = C = 7.96$ so the zero-cost collar is fairly priced. Show your calculated values for $d_1, d_2, N(d_1), N(-d_1)$ etc.

Question 2

A fund already holds stocks and a *zero-cost* equity collar is constructed by buying a put with $K_p = 430$ for a price $P = 7.96$ and selling a call with $K_c = 448.75$ and price $C = 7.96$.

Calculate and explain the payoffs if the stock price at maturity ends up at either $S_T = 460$ or 435 or 415.

Question 3

The current price of stock-Z is $S_0 = 100$ (which pays no dividends). Ms Sparkle, a fund manager, originally invested \$800,000 in stock-Z and is now worth \$1m. Over the next year Ms Sparkle is worried about increased volatility in the stock market and wants to lock in a minimum value for her holdings in stock-Z of \$900,000, in 1 year's time. Options traders' current view is that the volatility of stock-Z is $\sigma = 50\%$ p.a. and the risk-free rate is 3% (continuously compounded).

What will it cost Ms Sparkle to insure her portfolio? What is the cost in relation to the current value of Ms Sparkle's holdings of stock-Z?

Question 4
Given the scenario in question 3, Ms Sparkle thinks the cost of providing a floor is rather expensive, as it takes a big chunk out of her past gains of $200,000. She decides to sell a call today with strike $K_c = 120$.

(a) What is now the net cost of Ms Sparkle's position? Qualitatively, will Ms Sparkle's stocks and options positions give her a profit after 1 year, if stock-Z is worth $115 in a year's time?
(b) If Ms Sparkle decides today that she wants a zero-cost collar, what strike price will her written call have and what will her profit/loss be if in 1 year's time the stock-Z is worth $130. Ex-post, will Ms Sparkle be happy that she went for a zero-cost collar?

Question 5
Explain why the value of the equity in a firm is like a European call on the value of the firm's assets with a strike price equal to the face value of the bonds K_B issued by the firm. Consider the outcomes for equity and debt holders if the firm is solvent or insolvent at maturity of the bonds.

Question 6
Explain why an equity warrant is like a call option.

Question 7
How does a 'quanto' differ from a plain vanilla call option?

CHAPTER 31

Exotic Options

Aims

- To analyse various types of exotic option whose payoffs are path dependent.
- To use the BOPM and MCS to price various exotic (path dependent) options.
- To demonstrate the use of Asian (or average rate) options, barrier options and other exotics such as lookback, forward start and chooser options.

In this chapter we explain how some of the more exotic options contracts (e.g. on stocks, FX, and commodities) can be used in speculation and hedging. Closed-form solutions for the price of exotic options are often not possible and we demonstrate how some of these options are priced using binomial trees and Monte Carlo simulation (MCS).

It is difficult to say where plain vanilla options end and exotics begin. Generally speaking, the payoffs from exotics are more complex than those from vanilla options. As a result, exotics are largely OTC instruments and their payoffs may sometimes be 'structured' to an individual client's needs – hence the term *structured finance* is sometimes used to denote options with complex payoffs.

The payoff to a European option depends only on the price of the underlying at maturity, T (since it can only be exercised at maturity) and does not depend on the path taken to reach T. European options are said to be path independent. On the other hand, it may be profitable to exercise an American option prior to expiration and hence this is a *path-dependent* option. Many exotic options are path dependent. For example, an Asian (average price) call option has a payoff which depends on the difference between the average price S_{av} over the life of the option and the strike price. The Asian call payoff is $\max(S_{av} - K, 0)$ and the value of S_{av} depends on the path taken by the stock price over the life of the option. Similarly, some options

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

have a payoff which depend on the maximum or minimum value of the asset price over the life of the option – these are also path-dependent options.

Some path-dependent options have closed-form solutions for the options price, as long as one assumes that the underlying asset price is monitored continuously. In practice, payoffs to path-dependent options usually depend on observations of the underlying asset price *at discrete intervals* (e.g. closing prices), and closed-form solutions may not be possible. Path-dependent options (with discrete monitoring) can often be priced using either the BOPM or MCS.

31.1 THREE-PERIOD BOPM

31.1.1 European Plain Vanilla Call Option

We use the $n = 3$ period BOPM model. Assume the stock price can go up by 15% ($U = 1.15$) or down by 20% ($D = 0.8$) each period.[1] The initial stock price $S_0 = 100$, $r = 10\%$ per period and the strike is $K = 100$. The risk-neutral probabilities for 'up' and 'down' moves are:

$$q = (R - D)/(U - D) = 0.857143 \qquad 1 - q = 0.142857 \tag{31.1}$$

where $R = 1 + r$. The stock price tree is shown in Table 31.1:

To value a European option all we require are the four possible payoffs *at maturity*, which for a call option are max $[S(3)_i - K, 0]$ and for a put are max $[K - S(3)_i, 0]$. Using backward recursion through each node of the tree and risk neutral valuation RNV we obtain:

$$C = 26.02 \qquad P = 1.15$$

There are $8 (= 2^n)$ alternative paths for the stock price through the 3-period tree (*UUU, UUD, UDU*, etc.). However, as the price of a European call (or put) depends only on the outcomes for the stock price *at maturity* of the option contract, we can simplify the above recursive calculation. So instead of looking at all (the nodes across all) the eight possible paths, we can simplify the computation by using the following binomial formula with $n = 3$:

$$C = (1/R^n) \sum_{k=0}^{n} \binom{n}{k} q^k (1 - q)^{n-k} \max [SU^k D^{n-k} - K, 0] \tag{31.2}$$

In the above formula the sum is only over k, the number of up moves, which has only four values, 0, 1, 2 and 3. This is a consequence of path independence since we are only concerned with the payoffs at $n = 3$. For example, for the paths with two 'up moves' that is *UUD*, *UDU*

[1]For ease of exposition we do not set $U = e^{\sigma\sqrt{dt}} = 1/D$ in these examples.

TABLE 31.1 Stock price tree

Number of ups				
3				152.09
2			132.25	105.80
1		115	92	73.60
0	102	80	64	51.20
Time	0	1	2	3

and *DUU*, they have the same value for $S(3)$ and these three outcomes are given by the term $\binom{3}{2} = 3$. Hence for these three paths we obtain the term '$3q^2(1-q)(SU^2D - K)$' in the above formula and we do not have to deal with each of the three paths separately.

Although it is more time consuming, we could also value the plain vanilla European call option *as if* it were path dependent. To do this we follow the stock price along *each of the eight paths* to get the eight possible payoffs to the call at $n = 3$. Each of these paths $i = 1, 2, \ldots, 8$ has a payoff equal to $\max[S(3)_i - K, 0]$. The risk neutral probability for each path is:

$$q_i^* = q^k(1-q)^{n-k} \tag{31.3}$$

where k = number of 'up' moves and $(n - k)$ = number of 'down' moves. For example q_i^* for the 'three ups' *UUU*-path would equal q^3 while q_i^* for *each of* the three paths with 'two ups' (i.e. *UUD*, *DUU*, and *UDU*) is $q^2(1-q)$. The call premium is then the present value of the *expected payoffs along all eight paths* using risk-neutral probabilities $q_i^*(\mathrm{i} = 1, 2, \ldots, 8)$ and discounting using the (continuously compounded) risk-free rate:

$$C = e^{-rT} \sum_{i=1}^{8} q_i^* \ \max[S(3)_i - K, 0]. \tag{31.4}$$

Of course this gives us the same answer as the binomial formula of Equation (31.2), which is computationally easier and quicker. Unfortunately, for path-dependent options we need to calculate the outcomes along every possible path through the lattice, since the payoffs depend on the particular path taken – this can be computationally burdensome – for $n = 30$ periods (say).

31.2 ASIAN OPTIONS

Asian options have a payoff which is based on the average price over the life of the option. For example, an *Asian (average price) call option* has a payoff which depends on $\max[S_{av} - K, 0]$ where S_{av} is the average price over the life of the option.

An Asian average price currency option would be useful for a firm that wants to hedge the average level of its future receipts in foreign currency. The firm's *monthly* foreign currency receipts may fluctuate over the year so an Asian option is a cheap way to hedge, rather than using 12 vanilla calls with expiration dates in each of the months.

Average price Asian options have a payoff which depends on $\max[S_{av} - K, 0]$ for a call and $[K - S_{av}, 0]$ for a put. There are also *Asian average strike options* and here the average price replaces the strike price, so the payoff for a call is $\max[S_T - S_{av}, 0]$ and for a put is $\max[S_{av} - S_T, 0]$. Since the underlying principles are similar, we do not discuss the Asian average strike option.

The payoff from an Asian option (on a stock) may be based on the arithmetic or the geometric average of the underlying asset. For example, if the underlying price S for one path in a 3-period tree is 100, 115, 132, 152 then the payoff to a call based on the arithmetic average would be $[(100 + 115 + 132 + 152)/4] - K$ and for the geometric average it would be $[(100)(115)(132)(152)]^{1/4} - K$.[2] The geometric average is always less than the arithmetic average (except when all values of S are equal) and the difference between the two averages is greater, the greater the volatility of the price series.

Although analytic formulas (similar to Black–Scholes) for the price of an Asian option can be derived when the payoff is based on the geometric average, this is not the case for options based on the arithmetic average – which is unfortunate since the latter options are far more popular. It can be shown that (in a risk-neutral world) the *geometric average* of the asset price has the same probability distribution as the *asset price* itself, at maturity of the option, if the asset's expected growth rate is set equal to $(r - \delta - \sigma^2/6)$ rather than the standard $(r - \delta)$ and volatility is set at $\sigma/\sqrt{T}$ rather than σ. Hence we can price the geometric average option using the standard Black–Scholes formulas, with the above changes.

How can we price an *arithmetic* average option? It can be shown that the distribution of the arithmetic average is approximately lognormal and therefore we can apply Black's equation for pricing futures-options, with the following changes. We calculate the first two moments (mean, M and standard deviation, *stdv*) of the probability distribution of the arithmetic average (in a risk-neutral world) and fit a lognormal distribution to these moments. We then set

$$F_0 = M \quad \text{and} \quad \sigma^2 = (1/\mathrm{T}) \ln(stdv/M^2)$$

and use Black's equation for futures options (see Chapter 25, Equations 25.15–25.18). Numerical methods such as the BOPM and Monte Carlo simulation are often used to price Asian options.

Consider an *Asian (average price) call option* on stock price S, with $n = 3$ steps (using the same tree as above) which is reproduced in Table 31.2, for the eight possible paths for the stock price. The steps needed to price the Asian call using the BOPM are:

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[2]The option contract will stipulate exactly how the average is to be calculated, for example using daily, weekly, or monthly stock prices.

TABLE 31.2 Path in BOPM

Path number	Path	Number of ups	Prob.	S(0)	S(1)	S(2)	S(3)	S (average)
1	*UUU*	3	0.6297	100	115	132.25	152.09	124.83
2	*UUD*	2	0.1050	100	115	132.25	105.80	113.26
3	*UDU*	2	0.1050	100	115	92	105.80	103.20
4	*DUU*	2	0.1050	100	80	92	105.80	94.45
5	*UDD*	1	0.0175	100	115	92	73.60	95.15
6	*DUD*	1	0.0175	100	80	92	73.60	86.40
7	*DDU*	1	0.0175	100	80	64	73.60	79.40
8	*DDD*	0	0.0029	100	80	64	51.20	73.80

- Calculate average stock price $S_{av,i}$ at expiry, for each of the 8 possible paths $(i = 1, 2, \ldots, 8)$
- Calculate the option payoff for *each path*: $\max[S_{av,i} - K, 0]$
- The risk-neutral probability for a *particular path* is $q_i^* = q^k(1-q)^{n-k}$
- Weight each of the eight outcomes for the payoff by their respective probabilities q_i^* and sum them to give *the risk-neutral expected payoff*:

$$\sum_{i=1}^{8} q_i^* \max[S_{av,i} - K, 0].$$

- The call premium is then the PV of the expected payoff discounted at the risk-free rate:

$$C_{Asian} = e^{-rT}\left[\sum_{i=1}^{8} q_i^* \max[S_{av,i} - K, 0]\right]. \tag{31.5}$$

The eight final payoffs for the call option are shown in Table 31.3. The call premium for the Asian call option is $C = 13.05$. Similarly the price of an (average price) Asian put option is $P = 1.01$.

The payoffs and option premia for Asian *average strike options* using the BOPM are calculated in a similar way and are shown in the last two columns of Table 31.3. The problem with using the binomial model to price the Asian option is that at least 50 steps are required to get an accurate measure of the option price and this requires evaluating 2^{50} alternative paths! Even fast computers might not be quick enough to price the option in order to execute a deal.

TABLE 31.3 Payoff to asian options

Price call payoff	Price put payoff	Strike call payoff	Strike put payoff
24.83	0	27.25	0
13.26	0	0	7.45
3.20	0	2.60	0
0	5.55	11.35	0
0	4.85	0	21.55
0	13.60	0	12.80
0	20.60	0	5.80
0	26.20	0	22.60
13.05	1.01	13.99	1.17

Notes:
Asian average price options payoffs are, for a call, max[*S*(average) – *K*,0] and for a put, max[*K* – *S*(average),0].
Asian average strike options payoffs are, for a call, max[*S*(3) – *S*(average),0] and for a put, max[*S*(average) – *S*(3),0]

The plain vanilla call costs more at $C = 26.02$ than the Asian average call at $C = 13.05$. Why? For a plain vanilla call, the call premium increases the higher is the volatility σ of the underlying asset, S. The call premium for an Asian average price call (same underlying, strike, and maturity date) depends positively on the volatility of the *average* price, $S_{av} = (1/n)\sum_{i=1}^{n} S_i$, where n is the number of values used to compute the average. If S follows a Brownian motion and hence is identically and independently distributed (iid) then $\sigma_{av}^2 = \sigma^2/n$, which is smaller than σ^2. The Asian call premium that depends positively on σ_{av}^2, will therefore be lower than the call premium on a plain vanilla call (which depends positively on σ^2).

31.2.1 Monte Carlo Simulation (MCS)

An alternative and more efficient way to price an Asian option is to use MCS. Under risk-neutral valuation we 'replace' the real world growth rate of the stock μ with the risk-free rate r in the Brownian motion for the stock price. Assume the Asian option contract specifies that the underlying asset price S will be observed each trading day when calculating the average price over the whole year and:

$$S_0 = 100, K = 110, r = 0.10, \sigma = 0.40, T = 1 \text{ year}, dt = 1/252 = 0.004 \text{ (years)}$$

As we have seen, MCS produces a series of alternative paths for the stock price using a discrete approximation to a Brownian motion. The payoff for an *Asian (average price) call* option for each run-i of the MCS over $n = T/dt = 252$ periods is:

$$\text{Call payoff} = \max[S_{av,i} - K, 0] \tag{31.6}$$

where $S_{av,i}$ is the average stock price over the 252 simulated trading days. The price of the Asian option is the PV of the payoffs (discounted at the risk-free rate) over all m-runs of the MCS:

$$\hat{C} = e^{-rT}(1/m)\sum_{i=1}^{m} \max[S_{av,i} - K, 0] \tag{31.7}$$

We need to choose m large enough so that the error in estimating $\hat{C}$ is acceptable (we can obtain a standard error for our estimate of $\hat{C}$ as outlined in Chapter 26). Put in a more intuitive way, if we do $m + 1$ simulations then the calculated value for $\hat{C}$ should be 'close to' that found when using m simulations.

31.3 OTHER EXOTICS: LOOKBACKS, BARRIER, COMPOUND, AND CHOOSER

There is a virtually unlimited set of exotic options whose payoffs, time to maturity, strike price etc. can be 'structured' in the OTC market to suit the client's needs for either hedging or speculation. Needless to say the more 'exotic' the payoff structure, then generally speaking the more difficult they are to price and, often, numerical techniques are required. Below we briefly discuss some of the more common exotics.

31.3.1 Lookbacks (No-regrets) and Shout Options

Lookback options (on a stock) provide an investor with the opportunity to buy a stock at a potentially 'low' price (= long call option) or sell a stock at a potentially 'high' price (= long put option). For example, *a lookback (European) call* has the strike price set at *expiration,* at the lowest price $S_{\min}$ of the stock during the life of the option, so the payoff is $\max[S_T - S_{\min}, 0]$. Similarly, a *lookback (European) put* sets the strike price at expiration, equal to the highest price reached by the stock over the option's life, so the payoff is $\max[S_{\max} - S_T, 0]$.

These options are also referred to as *no-regrets options* since you would never regret not having exercised the option at an earlier date. The worst outcome possible with a lookback option is if it expires at-the-money ($S_T = S_{\min}$). There are also *fixed strike lookbacks* with payoffs for a call being $\max[S_{\max} - K, 0]$ and for a put, $\max[K - S_{\max}, 0]$.

We have the complete set of paths for the stock price in the BOPM (see Table 31.2) and the associated risk neutral probabilities q_i^* for each path. Hence, knowing the payoffs of the above lookbacks, it is easy to price them using the BOPM (or MCS). It should be obvious that

European lookbacks have payoffs which are more favourable than ordinary European options and hence will have higher call and put premia.

Shout options are European options that allow the holder to lock in a minimum payoff $S_t - K > 0$ at one point in time $t > 0$ during its life, but this payoff is not received until the expiration date, T. For example, suppose the stock price rises so that a call option is currently in-the-money. With a shout option the holder, who may fear a future fall in stock prices, can 'shout' that the current payoff $S_t - K$, will be the *minimum payoff at maturity*. The payoff to the shout (call) option at T is $\max\{S_t - K, S_T - K, 0\}$.

But if the holder shouts at time t, then the payoff to the shout option (at T) is $\max\{S_T - S_t, 0\} + S_t - K$. Given the above payoff, the value of the shout option is the present value of $S_t - K$ (discounted from time T) plus the Black–Scholes value of a European call with a strike price $K = S_t$.

To price a shout call option using the BOPM we work backwards through the tree. At each node we calculate the (recursive) value if the holder does not shout and the value $S_t - K$ if the holder shouts and we take the greater of the two. This is similar to pricing a standard American option.

31.3.2 Barrier Options

Barrier options either 'die' or 'come alive' before expiration. For example, a *knockout option* may specify that if the stock price rises or falls to a 'barrier level', the option will terminate on that date, and hence cannot be exercised at a later date. If the option is terminated when the stock price falls to a lower barrier, we have a *down-and-out option*, while if it terminates when the price rises to a higher barrier, they are *up-and-out-options*. Option premia are paid 'up front' (at $t = 0$).

Another variant of the barrier option is the *up-and-in-option*, whereby the option's 'life' does not begin until the stock price hits an upper barrier. The option premium is paid up front but the option cannot be exercised unless the stock price hits or crosses the upper barrier. Similarly a *down-and-in-option* is not 'activated' until the stock price hits (or crosses) the designated lower barrier.

31.3.2.1 Down-and-Out Put

Suppose you own a stock with current price $S_0 = \$100$ and you buy a 1-year plain vanilla (European) put with strike $K = 100$. If the stock price falls to $S_T = \$91$ at maturity, you obtain a cash payoff of $K - S_T = \$9$ from the put, which implies your stock+put portfolio is worth \$100 (the \$9 cash plus the value of the stock \$91) – you have provided a floor value for the stock of $K = 100$.

Consider instead purchasing a 1-year *down-and-in* (European) put with $K = 100$ and a 'knock-in' lower barrier set at $L = 90$. If the stock price falls *monotonically* to say $S_T = 91$ at maturity, then your option will not have knocked-in and you will receive no payoff – this is a worse outcome than for the plain vanilla put. If at maturity $S_T = 91$ but sometime over the past year the stock price fell below $L = 90$, then your down-and-in put would have 'come alive' earlier and will now payoff $K - S_T = \$9$ at maturity.

Hence for the down-and-in put to be worthwhile in providing a floor (at T), you must think that the stock price is going to be highly volatile over the next year, so it is likely that it would go through the lower barrier, $L = 90$, prior to expiration of the put. But why buy a down-and-in put when the vanilla put pays off in more states of the world? The answer is that a *down-and-in put* is cheaper than a vanilla put (with the same underlying, strike and maturity) since the down-and-in put doesn't come 'alive' until the lower barrier is hit. In other words you get what you pay for. With the down-and-in put, the payoff (at maturity) *may* be less than with the plain vanilla put, hence you pay less today for the down-and-in put.

31.3.2.2 Up-and-Out Put

Who might buy a European 'up-and-out' put? Consider a pension fund manager (Ms Long) who holds stocks that she wishes to 'cash in' after 1 year, or an oil producer (Mr Barrel) who will have barrels of oil 'ready for sale' in 1 year. Assume the stocks and oil are currently worth $S_0 = 100$. In both cases suppose they think prices are highly unlikely to cross an upper barrier $H = 110$ over the next year. Also, assume they are both willing to 'take a hit' of a price fall of up to 10% (in oil or stock prices, but no more), by the year-end. They could protect their position by buying a 1-year out-of-the-money (OTM) vanilla put with a strike of $K = 90$ – but this might be expensive, given the volatility in stock or oil prices.

Instead, suppose Ms Long buys a European *up-and-out put* with a strike of $K = 90$ and a 'knock-out' upper barrier of $H = 110$. If the stock price hits $H = 110$ on any day through the year, Ms Long loses her insurance (policy) on that day – that is, the put is now 'dead' and cannot be used to provide a floor of $K = 90$ if prices fall below that level in 1 year's time. But if prices do not rise above $H = 110$ on any day over the next year but they do fall below $S_T = 90$ by the end of the year, the up-and-out put 'remains alive' and provides a floor of $K = 90$. So there is more risk if you hold an up-and-out put rather than the vanilla put, but the up-and-out put costs you less.

31.3.2.3 Pricing Barrier Options

Pricing path-dependent barrier options using the BOPM or MCS is straightforward in principle. Again, consider the stock price lattice applied to down-and-out and down-and-in calls with the lower barrier set at $L = 90$, the upper barrier with $H = 110$ and the call is initially at-the-money, $S_0 = K = 100$ (Table 31.4).

Consider a *down-and-out call*. The payoff is $\max[S_T - K,\ 0]$ as long as S does not fall below $L = 90$. If any value of S along a particular path falls below L then the option knocks out (indicated by 'knock' in Table 31.4) and even if at expiration $S_T > K$, the payoff to this call is zero. Consider path number 4 (which is DUU) at $t = 1$ where $S = 80 < L = 90$. So even though

TABLE 31.4 Barrier options

						Barrier							
						90	90	90	90	110	110	110	110
						Down-out and down-in options Payoffs				Up-out and up-in options Payoffs			
Path number	Path	*S*(0)	*S*(1)	*S*(2)	*S*(3)	Out call [1]	In call [2]	Out put [3]	In put [4]	Out call [5]	In call [6]	Out put [7]	In put [8]
1	*UUU*	100	115.00	132.25	152.09	52.09	Knock	0	Knock	Knock	52.09	Knock	0
2	*UUD*	100	115.00	132.25	105.80	5.80	Knock	0	Knock	Knock	5.80	Knock	0
3	*UDU*	100	115.00	92.00	105.80	5.80	Knock	0	Knock	Knock	5.80	Knock	0
4	*DUU*	100	80.00	92.00	105.80	Knock	5.80	Knock	0	5.80	Knock	0	Knock
5	*UDD*	100	115.00	92.00	73.60	0	Knock	26.40	Knock	Knock	0	Knock	26.40
6	*DUD*	100	80.00	92.00	73.60	Knock	0	Knock	26.40	0	Knock	26.40	Knock
7	*DDU*	100	80.00	64.00	73.60	Knock	0	Knock	26.40	0	Knock	26.40	Knock
8	*DDD*	100	80.00	64.00	51.20	Knock	0	Knock	48.80	0	Knock	48.80	Knock
Value of the option						25.56	0.46	0.35	0.80	0.46	25.56	0.80	0.35

along this path, at expiry $S(3) = 105.8 > K = 100$, nevertheless this call has a zero payoff, since it has already 'died'. The payoffs for this call at $n = 3$, for all eight paths are shown in Table 31.4, column [1]. The call premium for the down-and-out call is simply the expected value of these eight payoffs (using risk-neutral probabilities q_i^*), discounted at the risk-free rate, which gives $C(\text{down-and-out}) = 25.56$.

Notice that for $L = 90$, the sum of the option premia on the down-and-out call and the down-and-in call (i.e. $25.56 + 0.46$), equals the premium for a plain vanilla European call, $C = 26.01$. This is because whenever the down-and-out call 'dies', the down-and-in call, 'comes alive', so if you hold both it is equivalent to holding a European vanilla call. This also demonstrates that the individual premia on the knock-out options are lower than the premia on the equivalent European vanilla options.

Take another example. If you want to insure a minimum value for a stock portfolio ($S_0 = 100$) using a put with a strike of $K = 100$, you might consider buying an up-and-out put with an upper barrier at $H = 110$ which costs $P_{U\&O} = 0.80$ – this is cheaper than buying a vanilla put, $P = 1.15$ (Table 31.4) – with the same strike and maturity date.

Also, a speculator who expects a strong bull market but with little prospect of a steep fall in stock prices, might consider purchasing a down-and-out call because it costs less than (an equivalent) vanilla call. If the stock price over the life of the option does not fall below the lower barrier but a bull market also ensues, then the speculator can make a handsome profit.

Note that for barrier options, the nodes in the BOPM lattice would need to be aligned with the exact time the stock price is 'checked' against the barrier (e.g. each day at 4 p.m.). Table 31.4 also shows the outcomes and premia for some other European barrier options.

It should be clear from the above that MCS can easily be adapted to price barrier options. Consider a European knock-out call option. We simulate the stock price using a Brownian motion (under risk-neutrality) and if on a particular run of the Monte Carlo a barrier is crossed, then the payoff at expiration is set to zero even if $S_T > K$. If the barrier is not crossed, the payoff is $\max(S_T - K, 0)$.

Excel and MATLAB files that price barrier options using the BOPM and MCS are on the website.

31.3.3 Compound Options

There are also options on options, known as *compound options*. Compound options have two strike prices and two exercise dates. For example, consider a call on a call. At T_1 the holder of the compound option can pay K_1 and take delivery of a call option, which gives her right to buy the *underlying asset* for K_2 at time T_2. The holder of the compound option will exercise at T_1 only if the value of this option exceeds K_1.

On 15 January an investor Ms Caution, decides that in 6 months' time (at $T = 15$ June) she wants to buy *a 2-year call option* (on Apple stock) with a strike K_1. But on 15 January she is worried that by 15 June, the price of cash-market 2-year call options on Apple may have increased (above K_1).

On 15 January she therefore wants to fix the *maximum price* $K_1 = \$10$ she will pay on 15 June for 'delivery' of the 2-year call option on Apple, but she also wants to be able to take advantage of lower call premiums in the cash-market on 15 June, should this occur. Hence, on 15 January Ms Caution buys a 'call on a call', with a strike price $K_1 = \$10$ and pays (Morgan Stanley) the compound-option call premium of $C_0^{comp} = \$1$ (say).

On 15 June if the cash-market price of 2-year call options on Apple stock (with strike K_2) being offered by Goldman's (say) is $C_{T_1}^{2yr} > K_1$, Ms Caution will exercise her compound option with Morgan Stanley and take delivery of a 2-year call option (on Apple) at the low price $K_1 = \$10$. Alternatively, on 15 June if 2-year call options on Apple from Goldman's cost $C_{T_1}^{2yr} < K_1$, she will not exercise the compound option with Morgan Stanley (and hence pay K_1) and instead buys a cash-market 2-year call option on Apple stock from Goldman's, at the lower price of $C_{T_1}^{2yr}$. Hence a compound call option provides insurance against a future increase in *call options prices*. There are also puts-on-puts, puts-on-calls and calls-on-puts – but 'enough already'. If we assume a GBM then European compound options have a complex closed-form solution that depends on integrals of the bivariate normal distribution (but we do not pursue that here).

31.3.4 Rainbow Options

Options can be structured to have a payoff based on the better or worse of two underlying assets and these are referred to as *min-max*, *rainbow options* or *alternative options*. For example, a rainbow call may pay off according to which of two underlying stocks (A or B) has the larger payoff at expiration.

$$\text{Payoff rainbow call} = \max[S_T^A - K_A, S_T^B - K_B, 0] \tag{31.8}$$

31.3.5 Chooser Option

Sometimes an exotic option can be priced analytically because it can be decomposed into two 'simpler options' that do have exact pricing formulas. A *chooser option* or *as-you-like-it option* allows an investor to choose at a (known) specific point in time $t < T$, whether the option is to be a call or a put. Once this choice has been made at t, the option remains as either a call or a put (to its expiration date, T). In a 'standard chooser', the call and put both have the same strike price and maturity and the choice has to be made at the prearranged time, t. 'Complex choosers' allow the holder to decide at *any* time before expiration whether to choose a call or put and the calls and puts may have different strikes and maturities. We only consider the standard chooser.

TABLE 31.5 Payoff to a chooser at T

Choice at t Chooser option	Payoff at T $S_T < K$	$S_T > K$
Call	0	$S_T - K$
Put	$K - S_T$	0

A chooser option is useful for speculating that there will either be a large rise (choose the call) or a large fall (choose the put) in the stock price between t and T. The prospective payoff at $t = 0$ is therefore like a straddle but the chooser option has a lower premium. The reason for the lower premium is that a straddle *always* has a positive (gross) payoff, if either the call or the put is in-the-money at expiration (i.e. $S_T > K$ or $S_T < K$). But with a 'chooser', after the choice is made at t, there is a possibility that the chooser option will end up out-of-the-money (e.g. if you choose a call at t and $S_T < K$ at expiration, then your chooser option has a zero payoff). The closer is the choice date to the maturity date of the options, the lower the probability of a wrong choice at t and hence the chooser becomes more like a straddle.

An analytic solution for pricing a chooser option (on a stock which pays no dividends) is possible because it can be replicated using a long call and a long put – hence the price of the chooser is simply the sum of the 'replication' call and put premia. The analysis is a little involved and requires several steps.

First calculate whether the investor (Ms Dizzy) will choose a call or put. At t, Ms Dizzy will choose the type of option which has the highest value. For example, she will choose a call if:

$$C(S_t, T - t, K) > P(S_t, T - t, K) \tag{31.9}$$

Substituting for P_t from put–call parity, $P_t = C_t - e^{-\delta(T-t)}S_t + Ke^{-r(T-t)}$ (where the underlying asset has a constant dividend yield, δ) this implies:

$$C_t > C_t - e^{-\delta(T-t)}S_t + Ke^{-r(T-t)} \Rightarrow S_t > Ke^{-(\delta-r)(T-t)} \tag{31.10}$$

Hence Ms Dizzy chooses the call at t whenever the stock price at t exceeds $K^* = Ke^{-(\delta-r)(T-t)}$. Given the choice at t, of either a call or put, the payoffs at maturity T consequent on this choice are given in Table 31.5.

To price the chooser option we need to consider a replication portfolio made up of 'plain vanilla' options that we can price using Black–Scholes. The price of the chooser option will then equal the price of this replication portfolio. The holder of the chooser at time t will choose the more valuable of the two options:

$$\max[C(S_t, K, T - t), P(S_t, K, T - t)] \tag{31.11}$$

Put–call parity at time t:

$$P(S_t, K, T-t) = C(S_t, K, T-t) + Ke^{-r(T-t)} - e^{-\delta(T-t)}S \tag{31.12}$$

Substituting in (31.11) for P:

$$\max[C(S_t, K, T-t), C(S_t, K, T-t) + Ke^{-r(T-t)} - e^{-\delta(T-t)}S] \tag{31.13}$$

Rearranging (31.13):

$$C(S_t, K, T-t) + \{e^{-\delta(T-t)}\max[0, e^{-(r-\delta)(T-t)}K - S_t]\} \tag{31.14}$$

The second term (in curly brackets) is the payoff from $e^{-\delta(T-t)}$ put options with strike $K^* = e^{-(r-\delta)(T-t)}K$ which matures at t. From (31.14) we see that the chooser is equivalent to the following replication portfolio at $t = 0$:

(a) Long one European call *with maturity* T and strike price K plus
(b) Long $e^{-\delta(T-t)}$ European puts *with maturity* t and strike price $K^* = e^{-(r-\delta)(T-t)}K$

Hence using Black–Scholes the price of the chooser V_{ch} (at $t = 0$) is

$$V_{ch} = C(S, K, T) + e^{-\delta(T-t)}P(S, K^*, t) \tag{31.15}$$

A straddle is a call and put with the same strike K and maturity $T(> t)$, so it differs from a chooser because for the chooser, the put matures at $t < T$. If $\delta = 0$ then the chooser has the same call as the straddle but the chooser has a put with a lower strike and a shorter maturity – hence the chooser costs less than the straddle.

31.4 SUMMARY

- Exotic (European) options have payoffs that are more complex than those from plain vanilla European options. Exotics are largely OTC instruments and their payoffs are structured to suit the needs of individual clients, in terms of maturity date, strike price, and different types of payoffs.
- Asian options have a payoff which is based on the average price of the underlying asset over the life of the option. They are useful when a corporate or an investor is paying or receiving periodic 'cash flows' (e.g. foreign currency).
- Barrier options either 'die' or 'come alive' before the expiration date, depending on whether the underlying asset price hits (or crosses) one or more barriers. These types of option cost less than the equivalent vanilla options (with the same underlying asset, strike price and time to maturity).

- Compound options are 'options on options'. For example, buying a 'call-on-a-call' from Morgan Stanley sets the *maximum price* K_1 that you pay at T_1 for 'delivery' of a 'second' *call option* on Apple stock (say). This deliverable call option (on Apple stock) will have its own strike K_2 and maturity date T_2 $(> T_1)$. Like all call options, a compound call option also allows the holder to take advantage of lower cash-market *call* premia should this occur at T_1. At expiration of the compound option at T_1, if the premium for a cash-market call option on Apple stock (with strike K_2 and maturity date T_2), quoted by Goldman is $C_2 < K_1$ then the compound option (with Morgan Stanley) is not exercised but instead the investor simply buys an identical cash-market call option from Goldman's at the lower price C_2.
- A chooser option allows the investor to choose at a specific point in time, whether the option is to be a call or a put.
- Lookback (European) options provide the investor with the opportunity to buy the stock (at maturity of the option) at the lowest price (lookback call) or to sell the stock at the highest price (lookback put) that has occurred over the life of the option.
- Many exotic (European) options are path dependent with a payoff at maturity that depends on *all the possible paths* taken by the underlying asset price and not just on the value of the underlying asset price on the expiration date of the option.
- For many exotic options closed-form solutions (like Black–Scholes) are not possible. Often, they are priced using numerical techniques such as binomial trees or Monte Carlo simulation or by solving a PDE by numerical methods (see Chapter 48).

EXERCISES

Question 1
Why might a US importer hedge using a long Asian (average price) call option on euros, with monthly averaging and maturity of one year?

Question 2
Explain the difference between path-dependent options and path-independent options and state why an American option is path dependent.

Question 3
Why might an investor holding stocks buy an up-and-out put. Intuitively, why is the price of an up-and-out put less than the price of a plain vanilla put?

Question 4
Why might a *speculator* buy an up-and-out put?

Question 5

Calculate the price of an (average price) Asian call option on a stock, using a *two-period* binomial model. Assume $S_0 = 100$ and the stock can go up 15% or down 10% per period. The risk-free rate $r = 5\%$ per period and $K = 95$. Include the current stock price when determining the average price.

Question 6

Explain the steps used to price a knock-out call option with an upper barrier $H = 110$ (which is greater than the current stock price, $S_1 = 100$) using Monte Carlo simulation. The maturity of the option is $T = 1$ year, $\sigma = 20\%$ p.a., $r = 3\%$ p.a. (continuously compounded) and the barrier condition is monitored every 0.01 years (approximately every 2.5 trading days).

Question 7

You are given the following information. Current stock price $S_1 = 100$, the drift rate of stock price, $\mu = 5\%$ p.a., risk-free rate $r = 0.03$ (continuously compounded), the volatility of stock returns $\sigma = 20\%$ p.a., $K = 105$ and the time to maturity $T = 1$ year. Assume that $dt = 0.01$ years (i.e. approximately 2.5 trading days) for the length of each timestep in the MCS.

Set out the steps you would take to price an Asian (average price) call option using Monte Carlo simulation. Assume $S_1 = 100$ is included in the averaging. (You can also set up the problem in Excel or some other convenient software.)

CHAPTER 32

Energy and Weather Derivatives

Aims

- To outline the main types of energy and weather derivatives.
- To show how energy derivatives are used for speculation and hedging 'price risk'.
- To show how weather derivatives are used to hedge 'volume risk' of energy producers and users.
- To discuss the use of catastrophe bonds.

There are large oil reserves in a number of countries around the globe, most noticeably the Organisation of the Petroleum Exporting Countries (e.g. the Middle East, Venezuela, Nigeria, etc.) and Russia, who together control about 60% of the world's oil reserves. Russia also has huge natural gas reserves.

Energy prices such as oil and natural gas are highly volatile. This means that large energy suppliers (e.g. in oil exploration and refineries such as BP, Shell, Exxon Mobile) and users of energy (e.g. airlines, transport, and manufacturing companies) face considerable uncertainty about the price they will receive or pay for energy in the future. To mitigate such price risk, consumers and producers can use either over-the-counter (OTC) or exchange traded energy contracts such as forwards, futures, options, and swaps. These 'commodity derivatives' are widely used in the energy sector – some are cash settled and some involve physical delivery of the underlying commodity. Electricity is a little different to oil and natural gas since it cannot be stored but its price can also vary tremendously on an hourly basis and derivatives contracts on electricity are also available.

The weather, in the form of abnormally high or low temperatures affects many industries. These include firms who have substantial heating costs in winter or costs of air conditioning in

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

summer, the leisure sector (hotels, ski resorts) and the agricultural sector (e.g. crop growers, vineyards, and orange growers). Abnormal variations in temperature can affect the volume of output produced by these industries which in turn affects their profits. However, derivative contracts based on 'temperature' can be used to mitigate these risks due to unforeseen 'volume effects'.

There are also weather derivatives based on the number of frost days in a month or the depth of snowfall at particular locations. Because the revenues and costs of various industries (e.g. flower growers, municipal authorities, and not least, ski resorts) are affected by frost and snowfall, weather derivatives can be used to mitigate the impact of abnormal levels of snow and frost-days, on the output of these sectors.

In life there is always the possibility of an unusually large natural catastrophe even though this may have a relatively low probability of occurrence – for example, very severe hurricanes, floods and earthquakes. Insurance claims against such events could cause a specific insurance company to go into liquidation, so there is a need to spread such risks across many participants – so called catastrophe (CAT) bonds are one way of doing this.

In this chapter we examine how derivatives are used to hedge (or insure) against price volatility found in spot energy markets and how weather derivatives are used to mitigate changes in profits which result from changes in the volume of output (e.g. in agriculture, energy supply and the hotel and leisure sectors), caused by abnormal changes in the weather.

32.1 ENERGY CONTRACTS

We will not give an exhaustive account of energy derivatives but instead concentrate on the main features of these contracts, so you get a flavour of what is on offer. There has been an active OTC and exchange-traded market in oil products since the early 1980s. OTC *forward contracts* set a known fixed price today, for future delivery at specific delivery points. Futures and options contracts are traded on crude oil and its refined products, gasoline, heating oil, and jet fuel. The key exchanges trading these contracts are the New York Mercantile Exchange, (NYMEX), the Chicago Mercantile Exchange (CME) and the International Commodities Exchange (ICE) in London. Traded options contracts are usually written on the futures price rather than the cash market (spot) price and are therefore 'futures options' – but we largely ignore this nuance in this chapter.

One crude oil futures contract is for delivery of 1,000 barrels, while one jet fuel, heating oil or gasoline futures contract is for delivery of 42,000 gallons (equivalent to 1,000 barrels). Some contracts require physical delivery (e.g. light sweet crude oil futures on NYMEX) while others must be cash settled (e.g. Brent crude oil futures on ICE, Gulf Coast Jet Fuel on CME) and do not allow delivery.

It is only recently that futures and option contracts on jet fuel have been introduced on CME. These trade Sunday to Friday 6 p.m.–5 p.m. on the Globex platform, with a 60-minute break each day (beginning at 5 p.m.). Monthly contracts are available for 36 consecutive months and one contract is for delivery of 42,000 gallons. These are ‘average price’ futures contracts as the floating price for each futures contract month is the arithmetic average of the daily high and low (‘Platts’ jet fuel) spot prices averaged over each business day in the month. Gulf Coast Jet Fuel options are European and also have a payoff which depends on the average price over the month – so they are Asian average price options.

There is also the possibility of avoiding delivery arrangements set by the exchange and instead using ‘exchange of futures for physicals’ (EFP) arrangements, prior to maturity. This is simply a bilateral agreement about location and price, for delivery between the party with the long futures position and the party with (an equal contract size) short position. The futures exchange (clearing house) must be notified of the EFP arrangement and the futures positions for both parties are then terminated. EFP might be used when the long wants delivery at a specific location that is different from that stipulated in the futures contract – if it cost more in transportation and insurance costs to deliver to the long’s desired location then the holder of the long futures position will pay an additional cash amount to the trader holding a short position.

There are also OTC and exchange traded contracts on natural gas. Delivery is ‘through the pipe’ at a specific geographical location, at a specific uniform rate through the month. The seller of natural gas (e.g. short futures position) is responsible for delivery at a specific delivery point, for example at the Henry Hub gas interconnector (in Louisiana) or at Zeebrugge (in Belgium) – a key European hub. The supplier of gas might be a separate company to the producer of the gas – particularly in deregulated gas markets such as the USA and UK. NYMEX and ICE trade futures (and options) contracts which (if not closed out) require physical delivery of 10,000 million British thermal units (mbtu) of natural gas. Of course, if you go to delivery and you need your gas in Boston, then you had better make sure you have purchased capacity in the pipe between the Louisiana delivery hub and Boston – because the pipe might be full! Options and swaps on natural gas are also available in the OTC market.

The electricity market is a bit different to oil and gas, as electricity cannot be stored. (Technically you can store it in a battery but this might require a very large battery.) In the US and UK electricity is produced primarily in gas-fired, coal-fired, and nuclear plants. The latter provide the base load and extra demand is met by gas and coal-fired plants (and some by wind turbines and solar). The transmission of electricity is costly and there are also transmission losses to consider, over long distances. So, electricity in the US is provided first to a specific region and any excess can then be sold in the wholesale market. Spikes in electricity consumption can be triggered by abnormally high temperatures in summer (particularly in the US where air conditioning is widely used) or abnormally low temperatures in winter. So there is considerable volatility in electricity prices on a daily basis.

There are active markets in the US and UK in OTC forwards, options and swaps on electricity prices and NYMEX and ICE trade a futures contract on the price of electricity. The contracts allow one party to receive a specified number of megawatt hours, at a specified price and location, during a particular month. For example, this could be a 5×8 contract, for Monday to Friday only during off-peak hours (11 p.m. to 7 a.m.) within a specified month.

There is also a 5×16 contract for peak hours (7 a.m. to 11 p.m.) from Monday to Friday or a 7×24 contract for delivery over 24 hours, for 7 days.

With an option contract on electricity you have precisely that – an option to take delivery at the strike price K or not. If the contract is for daily exercise, then with one day's notice you can choose whether you want to take delivery of electricity at a price K, for the next day. For an option with monthly exercise, you make a decision at the beginning of the month whether you will take delivery of electricity *each day* at the strike price K.

In electricity and natural gas derivatives markets you can also be a 'swinger'. You can purchase a *swing option* (also called a *take-and-pay option).* For an 'electric swinger', the option holder sets a maximum and minimum amount of power she will take on each day during a specific month and a maximum and minimum over the whole month. Each megawatt taken is at the strike price, K. You can then 'swing' the amount of power you choose to take each day (within the bounds set), although there is also usually a limit on the number of days on which you can *change* your rate of daily consumption.

At this point it is also worth mentioning another market which is likely to be of increasing importance in the future – carbon trading. This is a market where permits can be purchased by firms wanting to increase their output of greenhouse gases. The supply of permits is provided by firms who have been given an allocation of permits which exceed their output of greenhouse gases (see Finance Blog 32.1).

Finance Blog 32.1 Carbon Trading

Going green seems to be catching on. We have recycling of household rubbish, supermarkets charging for plastic bags or providing their customers with 'bags for life', houses with wind turbines and solar panels and holidaymakers paying to offset the carbon emissions of air travel. On a bigger scale we have the Kyoto, Paris, and Katowice conferences to reduce greenhouse gas emissions. What methods are available for reducing greenhouse gas emissions from industry?

One method is to tax emissions. The problem here is that you have to set the tax rate at a level you think will achieve your emissions target but given the technological complexities involved, you could set the tax rate too high or too low, thus undershooting or overshooting your target.

The method which seems to have the most adherents is carbon trading. This usually affects sectors such as power-generation, cement, ceramics, steel and paper industries. However, it is noticeable that the scheme does not cover air transport. The scheme works by setting allocations for emissions for each firm in the scheme. Firms which emit less than their allocation can sell the unused portion of their permits to the extreme polluters, via exchanges in London and Chicago. Although the US did not sign up to Kyoto it has a similar permit scheme limiting carbon dioxide, sulphur and nitrogen oxide emissions. In Europe it has been argued that the initial carbon permit allocation was too generous, so the price of carbon permits in the market was low and this provided little incentive for firms to curb their emissions.

It has been suggested that permits should not be allocated by government decree but should be auctioned. The higher the market price set when trading the permits, the higher the costs of pollution for specific firms and the greater incentive they have to innovate with clean technology or simply to reduce final output. Total carbon dioxide emissions are therefore set by the scheme but their allocation across firms is set by the market. At the moment, trading environmental permits are available in spot markets and on NYMEX there are futures and futures options on various spot market traded indices of pollution (e.g. CO2 emissions).

Source: Adapted from Cuthbertson and Nitzsche (2008).

32.2 HEDGING WITH ENERGY FUTURES

32.2.1 Running an Airline

Suppose you own a medium-size US airline 'EasyFly' and your marketing manager has told you the ticket prices you will be charging on all your routes, based on maintaining your competitive position in the market. If jet fuel remains at its current level it is estimated that EasyFly will make a handsome profit over the next 3 years. But that is a big 'if', in today's volatile oil market. If EasyFly does nothing and jet fuel remains at its current level (or falls) you will be a happy CEO and your shareholders will be smiling too. But if jet fuel rises in price, then despite all your efforts on the technical side (reliability, customer service, maintenance, staff costs, etc.) EasyFly will end up with lower profits and you may lose your job as CEO.

Here we demonstrate the use of a cross hedge so we assume EasyFly hedges some or all of its projected jet fuel costs, using *heating oil* (HO) futures contracts (traded on NYMEX) rather than futures on jet fuel prices. Since EasyFly fears a rise in spot jet fuel prices in the future it should go long (buy) heating oil futures today. If spot jet fuel prices do rise over the next year, so will the futures price on heating oil and EasyFly can close out its HO-futures contracts on NYMEX at a profit – the latter offsets the higher cost of EasyFly's spot jet fuel.

What practical issues are involved? EasyFly has to decide what quantity Q (gallons) of fuel to hedge and then calculate the appropriate number of futures contracts for each month:

$$N_F = \frac{\textit{Monetary value to be hedged } (= Q \times S)}{\textit{Contract size} \times S} \times \beta = \frac{Q}{\textit{Contract size}} \times \beta \tag{32.1}$$

Beta β, is the slope of a (OLS) regression with the change in the spot price of jet fuel as the dependent variable and the change in the futures price of heating oil as the independent variable. In practice either of the following equations are used to estimate beta: $\Delta S = \alpha + \beta \Delta F$, or

$\Delta S/S = \alpha + \beta(\Delta F/F)$. Since heating oil and jet fuel prices correspond to different 'commodities' this is a 'cross hedge'.

As you have to estimate the relationship between these two price changes, your hedge will be less than perfect – this is *basis risk*. However, over short horizons of say 1 year, the correlation coefficient (and 'beta') between changes in heating oil and jet fuel prices is quite high (the beta is around 0.9) and is reasonably stable over time – so, on average your hedge will work.

But basis risk is also affected by changes in the *convenience yield*, that is, the premium that holders of heating oil place on having the 'physical' heating oil available to satisfy their normal customers – this can cause a divergence between changes in the spot and futures prices at particular periods and could lead to an increase in the hedging error.

Suppose in January EasyFly decides to hedge 2 million gallons of jet fuel for each of the months of April, May, and June. The contract size for heating oil futures is 42,000 gallons (equivalent to 1,000 barrels) hence the number of futures contracts for each month of the hedge is 47.6 (48). So in January EasyFly goes long 48-April, 48-May, and 48-June contracts – taken together this is referred to as a strip hedge.

If EasyFly closes out each contract just before its maturity date this will provide a reasonably good hedge – any gains/losses on the futures contracts will closely offset any increased/lower costs of the spot jet fuel in April, May, and June. Effectively, EasyFly 'locks in' a known average purchase price for jet fuel, approximately given by the average of the current quoted futures prices in January for the April, May, and June futures contracts. EasyFly will not take delivery in the futures contracts because it requires physical jet fuel (at various airports around the country) and the futures contract it has used delivers heating oil to New York Harbour – also, planes don't fly well on heating oil!

32.2.2 Caps and Floors

The futures hedge 'locks in' a known effective price for jet fuel but does not allow Easyfly to gain from lower jet fuel prices should they occur. What if EasyFly wanted to set a cap (upper limit) on future jet fuel costs but also wanted to benefit from lower jet fuel prices should they occur?

Suppose the current spot price of jet fuel on 15 January is $S_0 = \$1.9$ per gallon and EasyFly will purchase jet fuel on 15 April in the cash market (e.g. at Atlanta airport). We know from earlier chapters that to set a maximum price payable, EasyFly needs to purchase a (European) call option (on jet fuel) at a specific strike price, $K_c = \$2.2$ per gallon (say), with an expiration date $T = 15$ April[1].

[1]For ease of exposition we assume the payoff on the option contract is based on the spot price of jet fuel at expiration of the option contract. In reality, the payoff is likely to be based on the average daily price of jet fuel over the contract month.

On 15 April, if jet fuel prices turn out to be high ($S_T > K_c$) and the option is cash settled, EasyFly receives cash of $S_T - K_c$. Hence, even though the spot price of the jet fuel EasyFly buys is high, the net cost of purchasing fuel on 15 April is $S_T - (S_T - K_c) = K_c$, so EasyFly effectively ends up paying a maximum price of $K_c = \$2.2$ (plus the cost of the call option premium). On the other hand, if the spot price of fuel turns out to be low (say $S_T = \$1.7$ per gallon) on 15 April, then EasyFly 'throws away' the call option (i.e. does not exercise the option) and buys fuel at the low spot price of \$1.7 (at Atlanta airport). For the insurance that the option provides, EasyFly has to pay the call premium C on 15 January.

The call option caps EasyFly's maximum payment for fuel at K_c for a *specific month* in the future – it is a caplet. If EasyFly wants to purchase fuel over several months ahead, then it need a series of calls – all with the same strike – but with different maturity dates, which match the dates of EasyFly's future spot purchases of fuel. This requires a strip of caplets, which are collectively known as *a cap* and are provided OTC by large banks and other financial institutions.

It should be fairly obvious that if you are a supplier of jet fuel (e.g. Exxon Mobile) and hold large physical reserves, then what you are worried about is a fall in the price of jet fuel, in the future. You can insure a floor value for your future jet fuel sales by buying a put option with strike $K_p = \$1.6$ say – this will guarantee you receive at least \$1.6 per gallon for your oil in the future. Of course if spot prices rise $S_T > K_p$, Exxon Mobile 'throws away' the put and simply sells its fuel at the high spot price. In these circumstances the put is (not surprisingly) known as a floorlet and a strip of puts is *a floor*.

The advantage of the put is that it gives Exxon Mobile the benefit of selling its jet fuel at high spot prices should they occur but also guarantees a floor price of $K_p = \$1.6$ should spot fuel prices fall sharply. In contrast, hedging a fall in fuel prices using a short futures position locks in the price you will pay in the future, so you cannot take advantage of lower spot fuel prices, should this occur.

32.2.3 Collar

Go back to the example of EasyFly that capped the price of its future purchases of fuel by buying a call with a strike, $K_c = \$2.2$. But buying the call may be expensive. Today, on 15 January EasyFly could offset some of the cost of the call premium C, by selling a put (with strike price $K_p < K_c$) and receiving the put premium P. If EasyFly, who will purchase fuel in the future, undertakes a 'buy call-sell put' position then this is known as a collar (and if the call and put premia exactly offset each other it is known as a zero-cost collar). We know what happens if fuel prices rise, the call sets the maximum effective cost of fuel purchases at K_c. But what happens if fuel prices fall substantially?

If the airline had not sold the put, it would directly benefit from very low spot prices S_T. But if EasyFly has sold a put and if $S_T < K_p$ on 15 April, it will have to pay out $(K_p - S_T)$ to the holder (buyer) of the put. If $S_T < K_p$, the *effective cost* of fuel for EasyFly is equal to the spot price of fuel plus Easyfly's cash payout on the put $= S_T + (K_p - S_T) = K_p$. Hence if $K_p = \$1.6$, the

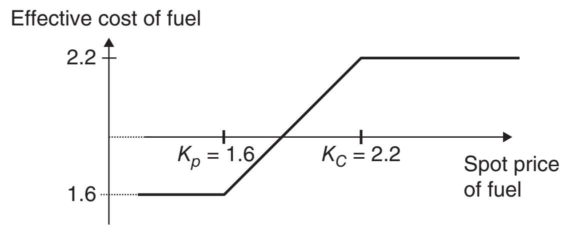

FIGURE 32.1 Collar trade by airline

minimum effective cost of the fuel by the airline will be \$1.6 per gallon (even if spot fuel prices are below \$1.6). So if EasyFly has to purchase airline fuel in the future and it also undertakes a collar trade, it will cap its effective cost at an upper level $K_c = \$2.2$, but also set a floor level it will pay for fuel of $K_p = \$1.6$. (Note that for the above to work as described, $K_p < K_c$.)

The effective cost (ignoring the cost of the call and the put) of the jet fuel to EasyFly is given in Figure 32.1. If the spot price turns out to be between $K_p = \$1.6$ and $K_c = \$2.2$, say $S_T = \$1.8$ then neither of the options is in-the-money and EasyFly purchases fuel at the spot price of \$1.8.

The key outcome of a collar trade for EasyFly is that it sets a maximum ($K_c = \$2.2$) and minimum price ($K_p = \$1.6$) for fuel paid by the airline.

32.3 ENERGY SWAPS

We have already discussed interest rate and currency swaps in previous chapters. Now we examine how the energy sector might use swaps to offset any price risks they face. Suppose in January a company such as Centrica in the UK agrees to supply natural gas (NG) to a large industrial customer-ABC (e.g. manufacturing firm) at a fixed price $S(fix)$, every month for a 6-month period beginning on 1 January. Centrica decides it will buy the gas in the spot market at whatever price it has to pay at the time of purchase (i.e. it pays a 'floating price', $S(float)$ to the gas producer who supplies the gas via the European hub at Zeebrugge in Belgium) – Figure 32.2.

Clearly, if the spot price of gas rises, the profit margin of Centrica will be squeezed (and it may even be committed to supplying the gas at a loss if the spot price in Zeebrugge $S(float)$ rises above $S(fix)$. How can it offset this price risk without renegotiating any of its existing contracts with its customers? It can undertake a receive-fixed, pay-floating swap from a swap dealer where $S(float)$ is the floating price used and X is the agreed fixed price (see the left-hand side of Figure 32.2). X is the swap rate. If the spot price of NG in Zeebrugge $S_T(float)$ at the end of any month is higher than X then Centrica receives a cash payment of $S_T(float) - X$ from the swap dealer. If Centrica buys gas in the cash market at $S_T(float)$, then the effective cost of Centrica's gas purchases is:

$$\text{Effective cost of gas} = S_T(float) - \{S_T(float) - X\} = X \tag{32.2}$$

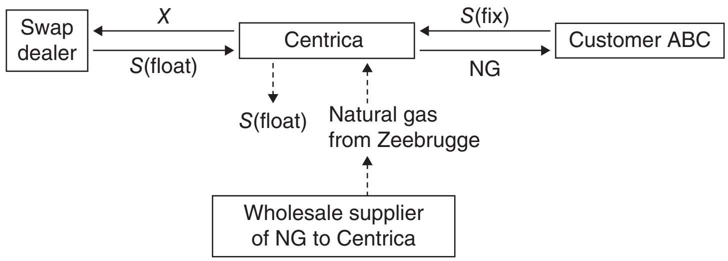

FIGURE 32.2 Natural gas – fixed for floating swap

The net result is that Centrica receives $S(fix)$ from its industrial customers and pays the swap dealer X. As long as $S(fix)$ is greater than X, Centrica will have locked in a fixed profit margin of $S(fix) - X$ on 1 January, on each unit of NG supplied over the next 6 months.

There are other important elements to the swap deal. There will be a notional quantity Q of NG in the swap, for example this might be $Q = 1$ million mbtu *per day* and the 'tenor' of the swap could be every month from January to June. The swap payments to Centrica would then be:

$$\text{Swap payments each month } \textit{to} \text{ Centrica} = (1 \text{ million}) \times \text{days} \times [S(\textit{float}) - X] \qquad (32.3)$$

where 'days' = number of days in the month. The swap contract may be cash settled each month and it is then a 'contract for differences' as only the net cash payment is made by one of the parties to the swap contract. If, in any month:

$S(float) > X$, then Centrica *receives* cash from the swap dealer.

$S(float) < X$, then Centrica *pays* cash to the swap dealer.

The floating index $S(float)$ will have to be decided upon and this spot price often goes under rather strange names; for example, for the Centrica deal it might be the 'front of month inside FERC' index. In fact, the floating payment $S(float)$ you receive from the swap dealer is based on the 'front of month' (FOM) price but Centrica might buy its NG at spot prices *throughout* the month, so there is some residual price risk remaining.

32.3.1 Pricing the Swap

The fixed price X for NG (say) quoted in the swap is the 'swap rate' and will vary depending on the maturity of the swap deal – so different swap rates will be quoted for 1-year, 2-year, … etc. swaps. Broadly speaking, the swap rate X is an average of the forward prices for NG over

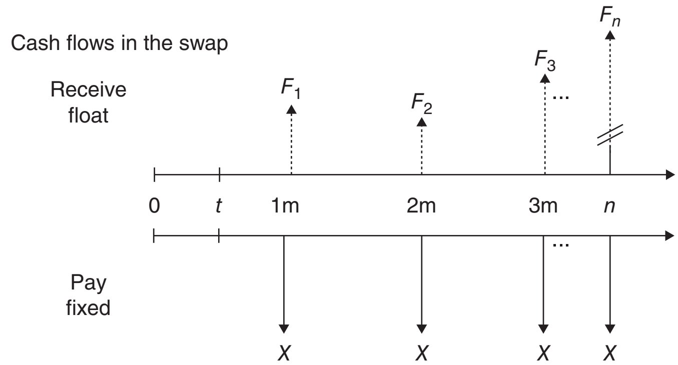

FIGURE 32.3 Pricing a commodity swap

the life of the swap – in terms of jargon, 'the swap is priced off the forward curve' (i.e. forward prices of NG at different maturities – see Figure 32.3).

The floating cash flows in the swap depend on future spot prices and our best guess (at $t = 0$) of these prices are the current quoted forward prices, F_k. The notional volume in the swap contract Vol_k can be different in different months. The risk-free discount rate is $d_k = e^{-r_k t_k}$ – using continuously compounded rates. The swap is priced at inception so that it has zero value to both parties:

$$\begin{aligned}\text{(Expected) Present value of floating receipts accruing at time-}k &= Vol_k F_k d_k \\ \text{Present value of fixed payments accruing at time-}k &= X(Vol_k d_k)\end{aligned}$$

The swap must have zero expected value at inception, so the fixed swap rate X quoted today is:

$$\sum_{k=1}^{n} Vol_k F_k d_k = X \sum_{k=1}^{n} Vol_k d_k$$

$$X = \frac{\sum_{k=1}^{n} Vol_k F_k d_k}{\sum_{k=1}^{n} Vol_k d_k} \tag{32.4}$$

Hence the swap rate X is determined by a (complex weighted) average of the forward prices of NG and current spot yields, over the life of the swap.

An OTC swap of the above type can be tailor-made by the swap dealer to suit Centrica. It can begin immediately or can have a delayed start (i.e. 'forward swap') or it can be made to apply only to specific months when Centrica feels the spot price for NG is likely to be abnormally high. The notional quantity in the swap (e.g. 1 million mbtu of NG per day), can be pre-set at different levels on particular days within a month. Alternatively, the notional quantity can be a fixed daily amount *within* any one month but with different daily levels applying for each separate month. These would then be referred to as 'roller coaster swaps'. With a swap, Centrica effectively locks in the effective price it pays for NG, over several months in the future, regardless of whether the spot price for NG in Zeebrugge turns out to be high or low.

Suppose Centrica is happy to receive $S(float) - X$ from the swap dealer when spot prices turn out to be high (and hence has an effective cost of buying NG of X). But when the floating price of NG is less than the fixed price X in the swap, Centrica only wants to *pay* the swap dealer 60% of $S(float) - X$. For example, if $S(float) = 80$ and $X = 100$ then the effective cost of buying NG would be $80 + 60\% \times 20 = 92$, whereas with a plain vanilla swap the effective cost would have been $X = 100$. So the '60% swap deal' allows Centrica to pay less when prices are low and hence participate in some of the benefit of low prices – not surprisingly this is known as a *participation swap*.

You may have noticed in Figure 32.2 that the swap dealer has taken on risk, since she is paying $S(float)$ to Centrica each month and receiving X-fixed from Centrica – so if NG prices rise in the future the swap dealer will experience losses. How does the swap dealer deal with this problem? Well, she may be lucky and find an offsetting deal with another party who wants to pay the swap dealer a floating price (with the same maturity and tenor as the original swap with Centrica). This may happen automatically or the swap dealer can shade her price quotes to encourage such swaps. But let us assume (realistically) that even after these offsetting swaps she still ends up as a net payer at a floating price.

We know that the swap dealer fears a rise in $S(float)$. But if there is a futures contract where the underlying asset is the spot price of NG, she can go long (an appropriate number of) futures contracts. Then if $S(float)$ increases in the future she can close out the futures at a profit and use this cash inflow to offset the higher floating payment in her swap deal. So in actual fact what derivatives markets are very good at providing is 'risk spreading' across a wide array of participants in these markets. The price risk is still there, but each party only holds 'just a bit' of it – it is an efficient 'risk sharing' mechanism. In short, without derivatives markets households' gas bills might be even higher. The nearby blog shows how electricity producers can 'lock in' a profit margin on their power generation plant.

Finance Blog 32.2 Spark Spread Swap

We all know that electricity can produce a spark and in the UK an electrician is called a 'sparky'. What can a 'spark' have to do with swaps? Suppose you run a gas-fired power plant ('Voltaire') which produces electricity. (Centrica also owns gas-fired power plants in the UK.) Voltaire's profit margin (per unit of electricity) is

$$\text{Profit margin of electric power plant} = S(electric) - S(NG)$$

(*continued*)

(*continued*)

where both prices are measured in say 'pounds per megawatt hour', £/MWh. (This simplified presentation avoids complications like the 'heat rate' which is the rate at which NG can be converted into electricity.) Voltaire may be willing to take on some 'price risk' due to the spot price of NG rising more than the price of electricity in the future, hence squeezing profit margins. But, if in any future periods $S(NG) > S(electric)$, Voltaire may temporarily shut down the plant and cease to provide electricity to the 'national grid' during these periods. (Of course many other factors may enter into the decision to temporarily close down the plant, not least the ease with which the plant can be up and running again.)

We can view the power plant as earning net receipts of $S(electric) - S(NG)$ (per megawatt hour) which will fluctuate as these two spot prices move up and down over time. Suppose Voltaire thinks that over the next few months either electricity or NG prices or both are likely to be more volatile than normal (e.g. due to highly variable weather conditions which affect electricity prices or risks on the supply of NG 'through the pipe' which affect spot NG prices). This means if Voltaire does nothing it faces great uncertainty about its future profit margins in selling electricity.

To avoid this uncertainty Voltaire takes out a swap and agrees to pay a swap dealer an amount $S(electric) - S(NG)$, in return for receiving a fixed payment X, from the swap dealer, over each of the next few months. The net result of running the power plant and having the swap position is that Voltaire has effectively fixed its profit margin at X (per megawatt of electricity sold). The agreed fixed payment X, in this particular swap is known as the *spark spread swap rate*. Once again swaps allow one party (Voltaire) to offset its risks from price fluctuations and to 'lock in' known cash receipts over the life of the spark spread swap.

Source: Adapted from Cuthbertson and Nitzsche (2008).

32.3.2 Crack Spread

The 'crack' you have probably heard of is the white powder 'celebs' allegedly snort to add zing to their lives.[2] Well 'crack' appears in energy markets too, in the form of the 'crack spread'. An oil refiner ('Heatforce') can convert crude oil (CO) into heating oil (HO) and its profit margin (per barrel) depends on the difference between these two prices. (I hope you can see the analogy with the spark spread). Clearly, the oil refiner is worried that in the future either heating oil prices fall or crude oil prices rise, or both – since this will hit its profit margin.

Heatforce could offset this risk by hedging using *separate* futures contracts on crude oil and heating oil – it would go long crude oil futures and would short heating oil futures. However, to save Heatforce the trouble of undertaking these two transactions itself, the futures market bundles them together and provides a futures contract on the crack spread. That is, the crack

[2]Or, if you have been to Ireland 'craic' (pronounced 'crack') means a friendly chat amongst friends, usually in pubs over a Guinness or Murphy's alcoholic beverage.

spread futures price moves (approximately) dollar-for-dollar with the cash market (spot) price differential $S_{HO} - S_{CO}$. Since Heatforce is worried about a fall in the (spot) crack spread, it will hedge by today, shorting crack spread *futures*. If the spot crack spread rises she makes higher profit margins but loses an equal amount on her short futures position. On the other hand if the spot crack spread falls her profit margins are squeezed but she makes an equal (dollar) amount when she closes out the short futures position.

There is a 1:1 crack spread futures contract which takes account of the fact that 1 barrel of crude oil can (physically) be refined to produce 1 barrel of heating oil – as outlined above. But it is also possible to take 3 barrels of crude oil and convert this into 2 barrels of gasoline (petrol) and 1 barrel of heating oil. This refiner faces price risk in the ratio 3:2:1 and it may not surprise you that there is a 3:2:1 crack-spread futures contract to hedge price risks for this type of refinery. Crack spread contracts are traded on CME and ICE.

32.4 WEATHER DERIVATIVES

Weather derivatives can be used by companies whose output and hence profits are affected by abnormal movements in 'the weather'. Whereas most futures contracts are used to hedge uncertain future price movements, broadly speaking weather futures are used to hedge uncertainty about the levels of 'output' in future periods (e.g. the number of gallons of heating oil you might sell to consumers in future months).

For the moment let's consider 'temperature' as the only 'weather variable' influencing revenues, costs, and profits (that is, we ignore such weather factors as the number of frost days, or the depth of snowfalls, which clearly affect some firms' output, like ski resorts). Energy producers and consumers are affected by temperature since this influences the amount of energy (heating oil, natural gas, electricity) used for heating in the winter months and (in the USA) the amount used for air conditioning in summer (see Weather Risk Management Association, `www.wrma.org`). The volume of agricultural production is also influenced by temperature (e.g. orange growers in Florida), as are leisure industries (e.g. hotel and holiday companies) and some manufacturers (e.g. of cold drinks or ice cream).

To keep things simple at this point, assume there are traded futures and options contracts, whose payoff depends on the temperature (at a particular point in time and at a particular geographical location), relative to an (arbitrary) average daily temperature which we take to be 65 °F. Let us also assume these derivatives contracts apply to a particular month (and we assume all months have 30 days).

Hence on 15 January, if on average market participants believe that the average *daily* temperature in June will be 70 °F then the current value of the June-weather futures contract will be $(70 - 65)$ °F $\times$ 30 days = 150 degree-days. The average cooling degree day (*CDD*) is 5 and the expected *cumulative CDDs* in June are 150 – this is because a higher temperature than 65 °F is likely to lead to more energy use for 'cooling', using air conditioning. Traders would say that the June contract is currently trading at 150 *CDD*. The futures contract multiple (on CME) is \$20 (per cumulative-CDD), so the June-futures price (on 15 January) is $\$20 \times 150$ $CDDs = \$3{,}000$.

As you are probably aware, one of the main topics of conversation of the average citizen is what the weather might be like over the coming months. So, one can only assume they might sometimes like to gamble on the weather. Clearly this is possible using traded weather futures. On 15 January assume the June weather futures is trading at 70 °F, equivalent to a price of \$3,000 but you think the (average) daily temperature in June will be 80 °F. As a 'weather speculator' you should buy June-weather futures at \$3,000. In June if your guess turns out to be correct, the June-futures contract (you purchased in January) will be priced at around $(80 - 65) \times 30$ days $\times \$20 = \$9{,}000$ and your profit on closing out will be \$6,000. The latter figure is simply the increase in the average daily temperature of 10 °F(= 80 − 70), scaled up by the 30 days and the \$20 per index point.

$$\text{Profit} = \text{Increase in temperature in futures contract} \times 30 \text{ days} \times \$20$$

In January the quoted price for the June-futures forecasts 150 $CDDs$ for June (i.e. an average daily temperature of 70 °F). Hence if you thought the average temperature in June would be 68 °F then as a speculator you would sell June-futures in January and hope to close out at a lower price in June.

What would an 'option contract on the weather' look like? Assume that on 15 January market participants expect the average daily temperature in June at a particular location (e.g. Portland Oregon) will be 70 °F, so cumulative $CDDs$ for June are $(70 - 65) \times 30 = 150$. A strike 'price' of $K = 150$ $CDDs$ for the June-options contract therefore corresponds to an expected average temperature in June of 70 °F. Let's take speculation first. If on 15 January, you think the average daily temperature in Portland in June will be higher than 70 °F then you might buy an option on CDD with a strike $K = 150$ CDD. If the average temperature in June actually turns out to be greater than 70 °F, the option will be in-the-money and can be exercised at a profit. If it is less than 70 °F, you simply lose your option premium.

32.4.1 Hedging and Insurance

Now let's consider hedging with weather derivatives. Suppose you run a number of large retail outlets in California (e.g. shops, offices, hotels), which use air conditioning. On 15 January you may be worried that the temperature in June will be abnormally high so your air conditioning (energy) costs will rise, due to an increase in usage (i.e. 'volume risk'), hence reducing your profits.

To offset some or all of this forecast extra energy cost you could buy June-weather futures on 15 January. If the temperature turns out to be abnormally high in June, you can then close out your futures position at a profit, which can be used to offset your higher air conditioning costs. If the temperature turns out to be lower than average in June, you close out your futures at a loss, but this is offset by lower energy costs. Here the aim of the hedge is to keep overall energy costs in June constant, regardless of the outcome for temperatures in June. If the hedge is successful the firm's spot purchases of energy in June plus any profits/loss on the futures will be close to the known June-futures invoice price quoted in January.

Of course you have to work out how much profit you will lose for each one degree Fahrenheit rise in temperature over that implied in the June-futures price (i.e. over 70 °F). Only then can you decide *how many* weather futures to purchase in order to offset your expected increase in energy costs (and loss in profits). So you need someone to provide you with a business scenario model, which shows how overall profits (in June) vary with temperature.

How would options work? On 15 January if you buy a June-call option 'on temperature' with a strike of $K = 150\ CDD$ this provides a cap on energy costs due to higher out-turn average daily temperatures in June of more than 70 °F, which causes increased use of air conditioning. If the June temperature turns out to be above 70 °F (i.e. the number of CDD in June is greater than $K = 150\ CDD$) then the money you make after exercising the call option will just offset your higher air conditioning costs, so your profits remain at their normal level. However, if the average daily temperature in June falls below 70 °F, (below the strike) your call option is worthless, but your energy costs fall, as you are using less air conditioning at lower temperatures.

You buy the 150-CDD June-call option on 15 January, then in June you have an asymmetric payoff. You obtain higher than average profits if temperatures turn out to be lower than 70 °F in June (because you use less air conditioning) and 'normal profits' if the temperature is abnormally high – as the call when exercised is in-the-money, which pays for your higher energy costs. Of course, to obtain this insurance you have to pay the call premium on 15 January.

So buying the June-call option on 15 January sets an effective maximum value of 'K' for your air-conditioning costs but allows you to take advantage of lower air-conditioning costs should temperatures in June be abnormally low. In contrast, when hedging on 15 January with a weather *futures contract* you make average profits, regardless of whether the temperature in June turns out to be high or low.

32.4.2 Contract Details

Now we have sorted out the general principles of speculation and hedging with weather derivatives, let us look at a few of the contractual and institutional details. The first OTC weather derivatives were introduced around 1997 and are based on the number of daily 'heating degree days' (HDD) or 'cooling degree days' (CDD), defined as:

$$\textit{Daily HDD} = \max(65 - \textit{Temp}, 0)$$

$$\textit{Daily CDD} = \max(\textit{Temp} - 65, 0)$$

where $Temp$ is the average of the highest and lowest temperature (°F) during the day at a specific location (e.g. at the Chicago O'Hare Airport weather station calculated by the Earth

Satellite Corporation). For example if $Temp = 70\ ^\circ\mathrm{F}$ then the daily $CDD = 5$ and the $HDD = 0$. The monthly *HDD* and *CDD* are simply the sum of the daily values. For example, the average daily temperature in Chicago in November is about 50 °F, which is 15 °F below the 'baseline' 65 °F set in weather contracts – giving an average daily $HDD = 15$. Suppose it has been abnormally cold in the Fall in Chicago so the actual out-turn values of *daily HDD* in November are somewhat lower than average, namely 20, 18, 20, 15, 19, 25, and the other 24 days have $HDD = 15$. Then the cumulative *HDD* out-turn for November is 477. Note that when temperatures are low (i.e. well below 65 °F), then *HDD* is high – a negative correlation. However, *CDD* increases with temperature (above 65 °F) – a positive correlation.

Lots of 'bespoke' weather futures are sold in the OTC market but some plain vanilla contracts are also available with 24-hour electronic trading (e.g. on the CME-Globex trading platform) – both weather futures and European options (on weather futures) are based on the monthly (cumulative) *HDD* or *CDD*. These products are also available as seasonal products. The winter season is November to March and the summer season May to September, although April can be added to the summer and October to the winter season, so all months can be covered. You can choose from 2 to 6 months in a seasonal strip contract, so you might choose say 3 months (December, January and March) from the winter season strip, as these may be the months when your (energy) needs are highest but also most volatile and hence may need to be hedged. The CME clearing house guarantees trading on Globex by requiring 'performance bond' deposits (i.e. collateral placed in a margin account) at each level in the clearing process – customer to broker, broker to clearing firm, and clearing firm to clearing house.

There are also weather derivatives which have payoffs that depend on the number of *frost days* in the month or on the *depth of snowfall* at particular geographical locations. These derivatives work along the same broad principles as weather derivatives based on temperature, as described above. Forwards and futures contracts written on frost days allow firms to protect their profits due to abnormal volume changes in their business, caused by abnormal severe frosts in particular months in particular locations (e.g. wine growers in Napa and Sonoma Valley in California who suffer if grapes are harmed by severe frosts, particularly in spring when the vines are most vulnerable; ditto flower growers in Holland).

The payout from snowfall derivatives is based on the number of inches of snow measured at specific locations at specific times of the day. If there is less or more daily snowfall than average in particular months, the 'snow derivative's' payoff at maturity will reflect this outcome. The most obvious case where snow derivatives can be used is by owners of businesses in ski resorts. If there is a lack of snow at particular ski resorts (and sometimes if there is too much snow and risk of avalanches), firms operating in this market can suffer abnormal losses. Also, excessive snowfalls can also severely disrupt the revenues and costs of transport companies, such as road haulage (the trucking business).

Options contracts allow you to purchase insurance. For example, in November a wine grower might buy a call option on 'frost days' in March. The call option pays out depending on the number of frost days (above the strike 'price' of a fixed level of frost days in the month) and this compensates wine growers for any deterioration in the grape harvest, due to frost. On the other hand, if the number of frost days in March is below the 'strike rate' then the call expires out-of-the-money but the grape grower (vigneron) has the benefit of a healthy crop of grapes to make excellent 'grand cru' wines (hopefully). The payment of the call premium is the cost of this insurance.

32.4.3 Pricing Options on Temperature

Temperature is uncorrelated with the (stock) market return, hence there is zero systematic risk and estimates of temperature from historical data can be assumed to apply in a risk-neutral world. Weather options can therefore be priced by using historical data on temperature to estimate the expected payoff and then discounting using the risk-free rate.

Suppose you have 60 years of historical data on cumulative *HDD* for *January* (at Chicago O'Hare Airport) and the histogram closely approximates a lognormal distribution from which you measure the mean $HDD = 820$ with a standard deviation of (the logarithm of *HDD*) = 0.10. In January 2018 you purchase a $T = 1$ year call option on *HDD* with a strike of $K = 800$, (when the risk-free rate is 3% p.a.). The OTC option contract stipulates that each cumulative *HDD* is \$1,000. Example 32.1 shows how to price the call.

EXAMPLE 32.1

Pricing a Call Option on *HDD*

Using insights from Black–Scholes and Black's model, it can be shown that if a stochastic variable $\ln V_T$ is normally distributed with mean (expected value) EV_T and standard deviation σ_V Then the expected payoff at T is:

$$E[\max(V_T - K, 0)] = (EV_T)N(d_1) - KN(d_2)$$

$$d_1 = \frac{\ln(EV_T/(K) + \sigma_V^2/2}{\sigma_V} \qquad d_2 = \frac{\ln(EV_T/(K) - \sigma_V^2/2}{\sigma_V} = d_1 - \sigma_V$$

Substituting $EV_T = 820$, $K = 800$, $\sigma = 0.10$ we have:

$$d_1 = 0.2969, \qquad d_2 = 0.1969, \qquad EV_T N(d_1) - KN(d_2) = 43.279$$

$$\textit{Price of call} = \$1{,}000 \exp(-0.03) \times 43.279 = \$42{,}000$$

32.5 REINSURANCE AND CAT BONDS

We have seen how weather derivatives can be used to hedge and provide insurance when there are abnormal weather conditions. In general OTC and exchange traded weather derivatives are used to hedge changes in temperature (or frost days or lack or excess of snowfall) which are slightly different from 'average'. These products are provided in the OTC derivatives market by large banks, specialist energy firms,[3] and insurance companies. On the other hand, extreme weather conditions such as hurricanes, earthquakes, and floods are generally dealt with via some form of explicit insurance contract with an insurance company.

Any single insurance company might hold a large number of catastrophe insurance contracts and may therefore be carrying a lot of risk, from low probability but highly costly events. To mitigate some of this risk an insurance company may reinsure say 70% of its risks with other companies, leaving it liable to only 30% of any claim (Lloyds of London is probably the oldest organisation providing such 'reinsurance'). Alternatively the insurance company can purchase a series of reinsurance contracts for *excess cost layers*. If the insurance company has \$100m exposure to hurricane damage in Florida then it may issue (separate) *excess-of-loss* reinsurance contracts to cover losses between \$40m and \$50m, \$50m, and \$60m etc. Between \$40m and \$50m of losses, the insurance company receives a dollar-for-dollar payoff from the reinsurer but receives nothing from this particular reinsurance contract if losses are outside this range. Hence, for the first reinsurance contract, the insurance company has the equivalent of a long bull spread – namely, a long call at a strike of \$40m and a short call at a strike of \$50m.

Reinsurance contracts are very useful but the insurance company can also issue bonds to cover catastrophic (CAT) risks. These have a liquid secondary market and can be sold to many investors thus 'spreading' the catastrophic risk more widely. CAT bonds pay a higher than average interest (coupon) but the holders agree to cover the 'excess-of-loss' insurance. For example, by issuing \$10m principal of CAT bonds an insurance company could cover losses between \$40m and \$50m by not re-paying some or all of the principal on the CAT bonds, should these losses occur. Alternatively it can make a much larger bond issue, with the covenant that any losses between \$40m and \$50m will be covered by a reduction in interest payments.

The demand for CAT bonds by investors arises from the fact that they pay higher coupons (or yield to maturity) and the returns on CAT bonds have an almost zero correlation with stock market returns and hence have no systematic risk. In a diversified stock portfolio specific risk will be small but adding CAT bonds can improve the (mean-variance) risk-return trade off in any already well diversified stock portfolio.

32.6 SUMMARY

- Forwards, futures, options, and swap contracts on the price of energy products (oil products, natural gas, electricity) are available OTC and as exchange traded products.

[3]The most infamous of these is the Enron Corporation of Houston Texas, which went bankrupt in 2001 after accounting scandals became public knowledge.

- Energy derivatives can be used to hedge (forwards, futures and swaps) and to provide insurance (options) against future changes in spot energy prices.
- Some contracts are cash settled and others involve delivery – although alternative delivery arrangements from those stipulated in the contract can be separately negotiated and this is known as 'exchange of futures for physicals' (EFP).
- Weather derivatives can be used to hedge (or provide insurance) against changes in the volume of energy used by industries whose profits depend on the weather.
- The most actively traded weather derivatives are based on temperature at particular locations at specific times but others depend on the number of frost days or the depth of snowfalls.
- The key exchanges for energy derivatives are the New York Mercantile Exchange (NYMEX), the Chicago Mercantile Exchange (CME) and the International Commodities Exchange (ICE) in London. There is also a very large OTC market.

EXERCISES

Question 1

You will have to purchase 560,000 barrels of (spot) jet fuel in 1 year's time. There is a futures contract available on heating oil which you are going to use to hedge your jet fuel purchases.

Contract size for heating oil is 42,000 barrels. Changes in jet fuel and heating oil prices have a beta of 0.9.

How would you implement the hedge and explain what happens if the spot price of heating oil either increases or decreases over the next year?

What is the effective price of your jet fuel?

Briefly explain why your strategy might not result in a 'perfect hedge'.

Question 2

It is 22 September and you have agreed to sell 10,000 mbtu of NG at S(fix) = 8.60 \$/mbtu to a commercial customer in early January. You do not currently have the NG. You will buy the NG in the cash market at the end of December but you are worried that NG prices will increase between September and the end of December.

On 22 September you decide to hedge your position by using an at-the-money (ATM) option (on NG futures). Assume the options are cash settled. ATM options are available which have a (fixed) strike price K, equal to the *current* futures price, F_0(Jan) = 8.20 \$/mbtu.

Derivatives (22 September)

The NG-futures for January delivery on NYMEX is F_0(Jan) = 8.20 \$/mbtu
ATM call costs, C_0 = \$0.15 (with expiry date of 28 December).

ATM put costs $P_0 = \$0.17$.

$$\text{Cash payoff to call (futures) option} = \max(0, F_T - K)$$

$$\text{Cash payoff to put (futures) option} = \max(0, K - F_T)$$

28 December: Price rise

On 28 December if you decide to exercise your (futures) option you face futures prices $F_1(\text{Jan}) = \$8.70$ and the spot price is also $S_1 = \$8.70$

28 December: Price fall

On 28 December if you decide to exercise your option you face futures prices $F_1(\text{Jan}) = \$7.70$ and the spot price is also $S_1 = \$7.70$.

What are the outcomes after using the required futures or options contract in these two cases? Briefly comment.

Question 3

In May, GasCorp has agreed to sell 10,000 mbtu *per day* of natural gas (NG) at $S(\text{fix}) = 10.5$ \$/mbtu to customers in the Boston area for each of the months in the next year from January–June. GasCorp does not currently have the NG. GasCorp will buy the NG in the spot (cash market) in each of the months January–June and hedge these purchases using a NG-swap. The current NG swap rate quoted by CitiGroupEnergy is $X = 10.0$ \$/mbtu.

(a) How would GasCorp hedge its January–June purchases using a swap?
(b) What is the payoff to GasCorp from the swap each month, if spot-NG prices turn out to be:

	Spot out-turn, S
Jan	11.85
Feb	11.87
Mar	11.65
Apr	9.69
May	9.49
Jun	9.6

(c) What is the 'effective average price' GasCorp pays for its NG purchases in each month?
(d) How might the swap dealer CitiGroupEnergy hedge its risk in the swap contract?

Question 4

You purchase a large amount of natural gas at the current cash-market price, for heating in the winter months (November to March). In July how might you use futures contracts to hedge your price and volume risk over the winter months? Explain.

Question 5.

Los Angeles: average temperatures in July and August

84 °F August Av. High 29 °C	
64 °F August Av. Low 18 °C	

In February, a large water park in Los Angeles, California owned by the DHockney Corporation called *ABiggerSplash,* knows that abnormally cool weather (between 65 °F and 70 °F average) over the summer months (July, August) leads to less customers and reduced profits. The 'normal' average temperature in July/August is 80 °F. The tick value for weather futures is \$100 per °F.

How can the water park mitigate such losses using weather futures?

Does it use futures on heating degree days (*HDD*) or cooling degree days (*CDD*)?

$$HDD = max\,(65 - Temp, 0)$$

$$CDD = max\,(Temp - 65, 0)$$

PART VIII

SWAPS

CHAPTER 33

Interest Rate Swaps

Aims

- To show how plain vanilla interest rate swaps can be used to convert uncertain future floating-rate interest cash flows into known fixed-rate cash flows (or vice versa).
- To examine the role of swap dealers, settlement procedures, pricing schedules and the termination of swap agreements.
- To demonstrate how cash payoffs in the swap are determined.
- To demonstrate the principle of *comparative advantage* – the source of gains in a swap.

Swaps are privately arranged contracts (i.e. OTC instruments) in which parties agree to exchange cash flows in the future according to a prearranged formula. Swap contracts originated in about 1981. The largest market is in interest rate swaps but currency swaps are also widely used.

The most common type of interest rate swap is a *plain vanilla fixed-for-floating rate* swap. Here one party agrees to make a *series* of fixed interest payments to the counterparty, and to receive *a series* of payments based on a variable (floating) interest rate (LIBOR). The interest payments are based on a stated notional principal of say $Q = \$100$m, but only the interest payments are exchanged each period, not the principal value – hence the use of 'notional'. The payment dates, day-count convention, maturity of the swap, and the floating rate to be used (usually LIBOR) are also determined at the outset of the contract. In a plain vanilla swap 'the fixed rate payer' knows exactly what the (dollar) interest payments will be on every payment date but the floating rate payer does not.

Why are swaps so popular? The first reason is that swaps can be used to reduce the cost of borrowing. Suppose Microsoft wants a fixed-rate loan with a principal $Q = \$100$m. Rather than

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

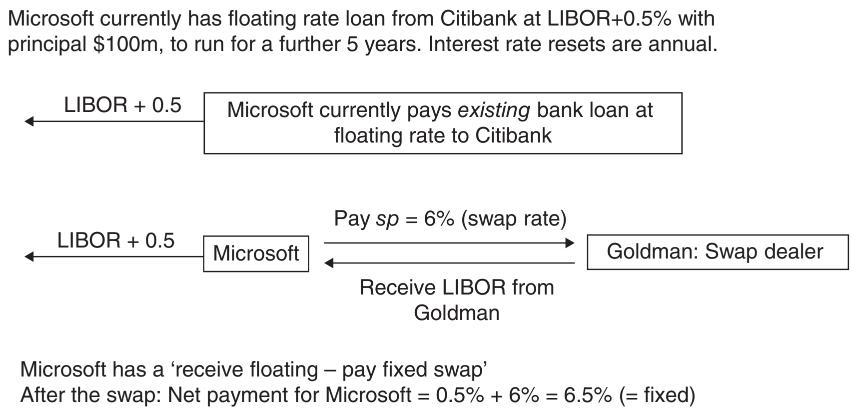

FIGURE 33.1 Interest rate swap

taking out a fixed-rate loan directly with Citibank at 6.6% p.a. (say), Microsoft may end up paying a lower fixed payment, if it obtains its fixed-rate loan indirectly.

Surprisingly, Microsoft first takes out what it does not want – a floating-rate loan at LIBOR+0.5% with Citibank (Figure 33.1). Then Microsoft goes to a swap dealer (e.g. Goldman Sachs) and agrees to swap LIBOR receipts for fixed rate payments. Microsoft agrees to receive floating rate (LIBOR) from Goldman and to pay Goldman a fixed rate of say 6%, on a (notional) principal of $100m in the swap. The fixed rate $sp = 6\%$ is the *swap rate*.

The LIBOR receipts from Goldman are simply passed on to Citibank. Effectively Microsoft now has a fixed-rate loan at 6.5% (rather than at 6.6%, if Microsoft had gone directly for a fixed rate loan from Citibank). The lower fixed loan rate via the swap occurs because of the principle of *comparative advantage* (see Appendix 33). It probably arises because of 'frictions' in credit markets. There may be an element of oligopoly amongst lending institutions or the market between different banks in different countries is fragmented, so that credit spreads on direct borrowing at fixed rates are relatively high. Swaps may then provide a method of circumventing this problem, providing a net gain to all parties in the swap (assuming no one defaults).

It may be immediately obvious to some readers that an interest rate swap is (analytically) nothing more than a *series* of forward rate agreements (FRAs), to exchange cash flows based on a floating interest rate in exchange for cash flows at a fixed rate, at various predetermined dates in the future. Since the swap is equivalent to a series of FRAs (or interest rate forward contracts) then what swaps offer is lower transactions costs than a strip of FRAs. Indeed, although the market in FRAs is liquid (competitive) at short horizons of up to 1–2 years, the swaps market is liquid at both short and long horizons out to 20–30 years.

