CHALMERS | GÖTEBORG UNIVERSITY

MASTER'S THESIS

Models for the Dynamics of Implied Volatility Surfaces

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GÖTEBORG UNIVERSITY
Göteborg, Sweden 2014

Thesis for the Degree of Master of Science (30 ECTS credits)

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Göteborg, March 2014

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D-MATH

Autumn semester 2013

MASTERS THESIS

of

Martin Andersson
Matriculation number: 10-729-630

Readers: Prof. Dr. J. Teichmann

Issue Date: December 23, 2013
Submission Date: December 23, 2013

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Abstract

In this thesis models for the dynamics of implied volatility surfaces are examined. In particular the focus is on models that have a joint dynamics for the implied volatility surface and the underlying of the options. A risk neutral drift condition for the option price surface is derived for models with a joint dynamics for the implied volatility surface and the underlying of the options. Two models that follow this type of dynamics have previously been proposed in the papers Carr and Wu [2010] and Carr and Wu [2013]. In this thesis it is shown that for any implied volatility surface which follows one of these models and fulfills a risk-neutral drift condition, the necessary condition on the large moneyness behavior of the surface to exclude static arbitrage cannot be fulfilled. Finally, for a range of models following this type of joint dynamics it is also shown that if an implied volatility surface fulfills a risk-neutral drift condition, then the condition on the large moneyness behavior of the surface to exclude static arbitrage cannot be fulfilled.

Sammanfattning

I den här uppsatsen undersöker vi modeller för dynamiken hos den implicita volatilitetsytan. Fokus ligger på modeller med en gemensam dynamik för implicita volatilitetsytan och optionernas underliggande tillgång. Ett villkor för den risk neutrala driften hos optionsprisytan är härlett för modeller med en gemensam dynamik för implicita volatilitetsytan och optionernas underliggande tillgång. Två modeller som följer denna typ av dynamik har tidigare blivit föreslagna i Carr and Wu [2010] och Carr and Wu [2013]. I den här uppsatsen visas det att för en implicit volatilitetsyta som följer en av dessa modellerna, och uppfyller ett villkor för den risk-neutrala driften, kan inte ett nödvändigt villkor på ytan för att utesluta statiskt arbitrage uppfyllas. Avslutningsvis visas det också att för en rad av modeller som följer denna typ av gemensam dynamik, och uppfyller ett villkor för den risk-neutrala driften, kan inte ett nödvändigt villkor på ytan för att utesluta statiskt arbitrage uppfyllas.

Acknowledgements

I would like to express my deep gratitude to Prof. Dr. Josef Teichmann for supervising this thesis and for the fruitful discussions we have had during the writing process.

To my girlfriend Helen Gillholm I would like to express my innermost appreciation for her everlasting moral support and high spirits. Helen, without you I would be lost.

I would also like to direct a special thank you to Erik Lindgren, Fredrik Fyring and Robert Johansson at SEB Merchant Bank for getting me started and introducing me to the topic.

Equally thankful I am to Stefan Zuber, Roman Maxymchuk and Martin Schwarb at Zürcher Kantonalbank for allowing me flexibility at work to finish my thesis during the autumn.

Last, but of course not least, I would like to thank my family and friends for all the encouragement you have given me throughout this process.

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List of Notations

η_t	Volatility of volatility process
μ_t	Drift process
ν_t	Instantaneous return variance process
ρ_t	Correlation process
σ_t	Spot volatility process
$\hat{\sigma}(K,T)$	Implied volatility surface
$\bar{\sigma}(K,T)$	Forward starting implied volatility
τ	Time to maturity
r_t	Risk-free interest rate
K	Strike price
T	Maturity
W_t	Standard Brownian motion
$\mathbb{Q}$	Risk-neutral probability measure

1 Introduction

In this chapter an introduction to the topic of the thesis is given. First the problem background is presented before the purpose and the research questions are stated. Finally, the disposition of the thesis is presented.

1.1 Problem Background

It has for a long time been widely acknowledged that the assumptions used in Black and Scholes [1973] and in Merton [1973] for the development of the Black-Scholes-Merton model are not realistic. The prices of traded European options are today instead quoted through the options' Black-Scholes implied volatility, i.e. the unique value for the volatility in the Black-Scholes formula such that the formula results in the observed market price. In the financial industry the implied volatility both presents attractive properties to exclude arbitrage and serves as a convenient mapping from the option price space to a single real number. For a fixed maturity the implied volatility can show a smile or a skew effect with respect to the strike price. This is attributed to fat tails or asymmetry in the return distribution of the underlying.

Bounds on option prices to exclude static arbitrage were presented already in Merton [1973]. The bounds can be divided into two different types, where the first type represents arbitrage between cash, the underlying, and options at a fixed strike and maturity and the second type represents arbitrage between options with different strikes and maturities. One attractive property of the implied volatility was shown in Hodges [1996] where it was shown that if a positive implied volatility is used to quote an option price the first type of arbitrage is always excluded. In Roper [2010] it is stated that *"a call price surface is free from static arbitrage if there are no arbitrage opportunities from trading in the surface"* [pp. 1]. Conditions for the call price surface to be free of static arbitrage can either be formed on the properties of the surface itself, [e.g. Roper, 2010], or formed on the result of different trading strategies in the surface such as bull and bear spreads, calendar spreads and butterfly spreads, [e.g. Carr and Madan, 2005]. Both sufficient and necessary conditions for the implied volatility surface to be free of static arbitrage have been derived in Roper [2010]. This is important for several reasons; first, if a theoretical model results in a parameterized implied volatility surface that do not fulfill these conditions it points to errors in the model; second, in stochastic implied volatility modeling it is important to know how to specify an initial implied volatility surface that is free of arbitrage.

In order to price and hedge derivatives in an arbitrage-free way there are many different approaches and models. A first branch of models uses the most common approach where the dynamics of the underlying is specified under a risk-neutral martingale measure. For example stochastic volatility models such as the Heston model, presented in Heston [1993], belong to this branch. A second branch of models is commonly referred to as market models for the implied volatility where instead the joint dynamics for the underlying and its options are specified. In interest rate modeling a similar approach leads to the well-known Heath-Jarrow-Morton drift conditions. Market models for the implied volatility commonly try to specify the continuous martingale component of the implied volatility surface and take the observed implied volatility as given in order to derive no-arbitrage drift restriction on the surface, [e.g. Schönbucher, 1999]. Both in Schweizer and Wissel [2008b] and in Carr and Wu [2013] the successfulness of the market model approach for implied volatilities is criticized. In particular Carr and Wu [2013] state that *"the knowledge of the initial implied volatility surface places unclear constraints on the specification of the continuous martingale component of its subsequent dynamics"* [pp. 6]. Instead a new approach was developed in Carr and Wu [2010], and later also in the revised version Carr and Wu [2013], where they start in a similar way as in Schönbucher [1999] and define a joint dynamics for the underlying and the implied volatility but instead of specifying the spot volatility process of the underlying Carr and Wu try to model the future dynamics of the implied volatility.

1.2 Purpose and Research Questions

In the light of the criticism of previously presented models for the implied volatility dynamics we in this thesis want to examine the new approach presented in Carr and Wu [2010] and Carr and Wu [2013]. Carr and Wu claim that their approach generates several unique features, which have not before been available in traditional literature, for example the computational times will be reduced drastically. The purpose of this thesis is therefore to evaluate if the models proposed by Carr and Wu are theoretically acceptable. For the models to be acceptable there has to exist at least one implied volatility surface, following the model dynamics, that is free from static arbitrage and fulfills the dynamic no-arbitrage conditions. From the given purpose the following research questions are formed

1) Are the models proposed by Carr and Wu free of arbitrage?

2) Can the model setup of Carr and Wu be extended to include a range of models with similar properties as the already proposed models?

1.3 Thesis Disposition

The disposition of the thesis is as follows. In Chapter 2 the theoretical framework of the thesis is presented thoroughly. In Chapter 3 the results are derived and in Chapter 4 the conclusions are presented.

2 Theoretical Framework

2.1 Introduction

This theoretical framework will give a background in the area of implied volatility surfaces. First, a short summary of the Black-Scholes approach will be given since the foundation of implied volatility surfaces is grounded in these findings. Second, local volatility is covered and linked to implied volatility before also implied volatility is defined and presented. Third, the large area of stochastic volatility is presented and the Heston and SABR models are described. Forth, the area of models for the dynamics of implied volatility surfaces and the Carr and Wu model named Vega-Gamma-Vanna-Volga are described in depth. Conditions for an arbitrage-free volatility surface are stated before finally the Kalman filter is presented.

2.2 Black, Scholes and Merton

In this section a short presentation of the Black Scholes model is given. This result will partly be used, particularly the Greeks, in later sections. First, the Black Scholes pricing equation is derived, then the Black Scholes formula and the Black Scholes Greeks are presented. In the whole section we are in a Black Scholes setting where the underlying stock price, S_t follows the dynamic

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard brownian motion. A closed form solution to this SDE can be written as

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}.$$

2.2.1 Black Scholes Equation

The Black Scholes equation can be derived from a hedging portfolio, Π, consisting of a short position in one option, $V(t, S_t)$, and long position in Δ number of stocks, S_t, i.e.

$$\Pi = -V_t + \Delta S_t.$$

Applying the Itô formula on the option V results in

$$dV_t = \left(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S_t \frac{\partial V}{\partial S} dW_t.$$

Now applying the Itô formula on the hedging portfolio Π gives us the equation

$$d\Pi = -\Big(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\Big)dt - \sigma S_t \frac{\partial V}{\partial S} dW_t + \Delta\Big(\mu S_t dt + \sigma S_t dW_t\Big)$$

To achieve a delta hedge, i.e. to hedge the randomness in the price of the underlying, we choose the quantity of stocks, Δ, in our portfolio such that

$$-\frac{\partial V}{\partial S} + \Delta = 0,$$

which means that the dW_t terms cancel out in the previous equation. For the portfolio to be free of arbitrage we require its rate of return to be equal to the risk free rate, r, and therefore we now have

$$\begin{aligned} d\Pi = \Big(-\mu S_t \frac{\partial V}{\partial S} - \frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S}\mu S_t\Big)dt = r\Pi dt \\ \iff \quad \Big(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\Big)dt = r\big(-V_t + \frac{\partial V}{\partial S} S_t\big)dt \\ \iff \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0. \end{aligned}$$

Hence, we have arrived at the Black Scholes pricing equation

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0. \tag{2.1}$$

2.2.2 Black Scholes Formula

The pricing formula for European call and put options in the Black Scholes settings can be derived using that there exists an equivalent martingale measure under which the discounted stock price, $\tilde{S}_t$ is a martingale, i.e.

$$\begin{aligned} d\tilde{S}_t = \sigma \tilde{S}_t dW_t \\ \tilde{S}_t = \tilde{S}_0 e^{\sigma W_t - \frac{\sigma^2}{2} t} \end{aligned}$$

We call this measure the risk-neutral measure $\mathbb{Q}$. Further we use that any option defined by a non-negative $\mathscr{F}_t$-measurable random variable h, which is square-integrable under $\mathbb{Q}$, is replicable and that the value at time t of any replicating portfolio is given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} h | \mathscr{F}_t\Big]. \tag{2.2}$$

Now, for a portfolio, V_t, with a payoff function $f(S_T)$ we have

$$
\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f(S_T)|\mathscr{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f(\tilde{S}_t e^{\sigma(W_T-W_t)-\frac{\sigma^2}{2}(T-t)})|\mathscr{F}_t\right] \\
&= \left\{S_t \text{ is } \mathscr{F}_t\text{-measurable, } W_T-W_t \text{ is independent of } \mathscr{F}_t\right\} \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f(\tilde{S}_t e^{\sigma(W_T-W_t)-\frac{\sigma^2}{2}(T-t)})\right] = \left\{W_T-W_t \sim N(0,T-t)\right\} \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty} f\big(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y\sqrt{(T-t)}}\big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy
\end{aligned}
$$

Entering the payoff function for a call, C, and a put, P, we arrive at the following pricing formulas

$$
\begin{aligned}
C(S,\sigma,t;K,T) &= S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \\
P(S,\sigma,t;K,T) &= Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+)
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
d_+ &= \frac{ln(\frac{S_t}{K}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\
d_- &= d_+ - \sigma\sqrt{T-t}.
\end{aligned}
$$

2.2.3 Black Scholes Greeks

To finish this section we here list some of the Greeks for calls and puts in the Black Scholes setup. The Greeks calculated from the pricing formula for the call option are: (with ϕ denoting the standard normal probability density function)

$$
\begin{aligned}
&\text{Delta:} && \Delta = \frac{\partial}{\partial S} = \Phi(d_+) \\
&\text{Gamma:} && \Gamma = \frac{\partial^2}{\partial S^2} = \frac{\phi(d_+)}{S\sigma\sqrt{T-t}} \\
&\text{Vega:} && \kappa = \frac{\partial}{\partial \sigma} = S\phi(d_+)\sqrt{T-t} \\
&\text{Theta:} && \Theta = \frac{\partial}{\partial t} = -\frac{S\phi(d_+)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_-) \\
&\text{Rho:} && \rho = \frac{\partial}{\partial r} = K(T-t)e^{-r(T-t)}\Phi(d_-).
\end{aligned}
\tag{2.4}
$$

For the put option the gamma and vega are the same as for the call. For the other Greeks we however have

$$
\begin{aligned}
&\text{Delta:} && \Delta = \Phi(d_+) - 1 \\
&\text{Theta:} && \Theta = -\frac{S\phi(d_+)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_-) \\
&\text{Rho:} && \rho = -K(T-t)e^{-r(T-t)}\Phi(-d_-).
\end{aligned}
\tag{2.5}
$$

