On Calibration and Simulation of Local Volatility Model with Stochastic Interest Rate

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Abstract

Local volatility model is a relatively simple way to capture volatility skew/smile. In spite of its drawbacks, it remains popular among practitioners for derivative pricing and hedging. For long-dated options or interest rate/equity hybrid products, in order to take into account the effect of stochastic interest rate on equity price volatility stochastic interest rate is often modelled together with stochastic equity price. Similar to local volatility model with deterministic interest rate, a forward Dupire PDE can be derived using Arrow-Debreu price method, which can then be shown to be equivalent to adding an additional correction term on top of Dupire forward PDE with deterministic interest rate. Calibrating a local volatility model by the forward Dupire PDE approach with adaptively mixed grids ensures both calibration accuracy and efficiency. Based on Malliavin calculus an accurate analytic approximation is also derived for the correction term incorporating impacts from both interest rate volatility and correlation, which integrates along a more likely straight line path for better accuracy. Eventually, the hybrid local volatility model can be calibrated in a two-step process, namely, calibrate local volatility model with deterministic interest rate and add adjustment for stochastic interest rate. Due to the lack of analytic solution and path-dependency nature of some products, Monte Carlo is a simple but flexible pricing method. In order to improve its convergence, we develop a scheme to combine merits of different simulation schemes and show its effectiveness.

Key words: Stochastic Interest Rate, Hybrid Local Volatility, Dupire Forward PDE, Arrow-Debreu Price, Finite Difference, Volatility Regularization, Predictor-Corrector

1. Introduction

In spite of some well-known problems, for example, wrong smile dynamics (c.g. Hagan et al. 2002) and forward skew decay (Gatheral, 2001), local volatility model remains a popular method for exotic derivative pricing and hedging because of its convenience.

Usually in a local volatility model the interest rate is assumed to be deterministic and Dupire's local volatility formula (Dupire, 1994) is then used to map market Black volatility surface to a local volatility surface. When the interest rate is random, the drift of the stock price becomes stochastic, which then contributes to the volatility of the stock price, especially in the long term. Without considering the impact of stochastic interest rate we consequently overestimate the local volatility of the stock price process. Intuitively, the volatility bias caused by the stochastic interest rate depends on both the interest rate volatility level as well as the correlation between the interest rate and equity price.

As such, for long-dated options and interest rate-equity hybrid products, it is necessary to model the correlation between interest rate and equity price adequately and take into account the impact of interest rate volatility on equity volatility. This requires the modeling of the joint evolution of interest rate and equity price. Although with the aid of Dupire's formula a local volatility model can be calibrated relatively easily, fast and robust calibration of a hybrid local volatility model is still a challenging task because of its higher dimensionality. In recent years some efforts have been devoted to deriving analytic approximations, for example, Balland (2005) and Benhamou et al. (2008), but some issues remain unaddressed, including calibration accuracy and stability. In this paper we seek to develop a faster and more accurate hybrid local volatility model with robust calibration and tackle some practical issues at the same time.

The rest of the paper is organized as follows. In the section 2, we first apply Arrow-Debreu price trick to derive the forward PDE and Dupire's formula for a local volatility model with stochastic interest rate, and then put forward a finite difference method to solve a transformed forward PDE to match a smoothed market implied Black volatility surface, and in the end derive an analytic approximation for the bias adjustment to the local volatility surface due to interest rate volatility. In order to improve stability and smoothness of the local volatility surface, we also propose three different volatility regularization forms. In the section 3, Monte Carlo simulation schemes for a hybrid local volatility model will be discussed. The section 4 demonstrates the significance of the impact of the interest rate volatility and correlation on equity option volatilities, and the efficiency and accuracy of the calibration method through numerical examples. Finally, the section 5 concludes the paper with further extension of the model and future research work.

2. Hybrid Local Volatility Model

A local volatility model with stochastic interest rate that follows a one-factor Gaussian model is governed by SDEs

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma(t, S_t)dW_t^S \tag{2.1}$$

and

$$dx(t) = -kx(t)dt + \sigma(t)dW_t^r \tag{2,2}$$

where the instantaneous short rate $r(t) = x(t) + \varphi(t)$ and $E[dW_t^S dW_t^r] = \rho dt$.

For the one-factor Hull-White model (c.g. Brigo and Mercurio, 2001) an analytic solution for the price of a zero coupon bond

$$P(t,T) = E\left[\exp\left(-\int_t^T r_s ds\right)\right] = \exp(A(t,T) - B(t,T)x_t) \tag{2.3}$$

where

$$B(t,T) = \int_t^T exp\left(-\int_t^u k(s)ds\right)du \tag{2.4}$$

and $A(t, T)$ is calibrated to the initial term structure $A(0,T) = lnP(0,T)$.

In particular, if the interest rate mean-reversion speed $k(t)$ is constant, we have

$$B(t,T) = \frac{1 - \exp(-k(T-t))}{k}$$

with $B(T,T) = 0$.

Note that the calibration of the one-factor Hull-White model short rate model can be done analytically, or numerically using trinomial trees, and here we won't discuss it in detail.

In the following, we derive the Dupire forward equation with stochastic interest rate under the spot risk-neutral measure. First, define the Arrow-Debreu price density

$$\phi(S,x,T) = E[\delta(S_T - S, x_T - x)\exp(-\int_0^T \left(x_t + \varphi(t)\right)dt)] \tag{2.5}$$

Changing from risk-neutral measure to T-forward measure by applying the Radon-Nikodym derivative

$$Z(t) = \frac{dQ^T}{dQ} = \frac{B(0)P(t,T)}{B(t)P(0,T)} = \frac{\exp(-\int_0^t r_s ds)P(t,T)}{P(0,T)} \tag{2.6}$$

yields

$$\phi(S,x,T) = P(0,T)E^T[\delta(S_T - S, x_T - x))]$$

Thus, Arrow-Debreu price is proportional to the probability density under the T-forward measure.

Under the risk-neutral measure, the Arrow-Debreu price density satisfies the 2-dimensional Fokker-Planck equation (c.g. Karatzas and Shreve, 1998)

$$\frac{\partial \phi(S,x,T)}{\partial T}+\frac{\partial}{\partial S}\big((x+\varphi(t)-q_t)S\phi(S,x,T)\big)+\frac{\partial}{\partial x}\big(-kx(t)\phi(S,x,T)\big)-\frac{1}{2}\frac{\partial^2}{\partial S^2}\big(\sigma(t,S_t)^2S^2\phi(S,x,T)\big)$$
$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\sigma(t)^2\phi(S,x,T)\big)-\frac{\partial^2}{\partial S\partial x}\big(\rho\sigma(t)\sigma(t,S_t)S\phi(S,x,T)\big)+\big(x+\varphi(t)\big)\phi(S,x,T)=0$$

with an initial condition

$$\phi(S,x,0)=\delta(S_0-S,x_0-x)$$

At the same time, a call option price can be expressed in terms of Arrow-Debreu price

$$C(T,K)=E[(S-K)^+D(T)]=\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\phi(S,x,T)dx\,dS \tag{2.7}$$

where stochastic discount factor

$$D(T)=\exp(-\int_0^T\big(x_t+\varphi(t)\big)dt)$$

is part of the Arrow-Debreu price.

Now differentiating the call option price w.r.t expiration and strike yields

$$\frac{\partial C(T,K)}{\partial T}=\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{\partial \phi(S,x,T)}{\partial T}dx\,dS$$

$$\begin{aligned}\frac{\partial C(T,K)}{\partial K}&=-\int_0^{+\infty}\int_{-\infty}^{+\infty}I_{\{S>K\}}\phi(S,x,T)dx\,dS+\int_0^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{dI_{\{S>K\}}}{dK}\phi(S,x,T)dx\,dS\\&=-\int_K^{+\infty}\int_{-\infty}^{+\infty}\phi(S,x,T)dx\,dS+\int_{-\infty}^{+\infty}(S-K)\delta(S-K)\phi(S,x,T)dx\\&=-\int_K^{+\infty}\int_{-\infty}^{+\infty}\phi(S,x,T)dx\,dS\end{aligned}$$

and

$$\frac{\partial^2 C(T,K)}{\partial K^2}=-\int_{-\infty}^{+\infty}\phi(K,x,T)dx$$

where

$$\frac{dI_{\{S>K\}}}{dK}=\delta(S-K)$$

Plugging the forward PDE into the theta of the call option price yields

$$
\begin{aligned}
\frac{\partial C(T,K)}{\partial T} = &\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{\partial}{\partial S}\big((x+\varphi(t)-q_t)S\phi(S,x,T)\big)dx\,dS \\
&+\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{\partial}{\partial x}\big(-kx(t)\phi(S,x,T)\big)dx\,dS \\
&-\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{1}{2}\frac{\partial^2}{\partial S^2}\big(\sigma(t,S_t)^2S^2\phi(S,x,T)\big)dx\,dS \\
&-\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\sigma(t)^2\phi(S,x,T)\big)dx\,dS \\
&-\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\frac{\partial^2}{\partial S\partial x}\big(\rho\sigma(t)\sigma(t,S_t)S\phi(S,x,T)\big)dx\,dS \\
&+\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)(x+\varphi(t))\phi(S,x,T)dx\,dS
\end{aligned}
$$