The intermediaries in a swap transaction are usually banks who act as swap dealers. They are usually members of the International Swaps and Derivatives Association (ISDA) which provides some standardisation in swap agreements via its *master swap agreement* which can then be adapted where necessary, to accommodate most customer requirements.

Swap dealers make profits via the bid–ask spread on the (fixed) swap rate and might also charge a small brokerage fee. Swap dealers take on many different swaps from many different counterparties but they will generally end up with a net open position (i.e. net payments or receipts at a floating rate). They usually hedge this position using interest rate (Eurodollar) futures contracts (and sometimes interest rate options).

33.1 USING INTEREST RATE SWAPS

Another reason you might use an interest rate swap is to remove existing interest rate risk. Suppose Microsoft has an existing long-term bank loan with Citibank which has been running for some time, on which it pays LIBOR+0.5%. Assume Microsoft now becomes worried about rising interest rates in the future, so it wants to switch to a fixed rate loan.

It could go back to Citibank and negotiate to switch from a floating rate loan to one where it pays known fixed (dollar) interest payments every year. But renegotiating loan contracts for large corporations is a tricky and expensive proposition – think of all those lawyers' fees when the covenants in the loan contract are altered, not to mention lengthy negotiations before the new loan is issued. It is easier and cheaper to keep the existing floating rate loan with Citibank but undertake a swap. This will result in Microsoft ending up with what it wants – namely, to have fixed known dollar interest payments. We have already dealt with this situation in Figure 33.1.

By using the swap, Microsoft has transformed an initial floating rate loan with Citibank into fixed rate payments at 6.5%. Microsoft effectively now has the equivalent of a fixed rate loan. If the maturity of the swap ends when the bank loan terminates and is for the same principal amount, and tenor, then Microsoft will have known fixed interest payments over the rest of the life of the loan. In a plain vanilla interest rate swap the floating rate is usually 'LIBOR-flat' (i.e. without a spread) and the swap dealer will then determine the appropriate fixed rate to charge in the swap.

Next, let us see how a swap can be used to reduce interest rate risk for a financial intermediary like a bank or Savings and Loan Association ('Building Society' in the UK and Savings Bank in Europe), so that the financial intermediary can 'lock in' a profit over future years. Assume Silverado bank has fixed rate receipts from its existing loans or housing mortgages, at say 12%, but raises much of its finance in the form of short-term floating rate deposits, at say LIBOR-1% (Figure 33.2). Income paid on deposits varies as market interest rates vary and this is a source of risk for the financial intermediary.

If the deposit rate is LIBOR-1% and LIBOR currently equals 11% the bank pays out 10% to depositors. But if its fixed rate mortgages and loans earn 12%, the financial intermediary currently earns a profit (maturity spread) of 2% p.a. However, the danger is that if LIBOR rises by more than 2% the S&L will make a loss – the source of the risk is future increases in LIBOR.

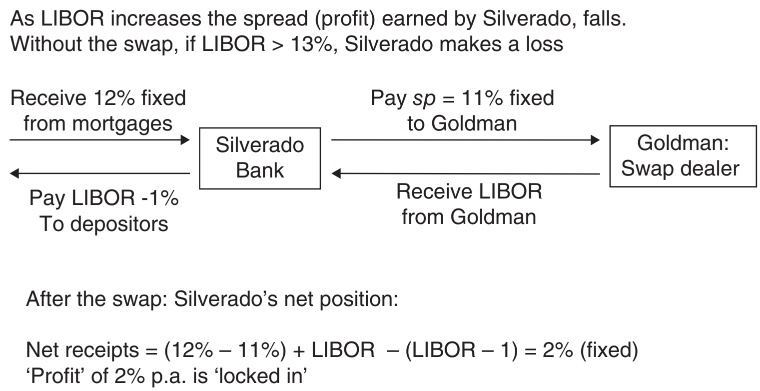

FIGURE 33.2 Interest rate risk of Silverado Bank

Silverado Bank can remove this LIBOR risk if it enters into a swap with Goldman to receive LIBOR and pay a fixed rate. Suppose Goldman sets the fixed rate payment in the swap at $sp = 11\%$. By entering into the swap Silverado is protected from any future rises in LIBOR rates. It now effectively has net fixed rate receipts of 2%, which is its fixed (risk-free) profit margin.

As discussed above, one reason for undertaking a swap is that some firms find it cheaper to borrow at say floating rates and then use a swap to create the effective fixed rate payments that they really want. This cost saving provides the financial incentive behind the expansion of the swaps market. An example of a swap agreement is shown in Finance Blog 33.1.

Finance Blog 33.1 ISDA Swap Agreements

The International Swaps and Derivatives Association (ISDA) in New York provides some standardisation in swap agreements via a number of *master agreements*. A stylised example of a confirmation agreement for a plain vanilla interest rate swap between Apple and the swap dealer JPMorgan-Chase is given below.

Dates

Trade date	15 February 2019
Effective date	20 February 2019
Termination date	20 March 2024
Business day convention	Following business day
Holiday calendar	US

Fixed leg

Fixed-rate payer	Apple
Notional principal	US$100m
Fixed rate	2.5% p.a.
Day-count convention	Actual/365
Payment dates	Each 20 February and 20 August Beginning 20 August 2019 and terminating 20 March 2024

Floating leg

Floating-rate payer	JPMorgan-Chase
Notional principal	US$100m
Floating rate	US$6-month LIBOR
Day-count convention	Actual/360
Payment dates	Each 20 February and 20 August Beginning 20 August 2019 and terminating 20 March 2024

The US calendar determines which days are deemed to be business days (e.g. not a weekend or US holiday). If a payment date occurs on a Saturday (say) the payment will take place on the Monday (or a Tuesday, if Monday is a US holiday). Note the different day-count conventions for the fixed and floating legs of the swap. A master agreement may cover all outstanding swap deals between the two parties and will also cover the legal arrangements in the event of a default by either side to the agreement.

33.2 CASH FLOWS IN A SWAP

Suppose on 15 March-01 ($t = 0$),[1] Microsoft enters into a 'receive-float, pay-fixed' swap – this is referred to as a '*long position*' in the swap (by convention). Suppose the swap rate $sp = 6\%$ on a notional principal of $Q = \$100$m, with a 6-month tenor and the swap ends on 15 March-04.

Floating rate payments are determined by LIBOR at the *beginning* of each (6-month) reference period but the *actual cash payment* occurs 6 months later. Hence the known value for LIBOR on 15 March-01, $L_{01} = 6.5\%$ is used to determine the floating cash payment on the 15 September-01. Similarly, whatever LIBOR turns out to be on 15 September-01 will determine the floating cash payment on 15 March-02 etc.

[1] We use the notation '01', '02' etc., to designate the 'generic' year in question (e.g. '01' could be the year 2019).

Day-count conventions differ, but we use the so-called 'money-market' day-count convention for the *floating leg* of the swap which is 'actual/360'. Hence we have $m_i = days_i/360$ and $days_i$= actual number of days between each repayment date. For a swap with a 6-month tenor actual $days_i$ would be around 180 but might vary slightly for each 6-month period. The first floating payment (on 15 September-01, at $t = 1$) is:

$$C_{FL,1} = L_{01}m_1Q = 0.065\ (184/360)\ \$100\text{m} = \$3{,}322{,}222$$

The cash flows for the fixed-leg $C_{X,i}$ are determined by the (fixed) swap rate. Hence, $C_{X,i} = (sp.h_iQ)$ where sp is the swap rate and h_i is the tenor in the fixed-leg. The convention for determining h_i for the fixed leg is set out in the swap contract – we assume the convention is $h = 180/360$ for each 6-month period in the fixed leg of the swap.[2] Hence the fixed cash flows are (always):

$$C_X = (sp.h.Q) = 0.06\ (180/360)\ \$100\text{m} = \$3\text{m}$$

The net receipts for the long position (receive floating-pay fixed) in the swap, on 15 September-01 is:

$$\textit{Net receipts 15 September-01} = \$3{,}322{,}222 - \$3{,}000{,}000 = \$322{,}222$$

Similarly the net receipts on 15 March-02 are determined by the out-turn LIBOR rate on 15 September-01 of 7% and the 181 days between September-01 and March-02:

$$\textit{Net receipts 15 March-02} = \$3{,}519{,}444 - \$3{,}000{,}000 = \$519{,}444$$

On 15 March-01 we do not know what the LIBOR rates will turn out to be on any of the reset dates $t = 1, 2, 3, \ldots, n-1$, until we actually reach these dates in 'real time'. But Table 33.1 assumes a set of out-turn LIBOR rates and calculates the resulting *ex-post* payments in the swap – that is, the swap payments that ensue *after* we observe the out-turn values for LIBOR.

[2]In practice, day-count conventions can be a little 'messy'. h_i could be different for each reset period (and be different from the day-count convention for the floating leg). However $h = 180/360$ is often used for each 6-month period in the pay-fixed leg of the swap, even though the number of days between different reset dates may actually differ from 180 and of course the '360' is arbitrary. But as long as you know the convention being used you can always calculate the fixed *dollar* cash flows. For example, when calculating *fixed cash flows*, the swap-market convention *in the US* is to use the '30/360' convention for either USD or Euro interest rate swaps – hence for our swap, using this US convention: h_i = (30/360 × Number of *months* between repayment dates). So for a 6-month tenor, we have $h = 180/360$. However, for sterling swaps the fixed leg has a '30/365' day count convention.

TABLE 33.1 Swap: cash pay-outs (ex-post)

Notional principal	100,000,000
Swap rate	6
Days in years (swap convention)	360
Tenor for fixed leg (days)	180
Tenor for floating leg is 'actual/360'	

Date	Days	LIBOR	LIBOR cash flow	Fixed cash flow	Receive float, pay fixed
15-Mar-01		6.5%			
15-Sep-01	184	7%	3,322,222	3,000,000	322,222
15-Mar-02	181	6.5%	3,519,444	3,000,000	519,444
15-Sep-02	184	6.25%	3,322,222	3,000,000	322,222
15-Mar-03	181	5.75%	3,142,361	3,000,000	142,361
15-Sep-03	184	5.25%	2,938,889	3,000,000	−61,111
15-Mar-04	182		2,654,167	3,000,000	−345,833

Note: Last reset date 15 March-01. First payoff in the swap is based on rates on 15 March but cash payment does not occur until 15 September.

33.3 SETTLEMENT AND PRICE QUOTES

Take the case of a US swap dealer who wishes to set the swap rate on a 10-year swap with the floating rate based on 6-month LIBOR (Table 33.2). She will have an indicative pricing schedule (for that day) where the fixed rate will be based on the yield on current 10-year Treasury notes (bonds). In fact this will usually be the par bond yield plus a spread.

For example on a 10-year swap, if the swap dealer agrees to pay fixed (to counterparty-A) and receive floating then she will quote $sp = 2.76\%$ (= 2.7 + 6 bps, swap spread) and receive 6-month LIBOR-flat. The swap spread reflects the 'normal' credit risk as perceived by the swap

TABLE 33.2 Indicative pricing schedule for swaps

Maturity	Current T-bond rate	Bank pays fixed	Bank received fixed
5 years	1.5%	5-year T-bond + 5 bp	5-year T-bond + 10 bp
10 years	2.7%	10-year T-bond + 6 bp	5-year T-bond + 12 bp
20 years	3.4%	20-year T-bond + 8 bp	5-year T-bond + 16 bp

dealer. Notice that no floating rates appear in the pricing schedule of Table 33.2 since the floating rate is understood to be 6-month LIBOR flat.[3]

The dealer will eventually hope to match the pay-fixed deal with an offsetting swap (from another counterparty-B) to receive-fixed (on the same notional principal and tenor) at 2.82% (= 2.7 + 12bps) – thus obtaining 6 bps bid–ask spread as profit. In our above stylised example, once the swap dealer finds counterparty-B, then she is perfectly hedged with certain net receipts of 6 bps (i.e. the mismatch risk is eliminated).

However, it may be difficult for the swap dealer to find a counterparty-B who has exactly the 'reverse wishes' of A. For example, if the maturity of A's swap is say 5 years but B will only enter into the swap for 3 years, then the swap dealer is a net floating rate receiver in years 4 and 5 and is subject to interest rate risk, in the last 2 years of the swap. She may then initially hedge her mismatched swaps book using (Eurodollar) futures contracts which mature in 4 years and 5 years, respectively.

The bid–ask spread reflects several factors, namely the degree of competition between dealers, the risk in (temporarily) holding a net open position on one leg of the swap deal, and a swap dealer's current inventory position of being either net long or short in the fixed (or floating legs) of all its outstanding swaps (i.e. the overall position of its swap book).

The bid–ask spreads on interest rate swaps in various currencies are given in Table 33.3. All the fixed rates are quoted against the appropriate LIBOR rate (usually 3 months or 6 months). As you can see the bid–ask spreads are rather small, reflecting the high degree of competition and liquidity in the swaps market.

TABLE 33.3 Swap rates (FT)

Oct 26	Euro		£		US$	
	Bid	Ask	Bid	Ask	Bid	Ask
1 year	1.38	1.43	0.82	0.85	0.37	0.40
5 year	2.13	2.18	2.10	2.15	1.47	1.50
10 year	2.78	2.83	3.18	3.23	2.67	2.70
20 year	3.13	3.18	3.74	3.82	3.41	3.44
25 year	3.06	3.11	3.79	3.92	3.52	3.55
30 year	2.94	2.99	3.80	3.93	3.58	3.61

[3]One practical point to note is that 6-month LIBOR is *quoted* assuming semi-annual payments with a 360-day year while US T-notes use semi-annual payments but with a 365-day year. Therefore, the LIBOR rate must be multiplied by (365/360) to put it on an equivalent basis to the T-bond rate. Hence, the *T-bond equivalent rate (to a LIBOR quoted rate) = LIBOR × (365/360)*. This need not concern us here.

33.4 TERMINATING A SWAP

Suppose a swap agreement has been in existence for some time and the *current value* of the swap *to Microsoft* who receives LIBOR and pays fixed, is \$1.2m and therefore the value to the swap dealer (Goldman) is −\$1.2m. (We will see how the mark-to-market value of the swap of \$1.2m is calculated in a later chapter.) Microsoft can terminate the swap by sale, assignment or buy-back.

In a sale, Microsoft simply finds a third party (e.g. Apple) to take over the fixed payments and LIBOR receipts (from Goldman) and Microsoft sells the swap (to Apple) for \$1.2m. Alternatively, if the value of the swap to Microsoft had been –\$1.4m (say) then Microsoft would have paid Apple \$1.4m to take on the swap. The swap dealer Goldman would have to approve the use of any third party in the deal (i.e. Apple) because of credit risk.

Microsoft could use a *buy-back* whereby Goldman would pay \$1.2m to Microsoft and the swap is terminated. Finally, Microsoft could undertake a *reversal,* that is enter a new swap where the cash flows exactly offset the cash flows in the original swap – the new swap (say with Morgan Stanley) will be a receive fixed-pay float with the same maturity, tenor, and notional principal as the original swap with Goldman.

33.5 COMPARATIVE ADVANTAGE

Interest rate swaps are undertaken because there are net reductions in the cost of borrowing for *both* parties to the swap. The swap dealer can also share in some of these gains. However, to keep things simple assume only two parties in the swap A and B.

A wishes to end up borrowing \$10m at a floating rate for 5 years

and

B wishes to end up borrowing \$10m at a fixed rate for 5 years.

The rates offered to A and B are shown in Table 33.4. 'A' has an *absolute advantage* in both markets since it can borrow *at both* floating and fixed rate at a lower cost than B (probably because B has a lower credit rating than A). Nevertheless, there is a net gain to *both* parties if they enter a swap. First look at the total cost to A and B if they directly borrow in their preferred form (i.e. A at floating and B at fixed) or non-preferred form (i.e. A at fixed and B at floating)

1. Total cost to A and B of direct borrowing in 'preferred form'

$$= B_X + A_F = 11.2\% + (L + 0.3\%) = L + 11.5\%$$

TABLE 33.4 Bank borrowing rates facing A and B

	Fixed	Floating
Company-A	10.00% (A_x)	LIBOR + 0.3%
Company-B	11.20% (B_x)	LIBOR + 1.0%
Absolute difference (B – A)	Δ(fixed) = 1.2	Δ(float) = 0.7
Net comparative advantage or 'quality spread differential'	NCA = Δ(fixed) – Δ(float) = 0.5 B has comparative advantage in borrowing at a floating rate. Hence B borrows from the bank at a floating rate.	

2. Total cost to A and B if they borrow in 'non-preferred form'

$$= A_X + B_F = 10\% + (L + 1\%) = L + 11\%$$

Hence the total cost is lower if they initially borrow in their non-preferred form:

$$\text{Net overall gain to A and B} = (B_X + A_F) - (A_X + B_F) = 0.5\%$$

Although there is a reduction in total cost using strategy (2), it results in A and B *not* having their *preferred* form of borrowing. However, a swap provides the mechanism to achieve their preferred form of borrowing and it has the advantage of lowering the overall cost of borrowing for both parties.

Looking at the overall gain in a slightly different way (Table 33.4), the key element is that B has *comparative advantage* when borrowing in the floating rate market, while A has comparative advantage when borrowing in the fixed rate market. (Comparative advantage is used in international trade theory to help explain why the UK exports some wine to France, even though the latter has an absolute cost advantage in producing wine.)

B has comparative advantage when borrowing in the floating rate market because B pays only 0.7% more than A does, whereas B pays (a larger) 1.2% more than A in the fixed rate market. (If you like, B pays 'less more' in the floating market than in the fixed rate market.)

B initially borrows floating and A borrows fixed. They then enter into a swap agreement whereby B agrees to pay A at a fixed rate and A pays B at a floating rate, so they both ultimately achieve their desired type of borrowing (i.e. B ends up paying fixed and A floating). A swap is advantageous to both parties if the net comparative advantage or quality spread differential is positive.

Net comparative advantage / Quality spread differential

$$\begin{aligned} NCA &= \textit{Difference in Fixed Rates} - \textit{Difference in Floating Rates} \\ &= (11.2\% - 10\%) - [(LIBOR + 1\%) - (LIBOR + 0.3\%)] \\ &= 1.2\% - 0.7\% = 0.5\% \end{aligned}$$

The 0.5% is the total gain from the swap, which is available to A and B. As long as A and B get some gain they will willingly enter the swap. We arbitrarily assume that the gain of 0.5% is split equally (0.25%) between A and B. (This split will depend on the relative bargaining power of A and B.) B has a comparative advantage in the floating rate market and hence issues \$10m floating rate debt at LIBOR + 1% while A issues fixed rate debt at 10% (Figure 33.3).

Here's how we work out the figures in the swap (which are the cash flows between A and B in Figure 33.3). We initially consider the swap from B's point of view (who ultimately wants to borrow fixed) and expects to gain 0.25% overall. We will find that this ensures that A also achieves a gain of 0.25%. This is what happens to B:

1. B initially borrows 'direct' from a bank at LIBOR + 1% (in which it has NCA)
2. B in 'leg1' of the swap agrees to receive LIBOR
3. Net payments by B so far are 1% (fixed)
4. But B must end up with a gain of 0.25%, hence

$$\begin{aligned} \text{B's } \textit{total} \text{ fixed interest payments} &= \text{'}\textit{Direct cost} \text{ of borrowing fixed} - \text{Swap gain'} \\ &= 11.2\% - 0.25\% = \mathbf{10.95}\% \end{aligned}$$

5. Hence in 'leg 2' of the swap B must pay fixed at $10.95\% - 1\% = \mathbf{9.95}\%$.

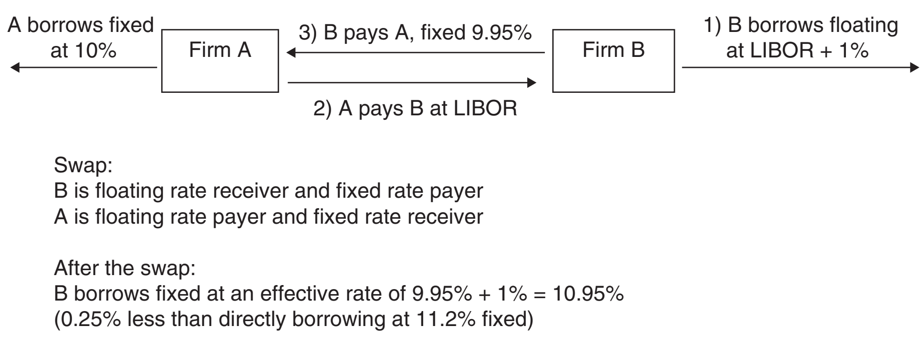

FIGURE 33.3 Interest rate swap (between A and B)

Out-turn after swap for cash flows of B:

B initially borrows at LIBOR from a bank and in the swap, receives floating and pays fixed:

- B pays LIBOR + 1% on its bank loan
- B receives LIBOR from A
- B pays 9.95% fixed to A.

Hence, the net result is that B ends up paying 10.95% fixed (= 9.95% + 1%) even though it started out with a LIBOR bank loan. Although B pays 10.95% fixed, this is 0.25% less than if it went directly to the bank and borrowed at the rate of 11.2% fixed (Table 33.4). Does the above also allow A to gain 0.25 from the swap? The cash flows for A are:

Out-turn after swap for cash flows of A:

Initially 'A', takes out a fixed rate bank loan and in the swap receives fixed and pays floating:

- 'A' pays 10% fixed on its bank loan
- 'A' receives 9.95% fixed from B
- 'A' pays LIBOR to B.

Hence 'A' ends up paying LIBOR + 0.05% (= LIBOR + 10% – 9.95%). 'A' has converted its fixed interest bank loan into a net floating rate payment. Also A's floating rate payment of LIBOR + 0.05% is 0.25% less than it would pay if it borrowed directly from the bank at LIBOR + 0.3% (see Table 33.4).

Hence in the swap, A agrees to pay B at LIBOR and B agrees to pay A fixed at 9.95% (Figure 33.3). The overall payments and receipts are:

- B takes out a bank loan of $10m at LIBOR + 1%
- 'A' takes out a bank loan of $10m at 10% fixed
- In the swap 'A' agrees to pay B at LIBOR on a notional $10m and
- B agrees to pay 'A' at 9.95% fixed, on notional $10m.

Both A and B gain by 0.25% each, compared with borrowing directly in their preferred form from the bank. It can be shown (but it is tedious) that if we put a swap dealer in the 'middle', who deals with both A and B then all three parties, A, B, and the swap dealer can each have a share in the 0.5% total gain (see Appendix 33).

33.6 SUMMARY

- Swap contracts allow one party to exchange periodic cash flows with another party. They are over-the-counter OTC instruments.
- A plain vanilla interest rate swap involves one party exchanging fixed interest payments for cash flows determined by a floating rate (LIBOR). The notional principal in an interest rate swap is not exchanged.
- Swaps enable a company with a bank loan at floating rates to effectively switch this to a fixed rate (or vice versa) – hence a swap can be used to eliminate interest rate risk of future cash flows.
- Swap dealers earn profits on the bid–ask spread of the swap deal.
- One reason for swaps is that it may be cheaper for a company to borrow at say floating and use a swap to convert this to a fixed rate loan, rather than to directly obtain a fixed rate loan from its correspondent bank.
- The principle of comparative advantage allows all the parties to the swap to obtain their desired cash flows (fixed or floating), at a lower cost than borrowing directly in their preferred form.

APPENDIX 33: COMPARATIVE ADVANTAGE WITH SWAP DEALER

We use the same figures as in the text which are reproduced here in Table 33.A.1.

Assume that the swap dealer takes part of the total gain of 0.5%. In Figure 33.A.1 the swap dealer breaks even on the floating rate, since she pays out and receives LIBOR. On the fixed leg the swap dealer receives 10% but only pays out 9.9%, which provides an overall gain of 0.1% for the swap dealer.

TABLE 33.A.1 Bank borrowing rates facing A and B

	Fixed	Floating
Company-A	10.00% (A_x)	LIBOR + 0.3%
Company-B	11.20% (B_x)	LIBOR + 1.0%
Absolute difference (B – A)	Δ(fixed) = 1.2	Δ(float) = 0.7
Net comparative advantage or 'quality spread differential'	NCA = Δ(fixed) – Δ(float) = 0.5 B has comparative advantage in borrowing at a floating rate. Hence B borrows from the bank at a floating rate.	

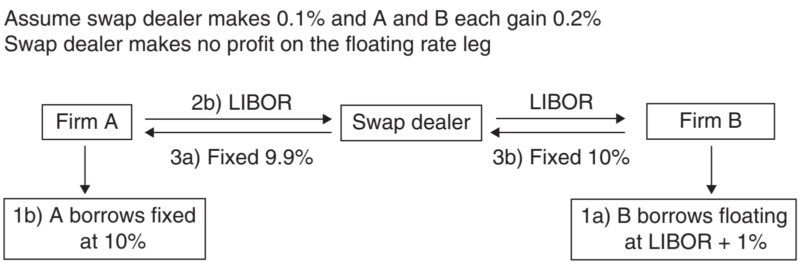

FIGURE 33.A.1 Swap dealer

Cash Flows A

'A' initially takes out a fixed rate bank loan and in the swap receives fixed and pays floating

- 'A' pays 10% fixed on the bank loan (as Table 33.A.1)
- 'A' receives 9.9% fixed from the swap dealer and
- pays LIBOR to the swap dealer.

The net effect is that A pays LIBOR + 0.1%, which is 0.2% less than going directly to the floating rate loan market. From Figure 33.A.1 it is easy to work out B's position.

Cash Flows B

B initially takes out a bank loan at LIBOR and in the swap receives floating and pays fixed

- B pays LIBOR + 1% on its bank loan (as Table 33.A.1)
- B receives LIBOR from the swap dealer and
- B pays 10% fixed to the swap dealer.

The net effect is that B pays 11% fixed (which is 0.2% better than borrowing directly at a fixed rate from its correspondent bank). The swap dealer gains 0.1% on the difference between its receipts and payments on the fixed legs of the two swaps. Note that the swap dealer is subject to potential default risk since either A or B could default, yet the swap dealer has to honour its commitment to the other party. Also note that LIBOR in the above example can take on any value and the swap will still be worthwhile to all parties – so they will willingly enter the swap.

EXERCISES

Question 1
You have a floating rate bank loan (at LIBOR plus a spread). Why might you use an interest rate swap?

Question 2
What is a forward rate agreement (FRA) and how does it relate to a plain vanilla interest rate swap?

Question 3
You are a swap dealer. What is the meaning of the term 'warehousing swaps'?

Question 4
Explain the difference between credit risk and market risk with reference to interest rate swaps. How can the market risk of interest rate swaps be hedged?

Question 5
The notional principal in a 'pay-fixed, receive-floating' interest rate swap with 3 further payments is \$20m. The fixed rate payer pays 11% p.a. and the floating rate payer, pays LIBOR. Payments in the swap are every 90 days. Today at time t, immediately after a payment date, LIBOR is 11.5% p.a. The LIBOR rates at $t + 90$ days, $t + 180$ days and $t + 270$ days, actually turn out to be 10.5% p.a., 10.2% p.a. and 9.6% p.a., respectively.

Show the schedule of actual payments in the swap. Assume actual/365 day-count convention for all interest rates.

Question 6
Consider the following loan rates facing firms A and B.

	Fixed	**Floating**
Firm-A	10% p.a.	LIBOR + 0.3% p.a.
Firm-B	11.2% p.a.	LIBOR + 1.0% p.a.

Firm-A can borrow more cheaply than firm-B in both the fixed and floating market. But firm-A wants to ultimately end up borrowing at a floating rate and B at a fixed rate. How might both A and B benefit by using the swaps market? Assume A and B deal directly with each other and split the gains from the swap, 50:50.

Question 7

Companies A and B have been offered the following rates per annum on a \$20 million, 5-year loan:

	Fixed rate	**Floating rate**
Company A	12.0%	LIBOR + 0.1%
Company B	13.4%	LIBOR + 0.6%

Company A wants to end up paying a floating rate while company B desires a fixed rate loan.

Design a swap that will net a bank, acting as intermediary, 0.1% p.a. and appear equally attractive to both companies. Assume the bank pays and receives LIBOR and split any gain in the swap equally between A and B (after the swap dealer has taken a 0.1% p.a. gain).

CHAPTER 34

Pricing Interest Rate Swaps

Aims

- To show that cash flows in a swap are equivalent to a *replication portfolio* consisting of a position in a fixed rate bond and a floating rate note (bond) (FRN).
- To demonstrate that the swap rate is calculated by assuming that the swap must have the same value to both parties at inception – otherwise the swap would not take place. The swap rate is determined solely by the term structure of interest rates.
- To show how arbitrage arguments can be invoked to demonstrate that the market value of an FRN is equal to the notional principal Q in the FRN (a) at inception t_0, and (b) just after, any reset date $t_i (i = 1, 2, \ldots, n)$.
- To show how to value an FRN at any date t (between payment dates), using two different methods the 'short method' and the 'forward rate method'. Both methods give the same outcome for the value of the FRN at any date t.
- To show that the value of a swap at any date t is the difference between the present value of the cash flows from a fixed rate bond and the present value of the cash flows from an FRN.
- To analyse how the mark-to-market value of a swap changes through time in response to changes in interest rates and the number of remaining payments in the swap.

A swap can be priced by considering the swap as a synthetic bond portfolio. When a pay-fixed, receive-float interest rate swap is initiated (at $t = 0$) each leg of the swap must have the same (present) value – otherwise both sides would not enter into the swap. This insight allows us to price the swap (i.e. determine the fixed swap rate, sp).

Over time ($t > 0$) the (mark-to-market) value of the swap to any one party can be positive or negative, as the present value (PV) of the fixed and floating rate payments in the swap alter

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

as interest rates change. Somewhat paradoxically *the PV of the variable rate (LIBOR) payments* remain relatively stable, even though these floating cash flows change over time. So it is the fixed leg of the swap which is the main source of changes in the mark-to-market value of the swap.

34.1 CASH FLOWS IN A SWAP

The payments in a receive-float, pay-fixed swap are shown in Figure 34.1 and look exactly like a long position in an FRN and a short position in a fixed rate bond.

If you have a 'receive-float, pay-fixed' swap, on a nominal principal $Q = \$100$m, your net cash receipts at each 6-month payment date, are $C_{FL} - C_X$. These are the same cash flows that would ensue from selling a fixed rate bond with maturity value[1] \$100m and using the proceeds to purchase (i.e. go long) a floating rate bond (also with maturity value \$100m). Both the floating and fixed rate bond are redeemed at maturity for $Q = \$100$m. In an interest rate swap the notional principal $Q = \$100$m is not exchanged at maturity, so there are no cash flows from this source in the swap. But the latter is also the case if we are long the FRN and short the fixed rate bond since the two principal amounts of $\$Q$ at maturity, cancel out (Figure 34.1). Hence, the net payment in the swap and the net payment from the long-short bond position are both $C_{FL} - C_X$ at maturity.

The replication portfolio of being long an FRN and short a fixed rate bond (both with maturity value Q) gives exactly the same (net) cash flows as the swap. You would not enter the swap unless the present value of floating payments V_{FL} equals the present value of the

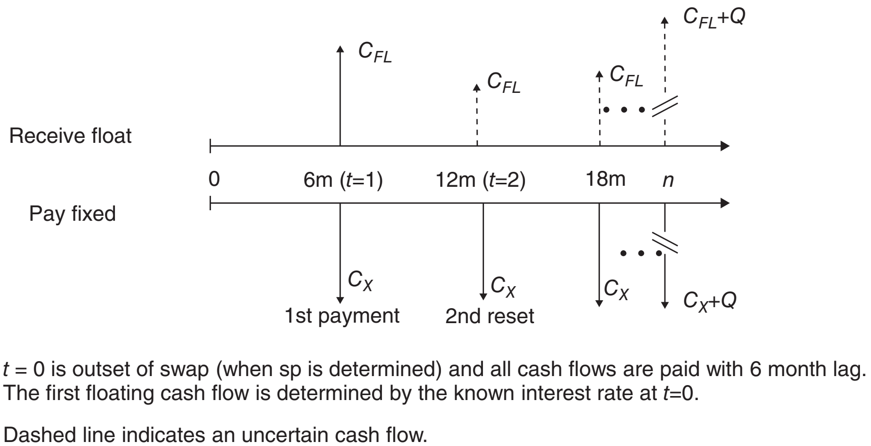

FIGURE 34.1 Cash flows in a receive-float, pay-fixed swap

[1] Also called the par, face, or principal value.

fixed payments V_{Fix}. Hence you only enter the swap if at $t = 0$, $V_{Fix} = V_{FL}$. As we see below, this result allows us to determine the swap rate. However, first we consider how to value the FRN.

34.2 FLOATING RATE NOTE (FRN)

An FRN is a bond with variable rate coupons based on future LIBOR rates. We can value the FRN in a number of different ways:

- Using arbitrage arguments.
- Using all forward rates to estimate *all (expected)* future floating cash flows. The value of the FRN is then the *present value* of these expected cash flows. This is the 'forward rate method'.
- Using a 'short method' to value the FRN which only uses the floating cash flow at the *next* payment date and the known principal Q of the FRN.

Not surprisingly, all three methods give the same answer for the value of the FRN (at any date) and all the methods are equivalent, although it is the arbitrage argument which really underlies the other two approaches.

It is also important to be clear about what point in time you are trying to value the FRN. When pricing a swap you need to find the value of the FRN at inception of the swap ($t = 0$). We will see that the present value of the FRN at $t = 0$ is just the principal in the FRN, which equals Q. After inception of the swap, interest rates may change (i.e. the yield curve will shift) and the market value of the FRN could increase or decrease – but we can still use the above three methods to value the FRN at $t > 0$.

34.2.1 Value of an FRN at $t = 0$

An FRN is a bond whose floating (coupon) payments are adjusted in line with prevailing market interest rates, which are determined at the *previous reset date*. Consider a notional principle of $\$Q$ and a LIBOR rate of L_{01} at time $t = 0$. At $t = 0$ the coupon payable at the first reset date $t = 1$, on the floating rate bond is known and equals $C_{FL,1} = L_{01} m_1 Q$ (where $m_1 = days/360$). Subsequent coupon payments on the 'floater' depend on future LIBOR rates at $t = 1, 2, 3, \ldots$ which are *not known* at $t = 0$. However, somewhat counter-intuitively it is shown in Appendix 34 that even though these future floating payments are uncertain, nevertheless the following propositions are true:

Proposition 1
At inception $t = 0$, all future receipts on an FRN have a present value equal to Q

Proposition 2

Immediately after any reset date t_i, all future floating receipts have a present value of Q

Proposition 1 allows us to price the swap at inception, $t = 0$. Proposition 2 helps in valuing the FRN (and hence the swap) after inception (i.e. $t > 0$). Hence the value of an FRN over time is like the stylised pattern in Figure 34.2 – its value equals Q at $t = 0$, it also equals Q just after any of the reset dates and at the maturity date, $t = n$. Between reset dates, the market value of the FRN can deviate from its notional value Q (Figure 34.2). But as interest rates generally do not change drastically over short periods, the value of the FRN does not deviate too far from Q, between reset dates.

A semi-intuitive way of showing that the (present) value of an FRN equals Q at $t = 0$ and just after each payment date is to value the FRN using current forward rates and to recursively calculate its value, as we move back in time from the final payment date (Figure 34.3).

For simplicity consider a 3-period FRN, which pays annual coupons (and we use compound rates). The forward rates are all known at $t = 0$. (They are calculated from the known spot rates at $t = 0$). The expected final floating payment is $C_{FL,3} = (1 + f_{23})Q$ but this has an *expected* present value *at* $t = 2$ of $PV_2 = (1 + f_{23})Q/(1 + f_{23}) = Q$. Hence at $t = 2$, the present value of the cash flow at $t = 3$ plus the cash flow at $t = 2$ is $PV_2 + C_{FL,2} = Q(1 + f_{12})$ – the latter has a present value at $t = 1$ of $PV_1 = (1 + f_{12})Q/(1 + f_{12}) = Q$. We have now shown that at $t = 1$, the present value of *both* the future cash flows at $t = 2$ and $t = 3$ are worth Q in total. At $t = 1$ we also receive a cash flow $C_{FL,1} = r_1Q$, so at $t = 1$ *all* (current and future) cash flows are worth $Q + r_1Q$ – but these have a present value at $t = 0$ of $(Q + r_1Q)/(1 + r_1) = Q$. Hence

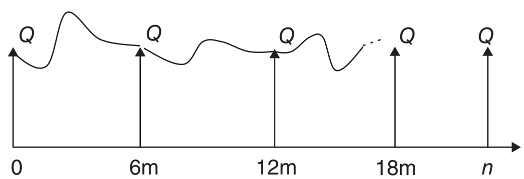

FIGURE 34.2 Value of a FRN over time

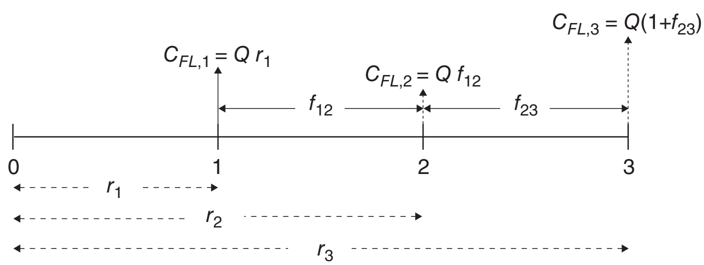

FIGURE 34.3 FRN, expected cash flows

the value of all the future cash flows (at $t = 1, 2, 3$) from the FRN have a value today (at $t = 0$) of Q. Also note from the above that just after the floating rate payment dates at $t = 1$ and $t = 2$, the (present) value of any remaining future cash flows, is also equal to Q. (See Appendix 34 for a formal proof of this using arbitrage arguments.)

34.3 PRICING A SWAP: SHORT METHOD

To price a plain vanilla swap at the outset ($t = 0$) means finding the fixed rate sp which makes the (present) value of the fixed rate bond equal to the (present) value of the FRN – or what is the same thing, that the fixed-for-floating swap has zero initial value.

For example, suppose you are a swap dealer who has to fix the swap rate sp for a 'new' fixed for floating (LIBOR) swap with a notional principal of \$$Q$. Let $C_{X,i} = (sp\, h_i\, Q)$ be the payment each period in the fixed leg of the swap and h_i the tenor in the fixed-leg (e.g. 180/360 or 181/360 etc.). The discount factors are $d_i = 1/[1 + r_i.t_i]$ where $r_i = LIBOR_i,\ t_i = days_i/360$ and $days_i$ is the *actual* number of days from t_0 to t_i.[2] The (present) value of the fixed leg at $t = 0$ is:

$$V_{Fix} = C_{X,1}\, d_1 + C_{X,2}\, d_2 + C_{X,3}\, d_3 + \ldots\ldots + C_{X,n}\, d_n + Q\, d_n \tag{34.1}$$

$$V_{Fix} = Q\left[sp \sum_{i=1}^{n} h_i d_i + d_n\right] \tag{34.2}$$

Note that if the notional principal in the swap is \$1, then the term in square brackets is the 'present value of the fixed payments in an interest rate swap, with \$1 notional principal'. Now we make use of our above proposition that the market value at t_0 of all the future floating (LIBOR) cash flows are today equal to the notional principal, so $V_{FL}(t_0) = Q$. The swap rate is that value of sp for which $V_{FL} = V_{Fix}$. Hence the swap rate is the solution to:

$$Q = Q\left[sp \sum_{i=1}^{n} h_i d_i + d_n\right] \tag{34.3}$$

The swap rate depends only on the term structure of spot rates at $t = 0$ (and not on Q):

$$sp_n = \frac{1 - d_n}{\sum_{i=1}^{n} h_i d_i} \tag{34.4}$$

This is the swap rate for an n-period swap. There are different quoted swap rates for swaps with different maturities. The '*swap spread*' is often defined as the difference between

[2]Note that even when the fixed payments are determined using the swap *convention* where $h_i = 180/360$ is the same for each reset period, nevertheless when calculating the discount factors $d_i = 1/[1 + r_i.t_i]$ we must use the *actual* number of days since the swap was initiated so, $t_i = days_i/360$.

the swap rate and the yield to maturity (y_n) of an n-period government bond, so the 'swap spread' $= sp_n - y_n$. A plot of the swap rate sp_n against n is often referred to as the '*swap rate curve*'.

Swap rates generally lie slightly above T-bond yields because swap rates reflect the creditworthiness of major banks that provide swaps (and also reflect liquidity in the market). In fact swap rates rather than government bond rates are often used as benchmark rates for pricing assets such as corporate bonds, mortgages etc., because the swaps' market is so liquid and virtually risk free. If the tenor h in the fixed leg of the swap is constant, then

$$sp_n = (1/h)\frac{(1-d_n)}{\sum_{i=1}^{n} d_i} \tag{34.5}$$

Table 34.1 shows the calculation of the (4-year) swap rate $sp_n = 5.3579\%$ p.a. from the spot yield curve on 15 March-01, using the above formula (with $h = 180/360$ for all periods in the fixed-leg of the swap).[3]

The discount rates are calculated as follows. For example, between 15 March-01 and 15 September-02 ($t = 3$) there are 549 days and the quoted spot rate (at $t = 0$) is $r_3 = 5.36\%$ p.a., so the discount factor is:

$$d_3 = 1/[1 + r_3.t_3] = 1/[1 + 0.0536(549/360)] = 0.924437 \tag{34.6}$$

How can we check that $sp = 5.3579\%$ is the correct swap rate? The PV of the fixed cash flows using $sp = 5.3579\%$ should equal the notional principal in the swap of \$100,000. We have $C_X = (sp.h.Q) = 0.053579(180/360)\$100\text{m} = \$2{,}367{,}497$ (payable every 6 months) and using the discount factors in Table 34.1:

$$V_{Fix} = C_X\left[\sum_{i=1}^{n} d_i\right] + Qd_n = \$100{,}000 \tag{34.7}$$

So, using $sp = 5.3579\%$ ensures that $V_{Fix} = \$100{,}000 = V_{FL}$ which is as expected. If the swap dealer is a fixed rate receiver (and floating rate payer) she will set the actual swap rate above 5.3579% to reflect the transactions cost of hedging her swaps book and the credit risk of the counterparty in the swap.

[3]As an aside note, we can use Equation (34.5) to calculate spot rates from swap rates. For example, suppose we have calculated 1-year and 2-year spot rates r_1, r_2 from T-bill prices then we can calculate the 3-year spot rate r_3 using Equation (34.5), if we observe (know) the current 3-year *swap rate*, sp_3. *If we also know the current market swap rates* sp_n for $n = 4, 5, \ldots\ldots n$ then we can use Equation (34.5) to calculate all the spot rates $r_4, r_5 \ldots\ldots r_n$ and hence construct the complete spot rate curve. This is often done in practice because the swaps market is very liquid and the *spot rate* curve is derived from observable *swap rates*.

TABLE 34.1 Calculation of swap rate

Today is 15 March-01

Notional principal: 100,000

Days in year (swap convention): 360

Days to 1st floating reset date: 184

6-month LIBOR on 15 March-01: 5.15%

Date	Days	Cum. days	Spot rates LIBOR	Discount factors LIBOR	Forward rates LIBOR	Floating cash flows	PV (Floating cash flows)	d*f*m
15-Mar-01								
15-Sep-01	184	184	5.15	0.9744	5.1500	2,632.2222	2,564.7133	0.0256
15-Mar-02	181	365	5.27	0.9493	5.2537	2,641.4436	2,507.4648	0.0251
15-Sep-02	184	549	5.36	0.9244	5.2576	2,687.2221	2,484.1663	0.0248
15-Mar-03	181	730	5.45	0.9005	5.2905	2,659.9635	2,395.2546	0.0240
15-Sep-03	184	914	5.54	0.8767	5.3102	2,714.1088	2,379.4313	0.0238
15-Mar-04	182	1096	5.65	0.8532	5.4376	102,749.0067	87,668.9698	0.0235
		1.Swap rate		0.0536		2. Swap rate		0.0536
		PV fixed CF		100,000		PV floating CF		100,000

34.4 PRICING A SWAP: FORWARD RATE METHOD

When using the ‘forward rate method’ to determine the swap rate we use *all* the forward rates to determine the present value of the floating rate payments V_{FL}, set this equal to the present value of the fixed rate payments V_{Fix} and then solve for the unknown swap rate, sp_n. Of course, we already know that the present value of the floating payments is equal to Q but the ‘forward rate method’ is useful in pricing other kinds of swap which we meet later – so we introduce the method here.

The floating coupon payments are $C_{FL,i} = f_i m_i Q$ where the forward rates f_i apply to the period between (t_{i-1}, t_i). Therefore $f_1 = L_{01}$ is actually the *known* LIBOR rate fixed at t_0 for the period (t_0, t_1). Also $m_i = days_i/360$ and $days_i = actual\ number\ of\ days$ between each repayment date. Spot rates are used to calculate forward rates. For example, the forward rate f_2 applicable to the period (t_1, t_2) is derived from the two spot rates r_1, r_2 using the arbitrage relationship:

$$(1 + r_1 m_1)(1 + f_2 m_{12}) = (1 + r_2 m_2), \qquad m_{12} = days_{12}/360 \tag{34.8}$$

For example, from Table 34.1 we see that $m_1 = 184$, $m_2 = 184 + 181 = 365$, $m_{12} = 181$, $r_1 = 5.15\%$ and $r_2 = 5.27\%$. Hence $f_2 = \{[1 + 0.0527\,(365/360)]/[1 + 0.0515\,(184/360)] - 1\}$ $360/181 = 0.052537$.[4] The present value of *all the floating payments*, $C_{FL,i} = f_i m_i Q$ at $t = 0$ is:

$$V_{FL}(t_0) = \sum_{i=1}^{n} C_{FL,i} d_i + Q d_n = Q\left[\sum_{i=1}^{n} f_i m_i d_i + d_n\right] \tag{34.9}$$

and from Table 34.2 this gives $V_{FL}(t_0) = \$100{,}000$ as expected, since we already know that at inception of the swap, the value of the floating leg equals Q. The PV of the fixed cash flows is $C_{X,i} = sp.h_i.Q$ hence, as before:

$$V_{Fix} = \left[\sum_{i=1}^{n} C_{X,i} d_i + Q\,d_n\right] = Q\left[sp\sum_{i=1}^{n} h_i d_i + d_n\right] \tag{34.10}$$

TABLE 34.2 New spot rates and discount factors

Today is 15 June-01

Notional principal: 100,000

Days in year (swap convention): 360

Days to 1st floating reset date: 184

6-month LIBOR on 15 March-01: 5.15%

Swap rate (set on 15 March-01): 5.36%

Date	Days	Cum. days	New LIBOR rate 15 Jun-01	Discount factors LIBOR	Fixed CF, $sp = 0.0536$	PV fixed cash flows	PV (floating cash flows)	Floating coupons	PV floating cash flows
15-Jun-01									
15-Sep-01	92	92	6.15	0.9845	2,679	2,637	5.1500	2,632	2,591
15-Mar-02	181	273	6.27	0.9546	2,679	2,557	6.2330	3,134	2,992
15-Sep-02	184	457	6.36	0.9253	2,679	2,479	6.1988	3,168	2,932
15-Mar-03	181	638	6.45	0.8974	2,679	2,404	6.1784	3,106	2,788
15-Sep-03	184	822	6.54	0.8701	2,679	2,331	6.1492	3,143	2,735
15-Mar-04	182	1004	6.65	0.8436	102,679	86,615	6.2182	3,144	87,007
PV fixed CF						99,024	PV floating CF		101,044
Value 'receive-float, pay-fixed' swap (15 June)						2,020			

[4]Equivalently, forward rates can also be derived from the LIBOR discount rates: $f_2 = [(d_1 - d_2)/d_2]\,(360/m_{12})$.

Equating V_{Fix} and V_{FL} and rearranging, we obtain an expression for the swap rate using the 'forward rate method':

$$sp_n = \frac{\sum_{i=1}^{n} d_i f_i m_i}{\sum_{i=1}^{n} h_i d_i} \tag{34.11}$$

The swap rate calculated using (34.11) is shown as '2. Swap rate' in Table 34.1 and naturally it gives the same numerical value of 5.36% as that derived in Equation (34.5), which uses only the term structure of spot (or discount) rates.

34.5 MARKET VALUE OF A SWAP

At inception of the swap on 15 March-01, the value of the swap is $V_{sp}(t_0) = V_{FL} - V_{Fix} = 0$ by construction. We now want to calculate the mark-to-market value of the swap sometime after its inception. The value of the swap changes over time as interest rates change and as the number of cash flows remaining changes. The value of a receive-float pay-fixed swap at $t > 0$ is the difference between the value of the FRN (floating leg) and the fixed coupon bond:

$$V_{sp}(t) = V_{FL}(t) - V_{Fix}(t) \tag{34.12}$$

Suppose on 15 June (i.e. after 3 months), spot rates (and hence forward rates) have all increased – what is the new value of the swap? To value the swap we need to find the new (present) values of the FRN and the fixed-rate bond, using the new higher spot rates. We begin by valuing the FRN and we can use two different methods:

(a) 'short method' – use only the next LIBOR cash flow (plus Q)
(b) 'forward rate method' – use forward rates as forecasts of all future LIBOR cash flows.

34.5.1 Value of FRN at $t > 0$ ('Short Method')

Suppose we are trying to value the swap at t (15 June), between $t = 0$ (15 March) and $t = 1$ (15 September-01) – Figure 34.4. From our earlier analysis we know that the present value of the floating leg equals Q immediately after any payment date. Between payment dates an FRN can have a value different to Q.

The floating cash flow at $t = 1$ (15 September) is known and depends on the LIBOR rate $L_{0,1}$ which was fixed on 15 March (at $t = 0$ and which pays out at $t = 1$). This cash flow is:

$$C_{FL,1} = Q[L_{0,1}(days_{0,1}/360)] = \$100\text{m}\ (0.0515)(184/360) = \$2{,}632 \tag{34.13}$$

Using proposition 2, the value of all future LIBOR cash flows accruing *after* t_1 (that is $C_{FL,2}, C_{FL,3}, \ldots . C_{FL,n}$ plus the final payment of Q at t_n), have a PV *at* $t = 1$ of Q. Hence the

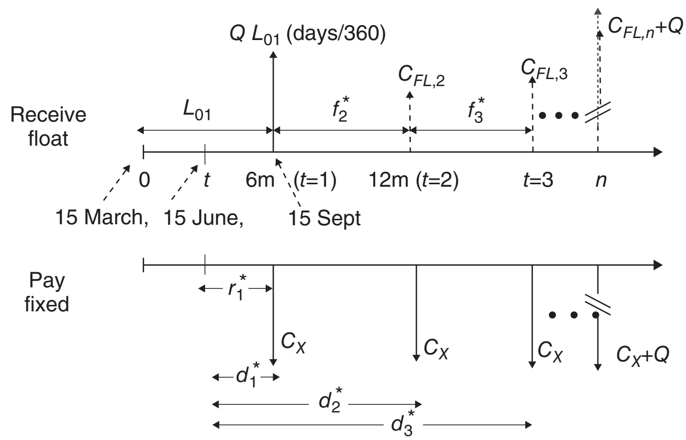

FIGURE 34.4 Value of swap at t, receive-float, pay-fixed

value of the FRN at t depends only on the present value of the *next* floating LIBOR payment $C_{FL,1}$ plus Q. The new spot rate applicable from 15 June to 15 September (92 days) is $r_1^* = 6.15\%$, so the new discount rate $d_1^* = 1/(1 + 0.0615\ (92/360)) = 0.984526525$ (Table 34.2).

Hence the (present) value of the FRN on 15 June is:

$$V_{FL}(t) = (C_{FL,1} + Q)d_1^* = (\$2{,}632 + \$100{,}000)\ 0.984526525 = \$101{,}044$$

At inception of the swap (15 March) the present value of the floating rate payments was \$100,000. Although the spot-LIBOR rate r_1^* has increased and the *next* cash flow of \$2,632 is unchanged, nevertheless the value of the FRN on 15 June has increased. This is because you now have less time to wait (92 days) for the first payment on 15 September (compared with 184 days at inception of the swap on 15 March-01).

The above formula with minor changes in notation applies to the value of the FRN between *any* two payment dates – its value depends only on the present value of the *next known* floating LIBOR payment C_{FL}, plus Q.

34.5.2 Value of FRN at $t > 0$ ('Forward Rate Method')

The value of the FRN at t (15 June) is the PV of *all* future LIBOR cash flows after time t. We can value the FRN by using the new forward rates at t to provide an estimate of expected future cash flows and discount these cash flows back to time t, using the new spot rates determined at t. This is the 'forward rate method'.

The value of the FRN between inception and the first payment date ($t_0 < t < t_1$) depends on: (i) the PV of the next LIBOR payment $C_{FL,1} = Q.L_{0,1}(days_{0,1}/360)$ which was fixed at

$t = 0$; plus (ii) PV of *all the future floating rate payments* $C_{FL,i} = f_i m_i Q_i$ at payment dates $t_i \geq 2$ ($i = 2, \ldots n$); plus (iii) the PV of Q. The 'new' interest rates at t are higher (Table 34.2) than at $t = 0$. All spot rates are now measured from time t and hence we have 'new' discount factors and new forward rates:

$$V_{FL}(t) = \left[\sum_{i=1}^{n} C^*_{FL,i} d^*_i + Q\, d^*_n\right] = Q\left[\sum_{i=1}^{n} f^*_i m^*_i d^*_i + d^*_n\right] = \$101{,}044 \qquad (34.14)$$

where $d^*_i = 1/[1 + r^*_i t^*_i]$, $r^*_i = LIBOR_{t,i}$ $t^*_i = days_{t,i}/360$, so the discount factors apply to the period from today t to the remaining payment dates, t_i. The PV of the FRN on 15 June is $V_{FL}(t) = \$101{,}044$ – not surprisingly this is the same value as found above using the 'short method'.

For more 'exotic' swaps than the plain vanilla swap considered here (and discussed in Chapter 35), we often cannot use the 'short method' and we need to value the floating leg using 'the forward rate method'.

34.5.3 Value of Fixed Leg at $t > 0$

Now let's value the fixed leg at t. The fixed payments are $C_X = (sp.h.Q) = 0.0536\ (180/360)$ \$100,000 = \$2,679. The only change to note is that discount rates now use the *new spot* rates at time t so that $t_i = days_{t,i}/360$ where $days_{t,i}$ is the actual number of days from t to t_i, that is from 15 June to the other reset dates (Figure 34.4), hence:

$$d^*_i = 1/[1 + r^*_i . t^*_i], \quad r^*_i = LIBOR_{t,i}, \quad t^*_i = days_{t,i}/360$$

The value of the fixed leg on 15 June-01 is calculated in Table 34.2:

$$V_{Fix}(t) = C_X \sum_{i=1}^{n} d^*_i + d^*_n = \$99{,}024 \qquad (34.15)$$

The PV of the fixed leg has fallen from \$100,000 (at inception on 15 March) to \$99,024 (on 15 June) because of the rise in spot rates which alters the discount factors d^*_i. The value of a receive-float, pay-fixed swap at time t is therefore:

$$V_{sp}(t) = V_{FL}(t) - V_{Fix}(t) = \$101{,}044 - \$99{,}024 = \$2{,}020 \qquad (34.16)$$

The rise in interest rates between 15 March and 15 June results in an increase in value of the receive-floating LIBOR-leg of the swap and a fall in the value of the 'pay-fixed' leg of the swap – so the 'receive-float pay-fixed swap' has increased in value and is worth \$2,020.

34.6 SWAP DELTA AND PVBP

In the above example the value of the swap changed because of (i) a change in the yield curve and (ii) because we moved through time and revalued the swap on 15 June. If we had valued the swap on, say, 15 April two years later and interest rates had not changed, the swap value would have changed simply because it has only two further cash flows remaining.

It is useful when hedging a swap to separate out the effect of interest rate changes and the 'passage of time', on the market value of the swap. To do so we consider what happens to the (present) value of the swap when all interest rates increase by the same small amount over a small time interval (say one day). If the change in interest rates is taken to be 1 bp then the resulting change in the value of the swap is known as the *swap delta*. The latter concept is useful when hedging a position in swaps – which we discuss in Chapter 39. The swap delta is a similar concept to the delta of an option. The value of the swap on 15 June (using the 'short method') gives:

$$V_{sp}(t) = Q[1 + L_{0,1}(days_{0,1}/360)]\, d_1^* - \left[C_X \sum_{i=1}^{n} d_i^* + Qd_n^* \right] = \$\mathbf{2{,}020} \tag{34.17}$$

where

$$d_i^* = 1/[1 + r_i^*.t_i^*], \quad r_i^* = LIBOR_{t,i}, \quad t_i^* = days_{t,i}/360.$$

What will happen to the value of the swap if *all* interest rates rise by 1 bp, over *the next day*? Higher spot and hence forward rates lead to an increase in the expected cash flows from the FRN and no change in the cash flows in the fixed leg, hence the net *cash receipts* of the receive-float pay-fixed swap, increase. But does this imply that the *present value* of these *net* cash flows increases?

All future floating LIBOR payments (except that at the next payment date) will increase but so will the discount rates applied to these higher floating cash flows, and in present value terms these two effects are largely offsetting so the value of the floating leg of the swap does not vary much (see Figure 34.2). But how much is this offset? Using the 'short method' we can easily see what's happening to the value of the FRN:

$$\text{Receive float: } V_{FL}(t) = Q[1 + L_{0,1}(days_{0,1}/360)]\, d_1^* \tag{34.18}$$

If interest rates increase, $V_{FL}(t)$ *will fall* the next day but not by much, as it only falls because of the increase in the (single) short-term LIBOR interest rate which implies d_1^* is smaller. On the other hand, although the fixed *dollar payments each period* do not change, there may be many of these, so their present value falls by a relatively large amount, after a rise in all spot interest rates. But as a fixed rate *payer* you are better off when the PV of your 'debt' is lower. Hence:

A rise in interest rates will increase the present value of a 'receive-float, pay-fixed' swap. Hence, a 'receive-float, pay-fixed swap' has a positive delta.

Another way of seeing how the value of a long maturity 'receive-float pay-fixed' swap changes after a rise in interest rates is to note that the floating-leg (FRN) has a small duration D_{FL} and the fixed-leg has a large duration, D_{Fix}. For each leg of the swap we have $dV = -V(D)dr$, hence for a parallel shift in the yield curve:

$$dV_{sp} = dV_{FL} - dV_{Fix} = -(V_{FL}D_{FL} + V_{Fix}D_{Fix})dr \qquad (34.19)$$

Since $D_{FL} < D_{Fix}$ then a rise in interest rates (usually) increases the value of the receive-float pay-fixed swap.

34.7 SUMMARY

- A *plain vanilla interest rate swap* involves one party exchanging a series of fixed cash flows for cash flows determined by a floating rate (LIBOR). The notional principal in the swap is not exchanged.
- Cash flows in a 'receive-float, pay-fixed swap' are equivalent to taking a long position in an FRN and a short position in a fixed rate bond. At inception, the two parties in the swap will not enter the swap unless the value of the fixed bond equals the value of the floating bond. Hence the swap rate is calculated by making the fixed leg of the swap equal to the value of the floating leg, at inception of the swap.
- The swap rate is determined by the term structure of spot rates, at inception of the swap.
- After a swap has been initiated, the value of the swap may become positive or negative (to one of the parties). The mark-to-market value of a swap at any time is the present value of the remaining future net cash flows in the swap.
- The mark-to-market value of a swap changes over time as spot interest rates (and hence forward rates) change and because the swap will have less cash flows outstanding. For a long-dated swap, it is the fixed-leg rather than the floating-leg whose *present value* changes by a large amount, as interest rates change.
- The change in the (present) value of a swap position after a 1 basis point change in all interest rates is known as the *swap delta* (or the present value of a basis point, PVBP). The swap delta is a useful concept when swap dealers try to hedge the whole of their swaps book.

APPENDIX 34: VALUE OF AN FRN USING ARBITRAGE

We wish to determine the present value V_{FL} of all future floating rate payments on an FRN at any reset *date* t_i $(i = 0, 1, 2, \ldots n)$. The FRN has a notional (face) value Q, which is fixed. Using an arbitrage argument, we show that V_{FL} must equal Q immediately after any reset date, including at t_0 when the swap is initiated. (Note that this analysis says nothing about the value of the FRN when we are between reset/payment dates.)

Suppose that *immediately after* any reset/payment date t_i we have $V_{FL}(t_i) > Q$. Consider the following arbitrage strategy:

1. Sell (short) an FRN at $V_{FL}(t_i)$ and invest \$$Q$ at LIBOR (in a risk-free bank deposit).
   Net cash inflow at t_i is $V_{FL}(t_i) - Q > 0$.
2. At the next payment date use the LIBOR receipts from the bank deposit to pay LIBOR on the short FRN.
   Roll over the principal Q of the deposit. Repeat this process until t_{n-1}.
   Net cash inflow $= 0$ at each payment date up to t_{n-1}.
3. At t_n use the LIBOR interest from the deposit account to pay LIBOR on the FRN and use the principal \$$Q$ from the deposit account to pay the principal on the FRN.
   Net cash inflow $= 0$.

Hence, if $V_{FL}(t_i) > Q$ at any payment date (including t_0) there is a risk-free arbitrage profit to be made, since you receive a net cash inflow at t_i and no net cash outflows after that date. This arbitrage strategy will result in bond arbitrageurs selling FRNs at time t_i, which will push the market price of the FRN down, until its market value equals Q (its par value), at which point arbitrage trades cease and equilibrium is restored. But the market price of the FRN is just the market's valuation of the future cash flows from the FRN so $V_{FL}(t_i) = Q$. Since the arbitrage is riskless, we expect $V_{FL}(t_i) = Q$ at all future payment/reset dates t_i $(i = 1, 2, \ldots n)$, and today at t_0.

EXERCISES

Question 1
What are the main causes of a change in the mark-to-market value of a 20-year pay-fixed, receive-floating (plain vanilla) interest rate swap? Explain.

Question 2
From a swap dealer's viewpoint, explain a 'matched' 20-year swap.

Question 3
Today a swap dealer agrees a receive-fixed, pay-floating 10-year interest rate swap on a notional principal of \$10m. What might cause the swap to have a positive value to the swap dealer, in 3 months' time? Explain.

Question 4
A swap dealer has to decide the swap rate to charge in a 'new' fixed-for-floating (LIBOR) swap on a notional principal of $Q = \$25\text{m}$. The swap's maturity is 2 years, with payments every 180 days. The term structure of LIBOR rates is 12% p.a. over 6 months, 12.25% p.a. over

1 year, 12.75% p.a. over 18 months and 13.02% p.a. over 2 years (all continuously compounded). Assume there are 360 days in a year.

Calculate the swap rate.

Briefly explain what factors determine the quoted swap rate.

Question 5

At inception, the swap rate is $sp = 6\%$ p.a. (simple rate) on a plain vanilla $Q = \$100$m swap.

The previous reset date was 15 January. The 6-month LIBOR rate on 15 January was 3.6% p.a. (simple rate). The tenor in the swap is 6 months.

It is now 15 March. The next (LIBOR) reset date is 15 July and the swap matures on the next 15 January (in 10 months' time).

Assume 6 months equals ½ year etc. and the yield curve is 'flat' at 5% p.a. (continuously compounded). Forward rates to calculate floating cash flows in the swap are 'simple rates', not continuously compounded.

Calculate the mark-to-market value of a receive-fixed, pay-floating swap *(on 15 March)* by considering the swap as a series of forward contracts.

Question 6

A $Q = \$100$m notional, interest rate swap has a remaining life of 10 months. Under the terms of the swap, 6-month LIBOR (floating) is exchanged for a fixed swap rate, $sp = 12\%$ p.a.

The yield curve is currently 'flat' at 10% p.a. (continuously compounded), which is equivalent to 10.254% p.a. (simple interest).

The 6-month LIBOR rate, 2 months ago, at the previous 'reset date' was 9.6% p.a. (simple rate) and the next payment dates are in 4 months and 10 months.

Consider the swap as a combination of a fixed and a floating bond and calculate the current value of the swap, to the party paying floating.

Question 7

It is immediately after a payment date on swap with a notional principal of $\$Q = \$100$m (annual payments). The swap rate $sp = 10\%$ p.a.. The swap is a receive-fixed, pay-floating interest rate swap with two remaining payment dates.

The current spot rates for the two periods are $r_1 = 5\%$ p.a. and $r_2 = 5.5\%$ p.a., respectively and the forward rate is $f_{12} = 6.0\%$ p.a. (compound rates).

Consider the swap as a bond portfolio and calculate the value of the swap to the party receiving-fixed.

CHAPTER 35

Other Interest Rate Swaps

Aims

- To price and value 'non-standard' interest rate swaps such as variable rate swaps, off-market swaps, zero-coupon swaps, swaps with variable notional principals and basis swaps.
- The swap rate on more complex interest rate swaps can be determined by making the present value of the fixed leg of the swap equal to the (expected) present value of the floating leg (at inception of the swap) – as for plain vanilla swaps. However, the methods used in calculating these present values differ for exotic swaps.
- To show how swap dealers hedge their interest rate swaps book using other fixed income assets such as FRAs, bonds, interest rate futures, and interest rate options.
- To demonstrate how to hedge the credit risk of a swap position using collateral, netting and credit enhancements.

35.1 SWAP DEALS

There are a wide variety of swap contracts which can be designed to suit particular customers' requirements. In a spread-to-LIBOR swap, the floating rate is not at LIBOR-flat but at LIBOR plus a spread. Since most floating-rate bank loans contain a spread over LIBOR, this swap can be constructed to exactly match the floating payments in a LIBOR bank loan. In an *off-market swap,* the fixed swap rate is set at whatever rate is chosen by the 'customer' (e.g. a corporate) – for example, this off-market swap rate might be chosen to exactly match the fixed rate on an existing bank loan. In a *zero-coupon swap* the fixed payment is not periodic but consists of one 'lump sum' payment at the maturity of the swap.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Some corporates may have fixed interest rate loans where the fixed payments *change over time* in a predetermined way – for example, the fixed rate might be 3% p.a. for the first 10 years followed by a fixed rate of 4% p.a. for a further 5 years. To match the payments in such a loan the corporate might use a swap to 'receive fixed' but the swap has a pre-agreed changing path for the (fixed) swap rate which matches that on the bank loan. Also, the corporate might pay floating (LIBOR-flat) in the swap, thus effectively converting her fixed rate loan (with different fixed rates over different time periods) to a variable rate loan.

In a *basis swap* both parties make floating-rate interest payments but they are linked to different floating rates, for example 90-day LIBOR against 180-day LIBOR or against the 90-day T-bill rate. This type of swap is also referred to as a *floating-floating swap*. It allows a corporate to exchange one set of floating-rate payments for a slightly different set of floating payments. Although LIBOR and T-bill rates (for the same maturities) tend to move together, the spread can change – which gives rise to net payments in the swap.

Another kind of basis swap is a *yield curve swap*. Here both parties pay floating but one might be based on a short rate (e.g. 3-month T-bill rate) and the other on the 20-year T-bond yield. If the yield curve is initially upward sloping but during the life of the swap it flattens out then this would benefit the party paying the 20-year rate and receiving the short rate.

A yield curve swap might be useful for a Savings and Loan Association (S&L) (or bank) to offset interest rate risk. An S&L has (mostly) short-term floating rate liabilities (e.g. deposits) and (some) floating rate long-term assets (e.g. variable rate mortgage receipts). The S&L will suffer if short rates rise relative to long rates (i.e. if the yield curve flattens) but it can offset this risk by entering into a yield curve swap, to receive cash flows based on a short rate and pay out cash flows based on a long rate.

Swaps may have variable notional principals. If the notional principal falls over the life of the swap this is an *amortising swap*. For example, if someone has floating loan payments linked to a mortgage where the principal on the 'mortgage' falls through time, then an amortising swap is appropriate. The converse, where the principal increases through time is known as an *accreting swap*. This is useful where, for example, a construction firm has building costs which increase by, say, $10m each year for 5 years and it has taken out a 'staggered LIBOR-loan' for $10m today, followed by an extra $10m principal in each succeeding year, over the construction period. The construction firm can convert this 'increasing principal' floating-rate loan into an 'increasing principal' fixed rate loan by entering into an accreting swap to 'receive-floating, pay-fixed', on a (notional) *variable* principal amount.

It is also possible to create a swap with a variable notional principal that first increases and then decreases over the life of the swap, to match the changing principal in a bank loan held by a corporate – this is known as a *roller-coaster* swap. A roller-coaster swap can be useful in hedging floating rate payments on a bank loan for project finance (e.g. for building an electricity generating station where the principal of the bank loan first increases (to pay the construction costs) and then decreases (after cash flows from the project come on stream).

Diff swaps or *quanto swaps* are cross-currency interest rate swaps. One party to the swap has cash flows determined by a foreign interest rate but these foreign cash flows are based on a notional principal in the home currency. They embed a fixed currency exchange rate in the swap deal.

For example, a US firm might enter a swap where it receives periodic cash flows based on (fixed) Euribor swap rates but calculated on a *dollar* principal amount. In the swap the US firm might pay USD-LIBOR (on the same dollar principal amount). The US firm therefore avoids any currency risk. The term 'diff swap' arises from the fact that payments in the swap depend on the difference between interest rates in two countries. The receipt of Euribor interest in the swap could be used to pay interest on an existing Euribor bank loan. The US firm then effectively has a fixed USD loan (and no exchange rate risk).

If you wish to enter into a swap at some time in the future at a swap rate fixed today, then a *forward swap* is appropriate. This is simply a forward contract on the swap rate. Now consider an option on a swap, which is known as a *swaption*. The swaption 'delivers' a (cash-market) swap at the expiration date of the option. If you choose to *receive the fixed payment* in the swap (and pay floating) it is called a *receiver swaption*. (The converse is a payer swaption.) Forward swaps and swaptions are discussed in Chapter 39.

We use the methods outlined in the previous chapter to value a variety of 'non-standard' interest rate swaps, these include:

- a swap with a variable notional principal
- a spread-to-LIBOR swap
- an off-market swap
- a zero-coupon swap
- a swap with a variable swap rate (for the 'fixed' payments).

35.2 PRICING NON-STANDARD SWAPS

35.2.1 Variable Notional Principal

Assume the day-count convention for the floating leg is $m_i = actual\ days_i/360$ and for the fixed leg it is $h_i = 180/360$. (We keep the subscript on h_i because in some swaps $h_i = actual\ days_i/(360\ or\ 365)$). In amortising, accreting and roller-coaster swaps the notional principal Q_i varies in a pre-specified manner, determined at the outset of the swap. We can price this swap in the same way as we priced a plain vanilla interest rate swap, the only changes being:

- We replace Q with Q_i, hence $C_{FL,i} = f_i\ m_i\ Q_i$ and $C_{X,i} = sp.h_i Q_i$.
- We have to value the floating leg using all the forward rates – we cannot use the 'short method' as we have a *variable* notional principal Q_i.

At inception of the swap the value of the floating and fixed legs are:

$$V_{FL} = \sum_{i=1}^{n} C_{FL,i} d_i + Q_n d_n \tag{35.1}$$

$$V_{Fix} = \sum_{i=1}^{n} C_{X,i} d_i + Q_n d_n \tag{35.2}$$

Hence equating the two present values and solving for the swap rate, sp:

$$sp = \frac{\sum_{i=1}^{n} (f_i m_i Q_i) d_i}{\sum_{i=1}^{n} Q_i h_i d_i} \tag{35.3}$$

Not surprisingly, the swap rate is determined by the predetermined values of the (changing) notional principal Q_i as well as the term structure of interest rates, reflected in d_i and f_i. Note that a plain vanilla swap has $Q_i = Q$ (fixed) and then (35.3) gives the swap rate for a plain vanilla swap, as derived in Chapter 34.

35.2.2 Spread-to-LIBOR Swap

Instead of LIBOR-flat in the floating leg, the swap contract may specify that the floating leg has an interest rate of (LIBOR + b), where b is a spread of say 30 bps ($b = 0.003$). We assume the notional value in the swap is fixed at Q. The value of the fixed leg is:

$$V_{Fix}(t_0) = \sum_{i=1}^{n} C_{X,i} d_i + Q_n d_n = (sp^* Q) \sum_{i=1}^{n} h_i d_i + Q d_n \tag{35.4}$$

The floating leg can be viewed as floating payments at LIBOR-flat with a PV of Q at inception, *plus* periodic constant payments at bQ, hence:

$$V_{FL}(t_0) = Q + bQ \sum_{i=1}^{n} m_i d_i \tag{35.5}$$

where we have assumed that the spread b uses the same day-count convention as the floating payments. Equating (35.4) and (35.5), the swap rate sp^*, which gives zero value for the 'spread-to-LIBOR' swap at inception, is:

$$sp_n^* = \frac{(1 - d_n)}{\sum_{i=1}^{n} h_i d_i} + \frac{b \sum_{i=1}^{n} m_i d_i}{\sum_{i=1}^{n} h_i d_i} = sp_n + \frac{b \sum_{i=1}^{n} m_i d_i}{\sum_{i=1}^{n} h_i d_i} \tag{35.6}$$

sp_n is the 'at-market' swap rate (at t_0) for a fixed-for-floating swap at LIBOR-flat. If you are paying fixed using the 'at-market' swap rate sp_n but receiving LIBOR-flat, the value of the swap to you is zero (by construction). But if you are paying fixed at sp_n and receiving LIBOR+b the value of this swap to you will be positive. Hence to make the value of this 'pay-fixed, receive-*LIBOR*+b' swap equal to zero, the fixed swap rate sp_n^* you pay in this 'spread-to-LIBOR' swap, is greater than the 'at-market' swap rate sp_n.

35.2.3 Zero-coupon Swap (Against LIBOR-flat)

In a zero-coupon swap there is a single fixed payment of Q_n^* at t_n which has present value $V_{Fix} = d_n Q_n^*$. We have to determine the value for Q_n^*, which makes the swap have zero value at inception of the swap.

The floating payments are the same as in a plain vanilla swap $C_{FL,i} = f_i \; m_i Q$, where the (notional) principal in the swap is Q. We know that the value of all these floating payments at t_0 is simply $V_{FL} = Q = \$100\text{m}$ (say). Suppose $d_n = 0.85$ (where $d_n = 1/[1 + r_n(days_n/360)]$). Equating $V_{Fix} = V_{FL}$ we have $Q_n^* = Q/d_n = \$100\text{m}/0.85 = \$117.647\text{m}$. The zero-coupon swap rate (per \$1 nominal principal) is then quoted as $sp_n^{zero} = 17.647\%$.

In other words, on a notional principal of \$100m the fixed leg of the swap has to pay out a single payment of \$117.674m at t_n while the floating leg receives LIBOR on each payment date. The zero-coupon swap rate is (defined as):

$$sp_n^{zero} = (Q^*/Q) - 1 = 0.17647. \tag{35.7}$$

It is easy to see that $sp_n^{zero} = (1/d_n) - 1 = r_n \; days_n/360$. Hence, the zero-coupon swap rate is the n-period LIBOR rate r_n (p.a.) adjusted by the maturity of the swap (= $days_n/360$).

In pricing our next set of swaps, so that the swap has the same value to both parties at inception, involves an *up-front payment* by one of the parties in the swap.

35.2.4 Off-market Swap (Against LIBOR-flat)

Suppose the current 'at market' swap rate (for an n-year swap) with $Q = \$100\text{m}$ is $sp = 6.2\%$, which implies that using $sp = 6.2\%$, $V_{Fix} = V_{FL} = \$100\text{m}$, so the swap has zero value (to both parties). How do you price a swap where the fixed-rate payer wants to pay fixed at $sp^* = 6\%$ (every period)? This is an 'off-market' swap. The present value of the fixed leg at $sp^* = 6\%$ is:

$$V_{Fix}(at \;\; sp^* = 6.0\%) = Q\left[\sum_{i=1}^{n}(sp^* h_i)d_i + d_n\right] \tag{35.8}$$

which must be less than $V_{Fix}(at \;\; sp = 6.2\%) = \$100\text{m}$. Suppose $V_{Fix}(at \; sp^* = 6\%) = \$99.5\text{m}$. Then the pay-fixed party in the off-market swap will deliver an up-front payment of \$100m − \$99.5m = \$0.5m to the swap counterparty. The 'pay-fixed' party in all future periods then pays fixed at $sp^* = 6\%$ and receives LIBOR-flat.

35.2.5 Swap with Changing Fixed Rates (Against LIBOR-flat)

It is a bit strange to think of the 'fixed rate' in a swap not being the same fixed rate *every period*. But at $t = 0$ someone might like to pay a known amount each period but this predetermined amount can be different in each period.

Suppose the current 'at market' swap rate (for an n-year swap) with $Q = \$100$m and with constant payments in the fixed leg is $sp = 6.2\%$. The fixed payments are $C_{X,i} = h_i\, sp\, Q$ and $V_{Fix}(sp = 6.2\%) = V_{FL} = \$100$m – so the swap has zero value to both parties.

The known 'fixed payments' in the swap with a known changing value for sp_i^* are $C_{X,i}^* = h_i\, sp_i^* Q$ where each different sp_i^* at each future reset date are all known *at* $t = 0$. The value of the fixed leg is:

$$V_{Fix}(with\ sp_i^*) = \sum_{i=1}^{n} C_{X,i}^*\, d_i + Q\, d_n \tag{35.9}$$

The numerical value of $V_{Fix}(with\ \ sp_i^*)$ is unlikely to equal \$100m and could be larger or smaller depending on the particular values chosen (at $t = 0$) for sp_i^*. Suppose $V_{Fix}(with\ \ sp_i^*) =$ \$99.4m. The 'pay-fixed at sp_i^* would pay an up-front payment of \$100m − \$99.4m = \$0.6m at $t = 0$ to the counterparty in the swap, then in all future periods would pay the (variable but) predetermined payments $C_{X,i}^* = h_i sp_i^* Q$ (and receive LIBOR each period).

35.2.6 Basis Swap

Consider an 18-month basis swap, with resets every 180 days, where you agree to pay the (variable) T-bill rate and receive LIBOR on a notional principal of $Q = \$100$m. This can be replicated with two plain vanilla swaps:

Pay T-bill rate and receive-fixed + pay-fixed and receive-LIBOR

Suppose the current term structure is:

Maturity	**T-bills**	**LIBOR**
$t_1 = 180/360$	r_1^{TB}	r_1^L
$t_2 = 360/360$	r_2^{TB}	r_2^L
$t_3 = 540/360$	r_3^{TB}	r_3^L

Pricing the two plain vanilla swaps so that they have zero value at inception gives:

(a) Pay variable T-bill rate and receive-fixed at sp^{TB}:

$$sp^{TB} = 2\frac{1 - d_3^{TB}}{\sum_{i=1}^{3} d_i^{TB}} \tag{35.10a}$$

where[1] $d_i^{TB} = 1/(1 + r_i^{TB} t_i)$

(b) Pay fixed at sp^L and receive LIBOR

$$sp^L = 2\frac{1 - d_3^L}{\sum_{i=1}^{3} d_i^L} \quad \text{where} \quad d_i^L = 1/(1 + r_i^L t_i) \tag{35.10b}$$

The term structure of LIBOR rates is usually slightly above that for T-bills (e.g. by 20–30 bps) because the former carries more credit risk, hence we might find that $sp^{TB} = 6\%$ p.a. and $sp^L = 5.6\%$ p.a. Now it is clear that the original basis swap of 'pay the T-bill rate and receive-LIBOR' would be priced so that one party would:

Pay the T-bill rate and receive (LIBOR – 0.4%).

35.2.7 Mark-to-market Value of Non-standard Swaps

After inception, all of the above swaps will generally have a positive or negative value to one of the parties in the swap. The mark-to-market value of these swaps can be calculated using the 'forward rate method'. All we need to do at time $t > 0$ is note the new spot rates (and hence discount rates) and calculate the new forward rates. The forward rates are used to determine the new floating rate payments $C_{FL,i}^* = f_i^* \ m_i^* Q_i$ remaining in the swap and these are discounted using the new spot rates at t, to give:

$$V_{FL}(t) = \left[\sum_{i=t+1}^{n} C_{FL,i}^* d_i^* + Q_i \ d_n^* \right] \tag{35.11}$$

[1]We assume the T-bill is priced using simple yield (and not the T-bill discount rate).

where the sum is from the next payment date $t + 1$ to n. The value of the (remaining) fixed payments at the new spot rates is straightforward:

$$V_{Fix}(t) = \sum_{i=t+1}^{n} C_{X,i} d_i^* + Q_i d_n^* \tag{35.12}$$

where $C_{X,i} = h_i sp.Q$. The current market value of a receive-floating, pay-fixed swap at any time after inception ($t > 0$) is:

$$V_{swap}(t) = V_{FL}(t) - V_{Fix}(t) \tag{35.13}$$

35.3 HEDGING INTEREST RATE SWAPS

A swap dealer will usually take on one side of a plain vanilla interest rate swap even if she cannot immediately find an offsetting swap. This is known as *'warehousing'*. If the dealer does warehouse an interest rate swap then she is exposed to interest rate risk. One way of hedging the *floating interest payments* is with a *series of* interest rate futures contracts. Eurodollar futures can be used and a series of Eurodollar futures is known as a Eurodollar strip.

35.3.1 Hedging Floating LIBOR Cash Flows Only

After aggregating all her interest rate swap deals, assume a swap dealer (Ms Float) currently has a net position to *receive-float, pay-fixed,* (in USD) which currently has positive value to the dealer. How can the swap dealer hedge against future changes in interest rates?

At first sight it may look as if she just has to hedge changes in LIBOR on the floating side since the fixed (coupon) payments are just that, *fixed*. Ms Float can hedge any *single future* LIBOR swap receipt by today, going long (buying) Eurodollar futures with a maturity date just after the LIBOR reset date. If interest rates fall in the future, Ms Float receives lower floating-rate cash flows from her swaps but the futures price will rise and the profit from closing out the futures offsets her lower LIBOR swap receipts. So today, *for each LIBOR reset date* in the swap Ms Float goes long N_F futures:

$$N_F = \frac{\textit{Notional Value of swap (\$Q)}}{\textit{Notional principal in futures contract}} \left[\frac{D(swap)}{D(futures)} \right] \beta_y \tag{35.14}$$

where *D(swap)* = ½. (say), the tenor of cash flows in the swap. If there are k remaining reset dates, she goes long a total of $k.N_F$ futures contracts today, using N_F contracts for each reset date – this is a strip hedge. Ms Float will then close out N_F contracts at each future reset date. However, this is not the hedge usually undertaken by swaps dealers, who invariably hedge the *present value* of the swap.

The above is fine if you just want to hedge the future floating cash flows – so, for example, the above futures hedge is correct for a company hedging cash flows from a floating rate bank loan or bank deposit. But if the swap dealer just hedges her LIBOR floating leg of the swap she is ignoring the fixed leg. Remember that although the fixed (coupon) payments in the swap are 'fixed' each period, their *present value is not* – and above we have not yet hedged the change in the present value of the fixed payments in the swap.

35.3.2 Hedging the Mark-to-market Value

The market value of the swap changes due to (a) interest rates changes and (b) the swap has fewer payment dates remaining. We now discuss a different type of hedge where the swap dealer chooses to hedge the *present value* of the swap – that is, the present value of the floating receipts minus the fixed payments. If the swap dealer hedges the mark-to-market (present) value of the swap then any changes in interest rates (over a short time horizon) will not alter the *present value* of the swap dealer's position. Because the hedge is only effective over a short time horizon, the hedge position must be periodically rebalanced.

Hedging the PV of the swap is a conceptually different approach to hedging only the future LIBOR cash flows in the swap. The choice between these two types of hedge depends on what you consider is the most useful hedge in the circumstances. For example, it is reasonable for a company with a floating rate loan just to hedge the upcoming floating rate payments using a strip of interest rate futures contracts – since these variable loan payments are the key cash-flow risk for the corporate. But it seems more sensible for the swap dealer to hedge changes in value for both legs of the swap. If the swap currently has a value of \$100m to the swap dealer, then with the hedge in place the swap dealer's position will still have \$100m value, after a change in interest rates (over a short time horizon). Also for regulatory purposes it is the mark-to-market value that is used in determining how much 'capital' (e.g. retained profits and equity capital) the swap dealer has to hold against the 'market risk' of her swap's book.

Suppose you are holding a receive-float, pay-fixed swap that currently has positive value. If over the next few days all interest rates (including forward rates) fall by 1 bp (parallel shift), the mark-to-market value of your swap position will fall. It is perhaps easiest at the outset to analyse what is going on by assuming that the PV of the floating payments remains unchanged. (Remember, as interest rates change, the PV of the floating payments stays close to $\$Q$.) As a fixed-rate payer in the swap, you are worse off after a fall in interest rates because the PV of your future fixed (coupon) payments increases – your liabilities have increased. Hence a fall in interest rates by 1 bp will result in a fall in the value of a receive-float, pay-fixed swap.

35.3.3 Delta of a Swap

The delta of a swap (by convention) is defined as the *dollar change* in present value of the swap for a 1 bp *increase* in *all* interest rates (i.e. an upward parallel shift in the yield curve).

The present value at time t of the net cash flows at each future swap payment date t_i is:

$$PV_i = (C_{FL,i} - C_X)d_i \tag{35.15}$$

where the floating cash flow depends on the appropriate forward rate and d_i is the discount factor (which depends on spot interest rates measured from today, t). The PV of the swap (with n remaining payments) is simply the sum of these PVs:

$$V_{swap} = \sum_{i=1}^{n} PV_i \tag{35.16}$$

At inception the swap has zero value. Suppose the pay-fixed swap has been in existence for some time and currently has a positive market value. To calculate the swap's delta we need to find the *change* in the present value of the swap due to a 1 bp (per annum) increase in interest rates (over a short time horizon). To do so we calculate the new present value of the swap with all interest rates 1 bp higher (using the 'forward rate method' for the FRN) – this is sometimes referred to as the 'perturbation method'.[2] We therefore undertake the following steps:

1. We know the current value of the swap, $V_{swap} = \sum_{i=1}^{n} PV_i$.
2. Set 'new' spot rates = existing spot rates *plus* 1 bp.
3. Calculate new discount factors, d_i^* and forward rates, f_i^*
4. Calculate new floating cash flows, $C_{FL,i}^* = f_i^* m_i Q$ at each payment date and keep the fixed cash flows C_X unchanged.
5. Calculate the new net cash flows $(C_{FL,i}^* - C_X)$.
6. Calculate the new PV of each net cash flow, $PV_i^* = (C_{FL,i}^* - C_X)d_i^*$ using the new discount (spot) rates.
7. Calculate the *change* in PV at each payment date, $PV_i^* - PV_i$.
8. Calculate change *in PV* of *all* future cash flows, $dV_{swap} = \sum_{i=1}^{n}(PV_i^* - PV_i)$.

The (dollar) change in the value of the swap dV_{swap} for a *1 bp increase* in all spot rates is referred to as the *swap's delta* (or the 'present value of a basis point', PVBP). A swap dealer will have a large number of interest rate swaps and it can aggregate all the fixed and floating cash flows at each payment date. The individual (net) cash flows from each swap may be positive or negative, as may their aggregated value at any payment date.

Suppose it is 15 April-01 and (to simplify matters) assume the aggregated swaps book only has payments every 6 months, with three further payments remaining. The next aggregated payments are on 15 September-01, 15 March-02, and 15 September-02. On 15 April-01 suppose the delta's for each of the three net cash flows are \$10,000, \$9,500, and \$8,400

[2]A similar approach is used to determine 'the Greeks' in the BOPM.

TABLE 35.1 Hedging the PV of the swap (receive-float, pay-fixed)

Today is 15 April-01

Three payment dates remaining: 15 September-01, 15 March-01, 15 September-02.

	15 September-01	15 March-01	15 September-02
Change in PV of net cash flows, for 1 bp change, $PV_i^* - PV$	\$10,000	\$9,500	\$8,400
Delta of one long Eurodollar futures	–\$25	–\$25	–\$25
Number of futures	400 (= \$10,000/\$25)	380 (= \$9,500/\$25)	336 (= \$8,400/\$25)
Aggregate delta for futures $= N_F \times \$25$	–\$10,000	–\$9,500	–\$8,400

Note: Today is 15 April-01. Three payment dates remaining: 15 September-01, 15 March-02, 15 September-02. Total N_F for the hedge = 400+380+340 = 1,120.

(Table 35.1) – giving a total swap delta of $\Delta_{sp} = \$27{,}900$. Since all the deltas are positive then, if interest rates increase (fall) over the next week by 1 bp, the present value of the swap position increases (falls) by \$27,900 (over the next week).[3]

35.3.4 Hedging the Present Value of a Swaps Book

We can hedge the (present) value of the cash flows in the swap using any fixed income assets such as bonds, interest rate futures, and interest rate options – since their market value changes as interest rates change and therefore they can be used to offset any changes in the PV of the swaps position. By far the most popular method uses Eurodollar futures because they are liquid out to long maturities of 20–30 years.