2.3 Local Volatility

The idea of Local Volatility (LV) is to make a simplifying assumption in order to make it possible to price exotic options consistent with the known prices of vanilla options, i.e. to find a stock price process compatible with the observed prices of vanilla options, while retaining model completeness for arbitrage free pricing and hedging. The simplifying assumption is that the local volatility is a deterministic function of the spot price and time. In Dupire [1994] it is stated that knowing all the prices of European options amount to knowing the probability densities of the stock spot price at different times conditional on the spot price's current value S_t. Using this result Dupire finds that under the risk-neutral measure there is a unique diffusion process, which is consistent with the distribution of European option prices. The local volatility function is then the corresponding unique diffusion coefficient $\sigma_{LV}(S_t, t)$. We now follow Gatheral [2006] and derive the Dupire formula. Under the following dynamic

$$\frac{dS_t}{S_t} = \mu_T dt + \sigma_{LV}(S_t, t; S_0) dW_t,$$

where $\mu_t = r_t - D_t$ is the risk-neutral drift, the undiscounted risk-neutral value of a European call option is given by

$$C(S_0, K, T) = \int_K^\infty \varphi(S_T, T; S_0)\big(S_T - K\big) dS_T. \tag{2.6}$$

$\varphi(S_T, T; S_0)$ is the probability density of the stock price at time T and hence the Fokker-Planck equation states

$$\frac{\partial \varphi}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_{LV}^2 S_T^2 \varphi) - \frac{\partial}{\partial S_T}(\mu S_T \varphi).$$

Further, when we differentiate (2.6) w.r.t to K two times we find

$$\begin{aligned}
\frac{\partial C}{\partial K} &= -\int_K^\infty \varphi(S_T, T; S_0) dS_T, \\
\frac{\partial^2 C}{\partial K^2} &= \varphi(K, T; S_0).
\end{aligned}$$

Now, if we differentiate (2.6) w.r.t. time we get

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \int_K^\infty \Big\{\frac{\partial}{\partial T}\varphi(S_T, T; S_0)\Big\}\big(S_T - K\big) dS_T \\
&= \int_K^\infty \Big\{\frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma_{LV}^2 S_T^2 \varphi) - \frac{\partial}{\partial S_T}(\mu S_T \varphi)\Big\}\big(S_T - K\big) dS_T \\
&= \Big\{\textit{two times integration by parts}\Big\} = \frac{\sigma_{LV}^2 K^2}{2}\varphi + \int_K^\infty K\varphi dS_T \qquad (2.7)\\
&= \frac{\sigma_{LV}^2 K^2}{2}\varphi + \mu(T)\Big(\int_K^\infty \varphi\big(S_T - K\big) dS_T \int_K^\infty \varphi K dS_T\Big) \\
&= \Big\{\textit{inserting the K derivatives}\Big\} = \frac{\sigma_{LV}^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + \mu(T)\Big(C - K\frac{\partial C}{\partial K}\Big).
\end{aligned}$$

If instead the option price is expressed as function of the forward price, $F_T = S_0 e^{\int_0^T \mu_t dt}$, a formula similar to (2.7) is also stated in Gatheral [2006] but without the drift term. Using this formula we find the simple expression

$$\sigma_{LV}^2(K,T,S_0) = \frac{\frac{\partial C(F_T,K,T)}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C(F_T,K,T)}{\partial K^2}}. \tag{2.8}$$

Note 1. A key feature of local volatility is that the σ_{LV} does not depend on the model we choose for the process of the underlying, e.g. stochastic volatility.

2.4 Implied Volatility

Since the Black Scholes model since long has been found to not be completely realistic, both researchers and practitioners have moved on. However, the model still serves as a common reference point for practitioners when quoting prices. This is done through the use of implied volatility, which maps the domain of the option prices to a single real number.

Definition 1 (Implied Volatility)**.** *The implied volatility ($\hat{\sigma}$) is the number entered for the volatility of the underlying into the Black Scholes formula such that the Black Scholes theoretical value of the contract is equal to the current market price, i.e.*

$$C(S,\sigma,t;K,T) = C_{BS}\big(S_t, \hat{\sigma}_t(K,T), t\big). \tag{2.9}$$

Further, in Gatheral [2006] a quantity which Gatheral names the forward starting implied volatility is defined. We will also adopt this quantity, however we follow a proposition presented in Keller-Ressel and Teichmann [2009] to avoid the circular definition given by Gatheral.

Proposition 1 (Proposition 2.1 in Keller-Ressel and Teichmann [2009])**.** *There exists a unique positive deterministic function, $\overline{\sigma}_{K,T}(t)$, s.t.*

$$\mathbb{E}\Big[C_{BS}\big(t,S_t,K,T;\overline{\sigma}_{K,T}(t)\big)\Big] = C(K,T), \qquad \forall t \in [0,T) \tag{2.10}$$

Proof. For $\sigma = 0$ we have $C_{BS}(t,S_t,K,T;\sigma) = (S_t - K)^+$. No arbitrage arguments exclude negative calendar spreads and therefore

$$\mathbb{E}\big[C_{BS}(t,S_T,K,T;0)\big] = E\big[(S_t-K)^+\big] \le E\big[(S_T-K)^+\big] = C(K,T).$$

For $\sigma \to \infty$ we have $C_{BS}(t,S_t,K,T;\sigma) \to S_t$ and therefore

$$\mathbb{E}\big[C_{BS}(t,S_T,K,T;\infty)\big] = E\big[S_t\big] = E\big[S_T\big] \ge C(K,T).$$

Now, since $\sigma \mapsto C_{BS}(t,S_t,K,T;\sigma)$ is continuous and a strictly monotone increasing function for any given S_t also $\sigma \mapsto \mathbb{E}\big[C_{BS}(t,S_t,K,T;\sigma)\big]$ have these properties and we can conclude that $\overline{\sigma}_{K,T}$ has a unique solution for each $t \in [0,T)$. □

Note 2. For $t = 0$, $\overline{\sigma}_{K,T}(0)$ is simply the Black-Scholes implied volatility, i.e. for $t = 0$, equation (2.10) becomes

$$C_{BS}\big(0, S_0, K, T; \overline{\sigma}_{K,T}(0)\big) = C\big(K, T\big).$$

Definition 2 (Black Scholes Forward Implied Variance)**.** *In line with proposition 1 we define the forward implied variance as*

$$\nu_{K,T}(t) = -\frac{\partial}{\partial t}\big(\overline{\sigma}_{K,T}(t)(T-t)\big).$$

We can now get the implied variance by

$$\hat{\sigma}^2(K,T) = \overline{\sigma}^2_{K,T}(0) = \frac{1}{T}\int_0^T \nu_{K,T}(t)\,dt. \tag{2.11}$$

Using the forward starting implied volatility we can relate the local or stochastic volatility to the implied volatility.

Proposition 2. *The square of implied volatility can be written as a time-average of weighted expectations of $\sigma(t, S_t)$.*

$$\hat{\sigma}^2(K,T) = \frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{G}_t}\big[\sigma^2(t,S_t)\big]\,dt \tag{2.12}$$

where the measures $\mathbb{G}_t$ are given by the Radon-Nikodym derivatives w.r.t. the pricing measure,

$$\frac{\partial \mathbb{G}_t}{\partial \mathbb{P}} = \frac{S_t^2\Gamma_{BS}(S_t, \overline{\sigma}_{K,T}(t))}{\mathbb{E}\big[S_t^2\Gamma_{BS}(S_t, \overline{\sigma}_{K,T}(t))\big]}. \tag{2.13}$$

Proof. We start by noting that from equation (2.4) we have $\Gamma_{BS} = \frac{\phi(d_+)}{S\sigma\sqrt{T-t}}$ and therefore we can write: (if $r = 0$)

$$\frac{\partial C_{BS}}{\partial t} = -\frac{S\phi(d_+)\sigma}{2\sqrt{T-t}} = -\frac{1}{2}\sigma^2 S_t^2 \Gamma_{BS},$$

$$\frac{\partial C_{BS}}{\partial \sigma} = S\phi(d_+)\sqrt{T-t} = \sigma S^2 (T-t)\Gamma_{BS}.$$

Now we define $f(t, S_t) = C_{BS}\big(t, S_t, K, T; \overline{\sigma}_{K,T}(t)\big)$ and apply Itô's formula so that we for any $\tau \in [0, T)$ have

$$f(T, S_T) - f(\tau, S_\tau) = \int_\tau^T \frac{\partial f}{\partial S} dS_t + \int_\tau^T \frac{\partial f}{\partial t} dt + \int_\tau^T \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 f}{\partial S^2} dt.$$

Taking expectation on both sides we first have for the left hand side $C(K,T) - \mathbb{E}\big[f(t, S_t)\big]$, which by Proposition 1 is equal to zero. Before we address the right hand side we note that from equation (2.4) and Definition 2 $f(t, S_t)$ satisfies

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\nu_{T,K}(t) S_t^2 \frac{\partial^2 f}{\partial S^2}.$$

Now on the right hand side we can interchange the integral and the expectation and since the dS_t-term contribution is zero we arrive at

$$0 = \frac{1}{2}\int_\tau^T \mathbb{E}\Big[\big(\sigma^2(t, S_t) - \nu_{K,T}(t)\big) S_t^2 \frac{\partial^2 f}{\partial S^2}\Big]\,dt.$$

Since τ is chosen arbitrary in $[0, T)$ the integrand must be zero $\forall\, t \in [0, T)$. Rewriting $\frac{\partial^2 f}{\partial S^2}$ as the BS-gamma evaluated at a volatility of $\overline{\sigma}_{K,T}(t)$ and rearranging the terms we find that

$$v_{K,T}(t) = \frac{\mathbb{E}\Big[\sigma^2(t, S_t) S_t^2 \Gamma_{BS}\big(S_t, \overline{\sigma}_{K,T}(t)\big)\Big]}{\mathbb{E}\Big[S_t^2 \Gamma_{BS}\big(S_t, \overline{\sigma}_{K,T}(t)\big)\Big]}.$$

Using the measure $\mathbb{G}_t$ from equation (2.13) we can write

$$v_{K,T}(t) = \mathbb{E}^{\mathbb{G}_t}\left[\sigma^2(t, S_t)\right],$$

and we are done. $\square$

2.5 Stochastic Volatility

In the Black Scholes model there is only one source of randomness, which is represented in the dynamics of the underlying. However, research has shown that the Black Scholes assumption of constant volatility is restrictive and unrealistic. In reality the volatility shows several distinct features such as being mean-reverting and auto-correlated. When trying to account for this behavior a range of different models have been developed. One branch of them is named stochastic volatility. This family of models has an additional source of randomness, which is represented in the volatility process. Stochastic volatility is represented by a underlying dynamic of the form

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sqrt{v_t} S_t dW_1 \\ dv_t &= \alpha(S_t, t, v_t) dt + \eta \sqrt{v_t} \beta(S_t, t, v_t) dW_2 \\ \langle dW_1, dW_2 \rangle &= \rho dt \end{aligned} \tag{2.14}$$

where $v(t)$ denotes the instantaneous return variance, η is the volatility of volatility and ρ is the correlation between W_1 and W_2. It is possible to derive a PDE that the value of a contract, V, must follow under a no arbitrage and stochastic volatility assumption [Gatheral, 2006, Wilmott, 2000]. Compared to when we derived the Black Scholes pricing equation we cannot hedge all randomness by only holding the underlying in this case. Here we also need to hedge the randomness arising from the non-constant volatility. We can achieve this by also holding another financial contract, V_1, on the same underlying. We want to achieve a delta hedge and a vega hedge by holding the quantities Δ and Δ_1 of S and V_1 respectively. Therefore we define the following portfolio

$$\Pi = V - \Delta S - \Delta_1 V_1.$$

Applying the Itô formula we derive

$$
\begin{aligned}
d\Pi =& \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta S\frac{\partial^2 V}{\partial S\partial \nu} + \frac{1}{2}\eta^2\nu\beta^2\frac{\partial^2 V}{\partial \nu^2}\right)dt\\
&\Delta_1\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 V_1}{\partial S^2} + \rho\eta\nu\beta S\frac{\partial^2 V_1}{\partial S\partial \nu} + \frac{1}{2}\eta^2\nu\beta^2\frac{\partial^2 V_1}{\partial \nu^2}\right)dt\\
&+\left(\frac{\partial V}{\partial S} - \Delta_1\frac{\partial V_1}{\partial S} - \Delta\right)dS + \left(\frac{\partial V}{\partial \nu} - \Delta_1\frac{\partial V_1}{\partial \nu}\right)d\nu.
\end{aligned}
$$

To achieve the delta and vega hedge and eliminate the randomness we set

$$
\begin{cases}
\frac{\partial V}{\partial \nu} - \Delta_1\frac{\partial V_1}{\partial \nu} = 0 \Longleftrightarrow \Delta_1 = \frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}}\\
\frac{\partial V}{\partial S} - \Delta - \Delta_1\frac{\partial V_1}{\partial S} = 0 \Longleftrightarrow \Delta = \frac{\partial V}{\partial S} - \frac{\partial V_1}{\partial S}\frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}}.
\end{cases}
$$

Using arbitrage arguments we now have

$$
\begin{aligned}
d\Pi =& \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta S\frac{\partial^2 V}{\partial S\partial \nu} + \frac{1}{2}\eta^2\nu\beta^2\frac{\partial^2 V}{\partial \nu^2}\right)dt\\
&\Delta_1\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 V_1}{\partial S^2} + \rho\eta\nu\beta S\frac{\partial^2 V_1}{\partial S\partial \nu} + \frac{1}{2}\eta^2\nu\beta^2\frac{\partial^2 V_1}{\partial \nu^2}\right)dt\\
=& r\Pi dt = \left(V - \Delta S - \Delta_1 V_1\right)dt = r\left(V - \frac{\partial V}{\partial S}S + \frac{\partial V_1}{\partial S}\frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}}S - \frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}}V_1\right)dt,
\end{aligned}
$$

which can be separated so that the left hand side only contains V terms and the right hand side only contains V_1 terms. Therefore both sides can only be functions of the variables S, ν, and t. This function can be arbitrary, but without loss of generality we can define it as $\alpha - \lambda\sqrt{\nu}\beta$. We arrive at the pricing PDE