Without loss of generality assuming

$$\lim_{x\to\pm\infty}\phi(S,x,T)=0$$

we obtain by integration by part

$$
\begin{aligned}
\frac{\partial C(T,K)}{\partial T} = &-\int_{-\infty}^{+\infty}\int_K^{+\infty}\big((x+\varphi(t)-q_t)S\phi(S,x,T)\big)dx\,dS-\frac{1}{2}\sigma(t,K)^2K^2\int_{-\infty}^{+\infty}\phi(K,x,T)dx \\
&-\int_K^{+\infty}(S-K)\int_{-\infty}^{+\infty}\frac{\partial^2}{\partial x\partial S}\big(\rho\sigma(t)\sigma(t,S_t)S\phi(S,x,T)\big)dx\,dS \\
&+\int_K^{+\infty}\int_{-\infty}^{+\infty}(S-K)\big(x+\varphi(t)\big)\phi(S,x,T)dx\,dS \\
&=K\int_{-\infty}^{+\infty}\int_K^{+\infty}(x+\varphi(t)-q_t)\phi(S,x,T)dx\,dS-q_t\int_{-\infty}^{+\infty}\int_K^{+\infty}(S-K)\phi(S,x,T)dx\,dS \\
&-\frac{1}{2}\sigma(t,K)^2K^2\int_{-\infty}^{+\infty}\phi(K,x,T)dx
\end{aligned}
$$

Plugging in derivatives of $C(T,K)$ w.r.t K results in Dupire's forward PDE with stochastic interest rate

$$\frac{\partial C(T,K)}{\partial T}=KE\big[(r_T-q_T)D(T)I_{\{S_T>K\}}\big]+\frac{1}{2}\sigma(t,K)^2K^2\frac{\partial^2 C(T,K)}{\partial K^2}-q_TC(T,K) \tag{2.8}$$

After some rearrangement we obtain the local volatility formula for the hybrid model

$$\sigma_{hyb}^2(T,K)=\frac{\frac{\partial C(T,K)}{\partial T}-KE[(r_T-q_T)D(T)I_{\{S_T>K\}}|F_0]+q_TC(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2.9}$$

which is similar to the Dupire formula.

2.1 Calibration of Local Volatility Model with Deterministic Interest Rate

As a special case where the interest rate is deterministic, the equation (2.9) reduces to Dupire's formula (Dupire, 1994)

$$\sigma_{det}^2(T,K) = \frac{\frac{\partial C(T,K)}{\partial T} + (f(0,T) - q_T)K\frac{\partial C(T,K)}{\partial K} + q_T C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2.10}$$

where the instantaneous short rate is replaced by instantaneous forward rate f(0,T).

Among practitioners there are a few popular ways to calibrate the local volatility surface. The first method is to use the above Dupire formula directly, which provides us a way to transform an implied Black volatility surface into a local volatility surface. In order to achieve stable local volatilities, the implied volatility surface must be first smoothed. In practice, option market quotes are only available for a few expiries and a few strikes, and thus it requires smooth and arbitrage-free interpolation along both time and strike in order to apply the Dupire's formula. Moreover, even if the implied volatility surface is smooth, the local volatility surface by a discretized Dupire's formula does not necessarily reprice market quoted options, depending on the choice of time step and the strike grid.

A second way is to price options using PDE pricing method and calibrate local volatility to match market quoted options

$$\frac{\partial C(S,t)}{\partial t} + (r_t - q_t)S\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma(t,S_t)^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = r_t C(S,t) \tag{2.11}$$

subject to the terminal condition C(S,T) = max(S_T-- K, 0).

Because the terminal condition and boundary conditions are strike and expiry dependent, each option requires a separate PDE. Each backward PDE has to be rolled back from its expiry to time 0 to price the option. Thus, this method is not computationally efficient enough for the calibration purpose.

A third approach is to apply a forward PDE to model the transition probability density,

$$\frac{\partial \phi(S,T)}{\partial T} + \frac{\partial}{\partial S}\big((r_t - q_t)S\phi(S,T)\big) - \frac{1}{2}\frac{\partial^2}{\partial S^2}\big(\sigma(t,S_t)^2 S^2 \phi(S,T)\big) + r_t\phi(S,T) = 0 \tag{2.12}$$

with initial condition $\phi(S,0)=\delta(S-S_0)$, and then value a European call at expiry T

$$C(T,K) = P(0,T)E[(S-K)^+] = P(0,T)\int_K^{+\infty}(S-K)\phi(S,T)dS$$

In this approach one can bootstrap the local volatility function parameters by rolling the PDE forward and value all the European options together, thereby greatly improving calibration efficiency.

A forth approach is to apply the Dupire forward PDE

$$\frac{\partial C(T,K)}{\partial T} + (r_T - q_T)K\frac{\partial C(T,K)}{\partial K} - \frac{1}{2}K^2\sigma(t,K)^2\frac{\partial^2 C(T,K)}{\partial K^2} + q_T C(T,K) = 0 \tag{2.13}$$

with initial condition $C(0,K)=(S_0-K)^+$. Similar to the forward PDE method it can value European options with any strike simultaneously and bootstrap the time dependent local volatilities by rolling the PDE forward step by step.

As a special case, Andreasen and Huge (2008) propose an implicit PDE approach to bootstrap the local volatility matrix assuming piecewise constant volatility along both the time and strike direction with a finite difference equation

$$\left(1 - \frac{1}{2}\Delta T_i \sigma_{det}^2(T_i, K) K^2 \frac{\partial^2}{\partial K^2} + \Delta T_i (r_T - q_T) K \frac{\partial}{\partial K} + \Delta T_i q_T\right) C(T_{i+1}, K) = C(T_i, K) \tag{2.14}$$

Based on this approach Abasto et al. (2013) propose a multi-step version to simplify the method and improve its accuracy.

Note that Dupire formula approach is a special case of the Dupire forward PDE method. Compared to the Dupire PDE method, it has a few drawbacks

(1) Because of instability, it requires pre-smoothing the Black volatility surface, which can potentially causes errors and arbitrages.
(2) It essentially approximates the time derivative of an option by a single time step, which results in additional discretization errors.
(3) Interpolation of the Black volatility surface along the time is inconsistent with the interpolation of local volatility surface along the time.

Because of these weaknesses we prefer the more flexible arbitrage-free Dupire forward PDE method over the Dupire formula, which allows repricing market quoted options, without losing too much efficiency. Thus, in this paper Dupire forward PDE method will be used to calibrate the local volatility matrix with deterministic interest rate.

2.1.1 Calibration using Finite Difference Method

The finite difference equation for the Dupire forward PDE can be written as

$$\left(1 - \frac{1}{2}\Delta T_i \sigma_{det}^2(T_{i+1}, K) K^2 \frac{\partial^2}{\partial K^2} + \Delta T_i \left(r_{T_i} - q_{T_i}\right) K \frac{\partial}{\partial K} + \Delta T_i q_{T_i}\right) C(T_{i+1}, K) = C(T_i, K) \tag{2.15}$$

with the initial condition

$$C(0, K) = (S_0 - K)^+$$

where r_T and q_T are assumed to be piecewise constant.

The purpose of calibration is to back out the local volatility σ_{det} given call option prices at two times, namely T_i and T_{i+1}. For robustness, we sample more data points from the option price curve than the knot points of the local volatility curve and accordingly minimize the sum of squared errors

$$\arg\min_{\sigma_{det}^2} \sum_{j=1}^{m} w_j (C\left(T_{i+1}, K_j\right) - \hat{C}(T_{i+1}, K_j))^2 \tag{2.16}$$

where one possible choice for weight w_j is the $1/vega^2$, which essentially transforms error in option price into error in implied volatility.

Also note that only grid points between K_{min} and K_{max} are included in the cost function and all data points close to the boundaries are excluded.

In order to calculate option prices, we can either roll forward the equation from T_i to T_{i+1} or roll back it from T_{i+1} to T_i. However, since it is well known that the explicit scheme suffers from numerical oscillation and instability, the implicit scheme or mixed scheme is preferred, i.e. given local volatilities and $C(T_i, K)$'s we obtain $C(T_{i+1}, K)$'s by solving the linear equations.

In the following we will discuss two different ways to numerically solve the difference equation, either with uniform grid or transformed grid, dependent on how far from the boundary.

2.1.1.1 Solve Difference Equation with Uniform Grid

In order to solve the PDE, we first derive the difference operators and then boundary conditions based on degenerated PDEs.

(1) Difference operators

Define central difference operators at K_j

$$\frac{\partial C(T,K)}{\partial K} \approx \frac{C\left(T,K_{j+1}\right) - C(T,K_{j-1})}{2\Delta K}$$

and

$$\frac{\partial^2 C(T,K)}{\partial K^2} \approx \frac{C\left(T,K_{j+1}\right) - 2C\left(T,K_j\right) + C(T,K_{j-1})}{(\Delta K)^2}$$

Therefore, in matrix notation the difference equation can be written as

$$A_{i+1}\, C_{i+1} = C_i$$

where the element of the tri-diagonal operator matrix A_{i+1} are

$$a_{j,j-1} = -\frac{1}{2}\Delta T_i \sigma_{det}^2\left(T_{i+1},K_j\right)K_j^2 \frac{1}{\left(\Delta K_j\right)^2} - \Delta T_i\left(r_{T_i} - q_{T_i}\right)K_j \frac{1}{2\Delta K_j}$$

$$a_{j,j} = 1 + \Delta T_i q_{T_i} + \Delta T_i \sigma_{det}^2\left(T_{i+1},K_j\right)K_j^2 \frac{1}{\left(\Delta K_j\right)^2}$$

$$a_{j,j+1} = -\frac{1}{2}\Delta T_i \sigma_{det}^2\left(T_{i+1},K_j\right)K_j^2 \frac{1}{\left(\Delta K_j\right)^2} + \Delta T_i\left(r_{T_i} - q_{T_i}\right)K_j \frac{1}{2\Delta K_j}$$

and in particular at boundaries

$$a_{0,0} = 1 + \Delta T_i q_{T_i} - \Delta T_i\left(r_{T_i} - q_{T_i}\right)K_0 \frac{1}{\Delta K_0}$$

$$a_{0,1} = \Delta T_i\left(r_{T_i} - q_{T_i}\right)K_0 \frac{1}{\Delta K_0}$$

$$a_{n-1,n-1} = 1 + \Delta T_i q_{T_i} + \Delta T_i\left(r_{T_i} - q_{T_i}\right)K_{n-1} \frac{1}{\Delta K_{n-1}}$$

$$a_{n-1,n-2} = -\Delta T_i\left(r_{T_i} - q_{T_i}\right)K_{n-1} \frac{1}{\Delta K_{n-1}}$$