When interest rates *increase* by 1 bp (per annum), the value of a receive-float, pay-fixed swap *increases* (i.e. positive delta), as the PV of the fixed payments is lower. To hedge we need to take a position in futures where a rise in interest rates leads to a loss on the futures – that is, today we take a long position in Eurodollar futures which have a 'delta' or 'tick value' of \$25. Table 35.1 shows that the number of contracts needed (for each reset date) is:

$$N_F = (\text{Swap's portfolio delta at each payment date})/\$25. \tag{35.17}$$

[3]Note that in practice not all the deltas will necessarily be positive, some of the deltas for the three net cash flows may be positive and some may be negative – depending on the aggregate net cash flows from all the swaps held by the swap dealer on these dates. Any negative cash flows reduce the overall swap's delta and of course the overall swap's delta may be negative.

Since we are only considering a potential change in interest rates over, say, 1 week, we can in principle use any interest rate futures contracts which mature after 1 week. But our rebalancing period might be longer than a week, in which case we might use longer dated Eurodollar contracts.

Consider implementing a strip hedge on 15 April-01. You take a long position in 400 September-01, 380 March-02 and 340 September-02 contracts – a total of 1,116 long contracts. Of course we could also use other Eurodollar strategies. For example, a stack hedge involves going long 1,116 September-01 contracts or 1,116 September-02 contracts.

If over the next few days (say) interest rates rise by 1 bp, then the (approximate) increase in the present value of the swaps position ($27,900) would be offset by the $27,900 losses on the long futures positions.[4] So the mark-to-market value of the hedge position would remain constant over a 'short' time horizon.

We have removed the interest rate risk on the PV of the swaps book. The swap dealer will feel happy holding less of a (regulatory) 'capital buffer' against the market risk of its swaps book, since it is hedged and the risk of losses from changes in the value of the swaps book is minimal (and the regulator will agree). From time to time the swap dealer will rebalance its position in 1,116 futures and calculate a new hedge position for N_F. This is because as interest rates change and as time moves on, this changes the delta of the swaps book.

35.3.5 Gamma and Convexity

Note that as swaps are equivalent to a 'long-short' bond portfolio, they have a non-linear (convex) response to interest rate changes. Our swaps 'delta hedge' is only effective for small changes in interest rates – or equivalently the hedge is only accurate over short time horizons. So the delta-hedged position should be rebalanced frequently (as we did when hedging options positions). If we thought there would be large changes in interest rates (over a short time period) then we require an estimate of the *swaps' gamma*. (This is a very similar concept to the convexity of a bond). As a Eurodollar futures contract has zero gamma we *cannot* use this to hedge the gamma of the swaps book.

To gamma hedge the swaps position, we require fixed income instruments that have a non-zero gamma – we could use FRAs, cash market bonds, options on T-bonds, or caps and floors. Using FRAs has the advantage that they have a zero vega and so do not introduce the additional complication of sensitivity to changes in the volatility (standard deviation) of interest rates. To see how this might work suppose the delta and gamma of our (total) swaps book are $\Delta_{sp} = 2{,}000$ and $\Gamma_{sp} = -30$ respectively. We first gamma hedge and then delta hedge our swap position:

1. Take positions in FRAs (and/or bonds) with a portfolio gamma $\Gamma_B = +30$, which then offsets the gamma of the swaps book. Suppose the FRAs used to obtain a zero gamma have a portfolio delta of $\Delta_B = -500$.

[4]Conversely, if interest rates fall, the value of the swaps position falls but the long futures can be closed out at a profit, so the hedge works in this case too.

2. The new delta for the 'swaps plus FRAs' is +1,500 (= 2,000 − 500). So if interest rates rise, the value of our current portfolio (i.e. swaps +FRAs) also rises.
3. Now take positions in Eurodollar futures to offset this 'new delta'. We go long 60 (= 1,500/25) Eurodollar futures. If interest rates rise there is a fall in value of our Eurodollar futures position and we close out at a loss, which balances the gain on our 'swaps plus FRAs'.

The mark-to-market swaps position would then be largely protected from both large and small changes in interest rates. While swap dealers delta-hedge their swaps position with futures, they often do not gamma hedge and are therefore subject to some price risk on their swaps book, if there are *large changes* in interest rates over a short time period.

35.3.6 Allocation of Cash Flows to Standard Payment Dates

In the above example we assumed all cash flows from the swaps occurred at specific fixed dates, 6 months apart. This is unrealistic, as swap dealers will have net cash flows in their swaps book on almost every day of the year. We therefore have a problem in aggregating cash flows, as our swaps do not have cash flows on exactly the same payment dates. Swap cash flows that fall between 'standard' payment dates need to be allocated to 'standard time buckets' to make the above calculations manageable.

Suppose it is 15 February ($t = 0$) and the 'standard' payment dates we have chosen (arbitrarily) are at t_1 (15 June) and t_2 (15 December). A *future* known cash flow in the swap C_t falls between these two standard payment dates ($t_1 < t < t_2$) – for example on 10 October.

How should we allocate C_t which accrues on 10 October, so that we end up with C_1 allocated to t_1 (15 June) and C_2 allocated to t_2 (15 December)? We use an allocation method that ensures the following:

1. PV of the allocated cash flows C_1, C_2 equals the PV of the actual cash flow C_t:

$$\begin{aligned} V_t &= V_1 + V_2 \\ C_t e^{-r_t t} &= C_1 e^{-r_1 t_1} + C_2 e^{-r_2 t_2} \end{aligned} \tag{35.18}$$

where (continuously compounded) interest rates are measured from today $t = 0$ (15 February, Figure 35.1).

2. The *change* in PV of the allocated cash flows C_1, C_2 equals the change in PV of the actual cash flow C_t (for a parallel shift in the yield curve):

$$\begin{aligned} dV_t = dV_1 + dV_2 &= (\partial V_1/\partial r_1)dr_1 + (\partial V_2/\partial r_2)dr_2 \\ (t\ \ C_t e^{-r_t t})dr &= [t_1(C_1\ \ e^{-r_1 t_1}) + t_2(C_2 e^{-r_2 t_2})]dr \end{aligned} \tag{35.19}$$

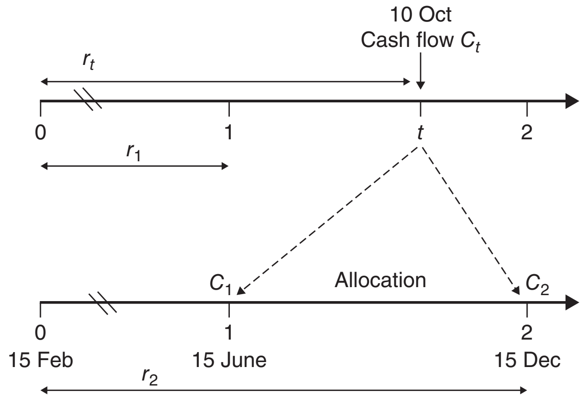

FIGURE 35.1 Allocation of cash flows

where $\partial V_i/\partial r_i = \partial(e^{-r_i t_i}C_i)/\partial r_i = -t_i(e^{-r_i t_i}C_i)$, (for $i = 1, 2$) and $dr_i = dr$ for a parallel shift in the yield curve.[5] Given current spot yields and the actual cash flow C_t, Equations (35.18) and (35.19) can be solved for the two unknowns, $(C_1,\ C_2)$ – the dollar amounts to allocate to each standard 'time-bucket' at t_1 and t_2. The coupons C_t include all cash flows in the swaps book which fall on 10 October. This allocation process has to be repeated for each cash flow that falls between any two standard payment dates.

35.4 CREDIT RISK

Hedging credit risk in the swap's book is a key issue for a swap dealer. In a swap between a dealer-D and company-A only one party will face default risk *at any one time*. This is because at time $t > 0$ if the value of the swap to the dealer is $V_D > 0$, then $V_A = -V_D < 0$. It is only if $V_D > 0$ and company-A defaults that there is a problem for the swap dealer. On the other hand if the swap dealer defaults when $V_D > 0$, then company-A is not harmed since the value of the swap to company-A is negative ($V_A < 0$). In this case company-A could simply discontinue its swap payments, although in practice it may not do so because of general reputation effects in the market.

As the value of the swap changes over the life of the swap then either party can in principle be subject to default risk at some time during the life of the swap. However, because the principal in an interest rate swap is not actually exchanged, the default risk in an interest rate swap

[5]For a single cash flow, the time to maturity equals the duration, $t_i = D_i$. Hence, the change in the PVs can also be written in terms of the familiar duration relationships: $dV_t = -V_t D_t dr$ for the actual cash flow and $dV_1 + dV_2 = -(V_1 D_1 + V_2 D_2)dr = -(D_p)dr$ where D_p is defined as the *portfolio* duration of the two cash flows.

is less than the default risk of holding a portfolio of bonds – where the principal payments do influence the market value of the bond portfolio.

Credit risks in a fixed-for-floating swap are difficult to assess but these risks are often managed by using so-called *credit enhancements*. The most common method to limit credit risk is to pledge some sort of collateral, such as a line of credit from a bank or securities (e.g. T-bonds) which are held 'in trust' by a third party. For example, if one party undergoes a downgrade in its Standard and Poor's or Moody's credit rating, the collateral can be increased.

Alternatively, marking-to-market occurs when periodically the swap's value is calculated and the party with a negative swap value pays the counterparty this amount in cash. The swap rate then has to be reset to give a zero current value for the swap, based on current interest rates. The new swap rate determines the fixed payments in the swap until the next 'mark-to-market' date, when the procedure is repeated. Clearly this procedure is similar to margin payments in a futures contract.

Netting is a fairly simple form of credit enhancement. If at $t > 0$ the swap dealer's position with company-Z is +\$5m and on another swap contract with company-Z it is minus \$4m then they can agree that the outstanding net position for the swap dealer is \$1m. Hence if company-Z defaults, the swap dealer is only exposed to \$1m credit risk rather than \$5m.

However, it must be pointed out that in the event of bankruptcy by company-Z, it is not always clear that the bankruptcy courts will legally accept such a deal. It is worth noting that in the UK the House of Lords deemed that UK Local Authorities (i.e. equivalent to US Municipal Authorities), acted *ultra vires* (i.e. *beyond the scope of authority)* in entering into swap contracts and these contracts then became null and void – even though Local Authorities had the funds to pay the losses on their swaps positions. Total losses from vanilla interest rate swap defaults have historically been very low (e.g. less than ½% of the principal value of all swaps entered into). This is probably because swap deals tend to involve large well-capitalised organisations with high credit ratings and a 'favourable reputation' to preserve.

35.5 SUMMARY

- We can adapt the techniques used for plain vanilla fixed-for-floating interest rate swaps, to price and value more complex interest rate swaps.
- The swap rate is calculated assuming the swap must have zero value (to both parties) at inception – otherwise the swap would not take place. The swap rate is calculated by making the value of the fixed leg of the swap equal to the value of the floating leg, at inception of the swap.
- After the swap has been initiated the value of the swap may be positive or negative (to one of the parties).
- Complex interest rate swaps such as a spread-to-LIBOR swap, an off-market swap, a zero-coupon swap, a swap with a time varying swap rate (for the 'fixed' payments) and a swap with a variable notional principal, can all be priced by using a replication bond portfolio (i.e. a fixed rate bond and an FRN) to represent the cash flows in the swap.

- Hedging the market risk of swaps is usually done by calculating the delta of the swaps book (for standard time buckets) and using interest rate futures to delta hedge the interest rate risk. If the swap dealer is worried about large changes in interest rates then she can also hedge the gamma of her swaps book using FRAs, bonds, or (fixed income) options (although using options also introduces vega risk). After gamma hedging, the swaps dealer can then use interest rate futures to hedge the swap's delta.
- Credit risk of a swap can be mitigated using collateral, netting, and credit enhancements.

EXERCISES

Question 1
Suppose a plain vanilla interest rate 'at-market' swap (which has zero value at inception) has a swap rate of $sp = 6\%$ p.a.. You want a 'receive-fixed, pay LIBOR' swap at an 'off-market' fixed swap rate of 6.1%. What will happen when you enter the swap and what will determine your future fixed payments?

Question 2
Suppose a plain vanilla fixed-for-floating (LIBOR) 'at-market' swap (which has zero value at inception) has a swap rate of 6% p.a. You want a 'spread-to-LIBOR' swap where you receive LIBOR+2% and pay fixed. Will you pay more or less than 6% p.a. fixed? What alternative arrangement might happen when you enter the swap?

Question 3
A bank pays 30-day LIBOR-1% on its deposits and receives 360-day LIBOR from its bank loans. How might it use a yield curve swap to hedge its interest rate risk over the next 2 years?

Question 4
Why does it make sense for a swap dealer to hedge the market-to-market value of her (interest rate) swaps book rather than hedge just the future floating rate payments?

Question 5
What is the delta of a (interest rate) swaps book?

Question 6
You have delta-hedged your (interest rate) swaps book on Monday and are going to reassess the hedge in a week's time. On Monday, have you eliminated all market risk and if not how might you improve your hedged positon?

Question 7
When is a swap dealer subject to credit risk (due to a default of the swap counterparty)?

CHAPTER 36

Currency Swaps

Aims

- To examine the reasons for undertaking a plain vanilla currency swap of fixed-USD for fixed-Euros.
- To show how a fixed-fixed USD-Euro currency swap can be replicated with a fixed rate bond in USD and a fixed rate bond in Euros.
- To demonstrate how swap rates in USD and Euros are determined in the fixed-fixed currency swap.
- To show how to value a 'fixed-fixed' USD-Euro currency swap at any time *t* after inception, using a replication portfolio of (a) a USD bond and Euro bond or (b) by considering the swap as a strip of FX-forward contracts.
- To show why currency swaps change in value over time.

36.1 USES

A plain vanilla currency swap involves two parties exchanging cash flows denominated in different currencies. One reason for undertaking a swap might be that a US firm ('UncleSam') with a subsidiary in Euroland selling hamburgers, might wish to borrow €100m at a fixed interest rate and eventually pay off the euro interest and principal with revenues from sales of its hamburgers in Euroland. This reduces the foreign exchange exposure of UncleSam as its payments on the Euroland debt are matched by receipts in euros.[1]

[1]Similarly a French spectacles manufacturer ('Eiffel') with a subsidiary in the US might wish to have dollar denominated debt and eventually pay off the interest and principal with dollar sales revenues from its US spectacles subsidiary. This reduces Eiffel's foreign exchange exposure and provides a possible swap counterparty to

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

However, it might be relatively expensive for UncleSam to raise euro finance from its correspondent euro bank (Société Générale, say), as it might not be 'well established' in this 'foreign' loan market. But, if UncleSam raises finance (relatively) cheaply using a fixed-rate USD loan (from its US correspondent bank, Citibank), it could then go to a swap dealer (Goldman) and agree to receive (fixed) USD interest and pay (fixed) Euro interest in the swap. Considering UncleSam's USD loan and the 'receive-USD pay-Euros' swap, effectively UncleSam now has a Euro liability (which can be paid off from the Euro receipts earned by its Euroland subsidiary).

'UncleSam' ultimately wants to borrow in euros but finds it cheaper to initially borrow in US dollars and then agree a currency swap to receive US dollars and pay euros.

Note that unlike interest rate swaps where the principal is 'notional' and is not exchanged either at the beginning or the end of the swap, this is not the case for currency swaps, where the principal amounts *are exchanged* at the outset and at the end of the swap.

36.1.1 Fixed-Fixed Currency Swap

The currency swap enables UncleSam to achieve its desired outcome. Let's see how this works out in detail. Suppose the current spot exchange rate is $1.4 per euro and UncleSam requires €100m to finance its expansion in Euroland. UncleSam borrows $140m from Citibank in New York (or it issues $140m of US bonds) at a fixed USD interest rate of 5.36%. The swap dealer Goldman then agrees to pay UncleSam the USD-swap rate of 5.36% in exchange for receiving the Euro-swap rate of 4.46% from UncleSam. The swap involves the following stages.

At $t = 0$ (Figure 36.1)

- UncleSam borrows $140m from Citibank in New York at a USD interest rate of 5.36% and passes this $140m to the swap dealer Goldman, who in return gives UncleSam €100m. (UncleSam uses this to invest in its Euroland hamburger subsidiary.)

At $t > 0$ (Figure 36.2)

- UncleSam's revenues from its Euroland subsidiary are used to pay Goldman €4.46m p.a. (4.46% of €100m) over the life of the swap.
- Goldman pays $7.84m p.a. (5.36% of $140m) in the swap to UncleSam, who then uses this to pay interest on its USD loan to Citibank.

Uncle Sam. However, Eiffel might find it cheaper to initially obtain a (fixed rate) euro loan from BNP Paribas and enter a swap with Goldman to pay-USD and receive-Euros. Also if Goldman captures both of these swaps, the FX risk to the swap dealer will be offsetting and Goldman will earn a 'spread' on each of the two swap deals.

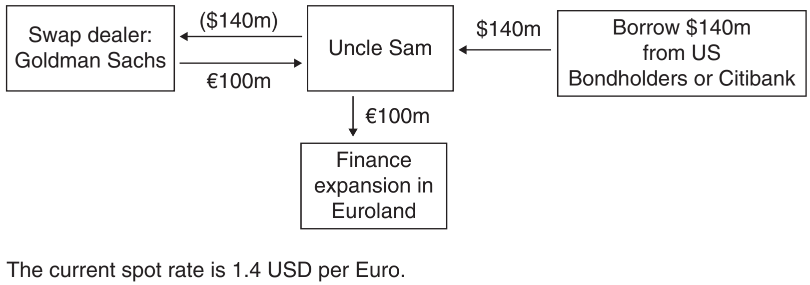

FIGURE 36.1 Principal cash flows at initiation of currency swap

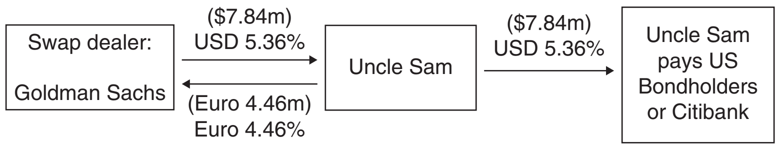

FIGURE 36.2 Cash flows in a currency swap

At maturity of the bank loan/swap

- UncleSam pays (€100m + €4.46m) to Goldman (from revenues earned in Euroland).
- Goldman pays ($140m + $7.84m) to UncleSam, who uses it to pay off the loan principal plus the last interest payment to Citibank.

Even though UncleSam *initially* borrows USD nevertheless after the swap, it is as if UncleSam has a Euro loan. This synthetic Euro loan may be at a lower interest rate than if UncleSam directly borrows in Euros from Société Générale (say).

In fact, an exchange of fixed cash flows in a foreign currency, for fixed cash flows in the home currency is not very common but it is useful to analyse this case. Example 36.1 illustrates how you can 'create' a fixed-fixed currency swap from two liquid fixed-for-floating swaps in two different currencies.

EXAMPLE 36.1

Creating a Fixed-Fixed Currency Swap in Practice

A fixed-fixed currency swap is usually 'created' from two other fixed-for-floating swaps. For example, a 'pay-Euro fixed, receive-USD fixed' currency swap can be constructed from:

(*continued*)

(*continued*)

(i) a 'pay-Euro fixed, receive-USD LIBOR' swap

and

(ii) a 'receive-USD fixed, pay-USD LIBOR swap.

The usual swap convention is to quote a fixed swap rate in the foreign currency against floating (LIBOR) in the home currency. For example, for a US company (UncleSam) this might be Euro fixed-swap rate against USD-LIBOR. Less often, both sides of the currency swap might be floating – this is a basis currency swap.

36.1.2 Swap as a Strip of Forward Contracts

UncleSam's receive-USD, pay-Euros currency swap could also be achieved by UncleSam entering into a strip of FX-forward contracts to pay euros and receive US dollars each year, at whatever today's current quoted *forward FX rates* happen to be, $F_1, F_2, \ldots$. The cash flows each year will be exchanged at these different forward FX rates. This approach provides a useful method of pricing and valuing the swap – as we see later.

From the swap dealer's viewpoint, the swap has Goldman paying USD-fixed and receiving cash flows in euros. Goldman will have many currency swaps on its books in different currencies and it is unlikely that it can find matching counterparties to all these different swaps. But Goldman can hedge all its swap net cash flows in any (one) currency, by using currency futures contracts, to offset any adverse future changes in spot exchange rates. (It can also use interest rate futures to hedge any interest rate risk, which affects the market value of the swap.) However, the largest risk to the swap dealer in a currency swap tends to come from volatile exchange rates rather than changing interest rates.

36.2 PRICING A FIXED-FIXED CURRENCY SWAP

UncleSam is receiving (fixed) USD cash flows in the swap and paying (fixed) Euro cash flows, at each payment date. How do we price this currency swap, that is, determine the fixed swap rate in US dolalrs and the swap rate in euros? Surprisingly, the two swap rates in the fixed-fixed *currency swap* are given by the two swap rates for plain vanilla *interest rate* swaps in each of the two countries (with the same tenor and maturity). Because the argument is a bit detailed note that the result we are looking for is as follows:

> *If the current (at-market) swap rate in a plain vanilla fixed-for-LIBOR interest rate swap in the US is $sp^{\$}$ and sp^{ε} is the equivalent rate for a Euro-zone plain vanilla interest rate swap, then these two swap rates are the correct swap rates for the fixed-fixed currency swap, US dollars for euros.*

In other words, if we know the (fixed-for-floating) *interest rate* swap rates in the two countries we do not have to make a separate calculation of the fixed-fixed *currency* swap rates. Let's see why the two swap rates for the fixed-fixed currency swap are the same as the two *interest rate swap rates,* in each of the two countries.

At the outset of the currency swap the notional principals are exchanged. UncleSam exchanges $Q^{€} = €100m$ for $Q^{\$} = \$140m$ (from Goldman, the swap dealer) at the current exchange rate $E_0 = 1.4$ $/€ – today, this is worth the same to both parties (i.e. UncleSam and Goldman), since $Q^{\$} = Q^{€} E_0 = \$140m$.

Table 36.1 shows the current USD spot interest rates (quoted on 15 March-01) from which we can calculate (in the usual way) the swap rate for a 3-year vanilla fixed-for-floating *interest rate swap* in USD and this is found to be $sp^{\$} = 5.35\%$ p.a. (per $1 notional principal). Given this swap rate, we know that the current value of all the future USD-LIBOR floating cash flows equal all future fixed-USD cash flows. Hence, given $sp^{\$} = 5.35\%$ all future fixed USD cash flows in a USD vanilla interest rate swap with a principal of $Q^{\$} = \$140m$ are worth $140m today.

Similarly, given the Euroland yield curve on 15 March-01, the Euro swap rate for a vanilla fixed-for-floating *Euro interest rate* swap is $sp^{€} = 4.46\%$ (per €1 notional principal) – Table 36.2.

Hence, given $sp^{€} = 4.46\%$ all future fixed Euro cash flows in a Euro vanilla interest rate swap with a principal of $Q^{€} = €100m$ are worth €100m today. But €100m today is worth $E_0 Q^{€} = \$140m$ today. Suppose we use the two current 'at-market' swap rates ($sp^{\$} = 5.35\%$,

TABLE 36.1 Swap rate for vanilla USD interest rate swap

Notional principal	$1m
Days in year (swap convention)	360
Days in 1st reset period (for 1st floating coupon)	184
LIBOR at last reset period (for 1st floating coupon)	5.15%

Date	Days	Cumulative days	Spot rates LIBOR	Discount factor LIBOR
15-Mar-01				
15-Sep-01	184	184	5.15	0.9744
15-Mar-02	181	365	5.27	0.9493
15-Sep-02	184	549	5.36	0.9244
15-Mar-03	181	730	5.45	0.9005
15-Sep-03	184	914	5.54	0.8767
15-Mar-04	182	1096	5.65	0.8532
				Swap rate = 0.0536

TABLE 36.2 Swap rate for vanilla Euro interest rate swap

Notional principal	€1m
Days in year (swap convention)	360
Days in 1st reset period (for 1st floating coupon)	184
LIBOR at last reset period (for 1st floating coupon)	4.15%

Date	Days	Cumulative days	Spot rates LIBOR	Discount factor LIBOR
15-Mar-01				
15-Sep-01	184	184	4.15	0.9792
15-Mar-02	181	365	4.27	0.9585
15-Sep-02	184	549	4.36	0.9377
15-Mar-03	181	730	4.45	0.9172
15-Sep-03	184	914	4.54	0.8966
15-Mar-04	182	1096	4.65	0.8760
				Swap rate = 0.0446

$sp^{\varepsilon} = 4.46\%$) from the *vanilla interest rate swaps in the US and the Euro-zone* as the two swap rates in a fixed-fixed *currency swap*. Then the present value of all the future Euro fixed cash flows in the currency swap is €100m, which today is equivalent to US$140m. But the latter is the same as the present value of all the future USD fixed cash flows in the currency swap. Hence, if the two swap rates in the fixed-fixed currency swap are quoted as $sp^{\$} = 5.35\%$ and $sp^{\varepsilon} = 4.46\%$ you would willingly enter the fixed-fixed currency swap, because the net value of the swap in US dollars (or euros) is zero.

36.3 VALUING A FIXED-FIXED CURRENCY SWAP

36.3.1 Currency Swap as a Bond Portfolio

The cash flows in the swap are shown in Figure 36.3. From 'UncleSam's' perspective the swap is equivalent to a synthetic position consisting of:

Long a fixed rate dollar denominated bond and short a fixed rate Euro denominated bond

$\Rightarrow$ *Receives \$'s fixed and pays €'s fixed*

The swap has zero value at inception. But the value of the swap at some time $t > 0$ depends on the value of the two fixed rate bonds. Consider valuing the swap at t before the first payment

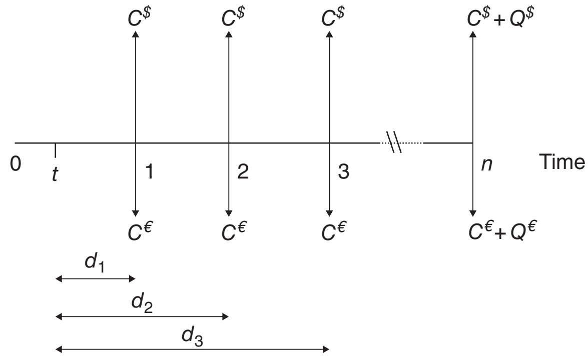

FIGURE 36.3 Value of currency swap at t

date ($t_0 < t < t_1$). The value (in USDs) of the 'receive-fixed USD, pay-fixed Euro' swap at time t is:

$$V_{sp}^{\$}(t) = V_{Fix}^{\$}(t) - E_t \ V_{Fix}^{\varepsilon}(t) \tag{36.1}$$

where E_t is the spot FX rate (USD per Euro) at time t and

$$V_{Fix}^{\$} = C^{\$} \sum_{i=1}^{n} d_i^{\$} + Q^{\$} d_n^{\$} \tag{36.2}$$

$$V_{Fix}^{\varepsilon} = C^{\varepsilon} \sum_{i=1}^{n} d_i^{\varepsilon} + Q^{\varepsilon} d_n^{\varepsilon} \tag{36.3}$$

where d_i are the discount factors (in the two countries). Assume $h^{\$} = h^{\varepsilon} = 180/360$ and $sp_0^{\$} = 5.36\%$ and $sp_0^{\varepsilon} = 4.46\%$, both of which were fixed at inception on 15 March-01. At $t = 0$ the future fixed cash flows in USD and Euros are:

$$C^{\$} = (sp_0^{\$}.h^{\$}Q^{\$}) = 0.0536 \ (180/360) \ \$140\text{m} = \$3.752\text{m}$$

$$C^{\varepsilon} = (sp_0^{\varepsilon}.h^{\varepsilon}Q^{\varepsilon}) = 0.0446 \ (180/360) \ €100\text{m} = €2.23\text{m}$$

The value of the swap to UncleSam will change if either USD-LIBOR or Euro-LIBOR interest rates change or the spot exchange rate changes. In the swap UncleSam has (fixed) future payments in euros, so a rise in the euro exchange rate against the US dollar will involve UncleSam paying out more USD and hence the USD value of the swap to UncleSam will fall (see Equation 36.1). However, a rise in euro interest rates will reduce the PV of UncleSam's euro payments (Equation 36.3) and hence increase the swap's USD value to UncleSam (Equation 36.1). Finally, if USD-LIBOR rates rise this will reduce the PV of the USD cash inflows and the value of the swap to UncleSam will fall. (The change in the value of the swap to the swap dealer is the opposite of that to UncleSam.)

The swap has zero value at inception on 15 March-01 and it terminates on 15 March-04. Suppose we want to value the swap to UncleSam on 15 September 02, when there are three payments left on 15 March-03, 15 September-03, and 15 March-04. Note that spot interest rates in the two countries and the spot-FX rate will be different on 15 September-02, than at inception of the swap at $t = 0$.

To simplify our calculations we assume that on 15 September the USD-LIBOR curve is flat at 5.5% and the Euro-LIBOR curve is flat at 4.5% (both continuously compounded) and the euro has appreciated from 1.4(\$/£) to $E_t = 1.5$(\$/£). We also assume the number of days between payment dates is exactly ½ year, so the new discount rates are $d_i = \exp(-r_i.t_i)$ where $t_i = 1/2$, 1 and 3/2.[2]

$$\begin{aligned} V^{\$}_{Fix}(t) &= \$3.752\text{m}\,[e^{-0.055(1/2)} + e^{-0.055(1)} + e^{-0.055(3/2)}] + \$140\text{m}\; e^{-0.055(3/2)} \\ &= \$3.752\text{m}\,[0.9729 + 0.9465 + 0.9208] + \$140\text{m}[0.9208] \\ &= \$139.5684\text{m} \end{aligned} \tag{36.4}$$

$$\begin{aligned} V^{\varepsilon}_{Fix}(t) &= €2.23\text{m}\,[e^{-0.045(1/2)} + e^{-0.045(1)} + e^{-0.045(3/2)}] + €100\text{m}\; e^{-0.045(3/2)} \\ &= €2.23\text{m}\,[0.9778 + 0.9560 + 0.9347] + €100\text{m}\;\;[0.9347] \\ &= €99.8668\text{m} \end{aligned} \tag{36.5}$$

$$V_{sp}(t) = V^{\$}_{Fix}(t) - E_t\;\; V^{\varepsilon}_{Fix}(t) = -\$10.2317\text{m} \tag{36.6}$$

Because of the rise in the euro since the initiation of the swap, the value of the swap to UncleSam has fallen because the value of the fixed cash payments UncleSam has to make in euros, require higher dollar payments.

36.3.2 Currency Swap as a Strip of Forward Contracts

'UncleSam' receives annual USD receipts of \$3.752m and the principal of \$140m at the end of the swap and pays out €2.23m annually with a repayment of principal of €100m at the termination of the swap. This is a series of forward contracts, to receive USD and pay Euros at each reset date t_i. The value of one of these forward cash flows in USDs is:

$$\text{USD} = (C^{\$} - F_i\;\; C^{\varepsilon}) \tag{36.7}$$

[2]The assumption of continuously compounded interest rates just make the calculations of the discount factors easier and more transparent than if we use the more complex day-count conventions and (simple) LIBOR rates. Of course, if we correctly switch between (simple) LIBOR rates and continuously compounded rates, we get the same value for the discount rates. Hence, not surprisingly, we obtain the same value for the swap, whatever interest rate conventions we use.

where F_i is the USD-Euro forward rate quoted on 15 September-02 for delivery at time t_i (6, 12, and 18 months ahead). In addition, the two principals $Q^{\varepsilon} = €100\text{m}$ and $Q^{\varepsilon} = \$140\text{m}$ are exchanged at maturity of the swap. The forward FX rates today (at time t) for delivery at horizons t_i are:

$$F_i = E_t \ e^{(r^{\$}-r^{\varepsilon})t_i} \tag{36.8}$$

The USD cash flows in (36.7) accruing at times t_i must be discounted back to time t, using USD risk-free rates, so each net cash flow has a present value in USD of:

$$\text{USD} = (C^{\$} - F_i C^{\varepsilon})e^{-r^{\$} t_i} \tag{36.9}$$

The spot rate on 15 September-02 is $E_t = 1.5(\$/€)$, interest rates are $r^{\$} = 5.5\%$ and $r^{\varepsilon} = 4.5\%$ so forward FX-rates on 15 September-02 are $F_1 = 1.507519$ $(= 1.5e^{0.01(1/2)})$, $F_2 = 1.515075$ $(= 1.5e^{0.01(1)})$ and $F_3 = 1.522670$ $(= 1.5e^{0.01(3/2)})$. The value of the 'receive-USD, pay-Euros' swap is given by the sum of the terms in (36.9) discounted at the USD interest rate $r^{\$} = 5.5\%$):

$$\begin{aligned}
[\$3.752\text{m} - F_1\ €2.23\text{m}]e^{-0.055(1/2)} &= \$379{,}647\\
[\$3.752\text{m} - F_2\ €2.23\text{m}]e^{-0.055(1)} &= \$353{,}401\\
[\$3.752\text{m} - F_3\ €2.23\text{m}]e^{-0.055(3/2)} &= \$328{,}219\\
[\$140\text{m} - F_3\ €100\text{m})]e^{-0.055(3/2)} &= -\$11{,}295{,}594\\
\text{Total} &= -\$10{,}234{,}327
\end{aligned}$$

So the receive-USD pay-Euros swap on 15 September-02 has a value to 'UncleSam' of *minus* \$10,231,702, which is the same as that found by considering the swap as a long-short bond position (see Appendix 36.B for an algebraic proof). Note that the change in value of the swap is mainly due to the change in the (present) value of the final payments of principal after the change in the exchange rate from 1.4(\$/€) on 15 March-01 to 1.5(\$/€), on 15 September-02. This large change in value would not occur in a plain vanilla *interest rate* swap because the principals are not exchanged (and its value does not depend on the exchange rate).

36.4 SUMMARY

- A plain vanilla 'fixed-fixed' currency swap involves the exchange of principals in two different currencies at the beginning and end of the swap *and* an exchange of foreign cash flows (Euros) for domestic cash flows (USD), over the life of the swap.
- A currency swap must have zero value at inception – this is how we price the swap. We ensure that the swap rate in domestic currency (USD) and the swap rate in the foreign currency (Euros) give a zero value (in USD or Euros) at inception for the currency swap.

- The two swap rates in a *fixed-fixed currency swap* are the same as the swap rates on *fixed-for-floating vanilla interest rate swaps* in the respective countries.
- Cash flows in a fixed-fixed currency swap are equivalent to taking a long and short position in a domestic and foreign bond, respectively. This 'synthetic swap' enables one to calculate the *value of a currency swap* during the remaining life of the swap. Equivalently, the swap can be viewed as a strip of forward-FX contracts. Both methods give the same market value for the swap, at any time after inception.
- Changes in interest rates in either country but especially changes in the exchange rate, lead to a change in the (mark-to-market) value of a currency swap.
- Swap dealers (usually banks) take on one side of a swap contract and they will try and match the cash flows incurred, with other swap counterparties.
- If the swap dealer cannot immediately find a matching counterparty to a currency swap she can hedge her net position in any currency. She can use currency futures contracts to hedge exchange rate risk and interest rate futures to hedge the interest rate risk in the two countries (which change the present value of the swap). The main risk to a swap dealer who does not have a matched swaps book in each currency comes from volatile exchange rates.

APPENDIX 36.A: PRICING A CURRENCY SWAP

Consider a currency swap to pay-fixed USD and receive-fixed Euros. How do we price this currency swap? Surprisingly, the swap rates for the fixed-fixed currency swap are equal to the swap rates for plain vanilla *interest rate* swaps in the two countries:

> *If the current (at-market) swap rate in a plain vanilla fixed-for-LIBOR interest rate swap in the US is $sp^{\$}$ and sp^{ε} is the equivalent rate for the Euro-zone interest rate swap, then these swap rates will correctly price a fixed-fixed currency swap.*

The notation which follows may look a little complex but all we are doing is calculating the present values for a fixed interest rate bond in USD and in Euros. The (present) value of the USD-leg and Euro-leg of a fixed-fixed currency swap (in their respective currencies) are:

$$V^{\$}_{Fix} = \sum_{i=1}^{n} C^{\$}_i d^{\$}_i + Q^{\$} \ d^{\$}_n \tag{36.A.1}$$

$$V^{\varepsilon}_{Fix} = \sum_{i=1}^{n} C^{\varepsilon}_i d^{\varepsilon}_i + Q^{\varepsilon} \ d^{\varepsilon}_n \tag{36.A.2}$$

The USD coupon payment is $C^{\$}_i = (sp^{\$}.h^{\$}_i Q^{\$})$, where $h^{\$}_i$ = day-count convention for the fixed-leg and the discount factor is $d^{\$}_i = 1/[1 + r^{\$}_i.t^{\$}_i]$, based on the USD-LIBOR rate,

$r_i^\$ = LIBOR_i^\$$ and $t_i^\$ = days_i^\$/360$ is the actual number of days from 0 to t_i. These definitions also apply to the Euro side of the swap which is based on the Euro-LIBOR term structure. At inception we know that the two present values in the currency swap must be equal (expressed in a common currency, here USD), otherwise the swap would not take place:

$$V_{Fix}^\$ = Q^\$ \left[sp^\$ \sum_{i=1}^{n} h_i^\$ d_i^\$ + d_n^\$ \right] = E_0 V_{Fix}^\varepsilon = E_0.Q^\varepsilon \left[sp^\varepsilon \sum_{i=1}^{n} h_i^\varepsilon d_i^\varepsilon + d_n^\varepsilon \right] \quad (36.A.3)$$

where E_0 is the USD-Euro spot exchange rate at inception of the swap. At the outset of the swap the notional principals are exchanged. Suppose the US firm wants to initially have $Q^\varepsilon =$ €100m funds available (e.g. to invest in a new plant in Europe) and the current exchange rate is $E_0 = 1.4(\$/£)$ so the US firm initially borrows $Q^\$ = Q^\varepsilon.\, E_0 = \$140$m (from Citibank, say). Substituting $Q^\$ = Q^\varepsilon.\, E_0$ in (36.A.3) we find that for the value of the swap to be zero we require:

$$sp^\$ \sum_{i=1}^{n} h_i^\$ d_i^\$ + d_n^\$ = sp^\varepsilon \sum_{i=1}^{n} h_i^\varepsilon d_i^\varepsilon + d_n^\varepsilon \quad (36.A.4)$$

The next bit of the argument is quite subtle and requires us to remember the formulas for vanilla *interest rate* swaps (i.e. fixed-for-floating) for each 'country' taken in turn. The left-hand side of (36.A.4) is the present value of cash flows from the fixed-leg of a vanilla fixed-for-floating *interest rate* swap in the US (per \$1 notional principal), with a current at-market swap rate of $sp^\$$. The value of the *fixed leg* in this swap must equal the value of the floating USD-LIBOR leg which we know equals \$1. So the left-hand side of (36.A.4) is equal to \$1. Hence a US vanilla *interest rate swap* with a principal of $Q^\$ = \$140$m has a present value of \$140m.

The right-hand side of (36.A.4) are cash flows from the fixed leg of a vanilla fixed-for-floating *interest rate* swap in the Euro-zone (per €1 notional principal) with a swap rate of sp^ε and we know that this fixed leg has a present value of €1. Hence a Euro *interest rate swap* with a principal of $Q^\varepsilon =$ €100m would have a present value of €100m. But €100m today is worth $E_0 Q^\varepsilon = \$140$m.

Hence, as long as we use the current at-market swap rates on vanilla interest rate swaps in the US and the Euro-zone and use these same swap rates in the fixed-fixed *currency swap,* both parties will enter the currency swap because the (present) value of the USD fixed leg of \$140m equals the present value of the Euro leg of \$140m ($= E_0 Q^\varepsilon$).

Hence, the at-market vanilla *interest* swap rates $sp^\$$ and sp^ε make the value of the two legs of the *currency* swap equal (in USD terms) and hence correctly price the fixed-fixed currency swap. However, if one party wants to use a different swap rate to $sp^\$$ or sp^ε in the currency swap then $V_{Fix}^\$ - E_0 V_{Fix}^\varepsilon$ would not be zero. There would then have to be an up-front exchange of cash at inception of the currency swap given by the difference $V_{Fix}^\$ - E_0 V_{Fix}^\varepsilon$, where the cash flows in each currency use the 'new' values of $sp^\$$ and sp^ε decided by the two counterparties.

APPENDIX 36.B: VALUATION OF A CURRENCY SWAP

We show that two different synthetic portfolios can be used to value a fixed-fixed currency swap: (i) a bond portfolio and (ii) a strip of forward FX contracts. Both give the same value for the swap.

Suppose a US (domestic) swap dealer has a 'receive-USD, pay-Euro' swap, which has been in existence for some time and we are valuing the swap at time t. The swap rates in the two currencies were determined at $t = 0$ and these determine the *remaining* fixed payments and we assume there are n payments remaining.

Currency Swap as a Synthetic Bond Portfolio

The US swap dealer can be considered to be long a US bond with fixed coupon receipts $C^{\$}_{X,i} = (sp^{\$}_0 h^{\$} Q^{\$})$ and short a foreign bond with payments $C^{\varepsilon}_{X,i} = (sp^{\varepsilon}_0 h^{\varepsilon} Q^{\varepsilon})$ and in addition, the principals $Q^{\$}$ and Q^{ε} are exchanged at the end of the swap ($t = n$). Spot interest rates (continuously compounded) are all measured from time t and the discount factors are $d_i = \exp(-r_i.t_i)$. The USD and Euro bonds have present values at t:

$$V^{\$}_{Fix}(t) = C^{\$} \sum_{i=1}^{n} e^{-r^{\$}_i t_i} + Q^{\$} \;\; e^{-r^{\$}_i t_i} \tag{36.B.1}$$

$$V^{\varepsilon}_{Fix}(t) = C^{\varepsilon} \sum_{i=1}^{n} e^{-r^{\varepsilon}_i t_i} + Q^{\varepsilon} \;\; e^{-r^{\varepsilon}_n t_n} \tag{36.B.2}$$

Converting from Euros into USD, at the current spot rate E_t gives the value of the swap at t:

$$V_{sp}(t) = V^{\$}_{Fix}(t) - E_t V^{\varepsilon}_{FL}(t) \tag{36.B.3}$$

$$V_{sp}(t) = C^{\$} \sum_{i=1}^{n} e^{-r^{\$}_i t_i} + Q^{\$} \;\; e^{-r^{\$}_i t_i} - E_t \left[C^{\varepsilon} \sum_{i=1}^{n} e^{-r^{\varepsilon}_i t_i} + Q^{\varepsilon} \;\; e^{-r^{\varepsilon}_n t_n} \right] \tag{36.B.4}$$

For example, in the main text Equations (36.4) to (36.6) give $C^{\$}_i = \$3.752$m, $C^{\varepsilon}_i = €2.23$m, $V^{\$}_{Fix}(t) = 139{,}586{,}430$, $V^{\varepsilon}_{Fix}(t) = 99{,}866{,}755$ and $V_{sp}(t) = -\$10{,}231{,}702$.

Currency Swap as a Series of Forward Contracts

At each payment date there is a receipt of USD, $C^{\$}_i = (sp^{\$}_0 h^{\$} Q^{\$})$ and a payment of Euros, $C^{\varepsilon}_i = (sp^{\varepsilon}_0 h^{\varepsilon} Q^{\varepsilon})$ with the latter being worth $(F_i C^{\varepsilon}_i)$ in USD, at time t_i. The PV of these USD net cash flows *discounted at USD interest rates* is:

$$\$(C^{\$}_i - F_i \;\; C^{\varepsilon}_i) e^{-r^{\$}_i t_i} \tag{36.B.5}$$

where today t, the forward rate for delivery at t_i is $F_i = E_t \; e^{(r^{\$}-r^{\varepsilon})t_i}$. Hence the USD value of all net cash flows to the 'receive-USD pay-Euros', viewed as a strip of forward contracts is:

$$\left[\sum_{i=1}^{n}(C_i^{\$} - F_i \; C_i^{\varepsilon}) + (Q^{\$} - F_n \; Q^{\varepsilon})\right] e^{-r_i^{\$} t_i} \tag{36.B.6}$$

It is easy to see that (36.B.4) and (36.B.6) are equivalent by substituting $F_i = E_t \; e^{(r^{\$}-r^{\varepsilon})t_i}$ in (36.B.4).

EXERCISES

Question 1
You are a US firm with a fixed-rate bank loan in sterling (GBP) with reset dates every 6 months for the next 5 years. How can you hedge your FX position using a currency swap?

Question 2
A US swap dealer has *a* series of net (GBP) fixed-sterling payments of £10m p.a. (and USD fixed receipts), over each of the next 5 years. What are the risks to the mark-to-market value of the swap dealer's position (apart from counterparty risk)?

Question 3
Consider a receive-USD, pay-Euros currency swap with a fixed swap-rate of $sp_E = 3\%$ p.a. in Euros and $sp_{US} = 5\%$ p.a. fixed in USDs.

At inception the swap principals in the two currencies were $\$Q = \$100$m and $€Q = €90$m. The tenor is annual (with an exchange having just taken place). The swap has been in existence for some time and currently has 2 more years to maturity. The current exchange rate is $S = 1.0$ Euros per USD. The current term structure of interest rates is flat in the United States and Euroland and the current interest rate $r_{us} = 6\%$ p.a. while the Euro interest rate is $r_E = 3\%$ p.a. (both continuously compounded).

Consider the swap as a *portfolio of bonds*. What is the market value of the receive-USD, pay-Euros swap?

Question 4
Consider a receive-USD, pay-Euros, currency swap with a fixed swap-rate of $sp_E = 3\%$ p.a. in Euros and $sp_{US} = 5\%$ p.a. fixed in USDs (both continuously compounded).

At inception the swap principals in the two currencies were $\$Q = \$100$m and $€Q = €90$m. The tenor is annual (with an exchange having just taken place). The swap has been in existence for some time and currently has 2 more years to maturity. The current exchange rate is $S = 1.0$ Euros per USD. The current term structure of interest rates is flat in the United States and

Euroland and the current interest rate $r_{us} = 6\%$ p.a. while the Euro interest rate is $r_E = 3\%$ p.a. (both continuously compounded).

(a) Calculate the 1-year and 2-year forward exchange rates.

(b) Consider the swap as a portfolio of *forward contracts*. What is the market value of the receive-USD, pay-Euros swap?

Question 5

A currency swap for Australian dollars and US dollars, with annual payments has a remaining life of 15 months with the next two payments in 3 months and 15 months. The fixed swap rates are $sp_{AUS} = 6\%$ p.a. and $sp_{US} = 4\%$ p.a. and the principals are AU\$50m and US\$30m.

The term structure of interest rates in both countries is flat with $r_{us} = 8\%$ p.a. and $r_{Aus} = 10\%$ p.a. (continuously compounded). The current exchange rate is $S = 0.65$ USD/AUD.

Consider the swap as a *portfolio of bonds*. What is the value of the swap to the party receiving USDs and paying AUDs?

Question 6

A currency swap for Euros and USDs, with annual payments, has a remaining life of 2 years. The fixed swap rates are $sp_E = 8\%$ p.a. and $sp_{US} = 11\%$ p.a. (simple interest) and the principals are €10m and US\$10m.

The term structure of interest rates in both countries is flat with $r_{US} = 11\%$ p.a. and $r_E = 8\%$ p.a. (continuously compounded). The current exchange rate is $S = 0.909090$ USD/Euro.

Consider the swap as a *portfolio of domestic and foreign bonds*.

What is the value of the receive-USD, pay-Euro swap?

CHAPTER 37

Equity Swaps

Aims

- To demonstrate how equity swaps allow an investor to gain temporary exposure to the stock market without actually buying or selling the stocks in her existing portfolio. The investor can then revert to her original stock portfolio at the termination of the swap.
- To describe how an 'asset swap' may involve a single equity, a basket of equities or a market index (e.g. S&P 500), either in the same currency or in different currencies – in the latter case the investor has to decide on whether to hedge the currency risk.
- To use arbitrage arguments to price an equity-for-LIBOR swap. Somewhat counter-intuitively receiving the return on equity and paying LIBOR is a fair swap, since the present values of the two legs are equal.
- To show how to price a domestic equity-for-domestic equity swap, or a swap of the return on a domestic equity for the return on a foreign equity.
- To demonstrate how to value an equity swap after inception of the swap. The value of an equity swap to one of the parties may be positive or negative and its value depends on the change in equity prices.

Equity swaps allow an investor to change her existing stock market risk without necessarily owning stocks or changing the amount of stocks she already holds. An equity swap is an OTC transaction which involves cash flows based on a notional principal amount. One leg of an equity swap is based on the return on a specific equity 'portfolio' such as a single stock, a basket of stocks or a stock index (e.g. S&P 500). The other leg of the swap often has cash flows based on an interest rate, either fixed or floating (LIBOR). The principal in the swap is not exchanged, hence the term 'notional principal'. Often the notional principal is fixed over the life of the equity swap but some equity swaps have a variable notional principal.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

A 'cross-currency equity swap' involves two different currencies, for example a US-based investor swapping USD-LIBOR for the return on the FTSE 100 stock index. The currency risk in the swap can be hedged as part of the swap deal.

An equity swap may also involve cash flows based on two different equity indices (i.e. 'two-index swaps' or 'relative performance swaps'). There are also 'outperformance swaps' where the equity leg may involve the maximum return on two or more equity indices or two or more baskets of equities.

Equity swaps can provide a low cost method of altering your current portfolio. Sometimes these cost savings arise from avoidance of withholding taxes or from reductions in transactions costs in illiquid markets or to get round certain market restrictions (e.g. short selling).

For example, a US investor who directly buys foreign stocks may pay a withholding tax (of say 30%) on dividends received – but an equity swap on a total return index avoids such a tax. Direct purchase of stocks in emerging markets may involve high transaction cost and credit risk of the counterparty (e.g. local brokerage firm), whereas equity swaps provide a way to minimise such costs, while securing exposure to emerging markets. If you hold stocks in a company in which you work and these have increased in value, you may want to lock in this gain – to do so you can use an equity swap where from today, you pay the return on your company stocks and receive a fixed rate of interest (from the swap dealer).

An 'equity linked deposit' offers a floor to the value of your investment but also pays out a fraction of any upside on a market index (e.g. S&P 500) – this is a structured product. A bank can 'structure' this product by purchasing a put on the S&P 500 (which sets a floor) and also buying an equity swap to receive the return on the S&P 500 (and pay a fixed rate of interest).

37.1 EQUITY-FOR-LIBOR: FIXED NOTIONAL PRINCIPAL

Equity swaps allow a portfolio manager (Ms Bright) to alter their exposure to the equity market without owning or transferring the equities she already holds. For example, suppose Ms Bright is holding (long-maturity) floating rate notes (FRNs) with a principal of $Q = \$100\text{m}$ and is normally quite happy with this portfolio. However, suppose she feels that the US stock market (S&P 500) will do rather better than usual over the next year and would like exposure of $50m to the US stock market. After one year she feels the stock market rally will be over and then wishes to return to her long-run desired portfolio of $100m in FRNs.

Ms Bright could accomplish this strategy by today (10 January) selling $50m of her FRNs, switching into an S&P 500 tracker fund (or ETF) and then reversing these trades one year later. This might entail high transactions costs – brokerage fees, commissions, bid–ask spreads, adverse market impact on prices etc. Instead, suppose Ms Bright enters a 'receive-equity returns, pay-LIBOR' swap on a notional principal of $Q/2 = \$50\text{m}$ with net cash flows paid every 3 months (the tenor in the swap). Each 3-month period, the cash flow from her portfolio of $\$Q = \$100\text{m}$ in FRNs *plus* her swap position is:

$$\begin{aligned}\text{Cash flow (3 months)} &= QL\ \ (days/360) + (Q/2)\ \ [R_{S\&P} - L(days/360)] \\ &= (Q/2)L(days/360) + (Q/2)R_{S\&P}\end{aligned}$$

TABLE 37.1 Ex-post cash flows, receive equity return and pay LIBOR

Notional principal = 100,000,000

Tenor: 3 months

Day-count LIBOR: days/360

Days in year: 360

Time	Days	LIBOR	S&P 500		Pay LIBOR	Receive S&P 500	Net receipts Swap
			Index	Return			
10 Jan-01			1,922				
10 April-01	91	5%	2,041	6.19%	1,263,889	6,191,467	4,927,578
10 July-01	91	5.35%	2,137	4.70%	1,352,361	4,703,577	3,351,216
10 Oct-01	92	5.50%	2,163	1.22%	1,405,556	1,216,659	−188,897
10 Jan-02	92	4.95%	2,268	4.85%	1,265,000	4,854,369	3,589,369

where $R_{S\&P}$ is the actual return *over 3 months* on the S&P 500 index and $L(days/360)$ is the *LIBOR* rate over the same 3-month period. (The *LIBOR* rate is fixed at time *t* and the payments take place '*days*' later.) Using the swap, Ms Bright has 'synthetically' changed the composition of her portfolio, which is now equivalent to holding \$50m in FRNs and having \$50m invested in the S&P 500. The equity swap will have lower transactions costs than directly buying and selling (an ETF on) the S&P 500 stock index and the FRNs – simply because the swap dealer is very efficient at structuring this swap – specialisation in capitalist economies tends to lower 'costs of production'.

Suppose the above equity swap is negotiated on 10 January-01, on a notional principal of \$100m when LIBOR is 5%. The first LIBOR payment 91 days later (10 April) will be (see Table 37.1):

$$\text{LIBOR cash flow} = \$100\text{m}\ (0.05)\ (91/360) = \$1{,}263{,}889.$$

Suppose the rise in the S&P 500 over this 90-day period is 6.19% so the equity cash flow (on 10 April) is:

$$\text{Equity cash flow} = \$100\text{m}\ \ (0.0619) = \$6{,}191{,}467.$$

Hence, Ms Bright who 'receives the S&P 500 and pays LIBOR' would have net receipts on 10 April of \$4,927,578 from the equity swap. Hence the portfolio manager receives (pays out) cash when the 3-month return on the S&P 500 index is higher (lower) than (3-month) LIBOR. Note that between April and July the S&P 500 *falls* and the portfolio manager (Ms Bright) *pays out* based on the fall in the S&P 500 and also *pays out* on LIBOR.

The net receipts from the equity swap clearly depend on the performance of the S&P 500 relative to LIBOR and the portfolio manager could win or lose on the swap, in terms of her net cash receipts over the life of the swap. But this is somewhat immaterial. The purpose of the equity swap is not to hedge her \$100m FRN portfolio but to synthetically transform it so she *replicates* a \$50m exposure to FRNs but also has a \$50m exposure to risky equity returns.

If the initial portfolio had consisted of fixed-coupon bonds then Ms Bright's equity swap might have involved paying a fixed rate of interest and receiving the return on the S&P 500 on a \$50m notional principal. This is an '*equity-for-fixed interest rate swap*'.

37.1.1 Double Exposure

Consider again the US portfolio manager (Ms Bright) currently holding $Q =$ \$100m in FRNs. Suppose over the next year, Ms Bright now would like a \$25m $(= Q/4)$ exposure to the S&P 500 and a \$25m $(= Q/4)$ exposure to the Russell 2000 index,[1] leaving a \$50m $(= Q/2)$ exposure to LIBOR. To achieve this she could agree two swaps with a life of 1 year and with cash flows determined every 3 months (say):

(a) *receive the return on S&P 500 and pay LIBOR on a notional of* $Q/4$
(b) *receive the return on Russell 2000 and pay LIBOR on a notional of* $Q/4$

The original portfolio of FRNs plus these two swaps results in net cash flows each period of:

$$= Q.L(days/360) + (Q/4)[(R_{S\&P} - L(days/360)] + (Q/4)[R_{Russell} - L(days/360)]$$
$$= (Q/2)\ \ L(days/360) + (Q/4)R_{S\&P} + (Q/4)R_{Russell}$$

Ms Bright has used the equity swap to synthetically change the composition of her existing portfolio, for a 1-year period. Note also that only a *single* OTC deal with one swap dealer would be required, since the swap dealer would 'bundle up' the two separate swaps. This is an example of *structured finance* in the OTC market.

37.2 UNHEDGED CROSS-CURRENCY EQUITY SWAP

This type of equity swap enables a US investor (Ms Bright) to gain exposure to foreign stocks (FTSE 100), including taking on exchange rate risk if she wishes. The alternative to using the swap would be to purchase the foreign equity directly but again this could involve high transactions cost (e.g. transactions taxes and possible dividend withholding taxes, bid–ask spreads etc.). Instead, Ms Bright, who is holding $Q =$ \$100m in FRNs, could agree to receive payments

[1]The Russell 2000 Index is a small-cap stock market index of the smallest 2,000 market-cap stocks in the Russell 3000 Index. The Russell 2000 is the most common benchmark for 'small cap' stocks. The S&P 500 index comprises large capitalisation stocks.

based on the return on the FTSE 100 (say) and pay 3-month US-LIBOR, on a notional principal of Q = \$100m. The expiry date of the swap might be in 1 year's time, with a tenor of 3 months. After entering the swap Ms Bright effectively has \$100m invested in both the FTSE 100 and the GBP-USD exchange rate, over the next year.

Suppose the GBP-USD exchange rate is currently $E_0 = 1.5$ (\$/£). Then \$100m is equivalent to £66.667m. If $R_{FTSE} = 4\%$ over the next 3 months then the sterling (GBP) investment increases to £69.333m but if sterling also depreciates by 2% over the next 3 months (to $E_1 = 1.47$ \$/£) then the USD value of the UK investment is \$101.92m (= £69.333 × 1.47), an increase of about 2%.

The swap is unhedged, so although the FTSE 100 increases by 4%, the USD swap receipts only increase by around 2%, because of the fall in sterling (GBP). Ms Bright will receive \$1.92m from the swap dealer at the first payment date (and will pay US-LIBOR). The net cash flows from the FRN *plus* the unhedged cross-currency swap are:

$$\begin{aligned} \textit{Net cash flows (USD)}^2 &\approx \$QL(days/360) + \$Q[R_{FTSE} + R_{FX}^{(\$/£)} - L(days/360)] \\ &= \$Q[R_{FTSE} + R_{FX}^{(\$/£)}] \end{aligned}$$

where $R_{FX}^{(\$/£)} = (E_1/E_0) - 1$ is the 'return' on the USD-Sterling exchange rate and is positive (negative) if sterling appreciates (depreciates) against the USD. Broadly speaking the effect of the swap is to synthetically 'convert' the \$100m FRN portfolio into a \$100m portfolio whose return depends on both the performance of the FTSE 100 and the USD-Sterling exchange rate.[3]

37.3 HEDGED CROSS-CURRENCY EQUITY SWAP

This works in a similar way to the unhedged swap except that at inception of the swap a fixed exchange rate is agreed (e.g. $E_0 = 1.5$ USD/GBP) which will apply to all currency conversions at each payment date and also to the notional principal. Effectively this means that there is no FX risk. Since $E_0 = 1.5$ USD/GDP, a \$100m notional principal in the swap amounts to £66.667m sterling. If $R_{FTSE} = 4\%$ over the next 3 months this becomes £69.333m. But at $t = 1$ this is converted at the agreed fixed exchange rate of 1.5 USD/GDP to give \$103.99 – a 4% increase on the initial USD investment of \$100m. The net cash flows from the initial holding of \$$Q$ in the FRN and the equity swap are:

$$\textit{Net cash flows } (USD) \approx \$(QR_{FTSE})$$

[2]First calculate the USD value at $t = 1$: $V_1^{\$} = V_1^{£}E_1 = V_0^{£}(1 + R_{FTSE})E_1 = (\$Q/E_0)(1 + R_{FTSE})E_1$. But $E_1/E_0 = 1 + R_{FX}^{\$/£}$ hence $V_1^{\$} = \$Q(1 + R_{FTSE})(1 + R_{FX}^{\$/£})$. Ignoring cross-product terms this becomes $\$Q[R_{FTSE} + R_{FX}^{(\$/£)}]$.

[3]If a *variable* notional principal is used then the starting point for calculating the sterling payments over the *second period* is £69.333m (not £66.667m).

Even if sterling falls to 1.47 ($/£) after 3 months, this does not affect the conversion from sterling to USD. The hedged cross-currency equity swap locks in the percentage return on the FTSE (in USD) regardless of what happens to the USD-GBP exchange rate.[4]

37.3.1 Hedging by the Swap Dealer

Although, in the above example, Ms Bright achieves her desired portfolio by using the equity swap (and hence presumably her desired risk-return trade off), the same cannot be said for the swap dealer. Clearly, in the above examples the swap dealer is exposed to risk due to US interest rates, the S&P 500, the FTSE 100 and the USD-Sterling exchange rate. These positions can be hedged using a 'strip' of appropriate futures contracts (i.e. a futures contract for each prospective reset date in the swap). The hedged cross-currency swap with the FTSE 100 involves both equity risk and exchange rate risk for the *swap dealer*, which can be hedged using stock index futures and currency futures. Of course, the swap dealer will net out all the positions in the whole of her swaps' book before initiating the different hedge positions. Alternatively, the swap dealer could hedge the *present value* of her swap positions – again using equity, interest rate and currency futures (see below for the calculation of the PV of equity swaps).

37.4 PRICING EQUITY SWAPS

In this section we:

- Price an equity-for-LIBOR swap and find the (present) value of this swap at any point during the life of the swap.
- Price an equity-for-fixed interest rate swap and find the value of this swap at any point during the life of the swap.
- Discuss the use of variable notional amounts in swap contracts.
- Price and then value an equity-for-equity swap, where the equities are in the same currency.
- Price and then value a cross-country, equity-for-equity swap, where the equities are in different countries.

37.4.1 Equity-for-LIBOR Swap: Fixed Notional

37.4.1.1 Pricing

Equity swaps with a fixed notional principal can be priced using arbitrage arguments. In an equity-for-LIBOR swap:

A fair swap rate is the exchange of the equity return for LIBOR at each payment date.

[4]We convert the sterling proceeds at $t = 1$ using the constant exchange rate E_0, hence as before $V_1^{\$} = V_1^{£} E_1 = V_0^{£}(1 + R_{FTSE})E_1 = (\$Q/E_0)(1 + R_{FTSE})E_1 = \$Q(1 + R_{FTSE})$ since $E_1 = E_0$ given the terms of the swap contract.

The above conclusion should initially seem counter-intuitive because on average equity returns are higher than LIBOR, so one might think that receiving the equity return and paying LIBOR is advantageous. But arbitrage arguments rule this out. (It's a bit like pricing stock options using arbitrage arguments, where we also find that the option premium does not depend on the 'real world' mean growth rate of the stock price.) In pricing this swap we assume any dividends on the stock or stock index are paid only at (the swap) payment dates or alternatively we assume the 'equity' is a (traded) 'total return' equity index.[5]

Essentially, the arbitrage argument replicates the cash flows in this swap by borrowing \$1 at LIBOR and investing this \$1 in equity at $t_0 = 0$ and at each future reset date. The present value of all of these \$1 investments in equity can be shown to equal the present value of \$1 invested at LIBOR. Hence a swap of the equity return for LIBOR is a 'fair deal' as both sides of the swap have the same present value. We now examine this argument in more detail.

Consider an investor who receives at time t_i the *per period* equity return z_i[6] between t_{i-1} and t_i and pays $LIBOR_i$ $(= r_i)$ plus a fixed spread s. The tenor for the LIBOR payments is $m_i = days_i/360$ and the *LIBOR* rate r_i is fixed at t_{i-1} with payment at t_i. Hence net receipts in the swap (on a notional principal of \$1) on any future payment dates $t_i = 1, 2, \ldots n$ are:

$$\text{Net receipts at } t_i = [1 + z_i] - [1 + (r_i + s)m_i] \tag{37.1}$$

To price this swap we need to find the value of s for which the *present value* of the net receipts is zero. Consider investing \$1 in equities or \$1 at *LIBOR* at each of the times $t_0, t_1 \ldots$ and t_{n-1}. This results in replicating the cash flows on the equity side and the *LIBOR* sides of the swap and we show that the PV (at $t = 0$) for these equity and *LIBOR* investments are equal when $s = 0$. So an exchange of equity returns for *LIBOR* (flat) is a 'fair swap' that both parties are willing to enter.

The economic intuition is as follows. Investors must be indifferent between the returns on two different investments *in risk-adjusted* terms otherwise they would buy one asset and sell the other, until the returns are equalised. So in risk-adjusted terms an investment in LIBOR or in individual equities or an equity index must be equivalent. Hence a swap of LIBOR (flat) for equity returns, or returns on one equity-A for returns on equity-B or returns on equity-A for returns on an equity index are all 'fair swaps'. Let's consider this for a swap to receive equity returns and pay LIBOR (flat).

Consider investing \$1 at $t = 0$ in an equity portfolio. This will be worth $\$1[1 + z_1]$ at $t = 1$, with $PV = \$1$ at $t = 0$ (discounting using the equity return). Similarly \$1 invested at *LIBOR* at $t = 0$ is worth $\$1\ [1 + r_1 m_1]$ at $t = 1$, with present value at $t = 0$ of \$1 (discounting at LIBOR).

Now move to $t = 1$ and invest \$1 in equity which will be worth $\$1[1 + z_2]$ at $t = 2$, which has a PV at $t = 1$ of \$1 and a PV *at* $t = 0$ of $(\$1 d_1)$, where d_1 = discount factor that applies between t_0 and t_1. Similarly, the cash flow from a \$1 investment at $t = 1$ *at LIBOR* gives \$1 $[1 + r_2 m_2]$ at $t = 2$, with PV *at* $t = 1$ of \$1 and PV at $t = 0$ of $(\$1 d_1)$ – the same PV as for the

[5]The percentage return on a 'total return index' equals the capital gain (percentage price change) plus the dividend yield. We also assume that the swap rate does not depend on the volatility of the stock return.
[6]Note that $(1 + z_1) \equiv S_1/S_0$ where S = price of stock (or value of equity index).

TABLE 37.2 Equity for LIBOR, cash flows and PV

Time	Equity return	PV (equity return) at $t = 0$	LIBOR payments	PV (LIBOR payments) at $t = 0$
$t = 0$				
$t = 1$	$1 + z_1$	1	$1 + r_1 m_1$	1
$t = 2$	$1 + z_2$	d_1	$1 + r_2 m_2$	d_1
$t = 3$	$1 + z_3$	d_2	$1 + r_3 m_3$	d_2
...	...	...	...	...
$t = n$	$1 + z_n$	d_{n-1}	$1 + r_n m_n$	d_{n-1}

equity investment. We can repeat this argument for all the other time periods to t_{n-1}. From Table 37.2 we see:

$$PV(t = 0, equity) = 1 + d_1 + d_2 + \dots \quad \dots + d_{n-1} \tag{37.2}$$

$$PV(t = 0, LIBOR) = 1 + d_1 + d_2 + \dots \quad \dots + d_{n-1} \tag{37.3}$$

We have *replicated* the payoffs in the swap by borrowing at LIBOR and using the proceeds to invest in equity and shown that a swap involving the equity return for LIBOR is 'fair', because they have the same present value.

37.4.1.2 Valuation

Although the equity-for-LIBOR swap has zero value at inception, its value can change over time as equity prices and interest rates change. We can easily value the equity swap at any date t, which lies between any two payment dates. For example, if $t_0 < t < t_1$ we show in Appendix 37 that the value of the 'receive-equity returns, pay-LIBOR' is (Figure 37.1):

$$V_t = (S_t/S_0) - d(t, t_1)(1 + r_1 m_1) \tag{37.4}$$

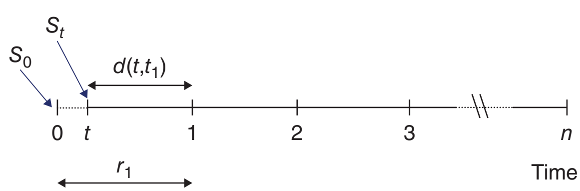

FIGURE 37.1 Value of equity swap at t

TABLE 37.3 Equity-for-fixed-interest, cash flows and PV

Time	Equity return	PV (equity return) at $t = 0$	Fixed interest	PV (fixed interest) at $t = 0$
$t = 0$				
$t = 1$	$1 + z_1$	1	$1 + sp\ m_1$	$d_1(1 + sp\, m_1)$
$t = 2$	$1 + z_2$	d_1	$1 + sp\ m_2$	$d_2(1 + sp\, m_2)$
$t = 3$	$1 + z_3$	d_2	$1 + sp\ m_3$	$d_3(1 + sp\, m_3)$
…	…	…	…	…
$t = n$	$1 + z_n$	d_{n-1}	$1 + sp\ m_n$	$d_n(1 + sp\, m_n)$

Hence the value of an equity-for-LIBOR swap (at $t > 0$) depends on the return on the stock since inception, as well as the (present value at t) of the next LIBOR payment (which is determined by r_1 (the LIBOR rate set at t_0 and payable at t_1).

37.4.2 Equity-for-Fixed-Interest Swap: Fixed Notional

37.4.2.1 Pricing

Consider receiving the equity return and paying a fixed interest rate 'sp' in the swap. We show that the fixed rate in this equity swap is the same as for a plain vanilla fixed-for-floating *interest rate* swap.

> *The swap rate for 'equity-for fixed interest rate' is the same as the 'swap rate' for a fixed-for-floating interest rate swap.*

In the swap the cash flows to equity are the same as above, repeated in columns 2 and 3 in Table 37.3.

The fixed-interest cash flows from \$1 invested at any time $t - 1$ are $(1 + sp.m_t)$, payable at t with present value at t_0 of $d_t(1 + sp.m_t)$, for $t = 1, 2, 3, \ldots, n$, where $d_t \equiv d(0, t)$ the discount factor that applies between $t = 0$ and t. The PV of the net cash flows in the swap are:

$$PV(t = 0, equity) = 1 + d_1 + d_2 + \ldots + d_{n-1} \tag{37.5}$$

$$PV(t = 0, fixed\ \ sp) = d_1(1 + sp.m_1) + d_2(1 + sp.m_2) + \ldots + d_n(1 + sp.m_n) \tag{37.6}$$

Equating these two present values and solving for sp (at $t = 0$):

$$sp = \frac{1 - d_n}{\sum_{i=1}^{n} d_i m_i} \tag{37.7}$$

Surprisingly, the *equity* swap rate in (37.7) is independent of the expected equity return (and the volatility of equity returns) and depends only on the term structure of interest rates. This is counterintuitive but we know it is correct because of the above arbitrage/replication argument.

Hence the swap rate for 'equity-for-fixed-interest rate' is the same as the 'swap rate' quoted on a fixed for floating *interest rate swap*. This should not be too much of a surprise as we have seen that 'equity-for-LIBOR' is a fair swap and we know that a 'fixed-for-floating (LIBOR)' interest rate swap, is also a fair swap. But a 'receive-equity, pay-LIBOR' swap plus a 'receive-LIBOR, pay-fixed interest rate' swap, is equivalent to 'a receive-equity, pay-fixed interest' swap, which must therefore also be a fair swap. The quoted swap rates for a vanilla interest rate swap and the equity-for-fixed interest swap must be equal, otherwise riskless arbitrage opportunities are possible.

37.4.2.2 Valuation

Consider the value of the equity-for-fixed-interest swap at t between any two payment dates (e.g. $t_0 < t < t_1$). The present value of the fixed cash flows at t is the same as above but all discount rates and day-count conventions now start from t (rather than $t_0 = 0$), hence, the fixed leg has:

$$PV_t(\mathit{fixed}, sp_o) = d_{t,1}(1 + sp_0.m_{t,1}) + d_{t,2}(1 + sp_0.m_{t,2}) + \ldots + d_{t,n}(1 + sp_0.m_{t,n}) \qquad (37.8)$$

Now consider the equity cash flows. For the *equity stream* at say t_2, this can be replicated by a \$1 investment at t_1 which gives a payout at t_2 of $(1 + z_2)$. At time t, this \$1 investment at t_1 is worth $d_{t,1}$. This argument applies to any payout after t_2, so the PV of all these \$1 equity investments is:

$$PV_t(\mathit{equity\ cash\ flows\ for}\ \ t \geq 2) = d_{t,1} + d_{t,2} + \ldots + d_{t,n-1}. \qquad (37.9)$$

Note that the last term is $d_{t,n-1}$ since the last \$1 investment to replicate the final payout at t_n takes place at t_{n-1}. Finally we have to deal with the time-t present value of the equity cash flow at t_1, which we have already found to be S_t/S_0. Hence the present value of all the equity cash flows in the swap, at t is:

$$PV_t(\mathit{equity\ cash\ flows}\ \ t_1\ \mathit{to}\ t_n) = (S_t/S_0) + d_{t,1} + d_{t,2} + \ldots + d_{t,n-1} \qquad (37.10)$$

The value at t of the 'receive-equity, pay-fixed interest' is the difference between Equation (37.10) and Equation (37.8), which (after some simplification) is:

$$V_t(\mathit{equity} - \mathit{fixed}) = (S_t/S_0) - d_{t,n} - sp_0[d_{t,1}.m_{t,1} + d_{t,2}.m_{t,2} + \ldots + d_{t,n}.m_{t,n}] \qquad (37.11)$$

Also, the above valuation formula is consistent with the determination of the swap rate sp_0 (Equation 37.7). This can be seen by setting $V_t(equity - fixed) = 0$, replacing all the $d_{t,i}$ by $d_{o,i}$, replacing S_t by S_0 and solving for sp_0.

37.4.3 Equity Swaps with Variable Notional Principals

This type of swap is structured so that the notional principal amount at say t_4 depends on the equity returns *realised* up to t_3. So the principal amount PA_3 applicable at t_3 is:

$$PA_3 = (1 + z_1)(1 + z_2)(1 + z_3) \tag{37.12}$$

Hence the net cash flow received at t_4 for an equity-for-LIBOR or equity-for-fixed interest swap are:

$$\text{Net cash flow, Equity-for-LIBOR at } t_4 = PA_3(1 + z_4) - PA_3(1 + r_4 m_4)$$
$$\text{Net cash flow, Equity-for-fixed at } t_4 = PA_3(1 + z_4) - PA_3(1 + sp_0 m_4)$$

It turns out that the first of the above cash flows can be replicated and it is found that a swap of 'equity-for-LIBOR' with a variable notional principal, is a fair one. But the second cash flow cannot be replicated and to value an 'equity-for-fixed interest' swap with a variable notional principal, we have to make assumptions about the path of equity prices (e.g. they follow a binomial path).[7]

37.4.4 Equity-for-Equity Swap (Same Currency): Fixed Notional

Consider a swap where one party receives the return on the S&P 500 and pays the return on the Russell 3000 index or, alternatively, a swap for the return on Microsoft in exchange for the return on Walmart. This type of swap applies to equity returns in the same currency.

37.4.4.1 Pricing

Using (slight modifications) of the above arguments for 'equity-for-fixed' or 'equity-for-floating' interest rates, it can be shown that equity-for-equity swaps are fair swaps (and hence require no up-front payment at inception).

A swap of equity returns-A for equity returns-B (in the same currency) is a fair swap.

[7] For more details see Chance and Rich (1995).

37.4.4.2 Valuation

Valuation of equity-equity swaps at time t (i.e. between payment dates) also follows similar arbitrage arguments described above. Suppose t lies between t_0 and t_1 then the value of the swap to receive the return on equity-A and pay the return on equity-B is:

$$V_t(\textit{equity-for-equity}) = (S_t^A / S_0^A) - (S_t^B / S_0^B) \tag{37.13}$$

where S_t = value of the respective portfolios/stocks at time t ($t_0 < t < t_1$).

37.4.5 Cross-country Equity-for-Equity Swap: Fixed Notional

37.4.5.1 Pricing

Suppose the swap involves receiving the return on the FTSE 100 (the foreign asset, S^f) and paying the return on the S&P 500 (domestic asset, S^d). The return to investing \$1 in the S&P 500, the domestic asset is:

$$1 + R_d \equiv S_1^{\mathrm{d}} / S_0^{\mathrm{d}} \tag{37.14}$$

The return in USDs, to investing \$1 at $t = 0$ in the FTSE 100 (the foreign asset, with price S^f) is:

$$1 + \text{Return USD, foreign investment} = (1 + R_f)(1 + R_{FX}^{\$/£}) = (S_1^f E_1^{\$/£})/(S_0^f E_0^{\$/£}) \tag{37.15}$$

where $E^{\$/£}$ is the USD-GBP spot FX-rate. To replicate the cash flows in both (37.14) and (37.15) requires \$1 at t_0. Hence the net payoff at t_1 to a cross-currency equity swap is:

$$\begin{aligned} \textit{Net cash flow} \ (\text{at } t_1) &= (1 + R_f)(1 + R_{FX}^{\$/£}) - (1 + R_d) \\ &= (S_1^f E_1^{\$/£})/(S_0^f E_0^{\$/£}) - (S_1^d / S_0^d) \end{aligned}$$

The payments at all future dates t_k takes the same form as above (with t_{k-1}, t_k replacing t_0, t_1, respectively). But as each of these cash flows can be replicated by a \$1 investment then:

A swap of the return on the FTSE for the return on the S&P 500 is a fair exchange.

37.4.5.2 Valuation

Between any two payment dates, say $t_0 < t < t_1$, the value of the swap is:

Value of cross-currency swap at t:

$$\begin{aligned} &= \frac{\textit{Value foreign asset in USD at } t}{\textit{Value foreign asset in USD at } t_0} - \frac{\textit{Value domestic asset in USD at } t}{\textit{Value domestic asset in USD at } t_0} \\ &= (S_t^f E_t^{\$/£})/(S_0^f E_0^{\$/£}) - S_t^d / S_0^d \end{aligned}$$

37.5 SUMMARY

- An equity swap is an exchange of cash flows in the future, where for one leg of the swap the cash flows are determined by the return on a stock or the return on a stock index. The other leg of the swap may involve floating or fixed interest payments or payments based on a domestic or foreign equity (index).
- The notional principal in an equity swap may be fixed or variable.
- If a swap involves foreign equities, the 'return' in the swap can either be hedged or unhedged, against changes in the spot FX-rate.
- Equity swaps are used by investment managers to alter the asset composition of their portfolio at low cost, since the desired exposure is obtained without actually buying or selling the assets in the existing portfolio.
- Equity swaps (with a fixed notional principal) can be priced using arbitrage arguments.
- A 'vanilla equity swap' is usually an exchange of 'equity returns for fixed interest' or 'equity returns for floating (LIBOR) interest'. Somewhat counter-intuitively, a fair exchange is the 'equity return for the fixed-interest swap rate' and 'equity returns for LIBOR', respectively.
- After inception, the value of a plain vanilla receive-equity, pay-LIBOR swap, can be either positive or negative (to one of the parties). Its value at t between any two payment dates (e.g. $t_0 < t < t_1$) depends on the equity price rise from the *last* reset date (S_t/S_0) and the present value of the *next* (known) LIBOR cash payment $d(t,t_1)(1 + r_1 m_1)$. The value of the swap is $V_t = (S_t/S_0) - d(t,t_1)(1 + r_1 m_1)$.
- Equity-for-equity swaps are fair swaps, if the equity returns are in the same currency (e.g. equity return on Microsoft swapped for the equity return on AT&T). Between any two payment dates (e.g. $t_0 < t < t_1$), the market value of a 'domestic equity for domestic equity' swap, depends on the difference in the return on the two equities (since the last reset date, t_0).
- A cross-currency equity swap might involve receiving the return on the S&P 500 and paying the return on the FTSE 100. This is a fair swap. The value of a cross-currency equity swap to a US resident (say) depends on the return on the foreign equity (in USD) and the return on the domestic equity (in USD) – the former depends on the change in the exchange rate – so there is FX risk in the swap.

APPENDIX 37: VALUATION OF EQUITY-FOR-LIBOR SWAP

Using replication and arbitrage arguments we show that the value of the 'receive-equity returns, pay-LIBOR' swap at any time t when the remaining payment dates are at $t_1, t_2, \ldots, t_n$ is given by (see Figure 37.A.1):

$$V_t = (S_t/S_0) - d(t,t_1)(1 + r_1 m_1) \tag{37.A.1}$$

First, we consider cash flows at t_2 and afterwards. Then we examine cash flows at the next payment date t_1 and find that it is only these cash flows that influence the value of the swap, at time t $(<t_1)$ – consistent with Equation (37.A.1).

As before, equity returns at t_2 may be replicated by a \$1 investment in *equity* at t_1, with PV at time t of $PV(t) = \$1.d(t, t_1)$. Similarly a \$1 investment in a *LIBOR deposit* at t_1 gives $\$1(1 + r_2 m_2)$ at t_2 with present value $PV(t) = d(t, t_1)$. Hence, the difference in the PVs at t of these two cash flows is zero. This is true for all cash flows at $t \geq 2$.

Hence it is only the PV of the cash flows at t_1 that determine the value of the swap at t. Between t_0 and t_1 the return on equity is $(1 + z_1)$ and the LIBOR payoff is $(1 + r_1 m_1)$. Hence at t, the present value of the long-short position is:

$$V_t(Equity - LIBOR) = PV_t(1 + z_1) - PV_t(1 + r_1 m_1) = PV_t(1 + z_1) - d(t, t_1)(1 + r_1 m_1) \tag{37.A.2}$$

where $PV_t(1 + z_1)$ is the present value at time t, of the cash flow $(1 + z_1)$ payable at the next reset date t_1.

Realised returns on equity will not be known until time t_1. But \$1 invested in equity at t_0, accrues to \$1 $(1 + z_1) \equiv S_1/S_0$ at t_1. We can replicate this cash flow by purchasing $N_0 = 1/S_0$ units of stock at t_0, since this will be worth $N_0 S_1 = S_1/S_0$ at t_1. Hence the time-t present value of $(1 + z_1)$ payable at t_1 is S_t/S_0:

$$V_t(Equity - LIBOR) = (S_t/S_0) - d(t, t_1)(1 + r_1 m_1) \tag{37.A.3}$$

Note that this formula is consistent with the swap having zero value at inception – since at $t = 0$, $d(0, t_1) = 1/(1 + r_1 m_1)$ and $S_t = S_0$, so from Equation (37.A.3), $V_t = 0$.

EXERCISES

Question 1
What are the key features of a plain vanilla equity swap?

Question 2
You are a US investor holding \$100m in stocks which mimic movements in the S&P 500 index. You predict that the UK equity market will increase faster than the US equity market over the next 2 years and that sterling (GBP) will appreciate. To take advantage of your forecasts, what kind of swap might you initiate?

Question 3
A 'market timer' switches between equities and cash (which earns interest given by the London Interbank Bid Rate, LIBID), depending on her forecast of whether the stock market will rise above the LIBOR rate.

How might she use equity swaps to time the market, if she thinks the rise in the stock market will exceed LIBID in each of the next 3 months?

Question 4

Why might a US investor holding US$100m in a deposit at LIBOR, with resets every 3 months, find an equity swap (on the S&P 500) useful over the next year? Explain why the investor might choose an equity swap with a principal of US$50m.

Question 5

How can a US investor holding $100m in a diversified portfolio of US stocks use an equity swap to facilitate a 25% exposure to the UK stock market, every 3 months for 1 year, without incurring exchange rate risk?

Question 6

Ms Gaga holds $100,000 in an equity swap, with a 6-month tenor, to receive the equity return and pay USD-LIBOR. The 6-month LIBOR rate 181 days ago was 4% p.a. and over the last 181 days the equity return was minus 1%. The day-count convention for LIBOR payments is actual/360. Today, what is the net payment in the swap?

PART IX

FIXED INCOME DERIVATIVES

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 38

T-Bond Option, Caps, Floors and Collar

Aims

- To examine the use of options on T-bonds (T-notes) and options on T-bond futures and Eurodollar futures.
- To show how interest rate caps (floors) can be used to set an effective upper (lower) limit on the interest rate to be paid (received) in the future on a LIBOR bank loan (deposit).
- To show how a collar is used to limit the maximum and minimum interest rates payable in the future, on an existing LIBOR bank loan or bank deposit.

Interest rate options have payoffs determined either by the interest rate itself (caps and floors) or the payoff is based on the *price* of an asset (e.g. bond price or bond futures price). There are a wide variety of interest rate derivatives which can be used either for speculation on the future direction of interest rates (or bond prices) or to hedge/insure against future changes in interest rates (or bond prices). Many interest rate derivatives are OTC instruments while some are traded on organised exchanges. In Chapters 40 and 41 we examine in more detail how fixed-income derivatives are priced but here we concentrate on describing how these derivatives are used.

38.1 OPTIONS ON T-BONDS AND EURODOLLARS

A European option on a T-bond gives the holder the right to buy or sell a T-bond at some time in the future at a certain known strike price K. The payoff to a long call option on the T-bond

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

is $\max(V_T - K, 0)$ where V_T is the market price of the bond at maturity of the option contract. A T-bond option can be used to speculate on future changes in long-term interest rates – for example, buy a call on a T-bond if you believe interest rates will fall in the future (i.e. bond prices will rise above K). A put on a T-bond can be used to insure a lower bound of K, for an existing portfolio of T-bonds if interest rates rise. If interest rates fall and bond prices rise (above K) you will not exercise the put but your bonds have a higher market price.

In practice, hedging or speculation is less costly if the underlying in the options contract is a futures contract rather than the cash market asset, such as a T-bond. Therefore the most popular exchange traded fixed-income options in the US are on T-bond futures, T-note futures and Eurodollar futures. These, of course, are futures options.

In an earlier chapter we noted that the value at expiration T, of a call on a T-bond futures option is $\max(F_T - K, 0)$ where F_T is the T-bond *futures price* at expiration of the option. If the option is exercised, the holder of a long call receives a long position in the futures contract and if the futures contract is immediately closed out there is a cash payout of $F_T - K$, where F_T is the Eurodollar futures price at expiration of the option's contract. (If the futures is not closed out then initial margin must be paid.)

A US T-bond futures call option quote might be denoted 'C = 3-10'. This implies C = 3-10/64% and on a futures contract size of \$100,000 of T-bonds, implies an invoice price for the option of $\$3,156.25$ ($= 0.0315625 \times 100,000$). A basis point (0.01%) change in the IMM futures index, implies a gain or loss on one T-bond futures contract of \$25. An option on a Eurodollar futures contract is priced similarly, so a call option with a quoted premium of $C = 0.50$ (50 basis points) implies that one call costs $50 \times \$25 = \$1{,}250$.

38.2 CAPLETS AND FLOORLETS

Interest rate options are widely used to either speculate on the future course of interest rates or to hedge/insure the interest payments/receipts on an underlying cash market position (e.g. LIBOR bank loan or bank deposit). Below we discuss:

- The mechanics of using interest rate options and how their payoffs are related to FRAs and interest rate swaps.
- Hedging a *single* interest rate payment or receipt, using a caplet or floorlet.
- Setting an upper limit on a *series* of interest rate payments, using a cap.
- Setting a lower limit on a *series* of interest rate receipts, using a floor.
- Establishing both a floor and a ceiling on a corporate or bank's (floating rate) borrowing costs by using a collar.

We have seen how an FRA 'locks in' an interest rate beginning at some point in the future. If you have a 90-day LIBOR loan with principal Q plus a long position in an FRA (with notional principal Q), then if interest rates either rise or fall, your effective borrowing rate on the loan remains constant.

$$\begin{aligned} \textit{Effective borrowing cost} &= \textit{Loan interest} - \textit{Payoff to FRA} \\ &= Q\{LIBOR_T - [LIBOR_T - FRA]\}(90/360) \\ &= Q\ \ FRA(90/360) \end{aligned}$$

38.2.1 Long Call (Caplet)

A caplet is like an FRA in that it sets an effective maximum interest rate you pay on a LIBOR bank loan. But if interest rates turn out to be low, you do not exercise the caplet and simply pay the lower rate of interest on your loan. So the caplet sets a maximum interest rate you will pay on the loan but allows you to take advantage of lower interest rates should they occur. A long position in a caplet has a \$-payoff at maturity T:

$$\textit{Payoff to caplet at } T = Q\max(0, LIBOR_T - K_{cap})\ \ (days/360)$$

where *days* is the tenor of the LIBOR rate in the caplet. Suppose the caplet is on 90-day LIBOR on a notional principal $Q = \$100\text{m}$, with strike $K_{cap} = 10\%$ p.a. and premium $C = \$250{,}000$ (i.e. equal to 0.25% of the \$100m principal). The option expires at $T = 30$ days. Using 'actual/360' day-count convention, the caplet has a cash payout (at T+90 days):

$$\textit{Caplet Payout at } T + 90 = \$100\text{m}\ \ \max(0, LIBOR_T - 0.10)\ \ (90/360)$$

The payoff is *determined* at expiration of the option at $T = 30$ but the cash payout does not take place until 90 days later (Figure 38.1).