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 V}{\partial S^2} + \rho\nu\beta S\frac{\partial^2 V}{\partial S\partial \nu} + \frac{1}{2}\eta^2\nu\beta^2\frac{\partial^2 V}{\partial \nu^2} + rS\frac{\partial V}{\partial S} - rV + \left(\alpha - \lambda\sqrt{\nu}\beta\right)\frac{\partial V}{\partial \nu} = 0. \tag{2.15}
$$

2.5.1 Heston Model

The Heston model was first proposed in Heston [1993] and is a lognormal model like the Black-Scholes model. However, instead of having a constant volatility, the volatility in the Heston model follows a CIR-process. In the original article Heston defined the spot asset price with the following diffusion process,

$$
dS(t) = \mu S dt + \sqrt{\nu(t)} S dW_1(t), \tag{2.16}
$$

where the variance process $\nu(t)$ is defined by the following Cox-Ingersoll-Ross square-root process,

$$
d\nu(t) = \kappa\big(\theta - \nu(t)\big)dt + \eta\sqrt{\nu(t)}dW_2(t). \tag{2.17}
$$

Here the $W_2(t)$ is a correlated Brownian motion with correlation parameter ρ to $W_1(t)$. We continue by entering the values corresponding to α and β into equation (2.15).

$$\frac{\partial V}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}+\rho\eta vS\frac{\partial^2 V}{\partial v\partial S}+\frac{1}{2}\eta^2 v\frac{\partial^2 V}{\partial v^2}+rS\frac{\partial V}{\partial S}-rV+\lambda(\overline{v}-v)\frac{\partial V}{\partial v}=0.$$

By defining K as the strike price, T as the maturity and $F_{t,T}$ as the time T forward price of the stock index and by using the change of variables $\tau = T-t$ and $x = ln(F_{t,T}/K)$ we arrive at the following pricing equation for a European call option C

$$\frac{\partial C}{\partial \tau}+\frac{1}{2}v\frac{\partial^2 C}{\partial x^2}+\rho\eta v\frac{\partial^2 C}{\partial x\partial \tau}+\frac{1}{2}\eta^2 v\frac{\partial^2 C}{\partial \tau^2}-\frac{1}{2}v\frac{\partial C}{\partial x}+\lambda(\overline{v}-v)\frac{\partial C}{\partial \tau}=0. \tag{2.18}$$

Solving the pricing equation

We follow Gatheral [2006] who bases his ansatz on Duffie, Pan and Singleton (2000). The ansatz is that the solution to (2.18) has the form

$$C(x,v,\tau)=K\left[e^x P_1(x,v,\tau)-P_0(x,v,\tau)\right],$$

which when entered in equation (2.18) results in the following equation

$$\begin{aligned}0=&-\frac{\partial P_0}{\partial \tau}+\frac{1}{2}v\frac{\partial^2 P_0}{\partial x^2}-\frac{1}{2}v\frac{\partial P_0}{\partial x}+\frac{1}{2}\eta^2 v\frac{\partial^2 P_0}{\partial v^2}+\rho\eta v\frac{\partial^2 P_0}{\partial x\partial v}+(\lambda\overline{v}-\lambda v)\frac{\partial P_0}{\partial v}\\ &-\frac{\partial P_1}{\partial \tau}+\frac{1}{2}v\frac{\partial^2 P_1}{\partial x^2}+\frac{1}{2}v\frac{\partial P_1}{\partial x}+\frac{1}{2}\eta^2 v\frac{\partial^2 P_1}{\partial v^2}+\rho\eta v\frac{\partial^2 P_1}{\partial x\partial v}+(\lambda\overline{v}-\lambda v+\rho\eta v)\frac{\partial P_1}{\partial v}\end{aligned} \tag{2.19}$$

with the terminal condition

$$\lim_{\tau\to 0}P_j(x,v,\tau)=\begin{Bmatrix}1 & if & x>0\\ 0 & if & x\leq 0\end{Bmatrix}=\theta(x),\quad for\quad j=0,1.$$

We can write the Fourier transform of P_j as

$$\tilde{P}(u,v,\tau)=\int_{-\infty}^{\infty}e^{-iux}P(x,v,\tau)dx \tag{2.20}$$

which then means that

$$\tilde{P}(u,v,0)=\int_{-\infty}^{\infty}e^{-iux}\theta(x)dx=\frac{1}{iu}.$$

The inverse transform is given by

$$P(x,v,\tau)=\int_{-\infty}^{\infty}\frac{e^{iux}}{2\pi}\tilde{P}(u,v,\tau)du. \tag{2.21}$$

Now we can substitute $\tilde{P}(u,v,\tau)$ into (2.19) and arrive at

$$
\begin{aligned}
0 = &-\frac{\partial \tilde{P}_0}{\partial \tau} - \frac{1}{2}u^2 v \tilde{P}_0 - \frac{1}{2} i u v \tilde{P}_0 + \frac{1}{2}\eta^2 v \frac{\partial^2 \tilde{P}_0}{\partial v^2} + \rho\eta i u v \frac{\partial \tilde{P}_0}{\partial v} + (\lambda\overline{v} - \lambda v)\frac{\partial \tilde{P}_0}{\partial v} \\
&-\frac{\partial \tilde{P}_1}{\partial \tau} - \frac{1}{2}u^2 v \tilde{P}_1 + \frac{1}{2} i u v \tilde{P}_1 + \frac{1}{2}\eta^2 v \frac{\partial^2 \tilde{P}_1}{\partial v^2} + \rho\eta i u v \frac{\partial \tilde{P}_1}{\partial v} + (\lambda\overline{v} - \lambda v + \rho\eta v)\frac{\partial \tilde{P}_1}{\partial v} \\
\Longleftrightarrow 0 = & \; v\left\{\left(-\frac{1}{2}u^2 - \frac{1}{2} i u\right)\tilde{P}_0 - \left(\lambda - \rho\eta i u\right)\frac{\partial \tilde{P}_0}{\partial v} + \frac{1}{2}\eta^2 \frac{\partial^2 \tilde{P}_0}{\partial v^2}\right\} + \lambda\overline{v}\frac{\partial \tilde{P}_0}{\partial v} - \frac{\partial \tilde{P}_0}{\partial \tau} \\
& v\left\{\left(-\frac{1}{2}u^2 + \frac{1}{2} i u\right)\tilde{P}_1 - \left(\lambda - \rho\eta i u - \rho\eta\right)\frac{\partial \tilde{P}_1}{\partial v} + \frac{1}{2}\eta^2 \frac{\partial^2 \tilde{P}_1}{\partial v^2}\right\} + \lambda\overline{v}\frac{\partial \tilde{P}_1}{\partial v} - \frac{\partial \tilde{P}_1}{\partial \tau}.
\end{aligned}
\tag{2.22}
$$

Continue with writing $\tilde{P}_j$ for $j = 0, 1$ as

$$
\tilde{P}_j(u, v, \tau) = e^{A_j(u,\tau)\overline{v} + B_j(u,\tau)v}\tilde{P}_j(u, v, 0) = \frac{1}{iu}e^{A_j(u,\tau)\overline{v} + B_j(u,\tau)v},
$$

which means that

$$
\begin{aligned}
\frac{\partial \tilde{P}_j}{\partial \tau} &= \left\{\overline{v}\frac{\partial A_j}{\partial \tau} + v\frac{\partial B_j}{\partial \tau}\right\}\tilde{P}_j \\
\frac{\partial \tilde{P}_j}{\partial v} &= B_j \tilde{P}_j \\
\frac{\partial^2 \tilde{P}_j}{\partial v^2} &= B_j^2 \tilde{P}_j.
\end{aligned}
$$

Equation (2.22) is satisfied if

$$
\frac{\partial A_j}{\partial \tau} = \lambda B_j,
$$

and if

$$
\begin{aligned}
\frac{\partial B_0}{\partial \tau} &= -\frac{u^2}{2} - \frac{iu}{2} - (\lambda - \rho\eta i u)B_0 + \frac{\eta^2}{2}B_0^2 = \frac{\eta^2}{2}(B_0 - r_0^+)(B_0 - r_0^-) \\
\frac{\partial B_1}{\partial \tau} &= -\frac{u^2}{2} + \frac{iu}{2} - (\lambda - \rho\eta i u - \rho\eta)B_1 + \frac{\eta^2}{2}B_1^2 = \frac{\eta^2}{2}(B_1 - r_1^+)(B_1 - r_1^-),
\end{aligned}
\tag{2.23}
$$

where

$$
\begin{aligned}
r_0^{\pm} &= \frac{(\lambda - \rho\eta i u) \pm \sqrt{(\lambda - \rho\eta i u)^2 - 4(-\frac{u^2}{2} - \frac{iu}{2})\frac{\eta^2}{2}}}{\eta^2} \\
r_1^{\pm} &= \frac{(\lambda - \rho\eta i u - \rho\eta) \pm \sqrt{(\lambda - \rho\eta i u - \rho\eta)^2 - 4(-\frac{u^2}{2} + \frac{iu}{2})\frac{\eta^2}{2}}}{\eta^2}.
\end{aligned}
$$

To find P_j we integrate equation (2.23) and find

$$B_0(u,\tau) = r_0^- \frac{1 - e^{-\sqrt{(\lambda-\rho\eta iu)^2 - 4(-\frac{u^2}{2} - \frac{iu}{2})\frac{\eta^2}{2}}\tau}}{1 - \frac{r_0^-}{r_0^+} e^{-\sqrt{(\lambda-\rho\eta iu)^2 - 4(-\frac{u^2}{2} - \frac{iu}{2})\frac{\eta^2}{2}}\tau}}$$

$$A_0(u,\tau) = \lambda\left\{ r_0^- \tau - \frac{2}{\eta^2} ln\left(\frac{1 - \frac{r_0^-}{r_0^+} e^{-\sqrt{(\lambda-\rho\eta iu)^2 - 4(-\frac{u^2}{2} - \frac{iu}{2})\frac{\eta^2}{2}}\tau}}{1 - \frac{r_0^-}{r_0^+}} \right) \right\},$$

$$B_1(u,\tau) = r_1^- \frac{1 - e^{-\sqrt{(\lambda-\rho\eta iu-\rho\eta)^2 - 4(-\frac{u^2}{2} + \frac{iu}{2})\frac{\eta^2}{2}}\tau}}{1 - \frac{r_1^-}{r_1^+} e^{-\sqrt{(\lambda-\rho\eta iu-\rho\eta)^2 - 4(-\frac{u^2}{2} + \frac{iu}{2})\frac{\eta^2}{2}}\tau}}$$

$$A_1(u,\tau) = \lambda\left\{ r_1^- \tau - \frac{2}{\eta^2} ln\left(\frac{1 - \frac{r_1^-}{r_1^+} e^{-\sqrt{(\lambda-\rho\eta iu-\rho\eta)^2 - 4(-\frac{u^2}{2} + \frac{iu}{2})\frac{\eta^2}{2}}\tau}}{1 - \frac{r_1^-}{r_1^+}} \right) \right\},$$

which we then can use together with the inverse Fourier transform (2.21) to calculate the pseudo probability as

$$P_j(x,v,\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left\{ \frac{e^{A_j(u,\tau)\overline{v} + B_j(u,\tau)v + iux}}{iu} \right\} du. \tag{2.24}$$

Finding the characteristic function

We continue to follow Gatheral [2006] and find the Heston characteristic function. The definition of the characteristic function is

$$\phi_T(u) = \mathbb{E}\left[e^{iux_T} | x_t = 0 \right].$$

If we define $k = ln(K/S_t) = -x$ and use that the probability P_0 from equation (2.24) is the probability that the final stock price is greater than the strike, we can find the probability density function $p(k)$ as

$$p(k) = -\frac{\partial P_0}{\partial k} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{A(u',\tau)\overline{v} + B(u',\tau)v - iu'k} du'.$$

We can now derive the characteristic function as

$$\begin{aligned} \phi_T(u) &= \int_{-\infty}^{\infty} p(k) e^{iuk} dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{A(u',\tau)\overline{v} + B(u',\tau)v} du' \int_{-\infty}^{\infty} e^{i(u-u')k} du \\ &= \int_{-\infty}^{\infty} \delta(u-u') e^{A(u',\tau)\overline{v} + B(u',\tau)v} du' \\ &= e^{A(u,\tau)\overline{v} + B(u,\tau)v}. \end{aligned} \tag{2.25}$$

Pricing using FFT

In Carr and Madan [1999] it is shown how the fast Fourier transform technique can be used to price options and in Kilin [2011] the FFT technique is compared with direct integration. To show how the Fourier transform technique work we follow Carr and Madan [1999]. First, for a call with maturity T we define k as the log-strike, $k = ln(K)$ and s_t as the log-price of the underlying, $s_t = ln(S_t)$. Further, we call the risk-neutral density of the log-price $q_T(s)$ and define the characteristic function as

$$\phi(u) \equiv \int_{-\infty}^{\infty} e^{ius} q_T(s) ds.$$

Using the risk-neutral density the call value can then be written as

$$C_T(k) \equiv \int_{k}^{\infty} e^{-rT}\left(e^s - e^k\right) q_T(s) ds.$$

However, the call price function is not square-integrable since $C_T(k) \to S_0$ as $k \to -\infty$. We therefore modify the call price by

$$c_T(k) \equiv e^{\alpha k} C_T(k), \quad \alpha > 0.$$

We define the Fourier transform

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk,$$

and the inverse Fourier transform

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v) dv. \tag{2.26}$$

We expand the expression for the Fourier transform $\psi_T(v)$ by entering the call price equation and derive

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \int_{k}^{\infty} e^{\alpha k - rT}(e^s - e^k) ds dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{s+\alpha k} - e^{(1+\alpha)k}) e^{ivk} dk ds \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \Big(\frac{e^{(\alpha+1+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha + 1 + iv} \Big) ds \\
&= \frac{e^{-rT} \phi_T\big(v - (\alpha+1)i\big)}{\alpha^2 + \alpha - v^2 i(2\alpha+1)v}.
\end{aligned} \tag{2.27}$$

The call price is calculated by entering equation (2.27) into equation (2.26). The FFT can then be applied to the integration in equation (2.26) since it is a direct Fourier transform.