(2) Boundary conditions

Different types of boundary conditions can be derived based on degenerated PDEs. Since at the upper and lower boundaries, K_{max} and K_{min}, respectively, the probability density is almost 0, i.e.

$$\frac{\partial^2 C(T,K)}{\partial K^2} \approx 0$$

the forward PDE degenerates to

$$\left(1 + \Delta T_i\left(r_{T_i} - q_{T_i}\right)K \frac{\partial}{\partial K} + \Delta T_i q_{T_i}\right) C(T_{i+1}, K) = C(T_i, K)$$

Moreover, as $K \to \infty$,

$$\frac{\partial C(T,K)}{\partial K} = -P(0,T) \int_K^{+\infty} \phi(S,T) dS \to 0$$

and

$$\frac{\partial^2 C(T,K)}{\partial K^2} = P(0,T)\phi(K,T) \to 0$$

and thus the PDE at boundary becomes

$$\frac{\partial C(T,K)}{\partial T} = -q_T C(T,K)$$

whose corresponding degenerated finite difference equation

$$\left(1 + \Delta T_i q_{T_i}\right)C(T_{i+1}, K) = C(T_i, K)$$

Similarly, at K = 0,

$$(r_T - q_T)K \frac{\partial C(T,K)}{\partial K} = 0$$

and thus naturally

$$\left(1 + \Delta T_i q_{T_i}\right)C(T_{i+1}, K) = C(T_i, K)$$

2.1.1.2 Solve Difference Equation with Transformed Grid

For short-dated options, price range is not that wide and boundary condition is not a big issue. However, when time-to-expiry increases, the impact of boundary condition becomes more significant.

In order to facilitate treatment of the boundaries, we can apply another transform (c.g. Pealat et al., 2011)

$$y = \frac{K}{K+S_0} \tag{2.17}$$

which assumes value between 0 and 1. By changing of variables, we have accordingly

$$\frac{\partial C(T,K)}{\partial T} = \frac{\partial c(T,y(K))}{\partial T}$$

$$\frac{\partial C(T,K)}{\partial K} = \frac{\partial c(T,y(K))}{\partial y} \frac{S_0}{(K+S_0)^2}$$

and

$$\frac{\partial^2 C(T,K)}{\partial K^2} = -2\frac{\partial c\big(T,y(K)\big)}{\partial y} \frac{S_0}{(K+S_0)^3} + \frac{\partial^2 c(T,y(K))}{\partial y^2} \frac{{S_0}^2}{(K+S_0)^4}$$

Substituting these back to the original PDE yields a transformed PDE

$$\begin{aligned}\frac{\partial c(T,y)}{\partial T} &+ [(r_T - q_T)y(1-y) + \sigma^2(T,K(y))y^2(1-y)]\frac{\partial c(T,y)}{\partial y} \\ &- \frac{1}{2}\sigma^2(T,K(y))y^2(1-y)^2\frac{\partial^2 c(T,y)}{\partial y^2} + q_T c(T,y) = 0\end{aligned}$$

with boundary conditions at *y*=0 and *y*=1

$$\frac{\partial^2 c(T,y)}{\partial y^2} = 0$$

and thus degenerated PDE

$$\frac{\partial c(T,y)}{\partial T} + q_T c(T,y) = 0$$

Its corresponding difference equation has difference operator

$$\begin{aligned}a_{j,j-1} = &-\frac{1}{2}\Delta T_i \sigma^2\big(T_{i+1}, y_j\big)y^2(1-y_j)^2\frac{1}{(\Delta y)^2} \\ &- \Delta T_i\left((r_{T_i} - q_{T_i})y_j(1-y_j) + \sigma^2\big(T_{i+1}, y_j\big){y_j}^2\big(1-y_j\big)\right)\frac{1}{2\Delta y}\end{aligned}$$

$$a_{j,j} = 1 + \Delta T_i q_{T_i} + \Delta T_i \sigma^2\big(T_{i+1}, y_j\big) y^2 (1 - y_j)^2 \frac{1}{(\Delta y)^2}$$

$$a_{j,j+1} = -\frac{1}{2}\Delta T_i \sigma^2\big(T_{i+1}, y_j\big) y^2 \big(1 - y_j\big)^2 \frac{1}{(\Delta y)^2} + \Delta T_i \left((r_{T_i} - q_{T_i}) y_j (1 - y_j) + \sigma^2\big(T_{i+1}, y_j\big) {y_j}^2 \big(1 - y_j\big) \right) \frac{1}{2\Delta y}$$

and in particular

$$a_{0,0} = 1 + \Delta T_i q_{T_i}$$

$$a_{n-1,n-1} = 1 + \Delta T_i q_{T_i}$$

Transformed PDE has wide range of the strike and thus it can potentially afford using more time steps per expiry. Thus, in our implementation we optimize the number of steps between T_i and T_{i+1}.

2.1.1.3 Combine Uniform Grid and Transformed Grid Adaptively

Change of variable helps calibrate longer dated option, and at the same time it may affect calibration accuracy of short-dated options. Because of the transformation

$$y = \frac{K}{K + S_0}$$

an evenly spaced grid in *y* results in an irregular grid in the strike space. As a result only 10% of the grid points fall into the range [-20% S_0, +20% S_0]. For short dated options this can greatly affect calibration accuracy because of too fewer data points between K_{min} and K_{max} and leads to inefficiency.

In order to address this issue, we can combine the two types of grids depending on the probability distribution of the terminal stock price. For short-dated options, the price range for 95% confidence is narrower and consequently the boundary conditions tend to have minimum impact on the volatilities for strikes between K_{min} and K_{max}. Thus, we can use uniform grid for short-dated options.

In contrast, the price range for long time-to-expiry can be much wider and the impact of boundary conditions cannot be neglected. In this case, we choose to use the transformed grid.

2.1.1.4 Finite Difference Schemes

In the above discussions we assume implicit scheme in order to avoid numerical instability. Applying Crank-Nicolson scheme, we have similarly

$$C(T_{i+1}, K) - C(T_i, K) = \frac{1}{2}\left(\frac{1}{2}\Delta T_i \sigma_{det}^2(T_{i+1}, K) K^2 \frac{\partial^2}{\partial K^2} - \Delta T_i \big(r_{T_i} - q_{T_i}\big) K \frac{\partial}{\partial K} - \Delta T_i q_{T_i}\right) \times \big(C(T_{i+1}, K) + C(T_i, K)\big)$$

By defining difference operator

$$A = \frac{1}{2}\Delta T_i \sigma_{det}^2(T_{i+1}, K)K^2 \frac{\partial^2}{\partial K^2} - \Delta T_i\left(r_{T_i} - q_{T_i}\right)K\frac{\partial}{\partial K} - \Delta T_i q_{T_i}$$

the difference equation can be simplified

$$\left(I - \frac{A}{2}\right)C(T_{i+1}, K) = (I + \frac{A}{2})C(T_i, K) \tag{2.18}$$

2.1.2 Interpolation of Black Volatility Surface

With change of variable, the original strikes are not necessary on the grid point anymore. Thus, we will have to interpolate implied volatility surface in such a way to ensure consistency between transformed grid points and original quotes.

A local volatility curve is assumed to be piecewise linear and flat beyond both K_{min} and K_{max}, and thus it has the same number of unknowns as the market quotes. Thus, we can choose to fit only the market quotes exactly, which, however, may fail or result in non-smooth local volatility surface or even unrealistic oscillation.

Alternatively, in order to calibrate a smooth local volatility surface we can sample more points from the implied Black volatility curve and then apply a least-squares fitting. This requires smooth interpolation of the implied Black volatilities. Without smoothing the implied Black volatility surface consistently over the expiry, the option prices might violate the calendar arbitrage condition and as a result the local volatility calibration tends to fail.

Moreover, for the local volatility to exist it requires the implied volatility interpolation to be free of arbitrages, including butterfly arbitrage and calendar arbitrage.

SABR (Hagan et al. 2002) and SVI (Gatheral, 2004) are two popular implied volatility interpolation approaches among practitioners. Empirically, we observe that the SABR method generally leads to better calibration. Furthermore, SVI can easily run into the local minimum issue and thus choosing sensible initial guesses become critical for stability.

In order to stabilize SVI calibration, we can shrink the model parameter space by either constraints or regularization. Based on each parameter's meaning, we restrict them, respectively,

$$0 \le a \le \max_i w_i$$

$$0 \le b \le \frac{4}{\tau(1 + |\rho|)}$$

$$-1 \le \rho \le 1$$

$$\min_i x_i \le m \le \max_i x_i$$

$$0 \le \sigma_{min} \le \sigma \le \sigma_{max}$$

As illustrated in Zeliade Systems (2008), when σ is equal to 0 the calibration becomes an ill-posed problem.

In contrast, SABR interpolation is smooth by design. In theory SABR is free of the butterfly arbitrage due to the consistent underlying stochastic processes, although different implementations of SABR formula may have probability leakage at the wings. Furthermore, because of calibration stability, it is less prone to calendar arbitrage as well.

Therefore, both intuitively and empirically SABR seems a better choice for interpolating implied volatility surface for the purpose of calibrating local volatility surface.

2.1.3 Volatility Extrapolation at Extreme Strikes

Because we use implicit scheme or mixed scheme, the call option prices $C(T_i, K)$ where $K > K_{max}$ or $K < K_{min}$ will affect the solved call option prices $C(T_{i+1}, K)$ with $K_{min} \leq K \leq K_{max}$. Thus, it is also important to extrapolate the market implied volatility surface, especially for long dated options.