If at expiration $LIBOR_T = 14\%$ the option has a payoff of $LIBOR_T - K_{cap} = 4\%$ p.a. (or 1% over 90 days). This profit of 4% p.a. offsets the higher borrowing rate of $LIBOR_T = 14\%$ giving an effective borrowing rate of 10% p.a., which equals the strike rate in the option contract. If at expiration $LIBOR_T = 8\% < K_{cap} = 10\%$ then the caplet is not exercised but you can take

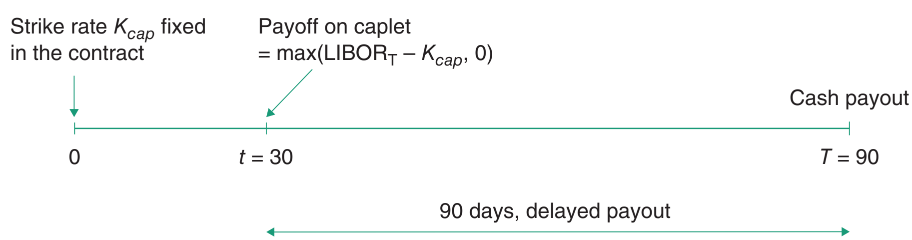

FIGURE 38.1 Payoff at maturity, 30-day caplet on 90-day LIBOR

advantage of the low borrowing cost of 8% p.a. The loan plus caplet ensures an effective maximum borrowing cost of $K_{cap} = 10\%$ but allows the 'upside' of borrowing at low rates should these occur. The caplet therefore provides insurance for the borrower:

$$\begin{aligned} \textit{Effective borrowing cost} &= Q \ \ LIBOR_T(90/360) - Q \ \ \{\max(0, LIBOR_T - K_{cap}) \ \ (days/360)\} \\ &= Q \ \ K_{cap} \ \ (days/360) \qquad\qquad \text{if } LIBOR_T > K_{cap} \\ &= Q \ \ LIBOR_T \ \ (\text{days}/360) \qquad\qquad \text{if } LIBOR_T \leq K_{cap} \end{aligned}$$

Of course, if the corporate at $T = 30$ days decides it does not want to borrow funds after all, the most it will have lost is the call premium and if interest rates have risen above $K_{cap} = 10\%$, then it will earn a profit from the call.

The effective borrowing cost of the loan as calculated above ignores the caplet premium of $C = \$250{,}000$ (which has to be paid at $t = 0$) and the fact that the cash payout on the option does not occur until T+90. Calculation of the correct effective rate on the loan requires the following steps. If 30-day LIBOR at $t = 0$ is $L_0 = 10\%$ then at T:

$$\textit{Caplet premium} + \textit{Interest at } T = \$250{,}000 \ \ [1 + 0.10(30/360)] = \$252{,}083$$

$$\textit{Effective amount borrowed at } T = \$100\text{m} - \$252{,}083 = \$99{,}748\text{m}$$

At T, if 90-day $LIBOR_T = 14\%$ then:

$$\textit{Caplet payout at } T + 90 = \$100\text{m} \ (0.14 - 0.10) \ (90/360) = \$1\text{m}$$

$$\textit{Unhedged interest on loan at } T + 90 = \$100\text{m} \ (0.14) \ (90/360) = \$3.5\text{m}$$

$$\textit{Unhedged cost} - \textit{caplet payout} = \$3.5\text{m} - \$1\text{m} = \$2.5\text{m}$$

$$\textit{Total amount paid out at } T + 90 \ (\textit{including principal}) = \$102.5\text{m}$$

The effective interest paid on borrowing \$99.748m at T and paying out \$102.5m at T+90 is:

$$\textit{Effective interest cost} = \left(\frac{102.5}{99.748}\right)^{\frac{365}{90}} - 1 = (1.02759)^{\frac{365}{90}} - 1 = 0.1167 \ \ (11.67\%)$$

The effective interest cost of 11.67% p.a. is reasonably close to that of the strike rate of $K_{cap} = 10\%$ (simple interest) in the option contract. The cost of the unhedged position is considerably higher than $K_{cap} = 10\%$ because the LIBOR rate at T is 14%:

$$\textit{Unhedged cost of loan} = (103.5/100)^{365/90} - 1 = 0.01497 \ (14.97\%)$$

With the caplet, whatever the out-turn LIBOR rate (at or above $K_{cap} = 10\%$), the cost of borrowing is a maximum of 11.67%. Of course, if $\text{LIBOR}_\text{T} < 10\%$, then the caplet is not

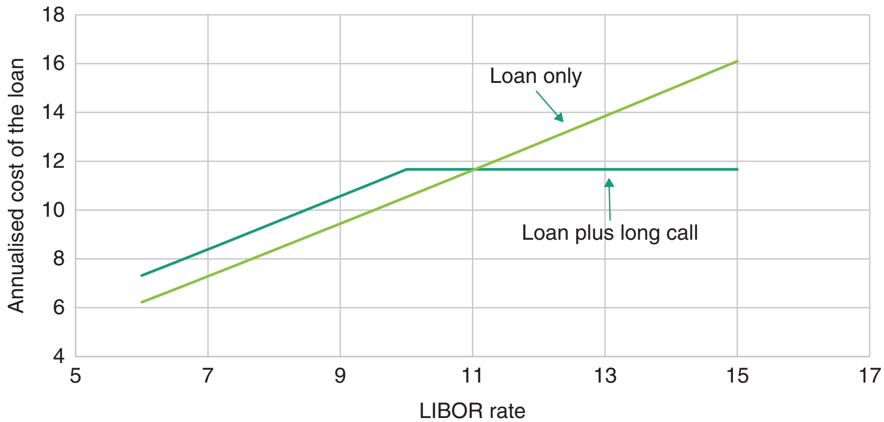

FIGURE 38.2 Loan + caplet

exercised and the interest cost of the loan is the low LIBOR rate. The cost of the unhedged position and the effective cost of borrowing with the long caplet are given in Figure 38.2. The payoff profile for the insured position is similar to a covered call.

38.2.2 Long Put (Floorlet)

Consider a corporate that will place \$100m in a deposit account in 90 days' time and the deposit account is on 180-day LIBOR. Alternatively, consider a bank that in 90 days time will issue a bank loan to a corporate, based on 180-day LIBOR (plus a spread) but has raised the finance with a fixed rate deposit. Both the corporate and the bank are vulnerable to a fall in 180-day LIBOR rates, in 90 days' time.

Both parties can 'lock in' a lower bound on their effective interest rate in 90 days' time, by today buying a put option (floorlet) on 180-day LIBOR, which has an expiration date in 90 days. If the strike $K_{FL} = 10\%$ the payoff from the floorlet in $T = 90$ days is:

$$\textit{Payoff from floorlet at } T = \mathrm{Q}\left\{\max(0, K_{FL} - LIBOR_T)\left(\frac{days}{360}\right)\right\}$$

where $days = 180$. But the corporate with the bank deposit will not receive the payoff from the floorlet until T+180. If at expiration 180-day $LIBOR_T = 8\% < K_{FL} = 10\%$, the payoff to the put is $(K_{FL} - LIBOR) = 2\%$ which offsets the lower deposit rate of 8% p.a. This effectively 'locks in' a minimum effective deposit rate of 8% + 2% which is equal to the strike rate $K_{FL} = 10\%$ – hence the put is known as a *floorlet*. If at T, 180-day $LIBOR_T = 11\% > K_{FL} = 10\%$, the put is not exercised but the corporate receives the high deposit rate of 11%. The floating rate bank deposit plus the long floorlet has a payoff profile of a long call – see Figure 38.3:

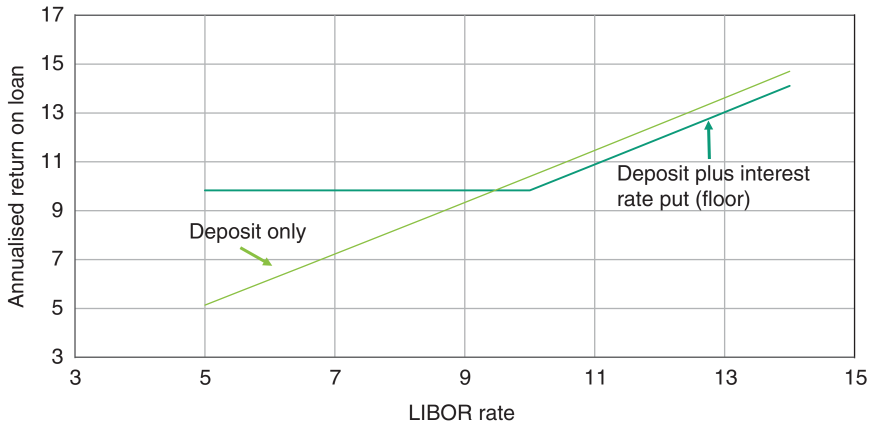

FIGURE 38.3 Deposit + floorlet

$$\begin{aligned} \textit{Effective deposit rate} &= Q\ \ LIBOR_T(90/360) + Q\{\max(0, K_{FL} - LIBOR_T)\ \ (days/360)\} \\ &= Q\ \ K_{FL}\ \ (days/360) \qquad\qquad \text{if } LIBOR_T < K_{FL} \\ &= Q\ \ LIBOR_T\ \ (\text{days}/360) \qquad\qquad \text{if } LIBOR_T \geq K_{FL} \end{aligned}$$

The effective return on the deposit plus floorlet depends on the exact timing of the cash flows. Suppose the put premium P = \$250,000 on a notional principal Q = \$100m. If 90-day LIBOR at $t = 0$ is $L_0 = 10\%$ then at T:

$$\textit{Floorlet premium} + \textit{Interest at}\ \ T = \$250{,}000[1 + 0.10(90/360)]) = \$256{,}250$$

$$\textit{Effective cash outflow at}\ \ T = \$100\text{m} + 256{,}250 = \$100{,}256, 250$$

If at expiration $LIBOR_T = 8\%$, then:

$$\textit{Interest received on deposit at}\ T + 180 = \$100\text{m}\ (0.08)(180/360) = \$4\text{m}$$

$$\textit{Payoff on floorlet at}\ T + 180 = \$100\text{m}\ (0.10 - 0.08)(180/360) = \$1\text{m}$$

$$\textit{Net cash payout (excluding principal)} = \$5\text{m}$$

$$\textit{Total amount received at}\ T + 180 = \$105\text{m}$$

$$\textit{Annualised return (deposit + floorlet)} = [105/100.25625]^{365/180} - 1 = 0.0983\ (9.83\%)$$

$$\textit{Annualised return unhedged} = [104/100]^{365/180} - 1 = 0.0828\ (8.28\%)$$

Note that the unhedged return is based on the actual bank deposit of \$100m and a final payment of \$104m, whereas for the deposit+floorlet we have to factor in the cost of the floorlet.

With the floorlet there is a minimum return of 9.83% (which is higher than the unhedged return on the deposit account of 8.28%). If $\mathrm{LIBOR_T}$ turns out to be higher than $K_{FL} = 10\%$, then the put is not exercised but the corporate earns a high interest rate on its deposit account.

The Excel file for the payoff to a caplet and floorlet can be found on the website.

Interest rate options are usually OTC instruments where the strike rate is usually chosen to be close to the *current* level of spot (or forward) interest rates. Note that unlike some other fixed-income derivative contracts (e.g. T-bill futures and T-bond futures) which have payoffs based on the *price* of the underlying asset, caplets and floorlets have payoffs based on interest rates.

38.3 INTEREST RATE CAP

So far we have used an OTC interest rate contract to 'lock in' either a maximum borrowing rate or a minimum deposit rate, at a *single* future date. However, a corporate or a bank may wish to lock in a *series* of payments or receipts over successive known future dates.

To obtain protection against a series of rising loan rates whilst also benefiting from any fall in rates, then today buy a cap.

A cap is nothing more than a series of individual caplets (calls) which mature on dates corresponding to future LIBOR reset dates on an existing loan. A cap is therefore a 'strip' of caplets. The cap premium is equal to the sum of the call premia for the separate caplets.

At each LIBOR reset date on the loan, the cap will have a payoff depending on whether LIBOR on the expiration date exceeds the strike rate in the cap contract. Actual cash payment takes place after the expiration date. For example, if the cap is written on 90-day LIBOR, payment will take place at T+90, based on the 90-day LIBOR rate on the expiration date.

Suppose on 10 April a corporate has a 1-year loan of \$20m, with interest payments based on 90-day LIBOR. The corporate fears a rise in interest rates over the year and wishes to cap its payments at the current 90-day LIBOR rate of $L_0 = 10\%$. It buys an interest rate cap for $C =$ \$60,000 with strike $K_{cap} = 10\%$[1] and expiration dates on 10 July, 10 October, etc. (assuming these are 'working days'). The cap premium at \$60,000 is 0.3% of the principal of \$20m.

$$\text{Net cash available 10 April} = \$20\text{m} - \$60{,}000 = \$19{,}940, 00$$

[1] It is also possible for each caplet to have a different strike rate for each reset date.

TABLE 38.1 Loan and cap

Interest cost	Per quarter	Per annum
with cap	2.61%	10.88%
without cap	2.82%	11.77%

Date	Days in quarter	LIBOR	Interest due	Cap payment	Principal repayment	Net cash flows	
						With cap	Without cap
10-Apr-00		10		–60,000	0	19,940,000	20,000,000
10-Jul-00	91	10.7	505,556	0	0	–505,556	–505,556
10-Oct-00	92	12.3	546,889	35,778	0	–511,111	–546,889
10-Jan-01	92	11.6	628,667	117,556	0	–511,111	–628,667
10-Apr-01	90		580,000	80,000	20,000,000	–20,500,000	–20,580,000

Note: First payment on the cap is based on 10 July but does not get paid until 10 October.

At each expiration date ($=T$) for the caplets, the cash payout 90 days later is:

$$\text{Payout at } (T+90) = \$20\text{m} \times \max\{0, LIBOR_T - 0.10\}\ (90/360)$$

Since $LIBOR = 10.7\%$ on 10 July this gives rise to a cap payout of \$35,778 on 10 October (Table 38.1). The net cash flow on 10 October of \$511,111 is the interest payment on the loan of \$546,889 less the cap payout of \$35,778. If interest rates rise over the year then the cap will lock in an effective (simple) interest cost for the loan of around $K_{cap} = 10\%$. Table 38.1 shows the (ex-post) cash payments in the cap, assuming that interest rates are always above 10% throughout the year.

The annualised borrowing rate with the cap in place is the internal rate of return from the net cash flows and depends on LIBOR rates over the life of the cap. The internal rate of return is the solution for y where the net cash flows are given in Table 38.1 (column 7):

$$\$19.94\text{m} = \frac{\$0.5056\text{m}}{(1+y)} + \frac{\$0.5111\text{m}}{(1+y)^2} + \frac{\$0.5111\text{m}}{(1+y)^3} + \frac{\$20\text{m} + \$0.5\text{m}}{(1+y)^4} \tag{38.1}$$

$$y = 2.615\% \text{ per quarter, annualised compound rate} = (1.02615)^4 - 1 = 10.88\% \text{ p.a.}$$

If the cap had not been purchased, the cash flows in the final column of Table 38.1 imply an unhedged cost (internal rate of return) of 11.77% p.a.

If LIBOR rates had fallen below 10% at any reset date, that caplet would not be exercised but the lower cash market LIBOR rate implies a lower effective borrowing cost for the corporate. A cap therefore fixes the maximum interest rates payable on a bank loan over the life of the cap but allows the corporate to take advantage of lower loan rates, should they occur.

38.4 INTEREST RATE FLOOR

A corporate with a series of future cash *inflows* which it wants to place on deposit would lose from a future fall in interest rates. Similarly, a bank which receives floating rate payments on its loans would lose from a fall in rates.

> *To obtain protection from a series of falling deposit rates, whilst also benefiting from any rise in LIBOR rates, buy a floor, today.*

A floor is a series of interest rate puts (floorlets), each one expiring on the reset date for interest paid on the bank deposit. The payoff from each floorlet on a notional principal \$$Q$ is[2]:

$$Q\max(0, K_{FL} - LIBOR_T)\ \ (days/360) \tag{38.2}$$

Suppose on 10 January a corporate places \$20m in a 1-year bank deposit tied to 90-day LIBOR and it purchases a floor with a strike rate $K_{FL} = 10\%$ at a premium of \$40,000, with the first reset date on 10 April. The payments with and without the floor, if we assume a fall in 90-day interest rates over the whole year, are given in Table 38.2.

The corporate has a cash outflow equal to the \$20m bank deposit and also pays \$40,000 for the floor, making a total cash outflow of \$20,040,000. On 10 January, 90-day LIBOR rates are $L_0 = 9.9\%$ hence:

$$\text{Deposit Interest (10 April)} = \$20\text{m } (0.099)(90/360) = \$500{,}500$$

TABLE 38.2 Deposit and floor

Interest cost	Per quarter	Per annum
with floor	2.48%	10.3%
without floor	2.32%	9.6%

Date	Days in quarter	LIBOR	Interest due	Floor payment	Principal repayment	Net cash flows	
						With floor	Without floor
10-Jan-00		9.9		−40,000	0	−20,040,000	−20,000,000
10-Apr-00	91	9.5	500,500	0	0	500,500	500,500
10-Jul-00	91	9	480,278	25,278	0	505,556	480,278
10-Oct-00	92	8	460,000	51,111	0	511,111	460,000
10-Jan-01	92		408,889	102,222	20,000,000	20,511,111	20,408,889

Note: First payment of the floor is based on 10 April but does not get paid until 10 July.

[2]The rate paid on deposits is LIBID but we continue to use LIBOR to avoid extra notation.

On 10 April LIBOR has fallen to $L_1 = 9.5\%$ hence:

$$\text{Payout from floorlet (on 10 July)} = \$20\text{m } (0.10 - 0.095)(91/360) = \$25{,}278$$

On 10 July based on $L_1 = 9.5\%$:

$$\text{Deposit Interest (10 July)} = \$20\text{m } (0.095)(91/360) = \$480{,}278$$

$$\text{Deposit interest + floorlet (10 July)} = 480{,}278 + 25{,}278 = \$505{,}556.$$

Cash flows with the floor are given in Table 38.2 (column 7) and imply an internal rate of return of 2.48% per 90 days or 10.3% p.a.$(= (1.0248)^4 - 1)$. If the floor had not been used then as LIBOR rates on the bank deposit fall, the unhedged return is 9.6% p.a. Hence the floor has boosted the corporate's effective interest rate by about 70 bps. Had LIBOR been above 10% over the year, the floor would not be exercised but the corporate would have the benefit of higher deposit interest rates (to offset against the floor premium it paid at the outset).

38.5 INTEREST RATE COLLAR

If a corporate is borrowing money at LIBOR, it can protect itself against future rises in LIBOR by buying a cap and it also benefits if interest rates fall, since its loan repayments will be low. The corporate may have to pay a hefty cap premium for this privilege. It can offset some of this cost by also *selling* a floor and hence receiving the floor (put) premium.

Suppose $K_{cap} = 10\%$ and $K_{FL} = 8\%$. Hence at LIBOR rates above 10%, the cap is exercised and the corporate's effective borrowing costs are a maximum of $K_{cap} = 10\%$ (Figure 38.4). When LIBOR falls below 8% the corporate can take advantage of lower loan rates but these

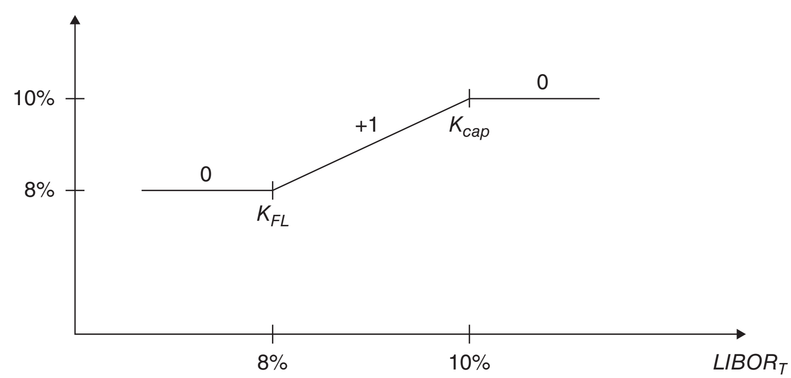

FIGURE 38.4 Collar

savings are offset because the corporate will have to pay $K_{FL} - LIBOR$ on the written floor. Hence for $LIBOR < K_{FL}$ (ignoring cash flows from the sale of the floor and the cost of the cap):

$$\text{Effective borrowing cost} = LIBOR_T + (K_{FL} - LIBOR_T) = K_{FL} = 8\% \tag{38.3}$$

The net effect of being long a cap and short a floor is to establish both a ceiling and a floor on the effective borrowing cost of the corporate's LIBOR loan – this combination is a collar.

A collar comprises a bank loan plus a long cap and short floor.
It establishes both a floor $K_{FL} = 8\%$ and a ceiling $K_{cap} = 10\%$ on the effective borrowing cost.

The effective cost of the loan plus a collar depends on the path for interest rates over the life of the collar and the strike rates chosen for the cap and the floor.

38.5.1 Payoffs from the Collar

We ignore the fact that the cap and floor premia are paid/received at $t = 0$ whereas the payoffs to the collar accrue at various times $t > 0$. At any expiration date, the option payoffs (ignoring the option premia) on the collar (for $K_{cap} = 10\%$ and $K_{FL} = 8\%$) and a 90/360 day-count convention are:

Payoff on the long cap and short floor

$$= \{\max[0, LIBOR_T - K_{cap}] - \max[0, K_{FL} - LIBOR_T]\} \;\; Q(90/360)$$

Effective borrowing cost with collar

$$\begin{aligned}
&= \{LIBOR_T + \{\max[0, LIBOR_T - K_{cap}] - \max[0, K_{FL} - LIBOR_T]\} \;\; Q(90/360) \\
&= Q\, K_{cap}(90/360) && \text{if } LIBOR_T > K_{cap} \\
&= Q\, K_{FL}(90/360) && \text{if } LIBOR_T < K_{FL} \\
&= Q\, LIBOR_T(90/360) && \text{if } K_{FL} < LIBOR_T < K_{cap}
\end{aligned}$$

It is clear from the above calculations that the collar involves a (simple) borrowing cost at each payment date of either $K_{cap} = 10\%$ or $K_{FL} = 8\%$ or $LIBOR_T$ if the latter is between 8% and 10%. The true effective borrowing cost requires the 'up-front' cost of the cap premium net of the receipts from the floor premium, to be taken into account. It is even possible to set the strike prices K_{cap} and K_{FL}, so that the premium received from the sale of the floor equals the premium paid for the cap. This is a *zero-cost collar*.

TABLE 38.3 Interest rate collar

Interest cost	Per quarter	Per annum
with collar	2.33%	9.65%
with cap	2.40%	9.95%
unhedged	2.40%	9.96%

Date	Days in quarter	LIBOR	Interest due	Cap payment	Floor payment	Principal repayment	Net cash flow		
							With collar only	With cap only	Unhedged
10-Apr-00		10.5	0	–350,000	500,000	0	100,150,000	99,650,000	100,000,000
10-Jul-00	91	11.5	2,654,167	0	0	0	–2,654,167	–2,654,167	–2,654,167
10-Oct-00	92	10	2,938,889	383,333	0	0	–2,555,556	–2,555,556	–2,938,889
10-Jan-01	92	9	2,555,556	0	0	0	–2,555,556	–2,555,556	–2,555,556
10-Apr-01	90	7.8	2,250,000	0	0	0	–2,250,000	–2,250,000	–2,250,000
10-Jul-01	91	7.7	1,971,667	0	–50,556	0	–2,022,222	–1,971,667	–1,971,667
10-Oct-01	92		1,967,778	0	–76,667	100,000,000	–102,044,444	–101,967,778	–101,967,778

The effective borrowing cost with the collar depends on the particular outcome for LIBOR at each reset date. An illustrative outcome is given in Table 38.3 for:

$$Q = \$100\text{m} \qquad C = \$350{,}000 \qquad P = \$500{,}000$$
$$K_{cap} = 10\%, \qquad K_{FL} = 8\% \qquad \text{90-day LIBOR}$$

The loan is for \$100m and the interest rate on 10 April-00 is 10.5%. The cap costs $C =$ \$500,000 and the written floor provides an immediate cash inflow of $P =$ \$350,000. Given the path of LIBOR assumed in Table 38.3, the cap is only exercised on 10 July-00 when $LIBOR =$ 11.5% and the caplet payout of \$383,333 is received on 10 October-00.

The floor is exercised on 10 April-01 and 10 July-01, when LIBOR falls below $K_{FL} = 8\%$ to 7.8% and 7.7% respectively (and payments are 90 days later). The interest paid on the corporate's loan at each reset date is based on LIBOR, 90 days earlier (column 4). The net cash flow to the corporate is given for the following three scenarios:

- With collar – Table 38.3, column 8
- With cap (only) – Table 38.3, column 9
- Unhedged – Table 38.3, column 10

The effective cost of borrowing (i.e. internal rate of return) for the unhedged strategy is 9.96% whereas that for the collar is lower at 9.65%. The important fact is that the collar limits the interest cost, should interest rates rise (above 10%) and the receipt of the put (floor) premium offsets some of the cost of the cap. However, as interest rates fall (below $K_{FL} = 8\%$) the short put involves a cash payout by the corporate, which offsets the lower interest payments on the company's bank loan. Thus the effective cost of hedging with the collar depends on the actual course of interest rates, until the expiration date of the collar.

If only a cap had been used then the effective cost of borrowing would have been 9.95% only just below the unhedged cost of 9.96%. This is because the cost of the cap is not offset by any future high cash payouts from the cap or by sufficiently low borrowing rates on the loan – given the path of interest rates we have chosen.

38.6 SUMMARY

- Call options on T-bonds and T-bond futures can be used to set a maximum purchase price for the underlying bond or bond future at expiration of the option, but you can also take advantage of low cash market prices should these occur. Similarly a put on T-bonds or T-bond futures can be used to set minimum selling prices for the T-bonds you already own but also allows you to take advantage of high T-bond prices should they occur in the future.

- A corporate with an outstanding LIBOR loan might buy an interest rate call (caplet) to provide insurance (i.e. maximum effective interest rate payable) but it also allows the corporate to take advantage of lower cash market interest rates should these occur in the future.
- An interest rate put (floorlet) might be used by a corporate who has funds in a (floating rate) deposit and fears a fall in interest rates in the future. It 'locks in' a minimum effective rate that will be paid on its bank deposit in the future. It also allows the corporate to take advantage of high deposit rates in the future, should they occur.
- A cap is used to protect against rising interest rates on a *series* of future loan payments. A cap is a series of individual caplets which each mature on dates corresponding to interest rate reset dates on a loan.
- A corporate with a *series* of cash inflows from funds held on deposit would lose from a future fall in rates. Similarly a bank which receives floating rate payments on its loans would lose from a fall in rates. To obtain a minimum effective rate on future cash inflows based on LIBOR, while also benefiting from any rise in rates, you would buy a floor.
- A collar consists of an existing LIBOR bank loan, then buying a cap (with a high strike price) and selling a floor (with a low strike price). This establishes both a ceiling and a floor on the effective borrowing costs for the corporate.

EXERCISES

Question 1
Explain how a European cap on 90-day LIBOR, with caplets which expire in 9 months and 1 year, and a strike of K_c is similar to an interest rate call option. Explain who might use this cap.

Question 2
You already have a bank deposit of $10m at a 90-day LIBOR floating rate. Explain how you can protect the future receipts from your bank deposit by using an interest rate floor with strike, K_{FL}. Explain the outcomes at an interest rate reset date.

Question 3
You have a LIBOR floating-rate bank loan. Explain how a collar can be used to mitigate interest rate risk. Explain with reference to outcomes at an interest rate reset date.

Question 4
You are pricing a cap on LIBOR with caplets expiring at the end of year-1, year-2, and year-3. Intuitively, what variables might influence the price of the cap. Briefly explain.

Question 5
What is the key difference between an FRA and a caplet? Explain with reference to holding a bank loan on which you pay LIBOR and then either taking a position in an FRA or in a caplet.

Explain with reference to outcomes at an interest rate reset date.

Question 6

On 1 June 2000, BigCorp takes out bank loan of \$10m at 90-LIBOR, with maturity date 1 June 2001 (when the principal is repaid). The current 90-day LIBOR rate is 10% p.a. At the same time BigCorp buys an interest rate cap (on 90-day LIBOR) with a strike rate $K_c = 10\%$ at a premium of \$15,000.

LIBOR turns out to be 10.5% on 1 September (+ 92 days) 10.8% on 1 December (+91days) and 11% on 1 March 2001 (+90 days). The cap payoff and loan interest payments use actual/360, day-count convention.

(a) Calculate the interest cost of the loan for each 3-month period.
(b) Calculate the cash flows from the cap for each 3-month period.
(c) Calculate net cash flows from the loan+cap for each 3-month period.
(d) Calculate the effective annual cost of (i) the loan (ii) loan+cap (using the internal rate of return IRR of the cash flows with and without the cap).

You may find it useful to use Excel for the calculations and fill in the entries in the following table.

Date	**Days in quarter**	**LIBOR**	**Interest due**	**Cap**	**Principal**	**With cap**	**Without cap**
01.06.00							
01.09.00	*92*						
01.12.00	*91*						
01.03.01	*90*						
01.06.01	*92*						

Question 7

Assume the stochastic behaviour of the short-term interest rate (90-day LIBOR) can be represented as:

$$r_{t+1} = r_t + \lambda(w - r_t)\mathrm{dt} + \sigma\sqrt{dt}\varepsilon_t \qquad \varepsilon_t \text{ is } niid(0, 1).$$

Current interest rate $r_1 = 10\%$ p.a., volatility of interest rates $\sigma = 0.007$ p.a. and the long-run equilibrium interest rate, $\omega = 8\%$ p.a. The rate of mean reversion $\lambda = 0.02$ (per period) and the time-step chosen is $dt = 0.01$ (years, approximately 3.5 days).

Using MCS, outline the steps required to calculate the price of a caplet (on 90-day LIBOR) with maturity $T = 1$ year and strike $K = 10\%$ p.a.

CHAPTER 39

Swaptions, Forward Swaps, and MBS

Aims

- To demonstrate how a payer swaption can be used to set an effective maximum swap rate that begins at expiration of the swaption but also allows the holder of the swaption to take advantage of lower (cash market) swap rates, should they occur in the future.
- To show how a long position in a forward swap locks in a known swap rate which will begin at expiration of the forward swap but the holder of the forward swap cannot take advantage of lower cash-market swap rates in the future should they occur.
- To analyse the use of various forms of mortgage-backed securities (MBS) and the interaction between interest rate changes, prepayment options, and the change in value of the MBS.
- To show how the Greeks can be used to hedge fixed income derivatives.

39.1 SWAPTIONS

Suppose you are thinking of taking out a 3-year pay-fixed (receive-float) vanilla interest rate swap in 2 years' time. You are worried that swap rates in 2 years' time will be high, implying you will pay a high fixed swap rate. If swap rates turn out to be low in 2 years' time then you will be happy to take advantage of this outcome. But you cannot know today what cash market swap rates will be in 2 years' time.

However, today, you would like to be able to fix the *maximum* swap rate you will pay in 2 years' time, whilst also being able to take advantage of low swap rates should they occur.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

You can achieve this desired outcome if today you buy a 3-year payer swaption which expires in $T = 2$ years' time with a strike rate $K_{sp} = 10\%$ p.a., which is the agreed fixed swap rate. You have then insured that the maximum swap rate you will pay in the future is K_{sp} but you can also take advantage of lower cash market swap rates should they occur, by not exercising the swaption and entering into a swap in the cash market in 2 years' time.

The 'underlying asset' in the swaption contract is a 3-year cash market swap with an agreed payment frequency (tenor) and notional principal. At expiration, the buyer of the swaption may either 'take delivery' and enter into a cash market swap, or the swaption can be cash settled. The strike K_{sp} is the agreed swap rate in the swaption. The swaption premium will be paid up front.

> *A swaption is an OTC option to enter into a cash market swap at the expiration date of the swaption, either as a fixed rate payer (at K_{sp}) and floating rate receiver (i.e. payer swaption) or vice versa (i.e. a receiver swaption).*

Suppose a US corporate thinks that it may need to borrow \$10m at a floating rate from its correspondent bank (Citibank) for 3 years, beginning in 2 years' time. However, it wishes to swap the floating rate payments for fixed rate payments, thus transforming the loan to a fixed rate. The corporate therefore needs a $Q = \$10$m swap (from Morgan Stanley, say), to pay fixed and receive floating *beginning in 2 years' time* and an agreement that the swap will last for a further 3 years (with annual payments).

Suppose the corporate thinks that interest rates will rise over the next 2 years and hence the cost of the *fixed* rate payments in the swap will be higher than at present. Rather than waiting for 2 years, the corporate can insure a maximum future swap rate by today purchasing (from Morgan Stanley) a 2-year European payer swaption, on a 3-year 'pay-fixed, receive-floating' swap, at a strike rate of $K_{sp} = 10\%$ (say). If cash market swap rates (quoted by JPMorgan, say) in 2 years' time are $sp_T = 11\% (> K_{sp} = 10\%)$ then the corporate will exercise the swaption and take delivery of a 3-year swap with a fixed swap rate equal to $K_{sp} = 10\%$, from Morgan Stanley.[1] A payer swaption therefore places a ceiling on the fixed rate payable in a swap at $K_{sp} = 10\%$. However, if swap rates turn out to be $sp_T = 9\%$ in 2 years' time then the corporate would not exercise its payer swaption and would enter a cash market swap (with JPMorgan) at the current (low) swap rate of 9%.

39.1.1 Expiration of the Swaption

What is the payoff to a swaption if the cash market swap rate at expiration ($T = 2$) turns out to be $sp_T = 11\%$? The swap underlying the swaption is for delivery of a 3-year swap, with a

[1] If exercised, the holder of the swaption can choose to cash settle or to enter into a new pay fixed swap with swap rate K_{sp}. We discuss cash settlement of the swaption below.

notional principal $Q = \$10\text{m}$ and a strike swap rate of $K_{sp} = 10\%$. Exercising the option allows one to enter into a swap to pay $K_{sp} = 10\%$ fixed (and receive LIBOR). But at $T = 2$ you could sell an otherwise identical 3-year swap in the cash market with fixed *receipts* at $sp_T = 11\%$ (and pay LIBOR). The LIBOR payments in each swap cancel each other out.

Hence the value (payoff) to the swaption at expiration is the present value of an annuity of $\$10m(sp_T - K_{sp})$, over 3 years (Figure 39.1):

$$V_{sp}(T = 2) = \$10\text{m}\ (\max[sp_T - K_{sp}, 0])\ AN \tag{39.1a}$$

$$AN = d_1 + d_2 + d_3 \tag{39.1b}$$

AN = present value of an annuity of \$1 paid at times t_i $(i = 1, 2, 3)$ where $d_i = 1/(1 + r_i t_i)$ using simple interest (or $d_i = e^{-r_i t_i}$, continuously compounded rates). r_i is the out-turn spot rate at T(the maturity date of the swaption (and ending $i = 1, 2$ or 3 years later).

If the payer swaption is in-the-money, $sp_T > K_{sp}$ and is cash settled then at expiration V_{sp} is received by the holder of the swaption. Alternatively, if at expiration the underlying swap rate is $sp_T = 9\% (< K_{sp} = 10\%)$ then the swaption would not be exercised but the corporate could then enter into a 'new' 3-year cash market swap at $T = 2$, at the 'low' cash market swap rate of $sp_T = 9\%$. Hence a payer swaption is an interest rate call where the payoff is the *annuity value* of $Q \max(sp_T - K_{sp}, 0)$, rather than simply a one-off payment of $Q \max(sp_T - K_{sp}, 0)$.

It should be obvious from the above that swaptions (either European or American) can also be used for speculation on the future value of the swap rate. You would buy a payer (receiver) swaption if you thought swap rates would rise (fall) in the future, relative to K_{sp}.

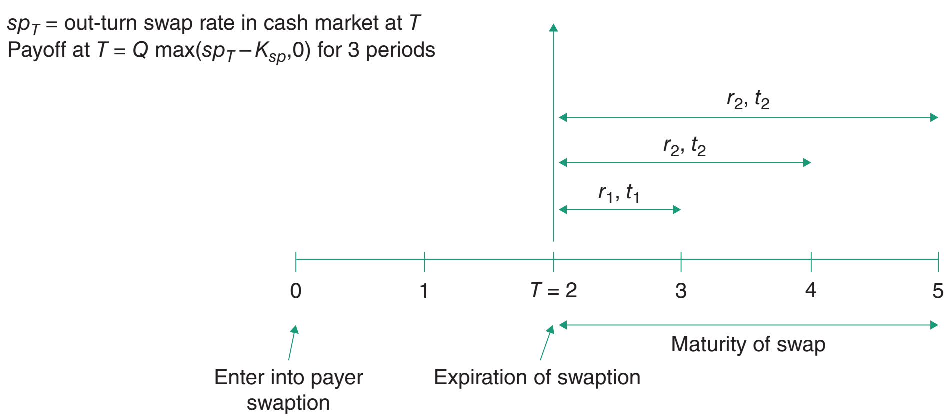

FIGURE 39.1 Payoff to European 3-year payer swaption

39.2 FORWARD SWAPS

A forward swap is a contract to enter into a swap at some future date, at a (forward) swap rate agreed today. Suppose you *go long* a 3-year forward swap with maturity $T = 2$ years, notional principal $Q = \$10$m, annual tenor and a forward swap rate sp^f. Then you have entered a contract today to 'pay-fixed at sp^f and receive-floating', on a swap beginning in 2 years and lasting for a further 3 years.

> *A long forward swap is a 'pay-fixed, receive-floating' swap that will commence in the future but at a forward swap rate sp^f agreed today.*

How do we price the forward swap at time $t = 0$? (Figure 39.2).

39.2.1 Pricing a Forward Swap

The forward swap rate at $t = 0$ is the *fixed* rate sp^f that makes the swap have zero *expected* value at $T = 2$ (and therefore also at $t = 0$). The floating leg of the swap (i.e. the FRN) is worth Q at $T = 2$. The fixed leg of the underlying swap has an *annual* coupon $C_X = sp^f Q$. The swap must have zero expected value at T:

$$\textit{Expected value of FRN at } T = \textit{Expected Value of fixed payments at } T$$

$$Q = C_X(d^f_{23} + d^f_{24} + d^f_{25}) + Q\ d^f_{25}$$

$$= sp^f Q(d^f_{23} + d^f_{24} + d^f_{25}) + Q\ d^f_{25} \tag{39.2}$$

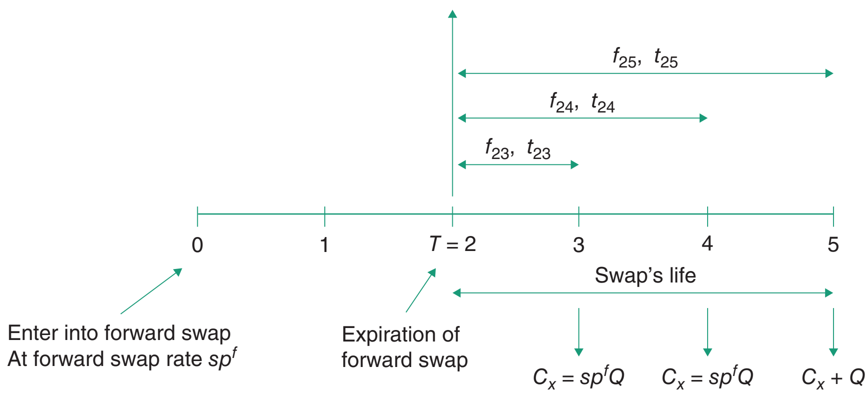

FIGURE 39.2 Two-year forward contract on a 3-year swap

where $d^f_{2i} = 1/(1 + f_{2i}t_{2i})$ and f_{2i} $(i = 3, 4, 5)$ are the (simple) *forward rates* quoted at $t = 0$ which apply to periods from $t = 2$ to $t = t_{2i}$. Hence the forward swap rate quoted today, for a swap with a maturity of $n = 3$ years, beginning in $T = 2$ years' time is:

$$sp^f = \frac{(1 - d^f_{25})}{\sum_{i=3}^{5} d^f_{2i}} \tag{39.3}$$

The forward swap rate depends only on the individual forward rates (from year-2 onwards). The above formula is of the same *form* as that used earlier to determine today's cash market swap rate, sp_0, except that for the forward swap we use forward rates beginning at $T = 2$ rather than spot interest rates beginning at $t = 0$.

39.2.2 Value of Forward Swap at Maturity

Assume $sp^f = 10\%$ for a 2-year forward swap on an underlying 3-year swap. Consider what happens at maturity ($T = 2$) of the forward swap, if the forward swap is cash settled. Assume that at $T = 2$ the cash-market swap rate $sp_T = 12\%$. The *value at expiration* V_{fs} to the fixed rate payer (i.e. long forward swap) is the 3-year annuity value of the cash flow $Q(sp_T - sp^f)$:

$$V_{fs}(T = 2) = Q(sp_T - sp^f)\,[AN] \tag{39.4a}$$

$$AN = [d_1 + d_2 + d_3] \tag{39.4b}$$

where (using simple rates) the discount factors are $d_i = 1/(1 + r_i t_i)$ and r_i $(i = 1, 2, 3)$ are the 1-year, 2-year and 3-year *out-turn* spot rates beginning at $T = 2$. Note that r_i are the actual spot rates (*ex-post*) known at T=2, which is now 'the present'. If the out-turn cash market swap rate $sp_T = 12\% > sp^f = 10\%$ then the long receives cash of V_{fs} at T= 2, from the swap dealer. If the out-turn swap rate $sp_T = 8\% < sp^f = 10\%$ then the holder of the long forward swap pays out V_{fs} to the swap dealer at T= 2.

The difference between a forward swap and a swaption is that the latter allows the holder to benefit from any fall in swap rates, while also putting a known ceiling on the swap rate payable in the future. For this privilege you pay the swaption premium. In contrast, the forward swap 'locks in' a swap rate sp^f payable in the future and this implies you cannot take advantage of low swap rates should they occur – but by way of 'compensation', the forward swap does not require an up-front payment (we ignore margin payments).

39.3 MORTGAGE-BACKED SECURITIES (MBS)

US financial institutions – primarily banks, savings and loan associations (S&Ls) and mortgage companies – issue fixed-rate mortgages with initial terms of between 15 and 30 years. However, these mortgages can be bundled up into a portfolio and sold to investors in the form of mortgaged-backed securities (MBS). This is *securitisation* – in the US and UK the MBS primary and secondary market is very large. The principal and interest payments of a MBS are often guaranteed by a government agency (e.g. in the US, the Government National Mortgage Association GNMA or 'Ginnie Mae'). Securitisation can also be based on future receipts from other 'assets' such as car loans, credit cards, recording royalties, telephone charges, or many other types of loan repayments – as we see in Chapter 43.

39.3.1 Mortgage Pass-throughs and Strips

There are a wide variety of ways that MBS can be structured and then sold on to investors (i.e. mutual funds, pension funds, hedge funds, sovereign wealth funds, and wealthy endowment funds – like those held by Harvard and Yale Universities).

The simplest MBS are *mortgage pass-throughs*. Here, the investor (periodically) receives a proportion of the interest and principal payments on the underlying mortgages. In a 'mortgage pass-through', the initial value of the underlying mortgages V_0 held by an investor is equal to the present value of the fixed (coupon) mortgage payments of $\$C$ (per period) plus the present value of any repayments of principal. The discount rates used to calculate the present value are often *risk-free* yields, since many mortgage payments are guaranteed by institutions such as GNMA. Otherwise the discount rate is a risk-free rate plus a spread, to reflect the credit risk of the mortgagees.

The pass-throughs are subject to considerable interest rate risk. Firstly, they are long maturity (high duration) 'bonds' and hence their (present) value (PV) rises (falls) when interest rates fall (rise) – this is a 'pure interest rate effect'.

Secondly, the pass-throughs are subject to *prepayment risk*. If interest rates fall, then some home owners will want to pay off their existing fixed-rate mortgages and refinance at new lower rates. Hence, the holder of a pass-through may find prepayments increasing but she loses some of the future interest payments on the mortgages, which no longer need to be paid (as the mortgage has been paid off early). The change in value of a mortgage *pass-through* to investors is therefore complex depending on the interaction of the pure interest rate effect and prepayments. Conversely, if market interest rates rise, the pure interest rate effect will reduce the PV of the mortgage pass-through but mortgagees may extend the life of the mortgage.

Mortgage pass-throughs can be split into interest-only (IO) and principal only (PO) strips. The investor can buy a share of either the IO or the PO strip, or both.

An interest only (IO) strip entitles the investor to receive only interest payments from the portfolio of mortgages.
A principal only (PO) strip entitles the investor to receive only the payments of principal.

To illustrate what might happen to the change in value of a pass-through (PT) we consider the separate effects on the present value (PV) of IO and PO strips, and use the relationship $PV_{PT} = PV_{IO} + PV_{PO}$.

When interest rates fall, the PV of any set of fixed interest payments will increase but any increase in prepayments will reduce the PV of an IO strip, as the prepayments imply that many future interest payments are now not required to be paid. The net effect is usually a fall in the value of IO strips, after a fall in interest rates, because of a high level of prepayments by mortgagees.

If interest rates rise, the PV of an IO strip will fall due to the pure interest rate effect but this may be offset by a lengthening of the life of the mortgage and hence the interest payments, so the overall effect may be an increase in value of the IO strip.

We can illustrate the effect of changes in interest rates on IO and PO strips and pass-throughs, by considering the following 'stylised' portfolio of mortgages.

Value of initial mortgage $P_0 = \$100{,}000$
Life of initial mortgage $n = 10$ years
Fixed rate agreed for mortgage repayments $r = 10\%$ p.a.

The fixed amount paid by the mortgagees \$$C$ p.a., must ensure that the present value of all the payments \$$C$ equals the current value of the mortgage advance, P_0. Hence, the required fixed annual mortgage payment \$$C$ is the solution to:

$$C \times [A_{n,r}] = P_0 \quad \text{where } A_{n,r} = \frac{1}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

$A_{n,r}$ is the PV of an annuity (of \$1 payable) over n-years, given the current (n-year) interest rate (yield), r. Solving the above equation gives $C = \$16{,}275$ p.a. In the first year, the interest accruing on the loan is $IO_1 = \$10{,}000 = rP_0$ which implies that $(C - rP_0)$ is \$6,275 can be used to reduce the initial principal, leaving an outstanding balance at the end of year-1, $P_1 =$ \$100,000 – \$6,275. For each succeeding year:

$$\text{Interest Payment in year-}t\text{: } IO_t = rP_{t-1}$$

$$\text{Principal repayment in year } t\text{: } PO_t = C - rP_{t-1}$$

P_t is calculated recursively:

$$P_t = P_{t-1} - PO_t = P_{t-1} - (C - rP_{t-1}) = (1+r)P_{t-1} - C \tag{39.5}$$

An investor (Mr Strip) who purchases an IO strip today, will be willing to pay the present value of the future mortgage *interest payments*. Suppose $y = 8\%$ p.a. is the current annual yield[2] and we also (somewhat arbitrarily) assume that with $y = 8\%(< 10\%)$, the *average life* of the mortgages in the securitised mortgage pool is 7 years because of early prepayments by some homeowners. The PV of these interest payments (Table 39.1, column 4) is:

$$PV_{IO}(8\%, 7\text{yr}) = d_1 \ IO_1 + d_2 \ IO_2 + \ldots \quad \ldots + d_7 \ IO_7 = \$41{,}733 \tag{39.6}$$

where using compound rates, $d_i = 1/(1+y)^i$. Similarly, an investor (Mr Principal) who purchases the principal only strip will be willing to pay the present value of the repayments of principal (Table 39.1, column 5) up to the end of year-7, plus the remaining principal $P_7 = \$40{,}472$ after early redemption by some mortgagees, hence:

$$PV_{PO}(8\%, 7\text{yr}) = d_1 \ PO_1 + d_2 \ PO_2 + \ldots + d_6 \ PO_6 + d_7 \ (PO_7 + P_7) = \$66{,}614 \tag{39.7}$$

The value of a ‘pass-through’ is simply the sum of the IO and PO strips, which is:

$$PV_{PT}(8\%, 7\text{ yr}) = PV_{IO} + PV_{PO} = \$108{,}347 \tag{39.8}$$

An investor in a mortgage pass-through has a claim on a set of coupon and principal payments. The reason MBS are derivative securities is that an investor in a MBS has implicitly *written (sold)* an American put option because the mortgagee has the right to redeem the mortgage. The strike price in the embedded put option is equal to the outstanding principal of the mortgage. Conversely, householders paying the mortgage have a long put which they will be more likely to exercise (i.e. remortgage), the lower are market interest rates in the future.

Let us consider what happens to the value of the IO and PO strips when interest rates change. We shall see that changes in value for the IO and PO strips depends crucially on how interest rates affect whether the prepayment option is exercised.

39.3.2 Interest Only Strips

If y falls to 7% and the prepayment date remains at the end of year-7 then PV_{IO} increases from \$41,733 to \$43,041 (i.e. pure interest rate effect). But if y falls to 7% and this moves the average prepayment date to the end of year-2,[3] then today the IO strip is only worth:

$$PV_{IO}(7\%, 2\text{yr}) = \frac{IO_1}{(1.07)} + \frac{IO_2}{(1.07)^2} = \$17{,}532 \tag{39.9}$$

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[2]Throughout, for pedagogic reasons we assume the yield curve is flat (i.e. currently at 8% p.a. for all maturities) and we also assume parallel shifts in the yield curve. This simplifies the calculations because we do not have to measure changes in value, when different spot yields change by different amounts.

[3]For illustrative purposes we take an extreme change in the average prepayment date.

TABLE 39.1 Mortgage-backed securities (MBS)

Years	Loan outstanding	Fixed payments	Interest accrual	Repayment of principal	Loan outstanding after payments		NPV	
							IO	PO
1	100,000	16,274.54	10,000	6,274.54	93,725.46			
2	93,725.46	16,274.54	9,372.55	6,901.99	86,823.47	NPV(7%, 2 years)	£17,532.14	£87,727.50
3	86,823.47	16,274.54	8,682.35	7,592.19	79,231.27			
4	79,231.27	16,274.54	7,923.13	8,351.41	70,879.86			
5	70,879.86	16,274.54	7,087.99	9,186.55	61,693.31			
6	61,693.31	16,274.54	6,169.33	10,105.21	51,588.10			
7	51,588.10	16,274.54	5,158.81	11,115.73	40,472.37	NPV(7%, 7 years) NPV(8%, 7 years) NPV(9%, 7 years)	£43,041.21 £41,732.57 £40,487.63	£69,871.16 £66,613.94 £63,561.14
8	40,472.37	16,274.54	4,047.24	12,227.30	28,245.07			
9	28,245.07	16,274.54	2,824.51	13,450.03	14,795.04			
10	14,795.04	16,274.54	1,479.50	14,795.04	0	NPV(9%, 10 years)	£44,444.24	£60,000.19

So although the fall in y from 8% to 7% raises the PV of each interest receipt, *the number* of interest payments has dropped from seven to two. Hence there is a substantial fall in PV_{IO} from \$41,733 to \$17,532, a fall of 60%.

Now suppose that y increases from 8% to 9%. If the average life of the mortgages remains at 7 years, then PV_{IO} will fall from \$41,733 to \$40,488. However, if the life of the average (fixed interest) mortgage is extended to 10 years, then there is an increase in PV_{IO} from \$41,733 to \$44,444 (6.5%), because of the increased *number* of interest payments (even though these are discounted at a higher rate). Hence, the change in value of PV_{IO} depends crucially on the interaction between the change in y ('pure interest rate effect') and the change in the average prepayment period (i.e. whether the prepayment option is exercised by the homeowner).

39.3.3 Principal Only Strips

There will be some early prepayments of principal simply because people move house for a whole variety of reasons (e.g. arrival of children, changing jobs etc.). But here we only concentrate on the *interaction* between interest rates and the prepayment option.

Suppose y falls from 8% to 7% or rises from 8% to 9%, with no change in the 7-year maturity of the underlying mortgages. Clearly, after an interest rate fall (rise) PV_{PO} rises (falls), ceteris paribus. However, if the fall in y from 8% to 7% brings the average prepayment date to the end of year-2, then:

$$PV_{PO}(7\%, 2\text{yr}) = \frac{PO_1}{(1.07)} + \frac{PO_2 + P_2}{(1.07)^2} = \$87{,}728 \tag{39.10}$$

where $P_2 = \$86{,}823$ (Table 39.1, column 6, row 2) is the 'large' prepayment of the remaining principal on the mortgages at the end of year-2. Hence PV_{PO} increases substantially from $PV_{PO}((8\%, 7\text{yr})) = \$66{,}614$ to $PV_{PO}(7\%, 2\text{yr}) = \$87{,}728$, that is, an increase of 31.7%.

Alternatively, if interest rates rise immediately from 8% to 9% and this pushes the prepayment date out to year-10, then PV_{PO} falls substantially from $PV_{PO}(8\%, 7\text{yr}) = \$66{,}614$ to $PV_{PO}(9\%, 10\text{yr}) = \$60{,}000$, that is, a fall of 9.9%.

From Table 39.1 we can calculate the PV of the mortgage pass-through using $PV_{PT} = PV_{IO} + PV_{PO}$ for our three cases:

$$PV_{PT}(8\%, 7\text{yr}) = \$108{,}347$$

$$PV_{PT}(7\%, 2\text{yr}) = \$105{,}260 \text{ (2.8\% fall)}$$

$$PV_{PT}(9\%, 10\text{yr}) = \$104{,}444 \text{ (3.6\% fall)}$$

Hence here, the value of the mortgage 'pass-through' is less sensitive to interest rate changes and consequent changes in the schedule of prepayments by homeowners, than are IO and PO strips, taken separately. The value of IO and PO strips are very sensitive to the complex interaction between changes in interest rates and any prepayments by homeowners, which also depend on the characteristics of the cohort of mortgage borrowers – such as age, marital status, number of children, geography, income, state of the (local) economy etc. Hence it is difficult to value and predict changes in IO and PO strips. While they (often) carry a government credit guarantee, there is no guarantee that their market value will remain unchanged and hence they are far from being 'safe investments' – as the events of the crash of 2008–9 illustrated.

39.4 HEDGING FIXED INCOME DERIVATIVES

The price of fixed income derivatives, such as options on T-bond futures or caps and floors, can change substantially due to non-parallel shifts in the yield curve or changes in the term structure of volatility. There is therefore not just 'one number' for the delta, gamma, and vega of a fixed income derivative – for example, you can have a delta for a cap with respect to the 3-month interest rate, the 6-month rate etc. Once you have calculated the type of shift in the yield curve you want to hedge against (e.g. for a parallel shift or a twist in the yield curve) then you can calculate the appropriate delta and gamma by recalculating the option price under this new interest rate scenario, using an option pricing formula such as Black's model – this is the 'perturbation approach'.

Suppose we want to calculate the delta of an option on a fixed income asset (e.g. option on a T-bond or option on Eurodollar futures). The delta is defined as the change in the option price due to a (small) change in spot yields (i.e. the 'zero curve'). Possible alternative ways of modelling and simulating changes in interest rates along the yield curve are:

- Calculate the change in the option price due to a parallel shift in all spot yields of 1 bp – this is sometimes referred to as a PV01, DV01, or PVBP – that is, the 'present value' or 'duration value' of a 1 bp (0.01%) parallel shift.
- Divide the zero curve (or forward curve) into a number of 'maturity buckets' and calculate the change in the option's price for a 1 bp change in *one* of these time buckets (e.g. for 0–3 month rates), holding the rest of the yield curve unchanged. Or alternatively calculate the change in the options price for a 2 bp change in one time bucket (e.g. 0–3 month rates) and a 1 bp change in another time bucket (e.g. 4–6 month rates) – to mimic a twist in the yield curve.
- Use a principal components analysis to decompose all spot rate changes across the yield curve into say the 'first 3 principal components'. Then calculate a delta for each of these three principal components (PC). The delta of the first-PC represents an (approximate) parallel shift in the yield curve, the delta for the second-PC represents a twist in the curve and the third-PC often can be interpreted as a possible 'bowing' of the yield curve.
- Calculate the change in the option's price due to small changes in all the 'yields' that have been used to construct the zero curve. The easiest way to think of this is that all

spot rates are changed by different amounts based on a trader's view of likely 'shifts' along the yield curve – this is like the 'bucket' case but with many more 'buckets'.

- A slightly more complex variant is to use the actual rates used to construct the yield curve. If the short end of the curve has been derived using yields based on zero-coupon bonds (bills) then these yields would be changed by a small amount. If the long end of the curve is constructed from spot yields derived from swap rates, then swap rates would be changed by a small amount.

Once we have our new spot rates we can calculate a new price for the (fixed income) option using an appropriate pricing formula (e.g. Black's model) and hence calculate the option's (overall) delta.

Calculating gamma is even more involved. Suppose there are five different spot rates r_i used to construct the zero curve and to price the option (e.g. swaption). Then there are at least five different gammas with respect to changes in these five different spot rates, which influence the change in the option's price df. (This assumes we ignore cross product terms $\partial^2 f/\partial r_i \partial r_j$ in the Taylor series expansion of df.) But once we have made our specific choice of a specific shift in the yield curve (e.g. parallel shift), the calculation of specific gammas using the perturbation method is straightforward, in principle.

The same procedure can be used to calculate vega by changing the volatility (of interest rates) by a small amount and calculating the change in the option premium. If the option premium depends on the volatility of interest rates at different horizons then when calculating the option's vega we can either assume an equal change in all volatilities or we can change volatilities by different amounts at different maturities (e.g. change the volatility of 3-month interest rates by a different amount than you change the volatility of 6-month rates).

39.4.1 Hedging Interest Rate Options and Swaps

Suppose a bank acts as a swap dealer and also buys and sells interest rate options such as caps and floors (to corporates and other banks). A long position in an interest rate cap increases in value as interest rates increase – it has a positive delta. So if a bank sells a cap (to a corporate) and interest rates rise, the bank loses – a short cap has a negative delta.

A cap is a series of caplets on an agreed notional principal. For simplicity assume Mega-Bank on 15 June 2019 holds a cap which consists of three caplets with maturity dates of September-2019, March-2020 and September-2020. Each caplet has a known price (on 15 June 2019). If we assume a 1 bp increase in interest rates (over 1 day say) then we can work out the change in value (over 1 day) of each caplet and hence of the cap itself. The cap will also have a gamma (with respect to interest rates) and a vega (with respect to the volatility of interest rates, σ).[4]

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[4]The gamma of a cap with price C is $\Gamma = \partial^2 C/\partial r^2$ and the vega is $\Lambda = \partial C/\partial \sigma$. A numerical value for these two Greeks can be directly obtained from a closed-form solution for C such as Black's model.

Assume MegaBank has positions in swaps and caps – how does the bank hedge these positions? We proceed as we did when hedging a portfolio of stock options (see Chapter 18) but here it is changes in interest rates and the volatility of interest rates that cause changes in the value of MegaBank's caps or swaps. We can easily calculate the total gamma and vega of MegaBank's 'swaps plus cap' portfolio. (The vega only arises from the caps, the vega of a swap is zero.)

Our first hedging method for MegaBank's 'swaps plus cap' can be sequential because we choose first to hedge with an interest rate option-X and then with a fixed income asset *with a zero vega* (e.g. a T-bond, which has non-zero gamma and delta but a zero vega). We first use option-X to hedge the vega, then we hedge the gamma and finally the delta.

First, we use an interest rate option-X (e.g. cap with a different strike or expiration date) to offset the vega of MegaBank's current cap position – so we are now vega neutral. This will lead to a new portfolio gamma, since option-X has a non-zero gamma. We can offset this new portfolio gamma by buying or selling T-bonds or FRAs (which have non-zero gammas). The important point is that achieving a gamma-neutral portfolio using T-bonds does not upset our newly acquired vega neutral position – as T-bonds have zero vegas.

However, including bonds in our hedge portfolio to achieve gamma neutrality, will alter the delta of our portfolio, which we now recalculate. Finally, we offset this remaining portfolio delta by using Eurodollar futures (which have zero gamma and vega). We are then vega, gamma, and delta neutral and we have been able to hedge sequentially – first with option-X (vega neutral), then with bonds (gamma neutral) and finally with Eurodollar futures (to achieve delta neutrality).

Our second hedging method uses two interest rate options (e.g. a mixture of caps or bond options or bond futures options). This hedge cannot be done sequentially because the two options both have non-zero gammas and non-zero vegas. To hedge MegaBank's 'swaps+caps' position we first use two other interest rate options X and Y (e.g. caps with different strikes and expiration dates) to *simultaneously hedge* the gamma and vega of our initial position. (This was done for stock options in Chapter 18.) MegaBank's portfolio now consists of 'swaps+caps+options-X+options-Y' and has a portfolio gamma and vega that are both zero. But MegaBank, by adding the two options X and Y, has changed the delta of its initial 'swap+caps' portfolio, so it now recalculates its new portfolio delta. MegaBank then 'neutralises' this new portfolio delta by taking offsetting positions in Eurodollar futures – which does not affect MegaBank's existing vega-gamma neutral position because Eurodollar futures contracts have zero vega and gamma.

39.5 SUMMARY

- A payer swaption is an option on a swap. If you hold a payer swaption with a strike K_{sp}, you have effectively placed an upper limit of K_{sp} on the fixed rate payable in a swap (which will start at the expiration date of the swaption contract). However, you can also take advantage of low cash market swap rates in the future by not exercising the swaption at maturity, if the out-turn swap rate sp_T is below K_{sp} and instead you purchase a cash market swap at the low swap rate, sp_T.

- A forward swap can be used by a corporate to 'lock in' a known swap rate that will begin at a specific time in the future. However, the corporate cannot then benefit from lower swap rates in the future should these occur.
- One of the largest and most active fixed income markets is for various forms of mortgage-backed securities (MBS). It is the prepayment possibility in these portfolios of mortgages that give them their 'embedded option' characteristics. The market value of mortgage pass-throughs, interest only and principal only strips are very sensitive to the interaction between changes in interest rates and the decision to prepay the principal early (or to extend the life of the mortgage).
- Calculating the Greeks for fixed income derivatives can be done by the perturbation method. For example, change the input (e.g. spot yields) by a small amount and recalculate the new option price (e.g. using Black's model or the BOPM) – this gives the option's delta for this specific set of spot rate changes. Other Greeks can be calculated in a similar way.
- A portfolio of options on fixed income assets (e.g. caps and floors) can be vega-hedged and gamma-hedged using other fixed income options (e.g. caps and floors with different strike prices and expiration dates) and finally delta-hedged using Eurodollar futures contracts.

EXERCISES

Question 1

A corporate treasurer has to borrow \$10m in 2 years' time, at 360-day LIBOR, for a further 2 years. She decides to take out a forward swap, to receive-LIBOR and pay-fixed, with tenor 1 year. The swap will begin in 2 years.

Current forward rates (continuously compounded) are $f_{12} = 5.0\%$, $f_{23} = 5.1\%$, $f_{24} = 5.2\%$, $f_{25} = 5.3\%$, $f_{13} = 5.05\%$, $f_{14} = 5.15\%$. Calculate and explain how the forward swap rate is determined.

Question 2

There is a forward swap, with swap rate sp^f and a swaption with strike K_{sp}. Assume both derivatives have maturity dates at T and the underlying swap has the same time to maturity, tenor, and notional principal for both derivatives.

What is the key difference between a forward swap, with swap rate sp^f, and a swaption with strike K_{sp}? Explain with reference to the payoffs at maturity.

Question 3

You hold a $Q = \$10\,\mathrm{m}$, bank deposit on 90-day LIBOR with four further resets in 90, 180, 270, and 360 days. Explain how you could reduce interest rate risk by using either a forward swap (with swap rate sp^f) or an interest rate 'floor' (with strike rate, K_{FL}). What is the key difference in outcomes at maturity for these two derivatives?

Question 4

How do mortgage pass-throughs differ from interest only (IO) and principal only (PO) strips?

Question 5

Explain why a principal only (PO) mortgage strip has a mark-to-market value to an investor that may be extremely volatile, with respect to changes in interest rates. What two key factors determine the change in value of a PO mortgage strip?

CHAPTER 40

Pricing Fixed Income Options: Black's Model and MCS

Aims

- To show how Black's model provides closed-form solutions for the price of European options on T-bonds, on T-bond futures, on caps, floors, collars and on European swaptions.
- To price fixed income options using Monte Carlo simulation (MCS).

In previous chapters we discussed hedging/insurance using options on T-bonds and Eurodollar futures and how caps, floors, collars, and swaptions are used to hedge/insure interest sensitive assets and liabilities such as floating rate bank deposits and loans. In this chapter we concentrate on how to price some of these derivatives. To price fixed income derivatives we can use:

- Black's model which gives closed form solutions
- MCS under risk-neutral valuation (RNV)
- BOPM model with an interest rate lattice (tree)
- Equilibrium term structure approach.

Black's model assumes the price of the underlying asset in the options contract has a lognormal distribution, *at maturity of the option*. MCS generates a path for the short-rate and prices the derivative under RNV. The BOPM uses a lattice for the 'short-rate' of interest. The equilibrium yield curve approach assumes a specific stochastic process for the interest rate and solves mathematically for the derivatives price – the BOPM and the equilibrium yield curve approach are dealt with in Chapters 41 and 49, respectively.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

40.1 BLACK'S MODEL: EUROPEAN OPTIONS

Black's (1976) model, which was originally used for pricing options on commodity futures can be adapted to give a closed-form solution for prices of *European* bond options, futures options, caps, floors, and European swaptions. The cash payout on an interest rate option may occur on the expiration date of the option, T. But note that for some options the *payoff* V_T is *determined* at T, but the actual *cash payout* is delayed to $T^* > T$. We adopt the following notation:

$$V_T = \text{cash value (price) of the ‘underlying asset’ at } T$$
$$F_0 = \text{forward price at } t = 0 \text{ of the underlying asset, for delivery at } T$$
$$T = \text{time to maturity of the option}$$
$$r = \text{interest rate (continuously compounded) for maturity, } T$$
$$K = \text{strike price}$$
$$\sigma = \text{volatility of the forward price.}$$

Black's model does not assume a GBM for the underlying but requires somewhat weaker assumptions namely, that at expiration $\ln V_T$ is normally distributed with standard deviation of $\sigma\sqrt{T}$ and $E(V_T) = F_0$, the forward price. If the European (fixed-income) option has a payoff, *which is paid at* T, then Black's formulas for the call and put premia are:

$$C = e^{-rT} \; [F\,N(d_1) - KN(d_2)] \tag{40.1}$$

$$P = e^{-rT}[KN(-d_2) - F_0N(-d_1)] \tag{40.2}$$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2(T/2)}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

If the option payoff is determined at T, but the *actual cash payment* is delayed until T^* then the above formulas for d_1, d_2 still apply (i.e. using T) but the discount factor is $e^{-r^*T^*}$ (where r^* is the interest rate for maturity T^*) .

40.1.1 European Bond Option

For a European option on a T-bond, the futures price of the bond is given by $F_{0,T} = [B_0 - PV(C)]e^{rT}$ where $PV(C)$ is the present value of the coupons on the T-bond, payable over the life of the option and B_0 is the current cash price of the bond. The volatility σ used in Black's model is for the forward bond price.

EXAMPLE 40.1

Price of European Bond Option

Consider a 9-month European call option with strike $K = 1{,}000$, on a 10-year T-bond with maturity value $M = \$1{,}000$ and semi-annual coupons. The current cash market bond price $B_0 = 980$ and over the next 9 months the present value of the coupons (discounted at the appropriate spot yields) is 20.

The 9-month spot yield is $r = 5\%$ p.a. (continuously compounded) and the volatility of the forward bond price is 8% p.a.

$$F_{0,T} = [B_0 - PV(C)]e^{rT} = [980 - 20]\, e^{0.05(9/12)} = 996.68$$

$$C = e^{-rT}\ [F\ \ N(d_1) - KN(d_2)] = 0.9632[996.68(0.4947) - 1{,}000(0.4671)] = \$25$$

40.1.2 Caps and Floors

Consider a caplet on 90-day LIBOR on a notional principal $\$Q$ with strike rate K_{cap}. The $tenor_k = 90/360$ and the caplet matures in $t_k = 2$ years. If the out-turn 90-day LIBOR rate at maturity of the cap is L_k the payoff at maturity is:

$$\textit{Payoff long caplet (at maturity)} = tenor_k\, Q \max(L_k - K_{cap}, 0) \tag{40.3}$$

The payoff is calculated at t_k but is paid at $t_{k+1} = t_k + tenor_k$. Note that r^* is continuously compounded, whereas K and f_k refer to *simple* annual rates. Assume $\ln L_k$ is normally distributed with standard deviation $\sigma_k\sqrt{t_k}$ then:

Invoice price of caplet that matures at t_k:

$$V(t_k)_{caplet} = tenor_k\, Q e^{-r^*\, t_{k+1}}[f_k N(d_1) - K_{cap}N(d_2)] \tag{40.4}$$

$$d_1 = \frac{\ln(f_k/K) + (\sigma_k^2/2)t_k}{\sigma_k\sqrt{t_k}} \quad d_2 = d_1 - \sigma_k\sqrt{t_k}$$

f_k = forward interest rate between t_k and t_{k+1}, calculated at $t = 0$

r^* = interest rate for maturity t_{k+1} (continuously compounded)

EXAMPLE 40.2

Pricing a Caplet

Suppose $Q = \$100{,}000$, *maturity* $t_k = 1$-*year*, $tenor_k = 90/360$, $f_k = 3\%$ p.a. (90-day simple rate, 90/360 day-count convention).

There is a flat term structure. Continuously compounded interest rates r (at all maturities) are given by, $1 + 0.03(90/360) = e^{r(90/360)}$, so $r = (360/90)\ \ln\ (1 + 0.03/4) = 0.0299$, $K_{cap} = 3.5\%$ p.a. and $\sigma = 20\%$ p.a. We have $d_1 = -1.4915$, $d_2 - 1.5915$. Using (40.4) the caplet price is:

$$V(t_k)_{caplet} = (90/360)(10{,}000)\ e^{-0.0299(1.25)}[0.03(0.0679) - 0.035\ \ (0.0557)] = 2.0773$$

40.1.3 Caps

Suppose we have a cap on a notional principal of $\$Q$ with n-reset dates, $t_1, t_2 \ldots\ldots t_n$ with the final *payment at* $T = t_{n+1}$. The tenor is the time between the reset dates (measured in years), $tenor_k = t_{k+1} - t_k$ (= 90/360 for example).

To price a cap, each caplet must be valued separately (see Equation 40.4) but this requires different forward volatilities σ_k^2 for each caplet. However, in practice cap premia are often quoted using *flat volatilities* – that is, σ_k^2 is assumed to be constant for all horizons k. The invoice price of a cap is the sum of the caplet premia that comprise the cap.

$$V_{cap} = \sum_{k=1}^{n} V(t_k)_{caplet}$$

40.1.4 Floorlet and Floors

The invoice price of a floorlet that matures at t_k is:

$$V(t_k)_{FL} = tenor_k Q\ \ e^{-r^* t_{k+1}}\ \ [\ K_{FL}N(-d_2) - f_kN(-d_1)\ ] \tag{40.5}$$

and the invoice price of a floor is the sum of the floorlet premia that make up the floor.

40.2 PRICING A CAPLET USING MCS

MCS can often provide a quick and conceptually easy method of valuing some fixed-income derivatives. For example, under RNV the price of a caplet is:

$$V_t = E^*[e^{-r_{av}(T-t)}\, V_T] \tag{40.6}$$

For example, consider pricing a caplet (on 90-day LIBOR) with a strike price $K_{cap} = 10\%$ and $T = 1$ year. To use MCS we need to generate a data series for the short-term LIBOR rate. Suppose we choose a discrete approximation to Vasicek's mean reverting model:

$$r_t - r_{t-1} = a(b - r_{t-1})dt + \sigma\sqrt{dt}\ \ \varepsilon_t$$

where ε – *niid(0,1)*. The (mean) long-run interest rate might be $b = 3\%$ p.a. (0.03) and the rate of convergence, $a = 0.20$. We take $dt = T/n = 0.01$, so we divide 1 year into $n = 100$ time units. Assume that the current short rate is $r_0 = 4\%$ p.a. (0.04) and the notional principal in the caplet is Q. MCS involves the following steps:

- Generate $n = 100$ observations on r_t and calculate its *average* value, $r_{av} = \sum_{t=1}^{n} r_t/n$
- Calculate the payoff of the caplet at expiration $= \max\{r_{100} - K_{cap}, 0)$
- The price of caplet for the first Monte Carlo run is $V^{(1)} = e^{-r_{av}(T+90/360)}\ \max\{r_{100} - K, 0\}$
- Repeat the above steps for $m = 10{,}000$ runs of the MCS then:

$$\textit{Invoice price of caplet} = Q(1/m)\ \sum_{i=1}^{m} V^{(i)} \tag{40.7}$$

It is easy to see how this method can be applied to other interest rate derivatives discussed above. For example, the value of a cap using MCS is simply the sum of the MCS prices of the individual caplets (with different maturity dates and strikes).

Option premia calculated using MCS are dependent on the specific stochastic model chosen for the short rate. Hence estimation issues arise. But the option can be priced using MCSs with alternative stochastic processes for the interest rate – and the sensitivity of alternative MCS option premia to alternative stochastic processes can be assessed.

40.3 EUROPEAN SWAPTION: BLACK'S MODEL

A 'pay-fixed, receive-floating' swap is equivalent to being short a fixed rate bond and long a floating rate bond. A *payer swaption* is an option to enter into an interest rate swap at maturity of the option contract T, to pay-fixed, receive-floating at an agreed swap rate K_{sp}, on a notional principal Q over the life of the swap. The underlying swap in the swaption contract begins at T and matures at $T_{sp} = T + n$ years. (A receiver swaption is an option to enter into a swap contract to receive-fixed, pay-floating at an agreed swap rate K_{sp}.) We use the following notation (Figure 40.1) and assume the swap rate at T is lognormal:

K_{sp} = swap rate in the option contract (simple rate, p.a.)

T = expiration of swaption (i.e. commencement of the swap)

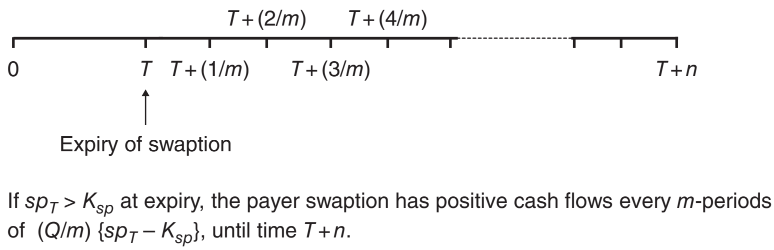

FIGURE 40.1 Payer swaption

n = maturity of underlying swap in the swaption contract

m = number of payments per year in the swap (= $1/tenor$)

sp_T = cash market swap rate (for an n-year swap) at T (simple rate, p.a.)

Q = notional principal for the underlying swap (in the swaption contract)

A payer swaption gives rise to *a series* of cash flows at expiration:

$$Q(tenor)\ \max(sp_T - K, 0) \tag{40.8}$$

which accrue at $t_1, t_2, \ldots\ldots t_n$ years from today (where $t_i = T + i/m$) – see Figure 40.1. The value today t_0, of any *one* of these cash flows received at time t_i is given by Black's formula:

$$V\ \text{(single cash flow)} = Q(tenor)e^{-r_i t_i}[sp^f\ \ N(d_1) - KN(d_2)] \tag{40.9}$$

$$d_1 = \frac{\ln(sp^f/K) + \sigma^2(T/2)}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T}$$

sp^f = *forward* swap rate to begin at T (for maturity at T_{sp}), simple rate, p.a.

r_i = (continuously compounded) interest rate for maturity t_i.

At t_0 we know the forward swap rate, sp^f. A payer swaption comprises a *series* of cash flows, hence its value (invoice price) today, at t_0 is:

$$\$V_{swpo}^{payer}(t_0) = Q(tenor)\,AN[sp^f\ \ N(d_1) - K_{sp}N(d_2)] \tag{40.10a}$$

$$AN = \left[\sum_{i=1}^{m.n} e^{-r_i t_i}\right] \tag{40.10b}$$

AN is the present value of an annuity of \$1 paid at t_i over $(m.n)$ periods.

EXAMPLE 40.3

Pricing a European Payer Swaption

Suppose $Q = \$100{,}000$, swaption maturity $T = 5$ years, with $sp^f = 4.04\%$ p.a. (flat term structure, simple rate), $r^* = 4\%$ (continuously compounded),[1] $K_{sp} = 4.2\%$ p.a. and $\sigma = 20\%$ p.a. The underlying swap has maturity $n = 3$ years and *tenor* = 180/360 ($m = 2$).

$$d_1 = \frac{\ln(4.04/4.2) + (0.2)^2(5/2)}{0.2\sqrt{5}} = 0.1369, \quad d_2 = 0.1369 - 0.2\sqrt{5} = -0.3103$$

$$AN = \left[\sum_{i=1}^{6} e^{-0.04t_i}\right] = 4.5829 \quad \text{where } t_i = 5 + i/2.$$

$$\begin{aligned} \$V_{swpo}^{payer}(t_0) &= Q(tenor)AN[sp^f \ N(d_1) - K_{sp}N(d_2)] \\ &= 100{,}000(180/360)\,4.5829\,[0.0404\,(0.5544) - 0.0402\,(0.3782)] = \$1{,}493.6 \end{aligned}$$

If we have a receiver swaption (i.e. receive-fixed, pay-floating) then the payoff is $Q(tenor)\ \max(K_{FL} - sp_T, 0)$ and the invoice price of the put is:

$$\$V_{swpo}^{receiver} = Q(tenor)\ AN\ [\ K_{sp}\ \ N(-d_2) - sp^f\, N(-d_1)\ ] \tag{40.11}$$

40.3.1 Limitations of Black's Formula

Black's model is widely used in practice because it provides a closed-form solution for *European option* premia on T-bonds (and T-bond futures), Eurocurrency futures and on swaps (i.e. swaptions). However, Black's model has its limitations. First, it is inconsistent to assume that at expiration of the option, the bond price, the bond futures price, the short rate, and the swap rate are *all* lognormally distributed.

Second, in reality a bond price or interest rates do not have a constant volatility over the life of the option but Black's formula assumes a constant volatility. For example, the volatility of the price of a bond approaches zero as the bond nears its maturity date. However, if the T-bond or T-bond futures option has a maturity T which is small relative to the 'life' of the underlying bond being delivered, then the constant volatility assumption may provide a reasonable approximation.

[1] If $r = 4\%$ is the continuously compounded rate and the yield curve is flat then the forward rate (simple rate, *tenor* = 180/360) is $sp^f = (1/tenor) * (\exp(r^*tenor) - 1) = 4.04\%$.

Note that Black's model only deals with European and not American or other 'more exotic' fixed income options. Some of the weaknesses of Black's model can be dealt with by using MCS – for example, we can incorporate stochastic volatility – but as we shall see in Chapter 41, the BOPM is also useful in pricing fixed income securities.

An Excel file on the website prices a caplet using MCS. MATLAB programs to implement Black's model for bond options, caps and floor and swaptions are also provided.

40.4 SUMMARY

- Black's (1976) model can be adapted to give a closed-form solution for pricing certain fixed-income options as long as we assume that at expiration, either interest rates or bond prices or futures prices are distributed lognormally. The latter cannot be true for all three 'prices' but this is often ignored in practice because of the tractability of Black's formula.
- Black's formula is used to price European options on T-bonds (and T-bond futures) as well as caps, floors, and swaptions.
- Monte Carlo simulation provides a flexible approach to pricing many types of fixed income options, including (some) path-dependent options. Of course, the resulting option prices will only be accurate if the stochastic process assumed for interest rates is a reasonable representation of their future behaviour. The robustness of the calculated MCS option premia can be assessed using alternative stochastic processes for interest rates.

EXERCISES

Question 1
Black's model can be applied to European options on T-bonds, T-bond futures, caps, floors, and swaptions. What key assumptions are required to apply Black's model?

Question 2
When pricing a caplet using MCS why do we not assume that the interest rate follows a geometric Brownian motion (GBM)?

Question 3
What are the key factors which determine the payoff at maturity from a 3-year payer swaption with notional principal Q, tenor of 90 days and a swap life of 2 years?

Question 4
Use Black's model to value a 6-month European put option on a 10-year bond with strike price $K = \$100$. The current price of the 10-year bond is $P_B = \$103$.
Present value of coupons on the bond (paid during the life of the option), $PV(C) = \$4$. The 1-year interest rate $r = 3\%$ p.a. (continuously compounded). The bond's (forward price) volatility is 5% p.a.
Show your calculations for $d_1, d_2, N(.)$, etc.

Question 5
Today, using Black's model, on a $T = 2$-year option, the implied price volatility for an underlying bond which matures 10 years from today is σ_{imp}. Suppose this implied volatility σ_{imp} is used today to price a $T = 9$-year option (on the same 10-year bond). Would the option price for the $T = 9$-year option be too high or too low?

Question 6
Use Black's model to calculate the price of a 9-month cap, on 90-day LIBOR, with strike $K_c = 5\%$ p.a. (actual/360, day count) and principal $Q = \$100{,}000$. The (interest-rate) volatility is $\sigma = 10\%$ p.a. and the 90-day forward rate (beginning in 9 months) is $f_k = 0.05$ (simple rate, actual/360).

The yield curve is flat at $r_s = 5\%$ p.a. (over 90 days, simple rate) so $(1 + r_s/4) = e^{r(1/4)}$ hence $\ln(1 + r_s/4) = r/4$, *which gives* $r = 4.969\%$ p.a. (continuously compounded).
Show your calculations for $d_1, d_2, N(.)$, etc.

Question 7
The underlying asset in a ($T = 3$-year) payer swaption with a strike of $K_{sp} = 7.5\%$ p.a. is an $N = 4$-year swap, with annual payments (tenor = 1 year) and principal $Q = \$100{,}000$. The volatility of the 4-year (forward) swap rate, $\sigma = 20\%$ p.a.
The yield curve is currently flat at $r = 8\%$ p.a. (continuous compounding). Forward rates at all maturities are 8% p.a. (continuous compounding) which gives a forward swap rate (simple rate) of $sp^f = 8.33\%$ p.a. ($= \exp(0.08) - 1$). The volatility of the (4-year) forward swap rate, $\sigma = 20\%$ p.a.
Show your calculations for $d_1, d_2, N(.)$, etc.

CHAPTER 41

Pricing Fixed Income Derivatives: BOPM

Aims

- To derive an interest rate lattice, which precludes risk-free arbitrage profits and is consistent with the current term structure of interest rates.
- To demonstrate how the BOPM is used to price European and American options such as caps, floors, collars, swaptions, callable bonds, and FRAs/FRNs with caps and floors.

In this chapter we price many different fixed income derivatives using the BOPM with an arbitrage-free lattice for the short-rate of interest. This 'no-arbitrage' approach ensures that the interest rate lattice is constructed so it is impossible to make risk-free profits by trading (different bonds) along the current yield curve. The lattice is *calibrated* so that it exactly mimics the current observed term structure – hence, the current term structure is an input to the BOPM and the output is the derivatives price.[1]

Pricing interest rate derivatives is more difficult than pricing options on stocks, currencies, commodities and futures contracts because some of the assumptions of the original Black–Scholes model are unlikely to hold for fixed income assets. In particular:

(i) the underlying stochastic process for short-term interest rates is more complex than for stock prices – for example, interest rates are mean reverting and therefore do not follow a geometric Brownian motion (GBM).

[1]Yet another alternative is the *equilibrium yield curve method*, which assumes a specific stochastic model for the short-term interest rate, whose parameters are estimated from past data. Given these parameters we then analytically solve for the derivatives price. This approach is discussed in Chapter 49.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

(ii) to price some interest rate derivatives we need not only the possible paths taken by a 90-day interest rate but the possible paths for interest rates with other maturities.
(iii) the volatility of interest rates may not be constant and may be different at different horizons.
(iv) if interest rates are stochastic, we should not discount the option's payoffs using a constant interest rate.

In Chapter 40 we noted that Black's model is widely used to price some fixed income derivatives such as European options on T-bonds, T-bond futures, Eurocurrency futures and swaptions because it provides a closed-form solution. But it has its limitations and in particular it does not price American and 'exotic' fixed income options. The BOPM is more flexible and can price many different fixed income derivatives – although being a numerical technique it is subject to estimation error and may require considerable computing power.

41.1 NO-ARBITRAGE APPROACH: BOPM

The BOPM can be used to price interest rate derivatives in much the same way as we demonstrated for stocks, but there are also some key differences. First, we require a lattice for the sequence of one-period spot rates (Figure 41.1) but this lattice must not allow any arbitrage profits to be made along the yield curve. This means that, given our chosen sequence of one-period spot rates, it must not be possible to sell a (zero-coupon) bond and use the proceeds to purchase other (zero-coupon) bonds and make risk-free profits.

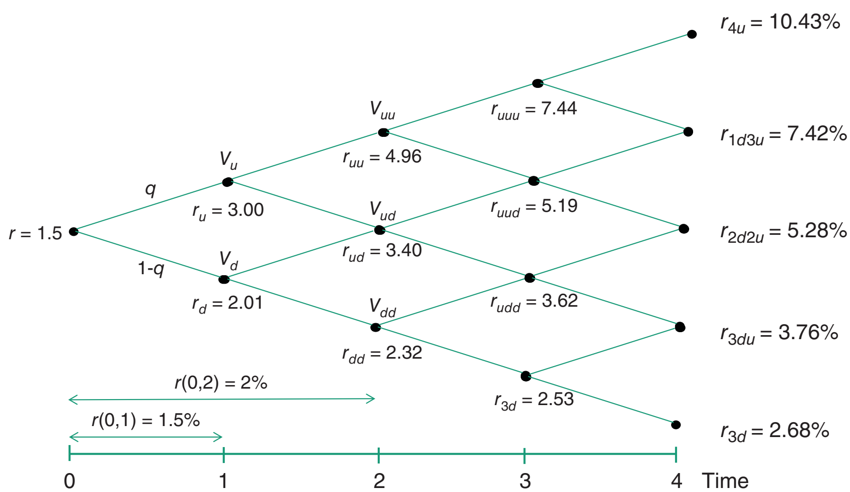

FIGURE 41.1 Short-rate lattice

The lattice in Figure 41.1 shows the possible one-period rates over four time periods and q is the *risk neutral* probability of an 'up' move. As noted in an earlier chapter, when pricing options, under risk-neutral valuation (RNV) we can *assign* a value $q = 1/2$. The size of the 'up' and 'down' movements in the one-period rates are then derived to be consistent both with the current observed term structure of 1-year, 2-year etc. spot rates and with empirical estimates of the term structure of *volatilities* of these spot rates. This is directly analogous to the calculation of U and D in the BOPM for stock options. However, for interest rates, the details of how to calculate the 'up' and 'down' movements are a little more complex. Assume for the moment that all the one-period rates in the lattice have been calculated and do not allow any risk-free arbitrage opportunities.

Terms such as V_{uu} (see Figure 41.1) denote the *value* of the derivative at this node. Also it can be shown that at each node such as uu (at $t = 2$), we know not only the one-period rate r_{uu}, say, but also the two-period rate (at node-uu) that is $r_{uu}(2, 4)$ and the three period rate, $r_{uu}(2, 5)$ etc., although the latter are not marked on the lattice of Figure 41.1.

41.1.1 Notation

What is important in the lattice (Figure 41.1) is the time period t along the horizontal axis and the *number* of 'up' moves. We could use the notation $r_{t,i}$ where t = time and $i = u$, *d, uu, dd, ud*, etc. However, this cannot be used when programming the BOPM. Therefore we designate i as the *number* of 'up' moves in the lattice. This implies that the following equivalent notation can be used:

$$r_{1d} = r_{10}, r_{1u} = r_{11} \text{ and } r_{2,dd} = r_{2,0},\ r_{2,ud} = r_{2,1}, r_{2,uu} = r_{2,2}$$

When using $r_{t,i}$ the i indicates the number of 'up' moves. A 'down' move is given the number zero, which geometrically is a horizontal move (Figure 41.2). Any of the one-period rates such as r_{10} and r_{11} are like one-period forward rates, when viewed from time zero.

Given the short-rate lattice in Figure 41.2, we can:

- demonstrate that no risk-free arbitrage profits are possible
- illustrate how the size of the 'up' and 'down' movements in the lattice can be made consistent with the current term structure of interest rates and volatilities.

41.1.2 Short-rate Lattice and the Term Structure

At each node (t, i) we have a short-rate $r_{t,i} \geq 0$ which is the *one-period rate* applicable from t to $t + 1$ (Figure 41.2). At time t there are a total of $t + 1$ nodes indexed by $i = 0$ to t. We do not assign *real world* probabilities for a transition to 'up' (or 'flat') for the short-rate. Because we are pricing an option, we assume RNV and *assign* risk-neutral probabilities for an 'up' move of $1/2$.

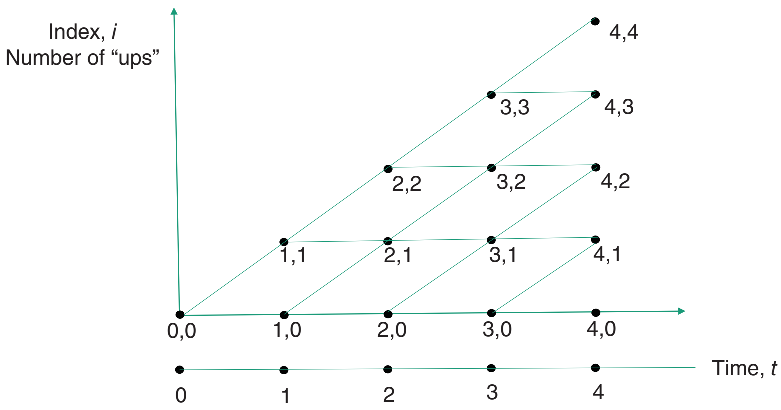

FIGURE 41.2 Short-rate lattice, $r_{t,i}$

Consider any interest rate security (e.g. 3-year zero-coupon bond) which has a value $V_{t,i}$ at node (t, i) somewhere in the middle of the lattice. Under RNV:

$$V_{t,i} = \frac{1}{(1 + r_{t,i})}\left(\frac{1}{2}V_{t+1,i+1} + \frac{1}{2}V_{t+1,i}\right) + D_{t,i} \tag{41.1}$$

where $D_{t,i}$ is the cash flow (e.g. coupon payment at node (t, i), which depends solely on t and i (i.e. it is not path dependent). We use this recursion to derive our short-rate lattice so that it is consistent with observed prices (and hence spot rates) of *zero-coupon bonds* of 1-year, 2-year etc. maturities.