2.5.2 SABR Model

The original proposal of the SABR model was given by in Hagan et al. [2002]. The model assumes that the forward asset price $F(t)$ and the instantaneous volatility $\sigma(t)$ follow the dynamics

$$
\begin{aligned}
dF(t) &= \sigma(t)F^{\beta}(t)dW_1(t)\\
d\sigma(t) &= \eta\sigma(t)dW_2(t)\\
d\langle W_1, W_2\rangle &= \rho dt,
\end{aligned}
$$

where $\eta > 0$ is the volatility of volatility and $\beta > 0$ is the leverage coefficient together with the initial conditions

$$
\begin{aligned}
F(0) &= F_0\\
\sigma(0) &= \sigma_0.
\end{aligned}
$$

Further, Hagan et al. [2002] describe how a closed-form algebraic formula for the implied volatility can be obtained as functions of today's forward price F_0 and the strike K. In Homescu [2011] the different choices for the parameters β and ρ are explained. A β equal to one corresponds to the log-normal with a flat skew and a β equal to zero corresponds to the normal model with a pronounced skew. For ρ the choice of $\rho > 0$ and $\rho < 0$ gives an inverse respectively a negative skew, while $\rho = 0$ gives a symmetric volatility smile given $\beta = 1$.

Implied Volatility

In Gatheral [2006] it is pointed out that since the SABR model is not mean reverting it is only good for short expirations. However, the model has an exact expression for the implied volatility smile when $\tau \to 0$. A correction to the formula of Hagan et al. [2002] is made in Obloj [2008] and we follow this corrected derivation here. With $x = ln(F_0/K)$ and $k = ln(K)$ we can write the implied volatility as

$$\hat{\sigma}(\alpha, \beta, \nu, \rho, k) = \hat{\sigma}_0(k)\big(1 + \hat{\sigma}_1(k)\tau\big) + O(\tau^2)$$

where

$$
\begin{aligned}
\hat{\sigma}_0(k) &= \frac{-\eta x}{ln\left(\frac{\sqrt{1-2\rho y+y^2}+y-\rho}{1-\rho}\right)}\\
\hat{\sigma}_1(k) &= \frac{(\beta-1)^2}{24}\frac{\sigma_0^2}{(F_0K)^{1-\beta}} + \frac{1}{4}\frac{\rho\eta\sigma_0\beta}{(F_0K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\eta^2\\
z &= \frac{(F_0^{1-\beta} - K^{1-\beta})\eta}{\sigma_0(1-\beta)}.
\end{aligned}
$$

2.6 Models for the Dynamics of Implied Volatility Surfaces

Models for the dynamics of implied volatility surfaces include for example market models for the implied volatility and in this thesis we focus on the new approach presented in Carr and Wu [2010] and Carr and Wu [2013], which also falls into this category. Market models for the implied volatility treat the implied volatility as a random process and try to model it based on option prices quoted in the market. A market model approach for the term-structure of the implied volatility was presented in Schönbucher [1999]. This approach was similar to what had previously been used for the term-structure of interest rates. In several steps the absences of arbitrage, the issues of modeling, and existence of a solution for market models for option prices have been addressed in Schweizer and Wissel [2008a,b] and Wissel [2008]. The main issues to achieve arbitrage free market models can be described as follows

1) At each fixed time the static arbitrage conditions must hold as well as the terminal condition given by the payoff function.

2) Dynamic arbitrage conditions poses restrictions on the model coefficients, typically on the drift.

3) For a proposed model the existence of a unique solution to the SDE system that constitutes the model dynamics must be showed.

Typically the dynamic conditions in 2) imply the static conditions in 1). Because of the dynamic drift conditions in 2) it is usually non-trivial to give sufficient conditions for 3).

2.6.1 Existence and Uniqueness of a Solution to an SDE system

In this section we present the existence and uniqueness result from Schönbucher [1999] which addresses issue 3) above. Schönbucher presents his result for a SDE-system of the form

$$d\hat{\sigma}(T,K) = u(T,K)dt + \gamma(T,K)dW_0 + \sum_{n=1}^{N} v_n(T,K)dW_n, \tag{2.28}$$

where T and K is the maturity and strike of an option. The implied volatility is here driven by the Brownian motion $W_1,...,W_N$ and a Brownian motion W_0 which is correlated, with correlation process $\gamma(T,K)$, to the Brownian motion process driving the underlying's price process. Using the N-dimensional Brownian motion $W = (W_1,...,W_N)^T$ and the volatility vector $v = (v_1,...,v_N)$ the SDE-system can be written as

$$d\hat{\sigma}(T,K) = u(T,K)dt + \gamma(T,K)dW_0 + v(T,K)dW.$$

For this system Schönbucher presents the following theorem

Theorem 1 (Theorem 2.1 in Schönbucher [1999])**.** *Let M be the number of traded options and let $X = (S,\hat{\sigma}_1,\hat{\sigma}_2,...,\hat{\sigma}_M) \in \mathbb{R}^{M+1}$ be the state vector. Let $T > 0$, and $u(\cdot,\cdot) : [0,T] \times \mathbb{R}^{M+1} \mapsto \mathbb{R}^{M+1}$ and $v(\cdot,\cdot) : [0,T] \times \mathbb{R}^{M+1} \mapsto \mathbb{R}^{M+1\times N+1}$ be measurable functions satisfying*

$$|u(t,x)| + |v(t,x)| \leq C(1+|x|)$$

for all $x \in \mathbb{R}^{M+1}$, $t \in [0,T]$ and some constant C, and

$$|u(t,x) - u(t,y)| + |v(t,x) - v(t,y)| \leq D|x-y|$$

for all $x, y \in \mathbb{R}^{M+1}$, $t \in [0,T]$ and some constant D. Then the stochastic differential equation $X(0) = X_0$ and

$$dX = u(t,X)dt + \sum_{i=0}^{N} v(t,X)dW_i$$

has a unique t-continuous solution $X(t;w) = (S(t;\omega), \hat{\sigma}_1(t;\omega), \hat{\sigma}_2(t;\omega), ..., \hat{\sigma}_M(t;\omega))$, each component of which is measurable, adapted and square-integrable. This solution is called a strong solution.

Schönbucher [1999] points out that since the price process of the underlying is included in the state variable X, the Lipschitz growth condition must also hold for the underlying. He states that it is enough if the diffusion of the implied volatility in (2.28) is Lipschitz continuous and if the underlying's volatility process is regular and a Lipschitz continuous function of the other processes in the state variable for the conditions of Theorem 1 to hold.

2.6.2 Existence Problems in Arbitrage-Free Market Models

This section follows the reasoning of Schweizer and Wissel [2008b] in order to explain why the arbitrage-free modeling of market models for implied volatility is commonly limited to the case of a single traded option C(K,T). Schweizer and Wissel [2008b] model the stock price process $(S_t)_{0 \leq t \leq T}$ and a set of call price processes $(C_t(K))_{0 \leq t \leq T}$ with strikes $K \in \mathcal{K} \subseteq (0,\infty)$ and fixed maturity T, by

$$C_t(K) = C_{BS}(S_t, K, (T-t)\chi_t(K))$$

with dynamics

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t, & S_0 &= s_0, \\ d\chi_t(K) &= u_t(K)\chi_t(K)dt + v_t(K)\chi_t(K)dW_t, & \chi_0(K) &= x_0(K) \end{aligned} \tag{2.29}$$

for $0 \leq t \leq T$. Here C_{BS} is the Black-Scholes formula and each $\chi(K) = \hat{\sigma}^2(K)$ is a positive process modeling the square of the implied volatility of C(K). In Schweizer and Wissel [2008b] it is shown that the existence of a common equivalent local martingale measure for the price processes S and $C(K)$ for all $K \in \mathcal{K}$ is essentially equivalent to the drift conditions

$$\begin{aligned} \mu_t &= -\sigma_t b_t, \\ u_t(K) &= \frac{1}{T-t}\left(1 - \frac{1}{\chi_t(K)}\left|\sigma_t + \frac{1}{2} ln\left(\frac{K}{S_t}\right) v_t(K)\right|^2\right) \\ &\quad + \left(\frac{1}{16}(T-t)\chi_t(K) + \frac{1}{4}\right)|v_t(K)|^2 - \left(\frac{\sigma_t}{2} + b_t\right)v_t(K) \end{aligned} \tag{2.30}$$

for all $K \in \mathcal{K}$ and a market price of risk process $b \in L^2_{loc}(\mathbb{R}^m)$. Note that it is the stock volatility σ and the processes $v(K)$ for all $K \in \mathcal{K}$, i.e. the volvol of $\chi(K)$, that determine the μ and the $u(K)$.

In order to illustrate some of the major difficulties Schweizer and Wissel [2008b] review the single option case. They consider a model as in (2.29) with μ and $u(K)$ following (2.30). In the case where $v(K)$ are nonzero constants and $\sigma \in L^2_{loc}(\mathbb{R}^m)$ the following two problems generally arise:

1) Because of the drift conditions a solution of (2.29) will in general only exist up to an explosion time.

2) Because of the factor $\frac{1}{T-t}$ the solution of (2.29) will typically explode at maturity.

For these reasons Schweizer and Wissel [2008b] state that it should not be expected that the system (2.29) does have a (non-exploding) solution on $[0, T]$ for a general specification of the coefficients $\sigma, v(K)$ and therefore there would not exist an arbitrage-free model with these coefficients. When including further options with different strikes it becomes unclear how to choose the $\sigma, v(K)$ since the static arbitrage conditions create constraints between the different options. Schweizer and Wissel [2008b] conclude that the static arbitrage conditions creates a complicated state space for modeling implied volatilities and instead propose a better suited parameterization, called local implied volatilities, with a simpler state space with the static arbitrage conditions included.

2.6.3 Schönbucher's Market Model for the Implied Volatility

In Schönbucher [1999] a market model for the implied volatility is presented. The model includes the underlying asset S, a set of European options and a risk-free asset with constant interest rate. Market completeness is achieved for this model since for each of the state variables, i.e. time, price of the underlying and implied volatility, there is one trade asset. Further, an advantage is that there is no need to specify the market price of risk process since it is implied in the observed option prices. Schönbucher [1999] proposes the following dynamics in order to model one implied volatility

$$\begin{aligned} dS &= rSdt + \sigma(\tau, \hat{\sigma}, S)dW_0 \\ d\hat{\sigma} &= udt + \gamma dW_0 + vdW_1. \end{aligned}$$

If we apply the Itô formula on the price of a call option $C(S_t, \hat{\sigma}, K, T)$ for this dynamics we find

$$dC = C_t dt + C_S dS + \frac{1}{2}\sigma^2 S^2 C_{SS} dt + C_{\hat{\sigma}} d\hat{\sigma} + \frac{1}{2} C_{\hat{\sigma}\hat{\sigma}} d\langle \hat{\sigma}, \hat{\sigma} \rangle + C_{S\hat{\sigma}} d\langle \hat{\sigma}, S \rangle.$$

From this equation Schönbucher then derives a no arbitrage condition for a European option. If we examine the previous equation closer we see that the drift component is

$$rCdt = C_t dt + rSC_S dt + \frac{1}{2}\sigma^2 S^2 C_{SS} dt + C_{\hat{\sigma}} udt + \frac{1}{2} C_{\hat{\sigma}\hat{\sigma}} v^2 dt + C_{S\hat{\sigma}} \gamma \sigma S dt.$$

This equation can be re-written in the following way

$$0 = \left(C_t + rSC_S + \frac{1}{2}\hat{\sigma}^2 S^2 C_{SS} - rC_S\right)dt + \left(\frac{1}{2}(\sigma^2 - \hat{\sigma}^2)S^2 C_{SS} + C_{\hat{\sigma}}u + \frac{1}{2}C_{\hat{\sigma}\hat{\sigma}}v^2 + \gamma\sigma SC_{S\hat{\sigma}}\right)dt.$$

We can see that the first part of this equation is in fact the Black-Scholes PDE which is equal to zero. Hence, the no arbitrage drift condition for a European option is

$$\hat{\sigma}u = \frac{1}{2\tau}\left(\hat{\sigma}^2 - \sigma^2\right) - \frac{1}{2}d_+ d_- v^2 + \frac{d_-}{\sqrt{\tau}}\sigma\gamma. \tag{2.31}$$

Schönbucher continues by using equation (2.31) to specify the spot volatility process by letting $t \to T$ and saying that no bubbles in the implied volatility can be allowed. This results in the following equation

$$\hat{\sigma}^2\sigma^2 - 2\gamma f\hat{\sigma}\sigma - \hat{\sigma}^4 + f^2 v^2 = 0$$

$$\Leftrightarrow \sigma = \frac{\gamma f}{\hat{\sigma}} \pm \sqrt{\hat{\sigma}^2 - \frac{f^2}{\hat{\sigma}^2}(v^2 - \gamma^2)}.$$

Schönbucher [1999] also sets up a model for the forward volatility to handle a set of options with maturities $T_1, T_2, ..., T_M$. This set up results in a drift condition that must hold for each forward volatility while the condition from equation (2.31) still must hold for the very first implied volatility.