Using SVI to extrapolate implied volatility will potential leads to negative variance at large strikes. For example, as $K \to \infty$ when $\rho = -1$

$$w(k) \to a$$

which might be negative.

It is well known that SABR formula leads to arbitrage, or negative probability density, at extreme strikes. Benaim et al. (2009) developed a functional “arbitrage-free” method to remedy the negative density of SABR in interest rates

$$P(K) = K^{\mu} \exp(a + bK + cK^2)$$

and

$$C(K) = K^{-v} \exp(a + b/K + c/K^2)$$

Unfortunately it does not really work for equities because the extrapolation will often explode. In order to fix the negative probability of the original SABR formula (Hagan et al., 2002) we adopt the higher order expansion by Paulot (2009), which is exact at the first order and has second-order correction for any strike.

2.1.4 Local Volatility Regularization for Stability

Because the set of price observations is discrete and finite, the type of problem is ill-posed. As such, it is well known that the calibration of local volatility models is instable, i.e. small changes and noise in the data may lead to substantial changes in the calibration results. When the coarse grid is discrete and bid and ask spreads are present, the instability can be even worse.

The calibration stability is of significant importance for computing Greeks and avoiding spurious P&L volatility. In order to tackle the ill-posedness of the calibration, many regularization techniques have been proposed to overcome the instability difficulties. The most common regularization techniques for volatility calibration include convex type regularization, for example, L2-norm, and entropy-based regularization.

With regularization, the objective function becomes

$$min \sum_{i=1}^{n} (\hat{C}_i - C_i)^2 + \lambda \cdot G(\sigma_{det}) \tag{2.19}$$

where $\hat{C}_i$ and C_i denote market option price and model option price at the strike K_i, respectively, and σ_{det} denotes the local volatility curve.

There are three types of penalty functions we consider imposing on the shape of the local volatility curve:

(1) Level

$$G(\sigma_{det}) = \sum_{i=1}^{n} \sigma_{det,i}^2$$

It essentially shrinks the solution to a zero volatility curve.

(2) Local slope

$$G(\sigma_{det}) = \sum_{i=1}^{n-1} (\sigma_{det,i+1} - \sigma_{det,i})^2$$

It effectively shrinks the solution to a flat volatility curve.

(3) Local convexity

$$G(\sigma_{det}) = \sum_{i=2}^{n-1} (\sigma_{det,i-1} - 2\sigma_{det,i} + \sigma_{det,i+1})^2$$

It shrinks the solution to a linear volatility curve.

In practice we observe that penalty on local slope and convexity has similar results, and it seems optimal to impose a mixture of penalty on both slope and convexity.

Another way to stabilize the calibration is to apply boundaries on the parameters, local volatilities in the current case, which is essentially equivalent to penalty on model parameters.

Local volatility regularization improves the robustness of the model calibration, especially for extreme strikes where the implied probability is so low and so is the sensitivity of the implied Black volatilities to the local volatilities. Through regularization we impose some sort of consistency on the shape of the local volatility curve.

2.2 Calibration of Hybrid Local Volatility Model

In the presence of stochastic interest rate, we have derived a similar PDE with an extra adjustment term

$$\left(1-\frac{1}{2}\Delta T_i\sigma_{hyb}^2(T_i,K)K^2\frac{\partial^2}{\partial K^2}+\Delta T_i(r_T-q_T)K\frac{\partial}{\partial K}+\Delta T_i q_T\right)C(T_{i+1},K)$$
$$-\Delta T_i K E[\frac{r_{Ti+1}-q_{Ti+1}}{B(T_{i+1})}I_{\{S_T>K\}}|\mathcal{F}_0]=C(T_i,K)$$

where the expectation term is strike dependent and is also a function of local volatility at T_i.

The expectation term can be evaluated by either PDE or Monte Carlo (c.g. Deelstra and Rayee, 2012), but that complicates the problem and the above implicit PDE approach does not have any advantage. Instead, in the following we will derive an analytic method to evaluate the hybrid local volatility adjustment.

Following Benhamou et al. (2008), we define the correction term

$$\sigma_{hyb}^2(T,K)-\sigma_{det}^2(T,K)=-\frac{f(0,T)\frac{\partial C(T,K)}{\partial K}+E[\frac{r_T}{B(T)}I_{\{S_T>K\}}|\mathcal{F}_0]}{0.5K\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2.20}$$

Changing the probability measure from risk-neutral to T-forward measure by Radon-Nikodym derivative

$$Z(T)=\frac{dQ^T}{dQ}=\frac{B(0)P(T,T)}{B(T)P(0,T)}=\frac{\exp(-\int_0^T r_t dt)}{P(0,T)}$$

yields

$$E\left[\frac{r_T}{B(T)}I_{\{S_T>K\}}\middle|F_0\right]=P(0,T)E^T\left[r_T I_{\{S_T>K\}}\middle|\mathcal{F}_0\right]$$

By definition, the correlation between the terminal interest rate and the indicator function

$$cov^T\left(r_T,I_{\{S_T>K\}}\right)=E^T\left[r_T I_{\{S_T>K\}}\middle|F_0\right]-E^T[r_T|F_0]E^T\left[I_{\{S_T>K\}}\middle|\mathcal{F}_0\right]$$
$$=E^T\left[r_T I_{\{S_T>K\}}\middle|\mathcal{F}_0\right]+\frac{f(0,T)}{P(0,T)}\frac{\partial C(T,K)}{\partial K} \tag{2.21}$$

where $r_T \equiv f(T,T)$ and $E^T[r_T|F_t]=f(t,T)$, $f(0,T)$ denotes the forward rate and

$$E^T\left[I_{\{S_T>K\}}\middle|F_0\right]=\frac{\partial}{\partial K}E^T[(S_T-K)^+|\mathcal{F}_0]=-\frac{1}{P(0,T)}\frac{\partial}{\partial K}C(T,K)$$

Thus, by plugging in these relationships the correction term can be simplified

$$\sigma_{hyb}^2(T,K)-\sigma_{det}^2(T,K)=-\frac{P(0,T)cov^T\left(r_T,I_{\{S_T>K\}}\right)}{0.5K\frac{\partial^2 C(T,K)}{\partial K^2}}=-\frac{P(0,T)cov^T\left(f(T,T),I_{\{F(T,T)>K\}}\right)}{0.5K\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2.22}$$

which can then be computed numerically.

2.2.1 Analytic approximation of the correction

Although the correction term can be computed using numerical methods in a straightforward way, in order to facilitate calibration we will derive analytic approximations using Malliavin calculus (c.g. Balland, 2005 and Benhamou et al., 2008).

In order to evaluate the covariance under the *T*-forward measure, we change measure to *T*-forward measure by applying drift adjustment. According to the Girsanov's theorem, under the *T*-forward measure the spot price and short rate are governed, respectively, by

$$\frac{dS_t}{S_t} = \left(r_t - q_t - \rho B(t,T)\sigma(t)\sigma(t,S_t)\right)dt + \sigma(t,S_t)dW_t^S \tag{2.23}$$

and

$$dx(t) = (-B(t,T)\sigma(t)^2 - kx(t))dt + \sigma(t)dW_t^r \tag{2.24}$$

The dependency between r_T and S_T results from both instantaneous correlation and the functional dependency of drift and local volatility on r_t.

By definition of forward rate and forward price, i.e.

$$f(t,T) = -\frac{dlnP(t,T)}{dT}$$

and

$$F(t,T) = \frac{S(t)}{P(t,T)}$$

they follow martingale processes in the *T*-forward measure

$$df(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r \tag{2.25}$$

and

$$\frac{dF(t,T)}{F(t,T)} = \sigma(t,S_t)dW_t^S + B(t,T)\sigma(t)dW_t^r \tag{2.26}$$

where $E[dW_t^S dW_t^r] = \rho dt$ and the local volatility function for *F*(t,T) depends on *f*(t,T),

$$B(t,T) = \int_t^T exp\left(-\int_t^u k(s)ds\right)du$$

and

$$\frac{\partial B(t,T)}{\partial T} = exp\left(-\int_t^T k(s)ds\right)$$

Moreover, define log forward price

$$X^T(t) = lnF(t,T)$$

which is governed by SDE

$$dX^T(t) = -0.5\left(\left(\sigma(t,S) + \rho B(t,T)\sigma(t)\right)^2 + (1-\rho^2)B(t,T)^2\sigma(t)^2\right)dt + \sigma(t,S_t)dW_t^S + B(t,T)\sigma(t)dW_t^r$$

Since a strong solution of $f(t,T)$ is given by

$$f(T,T) = f(0,T) + \int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r$$

we can write the covariance

$$cov^T\left(f(T,T), I_{\{F(T,T)>K\}}\right) = E^T\left[I_{\{F(T,T)>K\}}\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r\right]$$

Assuming the volatility function $\frac{\partial B(t,T)}{\partial T}\sigma(t)$ is hypo-elliptic, its Skorohod integral coincides with the Ito integral. By means of the integration by part (c.g. Nualart, 1995), the expectation of the product of the digital process $I_{\{F(t,T)<K\}}$ and the Skorohod integral is equal to the expectation of the integral

$$E^T\left[I_{\{F(T,T)>K\}}\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r\right] = E^T\left[\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)D_t^r I_{\{F(T,T)>K\}}dW_t^r\right]$$

where by chain rule the Malliavin derivative w.r.t the Brownian motion W_t^r of the indicative function

$$D_t^r I_{\{F(T,T)>K\}} = \delta(X^T(T) - lnK)\rho D_t^r lnF(T,T)$$

Thus, the covariance between forward rate and indicate function

$$cov^T\left(f(T,T), I_{\{F(T,T)>K\}}\right) = E^T\left[\delta(lnF(T,T) - lnK)\rho\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)D_t^r lnF(T,T)dW_t^r\right]$$

$$= E^T[\delta(X^T(T) - lnK)]E^T\left[\rho\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)E^T[D_t^r lnF(T,T)|X^T(T) = lnK]dW_t^r\right]$$

where we use the iterated expectation and the terminal probability density

$$E^T[\delta(X^T(T) - lnK)] = \frac{K}{P(0,T)}\frac{\partial^2 C(T,K)}{\partial K^2}$$