41.1.3 Arbitrage Opportunities

Is the lattice free from arbitrage opportunities? The answer is yes. To see this most simply, consider possible arbitrage opportunities over *one period* when you are currently at node (t, i). We know that:

$$V_{t,i} = \frac{1}{2}\left[\frac{V_{t+1,i} + V_{t+1,i+1}}{(1 + r_{t,i})}\right] \tag{41.2}$$

Profitable arbitrage over one period requires $V_{t,i} \leq 0$ and $V_{t+1,i} \geq 0$, $V_{t+1,i+1} \geq 0$ with one of these inequalities being strict. This is impossible since the coefficients on the V's in parentheses are both positive. Hence no arbitrage is possible over one period and by induction it can be shown that this is true over longer horizons.

41.1.4 The Lattice Meets the Data

The lattice of one-period rates must be representative of actual observed interest rates. But what does 'representative' mean? Consider the lattice at the end of year-1 (Figure 41.2). First, the 'spread' between r_d and r_u at t_1 should represent the *observed volatility* of these one-period rates, which when viewed from t_0 is the volatility of the *forward* rate between t_1 and t_2, which can be denoted σ_1.

Second, it would seem reasonable that our (one-year) spot rates in the lattice should also be consistent with the observed term structure, for 1-year, 2-year, 3-year etc., spot rates and zero-coupon bond prices. Let us see how this is accomplished in the Black–Derman–Toy (BDT) (1990) model. Today, we observe the current spot rates, r_i and the corresponding (zero-coupon) bond prices (with face value of \$1), where $P_{0,i} = \$1/(1 + r_i)^i$ for $i = 1, 2, 3, 4, 5$ and also the volatilities of the implicit forward rates (Table 41.1).

The quoted one-year bond price $P_{0,1} = 0.9852$ is determined solely by the observed 1-year spot rate $r = 1.5\%$. Our aim is to construct a lattice of one-period interest rates consistent with the observed data in Table 41.1. To do so we move *forward* through the lattice. First we need to find r_u and r_d (see Figure 41.2). In the BDT model, spot rates are assumed to be *lognormally distributed* so

$$\ln(r_u/r_d) = 2\sigma_1\sqrt{dt} \tag{41.3}$$

where $\sigma_1 = 20\%$ is the volatility of the observed forward rate between at t_1 and $dt = 1$-year. Hence σ_1 determines the 'gap' between r_u and r_d as one would expect. Once we fix r_d then r_u is determined by the above equation. We have to 'fix' r_d by trial-and-error (iteratively) so that r_d and r_u are consistent with the observed 2-year spot rate and with the observed current (two-year maturity) bond price $P_0(2) = 0.9612$. Suppose we arbitrarily choose $r_d = 2.01\%$, then from (41.3):

$$r_u = e^{2(0.20)(1)}\,(2.01\%) = 1.4918(2.01\%) = 3\%$$

Hence $r_u = 1.4918\,r_d$. With these 'trial values' of r_d and r_u we can now work out the price of a 2-year zero-coupon bond using Equation (41.1) (with $D_{t,i} = 0$):

$$V_u = \frac{1}{1.03}\left[\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\right] = 0.9709 \tag{41.4}$$

TABLE 41.1 Initial term structure and volatilities

Maturity of zero rates	Spot rates	Prices	Volatility
1 year	0.015	0.9852	$\sigma_1 = 0.2$
2 years	0.02	0.9612	$\sigma_2 = 0.19$
3 years	0.025	0.9286	$\sigma_3 = 0.18$
4 years	0.03	0.8885	$\sigma_4 = 0.17$
5 years	0.035	0.8420	

$$V_d = \frac{1}{1.0201}\left[\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\right] = 0.9803 \tag{41.5}$$

$$V_0(2) = \frac{1}{1.015}\left[\frac{1}{2}(0.9709) + \frac{1}{2}(0.9803)\right] = 0.9612 \tag{41.6}$$

Here our trial value of the 2-year bond price $V_0(2) = 0.9612$ is equal to the quoted bond price $P_0(2) = 0.9612$. Hence, our initial guess $r_d = 2.01\%$ is consistent with the observed 2-bond price and spot yield. If we had found that $V_0(2) \neq P_0(2)$, then another 'trial value' for r_d would be required. In practice, what we have are two (non-linear) equations (41.1) and (41.3) in two unknowns r_d and r_u which in general can be solved using some iterative algorithm (e.g. Excel's 'Solver'). Having obtained r_u and r_d we now move forward through the lattice, so:

$$r_{ud} = r_{dd}e^{2\sigma_2\sqrt{\Delta t}} \tag{41.7a}$$

$$r_{uu} = r_{ud}e^{2\sigma_2\sqrt{\Delta t}} \tag{41.7b}$$

Once we have 'fixed' r_{dd}, then we can use (41.7a) and (41.7b) with $\sigma_2 = 19\%$ (Table 41.1) to determine r_{ud} and r_{uu}. Again, we use the lattice to obtain a trial value for a 3-year coupon bond, $V_0(3)$ using backward recursion from t_2 (with r_u and r_d already known). We alter r_{dd} until the calculated $V_0(3)$ equals the quoted price of the 3-year zero, $P_0(3) = 0.9286$. Now our interest rate lattice is consistent with the observed term structure of interest rates (1.5%, 2%, …) and forward volatilities (20%, 19%, …) of Table 41.1. The resulting 'arbitrage free' lattice is shown in Table 41.2.

It is now straightforward to extend the lattice beyond $t = 3$ given spot rates and volatilities for 4-year, 5-year etc. zero coupon bonds. To obtain our no-arbitrage one-period lattice we have undertaken a sequence of separate 'backward recursions' to price each of the 1-year, 2-year etc. bonds and this is computationally intensive. However, it is possible to use a 'forward recursion' which requires just one 'sweep' through the lattice and is computationally easier (see Clewlow and Strickland 1998, Chapter 5).

TABLE 41.2 Short-term interest rate lattice

Number of ups					
4					0.1043
3				0.0744	0.0742
2			0.0496	0.0519	0.0528
1		0.0300	0.0340	0.0362	0.0376
0	0.0150	0.0201	0.0232	0.0253	0.0268
Time	0	1	2	3	4

41.2 PRICING A COUPON BOND

Consider using the no-arbitrage interest rate lattice of Table 41.2, to price a 5-year, 4% coupon bond, with face value $100.

The known payoff at $T = 5$ is the final coupon of $4 plus the principal of $100, making a total of $104 at all nodes at $T = 5$ (Table 41.3). To price the coupon bond we now simply work backwards through the lattice, discounting using the one-period rates in Table 41.2. For example, the price of the bond at node-(4,4) is:

$$V_{4,4} = \frac{(1/2)\ \$104 + (1/2)\ \$104}{(1 + 0.1043)} = \$94.18$$

Working our way backwards through the lattice finally gives the current value of the 5-year coupon bond as $P_{0,0}(T = 4) = 102.62$ (per $100 face value).[2]

41.3 PRICING OPTIONS

We can price interest rate derivatives such as European and American options on T-bonds, on T-bond futures, and options on Eurodollar futures using the no-arbitrage interest rate lattice (Table 41.2). Suppose the value of the underlying asset, at expiration of the option contract,

TABLE 41.3 Pricing a 4%, 5-year bond, $100 principal

Coupon rate = 0.04,

Coupon payment = 4

Number of ups						
5						104
4					94.18	104
3				96.61	100.81	104
2			98.02	100.77	102.78	104
1		101.57	102.97	103.89	104.23	104
0	102.62	106.75	106.65	106.18	105.29	104
Time	0	1	2	3	4	5

[2]Note that figures in all the tables are rounded.

T is Z_i (for all nodes-i at T). For example, the payoffs to a call option are $V_i = \max(Z_i - K, 0)$, where $i = uu, ud, dd$ for expiration at $T = 2$.

For a European option on a T-bond, Z_i are the T-bond prices in the lattice at maturity of the option contract. For an option on a bond future or on Eurodollar futures, Z_i are the futures prices at T. Having obtained the option payoffs at T, we then work backwards through the interest rate lattice, using $q = 1/2$ and discounting at the appropriate one-period rates in Table 41.2, to obtain the option value at $t = 0$. Some examples will make this clear.

41.3.1 European Call Option on a Bond

Consider pricing a 4-year European call option ($K = \$104$) on the 5-year coupon paying bond (in Table 41.3). The option expires at $T = 4$ and the value of the coupon paying bond at each of the nodes at $t = 4$ is given in Table 41.3. The call payoff at each node at $T = 4$, is:

$$\text{Payoff for the call (at } T = 4) = max(P_i^B - 104, 0)$$

which gives payoffs {0, 0, 0, 0.2299, 1.2882} which is given in Table 41.4.

The value of the call at node (3,0), say, using the one-period interest rates (in Table 41.2) to discount the outcomes is:

$$V_{3,0} = \frac{(1/2)\ 0.2299 + (1/2)\,1.2882}{1 + 0.0253} = \$0.7403$$

The other nodes are calculated in a similar fashion and we find that the value of the 4-year European call option (on a 5-year bond) with strike price $K = \$104$, is $V_{0,0} = 0.1262$ (per \$100 maturity/principal value of the bond).

TABLE 41.4 European call option ($T = 4$) on 5-year bond

Strike rate = 104

Number of ups					
4					0
3				0	0
2			0	0	0
1		0.0260	0.0536	0.1109	0.2299
0	0.1262	0.2302	0.4160	0.7403	1.2882
Time	0	1	2	3	4

41.3.2 American Call Option on a Bond

To price an American call option on the bond, we merely have to compare the intrinsic value of the option $IV_{t,i} = P^B_{t,i} - K$ at each node with the recursive call premium $V_{t,i}$ in Table 41.4. If $IV_{t,i} > V_{t,i}$ then early exercise is profitable and the entry $V_{t,i}$ is replaced by $(P^B_i - K)$. For example, at node-(3,0) the recursive value of the call is $V_{3,0}$ = \$0.7403 (Table 41.4) while the intrinsic value $P^B_{3,0} - K = 106.18 - 104 = 2.18$ (using Table 41.3). Therefore at node-(3,0) we 'replace' 0.7403 by the intrinsic value 2.18, which is the entry in square brackets in Table 41.5.

At node (3,1) the intrinsic value (using Table 41.3) is $P^B_{3,1} - K = -0.11$ and therefore a call value of 0.1109 is used at this node (Table 41.5). Our 'new' recursive call value at node-(2,0) discounted using $r_{2,0} = 0.0232$ (Table 41.2) is $V_{2,0} = 1.1188 = [(1/2)\ 2.18 + (1/2)\ 0.1109]/(1 + 0.0232)$ (Table 41.5). But at node-(2,0) the intrinsic value is $IV_{2,0} = (P^B_{2,0} - K) = 106.65 - 104 = 2.65$, which therefore 'replaces' the recursive value $V_{2,0} = 1.1188$ in Table 41.5. The value of the American call option on the bond is found to be $V^{Am}_{0,0} = 1.3653$ (Table 41.5), which of course must be at least as large as the price of the European call on the bond of $V^{Eur}_{0,0} = 0.1262$ (Table 41.4).

TABLE 41.5 American call option ($T = 4$) on 5-year bond

Strike rate = 104

$T = 4$

Call premiums are per \$100 notional

Number of ups					
4					0
3				0	0
2			0	0	0
1		0.0260	0.0536	0.1109	0.2299
		[0.0]	[0.0]	[0.0]	
0	1.3653	1.3255	1.1188	0.7403	1.2882
	[0.0]	[2.75]	[2.65]	[2.18]	
Time	0	1	2	3	4

Note:

(i) The payoff at $T = 4$ is $\max\{P^B_i - K, 0\}$ and is the same for a European or American call.

(ii) At each node we calculate the recursive value $V_i = [0.5V_u + 0.5V_d]/(1 + r_i)$ and the intrinsic value $IV_i = \max\{P^B_i - K, 0\}$. The intrinsic value is given in parentheses [.]. For all four nodes at $T = 3$ if $IV_i > V_i$ then we 'replace' V_i with IV_i in the lattice. We then calculate V_i for nodes at $T = 2$ using the larger of V_i and IV_i, etc.

41.4 PRICING A CALLABLE BOND

A callable bond is one where *the issuer* (e.g. company-X) can repurchase the bond at a fixed price at certain times over the life of the bond. We assume a (4%-coupon) 5-year bond with maturity (par) value \$100, can be 'called' in any of the years 1 to 4. Therefore company-X has a long call option on the bond, since it has the 'option' to recall it by paying the par value of the bond (if this turns out to be advantageous). However, if you are an investor (Ms Long) who holds a callable bond then this is equivalent to holding a conventional bond plus a *written* (embedded) American call option on the bond.

Assume that if company-X decides to call the bond then it pays the maturity (par) value of the bond ($M = \$100$) plus any coupon C accruing to Ms Long. Assume the strike price in the call option equals the par value of the bond ($K = M$), hence:

$$\textit{Value of conventional bond} + \textit{Value of written call} = \textit{Value of callable bond}$$

In general, company-X will 'call' (i.e. buy back) the bond if the yield to maturity falls below the coupon yield ($= C/M$). (Company-X can then refinance its debt by selling 'new' bonds at the lower current yield to maturity). Assume that the bond is called if the current market value of the bond is higher than its maturity value M (plus any coupon due), hence in the binomial lattice:

$$\begin{array}{ll} \textit{Bond is called by company-X if:} & V_i^B(\textit{recursive}) > M + C \\ \textit{Then replace } V_i^B(\textit{recursive}) \textit{ with} & M + C \textit{ in the lattice.} \end{array}$$

The payoffs in the lattice are discounted at the appropriate short-rate and we work backwards through the lattice, checking the above 'call provision' at each node.

At maturity of the bond ($t = 5$) the value of the bond is $M + C = \$104$ in all states. The bond can be called at $T = 4$. At nodes (4,0) and (4,1) the bond prices are $V^B(\textit{recursive}) =$ 105.29 and 104.23 (Table 41.3) $> K = M + C = 104$, so at these nodes the bond will be 'called' by company-X. Hence the value of the callable bond to Ms Long at these two nodes is 104 (Table 41.6).

At nodes-(4,2), (4,3), and (4,4) in Table 41.3, $V^B(\textit{recursive}) < K = 104$ and therefore in Table 41.6 these entries remain unchanged and equal to $V^B(\textit{recursive})$. On the other hand, at node-(3,0) in Table 41.6, $V^B_{3,0}(\textit{recursive}) = [(1/2)104 + (1/2)104]/1.0253 + 4 = 105.43 > K =$ 104, so the bond would be called and the entry in Table 41.6 is 104.

So, the bond is called at nodes (1,0), (2,0), (3,0), (4,0), and (4,1) and at these nodes we have entries of 104. Working backwards through the lattice we find at $V^B_{0,0,}(\textit{callable}) =$ 101.25. In addition, we have found that for a plain vanilla 4%-coupon, 5-year bond, $V^B_{0,0}(\textit{conventional}) = 102.62$ (Table 41.3) and a long (American) call on this bond = 1.3653 (Table 41.5) hence, the following relationship holds:

$$\textit{Value of a conventional bond} + \textit{Value of a written call} = \textit{Value of a callable bond}$$

$$102.62 + (-1.3653) = 101.25 \text{ (rounded)}$$

TABLE 41.6 Pricing a 5-year callable bond

Coupon rate = 0.04

Coupon payment = 4

Number of ups						
5						104
4					98.18	104
3				96.61	100.81	104
2			98.02	100.77	102.78	104
1		101.55	102.92	103.78	104	104
0	101.25	104	104	104	104	104
Time	0	1	2	3	4	5

Note: Bond can only be called in years 1 to 4.

It is also the case that the above relationship holds at *all* nodes across the three lattices of Table 41.3 (conventional bond), Table 41.5 (American call) and Table 41.6 (callable bond) and not just at node (0,0).

41.5 PRICING CAPS

41.5.1 Pricing a 2-year European Cap

To price a cap we first price the individual caplets. Consider a one-period ($T = 1$) caplet per \$1 of notional principal. Assume the cap payoffs are determined at t_1 but *paid out* at t_2. Hence, with a strike rate of $K = 2\%$ the payoffs to the caplet at t_1 using short-rates in Figure 41.1 (or Table 41.2) are:

$$V_u = \frac{max\{0,\ \ r_u - K\}}{[1 + r_u]} = \frac{0.03 - 0.02}{1.03} = 0.009709 \tag{41.8a}$$

$$V_d = \frac{max\{0,\ \ r_d - K\}}{[1 + r_d]} = \frac{0.0201 - 0.02}{1.0201} = 0.000098 \tag{41.8b}$$

Using backward recursion:

$$V_0 = \frac{qV_u + (1-q)V_d}{[1 + r(0,1)]} = \frac{(1/2)\ 0.009709 + \left(\frac{1}{2}\right)0.000098}{1.015} = 0.004831 \ \ \text{(per \$1 notional)} \tag{41.8c}$$

The 1-year caplet premium is 0.004831 (per \$1 notional) – about 0.48% of the principal. If the principal underlying the caplet is $Q = \$100{,}000$ then the invoice price of the (one-year) caplet is \$483.10. Pricing a two-period caplet (where cash payment takes place at t_3) is straightforward once we know V_{uu}, V_{ud}, and V_{dd}, which are given by:

$$V_i = \frac{max\{0, \ \ r_i - K\}}{[1 + r_i]} \tag{41.9}$$

where i = 'uu' or 'ud' or 'dd' for each of the three nodes at t_2 (Figure 41.1). The three payoffs at t_2 for $K = 2\%$ are given by $max\{0, r_i - K\} = \{2.96\%, 1.4\%, 0.32\%\}$ with $V_i = \{0.0282,$ $0.0135,\ 0.0031\}$ respectively. We then work backwards to obtain:

$$V_u = \frac{[qV_{uu} + (1-q)V_{ud}]}{[1 + r_u]} = \frac{\left[\left(\frac{1}{2}\right)\ 0.0282 + \left(\frac{1}{2}\right)\ 0.0135\right]}{1.03} = 0.0203$$

$$V_d = \frac{[qV_{ud} + (1-q)V_{dd}]}{[1 + r_d]} = \frac{\left[\left(\frac{1}{2}\right)\ 0.0135 + \left(\frac{1}{2}\right)0.0031\right]}{1.0201} = 0.0082$$

and the two-period caplet premium is:

$$V_0 = \frac{[qV_u + (1-q)V_d]}{[1 + r_0]} = \frac{\left[\left(\frac{1}{2}\right)0.0203 + \left(\frac{1}{2}\right)0.0082\right]}{1.015} = 0.01404 \text{ (per \$1 notional)}$$

Notice that although the above expressions look relatively complex, the premium V_0 for a two-period caplet, just depends on *all* the one-period rates in the lattice up to $T = 2$. The price of the one-period and two-period caplets are 0.004831 and 0.01404. So the price of a cap (per \$1 notional) with caplets expiring at $T = 1$ and $T = 2$ is 0.01887 (= 0.004831 + 0.01404). Pricing a European floorlet proceeds as above but the payoffs at each node-i are $max\{0, K - r_i\}/(1 + r_i)$.

41.5.2 Pricing an American Caplet

Suppose we are trying to price a 4-year American caplet. Again, we work backwards through the lattice. At all nodes at t_3 we calculate the 'recursive value' $V_{t,i}$ using Equation (41.2) and also the intrinsic values $IV_{t,i} = (r_{t,i} - K)/(1 + r_{t,i})$ and at each node we use $\max\{V_{t,i}, IV_{t,i}\}$. We then repeat the above procedure for the nodes at t_2, working backwards through the lattice at t_1 and t_0 (since it may be worthwhile to exercise immediately). The lattice entry at t_0 is then the value of the American caplet.

41.6 PRICING FRAS

The payoff in an FRA is based on the difference between the out-turn value for the interest rate r_i at the maturity date T and the agreed FRA rate, r_{FRA}.

In the BOPM the probability of k 'up' moves after n-time periods is given by the binomial probabilities $q_k^n = \binom{n}{k} \mathrm{q}^{\mathrm{k}}(1-\mathrm{q})^{\mathrm{n-k}}$. For example, for $q = 0.5$, the risk neutral probability of reaching node-ud, that is $k = 1$, 'ups' after $n = 2$ steps is $q_{ud}^2 = 0.5$ [$= (2!/1!1!)$ $(0.5)(0.5)$]. Similarly $q_{uu}^2 = 0.25$ and $q_{dd}^2 = 0.25$.

Assume the FRA is on a notional principal of $1 with maturity $T = 2$ *years*. The payoff is $(r_i - r_{FRA})$, which we assume is paid out at $T = 2$. The fair 'price' for the FRA is that value of r_{FRA} which makes the *expected value* of the payoff from the FRA (in a risk-neutral world) equal to zero, when viewed from t_0. Using the short-term rates in Figure 41.1 (or Table 41.2) we have:

Expected payoff: 1×2 FRA

$$q_{uu}^n(r_{uu} - r_{FRA}) + q_{ud}^n(r_{ud} - r_{FRA}) + q_{dd}^n(r_{dd} - r_{FRA}) = 0$$

$$0.25(4.96\% - r_{FRA}) + 0.5(3.4\% - r_{FRA}) + 0.25(2.32\% - r_{FRA}) = 0 \tag{41.10}$$

The sum of the risk-neutral probabilities is unity and the solution is:

$$r_{FRA} = \sum_i q_i^n \; r_i = 3.52\% \tag{41.11}$$

where for $n = 2$, $i = uu, ud, dd$

Therefore the FRA-rate r_{FRA} equals the expected value (using risk neutral probabilities) of the one-period (forward) rates, at expiration of the FRA contract.

41.6.1 Delayed Settlement FRA

The payoff to the FRA is determined at $T = 2$. But here, because interest is paid in arrears, the payment of the interest differential is due at t_3 (i.e. 'delayed settlement FRA'). The present value of this payment at $t = 2$, must equal zero, hence:

$$\frac{q_{uu}^n(r_{uu} - r_{FRA})}{(1 + r_{uu})} + \frac{q_{ud}^n(r_{ud} - r_{FRA})}{(1 + r_{ud})} + \frac{q_{dd}^n(r_{dd} - r_{FRA})}{(1 + r_{dd})} = 0$$

The solution for the delayed settlement r_{FRA}, using the short-rates in Table 41.2 at $t = 2$ is:

$$r_{FRA}^{delay} = \frac{\sum_i q_i^n \; r_i \;/\; (1 + r_i)}{\sum_i q_i^n / \; (1 + r_i)} = 0.0351 \quad (or\ 3.51\%) \tag{41.12}$$

where for $n = 2$, $i = uu, ud, dd$. Clearly one could easily extend the above to price FRAs with a maturity longer than $T = 2$.

41.7 PRICING A SWAPTION

Consider an option on a swap with expiration at $T = 2$, which delivers a 'pay-fixed, receive-floating' swap, at a swap rate K_{sp}. Assume the maturity of the deliverable swap is $N = 2$ years, that is the life of deliverable swap begins at $T = 2$ (at expiration of the swaption) and ends at $t = 4$). This is a payer swaption. How can we price this swaption at $t = 0$? The key element is to obtain the value of the payoffs on the swaption $\{V_{uu}, V_{ud}, V_{dd}\}$ for the 3 nodes at $T = 2$. Then all we need to do is apply backward recursion under RNV. The steps are given below.

41.7.1 Stage 1: Calculate Swap Rates

At each of the three nodes $(i = uu, ud, dd)$ at $T = 2$, calculate the (forward) swap rate sp_i $(T = 2)$ that would prevail on a 2-year swap, given the term structure at that node of the lattice (Figure 41.1). $sp_i(T = 2)$ is a *forward* swap rate because it applies to the underlying swap in the swaption, which 'begins' at maturity of the option. The (forward) swap rate at $T = 2$ is calculated so the present value (PV) of the floating leg of the swap equals the PV of the fixed leg. If the notional principal in the swap is $\$Q$ then the value of the floating leg (at $T = 2$) is $\$Q$ (see Chapter 34). The fixed leg of the deliverable 2-year swap has cash flows at $t = 3$ and $t = 4$. Equating present values of the floating and fixed legs, we can solve for the unknown (forward) swap rate sp_i at each of the three nodes ($i = uu$, ud, and dd) at $T = 2$:

$$Q = \frac{sp_i Q}{[1 + r_i(2,3)]} + \frac{Q(1 + sp_i)}{[1 + r_i(2,4)]^2}$$

Note that at node-uu (say) $r_{uu} = r_{uu}(2,3)$ is the one-period rate but $r_{uu}(2,4)$ is a '2-year' rate, which can be calculated from the one-period rates in Figure 41.1. At each of the three nodes ($i = uu$, ud, *nd* dd) at $T = 2$:

$$sp_i(T = 2) = \left[1 - \frac{1}{[1 + r_i(2,4)]^2}\right] AN_{2-4} \tag{41.13}$$

where $AN_i^{2-4} = \left[\frac{1}{[1+r_i(2,3)]} + \frac{1}{[1+r_i(2,4)]^2}\right]$

AN_i^{2-4} is the annuity value of \$1 paid at $t = 3$ and $t = 4$ (over the life of the underlying 2-year swap). We now have three values for the (forward) swap rate at $T = 2$, namely sp_{uu}, sp_{ud}, and sp_{dd}.

41.7.2 Stage 2: Calculate Swaption Payoffs

The payoffs on the swaption (Chapter 39) per $1 notional principal on the underlying swap at each node ($i = uu$, ud, dd) at $T = 2$, depends on the 2-year annuity value of $\max\{sp_i - K_{sp}, 0\}$ (Chapter 39):

$$V_i = \max\{sp_i - K_{sp}, 0\}\ AN_i^{2-4} \quad (\text{i} = \text{uu}, \text{ud}, \text{dd}) \tag{41.14}$$

We now have values V_{uu}, V_{ud} and V_{dd} for the swaption, at expiration ($T = 2$). Note that the V_i depend only on $r_i(2, 3)$, $r_i(2, 4)$, that is, on the term structure from $t = 2$ to the maturity date of the underlying cash-market swap at $t = 4$. This is fine because at each node-i, we know that we can construct the whole term structure from the one-period rates in the lattice.

41.7.3 Stage 3: Backward Recursion

It is now straightforward to price the swaption using backward recursion, discounting using the one-period rates (Figure 41.1). Given V_i ($i = uu, ud, dd$) we use the usual formulas to calculate V_u, V_d and then V_0. If the swaption premium $V_0 = 0.0140$ (per $1 notional principal) then for a deliverable swap with notional principal of $10m, the invoice price of the swaption is $140,000.

Notice that when we price a caplet using the BOPM, we only require the sequence of *one-period* rates in the lattice (Figure 41.1). But to price a swaption, on a 2-year underlying swap we require the term structure at each node, since we need both the one-period rates $r_i(2, 3)$ and the two-period rate $r_i(2, 4)$. This is because the underlying cash market swap has cash flows over several periods after expiration of the option (at $T = 2$).

41.8 PRICING FRNS WITH EMBEDDED OPTIONS

A plain vanilla FRN is a bond with variable coupon payments that depends on a floating interest rate (LIBOR, flat). A plain vanilla FRN with maturity (par) value of $Q = \$100$ has a market price equal to its par value when initially issued ($t = 0$) and its market value reverts to par at each reset date (see Chapter 34). The value of an FRN at each reset date is Q if the floating rate in the FRN is LIBOR (flat) but not if the floating rate is LIBOR plus a spread. We assume the reset dates for the FRN correspond to the nodes in the short-rate lattice. The payoff on an FRN is determined at each reset date, but cash is paid in arrears.

41.8.1 Capped FRN

Suppose we are considering a 5-year FRN, par value $Q = \$100$, which has a capped floating rate at $K_{cap} = 6\%$. At any reset date, if the LIBOR rate is above K_{cap} the investor holding the

FRN only receives $K_{cap} = 6\%$, rather than the higher LIBOR rate. Hence the market value of the FRN will be below its \$100 par value (at this node). A capped FRN is equivalent to a plain vanilla FRN plus an embedded written (short) cap.

$$\begin{aligned}\text{Payoff capped FRN} &= \text{Payoff vanilla FRN} + \text{written cap (with strike } K_{cap})\\ &= LIBOR - \max\{LIBOR - K_{cap}, 0\}\\ &= K_{cap} \quad \text{if } LIBOR > K_{cap}\\ &= LIBOR \quad \text{if } LIBOR \leq K_{cap}\end{aligned} \tag{41.15}$$

At maturity ($T = 5$) the FRN is worth \$100, its par value, at all nodes. Now consider each five nodes-i for a capped FRN at $t = 4$ (Table 41.7):

$$\begin{aligned}\text{If } r_i > K_{cap}:\ & \text{Coupon received (one period later)} = \$100K_{cap}\\ & V_i^{CapFRN} = \$100(1 + K_{cap})/(1 + r_i)\\ \text{If } r_i < K_{cap}:\ & \text{Coupon received (one period later)} = \$100\ r_i\\ & V_i^{CapFRN} = \$100(1 + r_i)/(1 + r_i) = \$100\end{aligned}$$

At node-(4,4) $r_i = 10.43\%$ (Table 41.2) which is greater than $K_{cap} = 6\%$, hence $V_i^{CapFRN} = 100(1 + K_{cap})/(1 + r_i) = 95.99$ (Table 41.7). Similarly at node-(4,3) $r_i > K_{cap}$ and $V_i^{CapFRN} = 98.67$. The nodes (4,0), (4,1), (4,2) all have $r_i < 6\%$ (Table 41.2) and the value of the FRN is \$100. At nodes $t = 3$ we calculate the 'recursive' value of the FRN: $V_{3,i}^{recursive} = [(1/2)V_{4,i+1} + (1/2)V_{4,i}]/(1 + r_{3,i})$, which excludes any coupon payments. We now see if the cap is operative at this node:

$$\begin{aligned}\text{At } t = 3, \text{if } r_i > K_{cap}:\ & V_{3,i}^{CapFRN} = V_{3,i}^{recursive} + \$100K/(1 + r_i)\\ \text{At } t = 3, \text{if } r_i < K_{cap}:\ & V_{3,i}^{CapFRN} = V_{3,i}^{recursive} + \$100r_i/(1 + r_i)\end{aligned}$$

TABLE 41.7 Pricing a 5-year capped FRN

Cap rate, $K = 0.06$

Number of ups					
4					95.99
3				96.18	98.67
2			97.88	99.37	100
1		98.82	99.70	100	100
0	99.35	99.85	100	100	100
Time	0	1	2	3	4

For example, at node-(3,3), $r_{3,3} = 7.44 > K_{cap} = 6\%$ hence (Table 41.7):

$$V_{3,3}^{Cap-FRN} = \frac{(1/2)\ 95.99 + (1/2)\ 98.67}{1.0744} + \frac{6}{1.0744} = 96.18$$

At node-(3,2), $r_{3,2} = 5.19 < K_{cap} = 6\%$, hence (Table 41.7):

$$V_{3,2}^{Cap-FRN} = \frac{(1/2)98.67 + (1/2)100}{1.0519} + \frac{5.19}{1.0519} = 99.37$$

We go backwards through the lattice to obtain the value of the capped FRN at $t = 0$ of 99.35. The embedded (long) cap must therefore be worth $100 - 99.35 = 0.65$.

41.8.2 Floored FRN

Using a similar procedure we can price an FRN with an embedded floor, $K_{FL} = 3\%$. If at $t = 4$, $r_i < K_{FL}$ then the holder of the FRN will receive the 'floor payment' of (\$100 K_{FL}) and the FRN will be valued above par at $V_i = \$100(1 + K_{FL})/(1 + r_i)$ – this occurs at nodes (1,0), (1,1), (2,0), (3,0), and (4,0) in Table 41.2 and hence at these nodes in Table 41.8, V^{FL-FRN} is above its par value.

If $r_i > K_{FL}$ then the floored-FRN pays a coupon based on the LIBOR rate = \$100(1 + r_i) and the FRN is valued at par (Table 41.8). Moving backwards through the lattice, the value of a floored FRN (at $t = 0$) is 102.19, hence the value of the embedded put is:

$$\textit{Value of embedded long put} = \textit{Value of floored FRN} - \textit{Value of vanilla FRN}$$

$$2.19 = 102.19 - 100 \tag{41.16}$$

From Equation (41.16), a 'floored FRN' with floor level K_{FL} is equivalent to a plain vanilla FRN (on LIBOR) with an embedded long put option on LIBOR, with strike equal to K_{FL}.

TABLE 41.8 Pricing a 5-year floored FRN

Floor rate = 0.03

Number of ups					
4					100
3				100	100
2			100	100	100
1		100.0007	100	100	100
0	102.19	101.44	100.96	100.62	100.31
Time	0	1	2	3	4

TABLE 41.9 Pricing a 5-year collared FRN

Cap rate = 0.06

Floor rate = 0.03

Number of ups					
4					95.99
3				96.18	98.67
2			97.88	99.37	100
1		98.82	99.70	100	100
0	101.54	101.29	100.96	100.62	100.31
Time	0	1	2	3	4

41.8.3 Collared FRN

Table 41.9 shows the values of a collared FRN with cap of 6% and floor of 3%. The entries at each node are either the par values of the vanilla FRN (when neither the cap nor floor are operative at this node) or the value of the capped FRN or the floored FRN (when these are operative). Working backwards through the lattice the collared-FRN is worth 101.54 at $t = 0$.

Hence we can see that:

$$\textit{Value of collared FRN} = \textit{Value of vanilla FRN} + \textit{Value of short call} + \textit{Value of long put}$$

$$101.54 = 100 + (-0.65) + 2.19$$

This completes our set of examples where we use the arbitrage-free short-rate lattice and the BOPM under RNV to price fixed income derivatives. The method is very flexible and can handle a wide variety of path-dependent interest rate options.

41.9 MORE LATTICES

Short-rates in the BDT model do not exhibit mean reversion, so if interest rates are very high (low) there is no tendency for them to return towards some long-run mean value. Use of a *trinomial lattice* rather than a binomial lattice can speed up computational time and can also make it easier to incorporate mean reversion in interest rates. (In addition, using a trinomial lattice can be shown to be equivalent to the finite difference method (see Chapter 48)).

A trinomial lattice for $t = 2$ periods and with each period equal to 1 year is shown in Figure 41.3. The (continuously compounded) short-rate is given at each node and the probabilities of an ‘up’, ‘flat’ and ‘down’ move are assumed to be $q_i = (0.25, 0.50, 0.25)$ respectively. Broadly speaking, these probabilities are chosen so that the lattice exhibits mean reversion and it matches the ‘real world’ variance of the short-rate.

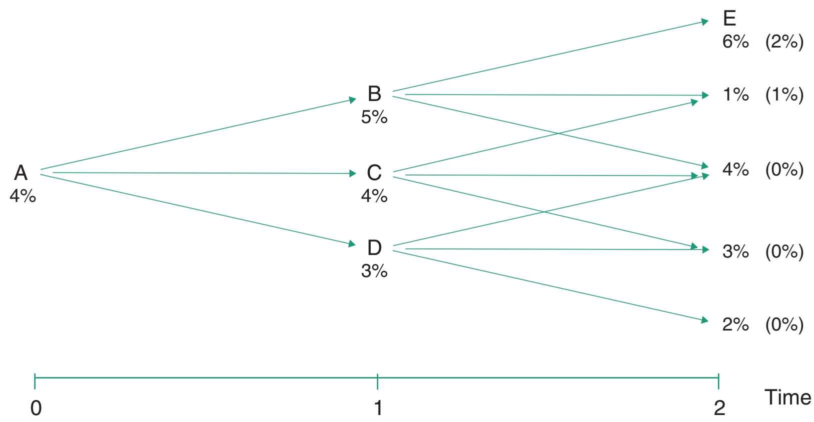

FIGURE 41.3 Trinomial-rate lattice

The trinomial lattice is used in exactly the same way as the binomial lattice. Consider pricing a 2-year European caplet (on 1-year LIBOR) with notional principal $Q = \$1\text{m}$ and $K = 4\%$. For simplicity assume the cash payouts actually occur at $t = 2$.[3] The cash payouts at the five nodes at $t = 2$ are $V_{2,i} = \max(r_{2,i} - K, 0)$ and are shown in parentheses in Figure 41.3. The value of the caplet at node B is:

$$V_B = \$1\text{m}[0.25(0.02) + 0.50(0.01) + 0.25(0.0)]\,\text{e}^{-0.05(1)} = \$9{,}512\text{ m} \tag{41.17}$$

We can calculate V_C and V_D in a similar fashion and then use these values to determine V_A, the premium for the caplet. Hull and White (1994) demonstrate how the trinomial lattice can be 'adjusted' so that it incorporates the empirically observed phenomenon of mean reversion for interest rates. For example, if interest rates follow the upper path A, B, E in Figure 41.3, then with mean reversion it would be more likely that the one-period rate would fall next period (towards its long-run mean value). This is accomplished in the lattice by increasing q_3 (the probability of a 'down' move) from 0.25 at node E (see Hull 2018 for more details). The Hull-White lattice also allows hedge parameters to be calculated so that options traders can establish vega, gamma, and delta neutral positions for their portfolios of interest rate options.[4]

41.10 SUMMARY

- The no-arbitrage approach begins with a model for the one-period short-rate $r_{t,i}$, which is *calibrated* to fit the current term structure. That is the one-period rates are

[3]If the actual cash payout is at $t = 3$ then the payoff (at $t = 2$) needs to be discounted at the one-period rates at each of the nodes at $t = 2$.
[4]Readers interested in the details of algorithms applied to various models of the short-rate to give alternative arbitrage free lattices and their use in pricing derivatives, should consult Clewlow and Strickland (2000).

consistent with observed prices of zero-coupon bonds and the volatility of forward rates. The resulting lattice for one-period rates does not allow arbitrage profits to be made (by buying and selling zero-coupon bonds).

- Bond prices and the price of many fixed income derivatives can then be found numerically, using the BOPM and backward recursion through the ‘no-arbitrage’ lattice.
- The BOPM applied to the no-arbitrage short-rate lattice is very flexible and can be used to price both European and American bond options and many other path-dependent fixed income derivatives (e.g. caps, floors, swaptions, callable bonds, FRAs and FRNs with caps and floors).
- Trinomial lattices increase computational speed. These lattices can also mimic mean reversion in short-rates, which more closely resembles their real world behaviour.

EXERCISES

Question 1

Intuitively, explain how one-period interest rates in the no-arbitrage interest rate lattice of the BDT model are related to the term structure of volatility. Consider a two-period tree to illustrate your answer.

Data for Questions 2–5

You are given the following data to be used in answering Questions 2–5. The interest rate lattice over three periods is:

			16% (0.216)
		14% (0.36)	
	12% (0.6)		13% (0.432)
10%		10% (0.48)	
	8% (0.4)		9.0 (0.288)
		7% (0.16)	
			6% (0.064)

The risk-neutral probability of an ‘up’ move is $q = 0.6$ and for a ‘down’ move is $(1 - q) = 0.4$. The figures in parentheses are the probability of reaching each node, times the number of ways to reach that node. They are therefore the BOPM terms:

$$q_k^n = \binom{n}{k} q^k (1 - q)^{n-k}$$

For $n = 2$, and $k = 1$ up-moves, the probability of being at node ‘*ud*’ (or equivalently ‘*du*’) is:

$$q_1^n = q_{ud}^n = \binom{2}{1} (0.6)^1 (0.4)^1 = 0.48$$

Similarly the probability of $k = 2$ up-moves (to reach node-*uu*) is:

$$q_2^n = q_{uu}^n = \binom{2}{2} (0.6)^2 (0.4)^0 = 0.36$$

Question 2
What is the invoice price of a 1-year European caplet with $K_c = 10\%$? The notional principal in the cap is $Q = \$1\text{m}$. Note that the payoff for a 1-year caplet is determined at $t = 1$ but the actual cash payout takes place at $t = 2$.

Question 3
What is the invoice price of a 2-year European caplet with $K_c = 10\%$? The notional principal in the caplet is $Q = \$1\text{m}$. What is the invoice price of a 2-year cap (with annual payments)?

Question 4
Qualitatively, explain how your analysis would change when pricing the 2-year cap, if the cap were an American style option.

Question 5
Using the above lattice, what is the FRA-rate r_{FRA} for a 2-year FRA where the cash payoff is determined at $t = 2$ but the actual payment is due at $t = 3$? This is a ‘delayed settlement FRA’.

PART X

CREDIT DERIVATIVES

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 42

Credit Default Swaps (CDS)

Aims

- To explain the operation, uses and pricing of credit default swaps (CDS).
- To analyse the use of the iTraxx (Europe) and CDX (USA) credit indices.
- To analyse forwards and options on the CDS spread.

Credit derivatives allow companies and financial institutions to hedge their credit risks and allow investors to hold assets whose value depends (in part) on the creditworthiness of companies and individuals. The increased use of credit derivatives was very rapid over the 2000–7 period, slowed down after the credit crisis of 2008 but has since substantially revived.

Banks issue loans to companies and individuals and if these default, the banks experience losses. There are two main ways banks can reduce their credit risk. First, they can 'bundle up' a portfolio of loans and sell bonds to investors (e.g. pension funds, insurance companies, sovereign wealth funds) who then receive the cash flows from the loans – this is securitisation, and is discussed in Chapter 43.

The second way a bank can reduce risk arising from a specific corporate loan is to retain the loans in its banking book but to buy protection against a default by the corporate. It can do this by purchasing a (single name) credit default swap (CDS), often from an insurance company. If the corporate defaults on its bank loan, the seller of the CDS (insurance company) will compensate the bank for any losses on the defaulting loan. There are also *basket CDS* where you can buy credit protection on a 'portfolio' of companies – this is a multi-name credit derivative.

In this chapter we explain how CDS operate and are priced, as well as the relationship between bond yields and the CDS spread. We discuss credit indices and basket CDS. We also show how forwards and options on the CDS spread can be used to hedge or insure against future changes in CDS spreads.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

42.1 CREDIT RISK AND CDS

Credit default swaps provide insurance against credit events. Suppose MegaBank owns a corporate bond (‘*the reference entity*’) with face value of Q = \$10m, maturity of 10 years and it is worried that the corporate might default over the next 5 years. MegaBank can buy a 5-year credit default swap with a financial institution (a swap dealer, such as Goldman Sachs). As a buyer of the CDS, MegaBank pays a fixed swap spread of say s = 80 bps per year to the seller (Goldman) over the next 5 years, unless a ‘credit event’ occurs, when the payments cease.

If the corporate bond does not default, (Megabank) the buyer of the CDS pays the seller \$80,000 $(= sQ)$ each year over the 5-year life of the swap – this is the annual cost of the insurance premium. If the bond experiences a ‘credit event’ (e.g. goes into default) the seller of the CDS (Goldman) agrees to buy the company bond from Megabank for \$10m (i.e. face value of the bond), even though it might only be worth \$2m in the ‘distressed debt market’. Alternatively, the credit default swap could be cash settled whereby MegaBank keeps the bond (e.g. worth \$2m) and the swap dealer (Goldman) pays Megabank \$8m. After a default, the buyer of the CDS does not pay any more premiums.

If the contract stipulates cash settlement, an independent calculations agent will poll dealers to determine the mid-market value of the cheapest deliverable corporate bond of the reference entity, for delivery on a specific date after the credit event. There will be several bonds which are eligible for delivery – usually they will all have the same seniority and approximately the same duration/maturity as the reference bond. The CDS buyer therefore has a ‘cheapest to deliver’ option. The *notional principal* in the CDS does not have to equal the face value of the corporate bonds that MegaBank holds – for example, if the notional principal in the CDS is less than the face value of its corporate bond then Megabank is less than fully insured.

Consider a CDS starting on 20 June 2019, with a quoted *CDS spread of* s = 80 bps annually, on a 5-year swap with a notional principal of \$10m on a ‘reference asset’ (Figure 42.1). Payments are usually made in arrears either each quarter, each half year or, as in our example, at

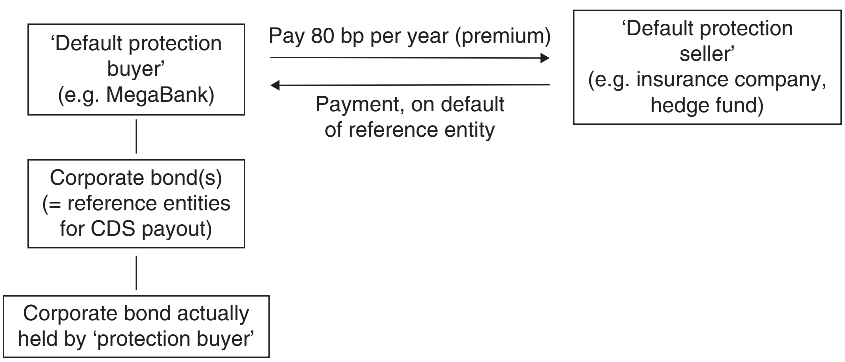

FIGURE 42.1 Credit default swap (CDS)

the end of each year. If there is no credit event, the buyer of the CDS will pay $80,000 on 20 June in each of the years 2021 to 2025. However, if there is a credit event on 20 September 2024, the annual payments by the buyer of protection will cease. But because payments are made in arrears, a final payment by the buyer is usually required. In our case, on 15 September the buyer of protection would have to pay approximately one quarter of the annual payment (= $20,000) for protection cover between 20 June and 20 September 2024.

Large universal banks and insurance companies are market makers in the CDS market and, for example, might quote a CDS spread at a bid of 260 bps and offer at 270 bps, on a new 5-year CDS. This means the swap dealer is prepared either to buy protection by paying 260 bps per year (i.e. 2.6% of the notional principal per year) or to sell protection and hence receive 270 bps per year. The swap dealer therefore makes 10 bps on a round trip transaction of buying and selling a 5-year CDS to two different counterparties (on the same notional principal).

Credit default swaps are used by banks when they issue loans to corporates and do not wish to increase their credit risk exposure. Suppose Bank-A already holds $100m of debt (i.e. bank loans and bonds) of corporate-Z and does not wish to increase its credit exposure any further but corporate-Z requires a further $10m loan. The bank can issue a further loan (or purchase a bond issued by corporate-Z) and immediately enter into a CDS with another counterparty (say Bank-B who possibly has little credit exposure to corporate-A). Here the CDS is being used to ‘spread’ the credit risk of corporate-Z amongst a wider set of financial institutions, than just Bank-A. Of course for Bank-A, protection from the credit event of company-Z defaulting, depends on the solvency of Bank-B, who sold the CDS contract to Bank-A.

42.2 SPECULATION WITH CDS

In contrast to CDS providing insurance, CDS can be bought or sold without owning the underlying bonds. Hence, traders such as hedge funds can speculate on credit events using CDS. The CDS spread widens as the perceived risk of default on the reference entity increases. Therefore a hedge fund can make a speculative profit if it correctly forecasts that a reference entity will become more or less risky.

For example, suppose today a hedge fund BigBucks thinks company-X will become more risky over the coming year. The hedge fund buys (from JPMorgan) a 5-year CDS (on a bond issued by company-X, the reference entity) with a notional principal of $10m at a spread of 200 bps.

If BigBucks’ forecast of an increase in credit risk turns out to be correct then the CDS spread might widen to 300 bps by the end of the year. The hedge fund might then sell a (new) CDS at 300 bps to another counterparty (e.g. pension fund). If there is no credit event over the remaining 4 years then the hedge fund will have locked in a net inflow of $100,000 p.a. for the next 4 years. This is the difference between the premium it receives of $300,000 p.a. (3%) from the pension fund and what it pays (to JPMorgan), namely $200,000 p.a. (2%). If there is a credit event, the hedge fund will have to pay out the notional principal to one side of the swap deal (pension fund), but will receive the same amount from the counterparty from which it initially purchased the CDS (JPMorgan). So from the end of the first year, the hedge fund is hedged against default but has secured a potential $100,000 p.a. profit for the next 4 years.

Note that the above speculative transaction (i.e. buy followed by sell) could take place over a very short horizon (e.g. 1 week) – as long as the CDS spread widens, the hedge fund can close out its long-short CDS positions at a profit. During the year, hedge funds will usually repeat these long-short trades many times, if they forecast that the credit risk of the reference entities will increase over short horizons.

If the hedge fund believes the reference entity will become less risky in the future then it will set up a potential speculative profit by undertaking the reverse trade (to that described above). For example, Megabucks selling a CDS today at 300 bps and buying a CDS at 200 bps in 1 year's time after a fall in the perceived riskiness of the reference entity ensues. Finance Blog 42.1 elaborates on the above points (which led to the collapse of Lehman Brothers in September 2008).

Finance Blog 42.1 Speculation Using CDS

Betting that a company (TrashUS) will go bust (i.e. insolvent or bankrupt) or simply become more risky in the future can be illustrated using a stylised and simplified example.

Suppose on 1 January 2017 a hedge fund (MegaBucks) thinks TrashUS will become 'more risky' in 1 month's time. The hedge fund (HF) today buys from Goldman a $100m, 5-year CDS for 100 bps with TrashUS as the reference entity. For ease of exposition, assume the payment of $1m p.a. to Goldman takes place today ($t = 0$) and at the *beginning* of each year for the following 4 years ($t = 1, 2, 3, 4$), but only if TrashUS does not default. Note that in this example, MegaBucks never holds the bonds of TrashUS, it only deals in CDS contracts, with TrashUS as the reference entity. Assume the HF is correct and in 1 month's time the CDS spread on TrashUS has increased to 120 bp.

After 1 month: 1 February 2017

On 1 February 2017, the HF sells a 5-year CDS (on $100m notional principal) at 120 bp to Legal and General (L&G), a pension fund. The HF now has a long-short position in CDS contracts.

Assume two cases: (a) TrashUS does not go into default; (b) TrashUS declares bankruptcy at $t = 4.5$ years.

Case (a) No default by TrashUS

HF pays out	5 yrs × $1m	= $5m	(100 bps to Goldman)
HF receives	5 yrs × $1.2m	= $6m	(120 bps from L&G)
Profit		= $1m	(over approximately 5 years)

The HF pays out \$1m each January and receives \$1.2m each February. This is a return of 20% over 1 month on its 'own funds' of \$5m – for each of the 5 years.[1] The above profit is very dependent on when TrashUS becomes more risky (or defaults). If the CDS spread does not move to 120 bp until February 2020 (say) then the HF could sell a 2-year CDS and receive $2 \times$ \$1.2m but it would have paid out \$5m in total (assuming no default of TrashUS).

Case (b) Default by TrashUS at 4.5 years

The cash flows up to the 4th year are as above – and the HF has made a profit of \$1m by February of year-4. The HF sold a CDS contract in February 2017 to L&G, so on default of TrashUS the HF pays \$100m to L&G and receives a TrashUS bond from L&G. But the HF in January 2017 bought a CDS contract from Goldman, so the HF passes the TrashUS bond (received from L&G) to Goldman and receives \$100m in exchange. The net cash flow to the HF when TrashUS defaults is zero because the HF has a long-short position in the CDS contract.

John Paulson's hedge fund earned profits in the region of \$15bn in the crash of 2008, in part due to large bets using CDS contracts. These CDS paid off when Lehman's and other corporates went into liquidation around September 2008. (By comparison note that George Soros made 'only' \$1bn in 1992 by betting that sterling would depreciate against the Deutsche mark.)

42.3 CONTRACT DETAILS

CDS are OTC instruments which first appeared around the end of the 1990s. By the end of 2007 there were around \$50trn notional outstanding in CDS contracts worldwide. However, since on average only around 0.2 per cent of investment grade companies default in any one year, most cash flows consist of the 'spread' payments from buyer to seller, which are much smaller than the notional outstanding. The most popular CDS contracts have maturities of 5 years but other maturities between 1 and 10 years are also available. Many US contracts mature on one of the following standard dates: 20 March, 20 June, 20 September, and 20 December.

Initially there was no centralised clearing house for CDS transactions but after the credit crunch of 2007–8 revealed a lack of transparency in the CDS market, many commentators and regulators advocated a centralised clearing process with collateral being posted – and this is now being implemented in many developed countries.

The reference entity in a CDS contract is usually a corporate or a sovereign entity (e.g. bonds issued by the Italian government) and the type of bond is often an unsubordinated (senior secured) corporate or government bond. If the 'reference entity' defaults, holders of

[1]In addition, the HF might borrow some of the $5 \times$ \$1m it pays out for the CDS contracts, thus giving it a higher return on its 'own funds' invested – this is leverage.

these bonds are paid before 'junior creditors'. The range of possible credit events is set out in the CDS contract and if there is any dispute it ultimately has to be settled in court. Credit events include bankruptcy of the reference entity or a failure to pay coupons on a bond or interest payments on a loan.

For European CDS on reference entities comprising Investment Grade bonds, a credit event generally also includes debt restructuring. A ratings downgrade (by Standard & Poor's, Moody's, or Fitch) is not classified as a credit event – because this might be open to manipulation. The *deliverable* in the CDS contract also has to be defined and may, for example, be limited to bonds or loans with a maximum maturity of 25 years and that the debt must not be 'subordinated' or 'callable'.

When a credit event occurs an auction may be held to fix the cash settlement price – this is a *credit-fixing* event. At the auction, large investment banks submit prices at which they would buy and sell the debt of the company which has suffered the credit event and is the reference entity in the CDS contract. For example, in 2008 Lehman's debt was valued at around 9% of face value – that is a 'haircut' of 91%. Washington Mutual's debt was at 57% of face value while Fannie Mae and Freddy Mac's debt was valued at the high rate of 95% of par (maturity) value – because Fannie and Freddie's debt is (effectively) underwritten by the US government.

42.4 PRICING AND VALUATION

The mid-market CDS spread (i.e. an average of bid and ask quotes) on *individual* reference entities can be calculated from estimates of survival and default probabilities. To illustrate CDS pricing, we initially assume default always occurs on a payment date (so there are no accruals issues), CDS payments are annual and occur *at the end* of each year. We use the following notation:

$$p = \text{hazard rate (assumed constant over life of the CDS)}$$

$$\pi_i = \text{probability of default } during \text{ the year}$$

$$v_i = \text{unconditional probability of survival to end of year-}i$$

$$d_i = \text{discount rate} = 1/(1+r_i)^i \qquad \text{or} \qquad d_i = \exp(-r_i t_i), \text{continuously compounded}$$

$$n = \text{remaining life of the CDS} \qquad s_n = \text{spread (decimal) on an } n\text{-year CDS}$$

$$R = \text{recovery rate (e.g. } R = 0.6) \qquad Q = \text{notional principal in the CDS contract}$$

Like a plain vanilla interest rate swap, when a CDS is initiated it must have zero present value to both parties – otherwise they would not enter the swap. Hence, the CDS spread on (an n-year) swap is that value s_n that makes the expected present value of future payments in the swap equal to the expected present value of future receipts. For the buyer of the CDS, the expected payments at time t_i are $v_i s_n Q$ and the expected receipts, given default, are $\pi_i\, Q(1-R)$. Equating the present value of expected receipts and payments gives the n-period swap rate as the solution to:

$$s_n Q \sum_{i=1}^{n} v_i d_i = Q(1-R) \sum_{i=1}^{n} \pi_i d_i \tag{42.1}$$

Hence the n-period CDS spread is:

$$s_n = (1-R)\frac{\sum_{i=1}^{n} \pi_i d_i}{\sum_{i=1}^{n} v_i d_i} \tag{42.2}$$

Suppose a 5-year CDS has the following characteristics:

Conditional probability of default, p	0.03 (3% p.a.)
Recovery rate, $R =$	0.40 (40% p.a.)
Notional principal, $Q =$	$\$10$m
Current (LIBOR) rate, r	0.04 (4% p.a., continuously compounded)

Assume that the yield curve is flat and if a default occurs it does so half-way through any particular year, which implies that accrual payments in the CDS (given default) are $(1/2)\,Q$. To determine the unknown swap rate s_n we have to equate the expected cash payments by the protection buyer with the expected cash payments by the protection seller.

42.4.1 Probability of Survival and Default

The hazard rate is the probability of default between time t and a small interval of time later $t + dt$, conditional on no earlier default. If viewed from today, suppose companies with bonds rated BB historically have a 1%, 3%, and 5% probability of defaulting at the end of year-1, year-2, and year-3 respectively. The probability of defaulting *during* the third year is $5\% - 3\% = 2\%$. This is the unconditional probability of default during the third year, as seen from today. The probability that the company will survive to the end of year-2 is $100 - 3 = 97\%$. Hence the probability that it will default during the third year conditional on no earlier default is $0.02/0.97 = 0.0206$ (2.06%) – this conditional probability over 1 year is the *hazard rate* (also called the default intensity).

Assume a constant hazard rate of $p = 3\%$ p.a., so the probability of survival[2] to the end of any year t_i is $v_i = e^{-p.t_i}$. For example, the probability of survival to the end of year-4 is $v_i = e^{-0.03(4)} = 0.8869$. The probability of default *during* year-4 equals the probability of survival to year-3 minus the probability of survival to year-4, which is 0.027 (= 0.9139-0.8869, Table 42.1).

[2]Note that $v_i \approx 1 - p.t_i$ which is (1 – probability of default at the end of year t_i).

TABLE 42.1 Unconditional default probabilities

Probability of default = 0.03

Time (years) t	Unconditional default probability $\pi = p \times v(t-1)$	Survival probability v
1	0.0296	0.9704
2	0.0287	0.9418
3	0.0278	0.9139
4	0.0270	0.8869
5	0.0262	0.8607

42.4.2 Cash Flows in the CDS

Given the survival probabilities v_i (Table 42.2, column 2), the expected payments for the protection buyer (given no default) are $v_i s_n Q$ (column 3) with present values $d_i v_i s_n Q$ (column 5), where $d_i = e^{-rt_i}$ and $t_i = 1, 2, \ldots 5$. The total present value of all of these payments is \$40.7281 s_n.

TABLE 42.2 PV of expected CDS spread payments (no default)

Conditional probability of default = 0.03
Risk-free LIBOR rate (continuously compounded) = 0.04
Notional principal, Q = \$10m

Time (years) t	Survival probability v	Expected payment $E(Payment) = Q \times v \times s$	Discount factor d	PV of expected payment $d \times E(Payment)$
1	0.9704	9.7045s	0.9608	9.3239s
2	0.9418	9.4176s	0.9231	8.6936s
3	0.9139	9.1393s	0.8869	8.1058s
4	0.8869	8.8692s	0.8521	7.5578s
5	0.8607	8.6071s	0.8187	7.0469s
Sum of PV of payoffs (no default)				40.7281s

$$\pi_i = \text{probability of default } \textit{during} \text{ the year}$$

Given the probability of default *during* the year π_i, the expected cash flows by the seller of the CDS are $\pi_i Q(1 - R)$ (Table 42.3, column 4). If defaults (only) occur half-way through each year then $d_i = e^{-rt_i}$ where $t_i = 0.5, 1.0, 1.5\ldots$ etc. For the CDS seller, the present value of expected cash flows (given default) are $d_i[\pi_i Q(1-R)]$ and the sum of these present values is \$0.759 (Table 42.3, column 6).

TABLE 42.3 PV of cash flows given default and recovery rate

Conditional probability of default = 0.03
Risk-free LIBOR rate (continuously compounded) = 0.04
Notional principal, Q = \$10m
Recovery rate, R = 0.4

Time (years) t	Unconditional default probability π	Recovery rate	E(Payoff) $= Q \times \pi \times (1-R)$	Discount factor d	PV of E(Payoff) = $d \times E$(Payoff)
0.5	0.0296	0.4	0.1773	0.9802	0.1738
1.5	0.0287	0.4	0.1721	0.9418	0.1621
2.5	0.0278	0.4	0.1670	0.9041	0.1510
3.5	0.0270	0.4	0.1621	0.8683	0.1407
4.5	0.0262	0.4	0.1573	0.8353	0.1314
Sum of PV given default (and recovery)					0.7590

It only remains to determine the accrual payments should a default occur half-way through any of the 5 years in the life of the CDS. The accrual payment is $Q/2$ and the expected cash flows are $(Q/2)\pi_i s_n$. The present value of the accrual payments is \$0.6325 s_n (Table 42.4, final column).

TABLE 42.4 PV of accrual cash flows CDS spread given default

Conditional probability of default = 0.03
Risk free LIBOR rate (continuously compounded) = 0.04
Notional principal, Q = \$10m
Accrual payment (given default) = 0.5

Time (years) t	Unconditional default probability π	Expected accrual E(Accrual) = $\pi \times (Q/2)$	Discount factor d	PV of expected accrual = $d \times E$(Accrual)
0.5	0.0296	0.1478s	0.9802	0.1448s
1.5	0.0287	0.1434s	0.9418	0.1351s
2.5	0.0278	0.1392s	0.9041	0.1258s
3.5	0.0270	0.1351s	0.8683	0.1173s
4.5	0.0262	0.1311s	0.8353	0.1095s
Sum of PV of accrual cash flows given default				0.6325s

Equating the expected cash flows with no default, to the expected cash flows given default:

$$\$40.7281\,s_n + \$0.6325\,s_n = \$0.759$$

gives a 5-year CDS spread of $s_n = 0.01835$ (1.84% p.a.). Hence the 'insurance premium' on the reference entity for a 5-year CDS with a notional principal of \$10m, is \$183,500 each year.

At inception, the value of the swap is zero and the swap rate is s_0 (say). If the CDS has been in existence for some time then its value could be positive or negative (to either the buyer or seller). The value of the CDS to *the seller* at $t > 0$ is the present value of the *remaining* expected swap premium receipts minus the expected payout in the event of default. Ignoring accrual payments this is:

$$V_{CDS}(seller)_t = s_0 Q \sum_{i=t}^{n} v_i d_i - Q(1-R) \sum_{i=t}^{n} \pi_i d_i$$

The summation goes from today $t > 0$ to the end of the swap's life, n. At $t > 0$ the default and survival probabilities as well as market interest rates used in the discount factors, will usually be different from those at $t = 0$ (when the swap was initiated).

42.4.3 Binary CDS

A binary CDS is similar to a standard CDS except that if default occurs a fixed dollar amount is paid, which is independent of the recovery rate. For example, a binary CDS might have a cash payout of $Q = \$10$m if there is a default. The only change in the above analysis is that for the cash flows given default, $\pi_i Q(1 - R)$, we set $R = 0$ (Table 42.5, column 4). The 'binary' CDS spread is the solution to:

$$\$40.7281\ s_n + \$0.6325\ s_n = 1.2649$$

Hence, $s_n = 0.0305$ (3.05% p.a.), which on a notional principal of \$10m is an annual payment of \$305,000.

42.4.4 Default Probabilities from CDS Spreads

When discussing the Black–Scholes option pricing formula we noted that by setting the *quoted* price of the option equal to the *theoretical* Black–Scholes price, we could derive the implied volatility of the underlying asset. Similarly, note that the theoretical CDS spread depends on probabilities of default π_i – see Equation (42.2). This means that if we have the term structure of quoted (mid-market) CDS spreads, then we can invert (42.2) to obtain the

TABLE 42.5 Binary CDS, PV of cash flows given default and recovery rate

Conditional probability of default = 0.03
Risk-free LIBOR rate (continuously compounded) = 0.04
Notional principal, Q = \$10m
Assume each payoff given default is \$$Q$, rather than \$$Q(1 - R)$, hence set Recovery rate = 0

Time (years) t	Unconditional default probability π	Recovery rate	$E(Payoff) = Q \times \pi \times (1 - R)$	Discount factor d	PV of E(Payoff) = $d \times E(Payoff)$
0.5	0.0296	0	0.2955	0.9802	0.2897
1.5	0.0287	0	0.2868	0.9418	0.2701
2.5	0.0278	0	0.2783	0.9041	0.2516
3.5	0.0270	0	0.2701	0.8683	0.2345
4.5	0.0262	0	0.2621	0.8353	0.2189
Sum of PV given default and recovery rate					1.2649

implicit conditional probability of default in each year. Today, we observe the mid-market quoted CDS swap rates $s_1, s_2, \ldots s_n$ for newly issued CDS (with different maturities) and using Equation (42.2), we can extract the conditional default probabilities $\pi_1, \pi_2, \ldots \pi_n$. In fact, these are risk-neutral default probabilities, not real-world default probabilities. (Risk neutral probabilities can also be estimated from bond prices and asset swaps but we do not pursue this here.)

42.5 BOND YIELDS AND THE CDS SPREAD

If you hold a corporate bond and also purchase protection via a CDS then you effectively hold a risk-free T-bond. For example, suppose you hold portfolio-A consisting of:

Portfolio-A

(a) 5-year corporate bond selling at (or near) its par (maturity) value which has a (par) yield $y_{corp}^{par} = 7\%$ p.a. and
(b) 5-year CDS on the corporate bond with a spread of $s = 2\%$ p.a.

Portfolio-A is free of credit risk, with a net return of 5% p.a. (ignoring the default risk of the *counterparty* in the CDS contract). Hence to prevent arbitrage profits, the following should hold:

Long a corporate bond and long CDS = Yield on risk-free T-bond

$$y_{corp}^{par} - s = y_{TB}^{par} \tag{42.3}$$

Hence:

$$s = y_{corp}^{par} - y_{TB}^{par} \tag{42.4}$$

where y_{corp}^{par} = *par* yield on a corporate bond, y_{TB}^{par} = *par* yield on a US Treasury bond and s is the CDS swap rate.

If the corporate bond does not default, portfolio-A gives a net return of 5% p.a. over 5 years consisting of receipts of $y_{corp}^{par} = 7\%$ less the CDS premium paid. If the corporate bond defaults after, say, 3 years then portfolio-A still earns 5% p.a. (net) over 5 years. This is made up of 5% p.a. net over the first 3 years from the corporate bond yield, minus the CDS spread payments. In year-3 after default, you receive the par value of the corporate bond from the CDS contract but you can invest this cash inflow at the risk-free rate for another 2 years.

42.5.1 Arbitrage Profits

Suppose the *quoted* corporate bond yield is $y_{corp}^{par} = 8\%$ (rather than 7% above) and the CDS swap rate is $s = 2\%$, then an investor could earn a risk-free arbitrage profit. The investor borrows \$$Q$ at the risk-free rate of 5%, buys the corporate bond (at par) yielding 8% and buys a CDS contract at a cost of 2%. If there is no default, the net return to this strategy is 8% from the corporate bond yield, less the cost of the CDS at 2% and less the cost of borrowing of 5%. This gives a known 1% p.a. arbitrage profit, if and until the corporate bond defaults. But if the corporate defaults, then the receipt of the principal (from the CDS contract) can be used to pay off the borrowing of \$$Q$ – so there is no risk to this arbitrage strategy even if the bond defaults.

Alternatively, suppose the corporate bond yield is $y_{corp}^{par} = 6\%$ (rather than 7% above). An arbitrageur would borrow the bond from her broker (JPMorgan), short-sell the bond, invest the cash proceeds at the risk-free rate of 5% and also simultaneously sell a CDS contract, receiving the premium of 2% p.a. The return to this arbitrage strategy is 5%+2% less the yield on the corporate bond of 6% (which has to be paid to the broker – JPMorgan), giving a risk-free arbitrage return of 1% p.a. These are not entirely riskless arbitrage opportunities but Equation (42.4) gives a reasonable first approximation to the relationship between corporate bond yields and CDS spreads.

42.6 CREDIT INDICES AND OTHER CDS CONTRACTS

Indices have been developed to track a 'portfolio' of CDS spreads. Two important ones are:

iTraxx Europe – a portfolio of 125 Investment Grade 'names' (corporates) in Europe
CDX NA IG – a portfolio of 125 Investment Grade companies in North America.

If some companies cease to be classified as investment grade they are dropped from the above indices and new 'investment grade' companies are added – this happens in March and September each year. There is an active market in CDS 'index protection' particularly for maturities of 3, 5, 7, and 10 years.

The quoted CDS index spread is (approximately) given by an average of the CDS spreads on the individual reference entities (companies), that make up the index. A quoted bid-price of 100 bps on a 5-year CDS index, implies that you can buy 5-year protection on a notional principal of \$1 million on each of the 125 reference entities in the index, for an annual payment of \$1.25m p.a. ($= 0.01 \times \$1\text{m} \times 125$). Simplifying somewhat, if a reference entity defaults, the protection buyer receives the usual CDS payoff and the annual payment is reduced by $\$1.25\text{m}/125 = \$10{,}000$. (Note that the precise way the contract works is more complex than this.)

42.6.1 Other CDS Contracts

As noted above, a *binary CDS* is similar to a regular CDS, except the payoff in default is the fixed dollar amount Q rather than $Q(1 - R)$. We can price a binary CDS using (42.2) by setting the recovery rate $R = 0$.

In a *basket CDS* there are a number of reference entities. A *first-to-default* CDS provides a payoff only when the first default occurs. Similarly, a kth-*to-default* CDS provides a payoff only when the kth default occurs. An *add-up basket* CDS pays off when *any* of the reference entities in the portfolio defaults. The *add-up basket* costs more than the kth-*to-default* CDS, since the former provides greater protection (for the buyer of the CDS). These basket CDS operate much like a regular CDS, so that after the specified default there is a settlement process and the swap is terminated – so there are no further payments by either party.

42.7 DERIVATIVES ON THE CDS SPREAD

There are forwards and options on the CDS spread. Forwards lock in a known future CDS spread, which will begin at some point in the future. An option on the CDS spread allows you to take advantage of the strike CDS spread K_s or the market CDS spread s_T, whichever is the most favourable 'price', at expiration of the option. If the company defaults before the maturity date of these derivatives contracts then they cease to exist (i.e. they 'knock out').

For example, on 15 January 2017 you might purchase a *forward CDS* with a maturity $T = 1$ year, on the 5-year CDS spread, at a forward 'price' of $s_0^f = 300$ bps. You then have an obligation to buy protection at 300 bps in 1 year's time on 15 January 2018. If the market (5-year) CDS spread on 15 January 2018 is $s_T = 400$ bps then you will be happy you took out the forward CDS – but if the CDS spread turns out to be $s_T = 200$ bps, then you may regret having taken out the forward contract. However, the latter is of no consequence because you initially bought the forward CDS on 15 January 2017 to 'lock in' a known CDS spread starting in 1 year's time. When you hedge with forwards on 15 January 2017 you do not have the luxury of hindsight. If the reference entity defaults before the end of the year, then the forward-CDS ceases to exist.

There are options which give you the right (but not the obligation) to enter into a credit default swap in the future, at a swap rate K_s which is fixed today – these are *credit default swaptions* or *CDS options*. On 15 January 2017 you might purchase a call option on a 5-year CDS where the call matures in one year ($T = 1$) and has a strike of $K_s = 240$ bps. If the (5-year) CDS spread on 15 January 2018, $s_T > K_s = 240$ bps then you exercise the call and effectively obtain a CDS where you pay the CDS rate $K_s = 240$, which is lower than the quoted 'cash market' CDS spread, s_T. If on 15 January 2018 the CDS spread $s_T < K_s = 240$ bps, you do not exercise the call option and instead you buy a CDS in the cash market at the lower quoted swap rate, s_T. Viewed as insurance, a 1-year call swaption on the 5-year CDS spread sets the maximum swap rate at $K_s = 240$ bps you will pay in 1 year's time, to enter a 5-year CDS swap but the swaption also allows you to take advantage of lower CDS rates should they occur.

The price of a call (default) swaption on the CDS spread is positively related to the current 'cash market' CDS spread s (i.e. the underlying asset in the option). Hence, you can hedge the credit risk from holding (cash market) corporate bonds by delta hedging with a credit default swaption.

A 1-year put option on the (5-year) CDS spread gives you the right to sell five-year protection on a corporate bond at a strike $K_s = 250$ bps (say), starting in 1 year. If the 5-year CDS spread for the corporate $s_T < K_s = 250$ bps in 1 year's time, the put option will be exercised. If you have *shorted* corporate bonds, you will lose if the corporate's perceived credit risk improves – as the bond price will rise. Today, you can offset this credit risk by buying a CDS put option.

For both call and put options on the CDS spread, the premiums must be paid up front and usually the option will cease to exist if the reference entity defaults before the expiration of the option. Clearly, calls and puts on the CDS spread can also be used to speculate on the future path of the CDS spread.

42.8 SUMMARY

- A (single name) credit default swap (CDS) is a marketable insurance contract, which pays out if the reference entity (e.g. a corporate bond) experiences a 'credit event' (e.g. bankruptcy). The buyer of a CDS is buying protection against possible default on a corporate bond – and the seller of a CDS (usually an insurance company) is the protection seller.
- The payment of the 'insurance premium' is known as the CDS spread. The CDS spread is usually paid in arrears. Payment ceases if the reference entity defaults – although this will usually involves a 'one-off' accrual payment by the protection buyer.
- The CDS spread s_n on an n-year CDS contract is determined by the (unconditional) probabilities of default π_i, the (unconditional) probabilities of survival v_i, the term structure of interest rates (as reflected in discount rates d_i) and the recovery rate R. Ignoring accrual payments the CDS spread is given by:

$$s_n = (1 - R)\frac{\sum_{i=1}^{n} \pi_i d_i}{\sum_{i=1}^{n} v_i d_i}$$

- If there is a default, the buyer of the CDS either (a) receives the face (maturity/par) value of the reference entity's bond, in return for 'delivering' the reference bond to the CDS counterparty or (b) the CDS is cash settled for $Q(1 - R)$, where R is the recovery rate.
- A binary CDS is like a standard CDS but the payout on default is a fixed amount of cash (e.g. the notional principal in the CDS, Q) which does not depend on the recovery rate.
- There are CDS contracts on indices comprising a portfolio of 125 investment grade corporate bonds – for example, the *European iTraxx* index and the *US, CDX NA IG* indices. If any one bond in the reference portfolio defaults, there is a payoff to the protection buyer equal to the notional principal of the bond in default (and the annual premium is reduced by 1/125 of the notional principal).
- Forwards and options are available on the CDS spread. A long forward contract locks in a known future CDS spread, which will begin at some point in the future. A (long) call on the CDS spread (which is a credit default swaption) sets a maximum credit default swap spread that will be paid in the future but allows the holder to take advantage of lower cash market CDS spreads at expiration of the call option, should they occur. A (long put) on the CDS spread sets a minimum value for the CDS swap spread you receive but allows you to take advantage of selling a CDS at a higher cash market spread, should this occur at expiration of the put option.

EXERCISES

Question 1
Explain why an investor who holds the bonds of company-Z might purchase a 5-year, CDS with a spread of 80 bps and notional principal of $10m, on the 'reference entity'. What are the possible outcomes?

Question 2
Bank-A has loans outstanding with company-Z of $100m. Company-Z asks bank-A for a further $20m loan. How can bank-A provide the additional loan to company-Z but use a CDS to reduce its credit exposure to company-Z? How does this 'diversify' the credit risk due to company-Z?

Question 3
For a European CDS, is a ratings downgrade (by Standard & Poor's, Moody's, Fitch) classified as a credit event?

Question 4
Explain two ways that a CDS can be settled after a credit event.

Question 5
Explain how and why a hedge fund might speculate on a credit event, using a 5-year CDS contract (on a bond issued by company-X) with a notional principal of $10m, when the CDS spread is 200 bps.
Consider the possible outcomes if the CDS spread widens to 300 bps after 3 months and consider two cases: (i) company-X does not default, or (ii) company-X defaults after 3 months.

Question 6
Qualitatively what are the main determinants of the CDS spread?

Question 7
The yield on a 5-year corporate bond-X is 7.5% p.a. and it is trading at par. The yield on a 5-year risk-free bond is 4% p.a. A 5-year CDS costs 400 bps per annum. How can you lock in a (risk-free) arbitrage opportunity?

CHAPTER 43

Securitisation, ABSs and CDOs

Aims

- To analyse the process of securitisation.
- To show how structured products such as asset backed securities (ABSs), collateralised debt obligations (CDOs) and ABS-CDOs give rise to credit enhancement.
- To outline the importance of ABS-CDOs in the 2008–9 credit crisis.
- To examine single tranche trading, synthetic CDOs, and total return swaps.

Banks can reduce the credit risk on their 'banking book' by securitisation. They 'bundle up' a portfolio of loans and sell securities which entitle the investor (e.g. a hedge fund or pension fund) to the promised future cash flows from these loans – so the credit risk is transferred from banks to the holders of the securitised assets. Often the cash flows from securitised assets are paid out in order of priority, known as tranches or a waterfall. Senior tranches get paid first and are the least risky and the equity tranche gets paid last and is the most risky. These securitised assets are known as asset backed securities (ABSs) and collateralised debt obligations (CDOs). They are multi-name credit derivatives since the loans in the securitised assets originate from many different companies (or individuals in the case of home mortgages or car loans). In this chapter we discuss how ABSs and ABS-CDOs are constructed, how they provide 'credit enhancement', and the risks posed by these securities particularly in the 2008–9 credit crunch.

43.1 ABSs AND ABS-CDOs

Securitisation is the term used when issuing marketable securities backed by cash flows from 'assets' which are often illiquid (e.g. bank loans). It became a very large market in the

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

2000–7 period, then faltered after the 2008–9 credit crunch but has substantially recovered in recent years.

ABSs depend on future promised cash flows from corporate loans by banks, aircraft leases, car loans (e.g. VW, General Motors), credit card receivables (most large banks), music royalties from a back catalogue (e.g. David Bowie's estate and Rod Stewart), telephone call charges (e.g. Telemex in Mexico) and football season tickets (e.g. Lazio, Real Madrid). If the underlying assets in the securitisation are cash flows from bonds issued by corporates or countries (i.e. fixed-income securities) then the ABS is referred to as a *collateralised debt obligation (CDO)*.

43.1.1 Special Purpose Vehicles

For example, suppose that MegaBank has made a series of long-term corporate loans and has therefore taken on large credit exposures. One way of reducing this exposure is for MegaBank to create a separate legal entity known as a special purpose vehicle (SPV) (or special investment vehicle, SIV), into which the loans are 'sold' – therefore they become 'off balance sheet' for the bank. If MegaBank gets into financial difficulties with its other activities (e.g. losses due to a rogue trader), the loans in the SPV cannot in principle be claimed by MegaBank's creditors. Similarly, if the loans in the SPV become insolvent, MegaBank's normal activities are (in principle) protected from these losses.

Often the SPV finances its initial loan purchases by issuing asset backed commercial paper (ABCP) to investors, with maturities between 1 and 12 months. The SPV is therefore initially financing its purchases of long-maturity assets (e.g. bank loans) using short-term ABCP. If there are no loan defaults, the SPV makes a profit on the positive long-short spread, over the period that it holds the loans. The aim of the SPV is to sell *asset backed securities* to 'final investors' (e.g. pension funds, insurance companies, private wealth funds, sovereign wealth funds or wealthy individuals), which then entitles these investors to the stream of promised future payments from the corporate loans.

After securitisation the default (credit) risk on the loans is spread across many investors, rather than just being held by MegaBank (or the SPV). MegaBank continues to collect the interest and repayments of principal on the loans (for a fee) but passes these cash flows to the owners of the ABS (e.g. pension funds) – hence the term *pass-through securities*.

For example, if the underlying assets are home loans to individuals, the ABS is referred to as a residential mortgage backed security (RMBS). In the US, the Government National Mortgage Association (GNMA or 'Ginnie Mae') bundles up home mortgages into relatively homogeneous 'pools' – for example, \$100m of 6%, 20-year conventional mortgages. GNMA then issues, say, 10,000 RMBSs, so each purchaser of a RMBS has a claim to \$10,000 of these mortgages and is entitled to receive 1/10,000 of all the payments of interest and principal. RMBSs are marketable and highly liquid. From an investor's point of view, purchasing such securities provides them with a higher yield than on T-bonds and allows them to take on exposure to mortgage loans. But they can at any time sell the RMBS in the secondary market to other investors (at whatever the current market price happens to be).

Investors in ABSs take on the default risk which arises if the underlying borrowers default on their payments. If you hold a *diversified portfolio* of ABSs then the overall default risk depends on the correlations between default risks for the different categories of asset backed securities (e.g. do most people who have credit card debt tend to default at the same time as (other) people, who owe money on car loans, home mortgages etc.?).

43.1.2 Tranches and a Waterfall

In the above example of a RMBS each investor has equal risk. Usually an ABS is structured so investors can decide whether they are willing to take 'the first hit' either from any defaults or from non-payment of interest – or, if they are rather more risk-averse, they can choose to be last to 'take the hit', if cash flows from the underlying assets are less than expected.

ABSs (or 'cash-CDOs') are a way of 'repackaging' credit risk to create 'tranches' of debt which have different seniority of payment and hence different risk characteristics. In this way, debt of average risk can be split into bundles of 'high' to 'low' risk and hence are structured to suit the risk appetite of different investor groups. For example, hedge funds may be willing to hold the 'high risk' part of an ABS but pension funds may only be willing to hold 'low risk' assets in the ABS.

A pool of bank loans (or corporate bonds), each with different default risks are placed in a portfolio, where the assets have an average promised yield of 8.5% (say). This portfolio of assets is sold by the originator (e.g. MegaBank) to an SPV. The loans might then be classified as prime, non-prime, or sub-prime by the SPV. Any one of these three 'classifications' (e.g. sub-prime loans) might themselves then be split into four further *tranches* (Figure 43.1).

The 'equity tranche' (of the underlying sub-prime loans) are then designated as having to take the first 5% of any losses due to a credit event (e.g. defaults) – this is a very risky tranche since if only 2.5% of the sub-prime loan portfolio defaults, this amounts to a 50% loss on the first tranche.[1] The equity tranche is often retained by the originator of the ABS (e.g. MegaBank) since it is difficult to sell to investors, but sometimes this tranche may be purchased by a hedge fund (possibly financed by a loan from MegaBank!).

The second, *mezzanine* tranche might be liable for the next 10% of losses, followed by the *senior* tranche which covers the next 15% of losses, leaving the fourth (*super-senior)* tranche liable for any losses greater than 30%, on the initial portfolio of loans. Tranches 2–4 of the ABS are sold to investors.

Representative promised yields on tranches 4 (least risky) to 1 (most risky) might be 6%, 8%, 16%, and 32% p.a. Cash flows from the underlying assets are first paid to tranche-4 (super-senior) until a return of 6% is obtained (on the principal in the tranche) and any remaining cash flows then go to tranche-3 until it secures its promised return of 8% and so on. So payments from the ABS are structured to be paid out in a known order of seniority (known as a *waterfall*).

[1] In the 2008 credit crunch these equity tranches were often referred to as 'toxic waste'.

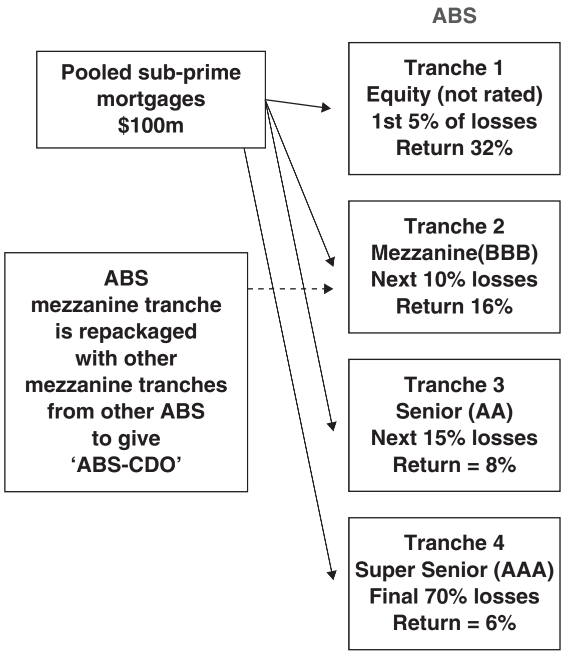

FIGURE 43.1 ABS

Tranche-4 might be rated AAA by the rating agencies (e.g. Standard & Poor's, Moody's or Fitch) since it will not be affected by credit events until the original portfolio has fallen in value by more than 30%. The ratings for the other tranches will depend on their perceived risk, which depends on the default risk for each bond in that tranche and the default correlations between the original issuers of these bonds/loans.

Hence ABSs are a means of creating some 'high quality' debt from 'average quality' debt, in the original portfolio of bonds/loans – as long as each tranche of the debt is correctly rated and gives the correct 'signals' to buyers of the ABS about their true risk exposure. Also, the originator of an ABS (an SPV) can trade the bonds held in the ABS portfolio providing (for example) that the agreed degree of diversification in the underlying bond portfolio is maintained.

The originator of the ABS sells the tranches to 'final investors' for more than they paid for the underlying bonds and also pass on the credit risk to investors. As defaults begin to rise, tranche-1 (the equity tranche) will fail to achieve its promised rate of return and if defaults continue these investors may not get all of their principal repaid. A very high level of defaults may result in the more senior tranches being affected.

43.1.3 Interest and Principal Repayments

All loan repayments can be split into a repayment of interest and principal. Cash flows are allocated to tranches via the waterfall and there is usually a separate waterfall for the interest and principal cash flows. On any payment date, any excess payment over the 'interest due' will constitute a repayment of the outstanding principal. Interest payments are allocated to the super-senior tranche first, until the super-senior tranche earns its promised return (on its outstanding principal amount). Next in line for interest payments is the senior tranche etc., until finally, if total interest payments are sufficient, the equity tranche is last to be paid.

Repayments of principal follow a similar pattern and first to be paid is the super-senior tranche and finally down to the equity tranche. If there are defaults on any principal repayments then the equity tranche takes the first 5% of losses and its outstanding principal is reduced accordingly. If losses are greater than 5% the equity tranche loses all of its designated principal amount and the mezzanine tranche takes the next 10% of losses of principal repayments etc. Hence cash flows go first to the super-senior tranche and last to the equity tranche and any losses of principal (defaults) are borne first by the equity tranche and last by the super senior tranche.

43.1.4 Rating Agencies

The ABS is structured so the super-senior tranche is rated AAA with the mezzanine tranche usually rated BBB and the equity tranche is unrated. The creator of the ABS will 'negotiate' with the rating agencies to determine what is required to obtain a AAA-rating. The ABS originator tries to make the super-senior tranche (of loans/corporate bonds) as large as possible, subject to it obtaining a AAA-rating from one of the rating agencies. This is because there is a large market for AAA-rated securities, partly because of regulatory requirements which require certain financial institutions to hold a minimum proportion of AAA-assets (e.g. pension funds, insurance companies). The SPV which creates the ABS makes a profit because the value-weighted average return on the underlying assets (e.g. loans and bonds), exceeds the weighted average return offered to all investors who hold the ABS tranches.

The mezzanine tranches of ABSs are often difficult to sell to investors. Clearly the risk depends in part on the characteristics of the underlying borrowers and on the macroeconomic environment. For example, in most cases, the only information for investors who purchased ABSs based on home loans was the borrowers' loan-to-value ratios and their FICO (Fair Isaac Corporation) scores (which represent the credit worthiness of the borrower and range from 300 to 850). But there is an incentive for property valuation assessors to put an artificially high value on a property so it then has a low loan-to-value ratio, which might increase repeat business from mortgage brokers. Also, some FICO scores were of doubtful quality given that a mortgage broker has an incentive to 'sell' mortgages to earn commissions. These conflicts of interest are known as 'agency problems' – the incentives of the two parties to the contract are not aligned. There may also be 'asymmetric information' if the mortgage broker has superior information to the borrower and does not disclose all relevant information – e.g. does not adequately explain that the interest charged may rise in future years, after an initial low 'teaser rate'.

43.1.5 ABS-CDO (Mezz-CDO, CDO-squared)

The risk in the mezzanine tranches of an ABS can be overcome to some extent by taking the mezzanine tranches from, say, 20 different ABSs (e.g. on mortgages, car loans, credit card

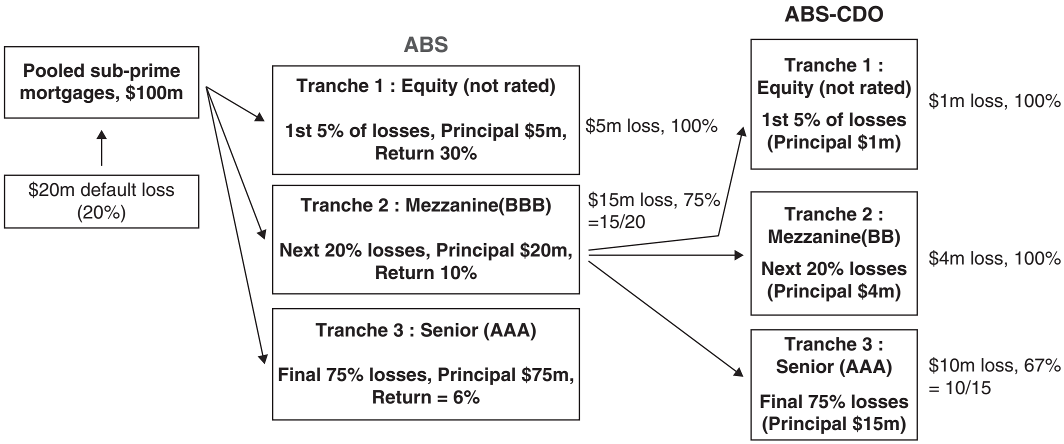

FIGURE 43.2 Leveraged losses

receivables etc.) and bundling them up into say, three further tranches (Figure 43.2) – this is an ABS-CDO (or Mezz-CDO or CDO-squared).