2.6.4 Carr and Wu's Vega-Gamma-Vanna-Volga Model

In Carr and Wu [2010] and Carr and Wu [2013] a model named the Vega-Gamma-Vanna-Volga (VGVV) model is presented. The name comes from that the model links the theta of the option to the four mentioned Greeks. Their approach is close to the approach of market models for the implied volatility surface since they also try to model implied volatility. However, the approach is also different compared to the market models since Carr and Wu only specify the dynamics of the implied volatility and leave the spot volatility process unspecified. The model is based on the condition that the discounted prices of options and the underlying all are martingales under the risk-neutral measure. The basic assumptions are that the implied volatility surface is driven by one single standard Brownian motion and that the underlying price dynamic is driven by a second correlated Brownian motion. We define the implied volatility $\hat{\sigma}(K,T)$ as a solution of the following SDE

$$d\hat{\sigma}(K,T) = \mu_t dt + \omega_t dW_1(t), \quad t \geq 0 \tag{2.32}$$

under the risk-neutral probability measure $\mathbb{Q}$. Further, by the Girsanov theorem there exists a standard Brownian motion, $(W_2)_{t\geq 0}$ under $\mathbb{Q}$ such that the stock price process S_t is a solution of the following SDE

$$dS_t = \sigma_t S_t dW_2(t), \quad t \geq 0 \tag{2.33}$$

where σ_t denotes the spot volatility. Finally, we let the stochastic process $\rho_t \in [-1,1]$ denote the correlation between the two Brownian motions, i.e.

$$d\langle W_1, W_2\rangle_t = \rho_t dt, \quad t \in [0,T]. \tag{2.34}$$

We now want to derive an equation that governs the implied volatility surface based on our definition of the implied volatility in Definition 1. We apply the Itô formula to the definition that states that $C_t(K,T) = C_{BS}(t, \hat{\sigma}(K,T), S_t)$, where C_{BS} refer to the Black-Scholes formula in equation (2.3) for the price of an European call option. Using the dynamic above we arrive at:

$$\begin{aligned} dC_t(K,T) =& \frac{\partial C_{BS}}{\partial S} dS_t + \frac{\partial C_{BS}}{\partial \sigma} d\hat{\sigma}_t(K,T) + \frac{\partial C_{BS}}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial S^2} d\langle S,S\rangle_t \\ &+ \frac{\partial^2 C_{BS}}{\partial S \partial \sigma} d\langle S, \hat{\sigma}(K,T)\rangle_t + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial \sigma^2} d\langle \hat{\sigma}(K,T), \hat{\sigma}(K,T)\rangle_t. \end{aligned} \tag{2.35}$$

For $t \geq 0$ we get

$$\begin{aligned} d\langle S,S\rangle_t &= \sigma_t^2 S_t^2 dt, \\ d\langle S, \hat{\sigma}(K,T)\rangle_t &= \sigma_t \omega_t \rho_t S_t dt, \\ d\langle \hat{\sigma}(K,T), \hat{\sigma}(K,T)\rangle_t &= \omega_t^2 dt, \end{aligned} \tag{2.36}$$

from the dynamics specified in (2.32 - 2.34). If we now enter this into equation (2.35) we find

$$\begin{aligned} dC_t(K,T) - \frac{\partial C_{BS}}{\partial S}\sigma_t S_t dW_2 - \frac{\partial C_{BS}}{\partial \sigma}\omega_t dW_1 = \\ \left[\mu_t \frac{\partial C_{BS}}{\partial \sigma} + \frac{\partial C_{BS}}{\partial t} + \frac{\sigma_t^2 S_t^2}{2}\frac{\partial^2 C_{BS}}{\partial S^2} + \sigma_t \omega_t \rho_t S_t \frac{\partial^2 C_{BS}}{\partial S \partial \sigma} + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial \sigma^2}\right] dt. \end{aligned} \tag{2.37}$$

Both $W_1(t)$ and $W_2(t)$ are martingales under the measure $\mathbb{Q}$ and for $C_t(K,T)$ to also be a martingale, and hence exclude arbitrage, we require the right hand side of equation (2.37), i.e. the drift component, to be equal to zero. We arrive at the fundamental equation that describes the VGVV-model

$$-\frac{\partial C_{BS}}{\partial t} = \mu_t \frac{\partial C_{BS}}{\partial \sigma} + \frac{\sigma_t^2 S_t^2}{2}\frac{\partial^2 C_{BS}}{\partial S^2} + \sigma_t \omega_t \rho_t S_t \frac{\partial^2 C_{BS}}{\partial S \partial \sigma} + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial \sigma^2}. \tag{2.38}$$

Note that this is not a PDE in the traditional sense since the function that solves the PDE is already known and also the coefficients are not deterministic but instead stochastic. We continue to relate this equation to the $\hat{\sigma}$ by entering the Black Scholes Greeks from equation (2.4) together with the expressions for $\frac{\partial^2 C_{BS}}{\partial S \partial \sigma}$ and $\frac{\partial^2 C_{BS}}{\partial \sigma^2}$. But first we note that since $\frac{\partial^2 C_{BS}}{\partial S^2} = \frac{\phi(d_+)}{S\sigma\sqrt{T-t}}$ we can write

$$\begin{aligned} \frac{\partial C_{BS}}{\partial t} &= -\frac{S^2\sigma^2}{2}\frac{\partial^2 C_{BS}}{\partial S^2}, \\ \frac{\partial C_{BS}}{\partial \sigma} &= S^2 \sigma (T-t)\frac{\partial^2 C_{BS}}{\partial S^2}, \\ \frac{\partial^2 C_{BS}}{\partial S \partial \sigma} &= -S d_- \sqrt{T-t}\frac{\partial^2 C_{BS}}{\partial S^2}, \\ \frac{\partial^2 C_{BS}}{\partial \sigma^2} &= S^2 (T-t)(d_-^2 + \sigma\sqrt{T-t}d_-)\frac{\partial^2 C_{BS}}{\partial S^2}. \end{aligned} \tag{2.39}$$

We enter this into equation (2.40) and then divide the whole equation with $S^2\frac{\partial^2 C_{BS}}{\partial S^2}$ which reveals:

$$\frac{\hat{\sigma}_t^2(K,T)-\sigma_t^2}{2}-\mu_t\hat{\sigma}_t(K,T)(T-t)-\rho_t\omega_t\sigma_t d_-\sqrt{T-t}-\frac{\omega_t^2}{2}\left[d_-^2+\hat{\sigma}_t(K,T)\sqrt{T-t}d_-\right](T-t). \tag{2.40}$$

This is the same equation as equation (2.31) which was presented in Schönbucher [1999] and this equation has to hold at every time in order to exclude arbitrage. In Carr and Wu [2010] and Carr and Wu [2013] the equation is used to introduce a total of three different dynamics for the implied volatility, which we now will continue and present.

Square-root Variance

The first parametric specification represents the implied volatility surface in terms of time to maturity and standardized moneyness defined as $z=\frac{ln(K/S_t)+\frac{1}{2}\hat{\sigma}^2\tau}{\hat{\sigma}\tau}$ $(=-d_-)$. This definition of moneyness can be interpreted as the number of standard deviations by which the log-strike exceeds the mean of the terminal log-stock price, i.e. $\mathbb{E}[ln(S_T)|S_t]$ under the probability measure that arises in the Black-Scholes model. Applying this change of variables results in the following equation

$$\frac{\hat{\sigma}_t^2(z,\tau)}{2}-\left[\mu_t\tau-\frac{\omega_t^2 z\tau^{3/2}}{2}\right]\hat{\sigma}_t(z,\tau)-\left[\frac{\sigma_t}{2}+\rho_t\omega_t z\sigma_t\sqrt{\tau}+\frac{\omega_t^2 z^2\tau}{2}\right]=0. \tag{2.41}$$

From here Carr and Wu [2010] propose the following mean-reverting dynamics with a deterministic square-root volatility

$$d\hat{\sigma}_t^2(K,T)=\kappa_t[\theta_t-\hat{\sigma}_t^2(K,T)]dt+2w_t e^{-\eta_t(T-t)}\hat{\sigma}_t(K,T)dW_1. \tag{2.42}$$

When applying this dynamics equation (2.41) is reduced to a quadratic equation in implied volatility

$$\begin{aligned}&(1+\kappa_t\tau)\hat{\sigma}_t^2(z,\tau)+(w_t^2e^{-2\eta_t\tau}\tau^{3/2}z)\hat{\sigma}_t(z,\tau)\\&\quad-\left[(\kappa_t\theta_t-w_t^2e^{-2\eta_t\tau})\tau+\sigma_t^2+2\rho_t w_t\sigma\sqrt{\tau}e^{-\eta_t\tau}z+w_t^2e^{-2\eta_t\tau}\tau z^2\right]=0.\end{aligned} \tag{2.43}$$

Given the parameters $(\kappa_t,\sigma_t,\eta_t,\theta_t,\rho_t,w_t)$ equation (2.43) can be solved and by taking the positive root as the solution the whole implied volatility surface is given. It should be noted that when $\tau=0$ the $\hat{\sigma}(z,0)=\sigma_t$ and that $\lim_{\tau\to\infty}\hat{\sigma}_t(z,\tau)=\theta$. This means that the smile/skew is flat for both short and long maturities. At intermediate maturities the smile $\hat{\sigma}_t(z,\tau)$ is a hyperbola. For $z=0$ the term-structure is given by

$$(1+\kappa_t\tau)\hat{\sigma}_t^2(0,\tau)-(\kappa_t\theta_t-w_t^2e^{-2\eta_t\tau})\tau-\sigma_t^2=0.$$

Lognormal Variance

The second parametric specification represents the implied volatility surface in terms of time to maturity and log-moneyness, defined as $x = ln(K/S_t)$, which results in the following equation

$$\frac{\sigma_t^2}{2} - \frac{\hat{\sigma}_t^2(x,\tau)}{2} + \left[\mu_t \hat{\sigma}_t(x,\tau) + \frac{\rho_t \omega_t \sigma_t}{2} \hat{\sigma}_t(x,\tau)\right]\tau + \frac{\rho_t \omega_t \sigma_t}{\hat{\sigma}_t(x,\tau)} x \\ - \frac{\omega_t^2}{8}\hat{\sigma}_t^2(x,\tau)\tau^2 + \frac{\omega_t^2}{2\hat{\sigma}_t^2(x,\tau)} x^2 = 0. \tag{2.44}$$

In this case Carr and Wu [2010] propose the following mean-reverting log-normal variance dynamics

$$d\hat{\sigma}_t^2(K,T) = \kappa_t\left[\theta_t - \hat{\sigma}_t^2(K,T)\right]dt + 2w_t e^{-\eta_t(T-t)}\hat{\sigma}_t^2(K,T)dW_1. \tag{2.45}$$

where κ, θ, w and η are non-negative stochastic processes that do not depend on K, T, or $\hat{\sigma}$. When applying this dynamics equation (2.44) is reduced to a biquadratic equation in the implied volatility

$$\frac{w_t^2}{4} e^{-2\eta_t\tau}\tau^2\hat{\sigma}_t^4(x,\tau) + \left[1 + \kappa_t\tau + w_t^2 e^{-2\eta_t\tau}\tau - \rho_t w_t \sigma_t e^{-\eta_t\tau}\tau\right]\hat{\sigma}_t^2(x,\tau) \\ - \left[\sigma_t^2 + \kappa_t\theta_t\tau + 2\rho_t w_t\sigma_t e^{-\eta_t\tau}x + w_t^2 e^{-2\eta_t\tau}x^2\right] = 0. \tag{2.46}$$

Given the parameters $(\kappa_t, \sigma_t, \eta_t, \theta_t, \rho_t, w_t)$ equation (2.46) can be solved and by taking the positive root as the solution the whole implied volatility surface is given. Equation (2.46) is a hyperbola in $\hat{\sigma}_t^2$ and x. In the limit when $\tau = 0$ we have

$$\hat{\sigma}_t^2(x,0) - \sigma_t^2 - 2\rho_t w_t \sigma_t x - w_t^2 x^2 = 0.$$

Proportional Volatility

A dynamics for the implied volatility with both the drift and the diffusion proportional to the volatility level is proposed in the revised paper Carr and Wu [2013]. The risk-neutral dynamics is written as

$$d\hat{\sigma}_t(K,T) = e^{-\eta_t(T-t)}\hat{\sigma}_t(K,T)(m_t dt + w_t dW_1), \quad w_t, \eta_t > 0, \tag{2.47}$$

where m_t, w_t and η_t are stochastic processes that do not depend on K, T or $\hat{\sigma}(K,T)$. This dynamics result in the following biquadratic equation for the implied volatility surface

$$0 = \frac{1}{4}e^{-2\eta_t\tau} w_t^2\tau^2\hat{\sigma}_t^4(x,\tau) + (1 - 2e^{-\eta_t\tau} m_t\tau - e^{-\eta_t\tau} w_t\rho_t\sigma_t\tau)\hat{\sigma}_t^2(x,\tau) \\ - (\sigma_t^2 + 2e^{-\eta_t\tau} w_t\rho_t\sigma_t x + e^{-2\eta_t\tau} w_t^2 x^2). \tag{2.48}$$

Given the parameters $(m_t, \sigma_t, \eta_t, \rho_t, w_t)$ equation (2.46) can be solved for the implied volatility surface by taking the positive root as the solution.

2.7 Conditions for an Arbitrage-Free Implied Volatility Surface

In this section we want to establish a limited number of conditions for an option price surface or an implied volatility surface to be arbitrage free based on the results in Carr and Madan [2005], Roper [2010], Lee [2004]. The conditions will be sufficient to exclude static arbitrage, i.e. arbitrage from trading assets within the price surface, and for the implied volatility surface the conditions will also be necessary given some mild requirements.

2.7.1 Conditions on Option Price Spreads

In this section we follow the approach in Carr and Madan [2005] where they work with option prices to derive sufficient conditions for an arbitrage-free price surface. To be able to define conditions for an arbitrage free price surface we first need to define possible trading strategies based on options that are priced in the price surface. We call these trading strategies for option spreads and proceed to define them in this section. Option spreads are long/short positions of equal sizes in options with different strikes and/or maturities written on the same underlying. There are several types of spreads and they are commonly grouped in to bull/bear spreads, calendar spreads and butterfly spreads after the differences in strike and/or maturity. To define sufficient conditions for exclusion of arbitrage based on different option spreads we use the following notation: C_{ij} is the price of a European call option with a strike K_i, $i = 0, ..., \infty$ and maturity T_j, $j = 0, ..., M$. Both the strikes and the maturities are increasing and positive sequences and the strikes go to infinity as i goes to infinity. For $i = 0$ we set $K_0 \equiv 0$ and for $j = 0$ we define $C_{i,0} = (S_0 - K_i)^+$.