Define conditional covariance

$$cov^T(f(T,T), X^T(T)|X^T(T) = lnK) = E^T\left[X^T(T)\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r\middle|X^T(T) = lnK\right]$$

and again by integration by part

$$cov^T(f(T,T), lnF(T,T)|X^T(T) = lnK) = E^T\left[\rho\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r\, D_t^r lnF(T,T)\middle|X^T(T) = lnK\right]$$

Substituting the two covariance formulas into the correction term gives rise to a simplified formula

$$\sigma_{hyb}^2(T,K) - \sigma_{det}^2(T,K) = -2cov^T(f(T,T), X^T(T)|X^T(T) = lnK) \tag{2.27}$$

Given the SDEs for *f*(t,T) and *F*(t,T) we can calculate the covariance through an integral

$$cov^T(f(T,T), X^T(T)|X^T(T) = lnK)$$

$$\approx E^T\left[\left[\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)dW_t^r \int_0^T \left(\sigma_{hyb}(t,S_t)dW_t^S + B(t,T)\sigma(t)dW_t^r\right)\right|X^T(T) = lnK\right]$$

$$= \rho\int_0^T \frac{\partial B(t,T)}{\partial T}\sigma(t)E^T[\sigma_{hyb}(t,S_t)|X^T(T) = lnK]dt + E^T\left[\int_0^T \frac{\partial B(t,T)}{\partial T}B(t,T)\sigma(t)^2 dt\right]$$

where additional assumptions are required to calculate the conditional expectation of local volatility $E^T[\sigma_{hyb}(t,S_t)|X^T(T) = lnK]$.

In general, by Taylor expansion the conditional expectation

$$E^T\left[\sigma_{hyb}(t,S_t)\middle|S_T = K\right] = \sigma_{hyb}(t, E^T[S_t|S_T = K]) + \tfrac{1}{2}\tfrac{d^2\sigma_{hyb}(t,S_t)}{dS_t^2}Var(S_t|S_T = K) + \cdots \tag{2.28}$$

Ignoring the convexity term, we obtain a first-order approximation

$$E^T\left[\sigma_{hyb}(t,S_t)\middle|S_T = K\right] \approx \sigma_{hyb}(t, E^T[S_t|S_T = K])$$

where $E^T[S_t|S_T = K]$ can be estimated according to Brownian bridge method.

In the *T*-forward measure, the log spot stock price follows

$$dlnS_t = (r_t - q_t - \rho B(t,T)\sigma(t)\sigma(t,S_t) - 0.5\sigma(t,S_t)^2)dt + \sigma(t,S_t)dW_t^S \tag{2.29}$$

When the drift is deterministic, we can decompose lnS_t

$$lnS_t = \int_0^t \mu(u)du + \int_0^t \sigma(t,S_t)dW_t^S$$

where $\mu(t)$ is a time-dependent drift function.

In the case where the local volatility function is constant, $\frac{lnS_t-\int_0^t \mu(u)du}{\sigma_S}$ is a Brownian motion and thus $E^T\left[lnS_t - \int_0^t \mu(u)du \middle| S_T = K\right]$ is a linear function of time t such that *lnS*(0) = *lnS₀* and *lnS*(T)=*lnK*.

If the local volatility function is not constant, but rather time-dependent, we change time to

$$\tau = \frac{T}{\int_0^T \sigma(u)^2 du}\int_0^t \sigma(u)^2 du$$

in such a way that the stochastic process has constant volatility in time scale of τ.

In a general case where the local volatility depends on the price level, we don't have a simple solution for the conditional expectation of the stock price. However, we can still apply the log-normal approximation

$$\sigma_{hyb}(t, E^T[S_t|S_T = K]) = \sigma_{hyb}\left(t, \ln(S_0) + \frac{\ln(K) - \ln(S_0)}{T}t\right)$$

which is consistent with the first order approximation of the implied volatility of a local volatility model by Berestycki et al. (2002). With this approximation we essentially calculate the conditional covariance along the straight line from the spot log price to the log strike.

Benhamou et al. (2008) and similarly Grzelak (2008) try to derive analytic approximation for the correction term using spot prices

$$cov^T(r_T, lnS_T|lnS_T = lnK)) \approx \rho \int_0^T \sigma_{hyb}(t,K)\frac{\partial B(t,T)}{\partial T}\sigma(t)dt \tag{2.30}$$

which includes only the dependency due to correlation, but not the drift impact, and approximates it along the path *S*(*t*) = *K*. Without the second term the impact of interest rate volatility is completely neglected in case where $\rho = 0$. In contrast, by transformation to the forward price we essentially incorporate the drift impact into the diffusion, and take into account both impacts on the equity thanks to correlation and interest rate volatility.

Bloch and Yukio (2008) derived the same approximation as ours with the adjustment purely due to interest rate volatility through variance and covariance analysis in the context of multi-currency. However, same as Benhamou et al. (2008) they estimate the conditional covariance along the path *S*(*t*) = *K*, which may overestimate (underestimate) the total volatility, and thus adjustment, for OTM (ITM) strikes.

Noticing that

$$\frac{\partial B(t,T)}{\partial T} = exp\left(-\int_t^T k(s)ds\right)$$

which is an exponentially decaying function. As a result, for fast mean-reverting interest rate models, the initial local volatility has smaller impact while the local volatility that is close to the end of the period has bigger impact. Hence, in this case approximating the covariance along the path $S(t) = K$ may not be a bad choice.

Finally, we have the correction term along the straight line path

$$\begin{aligned}
&\sigma_{hyb}^2(T,K) - \sigma_{det}^2(T,K) \\
&\quad = -2\rho \int_0^T \sigma_{hyb}\left(t, \ln(S_0) + \frac{\ln(K) - \ln(S_0)}{T} t\right) \frac{\partial B(t,T)}{\partial T} \sigma(t) dt \\
&\quad - 2 \int_0^T B(t,T) \frac{\partial B(t,T)}{\partial T} \sigma(t)^2 dt \\
&\quad = -2\rho \int_0^T \sigma_{hyb}\left(t, \ln(S_0) + \frac{\ln(K) - \ln(S_0)}{T} t\right) \exp(-k(T-t))\sigma(t) dt \\
&\quad -2 \int_0^T \frac{1-\exp(-k(T-t))}{k} \exp(-k(T-t))\sigma(t)^2 dt
\end{aligned} \tag{2.31}$$

and along the flat line path

$$\begin{aligned}
\sigma_{hyb}^2(T,K) - \sigma_{det}^2(T,K) &= -2\rho \int_0^T \sigma_{hyb}(t,K) \frac{\partial B(t,T)}{\partial T} \sigma(t) dt - 2 \int_0^T B(t,T) \frac{\partial B(t,T)}{\partial T} \sigma(t)^2 dt \\
&= -2\rho \int_0^T \sigma_{hyb}(t,K) \exp(-k(T-t))\sigma(t) dt \\
&\quad -2 \int_0^T \frac{1-\exp(-k(T-t))}{k} \exp(-k(T-t))\sigma(t)^2 dt
\end{aligned} \tag{2.32}$$

2.2.2 Numerical Solution for the Adjustment

In the above we have derived the forward PDE for the local volatility model and then analytic approximations of the correction term due to the stochastic interest rate.

In order to solve for the corrected local volatility $\sigma_{hyb}^2(T,K)$, we rewrite the correction term

$$\begin{aligned}
&\sigma_{hyb}^2(T,K) - \sigma_{det}^2(T,K) \\
&\quad = -2\rho \int_0^{T-\Delta t} \sigma_{hyb}\left(t, \ln(S_0) + \frac{\ln(K) - \ln(S_0)}{T} t\right) \exp(-k(T-t))\sigma(t) dt \\
&\quad - 2\rho\sigma_{hyb}(t, \ln(K))\sigma(T)\Delta t - 2 \int_0^T \frac{1-\exp(-k(T-t))}{k} \exp(-k(T-t))\sigma(t)^2 dt
\end{aligned}$$

In the following we will present a few different ways to numerically solve the above equation for the correction term. Since the methods are the same for straight-line path and flat-line path approximation, we will present below only for the latter case.

2.2.2.1 Fixed point method

Because the local volatility appears on both sides, we can apply a fixed-point method to solve for $\sigma_{hyb}^2(T, K)$ iteratively.

2.2.2.2 Quadratic equation

Assuming the local volatility function is piecewise constant and right continuous we can use a bootstrapping procedure to solve for the corrected local volatility function. To this end, the above equation can be rewritten as

$$\sigma_{hyb}^2(T_{i+1}, K) + 2\rho\sigma_{hyb}(T_{i+1}, K) \int_{Ti}^{Ti+1} \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)dt - \sigma_{det}^2(T_{i+1}, K) \\ + 2\rho \int_0^{T_i} \sigma_{hyb}(t, K) \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)dt + 2 \int_0^{T_{i+1}} B(t, T_{i+1}) \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)^2 dt = 0$$

which is a quadratic equation that can be solved easily. Because in most case

$$2\rho \int_{Ti}^{Ti+1} \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)dt > 0$$

only one of the quadratic equation is valid.

Let

$$b = 2\rho \int_{Ti}^{Ti+1} \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)dt$$

and

$$c = \sigma_{det}^2(T_{i+1}, K) - 2\rho \int_0^{T_i} \sigma_{hyb}(t, K) \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)dt - 2 \int_0^{T_{i+1}} B(t, T_{i+1}) \frac{\partial B(t, T_{i+1})}{\partial T} \sigma(t)^2 dt$$

Then, we have

$$\sigma_{hyb}(T_{i+1}, K) = \frac{\sqrt{b^2 + 4c} - b}{2}$$

If we assume other types of interpolation of the local volatility along the maturity, $\int_{Ti}^{Ti+1} \sigma_{hyb}(T_{i+1}, K) \frac{\partial B(t,T)}{\partial T} \sigma(t)dt$ will be evaluated differently but still as a function of $\sigma_{hyb}(T_{i+1},K)$.