In an ABS-CDO, the equity tranche might take the first 5% of losses, the mezzanine tranche the next 20 per cent and the senior tranche the final 75 per cent of losses. The senior tranche might be given a AAA-rating and the mezzanine tranche a BBB-rating. Once again, payments are structured to be paid out in a known order of seniority (*the waterfall)* and may result in the creation of highly rated debt from a debt-pool that had an initial overall low credit rating – this is credit enhancement. The difficulty from the investor's point of view is in assessing the true underlying risk of the rather complex tranches in an ABS-CDO – this was part of the cause of the sub-prime mortgage crisis in the USA which began in 2007–8.

43.2 CREDIT ENHANCEMENT

Credit enhancement is achieved by using the waterfall to place the underlying assets (e.g. bonds/loans) into several tranches where the senior tranche is structured to have a lower default risk (and hence higher credit rating) than the original loans/bonds themselves. To see how this works consider the simple case of two BBB-rated bonds on different companies (X and Y), each with a face value of $1. To simplify, assume the probability of default for X and Y are independent of each other (i.e. if X defaults in a particular year, this has no influence whether Y defaults).

Suppose the probability of default for each bond is $p = 10\%$ (in any year) and the correlation between defaults is zero (i.e. credit risks are independent). We now create two tranches, each containing $1 notional. The *junior tranche* does not pay out if (a) either one of the two bonds defaults or (b) if both bonds default. The *senior tranche* does not pay out, only if *both* bonds default simultaneously (in the same year). Given our independence assumption, the probabilities for the number of defaults (D = default, N = no default) from our two bonds are given by the binomial model:

$$\begin{aligned}
&p(2\ dflt, DD) && = 0.1 \times 0.1 && = 0.01\ (1\%) \\
&p(no\ dflt, NN) && = 0.9 \times 0.9 && = 0.81\ (81\%)^{2} \\
&p(1\ dflt, ND\ or\ DN) = 0.1(0.9) + 0.1(0.9) && = 0.09 + 0.09 && = 0.18\ (18\%)
\end{aligned}$$

The probability of *no cash flows* paid to the senior tranche is:

$$p(dflt, Senior) = p(2\,dflt, DD) = 0.1 \times 0.1 = 0.01\ (1\%) \tag{43.1}$$

The probability of *no cash flows* paid to the junior tranche is:

$$p(dflt\quad jnr) = \text{p(1 default or 2 defaults)} = 0.18 + 0.01 = 0.19\ (19\%) \tag{43.2}$$

The senior tranche has a lower probability of default (1%) than either of the two original bonds, which each have a probability of default of 10%. The senior tranche is genuinely less risky than the original bonds and the senior tranche may therefore warrant an AAA rating – this is *credit enhancement*. From our original \$2 bond portfolio we have created a \$1 face value senior tranche which is now AAA rated – we have 'structured' an increase in the degree of 'credit enhancement', covering 50% of the value of our original bond portfolio.

Notice that if the default correlation between the two bonds X and Y is $\rho = +1$ then there would be no credit enhancement – when one bond defaults so does the other, so the probability of default for the senior tranche would be 10%, the same as for the individual bonds. Clearly, the lower the default correlations the greater the credit enhancement. The difficulty in assessing the riskiness of an ABS is therefore in forecasting these default correlations.

We can see how further tranches in the waterfall can lead to greater credit enhancement. Assume again uncorrelated defaults and we have \$1 principal for each of *three* different BBB rated company bonds, each with a default probability of 10%. The probabilities for 0, 1, or 2 defaults using the binomial model are:

$$\begin{aligned}
p(0\ dflt|3\ trials) &= \{3!/(0!\ 3!)\}\{0.1^{0} \times 0.9^{(3-0)}\} = 0.9^{3} && = 0.729\ (72.9\%) \\
p(1\ dflt|3\ trials) &= \{3!/(1!\ 2!)\}\{0.1^{1} \times 0.9^{(3-1)}\} = 3 \times 0.1 \times 0.9^{2} && = 0.243\ (24.3\%) \\
p(2\ dflt|3\ trials) &= \{3!/(2!\ 1!)\}\{0.1^{2} \times 0.9^{(3-2)}\} = 3 \times 0.01 \times 0.9 && = 0.027\ (2.7\%)
\end{aligned}$$

[2] Alternatively we can use the binomial model directly: $p(1\,dflt|2\,\text{'}trials\text{'}) = \{2!/(1!\ 1!)\}\{0.1^{1} \times 0.9^{(2-1)}\} = 2 \times 0.09 = 0.18\ (18\%)$.

We now create three tranches with \$1 notional in each tranche. The senior tranche pays out unless there are (exactly) 3 defaults, the mezzanine tranche pays out unless there are 2 or 3 defaults and the equity tranche pays out only if there are zero defaults. Hence:

p(default for senior tranche: $p(3) = 0.1\%(= 0.1^3 = 0.001)$

p(default) for mezzanine tranche: p(2 or more defaults) $= p(2) + p(3) = 2.7\% + 0.1\% = 2.8$

p(default junior tranche): p(1 or more defaults) $= p(1) + p(2) + p(3)$

$$= 24.3 + 2.7 + 0.1 = 27.1\%.$$

Out of our original \$3 principal, we have structured two tranches with total face value of \$2 that have a probability of default, less than the 10% default probability of each of the original bonds. We have created a 66.6% credit enhancement since we have 2 tranches worth \$2 that have a lower risk of default than the original \$3 worth of (non-securitised) bonds.

43.3 LOSSES ON ABSs AND ABS-CDOs

Now return to our initial 'two-tranche' example. Suppose we now take two *independent junior tranches* from two different ABSs, one of which was structured from credit card loans and the other from telephone receipts, each with a probability of default of 19% (Equation 43.2). These two junior tranches (from two different ABSs) are now themselves structured into senior and junior tranches. This is an ABS-CDO.

The senior tranche of the ABS-CDO receives no payments only if both junior tranches of the two ABSs default. So the probability of default for the ABS-CDO senior tranche is $p(2\ \ defaults) = 0.19 \times 0.19 = 3.6\%$, which is substantially less than the probability of default of either of the two original junior ABS tranches (of 19%). Hence the senior ABS-CDO tranche will have a higher credit rating than either of the two junior ABS tranches and may be given an AAA rating.

43.3.1 Regulatory Arbitrage

Why would a bank allow its structured products division to issue ABSs while also allowing its trading desk to purchase ABSs issued by other banks? For a bank, purchase of AAA-rated tranches could probably be financed at LIBOR, but AAA-tranches earn about LIBOR+120 bps in normal times – hence a bank's trading desk might buy these senior tranches (from other banks or pension funds) to increase profits on its own trading desk. Also the regulatory capital requirements for bank mortgages that are held by an originating bank in its 'banking book' are usually higher than the capital required if instead they hold *ABS tranches* of mortgages, which are classified as being on the 'trading book'. Hence by originating and selling ABSs and then buying securitised ABSs (in the open market), banks might be able to reduce the levels of regulatory capital they have to hold – this is known as regulatory arbitrage. Hence in the run up to the 2008 crisis, one division of the bank might be securitising its bank loans and selling ABSs and ABS-CDOs (i.e. reducing credit risk), while the trading desk might be investing in them (i.e. increasing its credit risk). Hence, when the credit crunch hit in 2008, some banks found themselves in severe difficulties. We investigate this below.

Figure 43.2 represent a simplified ABS and ABS-CDO each with 3 tranches. The underlying assets in the ABS-CDO might comprise sub-prime mortgages from the mezzanine tranche of an ABS. We assume the total face value of all the underlying mortgages is \$100m. After the 'waterfall', the total of AAA-rated securities (senior tranches) in these *two* structures (the ABS and the ABS-CDO) is $75\% + (75\% \times 20\%) = 90\%$ (\$75m + \$15m).

Consider the ABS tranches (Figure 43.2). The junior tranche takes the first 5% (\$5m) hit and the ABS-Mezz tranche the *next* 20% of losses, which implies a maximum loss to the Mezz tranche of \$20m. The Mezz tranche absorbs losses from 5% up to 25% of the \$100m principal of the underlying bonds. The ABS-Senior tranche is likely to get its promised return of 6% if defaults on the underlying bonds are less than 25%.

However, the waterfall gives rise to levered losses. If a relatively small proportion of the underlying mortgages default then this leads to a relatively large loss in the equity and mezzanine tranches of the ABS and even substantial losses in the senior tranche of the ABS-CDO. We consider two cases – a loss of 10% and a loss of 20% in the underlying mortgages – and we investigate the consequences of these losses on the various tranches of the ABS and the ABS-CDO.

43.3.1.1 Case A: 10% Loss (\$10m) on Sub-prime Mortgages

Of the \$10m total loss on the mortgages, the ABS-equity tranche takes the first 'hit' of \$5m (= 100% of equity tranche principal), the ABS-Mezz tranche takes the remaining hit of \$5m which is $5m/20m = 25\%$ of the value of this tranche's principal amount (Table 43.1). The ABS-Senior tranche takes no hit (0% loss).

For the ABS-CDO equity-tranche the *maximum* loss is 5% of the \$20m principal of the ABS-Mezz tranche, which amounts to \$1m. For the ABS-CDO Mezz-tranche, the *maximum*

TABLE 43.1 Losses on ABS and ABS-CDO

Losses on subprime	ABS			ABS-CDO		
	Losses on equity tranche	Losses on mezzanine tranche	Losses on senior tranche	Losses on equity tranche	Losses on mezzanine tranche	Losses on senior tranche
10%	100%	25%	0%	100%	100%	0%
20%	100%	75%	0%	100%	100%	66.7%

loss is 20% of \$20m, that is \$4m. Hence the loss of \$5m on the ABS-Mezz-tranche, wipes out all the principal in both the ABS-CDO equity-tranche (\$1m principal, 100% loss) and ABS-CDO Mezz-tranche (\$4m principal, 100% loss). But the senior-tranche of the ABS-CDO remains intact if the losses on the underlying sub-prime mortgages are 10% or less.

43.3.1.2 Case B: 20% Loss (\$20m) on Sub-prime Mortgages

The loss on the mortgages is \$20m. The first \$5m loss accrues to the ABS equity tranche (i.e. 100% of principal of \$5m) and the remaining \$15m loss is absorbed by the ABS-Mezz tranche. But the latter has a principal of \$20m, so the loss is 15m/20m = 75% of the principal of the ABS-Mezz tranche. The ABS-Senior tranche takes no losses (Figure 43.2).

The loss for the ABS-Mezz tranche is \$15m. From the latter, the ABS-CDO equity tranche takes the first hit of \$1m (100% of its principal of \$1m) and the ABS-CDO Mezz tranche covers the next \$4m of losses (100% of its principal of \$4m). Hence the loss remaining for the ABS-CDO senior tranche is \$15m − \$5m = \$10m. But the principal in the ABS-CDO senior tranche is 75% × \$20m = \$15m. Hence the proportionate loss is 10/15 = 66.7% of principal value for the ABS-CDO senior tranche – quite substantial for a tranche that may have a credit rating of AAA.

Looking across any row of Table 43.1 we can see that losses are 'levered' as we move from the percentage losses of the underlying mortgages themselves – the percentage losses on some of the ABS and ABS-CDO tranches is much higher.

For example, a 20% loss in value of the underlying mortgages gives rise to a 75% loss in the ABS-Mezz tranche and 100% loss in the ABS-CDO Mezz tranche. There is zero loss in the ABS senior tranche but a 66.7% loss in the ABS-CDO senior tranche. Of course both the ABS and ABS-CDO junior tranches lose 100% of their value, as they take the first hit.

In July 2008 Merrill Lynch sold \$30.6bn AAA-rated senior tranches of CDO-squared to Loan Star funds for 22 cents on the dollar. When the value of this tranche fell to 16.5 cents on the dollar, Merrill had to (legally) take back these assets – this demonstrates the riskiness of these structured products and the complications surrounding whether they should appear 'off-balance sheet' or 'on-balance sheet'.

43.4 SUB-PRIME CRISIS 2007–8

In the summer of 2007 the US was hit by the sub-prime mortgage crisis. Since 2000 and particularly over the period 2004–7 many US home owners had been given loans on 100% (or more) of the estimated value of the house. Some could not meet repayments, house prices in the US fell by around 30% on average, so the collateral underlying these loans was suspect. In consequence, major banks such as Merrills, Citygroup and UBS took 'large losses' on sub-prime loans they held 'on-balance sheet' and sometimes even greater losses on asset backed securities (with underlying sub-prime loans) that their trading divisions had purchased in the open market. Many of these ABSs were also held by other investors around the world such as pension funds and insurance companies.

Securitisation means that banks and S&Ls no longer 'originate and hold' mortgages but 'originate and distribute'. Sub-prime refers to the fact that some mortgage pools contained mortgages issued to borrowers who had high credit risk (e.g. based on credit scoring models) and therefore had a high probability of default on repayments relative to 'normal' creditworthy borrowers. There is clearly an incentive for financial advisors to sell mortgages and hence earn commission. Prior to 2008 some mortgagees used 'self-certification', so their income was not checked by anyone and they may therefore have obtained mortgages that are very large in relation to their 'true' income. These were known as NINJA loans – as it was said that some of the mortgagees had 'no income, no job and assets'. Other mortgages known as '2/28' were used, where for the first 2 years of the mortgage there was a low 'teaser' rate of interest of say 3% p.a. but then the mortgage would switch to a high fixed rate (of say 6%) for the remaining 28 years.

Mortgages in some US states are 'non-recourse' – if the borrower defaults, the bank can only take possession of the house but not the borrower's other assets (e.g. stock portfolio). Hence you could have the bizarre situation where two houses currently worth $250,000, each have a mortgage of $300,000 but could only be sold in foreclosure for $200,000. It pays both mortgagees to default and purchase each other's houses, when they are in the foreclosure stage. As foreclosures increased (particularly if concentrated in particular neighbourhoods), this increased housing supply in specific local areas and put further downward pressure on house prices – exacerbating an already dire situation. When the housing market turned down in the US, partly as a result of the end of the 'teaser rates', investors in ABSs (sub-prime mortgages) suffered losses. Sometimes clauses in the securitisation documentation allowed investors to hand back their ABS tranches to the banks, if they had lost value very quickly.

In the crisis nobody wanted to purchase ABSs involving sub-prime mortgagees – this is because it is difficult to value assets that are not 'transparent'. Their value depends on the economic position of a 'pool' of mortgagees, their ability to repay and the value of the house relative to the loan outstanding. Uncertainty about which banks were extensively involved in the sub-prime home loan market led to a liquidity crisis, as banks became wary of lending to each other in the 1-month and 3-month interbank market. Interbank rates (LIBOR) increased even though the Fed lowered its discount rate. As Warren Buffett commented on the use of ABS/CDOs: 'One of the lessons that investors seem to have to learn over and over again, and will again in the future, is that not only can you not turn a toad into a prince by kissing it, but you cannot turn a toad into a prince by repackaging it' (*Financial Times*, 26 October 2007).

In the middle the crisis, November 2007 saw the 'retirement' of Stan O'Neal, the boss of Merrill Lynch, when they announced prospective losses on sub-prime of $8bn. This was closely followed by the departure of Chuck Prince, the boss of Citygroup, who initially announced losses of between $8bn and $11bn on its sub-prime portfolio of around $60bn. UBS head Huw Jenkins also 'stepped down'.

Some of the equity tranches of ABSs were purchased by hedge funds. But hedge funds tended to buy *on margin,* by borrowing funds from their prime broker (i.e. a bank) to buy the ABS-CDOs. In this case banks did not really spread the risks of their mortgage pools very much. If the ‘pool’ becomes ‘toxic waste’, the hedge fund cannot sell the ABS at a reasonable price and if the hedge fund is not able to pay back the interest and principal on its bank loan, then the bank itself is in trouble. If the hedge fund is a subsidiary of the bank (and the bank may have a part equity stake in the hedge fund) the bank may cross-subsidise any losses the hedge fund makes for some time, in order to preserve its ‘reputation’. Again this means that the bank still retains the risk of some of the ABSs it originated.

The senior tranches of sub-prime ABSs were given a AAA-rating by rating agencies and the latter have come under criticism because of their conflicts of interest. The ratings agencies are paid by the issuers of the ABS to provide a rating and hence may tend to give too high a rating in order to secure repeat business from the originator bank. Sometimes AAA-rated, ABSs turned out to be more dangerous than Class-A drugs – hence the term ‘toxic waste’.

The devastating consequences of the mix of credit risk and liquidity risk came to the fore in the UK in 2007–8 when the sub-prime mortgage crisis in the US spilled over into other countries – see Finance Blog 43.1.

Finance Blog 43.1 Liquidity and the Sub-prime Crisis 2007–8

The mortgage crisis spread to the UK in September 2007. A mortgage bank called Northern Rock financed a large proportion of its mortgage loans by borrowing in the 3-month and 6-month interbank market and rolling over this short-term financing. With the uncertainty caused by the US sub-prime crisis because some of the ‘toxic waste’ might have been held (or financed) by UK banks, the sterling interbank (LIBOR) market experienced a liquidity crisis – other banks would not lend to Northern Rock. As a consequence Northern Rock had to borrow directly from the Bank of England’s ‘lender of last resort facility’ (equivalent to the Fed discount window) at a penal rate – an amount estimated at between £20bn and £30bn, on a balance sheet of around £100bn. These loans to Northern Rock from the Bank of England (ultimately underwritten by the UK taxpayer) had as underlying collateral, Northern Rock’s mortgage book – so repayments to the Bank of England ultimately depended on the credit worthiness of Northern Rock’s mortgagees.

However, news of this emergency borrowing caused a run on Northern Rock with its depositors queuing up to withdraw their deposits. (The previous time there was a significant run on the UK banking system was the Bank of Glasgow in 1878 and before that the bank, Overend Gurney in 1866.) After a few days of panic by depositors in Northern Rock, the UK government (Treasury) agreed to underwrite all deposit accounts in Northern Rock, ‘while the crisis persists’. Meanwhile Northern Rock’s board of directors examined alternative bids by companies, such as the Virgin Group and some private equity groups, to purchase/take over the whole of Northern Rock. The latter was not successful. The possibility of breaking up the bank by selling off its branch network, its IT systems and its mortgage book to different bidders, was also discussed. Eventually the bank was ‘temporarily’ nationalised by the UK government.

The UK Treasury also announced an investigation into the deposit insurance scheme which in the UK then covered about 90% of deposits in each bank up to £33,000 ($45,000) per person – whereas US deposits were insured up to $100,000. The insured limit was eventually increased to £85,000 ($57,000). The Treasury also sought a change in European law which would allow the Bank of England not to have to disclose its 'lender of last resort' arrangements (for a period of time while it was pursuing this policy), so in future it could conduct 'secret negotiations' with potential buyers of distressed financial institutions. Northern Rock's share price fell by over 80% – at the time it was referred to as 'Northern Wreck'. The 'good parts' of the bank were eventually sold to the Virgin Group in 2011.

Source: Adapted from Cuthbertson and Nitzsche (2008).

43.5 SYNTHETIC CDOs

If the assets being securitised and placed in tranches are bonds issued by corporations or countries then the ABS is referred to as a 'cash CDO'. A long position in a corporate bond has the same credit risk as a short position in a CDS (written on the same bond) – if the company defaults both positions experience similar losses. The 'short' in a CDS contract receives periodic fixed payments but suffers a loss equal to the fall in value of the (reference) bond if it defaults – these cash flows are equivalent to a long position in a corporate bond. This implies that instead of forming a CDO from corporate bonds we can create a CDO using a portfolio of short positions in *credit default swaps* (CDS) – this is known as a *synthetic CDO*.

In a synthetic CDO, losses on the CDS are allocated to tranches. Suppose the total notional principal on a portfolio of CDS is $100m and there are three tranches. Suppose the tranches are as follows:

1. Tranche-1 takes the first $5 million of losses and has a promised return of 15% on the remaining tranche-1 principal.
2. Tranche-2 takes the next $30 million of losses and has a promised return of 200 bps (over LIBOR) on the remaining tranche-2 principal.
3. Tranche-3 takes the next $65 million of losses and has a promised return of 20 bps (over LIBOR) on the remaining tranche-3 principal.

The *synthetic CDO* consists of these three tranches of the CDS. Initially tranche-1 earns 15% on the $5m principal. If after say 1 year, $2m of losses occur on the portfolio of CDS then the notional principal of tranche-1 is reduced by $2m – so the 15% return is earned only on the remaining $3 million (rather than $5m). When losses reach $5 million, tranche-1 ceases to exist and tranche-2 takes any further losses, and so on. For example, when *total* losses reach $10 million, the notional principal of tranche-2 is reduced to $25m (=$30m – $5m) and it receives 200 bps per annum on the reduced principal.

Investors in a synthetic CDO have to immediately post the initial tranche principal as collateral and this earns LIBOR. When there are defaults and the 'short' CDS has to pay out funds, money is taken from the collateral of the tranche which has experienced the defaults. If there is any recovery of a proportion of the loans/bonds in default, these are usually used to reduce the principal on the most senior tranche. From that point on, the senior tranche earns its promised return based on its new lower principal value. However, the senior tranche does not actually *lose* its future claim on the principal that is retired – it just does not earn a return on it.

43.6 SINGLE TRANCHE TRADING

A single tranche trade is simply an agreement to buy or sell protection against losses on particular designated 'standard' tranches, where these standard tranches are created from a portfolio of 125 companies that are used in the CDX and iTraxx indices. (Note that single tranche trading is *not* a synthetic CDO, which requires a portfolio of assets.)

How are these tranches created? For example, in the case of single tranche trading on the CDX index, the equity tranche covers losses between 0% and 3%, the second tranche (mezzanine) covers losses between 3% and 7% and the other tranche 'coverage percentages' are shown in Table 43.2.

The price quotes for each tranche are in basis points and have a day-count convention of '30/360'. But the equity tranche is quoted differently. The equity tranche payer on the CDX-USA index pays 26% of the notional principal (in the CDX index) at the initiation of the equity single

TABLE 43.2 Five-year CDX and iTraxx single tranches

Tranche	Equity	Mezzanine	Tranche	Tranche	Tranche	Min. risk tranche
Panel A : CDX – USA						
Losses	0–3%	3–7%	7–10%	10–15%	15–30%	30–100%
Price quote	26% 400 bp	101 bp	20 bp	10 bp	4 bp	2 bp
Panel B: iTraxx – Europe						
Losses	0–3%	3–6%	6–9%	9–12%	12–22%	22–100%
Price quote	11% 400 bp	58 bp	14 bp	7 bp	3 bp	1 bp

Notes: Price quotes have day count '30/360' and are in basis points. The equity tranche is quoted differently. The equity tranche payer pays 26% CDX-USA (or 11% iTraxx-Europe) of the principal, on the CDX (or iTraxx) index, at the initiation of the equity single tranche and then pays 400 bp per year thereafter.

tranche trade and then pays 400 bps per year thereafter. For the equity tranche on the iTraxx (Europe) index the equivalent figures are 11% of the principal (in the iTraxx index), at the initiation of the equity single tranche trade and then 400 bps per year thereafter.

43.6.1 Trader-A: Buys Protection on 7–10% CDX-tranche

To see how single tranche trading works, suppose trader-A buys protection over the next 5 years from trader-B for the '7–10% tranche' on the CDX index for an amount of protection (notional principal) of \$9m, with a spread payment of 20 bps per year (Table 43.2). Payouts from B to A depend on default losses on the CDX index. If the cumulative loss is less than 7% of the portfolio principal, there is no payout and trader-A pays $0.2\% \times \$9\text{m} = \$18{,}000$ p.a. to trader-B (this is usually paid each quarter in arrears). Suppose at the end of year-2 cumulative losses increase from 7% to 8% (i.e. 1/3 of the total range of 7–10% for this tranche), then trader-A will receive \$3m (=1/3 of \$9m) and the tranche principal is reduced to \$6m.[3]

From this point on, the 20 bps per year is paid on the reduced principal. If at the end of year-3 cumulative losses hit 10% then trader-A receives an additional \$6m (and the 7–10% tranche principal is now zero = \$9m − \$3m − \$6m). Any subsequent losses over the final 2 years of the protection period, involve no further payments between these two counterparties.

Note that if you purchase protection on the *equity tranche* of CDX, the quote is different and equal to ' 26% p.a. + 400 bps '. This means the protection seller (trader-B) receives an initial up-front payment of (26% × \$9m) and then is entitled to receive 400 bps per year on any of the *remaining* tranche principal, throughout the life of the deal.

The pricing of 'CDS tranches' and synthetic CDOs is both technically and practically rather complex. This is because the quoted price depends on the expected default correlations between the underlying reference entities – these correlations are difficult to forecast, partly because we have so little data and partly because these correlations may not be constant over time.

To get some idea of what is happening, assume you have 100 reference entities in the synthetic structure. Suppose the probability of any reference entity defaulting (over say 5 years) is 2% (in any one year) and the credit default correlation is zero (e.g. defaults are independent). The binomial model tells us that the probability of one or more defaults over the 5 years is 86.74%, while the probability of 10 or more defaults is 0.0034%.[4] So a first-to-default CDS will cost a substantial amount, whereas a 10th-to-default CDS will cost much less.

So, if default correlations are low, the junior tranches of a synthetic CDO are very risky and therefore relatively expensive, but the senior tranches are very safe and it is relatively cheap

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[3]Although here we are describing single tranche trading and not a synthetic CDO, it is worth noting that this is also the way in which cash flows are calculated in a synthetic CDO.

[4]$p(1 \text{ or more defaults}|\ n = 100) = 1 - p(\text{zero defaults}) = 1 - (0.98)^{100} = 0.8674\ (86.74\%)$.
$p(10 \text{ or more defaults}|n = 100) = 1 - p(1 \text{ default}) – p(2 \text{ defaults}) \ldots. - p(9 \text{ defaults}) = 0.0034\%$,
where $p(\text{exactly } r \text{ defaults}|n = 100) = C_r^n p^r (1 - p)^{n-r}$. and p = probability of default.

to purchase protection. To take the other extreme, if the default correlation between all the reference entities is +1, the probability of one or more defaults is the same as the probability of exactly 2, 3, 4, ..., 99 or 100 defaults and all these probabilities equal 2%. So all tranches are equally risky and all have the same price. The price of the synthetic CDO depends on the correlations between defaults and these are difficult to forecast accurately.

43.7 TOTAL RETURN SWAP

A total return swap (TRS) can be used for hedging the total risk (i.e. market plus credit risk) of a corporate bond. We assume the payer in the TRS already owns the corporate bond. In a total return swap *the payer* agrees to pay the total return on an asset (e.g. 10-year corporate bond) and to receive LIBOR plus a spread (Figure 43.3). The total return *receiver* is the counterparty.

Suppose the reference asset is a 10-year corporate bond. Then a TRS with a life of 5 years and a notional principle of $10m, would involve periodic payments of the bond coupons (e.g. every 6 months) by the total return payer and receipt of LIBOR+30 bps (say) over the 5 years. LIBOR is determined on a coupon payment date and paid out at the next coupon (reset) date, as in a standard interest rate swap. At maturity of the TRS there is also a payment reflecting the change in value of the bond over the life of the TRS.

For example, if the 10-year bond increases in value by 10% over the 5-year life of the TRS, the total return payer will pay out $1m at the end of 5 years. But if the bond decreases in value by 10%, the total return receiver will pay $1m to the ‘TRS payer’. Before the maturity date of the total return swap there may be a default on the reference bond. At this point, the swap is usually terminated and the swap receiver makes a payment equal to ‘$10m less the market price of the bond’ at default.

There are variants on the above. For example, cash flows arising from any change in the market value of the bond may be made periodically rather than at maturity of the TRS. Also instead of a cash payment for the change in value of the bond at maturity of the TRS, there may be ‘physical settlement’ – the total return payer delivers the (reference) bond and receives cash equal to the notional principal of the TRS.

A TRS can be used for hedging credit risk, if the TRS-payer already owns the corporate bond. The credit risk is passed on to the TRS-receiver, while the TRS-payer has a net receipt of LIBOR+30 bps, independent of what happens to the bond price due to changes in credit risk. If the TRS-payer does not own the bond, the TRS-payer is effectively ‘short’ the corporate bond.

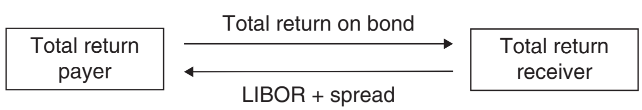

FIGURE 43.3 Total return swap

For example, the TRS-payer will have to pay the coupons each period and if the bond price rises by say 10%, the TRS-payer will pay out 10% of the (par) value of the bond (at maturity of the TRS).

The TRS-payer loses money if the counterparty (i.e. the TRS-receiver) defaults any time after the reference bond's price has declined, and the spread over LIBOR is compensation for this risk. The 'spread-over-LIBOR' is therefore determined by the credit quality of both the bond issuer and the receiver in the TRS, as well as the credit correlation between the two. The *receiver* in a TRS with principal $Q = \$10$m can also be viewed as owning a bond, financed by a loan at LIBOR+30 bps.

43.8 SUMMARY

- Asset backed securities (ABSs) are marketable securities where future payments depend on cash flows from a specific source (e.g. credit card receipts, rents from property or payments by mortgage holders). If the cash flows are from (underlying) fixed-income assets (e.g. mortgages, corporate loans) then the ABS is known as a collateralised debt obligation (CDO). The structuring of ABSs and CDOs is known as securitisation.
- In an ABS, cash flows are assigned to 'tranches' and each tranche receives its cash flows in a particular priority order ('the waterfall'). The senior tranche is paid first, followed by the mezzanine tranches and finally the equity tranche is paid (if funds are available). Hence the ABS is structured so the senior tranche has less credit risk than the other tranches (and less credit risk than the underlying loans/bonds) – this is 'credit enhancement'. The super-senior and senior tranches are usually given a AAA credit rating.
- The mezzanine tranches from several different ABSs can also be subject to 'a waterfall', which gives rise to an 'ABS-CDO' (also referred to as a CDO-squared).
- The default risk of ABSs and ABS-CDOs are difficult to forecast because they depend on the individual default risks of a large number of underlying borrowers and the correlation between defaults across these borrowers.
- Default by a relatively small proportion of the underlying assets (e.g. bank loans, mortgagees) can cause much larger proportionate losses for the mezzanine tranches of an ABS and even larger losses for the mezzanine tranches of an ABS-CDO. Even losses in the AAA-rated senior (or super-senior) tranches of both the ABS and ABS-CDOs are possible. This 'leverage effect' was the proximate cause of the difficulties faced by many banks in the credit crunch of 2008–9, when the US sub-prime home loan market collapsed.
- A 'synthetic-CDO' consists of tranches formed from short positions in credit default swap (CDS) contracts. As with 'cash-market' CDOs, each tranche of a synthetic-CDO takes losses in a specific order, the equity tranche takes the first losses (and has the highest promised return), followed by the mezzanine and senior tranches.

- In ‘single-tranche trading’ you can buy and sell ‘tranches’ of credit risk. For example, suppose you buy a 5-year, 7–10% single-tranche CDS on iTraxx. Then you are buying protection over the next 5 years on losses (for the 125 companies in the index) of between 7% and 10% of the agreed notional principal. This insurance against losses may cost you 200 bps per year of the notional principal (but payments cease once cumulative losses exceed 10% of the principal).
- The buyer of a total return swap (TRS) agrees *to pay* the total return on a reference bond – that is, the periodic coupon payments and any change in the market price of the bond, which takes place from inception and the maturity date of the TRS. Also, the buyer of the TRS receives LIBOR plus a spread (on the notional principal in the TRS contract). If an investor already holds a corporate bond, then a long position in a TRS can be used for hedging market and credit risk on the bond (i.e. ‘the reference entity’).

EXERCISES

Question 1
What is securitisation?

Question 2
What is tranching or a ‘waterfall’?

Question 3
Explain how an ABS-CDO (or CDO-squared) works.

Question 4
If the correlation between bond-defaults increases, explain what happens to the risk of the senior tranche of an ABS. Assume the senior tranche has only two bonds and each bond has a probability of default (over the next year) of 10%. In your answer, first assume zero correlation between two bond prices that are in the senior tranche and then a correlation coefficient of +1.

Question 5
You have two bonds where the probability of default (over the next year) for each bond is $p = 40\%$. Credit risks on the two bonds are independent so the correlation between defaults is zero. Each bond has par value of \$1. Structure these two bonds into a senior and a junior tranche and calculate the default probabilities of each tranche and demonstrate any ‘credit enhancement’.

Question 6
What is a total return swap and how can it be used to hedge a corporate bond?

Question 7
What is a sub-prime mortgage?

PART XI

MARKET RISK

CHAPTER 44

Value at Risk

Aims

- To explain the concept of value at risk (VaR).
- To measure VaR using the variance-covariance (VCV) approach.
- To outline methods used in forecasting volatility.
- To use backtesting to assess the accuracy of VaR forecasts.
- To outline the 'Basel internal models' approach in setting regulatory standards.

44.1 INTRODUCTION

Financial institutions, particularly large investment banks, hold positions in a wide variety of assets such as stocks, bonds, foreign exchange, as well as futures and options contracts. Because prices of these assets can move quickly and by large amounts, investment banks want to know what risk they are holding over the next 24 hours as well as over the next week or month. Once they know what their risk position is then they can either decide to maintain or reduce it or alternatively to hedge the risk – usually by using derivatives. Risk due to price changes is known as 'market risk' and measuring and monitoring changing market risk is the subject of this chapter.

For their own prudential reasons financial intermediaries need to measure the overall 'dollar' market risk of their portfolio, which is usually referred to as *value at risk* (VaR). In addition, the regulatory authorities use VaR to set minimum capital adequacy requirements for financial institutions. If a bank has a high VaR then the regulator may limit the amount of dividends paid to stockholders, so that the bank can increase its 'capital' via an increase in its retained profits. Alternatively, a bank can increase its 'capital' by issuing more stock to investors via a 'seasoned offering' (or 'rights issue' in the UK), which it often is reluctant to do because this

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

dilutes the claims on future dividends of existing stockholders. The possibility of having to increase ‘bank capital’ using either of these methods is supposed to act as a deterrent to banks increasing their level of market risk to unacceptable levels.

The interaction between investment decisions and risk calculations is now absolutely central in managing mutual funds, hedge funds, pension funds, and a bank’s marketable assets. The chief risk officer (CRO) in a financial institution has a key role in the risk management process.

In this chapter we examine how to measure market risk, mainly using stocks as our example. There are various ways to measure portfolio risk and we concentrate on the variance-covariance method (also called the delta-normal method). Since market risk varies over time we examine the accuracy of our forecast of VaR.

44.2 VALUE AT RISK (VAR)

In this section we use concepts from portfolio theory to provide us with a simple yet useful forecast of market risk, namely, value at risk – which is now the industry standard. Suppose you hold a portfolio of stocks. The return R_{t+1} *between today and tomorrow*, is the capital gain or loss on the stocks (plus any cash payments such as dividends). However, dividend payments are announced in advance, so the source of uncertainty and risk *over the next day* (say) is mainly due to price changes. For simplicity, assume dividend payments are zero so that the return on a stock is[1]:

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t} \tag{44.1}$$

Measuring returns over 1 day allows us to determine *daily* VaR for a stock portfolio. Suppose you currently hold \$3,030m in your stock portfolio and the CRO reports to the CEO of a bank, that over the next 24 hours (1 day) the VaR is \$100m, at a 95% confidence level (also referred to as a 5% ‘left-tail’ value or 5th percentile). This implies that:

Given your current stock portfolio, there is a 95% chance (probability) that you will lose less than \$100m over the next day (24 hrs).

Note that losing ‘less than’ \$100m also means you might make gains on your portfolio – in fact there is a 50% chance you will make gains (if daily stock returns are normally distributed with a zero mean) – Figure 44.1.

Note that the risk officer is not saying you *will lose* less than \$100m, only that there is a 95 out of 100 chance that you will lose less than \$100m. So, as a concept VaR involves a specific probability or confidence level and a specific time horizon over which we measure risk. We

[1] Alternatively, any dividend payments are added to the price at $t + 1$ and then R_{t+1} is the total return which equals the capital gain plus the dividend yield, $R_{t+1} = (P_{t+1} + D_{t+1} - P_t)/P_t = (P^*_{t+1} - P_t)/P_t$.

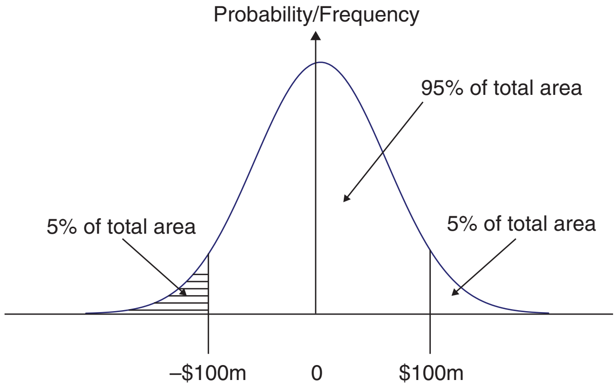

FIGURE 44.1 Normal distribution

cannot predict *exactly* how much you will lose (or gain) over any 24-hour period – since the world is risky – we can only provide a 'best guess' with a certain probability. This is not the only way we can express the daily VaR and an equivalent statement is:

Given your current stock portfolio then you expect to lose less than $100m (over any 24-hour period), in 19 out of the next 20 days.

You can see where 'probability' comes in, since '19 out of 20 days' is equivalent to '95 days out of 100', that is, 95% of the time. Hence, if you hold the $3,030m stock portfolio unchanged over the next 20 days, then on about 19 of those days you would lose less than $100m, hence:

If the daily portfolio VaR is $100m, at a 95% confidence level (5th percentile) then if you continue to hold your current stock portfolio, you expect to lose less than $100m in 19 of the next 20 days and to lose more than $100m on one day out of the next 20 days.

Note that even though the VaR concept is concerned with potential *losses* it is usually reported as a positive number. Much of the material discussed is based on the risk measurement methodology as set out in *RiskMetrics*™ (1996) which emanates from JPMorgan.

44.2.1 Measuring Risk

The riskiness of a *single asset* is summarised in the probability distribution of its returns. For daily stock returns the outcomes can take on many values and a convenient first assumption is that these outcomes are normally distributed. There is no *single* acceptable measure of the riskiness inherent in a particular statistical distribution, but for the normal distribution the standard deviation (or variance) is widely accepted as a measure of risk.

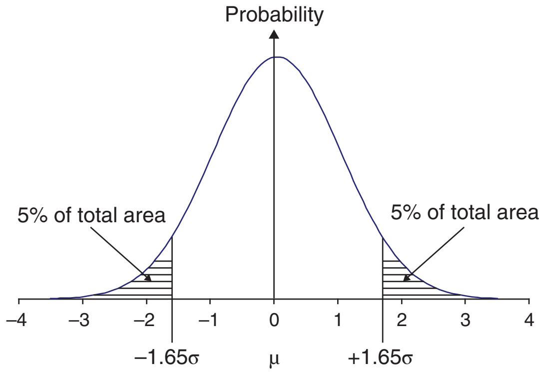

FIGURE 44.2 Normal distribution

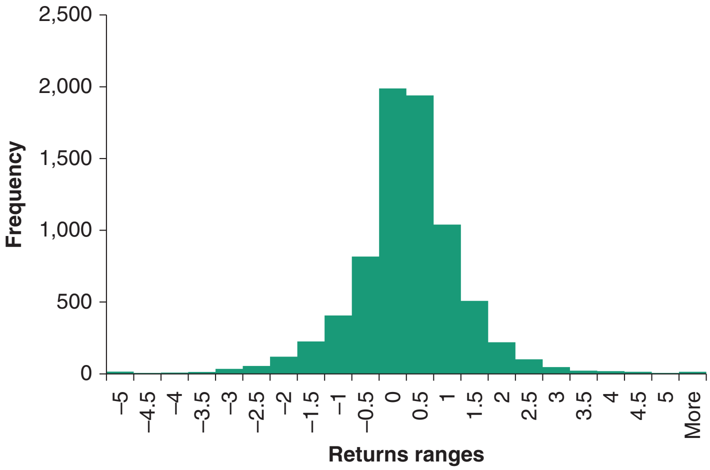

FIGURE 44.3 Daily returns S&P 500 (January 1990–March 2019)

For normally distributed returns we can be 90% certain that the actual return will equal the expected return μ plus or minus 1.65σ (see Figure 44.2) where σ is the standard deviation of the returns. Put another way, we expect the actual return to be *less than* $\mu - 1.65\sigma$ on 5% of occasions or greater than $\mu + 1.65\sigma$, also on 5% of occasions (e.g. 1 time in 20).

The mean daily return on a stock is small and close to zero. From a statistical point of view, there is so much 'noise' in daily returns (i.e. daily returns have a large standard deviation, relative to their mean return) so:

- when calculating *daily* VaR, we assume the mean return is zero.
- Hence, 1.65σ is a measure of the (per cent) 'downside risk' (at the 5th percentile).

If you hold a (net) position of $\$V_0$ in one asset (e.g. stock of AT&T), then the (dollar) VaR is:

$$VaR = \$V_0(1.65\sigma) \tag{44.2}$$

which is a basic ‘building block’ for calculating the VaR of more complex portfolios.

EXAMPLE 44.1

Calculation of Daily VaR

Suppose daily stock returns are normally distributed. Assume the mean daily return on AT&T stocks is $\mu = 0$ with standard deviation $\sigma = 0.005$ (0.5% *per day*). You hold $V_0 =$ \$115m in AT&T stocks. What is the daily \$-VaR of your position, at the 5th percentile and what does this mean?

If daily returns R are normally distributed then $\mu \pm 1.65\sigma$ has 5% of the probability distribution in each tail – leaving 90% of the probability in the ‘centre’ of the distribution. Hence 90% of daily returns lie between:

$$\mu + 1.65\sigma = +0.825\% \text{ and } \mu - 1.65\sigma = -0.825\%$$

Put another way, only 5% of the daily returns should exceed +0.825% and 5% should be smaller than –0.825%. Either of these cases should occur no more than 1 day in every 20 days (i.e. 5% of the time). The \$-VaR of a US investor holding \$115m in AT&T stock is:

$$VaR = \$V_0(1.65\sigma) = \$115\text{m } (0.825/100) = \$948{,}750$$

Hence in 95 days out of 100, you should not lose more than \$948,750 over any 24-hour period (assuming you maintain your initial position of \$115m in AT&T stocks). But 5% of the time (i.e. 5 days out of 100) you will lose more than \$948,750.

A more direct approach to calculating the daily VaR is to note that the dollar change in value of the stock is $dV = V_0R$, where V_0 = initial dollar holding in the stock, R = daily return on the stock. It then follows directly that $\sigma_{dV} = V_0\sigma_R$ and hence the VaR at the 5th percentile is:

$$\text{VaR} = 1.65\ \sigma_{dV} = V_0(1.65)\sigma_R.$$

44.2.2 Are Daily Returns Normally Distributed?

Empirically, daily returns (i.e. daily price *changes, dP/P*) for exchange rates, long-term bond prices, and stock prices are all *approximately* normally distributed but they do deviate somewhat from normality, in the following ways:

- return distributions often have fatter tails than the normal distribution
- returns are negatively skewed (i.e. there are more observations in the left-tail than the right-tail).

In Figure 44.3 we show the histogram of daily returns using the S&P 500 index of US stocks – the mean daily return is very close to zero. The histogram is broadly ‘bell shape’, like the normal distribution. But the empirical distribution has a thinner peak than the normal distribution and there are a few more occurrences of extreme negative and positive returns than would be associated with the normal curve (i.e. the empirical histogram has fatter tails than the normal distribution) – this is excess kurtosis. In addition, the empirical distribution is reasonably symmetric so there is no appreciable negative or positive skew. In fact, for daily returns on most *spot (cash market)* assets it is probably a *reasonable approximation* to assume normality.

44.2.3 Portfolio Risk

How can we measure the $-VaR of a *portfolio* of assets? First we present a ‘text book’ approach to this problem and then we use a more direct approach. Let’s take two assets for simplicity – stocks of AT&T and Microsoft, in proportions (or ‘weights’) ¼ and ¾ respectively. The return R_p on *portfolio* of stocks is:

$$R_p = w_1 R_1 + w_2 R_2 = (1/4)R_1 + (3/4)R_2 \tag{44.3}$$

The standard deviation of *returns* on the portfolio is given by the usual formula from portfolio theory (as found in standard text books):

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho\sigma_1\sigma_2} \tag{44.4}$$

where σ_i is the standard deviation of the return on stock-i and ρ is the correlation coefficient between the return on stock-1 (AT&T) and the return on stock-2 (Microsoft). So when we have a portfolio of stocks, consideration must be given to all the correlations between returns when calculating the *portfolio* standard deviation. (Note also that the term $\rho\sigma_1\sigma_2$ is the *covariance* between the two returns, usually written σ_{ij}.)

In Equation (44.4) σ_p is measured as a ‘per cent per day’ (expressed as a decimal) but the CEO of a bank is more interested in the *dollar*-VaR. By analogy with the single asset case the $-VaR for the portfolio is given by:

$$VaR_p = V_p(1.65)\sigma_p = V_p 1.65\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho\sigma_1\sigma_2} \tag{44.5}$$

where V_p is the total amount invested in the stock portfolio. It is more convenient if the VaR calculation uses the *dollar amount* in each stock. If the initial dollar amounts held in each stock are $V_1 = \$100$ and $V_2 = \$300$, then $V_p = V_1 + V_2 = \$400$ and $w_1 = V_1/V_p = 1/4$ and $w_2 = V_2/V_p = 3/4$. Substituting for w_1, w_2 in (44.5):

$$VaR_p = \sqrt{(V_1 1.65\sigma_1)^2 + (V_1 1.65\sigma_2)^2 + 2\rho(V_1 1.65\sigma_1)(V_2 1.65\sigma_2)}$$
$$= \sqrt{\mathrm{VaR}_1^2 + \mathrm{VaR}_2^2 + 2\rho(\mathrm{VaR}_1)(\mathrm{VaR}_2)} \tag{44.6}$$

This is rather convenient since the dollar *portfolio*-VaR uses the individual dollar VaR's for each stock. If the dollar amounts held in each stock (V_1, V_2) are positive, then the smaller the correlation between the two stock returns, the lower is VaR_p. This is the usual diversification effect in portfolio theory. However, also note that if you have *short-sold* say \$10m of stock-1 then $V_1 = -\$10m$ in Equation (44.6). So if stock-2 is held long then the larger is the *positive* correlation between the two stock returns, the smaller is portfolio risk and the VaR.

44.2.4 Worst-case VaR

Consider what would make VaR_p a rather large number, indicating high portfolio risk. If stock returns are perfectly positively correlated ($\rho = +1$) and you hold positive amounts in each stock $(V_1, V_2) > 0$, then this is a worst-case scenario, since there are no benefits from diversification. Thus the maximum value that VaR_p can take (for $V_1 > 0$, $V_2 > 0$), known as the worst-case VaR, is given by setting $\rho = +1$ in (44.6) which gives:

$$\text{Worst-case } VaR_p = V_p 1.65\,(w_1\sigma_1 + w_2\sigma_2) = VaR_1 + VaR_2 \tag{44.7}$$

Hence, the 'worst-case VaR' is simply the sum of the individual VaRs.[2] In the real world it is unlikely that the correlation between the returns of two stocks like AT&T and Microsoft is +1, so your 'best estimate of risk will usually be the diversified VaR' given by Equation (44.6). However, in 'crisis periods' it is often the case that correlations increase, and the worst-case VaR provides a 'high' figure which may be representative of these crisis periods (e.g. during the 1987, 2000–1 and 2008 stock market crashes and the bond price meltdown of 1997). So, risk managers usually look at both the diversified-VaR and the worst-case VaR. This gives them a feel for what would happen if historic correlations did not stay the same but increased in a crisis period.

44.2.5 Two Assets

Using matrix algebra, Equation (44.6) can be succinctly written as:

$$VaR_p = [\mathbf{ZCZ'}]^{1/2} \qquad C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{44.8}$$

where $Z = [VaR_1, VaR_2] = [V_1 1.65\sigma_1,\ V_2 1.65\sigma_2]$ is a row vector of the *individual* VaRs for each asset and C is the 2×2 correlation matrix.

[2]The worst-case VaR of Equation (44.7) also results from $V_1 > 0,\ V_2 < 0$ and $\rho = -1$.

When we have more than n-assets, VaR_p can be represented in exactly the same way but there are n entries in the Z-vector and the correlation matrix is $(n \times n)$. Statisticians provide estimates of the next day's forecast of 'volatilities' σ_i and correlation coefficients (in matrix **C)**, while traders provide the dollar amounts V_i they hold in each stock. So in practice, to calculate VaR_p with n-stocks we need:

1. the dollar position in each of the n-stocks
2. forecasts of the daily standard deviation and correlations.[3]

As mentioned above, when reporting *individual* VaRs these are always taken as positive, even though they represent a potential loss (which conventionally has a negative sign). In calculating VaR_p, the *sign* of the individual V_i must be incorporated in Z and if the stock is held long (i.e. you own it) then $V_i > 0$ but if the stock has been short-sold then $V_i < 0$. A simple example will clarify the issues involved.

EXAMPLE 44.2

Calculation of Portfolio VaR

Data: A US investor holds \$10m of AT&T stock and has short-sold \$5m of Cisco stock.
Daily volatility of AT&T, $\sigma_1 = 1.5\%$ and daily volatility of Cisco, $\sigma_2 = 1.0\%$
The correlation between AT&T returns and Cisco returns is $\rho = -0.1$

Questions: 1. What is the daily VaR?
2. What is the worst-case daily VaR?

Answer: Long position: $VaR_1 = (\$10\text{m})\ 1.65\ (1.5/100) = \$247{,}500$
Long position: $VaR_2 = (-\$5\text{m})1.65\ (1.0/100) = -\$82{,}500$

$$\begin{aligned} VaR_p &= \sqrt{VaR_1^2 + VaR_2^2 + 2\rho(VaR_1)(VaR_2)} \\ &= \sqrt{247{,}500^2 + 82{,}500^2 + 2(-0.1)(247{,}500)(-82{,}500)} \\ &= \$268{,}601 \end{aligned}$$

Note the two negative signs '–0.1' and '–82,500' reflecting the negative correlation and the short-sale of Cisco, respectively. These two 'factors' tend to increase portfolio risk and VaR. You obtain the same result if you use the matrix formulation in Equation (44.8). The worst-case VaR (assumes all stocks are held long and all correlation coefficients $\rho = +1$).

Worst-case $VaR_p = VaR_1 + VaR_2 = \$330{,}000$.

[3] If the number of assets in our portfolio is n, we have 'n standard deviations' and $n(n-1)/2$ distinct correlations. For example, for $n = 50$ that means 1,225 different correlation coefficients! Later we see how this very large number of correlations can be reduced by using the single index model (SIM).

44.2.6 VaR: Portfolio of Stocks

Now we calculate VaR using a more direct approach, which will be useful when we consider more complex portfolios (e.g. containing foreign assets and bonds) in the next chapter. For a portfolio of n-stocks, there is a *linear* relationship between the *dollar change* in the value of the portfolio dV_p and stock returns. Given $V_p = \sum_{i=1}^{n} N_i P_i$ where N_i = number of stocks-i, P_i = price of stock-i, then:

$$dV_p = \sum_{i=1}^{n} N_i dP_i = \sum_{i=1}^{n} V_i(dP_i/P_i) = \sum_{i=1}^{n} V_i R_i \tag{44.9}$$

where $V_i = N_i P_i$ is the dollar amount in each stock and $R_i = dP_i/P_i$ is the (proportionate) return on stock-i. The above holds exactly for stocks and is a reasonable approximation for several other securities (e.g. the return on a foreign stock measured in domestic currency, the return on a bond, FRAs, FRNs, forwards, and futures). It follows directly from Equation (44.9) that the standard deviation of the *dollar change* in portfolio value is:

$$\sigma_{dV_p} = \left[\sum_{i=1}^{n} V_i^2 \sigma_i^2 + \sum_{i \neq j} \sum V_i V_j \rho_{ij} \sigma_i \sigma_j \right]^{1/2} \tag{44.10}$$

Equation (44.10) can be used to calculate the standard deviation of the dollar change in portfolio value regardless of the form of the distribution of stock returns. But the standard deviation alone does not allow us to determine the 5th percentile lower 'cut-off' value, unless we assume a specific distribution for asset returns. In particular, if we add the assumption of (multivariate) normally distributed asset returns, then the diversified-VaR at the 5th percentile is $VaR_p = 1.65\sigma_{dV_p}$, which can be more compactly written (here, for $n = 3$ assets):

$$VaR_p = 1.65\ \sigma_{dV_p} = \sqrt{\mathbf{ZCZ'}} \tag{44.11}$$

where
$$\begin{aligned} VaR_i &= V_i(1.65)\sigma_i \\ Z &= [VaR_1, VaR_2, VaR_3] \\ C &= \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \quad \text{(3} \times \text{3 correlation matrix).} \end{aligned}$$

The linearity and normality assumptions are crucial here. The change in portfolio value is sometimes perfectly consistent with linearity (e.g. for a stock portfolio) and where it is not, we often impose linearity as an approximation (i.e. we use a first order Taylor series expansion for the change in portfolio value). Note that if the relationship between dV_p and asset returns is highly non-linear (e.g. for stock options), then even if the returns of the underlying assets (stocks) are themselves normally distributed, the distribution of the change in the option's value will *not be normally distributed*. Hence, for a portfolio of options we cannot simply use the '1.65 σ_{dV_p}' rule – we have to examine the complete distribution of outcomes, using MCS and other techniques.

Providing we assume linearity and (multivariate) normality, all we need to calculate VaR_p is the value of each asset holding V_i and forecasts for the correlation coefficients and the return volatilities – hence it is known as the 'variance-covariance' (VCV) method. The worst-case VaR occurs when all the n-assets are assumed to be held long (i.e. all $V_i > 0$) and all assets returns are perfectly positively correlated (hence **C** = the unit matrix where all elements are +1). Thus from (44.11):

$$\textit{Worst-case VaR} = VaR_1 + VaR_2 + \ldots + VaR_n$$

The VaR for a portfolio of three stocks is given in Table 44.1. Each individual VaR is calculated as $VaR_i = V_i(1.65\sigma_i)$ – this is in the fourth column. The worst-case VaR is simply the sum of the individual VaRs. The diversified VaR is calculated using the formula $\sqrt{ZCZ'}$. The diversified VaR is much smaller than the worst-case VaR because although all the correlation coefficients are positive, stock-2 has been short-sold (hence the '–10,000' in the second column) and this 'manufactures' a negative correlation between its return and the returns on stocks 1 and 3. Hence there is a strong diversification effect and the (diversified) VaR at 783 is much less than the worst-case VaR of 1,996.

An Excel spreadsheet on the website calculates the VaR for a portfolio of 10 stocks using matrix notation.

TABLE 44.1 VaR, portfolio of three stocks

Assets	Value	Std. dev.	VaR		Correlation matrix (= C)		
1	10,000	5.4180%	894		1	0.962	0.403
2	–10,000	3.0424%	502		0.902	1	0.61
3	10,000	3.6363%	600		0.403	0.61	1
			Individual VaRs	894	–502	600	

Worst case VaR = 1,996

Diversified VaR = 783

44.3 FORECASTING VOLATILITY

It is clear from the above analysis that to calculate daily-VaR using the variance-covariance approach we need to forecast tomorrow's stock return volatility. A simple forecasting scheme is to assume that tomorrow's forecast of daily volatility is a *simple moving average (SMA)* of past squared returns R_t^2, over the last 30 days (say):

$$\sigma_{t+1}^2 = (1/30)\sum_{i=0}^{29} R_{t-i}^2 \tag{44.12}$$

Note that in (44.12) we do not use the more usual term $(R_{t-i} - \overline{R})^2$ because we have assumed the mean daily return is zero. If today (= t) is Monday after the market has closed, then (44.12) gives a forecast of Tuesday's volatility based on the squared returns from Monday, last Friday's, Thursday's, etc., for the past 30 trading days. When 'tomorrow' arrives then on *Tuesday,* we make a new forecast. Now, our forecast for Wednesday includes Tuesday's, Monday's, etc. squared return (and we drop the final day's return from 31 days ago). Hence our forecast is continually updated as we 'move through time'.

A problem with the SMA is that it gives equal weight (of 1/30) to all of the past 30 day's squared returns, when forecasting *tomorrow's* volatility. It seems more intuitive when forecasting 'tomorrow' to give relatively greater weight to what has happened in the recent past then in the more distant one. This can be achieved by using an *exponentially weighted moving average (EWMA)* forecast:

$$\sigma_{t+1}^2 = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i R_{t-i}^2 \tag{44.13a}$$

where R is the daily return on the asset (with mean zero), λ is a 'weight' lying between 0 and 1 (e.g. $\lambda = 0.94$). The weights λ^i decline exponentially, so this forecasting scheme is known as an EWMA model. For example, for $\lambda = 0.94$, the λ^i weights decline as 0.94, 0.88, 0.83 , . . . , etc. and squared returns which occur further in the past are given less weight in forecasting tomorrow's variance. Actually, the above equation can be written in a simpler form:

$$\sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)R_t^2 \tag{44.13b}$$

Suppose you have 250 daily observations on squared returns R_t^2. To start the recursive forecast using (44.13b) we might (arbitrarily) take a simple average of the first 30 observations on R_t^2. If we denote this average as $\sigma_{30}^2 = \sum_1^{30} R_k^2/30$ then the forecast can be updated for day-31 using the 'new' value for R_{30}^2:

$$\sigma_{31}^2 = \lambda\,\sigma_{30}^2 + (1-\lambda)R_{30}^2 \tag{44.14}$$

The forecast σ_{31}^2 can then be updated daily until we get to day-100 after which the future values of σ_{101}^2, σ_{102}^2 etc. are independent of the initial value we used for σ_{30}. As we have 250

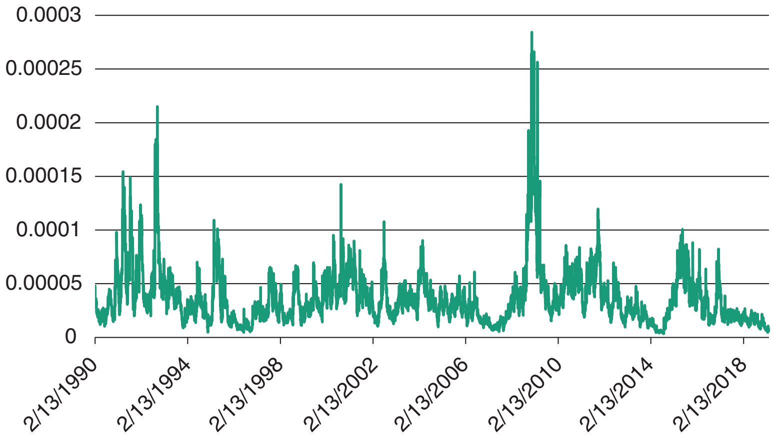

FIGURE 44.4 EWMA forecast – daily FX returns (Eurodollar)

days of historical data then we can continue the recursive forecast up to day-250. As additional information on daily returns becomes available our forecast of the variance (or standard deviation) changes as we move forward through time. It has been found that the EWMA forecasting equation gives better predictions than the SMA (based on the mean squared forecast errors).

The above EWMA formula when applied to daily changes in the spot exchange rate is shown in Figure 44.4 and you can see how the EWMA forecasts move by quite substantial amounts and this will directly influence our changing daily forecasts of the VaR of a portfolio of foreign assets, which depends on the exchange rate.

Banks and financial institutions want to forecast the VaR not only over 1 day but also over other horizons such as a week or a month. However, it would be time consuming to calculate the many volatilities (not to mention correlations) required for these different horizons. Changes in weekly volatility are not the same as movements in daily volatility. However, if we assume daily (continuously compounded, log) returns are independent over time (and identically distributed) we can 'scale them up' to give a forecast of weekly or monthly volatility by using the *'root T-rule'* ($\sqrt{\boldsymbol{T}} - rule$). For *iid* returns, the standard deviation over T-days is:

$$\sigma_T = \sqrt{T}\sigma \tag{44.15}$$

So given our EWMA forecast of tomorrow's daily standard deviation σ we simply scale it up by $\sqrt{T}$ – the number of trading days over which we want to calculate the VaR.

EXAMPLE 44.3

$\sqrt{T}$ – rule

You hold a \$100m stock portfolio. Tomorrow's forecast of daily portfolio volatility, $\sigma = 0.02$ (2% per day). What is your forecast for volatility over 25 (business) days and what is your estimate of the VaR over 25 days (at the 5th percentile), if you hold a \$100m portfolio?

Over 25 trading days (i.e. approximately 1 calendar month), the standard deviation of portfolio returns is $\sigma_{25} = \sqrt{25}\,\sigma = 5(0.02) = 0.10$ (i.e. 10% over 25 days). The VaR estimate over a 25-day horizon is therefore \$16.5m (= \$100m × 1.65 × 0.10).

Suppose each month has 25 trading days. Over the next 20 consecutive months you expect to lose more than \$16.5m in about one of these months (assuming you do not alter the composition of your current portfolio).

Forecasts for covariances use a similar EWMA scheme:

$$\sigma_{R_1R_2}(t+1) = (1-\lambda)R_{1t}R_{2t} + \lambda\sigma_{R_1R_2}(t) \tag{44.16}$$

A forecast for the correlation coefficient is derived from the separate forecasts for variances and covariances[4]:

$$\rho(R_1, R_2) = \sigma_{R_1R_2}/\sigma_{R_1}\sigma_{R_1} \tag{44.17}$$

44.4 BACKTESTING

Forecasts of daily standard deviations and correlations for each stock and hence for the daily VaR will change from day to day (even when asset holdings remain unchanged). We want to assess whether our forecasts of daily *portfolio*-VaR are accurate at the 1st percentile. From the standard normal distribution the 1st percentile 'cut off' point is –2.33. The (changing) daily forecast of portfolio VaR at the 1st percentile is therefore $VaR_{p,t+1} = V_p(2.33)\sigma_{p,t+1}$.

[4]Also note that the correlation coefficient between returns over a T-day horizon is the same as the one-day correlation coefficient: $\rho_T = \frac{\mathrm{cov}(R_{1T}, R_{2T})}{\sigma(R_{1T})\,\sigma(R_{2T})} = \frac{\rho\sigma_1\sigma_2 T}{(\sigma_1\sqrt{T})(\sigma_1\sqrt{T})} = \rho$

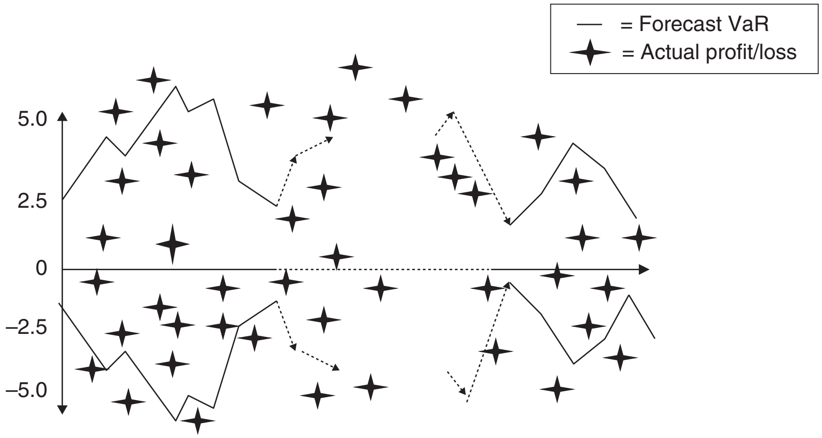

FIGURE 44.5 Backtesting

The historical out-turn for the *actual daily (overnight)* profit or loss (P/L) on the portfolio is $dV_p = \sum_{i=1}^{n} V_i R_i$ where R_i are actual returns on successive days.[5]

In Figure 44.5, the rolling one-day-ahead forecasts $VaR_{p,t+1}$ are the two symmetric solid lines. The actual daily P/L on the portfolio dV_p (assuming positions in the various assets are unchanged) are the 'stars'. The actual P/L figures reflect the profit/loss *that would have occurred* if the initial \$-position in each asset had been held over successive 24 hour periods over the past $n = 600$ days (say). The actual P/L does not reflect any position changes due to intraday trading.

If the forecasts of $VaR_{p,t+1}$ (at the 1st percentile) are adequate then we should observe about 1 out of 100 'stars' above the upper forecast-VaR solid line (and about 1 out of 100 below the lower solid line). If there are more than 1% of the 'stars' outside either of these lines, then our forecasts for σ_i and $\rho_{i,j}$ and hence for the portfolio-$VaR_{p,t+1}$, are an underestimate of the true portfolio risk. This procedure is known as *backtesting*.

In the hypothetical example in Figure 44.5 the *actual losses* over $n = 600$ days, exceed the forecast-$VaR_{p,t+1}$ about 9 times out of 600 daily observations, when for an accurate VaR model we expect on average about 6 (= 1% of 600) outliers. Hence the VaR-forecast slightly overstates the potential losses, at the 1st percentile. If a financial regulator (e.g. Bank of England in the UK and the Fed in the USA) found many more than 9 out of 100 'exceedances' (or 'outliers') then it might require a bank to hold more capital because of the poor performance of its risk model.

[5]Usually, when we calculate the daily profit/loss dV_p, the values for V_i are assumed to be held fixed each day at their *initial* values. This is the meaning of 'an unchanged portfolio'. It is as if you start each successive day with V_i held in each stock. In other words we do not 'cumulatively update' the value of each stock holding as returns change though time.

Example 44.4 tests whether the observed number of 'outliers' $X = 9$ is consistent with the assumption of normally distributed returns and accurate forecasts of volatilities and correlations that lie behind the VaR forecasts.

EXAMPLE 44.4

Backtesting

Percentile chosen for VaR outliers: $p = 0.01$ (1%)
Number (#) of days used (sample): $n = 600$
Actual number (#) of outlier losses: $X = 9$ (at the 1st percentile)
Expected # of outliers from VaR forecast: $EX = np = 6$
Stdv of # of outliers (for sample size n): $stdv(X) = \sqrt{np(1-p)} = 2.44$

Exact test using the binomial distribution (Excel)

Probability of finding 9 or more 'outliers' (if the VaR model is correct)

$$= 1 - BINOMDIST\,(8{,}600, 0.01,\ TRUE) = 0.152\ (15.2\%\ \text{probability})$$

'TRUE' in Excel gives the cumulative probability of between 0 and 8 outliers. So finding 9 (instead of the expected 6 outliers) could occur by chance with a reasonably high probability of 15.2%, even if your VaR model is 'correct/good'.

Using the normal approximation to the binomial distribution[6]

Using the normal distribution for our test, we can be '95% confident' that our VaR forecasts are 'statistically acceptable' if the number of outlier losses X is less than $6 + (1.65)\ 2.44 = 10$ (where 1.65 is the one-tail critical value for the $N(0,1)$ distribution at the 5th percentile). Hence, the observed number of outliers $X = 9$ lies within the acceptable statistical error band of 10 outliers which could occur just due to chance.

More formally, at 5% 'statistical significance level' we do not reject the null hypotheses that our VaR forecasts are acceptable. Repeating the above using the z-statistic, $z = (9-6)/2.44 = 1.23 < 1.65$ critical value for a one-tail test at 5% significance level.

The 'p-value' for $z = 1.23$ is 18.6% – this is the probability of observing 9 or more outliers, when the VaR model is correct (i.e. when the expected number of outliers $EX = 6$). We therefore do not reject the null hypothesis that our VaR model is 'accurate' (when predicting VaR losses at the 1st percentile level).

[6]Using the normal distribution as an approximation to the ('true') binomial distribution requires at least $np > 5$.

There are also ‘runs tests’ which check for independence in the VaR forecast outliers. If returns are independent over time as assumed in our VaR approach then any ‘outliers’ should appear in a random order over time. Put another way, we do not expect to see ‘bunching’ of outliers – that is, we should not observe five ‘outliers’, say, occurring in one specific week.

44.5 CAPITAL ADEQUACY

Financial firms (e.g. banks, mutual funds) hold positions in marketable assets and are therefore prone to losses. Prudential considerations imply that ‘capital’ (i.e. broadly speaking, the value of bank equity (stocks) held by investors in a bank plus cumulative retained profits) should be held as a ‘cushion’ against potential losses, otherwise the bank may become insolvent. The European Union, the Basel Committee (on Banking Supervision at the Bank for International Settlements, BIS) and the Federal Reserve Board (‘The Fed’) in the US have agreed that capital held by banks should reflect their market (and other) risks.

44.5.1 Basel Approach

The Basel ‘capital charge’ for market risk is based on the VaR of the bank’s (market) portfolio: the higher the forecast portfolio VaR, the higher is the amount of capital it has to hold to meet regulatory requirements (see Finance Blog 44.1). But how high? This is determined by specific rules negotiated via the BIS, but with some discretion by the home regulator (e.g. in the UK this is the Financial Policy Committee [FPC], which is part of the Bank of England and this function is often part of a country’s Central Bank responsibility).

Finance Blog 44.1 Bank Regulation for Market Risk

Broadly speaking the Basel rules have a *minimum* capital charge based on the VaR over a 10-day horizon and measured at the 1st percentile (rather than the 5th percentile). The 10-day VaR is calculated using the root-T rule applied to the forecast of daily portfolio volatility σ_p:

$$\textit{Basel}: \textit{Regulatory VaR} = k V_p \, 2.33(\sqrt{10}\sigma_p)$$

where the ‘multiplier’ k is chosen on a bank-by-bank basis with a minimum value of 3. The Basel-VaR determines the minimum capital that must be held by the bank against its 'market risks' (and there is also an adjustment for certain 'specific risks'). If the bank's VaR model looks a bit 'suspect' (i.e. when backtesting reveals many more outlier losses than predicted by the VaR forecasts, at the 1st percentile), then the 'home regulator' may increase a bank's capital requirement above this minimum level (i.e. increase $k > 3$). There is also a requirement to undertake a *stressed-VaR*. This involves the financial institution taking its current portfolio and calculating the losses that would have occurred, had this portfolio been held (unchanged) over a chosen past adverse historical period.

Regulators also operate a green, yellow, and red, 'traffic lights' approach with more severe penalties, the worse the performance of a bank's VaR model when backtested. There is also more attention paid to a measure of risk known as 'expected shortfall', which is the (dollar) amount you expect to lose on average, given that your losses are greater than your VaR forecast. The figure for the expected shortfall is therefore greater than the figure for the VaR amount.

44.6 SUMMARY

- If the daily VaR for a portfolio of stocks is \$10m, at a 95% confidence level (5th percentile) then you expect to lose no more than \$10m in 5 out of the next 100 days and therefore you expect to lose more than \$10m, on 5 days out of the next 100 days – assuming you hold an unchanged portfolio of stocks over the period.
- The (dollar) VaR for a stock portfolio is easily calculated using the variance-covariance method (VCV). The VCV approach assumes that each observed (daily) stock return is drawn independently from an identical distribution (i.e. same mean return and volatility) the return is normally distributed – in short, daily stock returns are assumed to be *niid*. In addition, daily returns across different stocks may be correlated – so returns are multivariate normal.
- Forecasts of variances and covariances (correlations) of returns are often generated using the EWMA method.
- The $\sqrt{T}$*-rule* is often used for 'scaling up' daily volatility forecasts, in order to forecast VaR over longer horizons of up to 1 month (25 trading days) – this requires an assumption that daily returns are identically and independently distributed over time (but does not require that they are normally distributed).
- Regulators check the accuracy of a bank's forecasts of portfolio-VaR by backtesting and set higher capital requirements when (a) the VaR forecast is high or (b) backtesting shows that the VaR forecasting model used is inaccurate.

EXERCISES

Question 1
Intuitively, why might you give more weight to recent movements in stock returns when forecasting tomorrow's volatility?

Question 2
Explain how you would assess whether your daily volatility forecasts for an individual stock return are accurate.

Question 3
A US investor holds \$10m of AT&T shares and has *short-sold* \$5m of Apple shares. Daily volatility of AT&T, $\sigma_1 = 1.5\%$ and daily volatility of Apple, $\sigma_2 = 1.0\%$. The correlation between AT&T returns and Apple returns is –0.1. What is the \$-VaR for this portfolio? What is the worst-case VaR?

Question 4
You have a portfolio consisting of £10,000 in each of 3 assets, 1, 2, and 3. You have calculated the daily standard deviations to be 5.418%, 3.0424%, 3.6363%, respectively. The correlation between returns on asset-1 and asset-2 is 0.962, between asset-1 and asset-3 is 0.403, and between asset-2 and asset-3 is 0.610.

(a) What is the VaR for this portfolio?
(b) What is the worse-case VaR?
(c) What is the VaR if £10,000 of asset-2 is short-sold? Explain this result compared with the outcomes in (a) and (b).

Question 5
Your forecast of *daily* volatility, $\sigma = 0.01$ (1% per day). How might you forecast volatility over 25 (business) days? What assumptions are you making?

Question 6
Explain the step used in 'backtesting' forecasts of daily VaR at the 5th percentile. Assume you have \$10m in each of two stocks and you have past daily data from 1 to T.

Question 7
Suppose the regulator for banks uses the 1st percentile to assess the VaR of a portfolio. What do you think the regulator would do if after backtesting the risk management system in your investment bank, found that actual losses exceeded your forecast-VaR losses 10 times in 175 forecasts?

CHAPTER 45

VaR: Other Portfolios

Aims

- To show how we can reduce the number of computations required to calculate VaR, while still retaining the linearity assumption and hence allowing the use of the variance-covariance (VCV) method.
- To show how the single index model (SIM) is used to simplify the calculation of the VaR for a portfolio containing a large number of stocks.
- To demonstrate how a portfolio of foreign stocks can be viewed as equivalent to holding the foreign market index and the foreign currency. This allows calculation of VaR in the domestic currency, using the VCV method.
- To show how representing a coupon paying bond as a series of zero-coupon bonds allows calculation of the VaR using the VCV method, for portfolios containing many bonds with different cash flows and durations.
- To show how the VCV method can be used to calculate the (approximate) VaR for a portfolio of options.

45.1 SINGLE INDEX MODEL

We use the single index model (SIM) to show how the volatility of a well-diversified portfolio of stocks can be measured by the volatility of the market return (e.g. the S&P 500) and the portfolio beta. This substantially reduces the computational burden when calculating VaR because the large number of correlations between individual stock returns can be encapsulated in the portfolio beta, and therefore do not have to be estimated. But it must be remembered that the SIM requires several simplifying assumptions – most notably that specific random events

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

that affect stock returns for one company are uncorrelated with random events which affect stock returns of another company.

45.1.1 Domestic Equity: SIM

Consider a portfolio consisting of n-stocks held in a specific country (e.g. USA). Forecasting the n variances and particularly the $n(n-1)/2$ correlations (covariances) across *all stocks* is computationally extremely burdensome, particularly for the covariance terms as there are so many of them. To ease the computational burden, we use the SIM since this allows all of the variances and covariances between returns to be subsumed into the n-asset betas (and a forecast for the variance of the market return). The SIM is:

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i \tag{45.1}$$

where R_i is the return on stock-i, R_m is the market return, β_i is the beta of stock-i. The ε_i are (usually) assumed to be *niid* and in addition we make the crucial assumption that there is no contemporaneous correlation across the random error terms for different stocks (at time t), $E(\varepsilon_i\varepsilon_j) = 0$. Hence, we assume that firm 'specific risk' (e.g. due to patent applications, IT failures, strikes, reputational effects, etc.) are uncorrelated across different firms. (Also, R_m is independent of ε_i.) An estimate of β_i can be obtained either from a 'risk measurement service' (e.g. investment bank) or by running a time series regression of R_i on R_m (using say one year of daily returns data). Using (45.1) the return on a portfolio of n-stocks can be represented as:

$$R_p = \sum_{i=1}^{n} w_i R_i = \alpha_p + \beta_p R_m + \varepsilon_p \tag{45.2}$$

where $\alpha_p \equiv \sum_{i=1}^{n} w_i\alpha_i$, $\beta_p \equiv \sum_{i=1}^{n} w_i\beta_i$, $\varepsilon_p \equiv \sum_{i=1}^{n} w_i\varepsilon_i$ and $w_i = V_i/V_p$ is the proportion of total portfolio value held in each stock. From Equation(45.2), the portfolio return depends *linearly* on the market return (given the portfolio beta β_p). It is easily shown that if $E(\varepsilon_i\varepsilon_j) = 0$, the variance of the portfolio return is:

$$\sigma_p^2 = \beta_p^2\sigma_m^2 + \sigma^2(\varepsilon_p) \tag{45.3a}$$

where $\sigma^2(\varepsilon_p) = \Sigma w_i^2\sigma^2(\varepsilon_i)$.

The specific risk of each stock $\sigma^2(\varepsilon_i)$ can be 'diversified away' when stocks are held as part of a well-diversified portfolio (with small amounts held in each stock). Hence the term $\Sigma w_i^2\sigma^2(\varepsilon_i)$ is small and can be ignored (see Appendix 45.B):

$$\sigma_p = \beta_p\sigma_m. \tag{45.3b}$$

Thus when using the SIM we only require estimates of β_i for the n-stocks and a forecast of σ_m, in order to calculate σ_p. We therefore dispense with the need to estimate n-values for the standard deviation of individual stock returns σ_i and more importantly the $n(n-1)/2$ values for the correlation coefficients ρ_{ij} between all of the stock returns. This considerably reduces the computational burden of estimating the portfolio-VaR, which is given by:

$$VaR_p = V_p 1.65(\beta_p \sigma_m) \tag{45.4}$$

The SIM is a 'factor model' with only one factor, the market return R_m. However, this approach could be extended within the VaR framework, by assuming several factors affect each stock return. For example, a widely used model is the so-called Fama–French three-factor model, where the return on any stock is determined by the market return R_m, the return on small (market cap) stocks minus large (market cap) stocks SMB and the return on high book-to-market minus low book-to-market stocks HML. Applying the SIM and the Fama–French three-factor model to each of the n-stock returns, implies that the forecast of *portfolio* variance σ_p only depends on estimates of the three factor betas, and forecasts of the three variances and six distinct covariances between the 'three factors' of the Fama–French model.

45.1.2 Foreign Assets

How do we deal with exchange rate risk when calculating VaR for a portfolio that contains foreign assets? Suppose a US investor (Mr Trump) holds a €100m diversified portfolio in German stocks which exactly mirror movements in the German stock market index, the DAX. So, Trump's German stock portfolio has a beta of 1 with respect to the DAX market index. The dollar value of Mr Trump's German portfolio is subject to changes in the DAX and changes in S, the Dollar-Euro spot FX-rate. In effect Mr Trump holds a two-asset portfolio, the foreign stock itself and the foreign currency. The percentage return on the German portfolio (in USD) is[1]:

$$\text{USD return: } R_p = R_{DAX} + R_S \tag{45.5}$$

where R_S = return on the spot-exchange rate, that is $R_S = dS/S$. If stocks in the DAX fall at the same time as the euro depreciates against the US dollar, then Mr Trump would lose in USDs on both counts – a 'double whammy' of losses on the DAX itself plus losses when euros are converted into dollars. This is because after depreciation of the euro, for every euro that Mr Trump has in the DAX, he obtains fewer dollars.

Hence a positive correlation between the DAX index and the euro exchange rate increases the riskiness of the US investor's portfolio, measured in US dollars. Conversely, a negative

[1] This relationship is exact for continuously compounded returns but is an approximation for 'standard' returns.

correlation between the DAX and the euro exchange rate implies a lower dollar risk for Mr Trump's German stock portfolio.

To measure the dollar-VaR for a US investor with stocks held in the DAX we have to modify our previous VCV approach. First, because we are interested in the dollar-VaR we have to convert Trump's initial position in Euros, into USD. If Mr Trump has €100m invested in the DAX and the *current* Dollar-Euro spot FX rate is $S_0 = 1.05$ (USD per Euro), then the initial USD position is $V_\$ = S_0(€100m) = \$105m$. The key equation (see Appendix 45.A) is:

$$dV_\$ \approx V_\$(R_{DAX} + R_S)$$

It follows directly that the *dollar-VaR* is then given by the usual formula:

$$\text{VaR}_\text{P} = \sqrt{\text{ZCZ}'} \tag{45.6}$$

where

$$\text{Z} = [VaR_{Dax}, VaR_s] = [V_\$ 1.65\sigma_{DAX}, V_\$ 1.65\sigma_S]$$

$V_\$$ = the initial \$-value of the position held in the DAX (\$105m)

σ_{DAX} = the volatility of the DAX index

σ_S = the volatility of the return on the Dollar-Euro'spot'exchange rate, $\sigma_{dS/S}$

The **Z**-vector 'contains' the *same* \$-value, $V_\$ = \$105m$ for both the DAX and the FX position. This is because for a US investor, holding the DAX is equivalent to holding both $V_\$$ in the DAX plus $V_\$$ in foreign exchange (euros). The correlation matrix **C** contains the correlation coefficient between the return on the DAX and the Euro-dollar spot exchange rate.

45.1.3 German and US Stock Portfolios

Suppose (as above) a Mr Trump (a US investor) holds a portfolio of German stocks but the beta of the German stocks is $\beta_p^G \neq 1$ and Mr Trump also holds a portfolio of US stocks with $\beta_p^{US} \neq 1$. Assume the US investor holds $V_\$^{US}$ dollars in US stocks and $V_\$^G$ *dollars* in German stocks (that is, $V_\$^G = SV_{Euro}^G$, where V_{Euro}^G is the Euro-value of the German stock portfolio and S is the current USD-Euro exchange rate). Assume the SIM holds *within* any single country, hence:

$$\sigma_G = \beta_p^G \sigma_{DAX} \qquad \sigma_{US} = \beta_p^{US} \sigma_{S\&P} \tag{45.7}$$

where $\sigma_{S\&P}$ = standard deviation of the S&P 500 US stock market index. The change in the *dollar* value of Mr Trump's portfolio is (approximately) linear in returns:

$$dV_p = V_\$^{US} R_{US} + V_\$^G (R_G + R_S) \tag{45.8}$$

where R_G = return on German stock portfolio and $R_S = dS/S$, the proportionate change in the spot FX rate. The USD-VaR for this portfolio is:

$$\begin{aligned} VaR_p &= \sqrt{ZCZ'} \\ Z &= [V_{\$}^{US}(1.65)\sigma_{US},\ V_{\$}^{G}(1.65)\sigma_G,\ V_{\$}^{G}\,(1.65)\,\sigma_S] \end{aligned} \tag{45.9}$$

$$\text{and } \ C = \begin{bmatrix} 1 & \rho_{US,G} & \rho_{US,S} \\ \rho_{G,US} & 1 & \rho_{G,S} \\ \rho_{S,US} & \rho_{S,G} & 1 \end{bmatrix}$$

This approach is easily extended to holding several foreign portfolios where for *each* foreign stock portfolio there are two entries in the Z vector – the VaR of the foreign stock portfolio and the VaR of the US-foreign currency, FX rate.

Suppose Mr Trump has a US stock portfolio as well as German and Brazilian stock portfolios (which are unhedged). Then there may still be some risk reduction (and hence a low portfolio VaR) if some of the spot FX rates have low (or negative) correlations with either the US-dollar exchange rate, or with stock market returns in different countries. If Mr Trump hedges his spot FX positions (using the forward market) then clearly any spot-FX volatilities and the correlation between spot FX-rates and foreign stock returns do not enter the calculation of VaR. But we cannot say unequivocally whether an unhedged or a FX-hedged 'world portfolio' has a lower risk – it depends on the interplay of the correlation coefficients in the two scenarios.

45.2 VaR FOR COUPON BONDS

Changes in bond prices and yields are not linearly related but we can use duration to give an *approximate* linear relationship. This then allows us to apply the variance-covariance approach to measure the VaR of a portfolio of coupon paying bonds. The easiest way to incorporate coupon paying bonds into the VCV approach is to note that for a portfolio of coupon paying bonds:

$$dV_p \approx -(V_p D_p)dy \qquad \text{hence} \qquad \sigma(dV_p) \approx V_p D_p \sigma(dy)$$

$$VaR_p = 1.65\,\sigma(dV_p) = V_p D_p (1.65)\sigma(dy) \tag{45.10}$$

where $D_p = \sum_{i=1}^{n} (V_i/V_p)D_i$ is the portfolio duration, V_i = dollar amount in each bond and V_p = total dollar amount in the portfolio of bonds, y = yield to maturity. One problem with the above approach is that it assumes *parallel shifts* in the yield curve – that is, all spot yields move by the same absolute amount – which is encapsulated in the change in the (single) yield (to maturity) dy. Also the duration-approximation is not accurate for large changes in yields.

45.2.1 VaR: Non-parallel Shifts in the Yield Curve

Can we improve on this approach and calculate the VaR for a portfolio of coupon paying bonds when there are non-parallel shifts in the yield curve? The way to do this is to consider a *single* coupon paying bond, with n-years to maturity, as a series of zero-coupon bonds. (This is extended to a *portfolio* of coupon paying bonds, below.) For a zero-coupon bond which has single cash flow CF_i at t_i, the current value (price) is:

$$V_i = CF_i \ e^{-y_i t_i} \tag{45.11}$$

y_i = (continuously compounded) *spot yield* for period t_i. The proportionate change in value (price) of the zero-coupon bond is the 'return' on the bond, that is $R_i = dV_i/V_i$. Differentiating (45.11) we have a linear relationship between the return on the zero-coupon bond and the (absolute) change in yield:

$$R_i \equiv dV_i/V_i = -t_i dy_i = -D_i dy_i \tag{45.12}$$

where for a zero-coupon bond $D_i = t_i$ (i.e. a zero-coupon bond's duration equals its maturity – measured in years). From (45.12):

$$\sigma_{R_i} = D_i \sigma_{dy_i}$$

A coupon paying bond is just a series of zero-coupon bonds and the *value/price* of a coupon bond with n-cash flows CF_i at time t_i is[2]:

$$V_p = \sum_{i=1}^{n} V_i \tag{45.13}$$

Hence:

$$dV_p = \sum_{i=1}^{n} dV_i = \sum_{i=1}^{n} V_i(dV_i/V_i) = \sum_{i=1}^{n} V_i R_i. \tag{45.14^3}$$

The dollar change in value of the coupon paying bond is (approximately) *linear* in the returns R_i of its constituent zero-coupon bonds (for small changes in yields). We have 'mapped'

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[2]Each cash flow CF_i consists of the sum of all the coupons C_i or 'coupons plus any repayments of principal' M_i, hence, $CF_i = C_i + M_i$ received at time t_i, from the coupon-paying bond held. For T-bonds the par value M_n is usually only paid at the maturity date but for corporate bonds with a 'sinking fund', the par value is often paid in instalments over the life of the bond – hence the use of M_i for payment of part of the outstanding principal at t_i.

[3]This can also be derived as follows. $V_p = \sum_{i=1}^{n} CF_i e^{-y_i t_i}$ hence:

$$dV_p = \sum_{i=1}^{n} CF_i(-t_i)e^{-y_i t_i} dy_i = \sum_{i=1}^{n} V_i(-D_i \, dy_i) = \sum_{i=1}^{n} V_i R_i$$

the non-linear 'yield-bond price relationship' into an (approximate) linear framework. In addition, if we assume yield changes dy_i are normally distributed, we can use Equation (45.14)[3] to calculate the VaR (at the 5th percentile) for a coupon paying bond:

$$VaR_p = 1.65\,\sigma_{dV_p} = \sqrt{\boldsymbol{ZCZ}'} \tag{45.15}$$

where $Z = \{VaR_1, VaR_2, \ldots, VaR_n\}$, $VaR_i = V_i(1.65)\sigma_{R_i}$, and C is the $(n \times n)$ correlation matrix of zero-coupon bond returns.

In general we do not forecast the volatility of bond returns σ_{R_i} directly, but instead forecast σ_{dy_i} using the EWMA model. Then $\sigma_{R_i} = t_i\sigma_{dy_i}$ provides a forecast of the standard deviation of (zero-coupon) bond *returns* for use in Equation (45.15). Similarly, we use the EWMA model to forecast the covariance $\sigma(dy_i, dy_j)$ between changes in spot yields from which we obtain the correlation coefficients (for use in the correlation matrix C):

$$\rho_{i,j} \equiv \rho(R_i,\ R_j) = \sigma(dy_i, dy_j)/(\sigma(dy_i).\sigma(dy_j))$$

All of the above equations can be applied to a *portfolio* consisting of m coupon paying bonds. The cash flows from the m-bonds at each time period t_i are aggregated to give the total coupon payments at this date, $CF_{i,Total} = \sum_{j=1}^{m} CF_{i,j}$. The current value of these coupons from all the bonds is then $V_{i,Total} = CF_{i,Total}\ e^{-y_i t_i}$ and the analysis follows as above.

EXAMPLE 45.1

VaR of Bond Portfolio

Total cash flows (coupons and any repayment of principal) from a portfolio of bonds are \$10,000 and \$20,000 at $t_5 = 5$ years and $t_7 = 7$ years, respectively. Hence the duration of these two cash flows are $D_5 = 5$ and $D_7 = 7$. The current 5-year and 7-year spot yields (continuously compounded) are $y_5 = 3\%$ and $y_7 = 4\%$, respectively.