Bull Spread

The bull spread, $Bull_{ij}$, are long/short positions of equal sizes in calls with the long position in the call with smaller strike and the short position in the call with larger strike. Here we define the position in each call to be of the size $\frac{1}{K_i - K_{i-1}}$ and therefore the bull spread is defined as:

$$Bull_{ij} \equiv \frac{C_{i-1,j} - C_{i,j}}{K_i - K_{i-1}}, \qquad i > 0. \tag{2.49}$$

Since $K_i - K_{i-1} > 0$ it is easily seen that the payoff of the bull spread is:

$$Bull_{ij}\Big|_{t=T_j} = \begin{cases} 0 & \text{for } S_{T_j} \leq K_{i-1} \\ \dfrac{S_{T_j} - K_{i-1}}{K_i - K_{i-1}} & \text{for } K_{i-1} \leq S_{T_j} < K_i \\ \dfrac{S_{T_j} - K_{i-1} - S_{T_j} + K_i}{K_i - K_{i-1}} = 1 & \text{for } K_i \leq S_{T_j}. \end{cases}$$

Hence, the bull spread payoff belongs to the interval $[0, 1]$.

Butterfly Spread

The definition of a butterfly spread, BS_{ij}, is to be long a call with strike K_{i-1}, be short $\frac{K_{i+1}-K_{i-1}}{K_{i+1}-K_i}$ calls with strike K_i and be long $\frac{K_i-K_{i-1}}{K_{i+1}-K_i}$ calls with strike K_{i+1}. Hence, the butterfly spread is

$$BS_{ij} \equiv C_{i-1,j} - \frac{K_{i+1}-K_{i-1}}{K_{i+1}-K_i} C_{i,j} + \frac{K_i - K_{i-1}}{K_{i+1}-K_i} C_{i+1,j}, \qquad i > 0. \tag{2.50}$$

Since the payoff is equal to

$$BS_{ij}\Big|_{t=T_j} = \begin{cases} 0 & \text{for } S_{T_j} \le K_{i-1} \\ S_{T_j} - K_{i-1} & \text{for } K_{i-1} \le S_{T_j} < K_i \\ S_{T_j} - K_{i-1} - \frac{K_{i+1}-K_{i-1}}{K_{i+1}-K_i}(S_{T_j} - K_i) = \frac{K_i - K_{i-1}}{K_{i+1}-K_i}(K_{i+1} - S_{T_j}) & \text{for } K_i \le S_{T_j} < K_{i+1} \\ S_{T_j} - K_{i-1} - \frac{K_{i+1}-K_{i-1}}{K_{i+1}-K_i}(S_{T_j} - K_i) + \frac{K_i - K_{i-1}}{K_{i+1}-K_i}(S_{T_j} - K_{i+1}) = 0 & \text{for } K_{i+1} \le S_{T_j}, \end{cases}$$

it is clear that

$$\begin{aligned} & C_{i-1,j} - \frac{K_{i+1}-K_{i-1}}{K_{i+1}-K_i} C_{i,j} + \frac{K_i - K_{i-1}}{K_{i+1}-K_i} C_{i+1,j} \ge 0 \\ \iff \quad & C_{i-1,j} - C_{i,j} \ge \frac{K_i - K_{i-1}}{K_{i+1}-K_i}(C_{i,j} - C_{i+1,j}). \end{aligned} \tag{2.51}$$

Calendar Spread

A calendar spread consists of two calls with the same strike K_i, one long with maturity T_{j+1} and one short with maturity T_j. Since it easily can be shown that the value of a European call is larger or equal to the intrinsic value of the same call (for an underlying that does not pay dividends), we require the calendar spread to be non-negative, i.e.

$$CS_{ij} \equiv C_{i,j+1} - C_{i,j} \ge 0, \qquad i,j \ge 0. \tag{2.52}$$

This means that there is a positive value to have the optionality during a longer time period.

Sufficient Conditions for No-Arbitrage

If we enter equation (2.49) in (2.51) we easily see that $Bull_{i,j} \ge Bull_{i+1,j}$ for $i,j \ge 0$. Defining $q_{i,j} \equiv Bull_{i,j} - Bull_{i+1,j}$ for $i = 1,\ldots,\infty$ we may see it as the marginal risk neutral probability that the stock price at T_j is equal to K_i since $\sum_{i=1}^{\infty} q_{i,j} = 1$ for all j if the conditions for the bull and butterfly spreads' payoffs hold. Further, we can for each discrete maturity, T_j, define the call price as

$$C_j = \sum_k (K_k - K)^+ q_{k,j}$$

and the risk-neutral probability measure, $\mathbb{Q}$, as

$$\mathbb{Q}_j(K) = \sum_{K_k \le K} q_{k,j}.$$

This measure is piecewise constant and increases from 0 to 1 as K increases from 0. All convex payoffs may be achieved through portfolios of calls with non-negative weights and by equation (2.52) it follows that all convex functions are priced higher when received at a later maturity. This then implies that the risk-neutral probability measure, $\mathbb{Q}_j$, is increasing in convex order w.r.t. j. Carr and Madan [2005] refer to a result by Kellerer (1972) that states that these conditions, i.e. that the measure is increasing in convex order, is equivalent to the existence of a Markov martingale with the same marginal distribution as the distribution raised from the option prices. In our case this means that there exists a Markov martingale, M_j such that

$$C_j(K) = E\big[(M_j - K)^+\big], \quad K > 0, \quad j = 0, 1, \ldots, M$$

It follows that the given call quotes are free of static arbitrage if the conditions in equations (2.49), (2.51) and (2.52) hold.

2.7.2 Conditions on Call Price Surfaces

Roper [2010] states conditions for both the call price surface and the implied volatility surface to be free of static arbitrage. The conditions for the call price surface to be free of static arbitrage are both necessary and sufficient and are now stated in the following theorem

Theorem 2 (Theorem 2.1 in Roper [2010])**.** *Let $s > 0$ be a constant.*

a) *Let $C : (0,\infty) \times [0,\infty) \mapsto \mathbb{R}$ satisfy the following conditions*

- **A1** *(Convexity in K) $C(\cdot,\tau)$ is a convex function, $\forall \tau \ge 0$.*
- **A2** *(Monoticity in τ) $C(K,\cdot)$ is non-decreasing, $\forall K \ge 0$.*
- **A3** *(Large strike limit) $\lim_{K\to\infty} C(K,\tau) = 0$, $\forall \tau \ge 0$.*
- **A4** *(Bounds) $(s-K)^+ \le C(K,\tau) \le s$, $\forall K > 0$, $\forall \tau 0$.*
- **A5** *(Expiry value) $C(K,0) = (s-K)^+$, $\forall K > 0$.*

Then

- **(i)** *the function $\hat{C} : [0,\infty) \times [0,\infty) \mapsto (R)$, $(K,\tau) \mapsto \begin{cases} s, & \text{if } K = 0 \\ C(K,\tau), & \text{if } K > 0 \end{cases}$ satisfy conditions A1-A5 but with $K \ge 0$ instead of $K > 0$.*
- **(ii)** *there exists a non-negative Markov martingale X with the property that*

$$\hat{C}(K,\tau) = \mathbb{E}\left[(X_\tau - K)^+ \middle| X_0 = s\right], \quad \textit{for all } K,\tau \ge 0$$

.

b) *All conditions in a) are necessary properties of $\hat{C}$ for it to be the conditional expectation of a call option under the assumption that X is a martingale.*

2.7.3 Conditions on Implied Volatility Surfaces

For the implied volatility surface to be free of static arbitrage Roper [2010] derives both sufficient and necessary conditions. Roper derives the conditions for the time-scaled volatility surface, i.e. $\Sigma = \sqrt{\tau}\hat{\sigma}$ using the log-moneyness $x = ln(K/S_t)$ and time to maturity τ. The following is Roper's theorem for sufficient conditions for a static arbitrage-free implied volatility surface.

Theorem 3 (Theorem 2.9 in Roper [2010])**.** *Let $s > 0$ and $\Sigma : \mathbb{R} \times [0,\infty) \mapsto \mathbb{R}$. Let Σ satisfy the following conditions*

IV1 *(Smoothness) for every $\tau > 0$, $\Sigma(\cdot,\tau)$ is twice differentiable.*

IV2 *(Positivity) for every $x \in \mathbb{R}$ and $\tau > 0$, $\Sigma(x,\tau) > 0$.*

IV3 *(Durrleman condition) for every $\tau > 0$ and $x \in \mathbb{R}$*

$$0 \le \left(1 - \frac{x\partial_x\Sigma}{\Sigma}\right)^2 - \frac{1}{4}\Sigma^2(\partial_x\Sigma)^2 + \Sigma\partial_{xx}^2\Sigma$$

IV4 *(Monotonicity in τ) for every $x \in \mathbb{R}$, $\Sigma(x,\cdot)$ is non-decreasing.*

IV5 *(Large moneyness behavior) for every $\tau > 0$, $\lim_{x\to\infty} d_+(x,\Sigma(x,\tau)) = -\infty$*

IV6 *(Value at maturity) for every $x \in \mathbb{R}$, $\Sigma(x,0) = 0$.*

Then

$$\hat{C} : [0,\infty) \times [0,\infty) \mapsto \mathbb{R},$$
$$(K,\tau) \mapsto \begin{cases} sC_{BS}(ln(K/s),\Sigma(ln(K/s),\tau)) & K > 0 \\ s, & K = 0 \end{cases}$$

is a call price surface parameterized by s that is free of static arbitrage. In particular, there exists a non-negative Markov martingale X with the property that $\hat{C}(K,\tau) = \mathbb{E}[(X_\tau - K)^+ | X_0 = s]$ for all $K,\tau \ge 0$

Roper [2010] can then show that four of the conditions are necessary given that the other two conditions hold. This is formulated in the following theorem

Theorem 4 (Theorem 2.15 in Roper [2010])**.** *Let $s > 0$ and $\Sigma : \mathbb{R} \times [0,\infty) \mapsto \mathbb{R}$. Let Σ satisfy the following conditions*

(1) *(Smoothness) for every $\tau > 0$, $\Sigma(\cdot,\tau)$ is twice differentiable.*

(2) *(Positivity) for every $x \in \mathbb{R}$ and $\tau > 0$, $\Sigma(x,\tau) > 0$.*

Let

$$\tilde{C} : [0,\infty) \times [0,\infty) \mapsto \mathbb{R},$$
$$(K,\tau) \mapsto \begin{cases} sC_{BS}(ln(K/s),\Sigma(ln(K/s),\tau)) & K > 0 \\ s, & K = 0 \end{cases}$$

Then if Σ violates any of conditions IV3-IV6 in Theorem 3, $\tilde{C}$ is not a call price surface free from static arbitrage.

2.7.4 Conditions at Extreme Strikes

In Lee [2004] the moment formula for implied volatility at extreme strikes was presented and proved. This is a general formula that uses the role of finite moments and it is not bounded to a particular model. Basically the formula means that the tails of the implied volatility skew can not be allowed to grow any faster than $\sqrt{\frac{2|x|}{T}}$. Lee states that the tail behavior of the implied volatility skew contains the same information as the tail behavior of option prices through the Black-Scholes formula. Further, Lee argues that option prices are bounded by moments since the option payoff can be dominated by a power payoff and that moments are bounded by option prices since a power payoff can be dominated by a mix of option payoffs. This means that the tail of the option prices contain as much information as the number of finite moments. The moments of the underlying are defined as

$$\tilde{p} := sup\{p : \mathbb{E}S_T^{1+p} < \infty\}$$
$$\tilde{q} := sup\{q : \mathbb{E}S_T^{-q} < \infty\}.$$

For $\hat{\sigma}, x$ and T we define the large strike and the small strike slope respectively and take the limsups of these coefficient as $x \to \pm\infty$. We have

$$\beta_R(T) := \limsup_{x\to\infty} \frac{\hat{\sigma}^2(x,T)}{|x|/T}$$
$$\beta_L(T) := \limsup_{x\to-\infty} \frac{\hat{\sigma}^2(x,T)}{|x|/T},$$

where β_R is referred to as the large strike tail slope and β_L is referred to as the small strike tail slope. Lee [2004] concludes that the β_R and β_L belong to the interval $[0,2]$ and only depend on the moments $\tilde{p}$ and $\tilde{q}$ according to the following moment formula

$$\tilde{p} = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2}$$
$$\tilde{q} = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}.$$

We can also rewrite these equations to find the large and small strike slopes as functions of the underlying's moments in the following way

$$\beta_R = 2 - 4(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p})$$
$$\beta_L = 2 - 4(\sqrt{\tilde{q}^2 + \tilde{q}} - \tilde{q}).$$

2.8 Kalman Filter

The Kalman filter is a recursive algorithm that combines the previous estimate with the current input data. One of the advantages with the Kalman filter is that only the last previous estimate is used which reduces the amount of data to store and it also means that the calculations are faster since only a small amount of data is used in each calculation step. [Haykin, 2001, Chapter 1] states that *"A key property of the Kalman filter is that it is the minimum mean-square (variance) estimator of the state of a linear dynamical system."* [pp. 20].

2.8.1 The Kalman Filter Algorithm

The Kalman filter uses a state-space representation where the state vector, denoted x_k for the discrete time k, is propagated using a transition matrix $F_{k+1;k}$ for the states. This equation is named the process equation and can be written as

$$x_{k+1} = F_{k+1;k} x_k + w_k, \tag{2.53}$$

where w_k represents error which is assumed to be independent normally distributed with zero mean and the following covariance

$$\mathbb{E}[w_n w_k^T] = \begin{cases} \mathbf{Q}_k & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases}$$

Further, the observations are denoted y_k and are linked to the states through the matrix H_k and the following measurement equation

$$y_k = H_k x_k + v_k, \tag{2.54}$$

where also the error v_k is assumed to be independent normally distributed with zero mean and the following covariance

$$\mathbb{E}[v_n v_k^T] = \begin{cases} \mathbf{R}_k & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases}$$

In [Haykin, 2001, Chapter 1] states the Kalman filter problem as *"Use the entire observed data, consisting of the vectors $y_1, y_2, ..., y_k$, to find for each $k \geq 1$ the minimum mean-square error estimate of the state x_i. The problem is called filtering if $i = k$, prediction if $i > k$, and smoothing if $1 \leq i < k$.* [pp. 3].