2.2.2.3 Finite difference method

Note that all the above formulas do not relying on the interpolation of $\sigma_{hyb}(T,K)$ along either the underlying or strike. Thus, we have the freedom to choose interpolation to better reprice market options.

Following Andreasen and Huge (2008), we can derive an implicit PDE approach to bootstrap the hybrid local volatility matrix assuming piecewise constant volatility along both the time and strike direction

$$\frac{1}{2}(\sigma_{hyb}^2(T,K) - 2E^T[cov^T(r_T, lnS_T)|S_T = K])K^2\frac{\partial^2 C(T,K)}{\partial K^2}$$
$$= \frac{\partial C(T,K)}{\partial T} + (r_T - q_T)K\frac{\partial C(T,K)}{\partial K} + q_T C(T,K)$$

whose finite difference equation then is then given by

$$\left(1 - \frac{1}{2}\Delta T_i \sigma_{hyb}^2(T_{i+1},K)K^2\frac{\partial^2}{\partial K^2} + E^T[cov^T(r_{Ti+1}, lnS_{Ti+1})|S_{Ti+1} = K])\Delta T_i K^2\frac{\partial^2}{\partial K^2}\right.$$
$$\left. + \Delta T_i(r_{Ti} - q_{Ti})K\frac{\partial}{\partial K} + \Delta T_i q_T\right)C(T_{i+1},K) = C(T_i,K)$$

where as derived earlier the expectation term

$$E^T[cov^T(r_T, lnS_T)|S_T = K] = \rho\int_0^T \sigma_{hyb}(t,K)\frac{\partial B(t,T)}{\partial T}\sigma(t)dt + \int_0^T B(t,T)\frac{\partial B(t,T)}{\partial T}\sigma(t)^2 dt$$

2.2.2.4 Monte Carlo Method

As pointed out by Gurrieri (2012), this type of analytical approximation method may suffer from loss of accuracy at high volatilities and/or large maturities, and works only for a specific class of 1-factor interest rate models. As a more generic method, we can use Monte Carlo method to bootstrap the hybrid local volatility function

$$\sigma_{hyb}^2(T_{i+1},K) = \frac{\frac{\partial C(T_{i+1},K)}{\partial T} - KE[\frac{r_{Ti+1} - q_{Ti+1}}{B(T_{i+1})}I_{\{S_{Ti+1}>K\}}|F_0] + q_T C(T_{i+1},K)}{0.5K^2\frac{\partial^2 C(T_{i+1},K)}{\partial K^2}} \quad (2.33)$$

where the *r.h.s* is a function of $\sigma_{hyb}^2(T_i,K)$, $0 \leq T < T_{i+1}$, and $\sigma_{hyb}^2(T_0,K) = \sigma_{det}^2(T_0,K)$. Compared to the aforementioned implicit FD equation, it is essentially an explicit FD equation.

Again, this bootstrapping procedure scheme can use either explicit or implicit scheme. If we adopt an implicit scheme, we have

$$\sigma_{hyb}^2(T_i,K) = \frac{\frac{\partial C(T_{i+1},K)}{\partial T} - KE[\frac{r_{Ti+1} - q_{Ti+1}}{B(T_{i+1})}I_{\{S_{Ti+1}>K\}}|F_0] + q_T C(T_{i+1},K)}{0.5K^2\frac{\partial^2 C(T_{i+1},K)}{\partial K^2}} \quad (2.34)$$

In the implicit scheme, it requires a fixed-point method with a few iterations because $\sigma_{hyb}^2(T_i,K)$ appears on both sides. Advantage of this method is that it might reduce calibration error.

As one would expect, this Monte Carlo method is slower than the analytical approximation, but it works with any interest rate model.

3. Simulation

The simulation of a hybrid model jointly evolves the stochastic interest rate and stock price, which can be carried out in two steps sequentially.

3.1 Simulation of Interest Rate

The solution of the SDE for the interest rate state variable $x(t)$

$$dx(t) = -kx(t)dt + \sigma(t)dW_t^r$$

is given by

$$x(t) = x_0 \exp(-kt) + \exp(-kt)\int_0^t \sigma(s)\exp(ks)\,dW_s^r$$
$$= x_0 \exp(-kt) + \exp(-kt)\sqrt{\int_0^t \sigma(s)^2 \exp(2ks)\,ds}\,Z$$

where Z is a standard Gaussian random number. This can be used to simulate $\Delta x(t_i) = x(t_i) - x(t_{i-1})$

$$\Delta x(t_i) \approx x(t_{i-1})(\exp(-k\Delta t_i) - 1) + \sigma(t_i)\sqrt{\frac{1-\exp(-2k\Delta t_i)}{2k}}\Delta W_t^r \qquad (3.1)$$

Then, the short rate will be given by $r(t_i) = x(t_i) + \varphi(t_i)$, and the numeraire is calculated by rolling forward

$$B(t_{i+1}) = B(t_i)\exp(-r(t_i)\Delta t_i)$$

3.2 Simulation of Stock Price

The stock price follows a SDE with local volatility

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma(t, S_t)dW_t^S$$

and defining $Y_t = ln(S_t)$ we obtain Y_t

$$dY_t = (r_t - q_t - 0.5\sigma_{loc}(t, Y_t)^2)dt + \sigma_{loc}(t, Y_t)dW_t^S$$

Since the local volatility function is piecewise constant along both time and underlying, it can be discretized

$$\Delta Y(t_i) \approx (r(t_i) - q(t_i) - 0.5\sigma_{loc}(t, Y_t)^2)\Delta t + \sigma_{loc}(t, Y_t)\Delta W_t^S \qquad (3.2)$$

which is usually called Log-Euler scheme.

If there is a interpolation knot point in the time interval $[t_i, t_{i+1}]$, we use average volatility

$$\bar{\sigma}(t_i, \exp(Y(t_i))) = \sqrt{\frac{1}{\Delta t_i}\int_{t_i}^{t_{i+1}} \sigma_{loc}^2(t, Y_t)du}$$

When the volatility is high or Δt is large, we can also apply Milstein scheme or predictor corrector scheme to correct for some bias.

Predictor-corrector schemes are designed to retain the numerical stability properties of similar implicit scheme, while avoiding the additional computational effort required for solving an algebraic equation in each step. Using the predictor-corrector scheme, we first apply Euler scheme to simulate the predictor

$$\hat{Y}(t_{i+1}) \approx Y(t_i) + (r(t_i) - q((t_i) - 0.5\sigma_{loc}(t, Y_t)^2)\Delta t + \sigma_{loc}(t, Y_t)\Delta W_t^S$$

where ΔW_t^S is correlated with ΔW_t^r

$$\Delta W_t^S = \rho \Delta W_t^r + \sqrt{1-\rho}\Delta W_t$$

with W_t an independent Brownian motion from W_t^r.

Then, the volatility is adjusted as the average

$$\sigma = 0.5(\sigma_{loc}\left(t_i, Y_{t_i}\right) + \sigma_{loc}\left(t_{i+1}, \hat{Y}_{t_{i+1}}\right)) \tag{3.3}$$

and accordingly the drift is adjusted to maintain no-arbitrage (Kloeden et al. 1994)

$$\mu = r(t_i) - q(t_i) - 0.5\frac{\sigma_{loc}(t, Y_t)^2 + \sigma_{loc}(t_{i+1}, \hat{Y}(t_{i+1}))^2}{2}$$

$$-0.25\left(\sigma_{loc}(t, Y_t)\frac{\partial \sigma_{loc}(t, Y_t)}{\partial Y_t} + \sigma_{loc}(t_{i+1}, \hat{Y}(t_{i+1}))\frac{\partial \sigma_{loc}(t_{i+1}, \hat{Y}(t_{i+1}))}{\partial Y_t}\right) \tag{3.4}$$

where the derivative can be calculated assuming the local volatility is piecewise log-linear.

Finally, we have

$$\Delta Y(t_i) \approx \mu \Delta t + \sigma \Delta W_t^S$$

Numerically we can show that predictor-corrector scheme works better than Euler, especially when the number of steps is very small. Note that in the case of flat volatility, it is the same as Euler scheme.

Alternatively, we can apply Milstein scheme to take into account the effect of price dependent volatility

$$\Delta Y_t = \int_t^{t+\Delta t} dY_s = \int_t^{t+\Delta t} r_s ds + \int_t^{t+\Delta t} q_s ds - 0.5\sigma_{loc}(t, Y_t)^2\Delta t + \sigma_{loc}(t, Y_t)\Delta W_t^S$$

$$+0.5\frac{\partial \sigma_{loc}(t, Y_t)}{\partial Y_t}\sigma_{loc}(t, Y_t)((\Delta W_t^S)^2 - \Delta t) \tag{3.5}$$

Three different simulation schemes are tested out to compare their convergence and accuracy, and predictor-corrector has the best performance.

3.3 Simulation of VIX

When we simulate stock prices using a local volatility model, more than often we need to calculate implied volatilities or VIX. Short term implied volatilities can be calculated using asymptotic approximations derived from heat kernel expansion method, for example, see Gatheral et al. (2009).

VIX is a volatility index created by CBOE to measure the expected 30-day volatility. Given an implied volatility curve, VIX can be approximated by a model-free method. In the following we present a simpler and faster to approximate VIX.