The daily standard deviation of the change in yields are 0.1% per day and 0.2% per day (respectively) and the correlation coefficient is $\rho = 0.95$. We want to calculate the 5-day VaR (at the 5th percentile). The value/price of the two zero-coupon bonds are:

$$V_5 \equiv CF_5\, e^{-y_5 t_5} = \$8,607.1 \qquad V_7 \equiv CF_7\, e^{-y_7 t_7} = \$15,115.7$$

The standard deviation of the return on each zero-coupon bond and the individual VaRs are:

$$\begin{aligned} \sigma_{R_5} &= D_5\,\sigma_{dy_5} = 0.5\%, \qquad\qquad \sigma_{R_7} = D_7\sigma_{dy} \\ VaR_5 &= \$8607.1\ (1.65)\ 5\ (0.001) = \$71 \\ VaR_7 &= \$15,115.7(1.65)\ 7\ (0.002) = \$349.2 \end{aligned}$$

(*continued*)

(*continued*)

The daily VaR for the bond portfolio is:

$$VaR_p = [VaR_5^2 + VaR_7^2 + 2\rho VaR_5 VaR_7]^{1/2}$$
$$= [(71)^2 + (349.2)^2 + 2(0.95)\,(71)\,(349.2)]^{1/2} = 417.2$$

In matrix notation $\boldsymbol{Z} = [VaR_5, VaR_7]$, $\boldsymbol{C} = 2 \times 2$ correlation matrix, with $\rho = 0.95$ and $VaR_p = \sqrt{\boldsymbol{ZCZ}'}$.

45.2.2 Mapping Cash Flows

A large investment bank may have coupon receipts on its bond portfolio virtually every week for the next 20–30 years. We cannot use current spot yields for every horizon t_i over the next 30 years – there would be too many cash flows to deal with. To reduce the scale of the problem 'standard vertices' of say 1, 3, 6, and 12 months and 2, 3, 4, 5, 7, 9, 10, 15, 20, and 30 years, are used. All cash flows are 'mapped' onto these standard vertices and bond return volatilities σ_i and correlations $\rho_{i,j}$ are provided only for these chosen vertices.

For example, suppose an actual cash flow of \$100m at $t = 6$ (Figure 45.1) has to be apportioned between the standard vertices at $t = 5$ and $t = 7$. One approach (see Appendix 45.C) is to allocate cash flows to adjacent standard vertices to ensure that:

- Total market value of the cash flows allocated to adjacent vertices (e.g. 5 and 7 years) equals the market value of the original cash flow (i.e. \$100m at $t = 6$ years).
- The two cash flows at $t = 5$ and $t = 7$ both have the same sign as the original cash flow.
- The volatility of (present) value of the two cash flows at $t = 5$ and $t = 7$ equal the volatility of the (present) value of the original cash flow at $t = 6$.

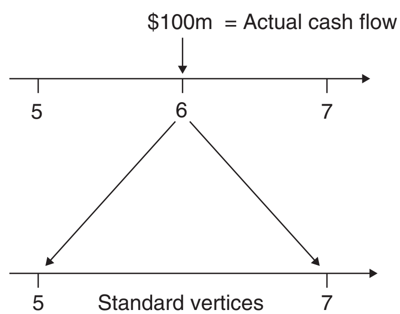

FIGURE 45.1 Mapping cash flows

45.2.3 Swaps

A pay-fixed, receive-floating interest rate swap is equivalent to a short position in a fixed rate bond and a long position in a FRN. The VaR for the fixed leg can therefore be analysed as a standard coupon paying bond. The value of the floating leg can be treated as a zero-coupon bond with a single known payment at the next payment date, discounted by the time to the next payment. Hence the VaR of the floating leg can also be calculated using the cash-flow mapping approach.

45.2.4 Principal Components Analysis

A useful alternative way of parsimoniously representing changes in say k different spot yields is to use principal components analysis (PCA). PCA 'represents' the changes in all the k spot yields by a few (usually about three) new variables called principal components (PCs).[4] When computing the VaR of a bond portfolio we then use these three principal components, in place of the many ($= k(k-1)/2$) correlations between the original k spot yields.

PCA is a statistical technique which takes a $(T \times k)$ matrix of changes in k-spot yields $\mathbf{X}$, and seeks to 'explain' the movement in *all* these yields using just the three principal components, $\mathbf{q}_i$ ($i = 1, 2, 3$). The $\mathbf{q}$'s are linear combinations (i.e. weighted averages) of the 'raw' spot rate data in the $\mathbf{X}$-matrix:

$$\mathbf{q}_i = \mathbf{X}\,\mathbf{c}_i \qquad (i = 1, 2, 3) \tag{45.16}$$

where $\mathbf{c}_i$ is an estimated constant. (It is in fact the eigenvalue corresponding to the largest eigenvector of the $(\mathbf{X}'\mathbf{X})$ matrix.) A major benefit of PCA is that the q's are orthogonal (uncorrelated) with each other and hence their correlation coefficients are zero (by construction).

It is generally found that about 95% of the variation in, say, 15 spot yields (e.g. for years 1, 2, 3, … 15) can be 'empirically explained' by about three PCs. The first PC is interpreted as representing parallel shifts in the yield curve, the second represents a twist in the yield curve, whereas the third (which explains the smallest proportion of the variation in the yield data) represents a 'bowing' in the yield curve (which occurs relatively infrequently). What is important for our VaR calculation is that the volatility of these 15 spot yields can be *'linearly mapped'* into only three variables – the three PCs, $\mathbf{q}_i$. The three PCs are uncorrelated with each other and this considerably reduces the dimensionality of the VaR calculations.

45.3 VaR: OPTIONS

The VaR for options can be dealt with in the VCV framework if we are willing to use the linear approximation given by the option's delta. The change in value of a *portfolio* of call and put

[4]This idea is similar qualitatively in 'reducing' the number of covariances in a portfolio of n-stocks to a much smaller number of variances and covariances by using the SIM or with the Fama–French three-factor model, as discussed above.

options on a specific stock (e.g. AT&T) but with different strikes and time to maturity, is given by the portfolio delta:

$$dV_{p,1} = \sum_{i=1}^{n} N_i(df_i) = \sum_{i=1}^{n} N_i\Delta_i(dS) = \Delta_{p,1} dS_1 \tag{45.17}$$

where df_i is the change in the price of the call or put options (on AT&T stock). If the options on AT&T all have the same delta Δ (i.e. same strike price and time to maturity) and there are N_{op} options held, then the portfolio delta simplifies to $\Delta_{p,1} = N_{op}\Delta$.

Now suppose we have options on two different stocks (Microsoft and AT&T) then:

$$dV_p = \Delta_{p1}S_1(dS_1/S_1) + \Delta_{p2}S_2(dS_2/S_2) = k_1R_1 + k_2R_2 \tag{45.18}$$

The change in value of the options portfolio is (approximately) linear in the two stock *returns* and hence we can apply the standard VCV approach as in the following example.

EXAMPLE 45.2

VaR for Options Portfolio

You hold 2,500 call options on Microsoft with $\Delta = 0.4$ and 10,000 call options on AT&T with $\Delta = 0.2$ giving portfolio deltas, $\Delta_1 = 2{,}500(0.4) = 1{,}000$ and $\Delta_2 = 10{,}000(0.2) = 2{,}000$. The current stock prices are $S_1 = 110$, $S_2 = 40$ with daily standard deviations $\sigma_1 = 2\%$ and $\sigma_2 = 1\%$ and correlation coefficient $\rho = 0.3$.

What is the 5-day VaR (at the 5th percentile) using the delta approximation? From Equation (45.18):

$$dV_p = 1000(110)R_1 + 2000(40)R_2$$
$$dV_p = 110R_1 + 80R_2 = V_1R_1 + V_2R_2$$

where dV_p is measured in thousands of dollars, hence:

$$VaR_1 = V_1(1.65)\sigma_1 = 3.63 \quad \text{and} \quad VaR_2 = V_2(1.65)\sigma_2 = 1.32$$
$$\mathbf{Z} = [VaR_1, VaR_2] = [3.63, \ \ 1.32]$$

$$VaR_p = \sqrt{ZCZ'} = [VaR_1^2 + VaR_2^2 + 2\rho\, VaR_1 VaR_2]^{1/2}$$
$$= [(3.63)^2 + (1.32)^2 + 2\,(0.3)\,3.63\,(1.32)]^{1/2} = 4.2183$$

The 5-day VaR (at the 5th percentile) is therefore:

$$VaR_p = \sqrt{5}(4.2183) = 9.4324 \ \ (\$9,432.4)$$

Options prices are actually non-linear functions of the underlying asset (e.g. stock price). Hence, if changes in the stock prices are large, the delta approximation and VCV approach may provide a very poor measure of VaR for options portfolios. Superior techniques such as MCS and historical simulation are therefore the usual methods used to calculate VaR for portfolios containing options (see Chapter 46).

45.4 SUMMARY

- When using the VCV method, for a well-diversified stock portfolio containing n-stocks, use of the SIM simplifies the calculation. This is because the n-variances and $n(n-1)/2$ correlations between the stock returns can be represented by the n-betas (that make up the portfolio beta) and the volatility of the market return. Therefore $\sigma_p = \beta_p \sigma_m$ and $VaR_p = V_p \beta_p (1.65) \sigma_m$.
- When a domestic investor holds a portfolio of foreign stocks she essentially holds a two-asset portfolio, the foreign stock portfolio and an equal amount in the foreign currency. The VCV approach can then be used to calculate the VaR.
- If we are willing to assume parallel shifts in the yield curve, the VaR of a portfolio of coupon-paying bonds can be reduced to the simple formula: $VaR_p = V_p D_p (1.65) \sigma(dy)$.
- For non-parallel shifts in the yield curve we can still apply the VCV approach to obtain the VaR for a portfolio of coupon-paying bonds (with n-cash flows). We model the coupon paying bonds as a set of zero-coupon bonds. The duration approximation allows the return on these zero-coupon bonds to be expressed as a linear function of changes in *all* of the spot yields. The VaR of the bond portfolio then depends on the volatility and correlations between all the spot yields at various maturities.
- In a large bond portfolio, coupon payments occur at many different time periods. To keep the VCV method tractable we have to 'map' these cash flows onto a smaller number of 'standard' time periods and calculate the VaR using only these standard time periods.
- The VaR for a portfolio of options can be analysed in the VCV framework using the options portfolio deltas. But this may be a poor approximation to the true VaR if the change in the underlying asset prices are large. This is because the true options prices are non-linear functions of the underlying asset prices and the delta approximation is only accurate for small changes in the underlying asset price.

APPENDIX 45.A: VaR FOR FOREIGN ASSETS

A US investor holds V^G_{Euro} Euros in German stocks, which in USD is $V^G_{\$} = S_0 V^G_{Euro}$, where S_0 is the current Euro-USD spot exchange rate. The USD change in value of this portfolio is:

$$dV \equiv V_1 - V_0 = V^G_{Euro}(1 + R_G)S_1 - V^G_{Euro}S_0 \tag{45.A.1}$$

Let $R_S = (S_1/S_0) - 1$ be the (proportionate) change in the spot exchange rate (i.e. the return on foreign exchange). Substituting $S_1 = (1 + R_S)S_0$ and rearranging:

$$dV = V^G_{Euro}(1 + R_G)(1 + R_S)S_0 - V^G_{Euro}S_0 = V^G_{\$}[(1 + R_G)(1 + R_S) - 1] \tag{45.A.2a}$$

$$dV \approx V^G_{\$}(R_G + R_S) \tag{45.A.2b}$$

where we have ignored the cross product terms, which are small. In (45.A.2b) we see that the USD investor is effectively holding equal dollar amounts $V^G_{\$}$ in German stocks and in the Euro-USD exchange rate. Hence it follows directly from (45.A.2):

$$\sigma_{dV} = V^G_{\$}\sqrt{(\sigma^2_G + \sigma^2_S + 2\rho_{G,S}\sigma_G\sigma_S)} \tag{45.A.3}$$

Therefore the USD-VaR for the US investor who holds the German portfolio is:

$$VaR_p(USD) = 1.65\sigma_{dV} = V^G_{\$}1.65\ \sqrt{(\sigma^2_G + \sigma^2_S + 2\rho_{G,S}\sigma_G\sigma_S)} \tag{45.A.4}$$

$$VaR_p = \sqrt{ZCZ'} \tag{45.A.5}$$

$$Z = [V^G_{\$}(1.65)\sigma_G,\ V^G_{\$}\,(1.65)\,\sigma_S] \quad \text{and} \quad C = \begin{bmatrix} 1 & \rho_{G,S} \\ \rho_{G,S} & 1 \end{bmatrix}$$

In addition, if we use the SIM then $\sigma_G = \beta_G\sigma_{Dax}$ where β_G is the portfolio-beta of the German stock portfolio and σ_{Dax} is the volatility of the German stock market index, the DAX.

APPENDIX 45.B: SINGLE INDEX MODEL (SIM)

Using the single index model (SIM) we show that the standard deviation of a portfolio of n-stocks is:

$$\sigma_p = \left[\sum_{i=1}^{n} w_i\beta_i\right]\sigma_m = \beta_p\sigma_m \tag{45.B.1}$$

where σ_p = portfolio standard deviation, σ_m = standard deviation of the market portfolio and $\beta_p = \sum_i w_i\beta_i$ is the 'beta' of the stock portfolio. The SIM for each stock return is:

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i \tag{45.B.2}$$

Assumptions:

$$E(\varepsilon_i) = 0 \tag{45.B.3a}$$

$$E(\varepsilon_i^2) = \sigma^2(\varepsilon_i), \tag{45.B.3b}$$

$$E(\varepsilon_i\varepsilon_j) = 0 \ \text{ for } \{\mathrm{i} \neq \mathrm{j}\} \tag{45.B.3c}$$

$$\mathrm{cov}(R_m, \varepsilon_i) = 0 \tag{45.B.3d}$$

It follows that :

$$ER_i = \alpha_i + \beta_i ER_m \tag{45.B.4a}$$

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2(\varepsilon_i) \tag{45.B.4b}$$

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 + E(\varepsilon_i \varepsilon_j) = \beta_i \beta_j \sigma_m^2 \quad \text{since} \quad E(\varepsilon_i \varepsilon_j) = 0 \text{ for } i \neq j \tag{45.B.4c}$$

The portfolio return and variance are:

$$R_p = \sum_i w_i R_i \quad \text{where} \quad w_i = V_i / V_p \quad \text{and} \quad V_p = \sum_{i=1}^{n} V_i \tag{45.B.5a}$$

$$\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_{i \neq j} \sum w_i w_j \sigma_{i,j} \tag{45.B.5b}$$

The formula for portfolio variance requires n-variances and $n(n-1)/2$ covariances. For example for $n = 150$ this amounts to 11,325, inputs to estimate. To reduce the number of inputs required we utilize the SIM. Substituting from Equation (45.B.2) in (45.B.5a):

$$R_p = \sum_i w_i R_i = \sum_i (w_i \alpha_i + w_i \beta_i R_m + w_i \varepsilon_i) = \alpha_p + \beta_p R_m + \varepsilon_p \tag{45.B.6}$$

$$\alpha_p = \sum_i w_i \alpha_i, \quad \beta_p = \sum_i w_i \beta_i, \text{ and } \varepsilon_p = \sum_i w_i \varepsilon_i.$$

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sigma^2(\varepsilon_p) \tag{45.B.7}$$

Equation (45.B.7) may be interpreted as, 'Total Portfolio Risk = Market Risk + Specific Risk'. Examining the specific risk term more closely and assuming $w_i = 1/n$, that is an equally weighted (i.e. well diversified) portfolio:

$$\sigma^2(\varepsilon_p) = \sum_i w_i^2 \sigma^2(\varepsilon_i) + \sum_{i \neq j} \sum w_i w_j E(\varepsilon_i \varepsilon_j) = \sum_i w_i^2 \sigma^2(\varepsilon_i) = (1/n) \overline{\sigma}^2(\varepsilon_i) \tag{45.B.8}$$

where we have used the key assumption of the SIM namely, $E(\varepsilon_i \varepsilon_j) = 0$. The *average* specific risk is defined as $\overline{\sigma}^2(\varepsilon_i) \equiv (1/n) \sum_i \sigma^2(\varepsilon_i)$ and as $n \to \infty$ this term goes to zero ($n = 35$ randomly selected stocks is usually sufficient to ensure this last term is relatively small). Hence under the SIM and assuming a well-diversified portfolio:

$$\sigma_p = \left[\sum_{i=1}^{n} w_i \beta_i \right] \sigma_m = \beta_p \sigma_m \tag{45.B.9}$$

A crucial assumption in the SIM is $E(\varepsilon_i\varepsilon_j) = 0$ – the covariance of firm specific 'random shocks' to stock-i and stock-j are *contemporaneously* uncorrelated, across all stocks. This is reasonable for stocks in different sectors (e.g. oil and IT) but not for stocks in the same sector (e.g. Shell and BP). However, in a well-diversified portfolio even though there may be some small positive correlations between some ε_i and ε_j for daily returns, nevertheless *portfolio* specific risk still falls quite rapidly as n increases and it is small relative to market (beta) risk. To see this note that from (45.B.8) that if $E(\varepsilon_i\varepsilon_j) \neq 0$ then:

$$\sigma^2(\varepsilon_p) = \frac{1}{n}\overline{\sigma}^2(\varepsilon_i) + \frac{n-1}{n}\overline{Cov(\varepsilon_i, \varepsilon_j)} \tag{45.B.10}$$

where $\overline{\sigma}^2(\varepsilon_i)$ is the *average variance* of the firm's specific risks and $\overline{Cov(\varepsilon_i, \varepsilon_j)}$ is the *average covariance of* the (contemporaneous) specific risks across firms:

$$\overline{\sigma}^2(\varepsilon_i) = \frac{1}{n}\sum_{i=1}^{n}\sigma^2(\varepsilon_i) \tag{45.B.11a}$$

and

$$\overline{Cov}(\varepsilon_i, \varepsilon_j) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ \text{for } i\neq j}}^{n} E(\varepsilon_i\varepsilon_j) \tag{45.B.11b}$$

From (45.B.11a) the variance term goes to zero as n increases so for large n, the variance of portfolio specific risk $\sigma^2(\varepsilon_p)$ equals the average covariance between specific risks across different stocks, $\overline{Cov(\varepsilon_i, \varepsilon_j)}$. In a well-diversified portfolio where stocks are held across many different sectors (and countries), the average covariance of 'specific risk' across firms is likely to be small relative to market risk, $\beta_p^2\sigma_m^2$. Hence Equation (45.B.9) is a reasonable approximation. Of course, if our portfolio is not well diversified (e.g. we hold only energy stocks) then the assumption of the SIM that $E(\varepsilon_i\varepsilon_j) = 0$ will be incorrect and $\overline{Cov(\varepsilon_i, \varepsilon_j)}$ may be quite large. In this case our simplified expression for portfolio risk $\sigma_p = \beta_p\sigma_m$ ceases to hold and *all* the $n(n-1)/2$ covariances $E(\varepsilon_i\varepsilon_j)$ are required for an accurate calculation of σ_p.

APPENDIX 45.C: CASH FLOW MAPPING

Mapping

A numerical example is the best way to illustrate this problem. Suppose we have an actual cash flow of \$100m at $t = 6$ years and we need to map this onto the 'vertices' at $t = 5$ and $t = 7$ (see Table 45.C.1).

TABLE 45.C.1 Data

Time/Vertices	Yield	Price volatility = (1.65σ)	Correlation matrix	
Year 5	$y_5 = 6.5\%$	$\sigma_5 = 0.3$	1	0.99
Year 7	$y_7 = 6.7\%$	$\sigma_7 = 0.6$	0.99	1

To find the proportions of the actual cash flow of \$100m at $t = 6$, which we should apportion to vertices $t = 5$ and $t = 7$ we proceed as follows:

1. The PV of the actual cash flow using y_6 is

$$V_6 = 100/(1 + y_6)^6 = \$68.15$$

2. The yield at $t = 6$ we take to be a linear interpolation of y_5 and y_7

$$y_6 = (1/2)\ 0.065 + (1/2)\ 0.067 = 0.066\ (6.6\%)$$

3. The volatility at $t = 6$ is a linear interpolation of σ_5 and σ_7:

$$\sigma_6 = (1/2)\ 0.3 + (1/2)\ 0.60 = 45$$

4. We apportion the cash flow at $t = 6$, between $t = 5$ and $t = 7$, to ensure positive cash flows at each vertex (V_5 and $V_7 > 0$) and that the volatility of $V_5 + V_7$ is equal to σ_6:

$$V_5 = \gamma V_6 \qquad \text{and} \qquad V_7 = (1 - \gamma)V_6 \tag{45.C.1}$$

$$\gamma^2\sigma_5^2 + (1 - \gamma)^2\sigma_7^2 + 2\gamma(1 - \gamma)\rho\sigma_5\sigma_7 = \sigma_6^2 = 0.45 \tag{45.C.2}$$

We have estimates of σ_5, σ_7, ρ (Table 45.C.1) and from Equation (45.C.2) we obtain two solutions $\gamma = 0.496$ or $\gamma = 3.38$ (see below). We ignore the solution $\gamma = 3.38$ since it violates positive cash flows at both vertices (i.e. $1 - \gamma < 0$). Hence for $\gamma = 0.496$:

$$V_5 = \gamma(\$68.15\text{m}) = \$33.80\text{m} \tag{45.C.3}$$

$$V_7 = (1 - \gamma)\ (\$68.15\text{m}) = \$34.35\text{m} \tag{45.C.4}$$

The results are summarised in Table 45.C.2.

Solution for γ

The left-hand side of (45.C.2) is a quadratic equation in γ of the form:

$$\alpha\gamma^2 + b\gamma + c = 0 \tag{45.C.5}$$

with $a = \sigma_5^2 + \sigma_7^2 - 2\rho\sigma_5\sigma_7$, $b = 2\rho\sigma_5\sigma_7 - 2$ and $c = \sigma_7^2 - \sigma_6^2$. The solution of Equation 45.C.5 for γ is: 0.496 which implies that 49.6 percent of the 6th year cash flow should be allocated to year-5.

TABLE 45.C.2 Mapping cash flow at $t = 6$ to vertices at $t = 5$ and $t = 7$

Actual cash flow: $100m at 6 years							
	(1) Term	(2) Actual cash flow	(3) Yield	(4) PV	(5) Price vol.	(6) Allocation weights, γ	(7) Flows
Vertex 5 years	5 years		$y_5 = 6.5\%$		0.3	0.496	$33.80m
Cash flow	6 years	$100m	$y_6 = 6.6\%$	$68.15m	0.45		
Vertex 7 years	7 years		$y_7 = 6.7\%$		0.6	0.504	$34.35m

EXERCISES

Question 1

Show how the single index model (SIM) can lead to considerable simplification when calculating the value at risk (VaR) over a 10-day horizon, for a portfolio of 20 domestic stocks with $10,000 in each stock. Clearly state any assumptions you make and their importance in determining the practical strengths and weaknesses of the SIM.

Question 2

What is 'mapping' and why is it useful in calculating the VaR for a portfolio consisting of coupon paying bonds?

Question 3

You are a US resident who holds a portfolio of UK stocks of £100m. The current USD/GBP exchange rate is 1.5 USD/GBP, the correlation between the return on the UK portfolio and the USD/GBP exchange rate is $\rho = 0.5$. The return on the FTSE All-Share index has a standard deviation of 1.896% per day and the volatility of the USD-GBP spot rate is $\sigma_{FX} = 3\%$ per day. Using the variance-covariance VCV method calculate the daily US dollar VaR at the 5th percentile.

Question 4

You are a US resident with €100m in the DAX-index and $100m face value in a US zero-coupon bond which matures in one year. The current spot FX rate is $S = 0.6$ (USD/Euro) and the US spot interest rate is $r = 3\%$ p.a. The *daily* standard deviations on the FX-rate, the DAX, and the US bond price are $\sigma_S = 3\%$, $\sigma_{DAX} = 2\%$, and $\sigma_B = 0.5\%$, respectively. The correlation coefficients are $\rho_{S,DAX} = -0.5$, $\rho_{S,B} = 0$, $\rho_{B,DAX} = 0.2$. Using the variance-covariance (VCV) method calculate the daily VaR of your portfolio (at the 5th percentile) and the worst-case VaR.

Question 5
A zero-coupon bond pays £1,000 in 2 years' time. You have the following information:

Current yield $y = 8.25\%$ p.a., standard deviation of *change* in yield $\sigma_{dy} = 1.009\%$ per day.

(i) Calculate the market value of the zero-coupon bond.
(ii) Calculate the daily VaR for this bond (at the 5th percentile).
(iii) Calculate the 10-day and 25-day VaR.

Question 6
You hold coupon bonds which pay \$10,000 in 1 year's time and \$110,000 in 2 years' time. The 1-year and 2-year spot yields are 3% p.a. and 4% p.a. respectively (continuously compounded). The volatility of the *daily change* in yields is $\sigma(dy_1) = 0.001$ (0.1% per day) and $\sigma(dy_2) = 0.002$ (0.2% per day).

The correlation between the change in the 1-year and 2-year yields is 0.8.

Use the variance-covariance method (VCV) to calculate the daily VaR of this coupon bond (at the 5th percentile).

Question 7
A coupon paying bond has the following cash flow profile

Year	**1**	**2**	**3**
Cash flow (\$)	400	450	500
Spot rate (% p.a.)	7.84	7.96	7.98
Stdv of bond price changes (% per day)	0.23	0.20	0.25

Spot yields are compound rates. The correlation between changes in the *price* (present value) of the 1-year, 2-year and 3-year cash-flows are $\rho_{12} = 0.8$, $\rho_{13} = 0.7$, and $\rho_{23} = 0.6$.

(a) Calculate the market value of each cash flow (i.e. price of the zero-coupon bonds).
(b) Calculate the (one day) VaR for each cash flow taken separately.
(c) Calculate the VaR of the coupon paying bond.

CHAPTER 46

VaR: Alternative Measures

Aims

- To examine non-parametric methods of estimating VaR such as historical simulation and bootstrapping procedures.
- To demonstrate how Monte Carlo simulation (MCS) is used to estimate VaR for portfolios containing options.
- To analyse variants on the traditional VaR approach such as stress-testing and extreme value theory.

46.1 HISTORICAL SIMULATION

When using the variance-covariance method to calculate portfolio-VaR, we assume returns are normally distributed – hence we can use the '1.65' scaling factor in our VaR calculations (for the 5th percentile). But if returns are actually non-normal using '1.65' is no longer valid and it would produce biased forecasts of VaR. Historical simulation (HS), directly uses actual historical data on returns to calculate VaR and does not assume a particular distribution for returns – we take whatever distribution is implicit in the historical data.

When we use the variance-covariance approach to measure risk we explicitly measure the variance (standard deviation) and the covariances (correlations) between each of the asset returns in our portfolio. These variance and covariances (correlations) are 'parameters'. However, in the HS approach, variances and covariances are not *explicitly* estimated – hence the term '*non-parametric*'. Instead these features of the data are 'encapsulated' in the time path of actual returns used in the HS approach.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

Currently hold \$100 in each of 2 stocks										
Day	1	2	3	...	700	701	702	703	...	1,000
R_1 (%)	+2	+1	+4	...	−3	−2	−1	−1		+2
R_2 (%)	+1	+2	0	...	−1	−5	−6	+15		−5
dVp(\$)	+3	+3	+4	...	−4	−7	−7	+14		−3

Order the 1,000 values for the \$-change in portfolio value in ascending order −12, −11, −11, −10, −9, −9, −8, −7, −7, −6 || −5, −4, −4, ..., 0, +2, +3, +3, +4, +5, ..., +8, ..., +14

VaR forecast for tomorrow at 1% tail (10th most negative) = −\$6

FIGURE 46.1 Historic simulation

The HS approach is straightforward and intuitive. Consider Figure 46.1, which shows daily returns over the past 1,000 days on two stocks, which could be AT&T and Exxon. In Figure 46.1 there are only a few representative numbers visible (out of the 1,000) but you can still make an informed guess about what these figures are telling you about volatilities and correlation. For example, *approximately,* what is the mean daily return of stock-1 and stock-2 and their standard deviations, and do the latter look as if they are changing over time? Very approximately the mean daily return on both stocks could be zero. The standard deviation of stock-1 looks to be around 2% per day and is fairly constant over time. For stock-2 the average volatility is somewhat higher and its volatility clearly varies over time, starting low in the early periods at around 1.5% and rising to quite a high level on days 701–3.

Next, by looking across the row 'R_1', we can describe the time series properties of the returns on stock-1. When stock-1 has positive returns they tend to stay positive for a while and when they become negative there is a tendency for them to remain negative – this means that stock returns R_1 are not independent over time – they are autocorrelated.[1]

Now consider the (contemporaneous) correlation *between* the two stock returns. Comparing the returns of stocks-1 and 2, at each point in time (i.e. looking down each column marked 1, 2, 3, ..., 1000) we can see that except for day-3, day-703 and day-1,000, the return on stock-1 moves *in the same direction* as the return on stock-2, so these stock returns are highly positively correlated. So even though we do not directly measure volatilities and correlations, they are there in the historical data. The historical simulation method makes use of this fact but without assuming the volatilities and correlations are constant over time and without imposing a specific distribution for returns (e.g. normal distribution), when calculating the VaR.

[1]In the variance-covariance method we would usually scale up daily stock return volatility using the root-T rule – but this requires that stock returns are independent over time (and identically distributed). The basic HS method also requires stock returns to be independent over time.

Suppose 'today' is day-1,000, and we currently hold \$100 in each stock. We want to forecast the dollar-VaR for tomorrow (i.e. day-1,001) at the 1st percentile. (Note the use of the 1st percentile here.) The HS approach asks the question:

If I had held my current position of \$100 in each stock, what would my profit and loss have looked like on each of the past 1,000 days?

To implement the HS approach requires the following steps:

1. Calculate the \$-change in value of your *current* portfolio over each of the last 1,000 days. Each of these *separate* 1,000 changes in value has occurred with equal probability since each represents one day out of the 1,000 days.
2. Order the 1,000 daily profits/loss figures dV_p from lowest (on the left) to highest (on the right). Some of the lower figures will be negative, indicating losses.[2]

The VaR at the 1st percentile is the value of dV_p which is 10th from the left (i.e. 1% of 1,000 data points corresponds to the 10th data point when placed in ascending order), so:

The dollar-VaR using HS (at 1st percentile) is \$6m.

From the above we can see that when we use actual historical data to calculate the HS-VaR we are implicitly incorporating the volatilities and correlations in the data, each day that we calculate the change in the portfolio value. These 'implicit' volatilities and correlations vary over time (e.g. the correlation between the two stock returns in Figure 46.1 is positive on days 1, 2, 700, 701, and 702 but is negative on the other days). The HS-VaR is based on the loss at a specific percentile and this estimate is subject to error, which is shown in Example 46.1.

EXAMPLE 46.1

Confidence Interval for Daily Forecast of HS-VaR

Suppose the estimate of HS-VaR is \$6m (with $n = 1{,}000$ days of data). When a known probability distribution is fitted to the histogram of dollar daily losses (which are positive numbers) and gains (which are negative numbers), the mean is estimated as $\mu = 0$, with standard deviation $\sigma = \$3\text{m}$. The known probability distribution which best fits the

(*continued*)

[2]This ordering ignores any autocorrelation in returns and essentially assumes independence of portfolio returns over time.

(*continued*)

empirical data could be a t-distribution or Pareto distribution, which both have fatter tails than the normal distribution. But for expositional purposes assume the empirical distribution of losses is normal. The $q = 99$th percentile of losses for this distribution (using Excel) is $x = NORMINV\,(q = 0.99,\ \ \mu = 0,\ \ \sigma = \$3\text{m}) = \$6.979\text{m}$.

The standard error of the HS-VaR percentile estimate is:

$$stde = \frac{1}{f(x)}\ \sqrt{(1-q)q/n}$$

where $f(x)$ is the probability density function (pdf) of the loss $x = \$6.979\text{m}$ at the 99th percentile, hence (using Excel):

$$f(x) = NORMDIST\,(x, \mu, \sigma, FALSE) = 0.00888$$

and $stde = \frac{1}{0.00888}\ \sqrt{0.01(0.99)/1{,}000} = \$0.3543\text{m}$

The 95% confidence interval for the HS-VaR estimate is $\$6\text{m} \pm 1.96\ \ (\$0.3543\text{ m}) = (\$6 \pm \$0.694)\text{ m}$.

This histogram for all 1,000 values of dV_p gives a picture of any skewness or kurtosis in the daily change in value of the portfolio – hence we can visually assess any deviations from the normality assumption. This is done using data on stock returns in Figure 46.2, for a portfolio of three stocks, with equal amounts of $10,000 held in each stock. These stocks are Bank of America, Boeing, and GM. From the histogram we can measure the VaR at any chosen percentile.

In Figure 46.2 you can clearly see fatter left and right tails, than would be found for the normal distribution. There is also some 'left skew' (i.e. more large negative returns than large positive returns). Because we are using an empirical histogram we are not assuming normality. The 5th percentile lower tail cut-off point in Figure 46.2 appears to be around $750. Indeed if we order all the daily changes in value from lowest to highest, the VaR from the historical simulation method at the 5th percentile is found to be $761.

Using the raw historical data on each of the three stock returns, we can calculate the sample standard deviations and correlation coefficients and hence the VaR using the variance-covariance (VCV) method. We find that holding $10,000 in each stock gives a *daily VaR* as follows:

$$\text{Historical Simulation Method} = \$761$$

$$\text{Variance-Covariance Method} = \$950$$

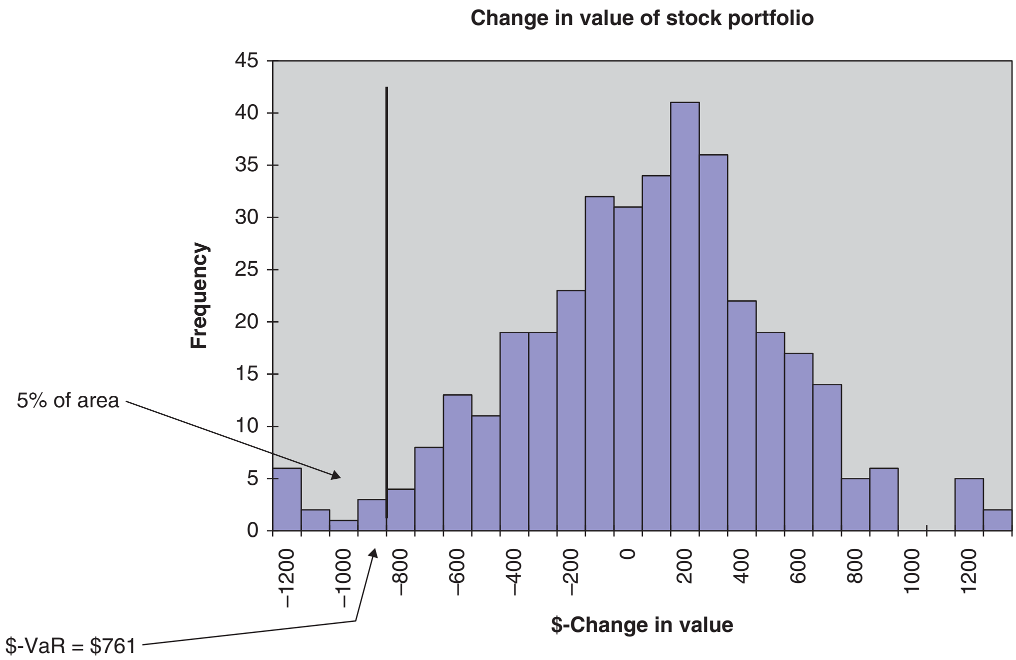

FIGURE 46.2 Historical Simulation

The HS-VaR of \$761 is 2.54% (= \$761/\$30,000) of the initial value of the portfolio and the VCV-VaR is \$950 (3.17% of portfolio value). So there is some difference between the two estimates of VaR which would probably be magnified if we are looking at VaR over a longer horizon than one day.

> *An Excel file calculating the VaR of a portfolio of stocks using historical simulation is available on the website.*

Why do the two methods, HS and the VCV approach, sometimes give different results? Which is the most accurate? The VCV method assumes that returns are independent (i.e. like a coin flip), that we have accurate forecasts of the standard deviations and correlations, and that all returns are normally distributed. The historical simulation method does not assume normality of stock returns but just takes whatever distribution exists in the historical data. It looks from the histogram in Figure 46.2 that the returns are not normally distributed so we might tend to favour the historical simulation approach. However, we cannot really say which estimate is the more accurate unless we do extensive 'backtesting' and compare the forecast errors of competing methods.

The HS method can be used to estimate VaR for any set of assets for which we have a reasonably long series of daily returns and hence daily profits and losses. So in principle the historical data might include the dollar change in value of a portfolio of calls or puts, bonds and foreign assets which can be added to give historical data series for the dollar change in the complete portfolio. There is an added difficulty with coupon paying bonds where calculating price changes may require many different daily changes in spot yields across the whole term structure – this is computationally burdensome.

Accurate estimates of the VaR at the 5th percentile and even more so at the 1st percentile, requires a long time series of data, and this is not always available for some assets. An obvious defect of the HS approach is the complete dependence of the results on the *particular* data set used. The data set used may contain 'unusual events' which are not thought likely to happen again in the immediate future (e.g. sterling leaving the ERM in 1992, the Asian crisis of 1997/8, the events of 9/11 in the USA, the crash of 2008/9, the European debt crisis of 2010). If so, the HS estimate of VaR in the forecast period may not be accurate.

You need quite a lot of data to accurately estimate the 1st percentile VaR from historical data. On the other hand, the longer is our historical data set the more the 'old' data may have an influence on our forecast of tomorrow's VaR, as the returns are all given equal weighting. Do we really want the forecast of tomorrow's VaR to be heavily influenced by stock returns that happened say 1,000 days ago (i.e. about 4 years ago when using trading days)?

46.2 BOOTSTRAPPING

A variant of the 'pure' historical simulation approach is the bootstrapping technique. As described above, the historical simulation approach consists of 'one draw' from the historical data set, where each of the 1,000 daily observations are each chosen once, in order to calculate the portfolio VaR. Suppose we take the view that we only want to use the most recent data to forecast tomorrow's VaR, where 'recent' is assumed to be the last 100 days. In other words, we think that historical data from more than 100 days ago is of absolutely no use in forecasting what is likely to happen tomorrow – so we discard it.

It is precisely because tail events are 'unusual' that we need quite a lot of data to reliably estimate the VaR. For example, if we are interested in the 1st percentile VaR then in our historical data set, this will only occur 1 in every 100 observations. So, to put an extreme case, suppose you only have 100 observations on *daily* portfolio returns and 99 of these have a *maximum negative* value of say 1% but the return on day-1 (i.e. 100 days ago) was *minus* 8%. Then if you currently hold \$100,000 in this portfolio, your *forecast* of tomorrow's VaR using the HS approach would be $0.08 \times \$100{,}000 = \$8{,}000$. Given the 'run' of actual historical returns do you really believe that \$8,000 is a good estimate of what might happen tomorrow? Well only your 'gut feeling' can answer this. The figure of \$8,000 seems too high a forecast, given the pattern in the historical data. Not only that, but the forecast for 2 days hence would be around $0.01 \times \$100{,}000 = \$1{,}000$ since the '–8%' would drop out of your moving 'historical window' of the past 100 days.

The dilemma is that you only want to use recent data. But unfortunately that limits the number of observations that are in the 1st percentile of the distribution, from which you are going to make your VaR forecast – in this case a single observation. The bootstrap tries to improve on this by randomly 'replaying history' with the fixed sample size of the last 100 days of real data and taking the lowest 1st percentile value for dV_p. We then repeat this a large number of times (say $B = 10{,}000$) and the best estimate of the VaR is taken to be the average of these 10,000 values of the 1st percentile VaRs.

46.2.1 Bootstrap VaR

In the 'basic bootstrap' approach the last $T = 100$ portfolio returns (i.e. the values from 901 to 1,000 in the last row of Figure 46.1) are re-sampled with an equal probability of choosing any of the 100 values (using draws from a uniform distribution). We sample with replacement – after each random draw we replace the number chosen in the table – so if we randomly draw the penultimate value +14, we replace it in the table before undertaking the next random draw.

We can think of each portfolio return for a *specific day* being numbered from $T = 1$ to 100, corresponding to the last 100 days of historical data. We then draw $T = 100$ numbers randomly with equal probability. For example, if the 100 random numbers drawn are $\{92, 70, 10, 87, 55 \ldots 92, 27, 70, 54, 10\}$ then we take the change in portfolio value corresponding to these days $\{dV_{92}, dV_{70}, dV_{10}, \ldots, dV_{92}, dV_{27}, dV_{70}, dV_{54}, dV_{10}\}$. Note that some 'days' appear more than once (e.g. 92, 70, 10) and others might not appear at all. We then choose the most *negative value* for the *change* in portfolio value, from this first random sample of 100 values of dV. This most negative value is the 1st percentile-VaR for our first 'random replay of the last 100 days'. For this first bootstrap, assume we obtain VaR $(j = 1) = \$500$ (loss is reported as a positive number).

Note that we have allowed any one day's returns to be randomly chosen more than once (or not at all). Although *on average* you pick any 'specific day' just once, because of randomness, some of the 100 days will not be picked at all and some will be picked more than once. We now repeat the above procedure for 1,000 'trials/runs/draws' to obtain 1,000 possible (alternative) values for the 1st percentile-VaR. The results might be:

$$VaR(j = 1) = \$500$$

$$VaR(j = 2) = \$300$$

$$\ldots$$

$$VaR(j = 999) = \$440$$

$$VaR(j = 1000) = \$400$$

All the above figures are likely to be losses, since you are always picking the lowest number (i.e. the 1st percentile) from a random sample of $T = 100$ random values. The average of the above 1,000 values might turn out to be $BVaR = \$450$ and this is our best forecast of the daily-VaR at the 1st percentile. This 'Bootstrap VaR' forecast, done on say Monday, predicts the VaR for Tuesday.

We can also plot the above 1,000 values of $VaR(j)$ in a histogram and the 'width' of this empirical distribution gives you an estimate of the standard deviation around your 'Bootstrap-VaR' central estimate:

$$\text{where } \sigma(VaR) = \sqrt{\textstyle\sum_{j=1}^{1,000} [VaR(j) - BVaR]^2/1,000}$$

You can order the 1,000 values of $VaR(j)$ from smallest to largest. A 95% confidence interval (i.e. 2.5% in each tail) or 'range' for your bootstrap VaR estimate is then the value of the ranked $VaR(j)$ in the 25th position (e.g. \$410) and in the 975th position (e.g. \$500). Hence your best bootstrap VaR estimate for Tuesday is $BVaR = \$450$ and the error band is \$410 to \$500.

At the end of Tuesday we drop the daily observation on the portfolio return for 100 days ago. We then add one extra day of returns data for the Tuesday, for each stock in our portfolio and recalculate the change in value of the portfolio for the Tuesday. Tuesday's change in portfolio value is now included in the 100 days of data for our bootstrap procedure, which give us the bootstrap VaR forecast for Wednesday. The bootstrap procedure mitigates the 'lack of data' problem when calculating the 1st percentile VaR using only $T = 100$ days of actual data and hence in principle we hope to obtain an improved estimate of the portfolio VaR.

The bootstrapping technique has similar advantages to the historical simulation approach but still suffers from the problem that the original $T = 100$ historical observations should be 'representative' of what might happen in the future. Also the 'larger simulated' set of data comes at a price. Since bootstrapping 'picks out' each of $T = 100$ daily returns *at random* (from the 100 historical 'days' of data), the simulated returns will 'lose' any of the serial correlation which happens to be in the historical returns (i.e. the basic bootstrap method implicitly assumes returns are independent over time). However, even this problem can be tackled by using a 'block bootstrap'.

46.2.2 Block Bootstrap

To explain the idea behind a block bootstrap, consider the following simple example. Suppose we undertake statistical tests on daily returns and find that serial correlation is positive but only first order (e.g. a moving average error of order one, MA(1)) – so only adjacent returns tend to be correlated. In this case, if we randomly draw the number 92 from the uniform distribution then we would revalue the portfolio not only with the returns on the 92nd day but also on the 91st and 93rd days.

If there is some first order positive serial correlation and the returns on the 92nd day are predominantly negative (positive) then there is a greater than 50% chance that the returns on the two adjacent days are also predominantly negative (positive). We then revalue the portfolio on each of these 3 days. We repeat this random selection in groups of 3 another

30 times to get approximately 100 returns and then choose the most negative value as our first estimate of VaR (at the 1st percentile). We then proceed as before to obtain 10,000 values of the 1st percentile-VaR, and average these 10,000 values, which is our best forecast for the bootstrap-VaR.

Since it is uncertain as to the degree of serial correlation in the data we can undertake a sensitivity analysis by recalculating the BVaR for different 'block lengths' (e.g. 3, 5, 12) to see if the estimated BVaR changes dramatically. There are also statistical tests available to choose the most appropriate block lengths.

As regulators allow financial institutions to develop and use their own 'internal models' then the HS approach has gained adherents, since although the raw data input requirements are greater than the parametric-VaR approach, the computation is straightforward. Bootstrap-VaR is perhaps done less often because of the need for re-sampling of the original data. These two non-parametric approaches are likely to be more accurate when return distributions are non-normal but in general these two approaches are often treated as complementary rather than as a substitute for the parametric-VCV approach.

46.3 MONTE CARLO SIMULATION

Monte Carlo simulation (MCS) is a very flexible parametric method which allows one to generate the whole distribution of portfolio returns and hence 'read off' the VaR at any desired percentile level. In contrast, the HS and bootstrap approaches can only 'reproduce' the outcomes that are inherent in the fixed set of historical data. MCS is particularly useful for calculating the VaR for portfolios which contain assets with non-linear payoffs, such as options. It is a parametric method because we assume a specific distribution for the underlying asset returns (e.g. returns on stocks, bonds, long-term interest rates, and spot-FX) and therefore the variances and correlations between these returns need to be *estimated,* before being used as inputs to the MCS.

However, in MCS one need not assume that the distribution is multivariate normal (although this is often assumed, in practice) and, for example, one could choose to generate the underlying asset returns from a fat-tailed distribution (e.g. Student's t-distribution, mixture normal) or one which has a skewed left tail (e.g. jump diffusion process). However, note that even if the underlying asset returns have a multivariate normal distribution, the distribution of the change in value of a portfolio of *options* will not be multivariate normal, because of the non-linear relationship between the underlying asset return and option premia – as in the Black–Scholes formula.

Within the MCS framework applied to options we can also choose whether to calculate the change in value of the portfolio of options by using the full valuation method (e.g. Black–Scholes formula), or using only the 'delta' or the 'delta+gamma' approximation or indeed, other methods of calculating option premia (e.g. assuming stochastic volatility). Of course, measuring VaR using the linear/delta method or the 'delta+gamma' approximation may not be accurate for options that are 'near-the-money', because these have a very pronounced non-linear response to changes in the underlying asset price.

We begin by showing how to measure the VaR of a single option using MCS and then move on to consider the VaR for portfolios which contains several options.

46.3.1 Single Asset: Long Call

Suppose you hold a long call on a stock with $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.6$ (60%) and with $T - t = 1$ year to maturity and the initial value of the call using Black–Scholes is $C_0 = 25.5$. What is the VaR of this call over a 30-day horizon? We simulate the stock price over 30 days assuming it follows a geometric Brownian motion (GBM) with the 'real world' mean return on the stock $\mu = 0.06$ (6% p.a.).

After 30 days, the simulated stock price is S_{30} and the 'new' call premium is given by the Black–Scholes equation $C_{30}^{(1)} = \text{BS}(S_{30},\ \ T - 30/252, \ldots)$, where we have assumed that the other inputs $r = 0.05$ and $\sigma = 0.6$ remain constant over the 30-day horizon. The dollar change in value for this first 'run' of the MCS is $dC^{(1)} = C_{30}^{(1)} - C_0$. We now repeat the above for say $m = 10{,}000$ runs and obtain $dC^{(i)} = C_{30}^{(i)} - C_0$ (for $i = 1$ to 10,000). Finally we order the values of $dC^{(i)}$ from lowest to highest or plot them in a histogram (Figure 46.3). The 5th (1st) percentile lower cut of point gives the VaR, which here is \$14.5 (\$17.9), that is about 57% (70%) of the initial call premium $C_0 = 25.5$.

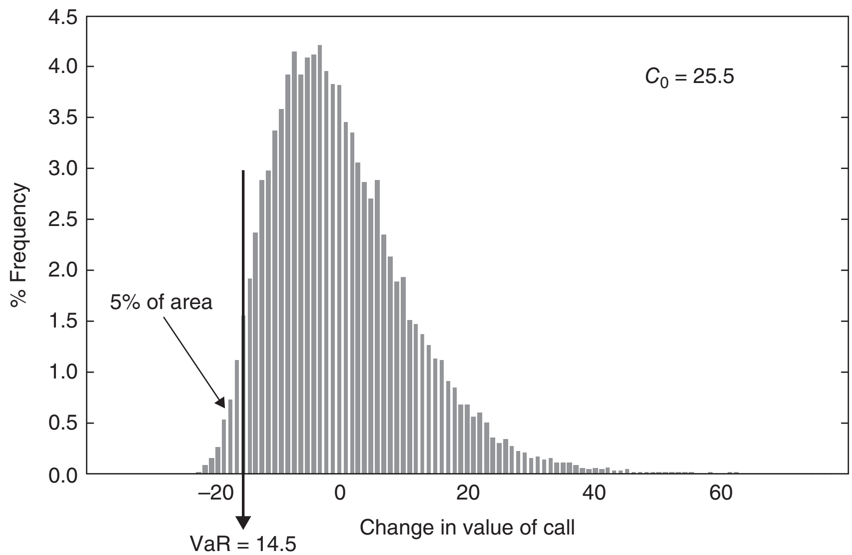

FIGURE 46.3 VaR of one long call option

Even though stock returns are assumed to be normally distributed, the histogram of the call premium (Figure 46.3) is non-normal because of the curvature of the relationship between the call premium and the stock price. We valued the option using a closed-form solution (Black–Scholes) which is a 'full valuation method'. We could instead have approximated the change in the option premium using either the linear 'delta approximation' or the 'delta+gamma' approximation. The choice here depends on the degree of accuracy versus the tractability of a particular valuation approach. We discuss this further below.

46.3.2 Options on Different Stocks

To make the algebra simple, assume we hold N_1 options (calls or puts) with option premiums P_1 on the same underlying stock (Exxon Mobile) with stock price S_1 and N_2 options with option premium P_2 *on a different* underlying stock (AT&T) with stock price S_2.

Both stocks are 'domestic' (e.g. US stocks). If $N_i > 0$ (< 0) then the options are held long (short). The Monte Carlo methodology to calculate the 30-day VaR at the 5th percentile, over a 30-day horizon, involves the following steps.

1. The *initial value* of the option's portfolio is:

$$V_p^0 = N_1 P_1 + N_2 P_2 \tag{46.1}$$

   Assume we value the options using the Black–Scholes formula. The initial two option premia $(i = 1, 2)$ are $P_i = BS_0(S_i, T_i, K_i, \sigma_i, r)$ where K_i is the strike price, T_i = time to maturity, σ_i is the annual stock return volatility and r is the risk-free rate. We assume the volatility and interest rate remain constant over the 30-day horizon.
2. Using statistical estimates of volatilities and correlations the following recursive equations:

$$S_{1t} = S_{1t-1} \exp[(\mu_1 - \sigma_1^2/2)dt + \sigma_1 z_{1t}\sqrt{dt}] \tag{46.2a}$$

$$S_{2t} = S_{2t-1} \exp[(\mu_2 - \sigma_2^2/2)dt + \sigma_2 z_{2t}\sqrt{dt}] \tag{46.2b}$$

   are used to generate the two stock price series over 30 trading days,[3] where $dt = 1/252$. This is the MCS. Each of the $z_i \sim niid(0, 1)$ but z_{1t} and z_{2t} are contemporaneously correlated with a correlation coefficient of ρ. This means the simulations mimic the correlation between the two stock returns found in the actual real world data. (The z_i are generated using the Choleski decomposition outlined in Chapter 26.)

[3]This equation gives an exact lognormal distribution for S for *any* size of step dt. Because of this we do not have to use the recursion over 30 separate days (with 30 separate draws of the $z's$) we can obtain S_{30} in 'one-step' using only one draw of z and using $S_{30} = S_0 \exp[(\mu - \sigma^2/2)dt + \sigma z\sqrt{dt}]$ where $dt = 30/252$. This speeds up the calculation of S_{30} compared with the 30-day recursion, where we use '30 draws' of z.

3. New Black–Scholes values for options prices and the portfolio value at $t = 30$ are:

$$P_i^{new} = BS(S_{i,t+30}, T_i - 30/252, \;\; K_i, \sigma_i, r) \;\; \text{for} \;\; i = 1, 2$$

$$V_p^{new} = N_1 P_1^{new} + N_2 P_2^{new} \tag{46.3}$$

Hence the first run of the MCS ($m = 1$), the change in value of the options portfolio is:

$$dV_p(m = 1) = V_p^{new} - V_p^0 \tag{46.4}$$

We repeat steps 2 and 3, $m = 10{,}000$ times and obtain 10,000 simulated values for the 30-day change in the \$-value of the options portfolio dV_p. We then locate the 5th percentile in the left tail of the distribution – this is the portfolio-VaR over 30 days.

46.3.3 Approximations: Delta and 'Delta+Gamma'

Above, we used the Black–Scholes equation to price the option – known as the 'full valuation method'. Alternatively, one can approximate the change in price of an option using the option's delta Δ_i (evaluated at the initial stock price at $t = 0$). Consider estimating the VaR over 30 days for one long call, using (i) the full valuation method (Black–Scholes), (ii) the delta approximation, and (iii) the 'delta plus gamma' approximation. Using the delta approximation:

$$dC_i^{new} = \Delta_i(S_{i,30} - S_{i,0}) \;\; \text{for } i = 1, 2 \tag{46.5}$$

where $S_{i,30}$ is the simulated value of the stock price at $t = 30$. Using Δ_i provides a (linear) first order approximation to the change in the call premium. The second-order *approximation* including the option's gamma is:

$$dC_i^{new} = \Delta_i(S_{i,30} - S_{i,0}) + 0.5 \;\; \Gamma_i \;\; (S_{i,30} - S_{i,0})^2 \tag{46.6}$$

A comparison of VaR outcomes is given in Table 46.1. At the 1st percentile, the full valuation method gives a VaR of \$17.9 (a fall of 30% of the initial call premium of \$25.5) while the delta approximation gives \$21.1 (a fall of 17%), which implies a substantial error. The 'delta+gamma' method gives \$17.8 (a fall of 30% of the initial call premium) which is very close to that for the full valuation method using Black–Scholes.

If we have a portfolio of options (on the same underlying stock-i), the delta and gamma in the above formula are the 'portfolio-delta' and 'portfolio-gamma' and the dollar change in value is:

$$dV_i^{new} = \Delta_p(S_{i,30} - S_{i,0}) + 0.5 \;\; \Gamma_p \;\; (S_{i,30} - S_{i,0})^2 \tag{46.7}$$

.

TABLE 46.1 Delta and delta+gamma approximation

Input	
Current stock price = \$100	Initial call premium = \$25.5
Volatility of stock = 60% p.a.	Strike price, K = \$100
Mean stock return = 6% p.a.	Interest rate, r = 5% p.a.
	VaR horizon = 30 days

Value at risk			
Percentile	**Full valuation (BS)**	**Delta approximation**	**Delta and gamma**
1st	\$ 17.9 (70.4%)	\$ 21.1 (82.7%)	\$ 17.8 (69.9%)
5th	\$ 14.5 (57.0%)	\$ 15.7 (61.8%)	\$ 13.9 (54.7%)

Notes: Figures in parentheses are the percentage of initial in value (i.e. \$VaR/Initial call premium).

For a portfolio of options *on different stocks* (e.g. options on AT&T, options on Exxon, etc.), the change in value for each option on stock-i is given in Equation (46.7) and the change in total portfolio value (i.e. containing all the options) of the n-different options is $dV_p = \sum_{i=1}^{n} dV_i^{new}$. In the MCS, the stock prices are simulated as in Equation (46.2) incorporating all the correlations between the n underlying stock returns, using the $(n \times n)$ Choleski decomposition.

The VaR of a portfolio of options using MCS in MATLAB is on the website.

The delta+gamma approximation can be used when the options being considered have no closed-form solution but we are able to get estimates of delta and gamma from numerical approaches to pricing the option (such as MCS or the BOPM).

In the above Monte Carlo simulation we assume normality of (log) stock returns. But if stock return distributions in reality have fat tails (or are left skewed) then the estimated VaR is likely to be an underestimate of the true VaR. We then need to use MCS, but perform sampling from return distributions that are more representative of the actual empirical data (e.g. Student's t-distribution which is symmetric but with fatter tails than the normal distribution).

46.4 ALTERNATIVE METHODS

VaR is an attempt to encapsulate in a single figure, the dollar-risk of a portfolio of assets, over a specific horizon with a given probability. The VaR can be calculated in a number of different ways. For example, the variance-covariance (VCV) parametric approach assumes (multivariate) normally distributed returns and uses specific forecasts of volatilities and correlations (e.g. EWMA). When the portfolio contains options then the ‘parametric’ Monte Carlo method is often used.

Clearly these approaches are subject to ‘model error’, since the estimated variances and covariances may be incorrect. In contrast, in the ‘non-parametric’ historical simulation method and bootstrapping we observe how our current portfolio value would have changed, given the actual historical data on returns. Non-parametric methods require a substantial amount of data and the results can sometimes be highly dependent on the sample of historical data chosen.

Let us examine some of the problems of the above approaches which are all based on forecasts derived from ‘averaging’ over a sample of recent data. For example, in the VCV approach a portfolio with market value $V_0 = \$606$m and $\sigma = 1\%$ (per day) has a VaR of \$10m over 1 day at the 5th percentile ($= V_0\ 1.65\sigma$). This indicates that for about 19 days out of 20, losses should not exceed \$10m and losses could exceed \$10m in about 1 day out of every 20. However the VaR figure gives no indication of what the *actual losses* on any specific day will be. If returns are (conditionally) normal then the cut off point for the 0.5% left-tail is 3.2 σ, giving a VaR of \$19.4m. So we can be ‘pretty sure’ losses will not exceed \$20m – in fact, we expect to lose more than \$20m, only 5 days in every 1,000 days. But even then, we cannot be absolutely sure of the *actual* dollar loss because:

- Even with the normal distribution, there is a small probability of a very large loss.
- Actual returns may have fatter tails than those of the normal distribution, so larger, more frequent dollar losses (on average) will occur (relative to those assuming normality).
- Correlations and volatilities can change very sharply. For example, within a 3-month period the daily correlation between daily returns on the Nikkei 225 and the FTSE 100 have varied from +0.9 to –0.9 and the daily volatility on the Nikkei 225 in a specific 3-month period varied between 0.7% to 1.8% per day.

All of the methods discussed (e.g. the VCV parametric approach, historical simulation, Monte Carlo simulation) generally assume the portfolio is held fixed over the holding period considered. This limitation becomes more unrealistic the longer the time horizon over which VaR is calculated. For example, while one may not be able to easily liquidate a position within a day (particularly in crisis periods), this is not necessarily true over periods longer than say 10 days.

Another crucial dimension which is missing in all the approaches, is the ‘bounce-back period’. On a mark-to-market basis, estimated cumulative 10-day losses might be large but if markets fully recover over the next 20 days, the cumulative actual losses over 30 days could be near zero. It may therefore be worthwhile to examine the VaR without using the $\sqrt{T}$ scaling factor, but instead trying to get an explicit estimate of volatilities over longer horizons (e.g. where returns might exhibit more mean reversion). However, it requires a substantial amount of data to accurately model and forecast volatilities over longer horizons.

46.4.1 Stress Testing

So far our VaR calculations, whether using the VCV approach or historical simulation/bootstrapping approaches, have their difficulties – the former assumes normally distributed returns and the latter require a relatively long run of past data. Both approaches assume that an average of recent past data is a good guide to what might happen in the future. Because of this dependence on a particular data set, they are often supplemented by stress tests. Stress testing estimates the sensitivity of a portfolio to specific 'extreme movements' in certain key returns. It tells you how much you will lose in a particular state of the world but usually gives no indication of how likely it is that this state of the world will actually occur.

Choice of 'extreme movements' may be based on particular crisis periods (e.g. 1987 stock market crash, the 1992 'crisis' when sterling left the ERM, the 1997–8 Russian bond default, the Asian currency crisis of 1997/8, the banking crisis of 2008–10, the Greek sovereign debt crisis of 2010–13). It follows that the implicit volatilities and correlations between asset returns are specific to the historical crisis periods chosen for the stress tests – and these are likely to be periods when correlations tend to increase dramatically.

Alternatively, the financial institution can simply revalue its existing portfolio using whatever returns it thinks 'reasonable and informative' – the danger here is outcomes which bear little or no relation to actual historical events or which mimic events which have never occurred and are unlikely to occur in the future – so there is a lot of judgement required when undertaking a 'reasonable' scenario.

Clearly, this type of stress testing is best done for relatively simple portfolios since otherwise the implicit correlations for the chosen scenario may be widely at variance with those in the historical data or even for any conceivable future event. But note another danger of stress testing – namely, that a portfolio might actually benefit from 'extreme' movements yet be vulnerable to 'small' movements. For example, the 'U' shaped payoff for a long straddle (buy a call and a put with the same strike and expiration date), indicates large profits if the underlying price moves by a large amount in either direction (prior to or at maturity) – but a loss of some of the option premia, if the underlying price moves by only a small amount.

In general, stress testing has several limitations and the same stress-testing scenario is unlikely to be informative for all institutions (portfolios). The choice of inputs for the stress test(s) will depend on the financial institution's or regulator's assessment of the likely source of the key 'prices' and sensitivities to changes in these prices, given its portfolio composition. For example, a commodities dealer is unlikely to find a stress test of a parallel shift in interest rates of 200 bps useful but would base her stress tests on movements in spot and derivatives prices of major commodities. One can also usefully turn the scenario approach 'on its head' and ask 'Given my portfolio, what set of *plausible* scenarios (for stock prices, interest rates, exchange rates, etc.) will yield very bad outcomes?'

Stress testing is therefore a useful complement to the usual VaR calculations but requires considerable judgement in its practical implementation. Because of the perceived failure of risk management techniques like VaR in the 2008–9 financial crisis, regulators have placed additional emphasis on stress testing, particularly for 'systemically important' banks.

46.4.2 Extreme Value Theory

Another method for calculating VaR which concentrates on the tails of the distribution is extreme value theory (EVT). This method is used in the natural sciences for estimating the expected maximum extreme 'loss', with a specific degree of confidence, over a given horizon (e.g. expected losses, at a given confidence level, from the failure of a dam or nuclear power plant within the next 5 years). In a very broad sense, the EVT approach is a mixture of the historical simulation approach together with the estimation of a smooth curve to represent the 'true' shape of *the tail* of the distribution. EVT uses only data in the extreme tail of losses and discards the rest of the data, namely small losses and all gains – as the latter data is assumed to be uninformative in forecasting extreme losses that might happen in the future.

Suppose we have data on actual daily changes in the value of a stock portfolio over the last 500 days. With EVT we have to choose where the tail of 'extreme losses' begins. Suppose we assume 'extreme losses' are those below the 5th percentile (left-tail) and this gives us 25 values for losses that are larger than $u = \$200$m, say.[4] If we plot a histogram of these worst outcomes it would probably look very 'lumpy' (Figure 46.4).

What EVT does is to fit a 'smooth curve' through this histogram of extreme values, without imposing any specific assumptions concerning the rest of the distribution. Having estimated the smooth curve, you can then 'read off' the VaR at any percentile in the tail (e.g. VaR at the 1st percentile as in Figure 46.4). There is an art in getting EVT to estimate the 'true' distribution

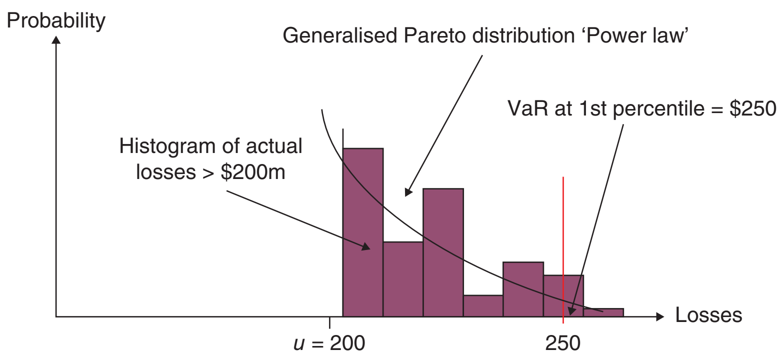

FIGURE 46.4 EVT

[4]The choice of $u = 200$ is a trade off between having data which we believe is truly in the left tail and having enough data points in the 'extreme tail' (of the histogram) to obtain a reasonable estimate for a smooth curve to represent the shape and position of the tail of the distribution.

for tail losses since the position and shape of the tail depends on how many of our empirical data points we choose to include as ‘extreme’.

> *On the website a MATLAB program estimates the EVT curve in Figure 46.4, and uses this to calculate the VaR at low percentiles (0.1%) in the extreme tail of the distribution of losses.*

Danielsson and de Vries (1997) report that for VaRs below 5th percentile, the VCV approach tends to underpredict the occurrence of tail events, the historical simulation approach tends to overpredict, while the EVT approach performs well. Hence the EVT approach may provide a more accurate VaR estimate, the ‘smaller’ the percentile of the distribution one is interested in.

46.5 SUMMARY

- The variance-covariance (VCV) method of forecasting daily, weekly or monthly VaR assumes all asset returns in a portfolio are *niid* – this is a reasonable assumption for a portfolio of domestic stocks, foreign stocks and futures contracts, a workable assumption for bonds, FRNs and swaps but is not accurate for portfolios containing options.
- The historical simulation method is reasonably accurate in predicting VaR between the 1st and 5th percentiles but it requires a substantial amount of data and is a little inflexible (e.g. it is difficult to do sensitivity analysis), compared with the standard parametric VCV method.
- The bootstrap approach allows a non-parametric calculation of VaR using only recent historical data – but it still relies on past data being representative of what might happen in the near future.
- VaR for portfolios containing ‘plain vanilla’ European options can be estimated using Monte Carlo simulation (MCS).
- The extreme value theory (EVT) approach uses data only from the extreme tail of the empirical loss distribution and estimates a smooth curve for the observed extreme losses, from which we can then obtain the VaR at any desired (low) percentile. EVT is more complex than other parametric methods and (like them) is subject to estimation error.
- Stress testing provides a complementary approach to measuring market risk to that provided by other methods such as VCV, EVT, MCS, HS, and bootstrapping. Much judgement is required when undertaking alternative scenarios and the *probability* of various outcomes (usually) cannot be assessed.

- Alternative approaches used in calculating VaR for large diverse international portfolios of stocks, bonds, swaps, futures, and options contracts are impressive. However, a large element of judgement and common sense is required when using and interpreting results from any of these approaches.

EXERCISES

Question 1
Briefly explain the difference between Monte Carlo simulation (MCS) and stress testing in assessing the riskiness of a portfolio of assets.

Question 2
In a Monte Carlo simulation indicate the steps involved in calculating the VaR over a 10-day horizon, for a long call option on a stock (assuming Black–Scholes holds). For expositional purposes assume $S_0 = 100$, $K = 100$, $r = 5\%$, $\sigma = 0.20$, $T = 1$ (year) and the mean return on the stock is $\mu = 12\%$ p.a.

Question 3
How would you use asset returns over the last 1,000 days to obtain the daily VaR at the 5th percentile, using historical simulation?

Question 4
Name two possible weaknesses when using historical simulation to obtain the daily VaR (at the 5th percentile) from returns over the last 1,000 days.

Question 5
A US investor is short \$10m in each of stock-1 and stock-2 and long \$10m in stock-3. The returns over the last 5-days on these three stocks are:

Days =	**1**	**2**	**3**	**4**	**5**
% Return-1	3	−2	8	−5	−1
% Return-2	2	1	3	4	3
% Return-3	−4	−5	−5	−2	−1

A US investor decides to use the historical simulation method with only the last 5 days of data to forecast her VaR for day-6, at the 20th percentile. What is the VaR?

Question 6
If daily stock returns are autocorrelated (with a first order autocorrelation coefficient of +0.2) how might this feature of the data be incorporated in a MCS?

Question 7
Does the (standard) historical simulation method incorporate the volatility and correlations between the assets that make up the portfolio, when estimating the portfolio VaR?

Question 8
When using MCS (and Brownian motion) to obtain the VaR of a portfolio of options on *different* underlying stocks, what key assumptions do you make?

Question 9
You hold one long European put (on 100 stocks of XYZ) and 50 stocks of ABC. Explain how you can use Monte Carlo simulation to obtain the 10-day VaR of this portfolio (at the 1st percentile)?

PART XII

PRICE DYNAMICS

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

CHAPTER 47

Asset Price Dynamics

Aims

- To explain a standard Wiener process and how this leads to a stochastic process for the stock price S, known as a geometric Brownian motion (GBM).
- To show how Ito's lemma can be used to move from a stochastic process for the stock price S to a stochastic differential equation (SDE) for any non-linear function $g(S, t)$.
- To explain the statistical relationship between a stochastic variable which is lognormal $\ln S_t$ and the variable itself, S_t. In particular, the relationship between their expected values and variances.

A great deal of analytic work in pricing derivative securities and in constructing hedge portfolios uses continuous time stochastic processes. Any variable (such as the stock price) which changes over time in a random way is said to be stochastic. In the real world we observe discrete changes in stock prices but if the time interval of observation is small enough, this approximates to a continuous time process. The Black–Scholes option pricing formula was derived using continuous time mathematics. The mathematics used is not much beyond simple calculus and the intuitive elements are stressed at each point in the argument. The aim is for the reader to get a feel for this approach rather than providing detailed proofs.

In the first section we examine a Weiner process, an Ito process, and a geometric Brownian motion (GMB) for the stock price S. We then derive a stochastic process for the logarithm of the stock price $\ln S_t$ and for the option premium $f(S, t)$ using continuous time mathematics and Ito's lemma. The resulting equations for $\ln S_t$ and $f(S, t)$ are both SDEs with the same source of randomness – the latter is important in deriving a *deterministic* partial differential equation PDE for the option premium $f(S, t)$, as we see in the next chapter. Finally we analyse the behaviour of the stock price when the logarithm of the stock price is assumed to be normally distributed – this relationship is often used in the continuous time literature.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

47.1 STOCHASTIC PROCESSES

In this section we model the behaviour of stock prices in continuous time and show how Ito's Lemma can be used to derive the stochastic process for the derivatives price, given a Brownian motion for the stock price.

47.1.1 Wiener Process

This is a basic building block in representing the stochastic behaviour of a 'cash-market' or 'spot' asset, such as the stock price. If a variable z follows a Wiener process over a short interval of time Δt then:

$$\Delta z = \varepsilon \sqrt{\Delta t} \tag{47.1}$$

where $\Delta z = z_{t+\Delta t} - z_t$ and $\varepsilon \sim niid(0,1)$ or $N(0,1)$ for short. It follows that:

$$\text{Expected value: } E(\Delta z) = 0 \tag{47.2a}$$

$$\text{Variance: } Var(\Delta z) = E(\Delta z)^2 = \Delta t \tag{47.2b}$$

$$\text{Standard deviation: } stdv(\Delta z) = \sqrt{\Delta t} \tag{47.2c}$$

Consider the change in z over a (long) period of time from 0 to T. We divide the period $\{0, T\}$ into $n = T/\Delta t$, small time intervals of length $\Delta t = 1/252$ (for example). Then:

$$\Delta z_T \equiv z_T - z_0 = \sqrt{\Delta t} \quad \sum_{i=1}^{n} \varepsilon_i$$

The ε_i are *niid* and therefore:

$$\text{Expected value: } E(\Delta z_T) = 0 \tag{47.3a}$$

$$\text{Variance: } Var(\Delta z_T) = n \;\; \Delta t = T \tag{47.3b}$$

$$\text{Standard deviation: } stdv(\Delta z_T) = \sqrt{T} \tag{47.3c}$$

$Var(\Delta z_T)$ depends only on the *time difference* from $t = 0$ *to* T. If we have two non-overlapping time intervals $\{t_1, t_2\}$ and $\{t_3, t_4\}$ then $z_2 - z_1$ and $z_4 - z_3$ are uncorrelated, as the ε_i's are independent (over time). A standard Wiener process dz is the limit as $\Delta t \to 0$:

$$dz = \varepsilon\sqrt{dt} \tag{47.4}$$

where dz and dt represent the usual notation in differential calculus. A Wiener process is sometimes referred to as a 'Brownian motion' and has the following properties:

Properties of a Wiener process

- for any $s < t$, $z(t) - z(s) \sim N(0, t-s)$
- for $0 < t_1 < t_2 < t_3 < t_4$, $\quad z_4 - z_3$ and $z_2 - z_1$ are uncorrelated
- $z_0 = 1$ with probability 1.

47.1.2 Generalised Wiener Process

The problem in using a standard Wiener process to represent stock prices is that the mean change $E(dz)$ is zero but we know that stock prices tend to increase over long periods of time. A slightly better model for stock prices is the 'generalised Wiener process':

$$dx = a\,dt + b\,dz \tag{47.5}$$

where 'a' and 'b' are constants. If we ignore the stochastic term dz, then:

$$dx/dt = \mathrm{a} \quad x_t = x_0 + a\,t \tag{47.6}$$

and x_t grows at a rate 'a' per period. In discrete time the generalised Wiener process is:

$$\Delta x = a \;\; \Delta t + b \;\; \Delta z \tag{47.7}$$

where $\Delta z = \varepsilon\sqrt{\Delta t}$

$$E(\Delta x) = a \;\; \Delta t \;\; \text{and} \tag{47.8a}$$

$$\mathrm{var}(\Delta x) = b^2 \;\; \Delta t \tag{47.8b}$$

The drift rate is 'a' (per period) and the variance rate is b^2 (per period). Over the period $\{0,T\}$:

$$E(\Delta x_T) = aT, \qquad \mathrm{var}(\Delta x_T) = b^2 T \tag{47.9}$$

47.1.3 Ito Process

A further generalisation of (47.7) is to allow 'a' and 'b' to depend on both the level of x and time, which gives rise to a stochastic differential equation SDE known as an Ito process:

$$dx = a(x,t) \;\; dt + b(x,t) \;\; dz \tag{47.10}$$

Here both the expected drift rate $a(x,t)$ and the variance rate $b(x,t)$ may vary over time and with the level of x.

47.2 GEOMETRIC BROWNIAN MOTION (GBM) AND ITO'S LEMMA

One would have thought that (47.5) would provide a reasonable characterisation of the behaviour of stock prices since it can have an upward drift rate ($a > 0$) plus a random element. In (47.5) the expected *absolute* change dx is a constant, a. But in finance we assume it is the expected *proportionate* change in the stock price dx/x that is constant. That is to say, we expect two identical stocks, one with a price of \$10 and one with a price of \$100 to both move by say 10% on average – that is, by \$1 and \$10, respectively – and not for both to change by \$1. Hence the stochastic behaviour of stock prices can be represented by a 'geometric Brownian motion (GBM)' using the *proportionate* change in the stock price:

$$dS/S = \mu\, dt + \sigma\, dz \qquad \text{where } dz = \varepsilon\sqrt{dt}\ . \tag{47.11}$$

A GBM is a specific form of Ito process with $a = \mu S, b = \sigma\ \ S \textit{ and } dS/S \sim N(\mu\ \ dt, \sigma^2 dt)$.

47.2.1 Ito's Lemma

Ito's lemma is a way of deriving the stochastic process for any *function* of a stochastic variable. Suppose we have some function $G(z)$ where z is a Wiener process and we require an expression for the differential of this function, $dG(z)$. From ordinary calculus, a Taylor series approximation for $dG(z)$ up to second-order terms in dz is:

$$dG(z) \approx \frac{\partial G}{\partial z} dz + \frac{1}{2}\frac{\partial^2 G}{\partial z^2}\ \ dz^2 \tag{47.12}$$

Note that $E(dz^2) = dt$ and in the above equation we now replace dz^2 by dt,[1] which gives us Ito's equation for the stochastic process $dG(z)$:

$$dG(z) = \frac{\partial G}{\partial z} dz + \frac{1}{2}\frac{\partial^2 G}{\partial z^2} dt \tag{47.13}$$

Above we used a simple heuristic derivation for the stochastic behaviour of dG. (A more detailed exposition is given in Appendix 47.) Note that although (47.12) is an approximation it can be shown that Ito's equation (47.13) is exact.

47.2.2 SDE for the Derivatives Price

Assume the stock price follows an Ito process:

$$dS = a(S,t)\, dt + b(S,t)\, dz \tag{47.14}$$

[1]We are cheating here since $dz^2 \neq dt$. It is the expectation $E(dz^2) = dt$. These are clearly not the same thing but this heuristic 'short-cut' is pedagogically useful at this point.

The SDE for the derivatives price $f(S,t)$ is a function of the stochastic variable S and time, t (which is deterministic). Ito's lemma gives the differential equation for $f(S,t)$ as (see Appendix 47):

$$df = \frac{\partial f}{\partial t}dt + \left[\frac{\partial f}{\partial S}dS + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}dt\right] \tag{47.15}$$

Substituting for dS from (47.14):

$$df = \left[\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial S} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right] dt + \left\{\frac{\partial f}{\partial S}b\right\} dz \tag{47.16}$$

The drift rate of the function $f(S,t)$ is given by the expression in the square brackets and its variance rate is the expression in the curly brackets. The key feature to note is that both dS (in Equation 47.14) and df depend on the *same* underlying source of uncertainty dz. Hence, it is possible to create a risk-free portfolio by combining the stock and the option. This portfolio must earn the risk-free rate – otherwise arbitrage profits can be made. This is the approach used in the BOPM to price an option and we use it here in a continuous time framework.

47.2.3 SDE for $d(\ln S)$

We now use Ito's lemma to derive the stochastic behaviour of $f(S)= \ln S$, given that dS/S follows a GBM. Note that the 'continuously compounded return', $R^c \equiv d(\ln S)$. Since $d(\ln S)$ using ordinary calculus, equals dS/S then it would seem from (47.11) that the expected value of $E[d(\ln S)]$ should equal μdt. However, this is not the case, since the transformation from dS/S to $d\ln S$ requires *stochastic* calculus and Ito's lemma. The GBM is:

$$dS = \mu S dt + \sigma S dz, \quad \text{where} \quad a = \mu S \text{ and } b = \sigma S. \tag{47.17}$$

Given $f(S) = \ln S$ then:

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$$

Substituting these expressions into Ito's equation (47.15) gives:

$$d(\ln S) = (\mu - \sigma^2/2)dt + \sigma dz \tag{47.18}$$

Using Ito's lemma the drift rate for $d(\ln S)$ is not μdt but $(\mu - \sigma^2/2)dt$. However, both dS/S and $d(\ln S)$ have the same variance, $\sigma^2 dt$. Hence given the GBM for $dS/S \sim N(\mu dt, \sigma^2 dt)$ the stochastic process for $d(\ln S)$ using Ito's lemma is:

$$d(\ln S) \sim N[(\mu - \sigma^2/2)dt, \quad \sigma^2 dt] \tag{47.19a}$$

$$\ln S_T \sim N[\ln S_0 + (\mu - \sigma^2/2)T, \quad \sigma^2 T] \tag{47.19b}$$

47.2.4 Two Stochastic Variables

What happens if we have an option whose payoff depends on two stochastic variables S_1 and S_2? For example, this could be an option with a payoff that depends on the highest of two stock prices at expiration of the option. If S_1 and S_2 both follow an Ito process then:

$$dS_1 = a_1(S_1, t)\ dt + b_1(S_1, t)\ dz_1$$
$$dS_2 = a_2(S_2, t)\ dt + b_2(S_2, t)\ dz_2$$

Suppose two Wiener processes dz_i have variances dt and correlation coefficient $-1 \leq \rho \leq 1$. Ito's lemma gives the SDE for the derivative security $f(S_1, S_2, t)$ which has a payoff depending on these *two* underlying variables:

$$df = \frac{\partial f}{\partial t}dt + \left(\frac{\partial f}{\partial S_1}dS_1 + \frac{\partial f}{\partial S_2}dS_2\right) + \frac{1}{2}\left(b_1^2\frac{\partial^2 f}{\partial S_1^2} + b_2^2\frac{\partial^2 f}{\partial S_2^2}\right)dt + \rho b_1 b_2\left(\frac{\partial^2 f}{\partial S_1 \partial S_2}\right)dt$$

The only 'new' term is the final one which accounts for the correlation between the two stock returns. We could also substitute for dS_1 and dS_2, which gives df as a direct function of the Wiener processes dz_1, dz_2. To hedge or 'remove' these two stochastic terms in the SDE for df requires delta hedging with *two* underlying assets S_1, S_2. As we see in the next chapter this gives a PDE (which is deterministic) and can be solved for the derivatives price either analytically or numerically.

47.3 DISTRIBUTION OF LOG STOCK PRICE AND STOCK PRICE

Below we examine the relationship between the statistical distribution for the 'logarithm of the stock price' ($\ln S$) and the distribution for the 'level of the stock price' S – the latter is needed because it is the expected stock price which determines the payoff to a (plain vanilla) option. In particular, if we know the mean and standard deviation of the logarithm of a variable $\ln S$, what is the mean and standard deviation for the level of S, itself. This is the standard statistical analysis of the lognormal distribution which is often used in continuous time finance.

Assume the continuously compounded (log) return $R_t^c \equiv \ln(S_t/S_{t-1})$ over a small interval of time $(t-1, t)$ is $N(\nu, \sigma^2)$ *and* obeys the following stochastic process (with $dt = 1$):

$$R_t^c \equiv \ln(S_t/S_{t-1}) = \nu + \sigma\,\varepsilon_t \tag{47.20}$$

where ε_t is an identically and independently distributed random variable with a zero-mean and is drawn from a symmetric distribution. Note that ε_t need not be normally distributed:),

$ER_t^c \equiv E[\ln(S_t/S_{t-1})] = \nu$ (per period) and $var(R_t^c) \equiv var[\ln(S_t/S_{t-1})] = \sigma^2$. From (47.20) the stock price S_t follows the stochastic process:

$$S_t = S_{t-1} e^{\nu + \sigma\,\varepsilon_t} \tag{47.21}$$

which is a non-linear function of $(\nu, \sigma, \varepsilon_t)$. What does (47.21) imply for the shape of the distribution for S_t and its expected (average) value? One way of thinking about this is to set $\nu = 0$, $\sigma = 0.1$, hence $S_t = S_{t-1} e^{0.10\varepsilon_t}$. To see how the exponential term influences the distribution of S_t consider an artificial scenario where $S_0 = 100$ and successive values of $\varepsilon_t = +1$, *always*.[2] The resulting price series is:

$$100, 110.52, 122.14, 134.99, 149.18, 164.87, 182.21, 201.38\ldots.$$

The gap between each successive value of S_t becomes larger – starting at 10.52, then 11.62 etc. – making a 'long' right tail for the distribution of S. Alternatively, if $\varepsilon_t = -1$ *always,* then the price series (starting at 100 and moving to the left) is:

$$\ldots.49.66, 54.88, 60.65, 67.03, 74.08, 81.87, 90.48, 100$$

Here, the gap between successive price changes gets smaller – starting at –9.52, then –8.61 etc. – which 'squashes' the distribution in the left tail (and S cannot fall below zero). Even though the distribution of ε_t is symmetric, the distribution of the price level S is truncated to the left and extended to the right and is clearly not symmetric. This tends to increase the expected value (mean) of the distribution for S, so that it exceeds the expected value of the distribution of $\ln S$ (i.e. $E \ln S = \nu = 0$). The distribution of $R_t^c \equiv \ln(S_t/S_{t-1})$ is symmetric by assumption (see Equation 47.20) but the distribution of $1 + R \equiv S_t/S_{t-1}$ is not symmetric, as argued above. We further illustrate the relationship between the mean of the distribution $E(\ln S)$ and ES, in Example 47.1.

EXAMPLE 47.1

Expected Value of ln*S* and *S*

Assume $R_t^c \equiv \ln(S_t/S_{t-1}) = \nu + \sigma\,\varepsilon_t$. For $S_0 = 100$, $\sigma = 0.10$ and $\nu = 0$, then $E\ln(S_t/S_{t-1}) = \nu = 0$. The mean (expected) value for the (level of the) stock price one period ahead is: $ES_1 = (S_0 e^{\nu}) E(e^{\sigma\,\varepsilon_1}) = 100E(e^{0.10\varepsilon_1})$.[3] Suppose we draw ε_t from a distribution where it takes

(*continued*)

[2]This is 'artificial' because ε_t is random and it is therefore highly unlikely to take successive values all of which are either +1 or –1 but we make this assumption for pedagogic reasons and relax this assumption below.
[3]Note that $ES_1 \neq (100)e^{0.10(E\varepsilon_1)} = 100$, as the expectations operator cannot be applied to a non-linear function.

(*continued*)

the value +2 or −2, with equal probability and therefore $E\varepsilon_t = 0$. The two equally likely outcomes for the stock price (in period-1) are:

$$S^+ = 100e^{0.10(+2)} = 122.14 \text{ and } S^- = 100e^{0.10(-2)} = 81.87$$

The expected stock price is: $ES = (122.14 + 81.87)/2 = 102.0$

Hence, if the initial stock price is 100 and the continuously compounded return has a mean (expected value) $\nu = 0$, then it is not the case that the expected (average) value of the stock price (in the next period) is equal to 100.

The average level of the stock price will be greater than 100 (in our case 102.0). This arises because of the exponential term $e^{\sigma\varepsilon_t}$. As the distribution of ε_t is symmetric then positive values for ε_t (e.g. +2) in a large sample of data will be accompanied by an approximate equal number of negative values (of −2). But the positive value of ε_t increases S by more than the negative value of ε_t decreases S, hence the average value of S is greater than 100.

We now assume ε_t is normally distributed and $\ln(S_t/S_{t-1}) \sim N(\upsilon,\ \sigma^2)$. It can be shown that the distribution of S_t has a mean:

$$ES_t = S_{t-1}\, e^{\nu} E(e^{\sigma\varepsilon_i}) = S_{t-1}\, e^{\nu} \int_{-\infty}^{+\infty} \phi(\varepsilon)\, e^{\sigma\varepsilon}\, d\varepsilon = S_{t-1} e^{\nu+\sigma^2/2}$$

where the mathematical formula for an $N(0,1)$ distribution is denoted $\phi(\varepsilon)$. For example, if $E[\ln(S_t/S_{t-1})] = \nu = 0.12$ p.a. and $\sigma = 0.15$ p.a., then $E(S_t)/S_{t-1} = e^{\nu}e^{\sigma^2/2} = 1.1275\ \ (1.0113) = 1.1403$. The mean of (S_t/S_{t-1}) is larger than $\nu = 12\%$ p.a. because S_t is a non-linear function of the stochastic variable, ε_t (Equation 47.21). ES_1 can also be calculated using a numerical technique. Set $S_0 = 100$(say) and draw $m = 1{,}000$ values of ε_t from the $N(0,1)$ distribution and calculate the mean using:

$$ES_1 = (100)e^{0.12}\left(\frac{1}{1{,}000}\right)\sum_{i=1}^{1{,}000} e^{0.15\varepsilon_i}$$

then ES_1 will be greater than S_0e^{ν}. More formally we can summarise the above lognormal distribution theory as follows. Assume $\ln(S_t/S_{t-1}) \sim N(\upsilon dt,\ \sigma^2 dt)$:

$$\ln(S_t/S_{t-1}) = \nu\, dt + \sigma\sqrt{dt}\,\varepsilon_t \qquad \varepsilon_t \sim niid(0,1) \tag{47.22a}$$

or

$$S_t = S_{t-1}e^{\nu\, dt+\sigma\sqrt{dt}\,\varepsilon_t} \tag{47.22b}$$

Then, the expected value and variance of $\ln(S_t/S_{t-1})$ and $\ln(S_T/S_0)$ are:

$$E[\ln(S_t/S_{t-1})] = \nu\, dt \text{ and } E[\ln(S_T/S_0)] = \nu\, T \tag{47.22c}$$

$$var[\ln(S_t/S_{t-1})] = \sigma^2 dt \text{ and } var[\ln(S_T/S_0)] = \sigma^2 T \tag{47.22d}$$

Using standard statistical distribution theory it can be shown that for the stock price:

$$E(S_t/S_{t-1}) = e^{(\nu+\sigma^2/2)dt} \quad \text{and} \quad E(S_T/S_0) = e^{(\nu+\sigma^2/2)T}$$

For completeness, also note that statistical distribution theory shows that the variance of the stock price at T is:

$$\text{var}(S_T) = S_0^2\; e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{47.23}$$

Hence, starting from the assumption that $\ln(S_t/S_{t-1}) \sim N(\nu dt, \sigma^2 dt)$ is *niid,* we have obtained the mean and variance for the level of the stock price, S_T.