Now, the recursive algorithm of the Kalman filter approach can be summarized in Table 2.1 below.

Table 2.1: The Kalman Filter recursive algorithm from [Haykin, 2001, pp. 10]

State-space model

$$x_{k+1} = F_{k+1;k} x_k + w_k,$$
$$y_k = H_k x_k + v_k,$$

where w_k and v_k are independent, zero-mean, Gaussian noise processes of covariance matrices Q_k and R_k, respectively.
Initialization: For $k = 0$, set

$$\hat{x}_0 = \mathbf{E}[x_0],$$
$$P_0 = \mathbf{E}\left[(x_0 - \mathbf{E}[x_0])(x_0 - \mathbf{E}[x_0])^T\right],$$

Computation: For $k = 1, 2, \ldots$, compute:

$$\hat{x}_k^- = F_{k+1;k}\hat{x}_{k-1}^-,$$

Error covariance propagation

$$P_k^- = F_{k+1;k}P_{k-1}F_{k+1;k}^T + Q_{k-1},$$

Kalman gain matrix

$$G_k = P_k^- H_k^T \left[H_k P_k^- H_k^T + R_k\right]^{-1},$$

State estimate update

$$\hat{x}_k^= \hat{x}_k^- + G_k(y_k - H_k\hat{x}_k^-),$$

Error covariance update

$$P_k = (I - G_k H_k)P_k^-.$$

2.8.2 The Unscented Kalman Filter

The Kalman filter only handles a state-space formulation of a linear dynamical system and to handle non-linear systems the unscented Kalman filter (UKF) has been developed. One great advantage of the UKF compared to other methods to handle non-linear system is that there is no need to calculate the Jacobians or Hessians for the UKF. The UKF uses Gaussian random variables for the propagation through the system dynamics and a deterministic sampling approach consisting in choosing a minimal set of sample points. The true mean and covariance of the Gaussian random variable are captured in the UKF. Also, for all non-linearities the posterior mean and covariance is captured, to the second order, when the Gaussian random variables are propagated through the system [Haykin, 2001, Chapter 7].

The UKF uses the unscented transformation. This means that we consider a random variable $x \in \mathbb{R}^L$ which we propagate through the non-linear function $y = f(x)$. If we define $\bar{x}$ as the mean of x and P_x as the covariance matrix we can form a matrix χ of $2L+1$ vectors

$$\begin{aligned}
\chi_0 &= \bar{x}, \\
\chi_i &= \bar{x} + \left(\sqrt{(L+\lambda)P_x}\right)_i, \qquad i = 1, \ldots, L, \\
\chi_i &= \bar{x} - \left(\sqrt{(L+\lambda)P_x}\right)_{i-L}, \quad i = L+1, \ldots, 2L,
\end{aligned}$$

where we call χ_i a sigma vector and where $\lambda = \alpha^2(L+\kappa) - L$ is a scaling parameter. There are a few choices for the parameters α, κ and β, however in [Haykin, 2001, Chapter 7] the following is stated *"The constant α determines the spread of the sigma points around $\bar{x}$, and is usually set* *to a small positive value (e.g.,* $1 \geq \alpha \geq 10^{-4}$*. The constant* κ *is a secondary scaling parameter, which is usually set to* $3-L$*, and* β *is used to incorporate prior knowledge of the distribution of* x *(for Gaussian distributions,* $\beta = 2$ *is optimal).* $\left(\sqrt{(L+\lambda)P_x}\right)_i$ *is the ith column of the matrix square root (e.g., lower-triangular Cholesky factorization)."* [pp. 229].

When the sigma vectors are propagated through the function f

$$\Upsilon_i = f(\chi), \quad i = 0, \ldots, 2L,$$

the mean can be estimated with a weighted sample mean

$$\bar{y}_k \approx \sum_{i=0}^{2L} W_i^{(m)} \Upsilon_i,$$

as well as the covariance as

$$P_y \approx \sum_{i=0}^{2L} W_i^{(c)} (\Upsilon_i - \bar{y})(\Upsilon_i - \bar{y})^T$$

where the weights W_i are defined as

$$\begin{aligned}
W_0^{(m)} &= \frac{\lambda}{L+\lambda}, \\
W_0^{(c)} &= \frac{\lambda}{L+\lambda} + 1 - \alpha^2 + \beta, \\
W_i^{(m)} &= W_i^(c) = \frac{1}{2(L+\lambda)}, \quad i = 1, \ldots, 2L.
\end{aligned}$$

The UKF is an extension of the unscented transformation where the state vector and the noise vectors are merged together, i.e. $x_k^a = [x_k^T \; w_k^T \; v_k^T]^T]$, and used in the recursive formulation in equation (2.53). The following table describes the whole recursive algorithm for the UKF from [Haykin, 2001, Chapter 7].

Table 2.2: The Unscented Kalman Filter recursive algorithm from [Haykin, 2001, pp. 232]

Initialize with

$$\hat{x}_0 = \mathbb{E}[x_0], \tag{2.55}$$

$$P_0 = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T], \tag{2.56}$$

$$\hat{x}_0^a = \mathbb{E}[x^a] = \begin{bmatrix} \hat{x}_0^T & 0 & 0 \end{bmatrix}^T, \tag{2.57}$$

$$P_0^a = \mathbb{E}[(x_0^a - \hat{x}_0^a)(x_0^a - \hat{x}_0^a)^T] = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q^w & 0 \\ 0 & 0 & R^v \end{bmatrix} \tag{2.58}$$

For $k \in \{1, \ldots, \infty\}$, calculate the sigma points

$$\chi_{k-1}^a = \left[\hat{x}_{k-1}^a \quad \hat{x}_{k-1}^a + \gamma\sqrt{P_{k-1}^a} \quad \hat{x}_{k-1}^a - \gamma\sqrt{P_{k-1}^a} \right] \tag{2.59}$$

The time update equations are

$$\chi_{k|k-1}^x = F(\chi_{k-1}^x, u_{k-1}, \chi_{k-1}^v), \tag{2.60}$$

$$\hat{x}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \chi_{x,k|k-1}^x \tag{2.61}$$

$$P_k^- = \sum_{i=0}^{2L} W_i^{(c)} (\chi_{x,k|k-1}^x - \hat{x}_k^-)(\chi_{x,k|k-1}^x - \hat{x}_k^-)^T \tag{2.62}$$

$$\Upsilon_{k|k-1} = H(\chi_{x,k|k-1}^x, \chi_{k-1}^n), \tag{2.63}$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \Upsilon_{i,k|k-1} \tag{2.64}$$

and the measurement update equations are

$$P_{\bar{y}_k \bar{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} (\Upsilon_{i,k|k-1} - \bar{y}_k^-)(\Upsilon_{i,k|k-1} - \bar{y}_k^-)^T \tag{2.65}$$

$$P_{x_k y_k} = \sum_{i=0}^{2L} W_i^{(c)} (\chi_{x,k|k-1}^x - \hat{x}_k^-)(\Upsilon_{i,k|k-1} - \bar{y}_k^-)^T \tag{2.66}$$

$$\mathcal{K}_k = P_{x_k y_k} P_{\bar{y}_k \bar{y}_k}^{-1} \tag{2.67}$$

$$\hat{x}_k = \hat{x}_k^- + \mathcal{K}_k (y_k - \hat{y}_k^-) \tag{2.68}$$

$$P_k = P_k^- - \mathcal{K}_k P_{\bar{y}_k \bar{y}_k} \mathcal{K}_k^T, \tag{2.69}$$

where

$$x^a = [x^T, w^T, v^T]^T, \quad \chi^a = [(\chi^x)^T, (\chi^w)^T, (\chi^v)^T]^T, \quad \gamma = \sqrt{L+\lambda}.$$

2.8.3 Unscented Kalman Filter Implementation by Carr and Wu

In Carr and Wu [2013] the authors apply the unscented Kalman filter to fit their model for the implied volatility surface. They assume that the implied volatility surface at any given time t is governed by the five covariates $(m_t, w_t, \eta_t, \sigma_t, \rho_t)$. These five covariates are treated as hidden states in the state-space formulation and the observed implied volatility surface is treated as the measurement with errors. In order to use the unscented Kalman filter some of the covariates first have to be transformed to span the whole real line. The covariates w_t, η_t, σ_t are defined to be strictly positive and therefore they are transformed by applying the natural logarithm. The last covariate ρ_t only takes values in $[-1, 1]$ and is therefore transformed by the function $ln(1+\rho_t)/ln(1-\rho_t)$. Carr and Wu assume that the state vector $X_t \in \mathbb{R}^5$, which contain the hidden states, is propagated as a random walk

$$X_t = X_{t-1} + \sqrt{\Sigma_x}\epsilon_t,$$

where the error ϵ_t is assumed to be normally distributed with zero mean and variance equal to one. One further assumption is that the covariates can have different variances but they are all independent of each others movements. Carr and Wu define the measurement equation on the logarithm of the implied volatility

$$y_t = h(X_t) + \sqrt{\Sigma_y} e_t,$$

where the error is assumed to be additive and normally distributed. In the case of Carr and Wu their data set contains weekly data of 40 different points on the implied volatility surface, i.e. $y_t \in \mathbb{R}^{40}$. With this set up Carr and Wu follow the algorithm described in Table 2.2. Finally, with the forecasted implied volatility $\bar{y}_t$ and the conditional covariance matrix $P_{\bar{y}_k,t}$ the quasi-log likelihood value for the observation can be calculated (assuming normally distributed forecast errors)

$$l_t(\theta) = -\frac{1}{2} log|P_{\bar{y}_k,t}| - \frac{1}{2}\left((y_t - \bar{y}_t)^T (P_{\bar{y}_k,t})^{-1} (y_t - \bar{y}_t)\right).$$

With the quasi-log likelihood value Carr and Wu can then choose the θ that maximize the sum of the log likelihood values

$$\theta \equiv \arg\max_\theta \mathcal{L}(\theta, \{y_t\}_{t=1}^N), \quad \text{with} \quad \mathcal{L}(\theta, \{y_t\}_{t=1}^N) = \sum_{t=1}^N l_t(\theta)$$

where N denotes the number of days in the data sample. When the parameters have been estimated on a training sample, e.g. a historical sample, the unscented Kalman filter can provide fast estimation of the implied volatility surface [Carr and Wu, 2013].

2.9 Summary

The presentation of the theoretical framework started at the Black Scholes model before the local volatility was introduced. Local volatility makes it possible to price and hedge options in a market where smiles or skews are present and does not depend on which model is chosen for the underlying price process. However, according to Hagan et al. [2002] the dynamic behavior of the skew/smile is wrong, i.e. when the price of the underlying decreases (increases) the local volatility model predicts that smiles shift to higher (lower) prices. After local volatility the definition of the implied volatility was given and stochastic volatility models were presented. Also these models present some unwanted behavior when it comes to the volatility surface. Gatheral [2006] states that all stochastic volatility models have the same implications on the volatility surface and they imply that the future implied volatility surfaces will have a shape similar to today's shape.

Models for the dynamics of implied volatility surfaces were presented next by first stating an existence and uniqueness theorem for the solution of a SDE system describing the joint dynamics of the implied volatility surface and the option's underlying asset. Problems of satisfying the dynamic drift conditions while still guaranteeing the existence of a solution were lifted before the market model for the implied volatility from Schönbucher [1999] was presented. The section finished with an in-depth presentation of Carr and Wu's Vega-Gamma-Vanna-Volga model where the models from both their papers were introduced.

Static arbitrage in an option price surface or in an implied volatility surface refers to trading strategies within a surface that result in arbitrage. Conditions to exclude static arbitrage were stated based on option spread payoffs, on the properties of the call price surface and on the properties of the implied volatility surface. In particular a theorem for necessary and sufficient conditions for an implied volatility surface to be free of static arbitrage was presented.

Finally, the recursive algorithm of the Kalman filter was presented. The Kalman filter uses a state-space representation and the algorithm combines the previous estimate of the state with the current observation to predict the new state. The Kalman filter only handles linear systems and therefore also the unscented Kalman filter was presented. The unscented Kalman filter uses normally distributed random variables for the propagation through the system and a deterministic sampling approach consisting of choosing a minimal set of sample points. In Carr and Wu [2013] the unscented Kalman filter has been implemented to estimate the levels of the different stochastic processes, drive their proposed models.

3 Result

3.1 Introduction

In this chapter we will present the results of the thesis. First, a risk neutral drift condition for the option price surface is derived given a joint dynamics between the implied volatility surface and the underlying of the options. Second, the infeasibility of the Lognormal Variance model and the Proportional Volatility model, proposed in Carr and Wu [2010] and in Carr and Wu [2013] respectively, are shown. This done by showing that an implied volatility surface that follows either of the dynamics and fulfills the risk neutral drift condition cannot also fulfill one of the necessary conditions to exclude static arbitrage in the initial implied volatility surface. Finally, we present a general statement for models that are represented by a joint dynamics between the implied volatility surface and the options' underlying.