By definition, VIX^2 is

$$VIX^2(t) = \frac{10000}{T-t} E\left[\int_t^T \sigma_{hyb}^2(u, S_t)du \middle| \mathcal{F}_t\right] \tag{3.6}$$

where T is usually 1M. Because the impact of stochastic interest rate on short-term options is very small, we ignore the impact of stochastic drift in this calculation.

A first order approximation is given by

$$VIX^2(t) \approx \frac{10000}{T-t} E\left[\int_t^T \sigma_{hyb}^2(t, S_t)du\right] = 10000\sigma_{hyb}^2(t, S_t)$$

Applying trapezoidal rule for numerical integration gives rise to

$$VIX^2(t) \approx 5000 \cdot E\left[\sigma_{hyb}^2(t, S_t) + \sigma_{hyb}^2(T, S_T)\right]$$

Assuming the probability distribution of $\ln S_T/S_t$ is approximated by a Gaussian distribution

$$N((r - q - 0.5\sigma_{hyb}^2(t, S_t), \sigma_{hyb}^2(t, S_t)(T-t))$$

and then by Taylor expansion the expectation

$$E\left[\sigma_{hyb}^2(T, S_T)\right] \approx \sigma_{hyb}^2(T, E[S_T]) + \frac{1}{2}\left(\left(\frac{\partial}{\partial S_T}\sigma_{hyb}(T, E[S_T])\right)^2 + \sigma_{hyb}(T, E[S_T])\frac{\partial^2}{\partial {S_T}^2}\sigma_{hyb}(T, E[S_T])\right)$$
$$\times E[S_T]^2(\exp\left(\sigma_{hyb}^2(t, S_t)(T-t)\right) - 1) \tag{3.7}$$

Further ignoring convexity of the local volatility curve, we obtain

$$E\left[\sigma_{hyb}^2(T, S_T)\right] \approx \sigma_{hyb}^2(T, E[S_T]) + \frac{1}{2}\left(E[S_T]\frac{\partial}{\partial S_T}\sigma_{hyb}(T, E[S_T])\right)^2 (\exp\left(\sigma_{hyb}^2(t, S_t)(T-t)\right) - 1)$$

$$\approx \sigma_{hyb}^2(T, E[S_T]) + \frac{1}{2}\left(\sigma_{hyb}(t, S_t)(T-t)E[S_T]\frac{\partial}{\partial S_T}\sigma_{hyb}(T, E[S_T])\right)^2 \tag{3.8}$$

where the second term is always positive and is small at the two wings and larger in the middle.

4. Implementation and Numerical Tests

The implementation of the hybrid local volatility model calibration follows a few steps:

(1) Use either SABR or SVI to smooth the implied volatility surface
(2) Calibrate the local volatility surface using finite difference method
(3) Calculate analytically the correction term due to stochastic interest rate

In a computer with Intel Core I5 3570 3.4GHz Quad Core CPU and 8Gb RAM, the entire calibration of a hybrid local volatility model can be completed within 0.2 second. Monte Carlo method with predictor-corrector scheme is then applied to price European options.

Using the SPX 500 implied volatility surface on 1/29/2016 we first demonstrate the impact on the model implied Black volatility from both interest rate volatility and correlation between SPX and interest rate.

The Table 4.1 shows the market implied Black volatility surface for SPX as of 1/29/2016, which is a typical volatility surface around 20%.

Table 4.1 Market implied Black volatility surface

Expiry/Strike	50%	60%	70%	80%	90%	95%	100%	105%	110%	120%	130%	150%
6/16/2016	43.69%	38.60%	33.71%	28.82%	23.82%	21.26%	18.67%	16.10%	13.75%	10.60%	9.50%	9.89%
9/16/2016	39.54%	35.21%	31.08%	27.01%	22.96%	20.91%	18.85%	16.81%	14.87%	11.82%	10.28%	9.68%
12/16/2016	37.24%	33.32%	29.60%	26.01%	22.46%	20.68%	18.91%	17.15%	15.47%	12.63%	10.77%	9.26%
6/15/2017	34.70%	31.34%	28.15%	25.11%	22.17%	20.73%	19.31%	17.92%	16.56%	14.11%	12.36%	10.85%
12/15/2017	32.86%	29.89%	27.13%	24.50%	21.99%	20.78%	19.59%	18.43%	17.32%	15.29%	13.71%	12.19%
6/15/2018	31.67%	29.01%	26.54%	24.22%	22.02%	20.97%	19.95%	18.97%	18.02%	16.31%	14.88%	13.20%
12/21/2018	30.85%	28.43%	26.19%	24.10%	22.14%	21.20%	20.30%	19.43%	18.60%	17.09%	15.81%	14.16%
12/20/2019	29.81%	27.88%	26.07%	24.36%	22.75%	21.99%	21.25%	20.54%	19.87%	18.63%	17.54%	15.91%
12/18/2020	29.73%	28.02%	26.43%	24.94%	23.56%	22.90%	22.27%	21.67%	21.09%	20.03%	19.08%	17.56%
1/27/2021	29.77%	28.08%	26.51%	25.03%	23.66%	23.01%	22.39%	21.79%	21.23%	20.18%	19.24%	17.73%

In order to see how interest rate volatility and correlation affect the SPX option Black volatilities, we calibrate the local volatility model with deterministic interest rate and then increase the volatility of the stochastic interest rate as well as correlation.

Table 4.2 Model implied Black volatility bias with stochastic interest rate and ρ=0

Expiry/Strike	50%	60%	70%	80%	90%	95%	100%	105%	110%	120%	130%	150%
6/16/2016	2.66%	-0.08%	-0.99%	-0.90%	-0.31%	-0.10%	0.08%	0.10%	-0.25%	-3.46%		
9/16/2016	3.26%	1.25%	0.37%	-0.02%	-0.07%	-0.03%	0.01%	0.05%	-0.02%	-0.80%	-1.50%	
12/16/2016	2.11%	0.92%	0.34%	0.06%	-0.04%	-0.02%	0.05%	0.10%	0.08%	-0.36%	-1.15%	
6/15/2017	1.37%	0.47%	0.09%	-0.04%	-0.03%	-0.03%	0.00%	0.04%	0.08%	-0.05%	-0.49%	-0.86%
12/15/2017	1.27%	0.65%	0.31%	0.07%	0.00%	0.03%	0.06%	0.10%	0.13%	0.17%	0.01%	-0.80%
6/15/2018	0.91%	0.45%	0.19%	0.04%	0.04%	0.08%	0.12%	0.16%	0.19%	0.19%	0.09%	-0.41%
12/21/2018	0.80%	0.38%	0.12%	0.01%	0.03%	0.07%	0.12%	0.17%	0.20%	0.24%	0.21%	-0.13%
12/20/2019	0.72%	0.40%	0.26%	0.25%	0.26%	0.28%	0.31%	0.35%	0.38%	0.41%	0.37%	0.06%
12/18/2020	0.57%	0.28%	0.20%	0.21%	0.25%	0.29%	0.32%	0.34%	0.36%	0.39%	0.38%	0.25%
1/27/2021	0.55%	0.32%	0.22%	0.23%	0.28%	0.31%	0.34%	0.36%	0.39%	0.42%	0.42%	0.29%

Table 4.3 Model implied Black volatility bias with stochastic interest rate and ρ=0.3

Expiry/Strike	50%	60%	70%	80%	90%	95%	100%	105%	110%	120%	130%	150%
6/16/2016	2.68%	-0.04%	-0.94%	-0.85%	-0.25%	-0.04%	0.14%	0.17%	-0.17%	-3.23%		
9/16/2016	3.34%	1.35%	0.48%	0.09%	0.04%	0.09%	0.13%	0.18%	0.11%	-0.66%	-1.38%	
12/16/2016	2.25%	1.08%	0.50%	0.22%	0.13%	0.15%	0.22%	0.28%	0.26%	-0.18%	-0.96%	
6/15/2017	1.61%	0.73%	0.35%	0.23%	0.25%	0.25%	0.27%	0.32%	0.36%	0.24%	-0.21%	-0.79%
12/15/2017	1.61%	1.01%	0.67%	0.43%	0.38%	0.40%	0.44%	0.48%	0.51%	0.55%	0.39%	-0.47%
6/15/2018	1.36%	0.90%	0.65%	0.51%	0.51%	0.55%	0.60%	0.64%	0.67%	0.66%	0.57%	0.05%
12/21/2018	1.33%	0.94%	0.68%	0.58%	0.61%	0.65%	0.69%	0.74%	0.77%	0.81%	0.79%	0.44%
12/20/2019	1.45%	1.14%	1.00%	0.99%	1.01%	1.03%	1.05%	1.09%	1.12%	1.16%	1.12%	0.82%
12/18/2020	1.47%	1.18%	1.11%	1.11%	1.16%	1.19%	1.22%	1.24%	1.27%	1.29%	1.29%	1.16%
1/27/2021	1.47%	1.24%	1.15%	1.15%	1.20%	1.23%	1.26%	1.28%	1.30%	1.34%	1.34%	1.21%

In the case where ρ =0 the implied volatility bias caused by interest rate volatility increases with the time-to-expiry and reaches about 34bps for 5Y options as shown in Table 4.2. When we increase the correlation ρ to the level 0.3, the implied volatility bias further increases across the volatility surface and reaches about 126bps for 5Y options as in Table 4.3. This illustrates the significant volatility contribution of the interest rate volatility to long term equity option volatilities by both interest rate volatility and correlation between interest rate and equity return. As derived earlier, the impact monotonically increases as interest rate volatility and/or the correlation increases.

After calibrating the hybrid local volatility model, we use Monte Carlo method to price European options and calculate the model implied Black volatilities. By comparing the market Black volatilities with model implied Black volatilities, we can check the calibration accuracy. The table below shows the calibration error in percentage across expiries and relative strikes as of 1/29/2016.