47.4 SUMMARY

- An Ito process is one where the (absolute) change in a stochastic variable x over a short interval of time, is normally distributed. The drift rate and variance may be functions of x and time.
- The stochastic behaviour for the proportionate change in stock prices (over a small interval of time) can be represented by a continuous time process with positive drift, known as geometric Brownian motion (GBM), which is a specific form of Ito process.
- Ito's lemma can be used to determine the stochastic process for any non-linear function $f(S, t)$, given the stochastic process for S.
- If the continuously compounded return is normally distributed $\ln(S_t/S_{t-1}) \sim N(\nu dt, \sigma^2 dt)$ then statistical distribution theory implies that the distribution of (S_t/S_{t-1}) is not symmetric and has a long right tail and a truncated left tail. The distribution of (S_t/S_{t-1}) is lognormal, with mean $E(S_t/S_{t-1}) = e^{(\nu+\sigma^2/2)dt}$.

APPENDIX 47: ITO'S LEMMA

In moving from a GBM for dS/S with drift μ to a SDE for $d(\ln S)$ requires the use of Ito's lemma and the drift term for $d(\ln S)$ is $\mu - \sigma^2/2$. Below we give a heuristic proof of Ito's lemma. An Ito process (for $dt \to 0$) is:

$$dx = a(x,t)\; dt + b(x,t)\, dz, \quad \text{where} \quad dz = \varepsilon\sqrt{dt}, \;\; \varepsilon \sim niid(0,1) \tag{47.A.1}$$

Below we show that any function $f(x,t)$ satisfies Ito's equation:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}a + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2\right)dt + \frac{\partial f}{\partial x}b\ \ dz \tag{47.A.2}$$

From (47.A.1), $(dx)^2$ is of order dt. Expanding $f(x,t)$ as a Taylor series (ignoring terms of higher order than dt):

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 \tag{47.A.3}$$

Substituting for dx and dx^2 from (47.A.1):

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(a\ \ dt + b\ \ dz) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}[a\ \ dt + b\ \ dz]^2 \tag{47.A.4}$$

The term in square brackets is:

$$[a\ \ dt + b\ \ dz]^2 = (a\ \ dt)^2 + 2ab\ \ (dt\ \ dz) + (b\ \ dz)^2 = O_p dt^2 + O_p dt^{3/2} + b^2\varepsilon^2 dt \tag{47.A.5}$$

where $O_p(.)$ represents the 'order in probability'. The first two terms are of higher order than $O_p(dt)$, while the final term is of order dt and must be retained. Also, $\varepsilon \sim N(0,1)$, so that $Var(\varepsilon) = E\varepsilon^2 - (E\varepsilon)^2 = 1$. But $E\varepsilon = 0$, so $E\varepsilon^2 = 1$ and therefore $E(\varepsilon^2 dt) = dt$. It can also be shown that $Var(\varepsilon^2\ \ dt)$ is of order dt^2 and as a result, '$\varepsilon^2 dt$' *itself*, becomes non-stochastic with expected value, dt. Thus as $dt \to 0$:

$$(b\ \ dz)^2 \to b^2\ \ dt \tag{47.A.6}$$

$$[a\ \ dt + b\ \ dz]^2 \to b^2\ \ dt \tag{47.A.7}$$

This is the basis of our 'simplification' of the proof, so we simply 'replace' the stochastic term $(bdz)^2$ by $b^2 dt$. Substituting from Equation (47.A.7) in (47.A.4) and letting $dt \to 0$ gives Ito's equation (47.A.2).

EXERCISES

Question 1
Do you think that the number of centimetres of rainfall per month at a certain place are (statistically) independent over time? Explain.

Question 2
Explain the basic properties of a standard Wiener process.

Question 3
Explain a geometric Brownian motion (GBM) for stock prices.

Question 4
If the change in the stock price follows a GBM, $dS = \mu \ S \ dt + \sigma S \ dz$, use Ito's lemma to show that $d\ln(S) = (\mu - \sigma^2/2)dt + \sigma\, dz$.

Question 5
The continuously compounded stock return follows the stochastic process:

$$R_i^c \equiv \ln(S_t/S_{t-1}) = \nu + \sigma\varepsilon_t \tag{1}$$

ε_t is an identically and independently distributed random variable from a *binomial distribution*, where it takes the value +1 or −1, with equal probability and therefore $E\varepsilon_t = 0$. The mean of the continuously compounded return equals ν (per period) and its variance is σ^2. Assume $S_0 = 100$, $\nu = 0$ and $\sigma = 0.1$, so that from (1) $S_t = S_{t-1}e^{0.10\varepsilon_t}$ and the stock price after 2 periods is determined by:

$$S_2 = S_0 e^{0.10(\varepsilon_1+\varepsilon_2)} \tag{2}$$

Calculate the four possible out-turn values for the combination $(\varepsilon_1 + \varepsilon_2)$, the resulting four possible outcomes for S_2 and the expected value ES_2. Briefly comment on the distribution of $R_2^c \equiv \ln(S_2/S_0)$ and the resulting distribution of (S_2/S_0).

CHAPTER 48

Black–Scholes PDE

Aims

- To show how a no-arbitrage 'replication portfolio' can be constructed from stocks and options which results in a *deterministic* partial differential equation (PDE) for the dynamics of an option price.
- Solving this PDE for a European option gives the Black–Scholes closed-form solutions for call and put premia.
- To show how European options can be priced by solving the Black–Sholes PDE numerically, using finite difference methods.

We show how a stochastic differential equation (SDE) for S and for the option premium $f(S,t)$, can be 'combined' to give a purely *deterministic* partial differential equation (PDE) for the option premium $f(S,t)$ – this is the 'Black–Scholes PDE'. This PDE can be solved (by various methods) to give an analytic solution for European option prices – this is the famous Black–Scholes–Merton closed-form solution for call (or put) premia.

Then we show how finite difference methods can be used to solve the PDE numerically and provide an estimate of call and put premia. After completing this chapter, the reader should have a good basic knowledge of the continuous time approach and feel confident in consulting more advanced texts in this area.

48.1 RISK-NEUTRAL VALUATION AND BLACK–SCHOLES PDE

In this section our aim is to show how a derivative's price can be replicated using stocks and bonds, which eliminates any uncertainty represented by the stochastic variable, $dz = \varepsilon\sqrt{dt}$.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

This results in a partial differential equation (PDE) for the price of the derivative which is *deterministic* and can be interpreted in terms of risk-neutral valuation. This Black–Scholes PDE can be solved analytically for European call and put premia.

48.1.1 Black–Scholes PDE

Below we outline the steps needed to derive and solve the Black–Scholes PDE for a European option (on a non-dividend paying stock) – further details are in Appendix 48. Assume S follows a GBM:

$$dS = \mu S\,dt + \sigma S\,dz \tag{48.1}$$

The price B of a risk-free (zero-coupon) bond with a constant interest rate follows the process:

$$dB = rB\,dt \tag{48.2}$$

The price of a derivative security with underlying asset S is denoted $f(S,t)$. Ito's lemma allows us to obtain a SDE for $f(S,t)$ which also depends on the uncertainty represented by dz. From Ito's lemma using $a = \mu S$ and $b = \sigma S$, the dynamics of the derivative's price is (see Chapter 47):

$$df = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(\mu S) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right]dt + \frac{\partial f}{\partial S}(\sigma S)dz \tag{48.3}$$

The SDE for the stock price and the derivatives price both depend on the same source of uncertainty, dz. Over a small interval of time, portfolio-A consisting of $N_S = (\partial f/\partial S) = \Delta$-stocks and holdings of bonds, can be made to replicate the payoff from the derivative security. The resulting equation (see Appendix 48) for the change in price of the derivative security does not depend on dz and is deterministic. This equation is known as the *Black–Scholes PDE*:

$$\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(rS) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right] = rf(S,t) \tag{48.4}$$

In Equation (48.4) the term in square brackets is similar to that in (48.3) but μ has been replaced by r. Therefore the solution to (48.4) for the option price does not depend on μ – this is risk-neutral valuation (RNV), which is a consequence of no-arbitrage possibilities. When a stock pays dividends at a rate δ the Black–Scholes PDE is:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(r-\delta)S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2 = rf(S,t) \tag{48.5}$$

The Black–Scholes PDE applies to *any* derivative security whose payoff depends on one underlying asset (and time). It therefore holds for plain vanilla European calls and puts, American options, and exotic options such as standard barrier options and lookback options. Sometimes the PDE (together with the boundary conditions for the option) results in a closed-form solution for the options price but if this is not possible, the PDE can be solved using finite difference methods.

Complex algebra is required to obtain the closed-form solution to (48.5) for the Black–Scholes call (or put) premium on a European option. Here we simply note that the solution to (48.5) requires specific boundary conditions. For example, for European calls and puts these are:

$$\text{Call: } f(0,t) = 0 \quad \text{and} \quad f(S,T) = \max(S_T - K, 0) \tag{48.6}$$

$$\text{Put: } f(\infty,t) = 0 \quad \text{and} \quad f(S,T) = \max(K - S_T, 0) \tag{48.7}$$

These boundary conditions when used with the PDE give the Black–Scholes closed-form solutions for European option prices:

$$C = S\ N(d_1) - K\ e^{-r(T-t)}\ N(d_2) \tag{48.8a}$$

$$P = Ke^{-r(T-t)}N(-d_2) - S\ N(-d_1) \tag{48.8b}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \qquad d_2 = d_1 - \sigma\sqrt{(T-t)}$$

There are several solution methods available to solve the Black–Scholes PDE. In one method you take the Black–Scholes PDE (48.4) and transform it into a simpler PDE known (in physics) as the 'heat equation'. (So called, because it explains the diffusion of heat through a metal bar with a heat source at one end.) The solution to the heat equation then gives the Black–Scholes formulas for European call and put premia.

Another method to solve for the option price involves integration and we outline the steps below to obtain the Black–Scholes call premium. Assume $\ln S_T \sim N(\eta, \sigma^2 T)$ where $\eta = \ln S + (r - \sigma^2/2)T$ and the density function for S_T is:

$$g(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi T}} \exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \eta}{\sigma\sqrt{T}}\right)^2\right]$$

The call premium under risk-neutral pricing can written in terms of conditional expectations:

$$C = e^{-rT}\hat{E}(S_T \,\mathrm{x}\, Ind) - K e^{-rT}\ prob(S_T > K) \tag{48.9}$$

where Ind is an indicator function, that takes the value 1 if $S_T > K$ and zero otherwise. The first and second terms in (48.9) can be written:

$$\frac{e^{-rT}\ S_T}{\sigma\sqrt{2\pi T}}\int_K^{\infty} g(S_T)\,dS_T \tag{48.10a}$$

$$\frac{1}{\sigma\sqrt{2\pi T}}\int_{\ln K}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{\ln S_T-\eta}{\sigma\sqrt{T}}\right)^2\right]\ d(\ln S_T) \tag{48.10b}$$

The integrals in (48.10) are not straightforward to evaluate but they result in the terms $S\ N(d_1)$ and $Ke^{-rT}N(d_2)$ which implies $C = S\ N(d_1) - Ke^{-rT}N(d_2)$. The route to obtaining the Black–Scholes option pricing formula was a rather tortuous one (Finance Blog 48.1) – see Sudaram and Das (2015).

Finance Blog 48.1 Particle Finance, PDE (Pretty Damn Easy?)

The mathematics used to explain option prices has led to the subject area being dubbed 'particle finance' because much of the continuous time mathematics was initially used to explain physical phenomena. It all started in 1828 when *Mr Robert Brown* published a paper on his observations of pollen grains suspended in water. He noticed that the pollen grains were in continuous random motion. The puzzle was that the individual random molecules in the water seemed too small, to move 'the large' grains of pollen. The explanation lay in the mathematics of what later became known as Brownian motion.

In a Brownian motion, the speed of 'molecules' is normally distributed and hence symmetric. One might expect the pollen grains to remain in the same position as they are 'hit' with equal probability from the right then the left. However, if a set of unusually high velocity molecules (i.e. from the tail of the normal distribution) hit the particle, the pollen will move in one direction. Since the motion of the molecules is normally distributed, the next few 'hits' are likely to be small and random so the pollen stays around its *new* position.

Around 1900, it was a PhD student of the French mathematician Poincaré called *Louis Bachelier*, who first used Brownian motion to explain random movements in stock prices. Mathematically, the difficulty with Brownian motion is that the path of the pollen grains is described by a zigzag path which is *discontinuous* (even as $dt \to 0$). In contrast, normal calculus assumes the function to be differentiated, is everywhere continuous.

It was *Einstein* in 1905 who eventually unravelled the mathematics of the problem of why the pollen grains move randomly. This 'diffusion process' as it became known, can also explain why smoke particles from a cigarette gradually fan out in all directions (i.e. positive drift). Hence it provides an explanation of why 'smoke gets in your eyes' even if you are standing at the other end of a long room from a smoker.

It was *Kiyoshi Ito* in 1951 who took the mathematics of diffusion processes further, since he established the 'rules of differentiation' for stochastic variables. This enabled Black, Scholes, and Merton in the early 1970s to derive the SDE for an option, given the underlying asset (e.g. stock price) followed a Brownian motion. It was then possible to combine the SDE for the option and the SDE for the underlying asset, to create a risk-free portfolio which resulted in a *deterministic* PDE for the option premium. Given the PDE for the option price, there was a set of standard solution methods already in use in the physical sciences (i.e. 'heat equation'), to provide a closed-form solution for European option premia.

Early studies of the behaviour of 'particles' by physicists considerably helped in solving the option pricing problem. Other mathematical theorems from the physical sciences have also been used in continuous time finance. For example, the *Feynman-Kac* formula, due in part to the Nobel prize winning physicist Richard Feynman, and also the mathematical theorem of *Girsanov,* are both used in solving option pricing problems.

Source: Adapted from Cuthbertson and Nitzsche (2001).

Although Black–Scholes was a major breakthrough, it is not always the case that a closed-form solution for the PDE is possible. What happens then? First, it is often possible to obtain a *numerical* solution to the Black–Scholes PDE (e.g. using finite difference methods). Alternatively we can use other numerical methods such as Monte Carlo simulation or the BOPM (under RNV) to price an option.

48.1.2 Does a Forward Contract Obey the Black–Scholes PDE?

The price of *any* derivative security $f(S, t)$ should satisfy the Black–Scholes PDE. We can illustrate this by considering a forward contract on a (non-dividend paying) stock. Using a no-arbitrage approach, we established that the *value* of a forward contract f at time t is:

$$f = S - Ke^{-r(T-t)} \tag{48.11}$$

where K = delivery price (= price F_0 established at $t = 0$) and $T - t$ = time to maturity. Does the above expression satisfy the Black–Scholes PDE? We have:

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)} \qquad \frac{\partial f}{\partial S} = 1 \qquad \frac{\partial^2 f}{\partial S^2} = 0 \tag{48.12}$$

Substituting the above equations in the PDE (48.4):

$$r\ [-Ke^{-r(T-t)} + S] = rf \tag{48.13}$$

And using (48.11) in (48.13) we see that forward price satisfies the Black–Scholes PDE.[1]

[1] Also note that it is trivial to show that a zero coupon bond (with face value of $1) and $f(S, t) = e^{-r(T-t)}$ satisfies the Black–Scholes PDE. This is left as a simple exercise for the reader.

48.2 FINITE DIFFERENCE METHODS

Finite difference methods provide an approximation to the continuous time PDE for the derivatives price, which can then be solved numerically. Consider the premium $f(S,t)$ on a European put option on a (dividend paying) stock. The Black–Scholes PDE is:

$$\frac{\partial f}{\partial t} + (r-\delta)S\ \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - r\ f = 0 \qquad (48.14)$$

The *explicit finite difference method* approximates the above PDE and gives a set of equations that can be solved numerically for the option premium f. To solve (48.14) we require some initial known values on the edges of the grid and these are given by the boundary conditions for the value of the put (e.g. the payoff at maturity, or when the stock price is zero). To illustrate the method consider the grid in Figure 48.1 which has time differences of Δt along the horizontal axis and stock price changes of ΔS on the vertical axis.

Each node on the grid will come to represent the option premium $f(S,t) = f_{i,j}$ where the index-i represents a particular time (x-axis) and the index-j represents the value of S (y-axis), hence:

$$f_{i,j} = f[i(\Delta t), \quad j(\Delta S)] \qquad (48.15)$$

The life of the option is T years. The time axis is divided into N equally spaced intervals of length $\Delta t = T/N$, so there are $N+1$ time periods:

$$0, \quad \Delta t, \quad 2\Delta t, \quad 3\Delta t, \quad \quad \ldots. \quad T$$

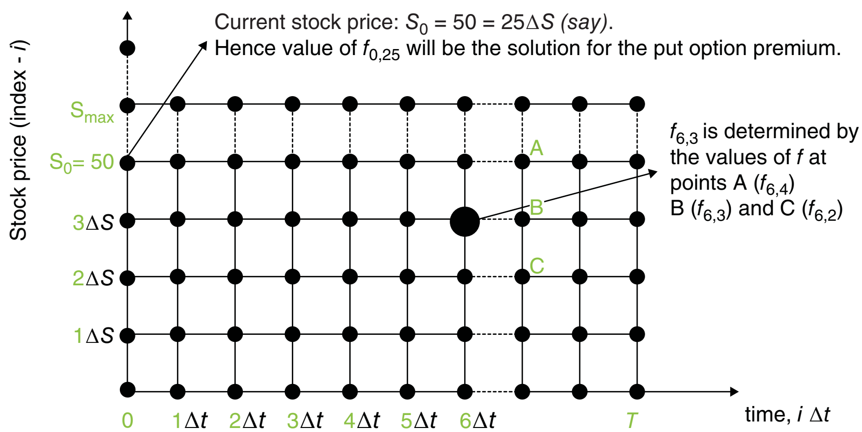

FIGURE 48.1 Finite difference

We now set the maximum value for S on the grid so that the put (with a strike of K) will have virtually zero value (this is an upper boundary for S) – for example, if $K = 50$ then an upper boundary might be $S_{max} = 100$. Now consider M equally spaced time intervals for S so that $\Delta S = S_{max}/M$ so the $M + 1$ stock prices in the grid are:

$$0, \quad \Delta S, \quad 2\Delta S, \quad 3\Delta S, \qquad \ldots . \quad S_{max}$$

One of the nodes on the (left) vertical axis of Figure 48.1 will coincide with the *current* stock price $S_0 = 50$ (say) and the solved value for $f_{i,j}$ at this *same* node will be the option premium using the finite difference method. Now we are in a position to provide an approximation to the (differential) terms in the PDE (48.14). In the middle of the lattice the derivative $\partial f/\partial S$ can be approximated in three different ways (see Figure 48.2).

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S} \quad \text{forward difference} \tag{48.16a}$$

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \quad \text{backward difference} \tag{48.16b}$$

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\ \Delta S} \quad \text{central difference} \tag{48.16c}$$

The other derivatives are (Figure 48.3):

$$\frac{\partial^2 f}{\partial S^2} = \left[\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S}\right] \Big/ \Delta S = \frac{f_{i,j+1} - 2f_{i,j-1} + f_{i,j}}{(\Delta S)^2} \tag{48.17a}$$

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{48.17b}$$

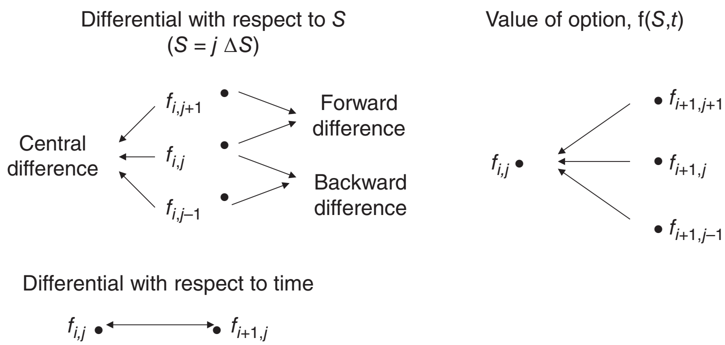

FIGURE 48.2 Use of grid points

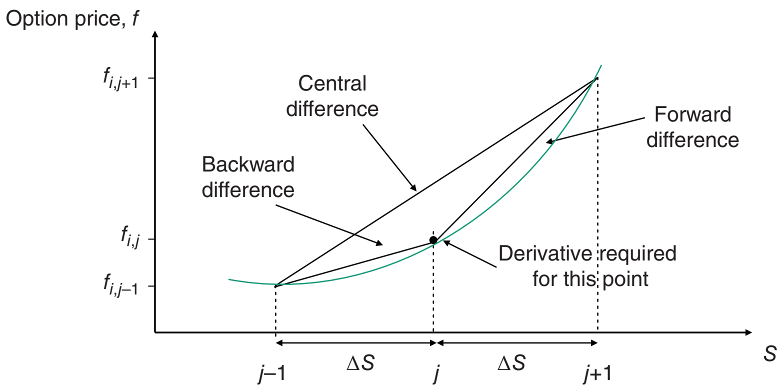

FIGURE 48.3 Approximations for $\partial f/\partial S$

It is only the derivative $\partial f/\partial t$ that involves time. The explicit finite difference method simplifies the solution method by assuming that $\partial f/\partial S$ and $\partial^2 f/\partial S^2$ at the point (i,j) on the grid are the same as at point $(i+1,j)$ so the central difference equation [16c] and equation [17a] become:

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\ \Delta S} \tag{48.18a}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} - 2f_{i+1,j-1} + f_{i+1,j}}{(\Delta S)^2} \tag{48.18b}$$

Substituting (48.17b), (48.18a), and (48.18b) in the Black–Scholes PDE (48.14) and rearranging:

$$f_{i,j} = A_j f_{i+1,j-1} + B_j f_{i+1,j} + C_j f_{i+1,j+1} \tag{48.19}$$

$$A_j = \frac{1}{(1+r\Delta t)}[-(1/2)(r-\delta)j\Delta t + (1/2)\sigma^2 j^2 \Delta t], \tag{48.20a}$$

$$B_j = \frac{1}{(1+r\Delta t)}[1-\sigma^2 j^2 \Delta t], \tag{48.20b}$$

$$C_j = \frac{1}{(1+r\Delta t)}[(1/2)(r-\delta)j\Delta t + (1/2)\sigma^2 j^2 \Delta t] \tag{48.20c}$$

From (48.19) we can calculate the value of $f_{i,j}$ once we know the values of f for the three nodes at time $(i+1)$ – Figure 48.2. Hence once we have the terminal conditions for the option, we solve for $f_{i,j}$ by working backwards through the grid (i.e. similar to solving the BOPM recursively). This is known as the *explicit finite difference method*. We know the value of the put for all nodes along the right boundary is $\max(K - S_T, 0)$, the value along the bottom boundary ($S = 0$) is K and along the top boundary (i.e. $S_{\max} = 100$) the value of the put is zero.

Value of put at time T

$$f_{N,j} = \max(K - j\Delta S, 0) \qquad j = 0, 1, 2, \ldots, M \tag{48.21a}$$

Value of put when $S = 0$

$$f_{i,0} = K \qquad i = 0, 1, 2, \ldots, N \tag{48.21b}$$

Value of put when $S \to \infty (S = S_{\max})$

$$f_{i,M} = 0 \qquad i = 0, 1, 2, \ldots, N \tag{48.21c}$$

Equation (48.19) can then be used to determine the option premium $f_{i,j} = f(i\Delta t, j\Delta S)$ at *all points along the left hand edge* of the grid, by working backwards from T. The value of $f_{i,j}$ which corresponds to the same node as the current stock price is the solution for the option premium.

The main problem with the explicit finite difference method is that it may not converge. There are a wide variety of other numerical methods available (e.g. implicit finite difference method, Crank-Nicholson, Hopscotch etc.) which can overcome such problems (see Hull 2018).

Pricing an American put option is easily incorporated into the explicit finite difference method. Each time we calculate the grid value $f_{i,j}$ we check to see if $f_{i,j} < K - S = K - j\ \Delta S$. If $f_{i,j} < K - S$, then we replace $f_{i,j}$ at that node with $K - S$. We then repeat this procedure for all the nodes.

We could also use finite difference methods based on an approximation to the differential equation for $d(\ln S)$ (rather than for dS) and this turns out to be slightly easier computationally. Although finite difference methods can be used for valuing American and European options they are more difficult to use when handling path-dependent options (e.g. where the payoff from the option depends on the past history of the underlying asset price). We then have to move to a 'higher order' problem since we have another state variable (e.g. the average value of the underlying for an Asian option). We therefore have to index the option value $f_{i,j}^k$ where $k =$ new state variable. It is also the case that some finite difference methods are unstable and sensitive to rounding errors in the computational procedure. Because of these difficulties it is worth noting that analytic *approximations* for some options' prices are often available (e.g. for American options, Asian average price options).

48.3 SUMMARY

- The stochastic process for the underlying stock S and for the option premium $f(S, t)$ have a common stochastic term dz, which can be eliminated by using delta hedging. This results in the *deterministic* Black–Scholes PDE for the option premium. The PDE depends on the risk-free rate but is independent of the actual growth rate of the stock price, μ – this is risk-neutral valuation.
- In some cases the Black–Scholes PDE can be solved to give a closed-form solution for the option premium (e.g. for several types of European calls and puts).
- The Black–Scholes PDE can often be solved using numerical methods (e.g. finite difference methods). The procedure is similar to that used in the BOPM in that the PDE is initially 'anchored' by the boundary conditions implicit in the option contract and then the PDE is solved recursively.
- Generally speaking, numerical techniques to solve the Black–Scholes PDE for many European options is usually accurate and speedy, even if the model involves more than three variables or 'dimensions' (e.g. time and two underlying stochastic variables S_1 and S_2). When the dimension of the problem is greater than three, then MCS tends to be a more efficient method of pricing a European option.

APPENDIX 48: DERIVATION OF BLACK–SCHOLES PDE

We derive the Black–Scholes PDE for the derivative security $f(S, t)$ (on a stock which pays no dividends), using two alternative replication portfolios, over short intervals of time. The analysis closely follows that used in earlier chapters to price options using the BOPM. Here we simply repeat that analysis in a continuous time framework.

Case A: Replication using stocks and (zero coupon) bonds

We replicate the price dynamics of the derivative security (e.g. call or put) using stocks and zero-coupon bonds (risk-free asset). Assume the stock price follows a GBM and the bond price is deterministic:

$$dS = \mu S dt + \sigma S dz \tag{48.A.1}$$

$$dB = r \ B \ dt \tag{48.A.2}$$

From Ito's lemma, the price of the derivative security $f(S, t)$ follows the SDE:

$$df = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(\mu S) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right] dt + \frac{\partial f}{\partial S}(\sigma S) dz \tag{48.A.3}$$

In our replication portfolio we hold N_S stocks and N_B bonds. The value of the replication portfolio V is set equal to the value of the derivative security (at any time t):

$$V = N_S S + N_B B = f(S, t) \tag{48.A.4}$$

Hence the number of bonds held is:

$$N_B = \frac{1}{B}\,[f(S, t) - N_S S] \tag{48.A.5}$$

The instantaneous change in the value of the replication portfolio is:

$$dV = N_S dS + N_B dB \tag{48.A.6}$$

Substituting from (48.A.1) and (48.A.2) in (48.A.6) gives:

$$dV = N_S(\mu S\,dt + \sigma S\,dz) + N_B r \;\; B \;\; dt \tag{48.A.7}$$

We require the change in value of the replication portfolio to equal the change in value of the derivative itself:

$$dV = df \tag{48.A.8}$$

Equating the coefficients on dz in Equation (48.A.7) and (48.A.3) implies:

$$N_S = \partial f/\partial S \equiv \Delta \tag{48.A.9}$$

Thus the number of stocks to hold at time t equals the option's delta, $\Delta = \partial f/\partial S$. From Equations (48.A.5) and (48.A.9):

$$N_B = \frac{1}{B}\left[f(S, t) - \frac{\partial f}{\partial S} S\right] \tag{48.A.10}$$

It remains to equate the coefficients on dt. Substituting for N_B in (48.A.7), using $N_S = \partial f/\partial S$ and then equating with (48.A.3) for terms in dt gives:

$$\frac{\partial f}{\partial S}\mu \;\; S + \left[\frac{1}{B}f(S, t) - \frac{\partial f}{\partial S}S\right] rB = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu \;\; S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right] \tag{48.A.11}$$

Rearranging (48.A.11) we have the Black–Scholes PDE for the derivative security $f(S, t)$ (on a stock which pays no dividends):

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2 = rf(S, t) \tag{48.A.12a}$$

The PDE equation (48.A.12a) is deterministic and independent of μ the growth rate of the stock price, so the option price does not depend on μ. For a stock which pays dividends, the growth of the stock price (in a risk-neutral world) is not equal to r but to $r-\delta$, where δ is the growth rate of dividends, hence the PDE becomes:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(r-\delta)S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2 = rf(S,t) \tag{48.A.12b}$$

Case B: Risk-free Portfolio-B (Stock Pays No Dividends)

The Black–Scholes PDE can be derived in a simpler fashion if we assume we already know how to construct a risk-free portfolio. Assume the risk-free Portfolio-B consists of a long position in N_S stocks (where $N_S = \partial f/\partial S = \Delta$) and a short position in the derivative security ($N_f = -1$) with current price f. The value of portfolio-B is:

$$V = -f + \left(\frac{\partial f}{\partial S}\right)S \tag{48.A.13}$$

$$dV = -df + \frac{\partial f}{\partial S}dS \tag{48.A.14}$$

Substitute in (48.A.14) for dS from Equation (48.A.1) and for df from Ito's equation (48.A.3). Crucially, the term in dz cancels out and we are left with:

$$dV = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right]dt \tag{48.A.15}$$

The change in value of portfolio-B is deterministic so risk-free arbitrage profits are possible unless portfolio-B earns the risk-free rate:

$$dV = V r\,dt \tag{48.A.16}$$

Substituting from Equation (48.A.15) and (48.A.13) in (48.A.16):

$$\left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right]dt = \left[-f + \frac{\partial f}{\partial S}S\right]r\,dt \tag{48.A.17}$$

A simple rearrangement of (48.A.17) gives the Black–Scholes PDE, Equation (48.A.12).

Dividend Paying Stock

If we hold $N_s = \partial f/\partial S$ stocks which pay a continuous dividend at the (proportionate) rate δ, then the (dollar) dividend paid out over time interval dt is $\delta(\partial f/\partial S)S\,dt$, which needs to be added to the right hand side of (48.A.15). The rest of the analysis is unchanged and the Black–Scholes PDE for an option on a dividend paying stock is:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(r - \delta)S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2 = rf(S, t) \tag{48.A.18}$$

EXERCISES

Question 1

Use Ito's lemma to shown that if dS/S follows a GBM with drift rate μ and variance, σ then the futures price $F = f(S, t) = Se^{r(T-t)}$ follows a GBM with drift rate $\mu - r$ (where r is the risk-free rate) and variance, σ^2.

Question 2

If the change in the stock price follows a GBM, $dS = \mu S dt + \sigma S dz$, use Ito's lemma to show that $f = S^2$ follows $df = (2\mu + \sigma^2)f \;\; dt + 2\sigma f \;\; dz$.

Question 3

What assumptions are made in moving from an SDE for the option price to the Black–Scholes PDE?

Question 4

What is a partial differential equation (PDE)? What is the key difference between a SDE and a PDE?

Question 5

What are the key strengths and weaknesses in calculating European option premia using finite difference methods? What role is played by boundary conditions?

CHAPTER 49

Equilibrium Models: Term Structure

Aims

- To show how bond prices and option prices can be derived from a model of the short-rate, using risk-neutral valuation.
- To investigate the properties of different time-series models of the short-rate.
- To demonstrate how continuous time models of the term structure can be used to price bonds and options on bonds.

Black's model for pricing fixed income derivatives such as European options on bonds, on bond futures, as well as caps, floors, and swaptions, assumes the underlying variable is lognormal at expiration of the option. Black's model cannot price derivatives which depend on the evolution of interest rates through time – such as American style options, callable bonds and other path-dependent options.

The *equilibrium yield curve approach* assumes a specific continuous time stochastic process for the one period (short) rate. The parameters of this process are then estimated from historical data. Bond prices and fixed income derivatives prices can then be derived mathematically and these prices depend directly on the estimated parameters of the stochastic process for the short-rate. Hence, in the equilibrium approach the current term structure is an *output* from the short-rate model. (In contrast, in the no-arbitrage BOPM applied to fixed-income assets the current term structure is used as an *input* to calibrate the lattice for the short-rate.) The sequence of steps in the equilibrium model approach are:

- Choose a continuous time formulation for the short-rate and estimate the unknown parameters of this process, from historical data.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

- Mathematically, derive the whole term structure from the chosen short-rate model.
- Use continuous-time mathematics to establish a PDE for the price of the fixed income derivative and solve the PDE to give a (closed-form) solution for the derivative's price (which then depends on the estimated parameters of the short-rate model).

It is worth noting at the outset that if the estimated time series model of the short-rate does not closely 'fit' the current observed behavior of interest rates, then the closed-form solution for the price of the derivative may not be accurate. However, it is not easy to empirically uncover what is the best statistical representation of the short-rate. This is a major practical drawback of this approach. However, we can still take a no-arbitrage approach, based on continuous time models for the short-rate, if we adapt these short-rate equations to fit today's term structure. This is sometimes possible by making the drift rate in the short-rate equation a function of time (e.g. Ho and Lee 1986 and Hull and White 1990). The mathematics of equilibrium models quickly becomes complex so we only present a heuristic overview of the approach.

49.1 RISK-NEUTRAL VALUATION

The instantaneous short-rate $r(t)$ is the rate which applies over an infinitesimally short period of time (i.e. t to $t + dt$ as $dt \to 0$). Derivatives prices depend on the process for $r(t)$ in a risk-neutral world, so that if a derivative has a payoff V_T at time T then the current value V_t of the derivative is:

$$V_t = E^*[e^{-r_{av}(T-t)} V_T] \tag{49.1}$$

where E^* is the expected value in a risk-neutral world, r_{av} is the average value of the short-rate (between t and T). We can use Equation 49.1 to price bonds and other fixed income assets (which depend on interest rates). For example, to value a zero-coupon bond which pays $V_T = \$1$ at T, we have:

$$P(t, T) = E^* \, e^{-r_{av}(T-t)} \tag{49.2}$$

If $R(t, T)$ is the observed continuously compounded interest rate on this 'zero' then it is known at time t, hence:

$$P(t, T) = e^{-R(t,T)(T-t)} \tag{49.3}$$

It follows that:

$$R(t, T) = - \frac{1}{(T - t)} \ln P(t, T) \tag{49.4a}$$

$$R(t, T) = - \frac{1}{(T - t)} \ln[E^* e^{-r_{av}(T-t)}] \tag{49.4b}$$

If we know the stochastic process for the short-rate r in a risk-neutral world, then we can use (49.4b) to determine all the (continuously compounded) zero-coupon rates $R(t, T)$ – that is, the complete term structure. First, we examine alternative plausible models for the instantaneous short-rate rate.

49.2 MODELS OF THE SHORT-RATE

The risk-neutral process for the instantaneous short-rate $r(t)$ is usually an Ito process:

$$dr(t) = \mu(r,t)\ dt + \sigma(r,t)\ dz \tag{49.5}$$

where dz is a Wiener process. We know today's short-rate, $r(0)$. Note that (49.5) is a 'one-factor model' since there is only one source of uncertainty, $dz = \varepsilon\sqrt{dt}$. The crucial issue is how to choose a simplified tractable version of (49.5) which adequately represents the actual stochastic behaviour of the short-rate. Whether or not the short-rate exhibits (very slow) mean reversion or is non-stationary is a contested empirical issue (particularly since the advent of the literature on cointegration – see Cuthbertson and Nitzsche 2004). Hence, models of the short-rate both with and without mean reversion are used. Some alternatives are given below.

49.2.1 Rendleman–Bartter (1980)

Here μ and σ in Equation (49.5) are assumed to be constant, hence r follows a GBM. But a GBM may give rise to negative values for the short-rate – so this model is somewhat unrealistic since observed nominal rates are hardly ever negative.[1]

49.2.2 Ho–Lee (1986)

The Ho–Lee model has no mean-reversion but allows the drift rate to depend on t:

$$dr = \theta(t)dt + \sigma dz \tag{49.6}$$

The function $\theta(t)$ is chosen so that the resulting forward rate curve fits the empirically observed term structure of forward rates. The function $\theta(t)$ is the drift and represents the average direction in which the short-rate moves over time (and is independent of the level of r so there is no mean-reversion). It can be shown that (approximately) $\theta(t) = F_t(0, t)$ where $F_t(0, t) = \partial F(0, t)/\partial t$ and $F(0, t)$ is the instantaneous forward rate. Hence *the slope* of the forward curve determines the direction the short-rate moves at time t.

[1] In the low interest rate environment since 2010, some short-rates are negative (e.g. in Switzerland, Germany and Japan).

With the Ho–Lee model, the price of zero-coupon bonds and European options on zero-coupon bonds can be determined analytically – see below. The Ho–Lee model can result in negative values for $r(t)$ and in this respect the Hull–White model is an improvement.

49.2.3 Hull–White (1990)

In contrast to the Ho–Lee model, the Hull–White model exhibits mean-reversion:

$$dr = [\theta(t) - a\ \ r]dt + \sigma dz = a[(\theta(t)/a) - r]dt + \sigma dz \tag{49.7}$$

It is the term in square brackets that produces mean-reversion – the short-rate r is pulled towards its equilibrium value $r^{eqm} = \theta(t)/a$. Thus if there were no stochastic shocks r would move towards its long-run value of $\theta(t)/a$, either from below if $\theta(t) > a\ \ r$ or from above if $\theta(t) < a\ \ r$. The parameter $\theta(t)$ is determined by the initial term structure and it can be shown that:

$$\theta(t) = F_t(0,t) + a\,F(0,t) + (\sigma^2/2a)(1 - e^{-2\,a\,t})$$

If we ignore the final term (which is small) then substituting $\theta(t)$ in 49.7, the drift of the short-rate at time t is $F_t(0,t) + a[F(0,t) - r]$. Hence, the path of the short-rate is determined by the slope of the initial forward rate curve, plus a mean-reverting component determined by the parameter a.

49.2.4 Black–Derman–Toy (1990)

This is the same as the Ho–Lee equation but with $d(\ln r)$ as the dependent variable and there is no mean-reversion:

$$d(\ln r) = \theta(t)dt + \sigma dz \tag{49.8}$$

Using Ito's lemma the equation for dr is:

$$dr = [\theta(t) + (1/2)\sigma^2]dt + \sigma dz \tag{49.9}$$

49.2.5 Black–Karasinski (1991)

This approach takes the Black–Derman–Toy model and adds mean-reversion. If there are no stochastic shocks, $\ln r$ approaches its equilibrium value $\ln r = \theta(t)/a$.

$$d(\ln r) = (\theta(t) - a\ln r) + \sigma dz \tag{49.10}$$

The advantage of this model (unlike Ho–Lee and Hull–White) is that interest rates cannot be negative, but its weakness is the difficulty in obtaining closed-form solutions for fixed income assets, even for simple cases such as the price of a zero-coupon bond.

49.2.6 Vasicek (1977)

This model has mean-reversion with a fixed value for the equilibrium level of interest rates of $r^{eqm} = b$. The short-rate is assumed to be normally distributed and hence it is possible for r to be negative:

$$dr = a[b - r]dt + \sigma dz \tag{49.11}$$

49.2.7 Cox–Ingersoll–Ross (CIR 1985)

This has a fixed equilibrium value for the short-rate $r^{eqm} = b$ (as in the Vasicek model) and the standard deviation of the short-rate is proportional to $\sqrt{r}$. Hence as $r \to 0$, the standard deviation approaches zero and short-rates are always non-negative. The risk-neutral process for r is:

$$dr = a[b - r]\,dt + \sigma\sqrt{r}\,dz \tag{49.12}$$

In the 'equilibrium model approach', the parameters in the above equations are estimated from time series data. For example, in the CIR model, econometric estimates of a, b and σ are required.

49.3 PRICING USING CONTINUOUS TIME MODELS

The above equations are one-factor models of the short-rate, as there is only one source of uncertainty. It is usually mathematically fairly difficult (although not impossible) to obtain a closed-form solution for a derivative security, when the underlying short-rate is governed by one of the above dynamic equations. We therefore consider a few simple cases which have analytic solutions but as the reader should note, these are largely for illustrative purposes only.

All the solutions take the short-rate equation as given and this leads to a PDE for the derivative's price. The solution to this PDE then gives the (closed-form) solution for the price of the derivative. The price of the derivative depends on the estimates of the parameters of the short-rate equation and the current value of the underlying asset (e.g. either the current short-rate or the zero coupon bond price), as well as the time to maturity $T - t$ of the derivative security. When closed-form solutions are not possible, the PDE can be solved for the derivatives price using numerical methods (e.g. finite-difference methods).

49.3.1 Black–Scholes

Suppose we have a fixed income derivative security with price $f(r, t)$ and the derivative has no payments until the terminal date T (e.g. European style option). It can be shown that if $r(t)$ follows an Ito process (49.5), then the Black–Scholes PDE is (see Chapter 48) is:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2 - rf = 0 \tag{49.13}$$

where μ and σ may be functions of r and t. The payoff for the derivative at maturity T is assumed known – for example, for a call $f_T = \max(0, V_T - K)$ where V_T = price of the underlying asset.

49.3.1.1 Solution 1: Zero-coupon Bond, Constant Short-rate

We price a zero-coupon bond using the Black–Scholes PDE, under the assumption of either constant or stochastic interest rates. A constant short-rate implies $\partial f/\partial r = 0$ and $\sigma^2 = 0$, so (49.13) becomes:

$$\frac{\partial f}{\partial t} - r\ f = 0 \ \text{ or } \ d(\ln f) = r\ \ dt \tag{49.14}$$

This has a solution:

$$\ln f(t) = rt + k \tag{49.15}$$

where k is an arbitrary constant. The boundary condition is $f(T) = M$, the maturity value of the bond, which using (49.15) gives:

$$\ln f(T) = rT + k = \ln M \tag{49.16}$$

hence $k = \ln M - r\ \ T$. Substituting in Equation (49.15) for k gives:

$$\ln f(t, T) = -r\ \ (T - t) + \ln M \tag{49.17a}$$

Therefore the price of a zero coupon bond with face value M and maturity $T - t$, when the interest rate is constant is:

$$f(t, T) = Me^{-r(T-t)} \tag{49.17b}$$

49.3.1.2 Solution 2: Zero-coupon Bond, Stochastic Interest Rates

Suppose the short-rate follows the Ho–Lee model:

$$dr = a\ \ dt + \sigma dz \tag{49.18}$$

where a and σ are estimated parameters. First we *assume* a solution for the price of the zero coupon bond $f(t, T)$ (with maturity value \$1) of the form:

$$f(t, T) = k(t, T)e^{-r(T-t)} \tag{49.19}$$

Hence: $\frac{\partial f}{\partial r} = -k(T - t)e^{-r(T-t)}$, $\frac{\partial^2 f}{\partial r^2} = +k(T - t)^2 e^{-r(T-t)}$, $\frac{\partial f}{\partial t} = k\,r\,e^{-r(T-t)} + \frac{\partial k(t,T)}{\partial t} e^{-r(T-t)}$

Substitution in the Black–Scholes PDE, Equation (49.13), with $\mu = a$ gives:

$$\frac{\partial k(t,T)}{\partial t} = (T-t)\ \ k\ \ a - \frac{1}{2}(T-t)^2\ \ \sigma^2 k \tag{49.20a}$$

$$\frac{d\ln k}{dt} = \left[(T-t)\ \ a - \frac{1}{2}(T-t)^2\sigma^2\right] \tag{49.20b}$$

We now integrate (49.20b) over (t, T) and incorporate the boundary condition, $k(T,T) = \$1$ to give:

$$\ln k(t,T) = -(1/2)\ \ (T-t)^2 a + (1/6)(T-t)^3\sigma^2 \tag{49.21}$$

When (49.21) is substituted in (49.19) we have an explicit expression for the price $f(r,t)$ of a zero-coupon bond which depends on the time to maturity $T-t$ and the parameters of the short-rate process. The price of the zero-coupon bond will be consistent with no arbitrage along the yield curve (given the assumed Ho–Lee process for the short-rate rate). Substituting for $f(r,t)$ from (49.19) in (49.4b):

$$R(t,T) = \frac{1}{(T-t)}\ln f(t,T) = \frac{-1}{(T-t)}\ln k(t,T) + r(t) \tag{49.22}$$

Hence, in the Ho–Lee model an instantaneous change in the short-rate rate leads to *an equal change* in all long rates, that is, a parallel shift in the yield curve. This is a rather restrictive term structure model which could be made more flexible (and realistic) by using an alternative stochastic process for the short-rate rate, as illustrated below.

49.4 BOND PRICES AND DERIVATIVE PRICES

The Vasicek model incorporates mean reversion and it can be shown that the price of a zero-coupon bond is:

$$P(t,T) = k(t,T)e^{-B(t,T)\,r(t)} \tag{49.23}$$

where k and B depend on the parameters of the short-rate equation:

$$k(t,T) = \exp\left[\frac{[B(t,T) - T + t](a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right] \tag{49.24a}$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{49.24b}$$

These expressions for k and B are more complex than those for the Ho–Lee model and they allow the yield curve (at time t) to be either (smoothly) upward or downward sloping or to exhibit a 'humped shape'. The expression for long-rates is:

$$R(t,T) = -\frac{1}{(T-t)}\ln P(t,T) = \frac{1}{(T-t)}\ln k(t,T) + \frac{1}{(T-t)}B(t,T).r(t) \tag{49.25}$$

It is clear from the coefficient on $r(t)$ in the above equation that when $r(t)$ changes by 1%, long-rates do not all move by the same amount as the short-rate, so shifts in the yield curve are not constrained to be parallel.

49.4.1 Call Option on a Zero-coupon Bond

It requires some complex continuous time mathematics to derive the closed-form solution for an interest rate *derivative security* based on the Vasicek, Ho–Lee, and Hull–White models. But Jamshidian (1989) has shown how a European call or put on a zero-coupon bond (or a coupon paying bond) can be priced. For example, for a call option on a zero-coupon bond that matures at time s, the call premium has a closed-form solution of the form :

$$Call(0,T) = Q\ \ P(0,s)\ \ N(h) - K\ \ P(0,T)\ \ N(h-\phi) \tag{49.26}$$

where Q = (principal) value of the bond, K = strike price of the bond, $P(0,s)$ is the price of a deliverable bond with maturity value of \$1 at any time s (including $s = T$), $N(.)$ is the cumulative normal distribution and T is the maturity of the option. The terms h and ϕ are complex expressions which depend on (K, T, t, Q) and the price of bonds of maturity s and T that is, $P(0,s)$ and $P(0,T)$ – as well as the parameters of the short-rate equation. The price of a put option is:

$$Put(0,T) = K\ \ P(0,T)\ \ N(-h+\phi) - Q\ \ P(0,s)\ \ N(-h) \tag{49.27}$$

49.4.2 Option on Coupon Paying Bond

The CIR and Hull–White short-rate models can also be used to price European call and put options on zero-coupon bonds and can be adapted to price options on coupon paying bonds (Jamshidian 1989). For all one-factor models of the short-rate, zero-coupon bond prices of all maturities move up and down in unison as the short-rate changes. This allows European options on coupon-bonds to be expressed as the sum of the prices of options on zero-coupon bonds. A European swaption can be viewed as an option on a coupon paying bond and hence can be valued using one-factor models of the short-rate.

A further development is to assume the short-rate is influenced by two stochastic factors, for example by assuming that σ is also stochastic. Closed-form solutions for bond prices and interest rate derivatives can often be derived based on two-factor models (Longstaff-Schwartz 1992, Hull and White 1994).

49.4.3 Hedging Using One-factor Models

Although one-factor models are usually accurate enough for pricing bonds and some European options, we cannot rely on accurate hedging of interest rate options, since one-factor models usually assume restrictive shifts in the yield curve, which are not realistic. The price of fixed-income derivatives can change substantially due to non-parallel shifts in the yield curve and due to changes in the term structure of volatility.[2]

As noted in an earlier chapter, there is not just one number for the delta, gamma, and vega of a fixed income derivative – because of different changes in spot rates and volatilities, along the yield curve. To put in place a hedge you first have to decide what type of shift in the yield curve you want to hedge against (e.g. a parallel shift or a twist in the yield curve). You then calculate the appropriate delta and gamma by changing each of the spot rates by the required amount and recalculating the derivatives (option) price under this new scenario – this is the 'perturbation approach' to obtaining the Greeks.

For example, the '5-year delta' is the change in the value of your derivatives position for a small (1 bp) change in the 5-year spot yield. You can repeat this for spot rates at other maturities (e.g. 1 bp change in the 1-year spot rate) to give the derivatives portfolio deltas, for changes in several spot rates, along the yield curve. These changes in spot yields also imply changes in the forward curve and if the derivatives price depends on forward rates (e.g. caps and floors) these can also be incorporated in the calculation of delta.

Having obtained the deltas of your derivatives portfolio with respect to changes in several spot rates, you can then choose hedging instruments to offset these effects (e.g. using interest rate futures and other fixed-income options). The same procedure can be used to calculate the derivatives portfolio vega by changing volatilities along the yield curve, each by a small amount and calculating the change in the value of the derivative.

49.5 SUMMARY

- In the equilibrium model approach, the parameters of the chosen short-rate model are estimated. Continuous time mathematics is then used to derive a closed-form solution for bond prices and prices of fixed income derivatives (e.g. options on bonds, caps and floors etc.), which depend on the estimated parameters of the chosen short-rate equation.
- The main problem with the equilibrium approach is that small errors in estimating the short-rate parameters can lead to large errors in the price of the derivative security.

[2]It may seem strange to assume changes in the yield curve that are not consistent with the short-rate equation used to price the derivative. But this is what we do when using the Black–Scholes model for pricing stock options and then vega hedging – the pricing model assumes a constant volatility but vega-hedging assumes volatility changes over time. In short, all models are approximations and their usefulness depends on the problem at hand and the size of any ensuing errors.

- Different models for the short-rate give different outcomes for the response of long-rates to changes in the short-rate – this is the 'equilibrium' term structure equation. Some short-rate models are restrictive, allowing only parallel shifts in the yield curve.
- When hedging, if we assume non-parallel shifts in the yield curve then we need to calculate the portfolio delta (or gamma or vega) of the fixed-income derivative with respect to specific changes in spot rates (or volatilities) along the yield curve – usually using a 'perturbation approach'.
- The gamma and vega risk on a portfolio of fixed income options (on the same underlying) can be hedged using other fixed income options (e.g. caps, floors, options on T-bonds and on T-bond futures) with different maturities and strike prices. Finally, interest rate futures can be used to hedge any residual delta-risk with respect to changes in interest rates.

EXERCISES

Question 1
In the 'equilibrium yield curve' approach, explain why the *estimated* equation for the (one-period) short-rate is so important.

Question 2
In the equilibrium yield curve approach, derivatives are priced assuming a risk-neutral world. What is the key equation connecting a fixed income derivative's value today and the payoff to the derivative at maturity, T.

Question 3
Explain why you might not want to use a geometric Brownian motion, $dr(t) = \mu(r, t)dt + \sigma(r, t)dz$, as a realistic model of the short-rate.

Question 4
Write down the Vasicek equation for the short-rate and explain how mean reversion is incorporated in the equation.

Question 5
What advantages does the Cox–Ingersoll–Ross (CIR) model for the short-rate have over the Vasicek model?

Glossary

A

Abnormal profits The extra return over and above that which is compensation for the riskiness of an investment.

ABS (asset-backed security) Cash flows from assets (e.g. bonds, loans) which are bundled together and then sold as an ABS.

ABS-CDO Cash flows from several ABSs are combined, into tranches, which determine the order of payment.

Accrued interest Interest earned on a bond since the last coupon payment.

Active portfolio management Investment strategies which aim to achieve abnormal returns.

Agency costs Costs associated with monitoring contracts.

American option Option which may be exercised at any time up to the expiry date.

Anomaly Returns which cannot be explained by known asset pricing models.

Arbitrage (arbitrageur) An investment strategy that enables a risk-free profit to be made by an arbitrageur. Usually involves trading in two (or more) securities.

ARCH / GARCH model A model of time varying conditional volatility estimated using historical data.

Asian option Option whose payoff depends on the average price of the underlying asset over the life of the option.

Ask (offer) price The price at which a market maker/dealer offers to sell a security.

Asset General term for anything with economic value.

Asset allocation Decision on how to split your wealth into different asset classes, for instance stocks, bonds, and cash.

Asymmetric information Situation where one party has more information than another party.

At-the-money option An option with a strike price (or the present value of the strike price) equal to the current market price of the underlying security.

Average price option An option whose payoff depends on an average of the underlying asset price (relative to the strike price), over the life of the option.

Average strike option An option whose payoff depends on the final asset price relative to an average of the asset price, over the life of the option.

Derivatives: Theory and Practice, First Edition. Keith Cuthbertson, Dirk Nitzsche and Niall O'Sullivan.

B

Backtesting A method used to assess forecasting accuracy, particularly in value at risk (VaR).

Backwardation A condition in which the forward/futures price is below the current spot price. Also referred to as selling at a discount – see Contango.

Barrier option An option whose final payoff at maturity depends on whether the underlying asset has reached or crossed a predetermined value (the barrier).

Basis The difference between the spot price of the underlying asset and the futures price.

Basis point (bp) 1/100 of 1% (0.01 per cent, 0.001 as a decimal).

Basis risk The risk to a hedger associated with variation in the basis over time.

Basis swap A swap where cash flows are exchanged based on two different floating interest rates.

Basket credit default swap A credit default swap on several reference entities.

Bear market A market in which prices are falling.

Bear spread (with calls) Describes the payoff profile of selling a call with a low strike price and buying a call with a higher strike price. Both calls have the same maturity and underlying asset.

Bermudan option An option that can only be exercised on specific dates.

Best execution Duty of a broker-dealer to provide the lowest available price (when buying stock) or the highest available price (when selling stock) for customers.

Beta A measure of the responsiveness of a stock's return to changes in the market return – a measure of the systematic (market) risk of the stock.

Bid price The price a market maker/dealer pays to buy a security.

bid-ask (offer) spread Difference between the bid and ask (offer) price – see Bid price, Ask (offer) price.

Binary credit default swap A credit default swap (CDS) with a fixed dollar payoff if default occurs.

Black's model A model to price European options on futures contracts. Can be applied whenever the underlying asset price at maturity is lognormal.

Black–Scholes model A model for pricing European options.

Bond market Market where long-term cash-market fixed income securities are traded.

Bootstrapping Statistical technique to extract a 'representative' sample from a given data set. The sample is generated by repeated re-sampling of the original observations, usually with replacement. Alternatively, a procedure for calculating spot yields.

BOPM (Binomial Option Pricing Model) In each short period of time only two outcomes for the asset price are possible.

Broker Agents who find best buying and selling prices and also provide IT services, facilitate short-sales and margin finance. Usually they are large investment banks.

Brownian motion A specific stochastic process which describes the random path of a variable (e.g. stock price) over small intervals of time.

Bull market Market in which prices are rising.

Bull spread (with calls) Describes the payoff profile from buying a call with a low strike price and selling a call which has the same maturity but with a higher strike price.

Butterfly Payoff profile engineered by buying a low price call and a high price call and selling two calls with an intermediate strike price. (Can also use puts.)

C

Calendar spread A long–short position in two options with different times to maturity.

Call option A European option giving the holder the right (but not the obligation) to buy the underlying security on a specified (expiry) date at a (strike) price agreed at the outset of the contract. An American option can be exercised at any time on or before the expiry date.

Call premium Price of a call option.

Callable bond Bond which can be redeemed at a predetermined price by the company that issues it, before its maturity date.

Cap Set of interest rate caplets which have different maturity dates.

Capital Asset Pricing Model (CAPM) An equilibrium model which states that the expected excess return on a stock equals the stock's beta multiplied by the excess return on the market portfolio.

Caplet Call option, which pays off if the floating interest rate at expiry (e.g. LIBOR) exceeds the strike (interest) rate.

Cash flow mapping Process of converting actual cash flows at different dates into cash flows at standardized dates in order to simplify the calculation of value at risk.

Cash settlement A procedure applicable to certain futures and options contracts wherein a cash transfer is employed at contract settlement rather than the actual delivery of the asset underlying the derivatives contract.

Cash-and-carry arbitrage A (riskless) arbitrage trading strategy that determines forward/futures prices.

CDD (cooling degree days) The average of the highest and lowest temperatures (midnight to midnight) at a particular location if the temperature is greater than 65 °F. If the temperature is less than 65 °F, CDD is zero.

CDO (collateralised debt obligation) A combination of debt instruments which are tranched so that the order of cash payments are in the form of a 'waterfall', depending on the seniority of the holder of the CDO.

CDO-squared A security formed by splitting a particular tranche of a CDO into several other tranches, with different levels of seniority.

CDS, (credit default swap) The holder of a CDS, receives the face value of a reference bond, if the reference entity defaults.

CDS spread Amount that must be paid each year (bps) by the holder of a CDS.

Cheapest to-deliver (CTD) Refers to the cheapest bond which can be delivered by the short in a T-bond futures contract on the CME. The exchange denotes the set of bonds which are eligible for delivery.

Chooser option The holder has the right to choose whether the option should be a call or put, at some point during the option's life.

Clean price Quoted price of a bond which excludes accrued interest or rebate interest.

Clean price of a bond The clean price is the quoted price. The cash price ('dirty price') actually paid is the clean price plus accrued interest.

Clearing house A firm associated with an options or futures exchange, that guarantees contract performance and otherwise facilitates trading.

Closed-form solution A mathematical solution of the form $y = f(x, z)$.

Closing out Selling an asset you already hold or buying an asset you have previously shorted (sold).

Closing Price Price of the last trade on a particular day for a specific security.

CMO (collateralised mortgage obligation) A mortgage-backed security where the cash flows from the mortgages may be split between interest only payments and payments which reduce the principal of the mortgage.

Collar Holding stocks and buying a low strike put and selling a high strike call.

Collateral The value of any asset held against the possible fall in value of another asset.

Commercial paper A form of zero-coupon bond issued by companies to raise funds. Maturity 7 days to 2 years. Active secondary market.

Commodity Futures Trading Commission The US regulator for trading futures contracts.

Commodity swap A swap where the two cash flows to be exchanged are based on commodity prices.

Compound option An option on an option.

Consol See Perpetuity.

Consumption asset An asset held primarily for use in the production process (e.g. wheat, oil, natural gas, electricity).

Contango A condition in which the forward price is above either the current or expected future spot price. See also Backwardation.

Contract A binding agreement between two parties.

Convenience yield An implicit return earned by the holder of a consumption asset, because she wishes to be in a position to supply her customers in the future.

Conversion factor Used to adjust the value of a deliverable Treasury note or Treasury bond in a futures contract.

Convertible bond A corporate bond which gives the holder the right but not the obligation to convert all or part of the bondholding into common stock, on specified dates and on specified terms.

Convexity Measures the curvature of the price–yield relationship for bonds.

Corporation Form of ownership where the company is owned by its shareholders. In the case of bankruptcy the personal assets of the shareholders cannot be used to pay off any residual debt of the company.

Correlation A measure of the (linear) dependence between two variables. The correlation coefficient lies between +1 and –1.

Cost of carry The cost of holding a spot asset between two time periods – may involve a storage cost, an interest cost net of any cash flows accruing to the spot asset (e.g. dividends) and any convenience yield on the spot asset (i.e. consumption commodity).

Counterparty The trader/agent on the other side of a financial transaction.

Coupon Interest paid on a bond, usually in two equal, semi-annual instalments. Sometimes expressed as a cash amount and sometimes as a percentage of the nominal (par) value of the bond – then it is called the coupon *rate*.

Coupon stripping Process which refers to selling the coupons from a bond to various counterparties. This creates a series of zero-coupon bonds. The final payment of principal (maturity) on the bond may also be sold separately.

Covariance A measure of linear dependence between two variables. A positive covariance implies two variables tend to move in the same direction. The value of the covariance depends on the units used to measure the two variables.

Covered call A short position in a call with a long position in the underlying asset.

Covered interest rate parity Relationship which describes a no-arbitrage condition in the foreign exchange market. The interest differential in favour of country-A is equal to the forward discount of country-A's currency.

Crack spread Difference between the spot price of heating oil and the spot price of crude oil. Derivatives are written on the crack spread so that oil refiners can offset risk to their profit margins from changes in these two spot prices.

Credit derivatives A derivative where the payoff depends on a 'credit event' (e.g. failure to meet several contractual periodic payments or outright default).

Credit event An event such as default or debt restructuring which triggers a payout on a CDS.

Credit risk Describes the general risk that the counterparty will default and not honour the contract – see also Default risk.

Cross hedge A futures hedge in which the asset underlying the futures contract differs from the asset being hedged.

Cross rate An exchange rate between two currencies that is implied by their exchange rates with a third currency. For example, dollar-sterling and dollar-euro gives rise to the cross-rate, sterling-euro.

Cum-dividend An asset purchased 'cum-dividend' gives the purchaser the right to the next interest (or dividend) payment – see Ex-dividend.

Currency swap In a plain vanilla currency swap two parties agree to exchange two currencies at recurrent intervals based on agreed (fixed) interest rates in the two currencies and on agreed notional principal amounts. The principal amounts in the currency swap are also exchanged at the beginning and end of the swap.

D

Daily price limits Barriers which indicate the cessation of trading for that day, if certain price limits are reached.

Day count Convention about the number of days used for calculating interest. The day-count convention sets the number of days to payment and the number of days to represent one year. An example is the actual/360 day-count convention.

Day trading Buying and selling the same securities within the same day. Closing out the position before the end of the trading day.

Dealer See Market maker.

Debt Borrow funds on which cash repayments are required in future periods. Debt holders (e.g. bondholders) can place a firm into liquidation.

Default risk The risk that one counterparty will fail to honour its part of an agreed set of financial transactions.

Delivery day Day when the underlying 'asset' in a derivative contract has to be delivered and the long position in the contract pays the short (i.e. trader with an outstanding short position) in cash.

Delta The change in price of a derivative with respect to the change in price of the underlying asset.

Delta hedge To create a hedge (riskless) position which takes into account the sensitivity of an option's premium to changes in the price of the underlying security on which the option is based.

Delta neutral portfolio A portfolio whose value is not affected by (small) changes in the value of the underlying asset, over short periods of time.

Derivative security An asset whose price is dependent, or contingent, upon the price of the underlying asset in the derivative contract.

Dirty price The price of a bond including accrued interest.

Discount broker Firm which offers a limited brokerage service at reduced prices compared to brokers who offer a full range of services.

Discount rate (on Treasury bills) The difference between the redemption (maturity/par) value and the purchase price of a T-bill, expressed as a percentage of the par-value of the T-bill.

Diversification A process of adding assets to a portfolio, whose returns have a less than perfect positive correlation with each other – this helps reduce the overall risk (standard deviation) of portfolio returns.

Dividend yield The ratio of the (annual) dividend per share to the share price.

Dividends Cash payments made to shareholders of a firm. They can vary over time and are not guaranteed.

Down-and-in option An option that comes alive when the underlying asset price falls to a pre-specified level.

Down-and-out option An option that dies when the underlying asset price falls to a pre-specified level.

Duration A measure of the sensitivity of a bond's percentage price change to the change in a market yield. The percentage change in the bond price (approximately) equals the duration multiplied by the absolute change in the yield.

Dynamic hedging A strategy in which the risk of an option's position is offset by continuous trading in the underlying asset or appropriate futures contract.

E

Efficient market A market in which asset prices reflect the 'true' or 'fair value' of the asset. The fair value is often determined by calculating the discounted present value of the expected future cash flows accruing to the asset.

Embedded option An option which is 'part of' a security. For example, a convertible bond has an embedded call option to convert the bond into stocks at possible future dates, at a predetermined price.

Equity (value) The share capital in a company = number of shares × current price. The shares themselves are also referred to as stock or equities.

Equity market Any market where shares of companies are traded.

Equity swap A swap where the return on an equity index is (usually) exchanged for a (fixed or floating) rate of interest.

EURIBOR Interbank market rate for borrowing and lending between banks in the Eurozone.

Eurodollar A dollar-denominated deposit in a bank located outside the USA.

Eurodollar futures contract A futures contract written on a Eurodollar deposit.

Eurodollar market Any market where Eurodollar deposits/loans are traded.

European option Option which may only be exercised on the expiry date.

Excess return Return on an asset minus the return on a risk-free asset.

Ex-dividend (xd) Date on which a holder of stock/bond becomes entitled to receive the next cash payment (e.g. Dividend/coupon), after which the asset trades 'ex-dividend'.

Ex-dividend date An investor who owns the stock before the ex-dividend date will receive the next dividend payment.

Exercise price The price at which an option holder may buy or sell the underlying asset, if the option is exercised.

Exotic option Generic term for options with relatively complex payoffs which can be path dependent. Examples of exotic options include Asian, lookback, barrier, and chooser options.

Expectations theory A theory of the term structure of interest rates in which forward rates of interest represent the market's unbiased expectation of future (spot) interest rates.

Expiry date Date when an option expires and the underlying asset has to be delivered (if the contract has not been closed out).

Exponential weighted moving average Statistical method which calculates the volatility of an asset's return, allowing the forecast of volatility to change over time.

F

Face value See Nominal value.

Financial engineering The process of designing new financial instruments. Often describes a combination of different derivatives (e.g. calls, puts, futures) to create certain desired payoffs. See Synthetic securities.

Financial futures contract A futures contract written on a financial asset such as a bond, stock, stock index, interest rate or two currencies (FX).

Financial instrument An umbrella term used to refer to all types of securities.

Flat yield (Interest yield or running yield.) The annual coupon payment on a bond as a percentage of the market price of the bond.

Floor Set of floorlets, with different maturity dates and often different strike prices.

Floorlet Put option, which pays off if the floating interest rate at expiry (e.g. LIBOR) is below the strike (interest) rate.

Foreign currency option An option contract to exchange one currency for another.

Forward contract An agreement between two parties to buy or sell an underlying asset at a known future date, at a price agreed today. Forward contracts usually go to delivery.

Forward exchange rate Rate of exchange of one currency for another, agreed today but executed in the future.

Forward interest rate Interest rate applicable between two dates in the future.

Forward points Difference between the forward and spot rate for FX.

Forward swap (deferred swap) An agreement today, to enter into a swap at some point in the future, at a fixed rate of interest agreed today, known as the forward swap rate.

FRA (forward rate agreement) The parties involved agree to pay the difference in cash flows between the out-turn rate of interest at some point in the future (e.g. LIBOR) and the rate of interest agreed at the outset of the contract (the FRA rate), based on an agreed notional principal amount.

FRN (floating rate note) An interest-bearing security where the coupons paid are based on what interest rates turn out to be at specified future dates.

Futures contract A contract between two parties to trade a specific asset in the future for a known price determined at contract inception. A key difference between a forward and a futures contract is that the latter can be easily 'closed out'.

Futures option An option where the underlying asset is a futures contract.

Futures price A price agreed today for delivery of an underlying asset, at some point in the future.

G

Gamma The change in an option's delta due to a small change in the underlying asset price.

Gamma-neutral A portfolio of options which will not change in value for a relatively large change in the underlying asset price.

Garman-Kohlhagen formula Valuation formula for European-style foreign currency options.

Geometric Brownian motion Stochastic process for the growth in an asset price, which follows a generalised Wiener process.

Gilts, gilt-edged securities Sterling, marketable, interest-bearing bonds issued by the United Kingdom Government.

Gordon Growth Model Economic model to calculate the value of a firm's stock. It assumes that the value of a firm's stock is determined by the level of current dividends, the future dividend growth rate, and the (risk-adjusted) discount rate.

Greeks Summary statistics to calculate (the approximate) change in the price of an option, as the variables which determine the option price change (e.g. option's delta, gamma, theta, vega).

H

Haircut The small commission a broker takes for organising a transaction for a client (e.g. to allow the client to borrow stock for short-sales, or to undertake a repo transaction). 'Haircut' has several different meanings.

Hazard rate The probability of default over a short time period, conditional on no earlier default.

HDD (heating degree days) The average of the highest and lowest temperatures (midnight to midnight) at a particular location if the temperature is less than 65 °F. If the temperature is greater than 65 °F, HDD is zero.

Hedge A transaction in which a trader tries to protect an existing risky position by taking an offsetting position in another asset.

Hedge fund Actively managed funds which usually use highly leveraged transactions and derivatives in their investment strategies.

Hedge ratio The number of securities-A required to offset any change in the value of existing securities-B, which are currently held.

Historical volatility The magnitude of historical price fluctuations, often measured by the standard deviation of asset returns – see Volatility.

Holding period return The rate of return over a specific period of time – includes capital gains and any cash payments (over the holding period) from the security, usually expressed as a proportion of the current value of the asset.

I

IMM index An index used to determine the change in value of an interest rate futures contract.

Implied volatility The market's view of the volatility of an asset return (e.g. a stock) that is reflected in the current option's price. It is that value for volatility, which makes the option's quoted price equal to the 'theoretical price' (e.g. as given by Black–Scholes).

Index arbitrage An arbitrage strategy using a stock index futures contract and the actual stock index underlying the futures contract.

Index futures A futures contract on a (stock) price index (e.g. on the S&P 500 index).

Index option An option's contract on a (stock) price index.

Index tracking A form of passive portfolio management aiming to replicate the movements of a specific index of securities (e.g. S&P 500).

Initial margin A 'good faith deposit' used to guarantee that two parties to a contract, will honour the terms of the contract. For example, when initially either buying or selling a futures contract you must place an initial margin with the clearing house. The margin might be paid in cash or Treasury bills.

Interbank market An informal network of banks that lend and borrow from each other in various currencies, from overnight to one year.

Intercommodity spread A long and short position in two different futures contracts with different underlying assets, but with the same delivery date.

Inter-Dealer Broker (IDB) Firm which obtains price quotes on a 'no-names' basis.

Interest rate futures Futures contracts written on fixed income securities such as Treasury bills, notes, and bonds. There are also futures written on interest rates (e.g. Eurodollar futures).

Interest rate option Option where the payoff depends on the level of some interest rate in the future (relative to the strike 'price/rate' in the option contract).

Interest rate parity See Covered interest rate parity, Uncovered interest rate parity.

Interest rate swap A contract where a series of floating-rate (variable) interest payments are exchanged for a series of fixed-rate payments (or vice versa).

Internal rate of return (IRR) The constant rate of return which just allows a project/investment to break even. It is the single discount rate which equates the present value of the costs of the investment with the present value of the future cash flows from the investment.

In-the-money option A call (put) option where the underlying asset price is greater (less) than the (discounted present value of) the option's strike price.

Intrinsic value (of an option) For a call option, the amount by which the current spot price is above the strike price or zero, whichever is the greatest. For a put option the amount by which the current spot price is below the strike price, or zero, whichever is the greatest.

IO (Interest only) A mortgage-backed security where the holder receives only the interest payments from the underlying mortgage pool.

ISDA (International Swaps and Derivatives Association) An institution which oversees over-the-counter derivatives, including the creation of master agreements.

ITM (in-the-money) option A call option where the current spot price is greater than the strike price or a put option where the current spot price is below the strike price.

Ito's lemma An equation which enables one to move from a stochastic process for the underlying asset (e.g. S) to the stochastic process for some function of the underlying asset (e.g. $\ln S$).

J

Jensen's alpha Measure of the abnormal return on a stock after taking account of the market risk of the stock.

Jump diffusion process A random series for asset prices which experience random sudden (large) jumps.

L

LEAPS Long-term Equity Anticipation Securities. They are long maturity options on stocks or stock indices.

Leverage Increases both the expected return on an investment strategy and the volatility of possible outcomes. Can be accomplished by borrowing or using derivatives.

Local A trader on the floor of a futures exchange who trades on her own account.

London Interbank Bid Rate (LIBID) The interest rate at which a large AA-rated bank is willing to accept funds (in a particular currency) from another AA-rated bank.

London Interbank Offer Rate (LIBOR) The interest rate at which a large AA-rated bank will lend funds (in a particular currency) to another AA-rated bank.

Long position If you purchase a primary or derivative security then you are said to 'go long'.

Long-short hedge A position which involves a long position in one security (e.g. stocks) and a short position in another security (e.g. a futures contract on the stock) in order to hedge the long position.

Lookback option Option whose payoff depends on the maximum or minimum value of the asset price, over the life of the option.

M

Margin Collateral that must be posted to allow you to transact in a futures or options contract, in order to insure the clearing house against default risk.

Margin call A request for additional payments into the margin account if the value of the mark-to-market positions fall below the maintenance margin, set by the clearing house.

Market maker A trader who quotes buying and selling prices at which she is willing to trade in specific stated amounts.

Market risk Risk which cannot be diversified away. Proportion of the asset's total risk which relates to movements in the overall market or to general changes in the economy (e.g. oil prices, interest rates – see Systematic risk.

Market timing Form of active portfolio management which shifts funds into (out of) the (stock or bond) market when the market is predicted to rise (fall).

Mark-to-market The act of revaluing asset positions using current market prices.

Maturity date Date on which an asset is redeemed. For bonds this involves a cash payment (the principal/maturity/par value). For a derivative this might involve a cash payment or delivery of the underlying asset.

MBS (mortgage-backed security) A security where the holder receives cash flows from a pool of mortgages.

MCS (Monte Carlo simulation) A technique to simulate a random variable according to a known distribution.

Mean reversion Process which describes the path of a time series which tends to return to its long-run average value.

Mezzanine tranche A tranche in a CDO, which takes losses after the equity tranche but before all the senior tranches.

Mid-price Average of the bid and offer price of an asset.

Money market The market for borrowing and lending funds with less than one year to maturity.

Moral hazard Economic concept often observed in insurance markets where the likelihood of a claim being made increases after the insurance has been taken out.

Mutual funds A fund management company that buys/sells a portfolio of assets (e.g. stocks, bonds, real estate, commodities).

N

Naked position A risky position in an asset – either net long or net short.

Net present value (NPV) Value today of 'future positive cash flows less negative cash flows' from an asset.

Netting Calculating the net position for assets and liabilities with a specific counterparty, which then determines collateral requirements.

NINJA Person with no income, no job, and no assets.

No-arbitrage interest rates A set of interest rates which do not allow any risk-free arbitrage profits to be made.

No-arbitrage profits A set of asset prices which does not allow risk-free profits to be made.

Nominal (value) (Face or par value.) The fixed amount in a contract which is used to determine cash flows. For example, the dollar amount on which interest payments are calculated in an interest rate swap contract.

Normal distribution Bell-shaped, symmetric probability distribution for a continuous random variable.

Notional principal The principal used to calculate (dollar) interest payments in a fixed income asset such as a plain vanilla interest rate swap, an FRA, cap, floor, or swaption. The principal itself is (usually) not swapped.

O

OCC (Options Clearing Corporation) See Clearing house.

Offer price See Ask price.

OIS An overnight index swap.

Open interest Total number of futures or option contracts (of a specific type) which have not been closed out (or reached expiry/maturity).

Option Contract which gives the purchaser the right, but not the obligation, to buy (a 'call' option), or to sell (a 'put' option), a specified amount of a commodity or financial asset at a specified price, by or on a specified date.

Option premium Price of an option.

Options Clearing Corporation The organisation that serves as the clearing house for options traded on US exchanges.

Ordinary shares Ordinary shares represent a claim on the profits of a firm. Ordinary shareholders have voting rights and the shareholders are the owners of the firm.

Out-of-the-money A call (put) option where the underlying asset price is below (above) the option's strike price (or present value of the strike price).

Over-the-counter (OTC) instrument Transaction between two counterparties (often two banks or a bank and a corporate), the details of which are directly negotiated between the issuer and purchaser.

P

P/E ratio (price/earnings ratio) A company's current share price divided by its earnings per share (measured over some recent historic period).

Par See Nominal value.

Par value The principal amount of a bond, payable at maturity.

Par yield The coupon rate on the bond which makes the market price equal to the principal value.

Parallel shift A movement in the yield curve, where rates at all maturities move up or down by the same amount.

Parisian option Barrier option where the underlying asset price has to be above or below a pre-specified barrier for a pre-specified period of time, before the option is knocked-in or knocked-out.

Passive portfolio management See index tracking.

Path-dependent option An option whose payoff depends upon the complete path of the underlying asset (and not just the price of the underlying asset at maturity of the option).

Payoff The cash value of the option, at maturity of the option.

Performance measures A 'statistic' to measure the performance of a portfolio relative to some benchmark portfolio. Differences in risk of the different portfolios are usually incorporated in the performance measure.

Perpetuity Fixed income security which is never redeemed by the issuer and pays coupons for ever.

Pit An area on the trading floor of a futures or options exchange where contracts are traded by 'open outcry'.

Plain vanilla A standard cash market or derivative security.

Plain vanilla swap A term which describes a basic fixed-for-floating interest rate swap or a simple currency swap.

PO (principal only) A mortgage-backed security where the holder receives only the principal payments from the pool of mortgagees.

Portfolio insurance A strategy in which a portfolio of stocks and futures contracts mimics the price movements of a portfolio of stocks and put options. Portfolio insurance is designed to ensure a minimum future value for an equity portfolio but also allows upside capture.

Position limit The maximum amount held in a particular asset or set of assets. This limit might be set by the individual trader, a broker or the exchange itself (e.g. CME Group).

Position trader Trader who holds speculative positions over horizons of 1 day to 1 month or even longer.

Present value Value today of all future cash receipts less cash payments.

Primary market Market where new issues of securities are offered to the public.

Principal Par, maturity, face value of an debt instrument (e.g. bond) which is paid at maturity.

Principal agent problem Describes a conflict of interest which can arise between different agents in or connected with an organisation (e.g. shareholders and directors).

Program trading The use of computers to simulate real-time data in order to detect arbitrage opportunities.

Protective put Holding the underlying asset and a long position in a put option (on the underlying asset).

Pure discount bond See Zero-coupon bond.

Put option A derivative security giving the buyer the right to sell an underlying asset at a known strike price on or before a specified maturity date.

Put premium Price of a put option.

Put–call parity A pricing relation between put and call premia, the price of the underlying asset and a risk-free amount of cash, that ensures no arbitrage profits can be made.

Q

Quanto An option where the payoff is determined in one currency but is paid in another currency.

R

Rainbow option An option where the payoff is determined by two or more underlying assets.

Random walk (model) A model where the changes in a variable (e.g. stock price, exchange rate) are random and independent over time.

Rebate interest Interest due to be paid by the seller of a bond to a purchaser when the bond is purchased without the right to the forthcoming interest payment (i.e. ex-dividend).

Redemption yield See Yield to maturity (YTM).

Reference entity A named company on which a credit default swap (CDS) is written.

Replication portfolio See Synthetic securities.

Repo (repurchase agreement) A form of collateralised borrowing. One party sells securities (e.g. T-bills) to another and at the same time commits to repurchase the securities on a specified future date, at a specified price, which is higher than the initial selling price. The difference between the two prices (expressed as a percentage) is the cost of borrowing in the repo market.

Repo rate The cost of collateralised borrowing using a repo.

Reset dates Dates on which a floating rate is realised.

Reverse repo A form of collateralised lending. A reverse repo is a repo transaction as seen from the point of view of the party who initially buys the securities and hence lends money.

Rho Change in an option's price for a small change in the risk-free rate.

Risk aversion A person who will only take part in a gamble in which the expected monetary outcome for them is positive. For example, if you are willing to pay less than $1 to enter a bet on the toss of a (fair) coin with a $1 receipt for 'heads' and a $1 payout for 'tails' then you are risk averse.

Risk management Set of techniques for measuring and controlling risk.

Risk neutral A situation in which an investor is indifferent between a risky monetary outcome and an equal amount that is certain. A risk-neutral investor will take part in a gamble in which the expected monetary outcome is zero. For example, if you are willing to pay $1 to enter a bet on the toss of a (fair) coin with a receipt of $1 for 'heads' and a payout of $1 for 'tails' then you are a risk-neutral investor.

Risk-neutral valuation (RNV) Describes the no-arbitrage approach to option pricing. To correctly price the option, we can assume that the underlying asset (e.g. stock) grows at the risk-free rate.

Risk-free rate The rate of return on an investment which is known with certainty.

RiskMetrics™ Methodology originally proposed by JPMorgan to measure the price risk of a portfolio of cash market and derivative securities.

Running yield See Flat yield.

S

Secondary market Market where securities are traded once they have been issued.

Securities and Exchange Commission (SEC) A Federal agency charged with the regulation of US security and options markets.

Securitisation A portfolio of different assets is created and then sold to final investors.

Settlement date Date on which the ownership of an asset passes from one party to the other.

Settlement price The futures price established at the end of each trading day upon which daily mark-to-market of margin positions is based. The settlement price is usually an average of the last few trades of the day.

Sharpe ratio Reward to variability ratio. Risk-adjusted measure of portfolio performance – it is the average excess return divided by the standard deviation of returns.

Short position Position where a market maker (dealer) has sold more of an asset than she has purchased.

Short rate An interest rate which applies over a very short period of time.

Short-selling A transaction in which a security is borrowed (from a broker) and sold in the market, with an obligation to return the borrowed security at a later date. Collateral in the form of margin payments (to the broker) are required.

Shout option The holder of the option can lock in a minimum payoff, at one point during the option's life.

Single index model Linear model which describes the relationship between the (excess) return on an individual stock (or portfolio of stocks) and the (appropriate) market return (e.g. S&P 500).

Spark spread Difference between the spot price of electricity and the spot price of natural gas which is used to produce electricity (the two are connected via the heat rate). Derivatives are written on the spark spread so that electricity generating companies can offset risk to their profit margins from changes in these two spot prices.

Specific risk Risk which is specific to the firm, such as strikes, patent disputes, legal challenges etc. Specific risk is relatively small (near zero) in a well-diversified portfolio of many assets (e.g. randomly chosen stocks). Also called unsystematic risk and diversifiable risk.

Speculation Taking risky bets on the future value of an asset.

Speculator An investor who takes a risky position in an asset.

Spot (cash) market The market for assets that entail immediate (or near immediate) delivery and (near) immediate cash payment.

Spot (interest) rate An interest rate which applies from today to one specific date in the future.

Spot price The current price of an asset traded in the spot (or cash) market which is for (near) immediate delivery and payment.

Spot volatilities When pricing a cap, different (interest rate) volatilities are used to price each caplet.

Spread (yield spread) Difference between two 'prices'. For example, the difference between a dealer's buying price and selling price for the same asset or the difference between the yield on corporate and government bonds (yield spread).

Spread trading A trade involving two or more assets, usually involving derivatives, in order to speculate on the direction the spot (cash market) asset will move (e.g. bull spread).

Stack and roll hedge When the hedge period is long, short dated futures contracts are used and rolled over into the next set of short dated futures contracts.

Static hedge A strategy by which the value of an existing cash market position is maintained by using derivatives but the initial hedged position is not rebalanced.

Stochastic process An equation describing the random behaviour of a variable.

Stock index A weighted index of individual stock prices.

Stock index option An option giving the owner the right to buy or sell a 'stock index' at the known strike price. These option contracts are cash-settled – there is no delivery of the underlying stocks in the index.

Stock option An option on a stock.

Straddle Payoff profile of a call and put with the same strike price and time to maturity. 'V-shaped' or 'inverted V-shaped' payoff.

Stress testing Calculating the change in value of a portfolio consequent on extreme market movements.

Stressed VaR Calculating the VaR using historical simulation over a chosen 'crisis period'.

Strike price Price at which the option holder has the right to buy or sell the underlying commodity or financial asset, if the holder chooses to exercise the option.

Strips market This is the secondary market which trades the individual coupons that have been 'stripped' (i.e. legally separated) from a coupon paying bond.

Subordinated debt Ranks behind other bondholder claims if the firm goes bankrupt.

Sub-prime mortgage A mortgage given to individuals with high credit risk.

Swap An exchange of cash flows in the future, according to a prearranged contract, determined at initiation of the swap.

Swap dealer Financial intermediary who provides swaps to counterparties (often corporates).

Swap rate The fixed (interest) rate determined at the initiation of an interest rate swap.

Swaption An option where the underlying asset is a swap contract (e.g. a fixed for floating swap).

Synthetic CDO A CDO created from selling a portfolio of credit default swaps.

Synthetic security A structured or financially engineered product which replicates the same cash flows as another asset, but uses different financial instruments (e.g. a long forward contract on a stock which is replicated by purchasing the stock (on the NYSE) using borrowed funds.

T

Tailing the hedge An adjustment in the number of futures contracts needed to hedge, because of daily marking-to-market of the futures contracts.

TED spread The difference between 90-day LIBOR and the 90-day, Treasury bill rate.

Tenor The frequency of payments in a financial contract.

Term repo Repo trades with a fixed maturity date (greater than 1 day).

Term structure of interest rates The relation between yields and the time to maturity of bonds of a similar risk class (e.g. all government bonds or all corporate bonds with a particular credit rating). Also called the yield curve.

Theta The change in the options price over a small period of time.

Tick Units in which minimum price movements are usually recorded and measured.

Time decay Change in the price of an option as the option approaches maturity (all other things held constant).

Time value (of an option) The amount by which the option premium exceeds its intrinsic value.

Total return swap The coupon and principal payments on a bond are exchanged for cash flows based on LIBOR plus a spread.

Tranches Parts of a CDO, which suffer losses in a pre-specified order.

Treasury bill Instrument of up to 12 months maturity (but normally less), issued by governments. It is a discount instrument – its initial selling price is below its maturity (par/ face) value. There are no intermediate cash flows.

Treasury bond Debt security issued by a government with maturity in excess of 12 months. Treasury bonds usually have periodic coupon payments, payable every 6 months.

Treasury note US Treasury bonds with maturity of between 1 and 7 years.

Tree A discreet representation of the possible stochastic changes in an underlying asset price. Binomial or trinomial trees can be used to price options.

U

Underlying (asset) Specific asset on which a derivative contract is based (e.g. Apple stock, stock index, T-bond, LIBOR rate, etc.).

Up-and-in option An option that comes alive when the price of the underlying asset reaches or crosses a pre-specified upper boundary.

Up-and-out option An option that dies when the price of the underlying asset reaches a pre-specified upper boundary.

V

Value at risk Maximum expected 'dollar' loss over a specific time horizon at a pre-specified probability (percentile) level.

Variance A measure of the dispersion of a random variable around its expected return. The square root of the variance is the standard error, often referred to as 'volatility'.

Variance–covariance matrix (VCV) A matrix containing variances on the diagonal and covariances in the off-diagonal positions. Used in calculating VaR for large portfolios of assets.

Variance swap A swap where one party pays a fixed predetermined variance rate and the other pays the realised variance rate, over specific time periods (tenor), with payments based on a predetermined notional principal.

Variation margin If the value in the margin account falls below a pre-specified level called the maintenance margin, then additional funds must be added to the margin account so that the amount in the margin account is made up to the predetermined initial margin.

Vega (also known as lambda, kappa and sigma) A measure of the sensitivity of the call/put premium to small changes in the standard deviation of the (log) underlying asset price.

Vega-neutral portfolio A portfolio of options whose value does not change (much) for a large change in the price of the underlying asset.

VIX An index which tracks the volatility of the S&P 500 stock index.

Volatility Measure of the variability in asset returns/prices. Often measured by the standard deviation and taken as one measure of risk. See Implied volatility, Variance.

Volatility skew A plot of implied volatility against different strike prices for options on the same underlying asset and with the same time to maturity. This graph of implied volatility against the strike price produces a 'non-symmetrical-smile' or 'skew'.

Volatility smile The relationship between implied volatilities calculated from the quoted prices of a set of options (on the same underlying asset, with the same expiry date) and the different strike prices of these options. For options on foreign currencies, the graph of implied volatility against the different strike prices is in the shape of a 'smile'.

Volatility surface A graph or table showing implied volatilities for different strike prices and times to maturity (for an option on the same underlying asset).

Volatility term structure The different values for implied volatility calculated from options with different maturity dates (but with the same strike prices and underlying asset).

W

Warehousing Denotes the situation where a financial institution carries an open position on its books until a suitable offsetting transaction can be found with another counterparty.

Warrants Instrument which gives the holder the right (but not the obligation) to buy shares directly from the company at a fixed price, at some time(s) in the future. A type of call option. If exercised, the company must issue more stocks.

Waterfall A set of rules to determine the order in which different tranches in a CDO will receive cash flows.

Weather derivative Derivative where the payoff depends on the weather (e.g. temperature, inches of snowfall, number of frost days) at a particular geographic location over a specific time period.

Wiener process A stochastic process, where changes in a variable over a short period of time are independent, normally distributed and have zero-mean and a variance proportional to the time period.

Writer The seller of an option contract. The writer of a call or put option has a short position and has to post margin payments with the clearing house.

Y

Yield The return on security over a specific period of time.

Yield curve See Term structure of interest rates.

Yield-to-maturity A single value for the yield, at a specific point in time, which when used as a discount rate makes the present value of the remaining coupon payments and the present value of the bond's maturity value equal to the quoted price of the bond. The YTM is the 'internal rate of return' of the bond.

Z

Zero-coupon bond A bond which does not pay any coupons and the holder only receives a single payment (i.e. the bond's par or face value or maturity value) on the maturity date of the bond. A zero-coupon bond sells at a discount to its face (par/maturity) value.

Zero-coupon yield curve Spot rates plotted against time to maturity. In principle the spot rates are calculated from quoted prices of zero-coupon bonds.

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