Table 4.4 Hybrid local volatility model calibration errors

Expiry/Strike	50%	60%	70%	80%	90%	95%	100%	105%	110%	120%	130%	150%
6/16/2016	2.55%	-0.11%	-0.97%	-0.86%	-0.26%	-0.05%	0.13%	0.16%	-0.19%	-3.40%		
9/16/2016	3.17%	1.23%	0.39%	0.02%	-0.03%	0.02%	0.05%	0.10%	0.02%	-0.76%	-1.53%	
12/16/2016	2.04%	0.91%	0.35%	0.08%	-0.01%	0.00%	0.08%	0.13%	0.11%	-0.34%	-1.17%	
6/15/2017	1.34%	0.47%	0.10%	-0.02%	-0.01%	-0.01%	0.01%	0.06%	0.09%	-0.05%	-0.55%	-1.10%
12/15/2017	1.22%	0.63%	0.30%	0.05%	-0.01%	0.01%	0.04%	0.08%	0.11%	0.13%	-0.08%	-1.00%
6/15/2018	0.86%	0.41%	0.15%	-0.01%	-0.01%	0.03%	0.07%	0.10%	0.12%	0.10%	-0.03%	-0.62%
12/21/2018	0.72%	0.32%	0.05%	-0.08%	-0.06%	-0.02%	0.01%	0.06%	0.08%	0.10%	0.04%	-0.42%
12/20/2019	0.58%	0.25%	0.09%	0.06%	0.07%	0.08%	0.11%	0.14%	0.16%	0.17%	0.09%	-0.34%
12/18/2020	0.38%	0.06%	-0.06%	-0.07%	-0.04%	-0.02%	0.02%	0.03%	0.05%	0.04%	0.00%	-0.25%
1/27/2021	0.35%	0.09%	-0.03%	-0.06%	-0.03%	0.00%	0.03%	0.04%	0.05%	0.06%	0.02%	-0.22%

Except for extreme strikes the majority of calibration errors are within 10bps, which is much lower than the bid-ask spread in term of Black volatility. For extreme strikes the error is partially due to the Monte Carlo pricing error.

The regularization on the local volatility surface helps improve the stability of the model calibration, in terms of both the smoothness of local volatility curves along the strike and expiration and recalibration after perturbation. The regularization method helps stabilize the calibration as shown in the graph below which compares the local volatility curve for the case with and without regularization.

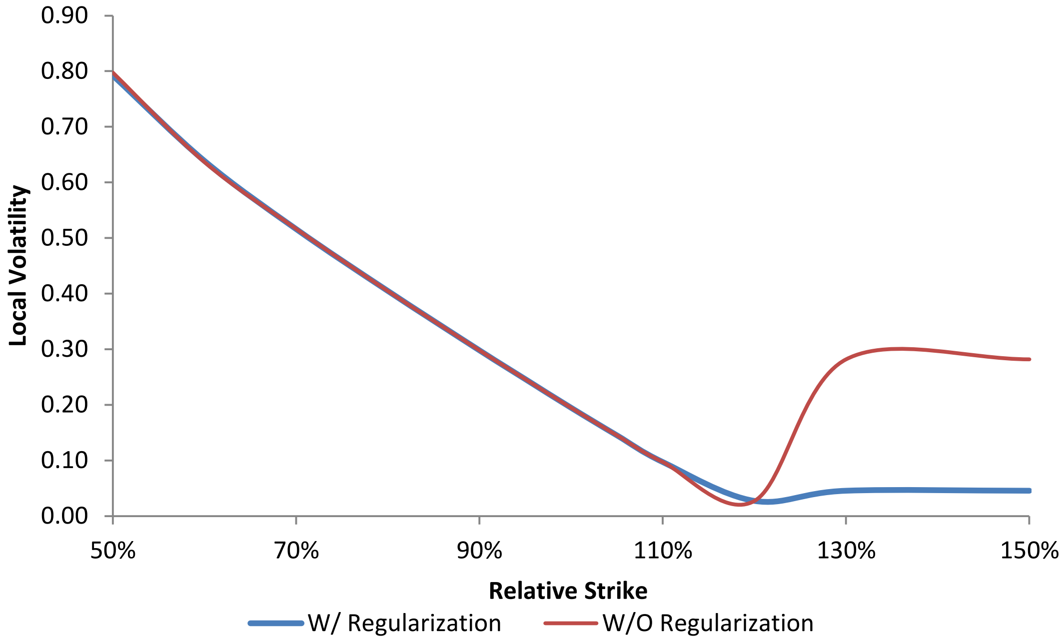

Figure 4.1 Compare local volatility curve with and without regularization

Local volatility regularization not only smooths the local volatility curve but also improve the calibration accuracy in most cases by avoiding overfitting to local noises in similar way to Ridge regression. The robustness of local volatility calibration is in particular important for computing Greeks. If the local volatility curve swings after perturbing the market volatility or other inputs, it leads to spurious risk exposure.

5. Conclusion

In this paper we derived the hybrid local volatility model with stochastic interest rate, applied finite difference method to solve the Dupire forward PDE with both uniform grid and transformed grid, and also developed analytic approximate adjustment to correct local volatilities. The numerical results show significant bias can be caused by interest rate volatility and correlation without taking into account stochastic interest rate, especially for long term options. Through numerical examples we also demonstrate that the efficiency, accuracy, and robustness of the calibration methods are satisfactory in practice. Although throughout the paper we assume the interest rate is one-factor Gaussian, the model can be easily extended to multi-factor models, for example, two-factor Gaussian model. Moreover, jumps are usually added to the local volatility model as a resort to the forward skew issue, and the hybrid local volatility model can incorporate jumps that are independent of the interest rate in an almost straightforward way.

References

Abasto, F. Damian, Bernhard Hienzsch, and Mark P. Kust (2013) Monte Carlo Pricing with Local Volatility Grids. Available at SSRN: https://ssrn.com/abstract=2252916 or http://dx.doi.org/10.2139/ssrn.2252916

Albani, V., A. De Cezaro and J. P. Zubelli (2013) Convex Regularization of Local Volatility Estimation in a Discrete Setting. Available at SSRN: https://ssrn.com/abstract=2308138

Andersen, B. G. Leif and Rupert Brotherton-Ratcliffe (1998) The equity option volatility smile: an implicit finite-difference approach, *Risk*.

Andreasen, J. and B. Huge (2011) Volatility Interpolation, *Risk*, March 2011.

Balland, P. (2005) Stochastic Volatility Model with Interest Rate Volatility, *ICBI Global Derivative Proceedings*.

Benaim, S., M. Dodgson, and D Kainth (2009). An arbitrage-free method for smile extrapolation. *Royal Bank of Scotland.*

Berestycki, H., J. and I. Florent (2002) Asymptotics and calibration of local volatility models, *Quantitative Finance*, Vol 2, 61-69.

Benhamou, E. , A. Gruz and A. Rivoira(2008) Stochastic interest rate for local volatility hybrids models. Available at SSRN: https://ssrn.com/abstract=1107711. or http://dx.doi.org/10.2139/ssrn.1107711

Bloch, Daniel and Yukio Nakashima (2008) Multi-Currency Local Volatility Model. Available at SSRN: https://ssrn.com/abstract=1153337 or http://dx.doi.org/10.2139/ssrn.1153337

Fabio Mercurio (2006). Interest Rate Models — Theory and Practice: with Smile, Inflation and Credit. Springer Verlag.

Deelstra Griselda and Gregory Rayee(2013) Local Volatility pricing Models for Long-dated FX Derivatives. Applied Mathematical Finance, 20(4): 380-402.

Bruno Dupire (1994) Pricing with a smile. *Risk*, 7(1):18–20.

Gatheral, Jim (2001) Lecture 3: Asymptotics and Dynamics of the Volatility Skew, Case Studies in Financial Modelling Course Notes, Courant Institute of Mathematical Sciences, Fall Term, 2001.

Gatheral, J. (2004) A Parsimonious Arbitrage-Free Implied Volatility Parameterization with Application to the Valuation of Volatility Derivatives. Global Derivatives and Risk Management.

Gatheral, Jim (2006) The Volatility Surface: A Practitioner's Guide, John Wiley & Sons Inc.

Gatheral, Jim, Elton P. Hsu, Peter Laurence, Cheng Ouyang and Tai-Ho Wang (2009) Asymptotics of implied volatility in local volatility models. Available at SSRN: https://ssrn.com/abstract=1542077

Grzelak, L.A., N. Borovykh, S. van Weeren, and C. W. Oosterlee (2008) Incorporating an Interest Rate Smile in an Equity Local Volatility Model. Available at SSRN: https://ssrn.com/abstract=1295882

Gurrieri, Sebastien (2012) Monte Carlo Calibration of Hybrid Local Volatility Models. Available at SSRN: https://ssrn.com/abstract=2112819

Hagan, Patrick S., Deep Kumar, Andrew Lesniewski , Diana Woodward (2002) Managing Smile Risk, *Wilmott* 1:84-108.

Karatzas, I. and Shreve, S. E. (1998). Methods of Mathematical Finance, volume 39. Springer Science & Business Media.

Kloeden, P. E., E. Platen, H. Schurz (1994) Numerical Solution of SDE through Computer Experiments. Springer.

Lee, Roger (2002) Implied Volatility: Statics, Dynamics, and Probabilistic Interpretation. *Recent Advances in Applied Probability*.

Nualart, D.(1995) Malliavin Calculus and Related Topics. (Probability and its Applications) Berlin Heidelberg New York: Springer.

Overhuas, Marcus, Ana Bermudez, Hans Buehler, Andrew Ferraris, Christopher Jordinson, and Aziz Lamnouar (2007) Equity Hybrid Derivatives. John Wiley and Sons Ltd.

Louis Paulot (2009) Asymptotic implied volatility at the second order with application to the SABR model, Misys.

Pealat, Guillaume and Daniel J. Duffy (2011) The Alternating Direction Explicit (ADE) Method for One-Factor Problems. *Wilmott Magazine*, 54-60.