3.2 The Vega-Gamma-Vanna-Volga Model

In this section we will derive a risk neutral drift condition for the option price surface. Our market will contain a set of European options, the underlying of the options, and a risk-free asset with zero interest rate. We consider a general model with a probability space $(\Omega, (\mathscr{F}_t)_{(t\geq 0)}, \mathbb{Q})$, where the filtration $(\mathscr{F}_t)_{(t\geq 0)}$ is generated by two correlated Brownian motions $W^{(1)}, W^{(2)}$ and satisfies the usual conditions. $\mathbb{Q}$ is the risk neutral measure. The joint dynamics between the implied volatility surface, denoted $\hat{\sigma}$, and the underlying of the options, denoted S_t, is given by

$$\begin{aligned} d\hat{\sigma}_t(K,T) &= \mu_t(K,T)dt + \omega_t(K,T)dW_t^{(1)}, \\ dS_t &= \sigma_t S_t dW_t^{(2)}, \\ d\langle W^{(1)}, W^{(2)}\rangle_t &= \rho_t dt, \end{aligned} \tag{3.1}$$

and we assume that $\hat{\sigma}_t$ and S_t are solutions to the SDEs under the risk neutral measure $\mathbb{Q}$. σ_t denotes the spot volatility process of the underlying and we also assume that the implied volatility and the underlying are driven by the two correlated Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$ with a correlation process $\rho_t \in [-1,1]$. This type of dynamics have previously been proposed in for example Schönbucher [1999], Carr and Wu [2010], and Carr and Wu [2013] where also risk neutral drift conditions for the call price surface have been stated. We will now derive the drift condition for an option price surface quoted using log-moneyness defined as $x = ln(K/S_t)$.

Theorem 5. *For a model of the form (3.1) the risk neutral drift condition for the option price* *surface is given by the equation*

$$
\begin{aligned}
0 = \hat{\sigma}^2(K,T) - \sigma^2 - 2(T-t)\mu(K,T)\hat{\sigma}(K,T) - \frac{2\rho\sigma x\omega(K,T)}{\hat{\sigma}(K,T)} \\
- \rho\sigma(T-t)\omega(K,T)\hat{\sigma}(K,T) - \frac{x^2\omega^2(K,T)}{\hat{\sigma}^2(K,T)} + \frac{1}{4}(T-t)^2\omega^2(K,T)\hat{\sigma}^2(K,T).
\end{aligned}
\tag{3.2}
$$

Proof. Let $C(K,T)$ denote price of a call option with strike K and maturity T written on the underlying S_t and let $P(K_1,T_1)$ denote the price of a reference put option with strike K_1 and maturity T_1 written on the same underlying. Denote the Black-Scholes formula for the call and put price with $C^{BS}(S_t,\hat{\sigma}(K,T),K,T)$ and $P^{BS}(S_t,\hat{\sigma}(K,T),K,T)$ respectively. Per the definition of the implied volatility the following holds

$$
\begin{aligned}
C(K,T) &= C^{BS}(S_t,\hat{\sigma}(K,T),K,T), \\
P(K_1,T_1) &= P^{BS}(S_t,\hat{\sigma}(K_1,T_1),K_1,T_1).
\end{aligned}
$$

Now, form a portfolio, Π, consisting of one call , Δ number of the underlying and Δ_1 number of the put option. This portfolio's exposure to $W_t^{(1)}$ and $W_t^{(2)}$ can be hedged away by choosing the quantities Δ and Δ_1 in a specific way. First, apply Itô's formula on the portfolio Π

$$
\begin{aligned}
d\Pi = &\left(\frac{\partial C^{BS}}{\partial t} + \mu_t(K,T)\frac{\partial C^{BS}}{\partial \sigma} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 C^{BS}}{\partial S^2} + \rho_t\omega_t(K,T)\sigma_t S\frac{\partial^2 C^{BS}}{\partial S\partial\sigma} + \frac{1}{2}\omega_t^2(K,T)\frac{\partial^2 C^{BS}}{\partial \sigma^2}\right)dt \\
+\Delta_1 &\left(\frac{\partial P^{BS}}{\partial t} + \mu_t(K_1,T_1)\frac{\partial P^{BS}}{\partial \sigma} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 P^{BS}}{\partial S^2} + \rho_t\omega_t(K_1,T_1)\sigma_t S\frac{\partial^2 P^{BS}}{\partial S\partial\sigma} + \frac{1}{2}\omega_t^2(K_1,T_1)\frac{\partial^2 P^{BS}}{\partial \sigma^2}\right)dt \\
+ &\left(\omega_t(K,T)\frac{\partial C^{BS}}{\partial\sigma} - \Delta_1\omega_t(K_1,T_1)\frac{\partial P^{BS}}{\partial\sigma}\right)dW_t^{(1)} + \left(\frac{\partial C^{BS}}{\partial S} - \Delta_1\left(1 + \frac{\partial P^{BS}}{\partial S}\right) - \Delta\right)dW_t^{(2)}.
\end{aligned}
$$

Second, choose Δ and Δ_1 so that both of the last two terms cancel out. With the assumption of an interest rate equal to zero, the no dynamic arbitrage condition requires the drift of both options to be zero, i.e.

$$
\frac{\partial C^{BS}}{\partial t} + \mu_t(K,T)\frac{\partial C^{BS}}{\partial \sigma} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 C^{BS}}{\partial S^2} + \rho_t\omega_t(K,T)\sigma_t S\frac{\partial^2 C^{BS}}{\partial S\partial\sigma} + \frac{1}{2}\omega_t^2(K,T)\frac{\partial^2 C^{BS}}{\partial \sigma^2} = 0
$$

must hold. Noting that all derivatives are evaluated at the implied volatility $\hat{\sigma}(K,T)$ and using that the theta, vega, vanna and volga from the Black-Scholes model can be rewritten as functions of the gamma it follows that

$$
\begin{aligned}
\frac{\partial C^{BS}}{\partial t} &= -\frac{1}{2}\hat{\sigma}^2(K,T)S^2\frac{\partial^2 C^{BS}}{\partial S^2}, & \frac{\partial^2 C^{BS}}{\partial S\partial\sigma} &= \frac{x + \frac{\hat{\sigma}^2(K,T)}{2}(T-t)}{\hat{\sigma}(K,T)S}S^2\frac{\partial^2 C^{BS}}{\partial S^2}, \\
\frac{\partial C^{BS}}{\partial \sigma} &= \hat{\sigma}(K,T)(T-t)S^2\frac{\partial^2 C^{BS}}{\partial S^2}, & \frac{\partial^2 C^{BS}}{\partial\sigma^2} &= \left(\frac{x^2}{\hat{\sigma}^2(K,T)} - \frac{\hat{\sigma}^2(K,T)(T-t)^2}{4}\right)S^2\frac{\partial^2 C^{BS}}{\partial S^2}.
\end{aligned}
$$

The previous equation can now be rewritten by entering the theta, vega, vanna and volga into the equation and dividing by $-\frac{S^2}{2}\frac{\partial^2 C^{BS}}{\partial S^2}$ which results in

$$
\begin{aligned}
0 = \hat{\sigma}^2(K,T) - \sigma^2 - 2(T-t)\mu(K,T)\hat{\sigma}(K,T) - \frac{2\rho\sigma x\omega(K,T)}{\hat{\sigma}(K,T)} \\
- \rho\sigma(T-t)\omega(K,T)\hat{\sigma}(K,T) - \frac{x^2\omega^2(K,T)}{\hat{\sigma}^2(K,T)} + \frac{1}{4}(T-t)^2\omega^2(K,T)\hat{\sigma}^2(K,T),
\end{aligned}
$$

and the proof is finished. □

3.3 The Dynamics Proposed in Carr and Wu [2010]

Carr and Wu [2010] have proposed a model for the dynamics of implied volatility surfaces. They named it the Lognormal Variance model and proposed the following mean-reverting dynamics with log-normal variance

$$
d\hat{\sigma}_t^2(K,T) = \kappa_t\left[\theta_t - \hat{\sigma}_t^2(K,T)\right]dt + 2w_t e^{-\eta_t(T-t)}\hat{\sigma}_t^2(K,T)dW_1, \tag{3.3}
$$

where w_t, η_t, κ_t and θ_t are non-negative stochastic processes that do not depend on K, T and $\hat{\sigma}$. According to Carr and Wu [2010] the μ_t and ω_t in equation (3.1) is represented by

$$
\begin{aligned}
\mu_t &= \frac{1}{2}\left(\frac{\kappa_t\theta}{\hat{\sigma}_t(K,T)} - \left(\kappa_t + w_t^2 e^{-2\eta_t(T-t)}\right)\hat{\sigma}_t(K,T)\right), \\
\omega_t &= w_t e^{-\eta_t(T-t)}\hat{\sigma}(K,T).
\end{aligned}
$$

However, we will now show that if the processes w_t, η_t, κ_t and θ_t do not dependent on K and T, the implied volatility surface in the model cannot be free of arbitrage.

Theorem 6 (Arbitrage in the Lognormal Variance Model of Carr and Wu)**.** *Assume $\hat{\sigma}_t$ is an implied volatility surface following the lognormal volatility model where the dynamics are described by equations (3.1) and (3.3) with generic assumptions on the processes w_t, η_t, κ_t and θ_t. Then for any $\hat{\sigma}_t$ that fulfills the risk-neutral drift condition in Theorem 5, the necessary condition on the large moneyness behavior of $\hat{\sigma}_t$ to exclude static arbitrage in Theorem 4 cannot be fulfilled. In other words, w_t, η_t, κ_t and θ_t cannot all be chosen independent of K and T.*

Proof. For the Lognormal Variance model the drift condition from Theorem 5 is

$$
\begin{aligned}
\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2\hat{\sigma}_t^4 + \left[1 + \kappa_t(T-t) + w_t^2 e^{-2\eta_t(T-t)}(T-t) - \rho_t w_t\sigma_t e^{-\eta_t(T-t)}(T-t)\right]\hat{\sigma}_t^2 \\
- \left[\sigma_t^2 + \kappa_t\theta_t(T-t) + 2\rho_t w_t\sigma_t e^{-\eta_t(T-t)}x + w_t^2 e^{-2\eta_t(T-t)}x^2\right] = 0.
\end{aligned}
$$

Solving this biquadratic equation for the implied volatility by taking the positive root gives

$$\begin{aligned}\hat{\sigma}_t = \Bigg(&-\frac{1+\kappa_t(T-t)+w_t^2 e^{-2\eta_t(T-t)}(T-t)-\rho_t w_t \sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2} \\ &+\Bigg[\left(\frac{1+\kappa_t(T-t)+w_t^2 e^{-2\eta_t(T-t)}(T-t)-\rho_t w_t \sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2}\right)^2 \\ &+\frac{\sigma_t^2+\kappa_t\theta_t(T-t)}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\Bigg]^{1/2}\Bigg)^{1/2}.\end{aligned} \tag{3.4}$$

In order to simplify, lets introduce the following notation

$$\begin{aligned} a &= \frac{1+\kappa_t(T-t)+w_t^2 e^{-2\eta_t(T-t)}(T-t)-\rho_t w_t \sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2}, \\ b &= \frac{\sigma_t^2+\kappa_t\theta_t(T-t)}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}, \end{aligned}$$

where a and b do not depend on the log-moneyness x. Using this notation the implied volatility can now be written as

$$\hat{\sigma}_t = \left(-a+\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}\right)^{1/2}.$$

If it first can be asserted that the second derivative of the implied volatility surface exists for every time to maturity larger than zero, $\tau = T-t > 0$, and that the implied volatility surface is strictly positive for every $x \in \mathbb{R}$ and $\tau > 0$ then Theorem 4 applies and the condition on the large moneyness behavior of $\hat{\sigma}_t$ is a necessary condition for $\hat{\sigma}_t$ to be free of static arbitrage. By deriving the first and second derivative with respect to x as

$$\frac{\partial}{\partial x}\hat{\sigma}_t = \frac{2}{\hat{\sigma}_t}\frac{\frac{\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}},$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\hat{\sigma}_t = &-\frac{4}{\hat{\sigma}_t^3}\frac{\left(\frac{\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}\right)^2}{a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}} \\ &+\frac{2}{\hat{\sigma}_t}\frac{\frac{1}{(T-t)^2}}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}} \\ &-\frac{8}{\hat{\sigma}_t}\frac{\left(\frac{\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}\right)^2}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{3/2}}.\end{aligned}$$

it is clear that the second derivative exists for all $\tau > 0$. Second, from equation (3.4) it is seen that the implied volatility will be strictly positive if

$$0 < \frac{\sigma_t^2 + \kappa_t \theta_t (T-t)}{\frac{w_t^2}{4} e^{-2\eta_t (T-t)} (T-t)^2} + \frac{8\rho_t \sigma_t x}{w_t e^{-\eta_t (T-t)} (T-t)^2} + \frac{4x^2}{(T-t)^2}.$$

But this expression's minimum is equal to $\frac{\sigma_t^2+\kappa_t\theta_t(T-t)}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}$ for $x = -\frac{2\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}}$. Therefore the implied volatility must be strictly positive for all $x \in \mathbb{R}$ and $\tau > 0$ since σ_t is strictly positive and κ_t, θ_t and η_t are nonnegative. Now the implied volatility surface satisfy the requirements of Theorem 4 and therefore all the necessary conditions from the theorem must hold for the surface. In particular the condition

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty} \frac{-x + \frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}} = -\infty,$$

must hold. But for $t = 0$ the limit behavior when $x \to \infty$ of the implied volatility surface in equation (3.4) follows

$$\hat{\sigma}_0 = \sqrt{\frac{2x}{T}} + \mathcal{O}(x^{-1/4}) \quad \text{as } x \to \infty,$$

and hence

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty} \frac{-x + \frac{1}{2}\hat{\sigma}_0^2 T}{\hat{\sigma}_0\sqrt{T}} = \lim_{x\to\infty} \frac{-x+x}{\sqrt{2x}} = 0.$$

Therefore, when w_t, η_t, κ_t and θ_t are non-negative stochastic processes that do not depend on K, T and $\hat{\sigma}$ in the dynamics given by equations (3.1) and (3.3) the necessary condition on the large moneyness behavior of $\hat{\sigma}$ cannot be fulfilled and the initial implied volatility surface is not free of arbitrage. □

3.3.1 Negatively Priced Bull Spread

Carr and Wu [2010] used long time series of option prices together with the unscented Kalman filter to find the levels of all the stochastic processes σ_t, ρ_t, w_t, η_t, κ_t and θ_t at the same point in time. To illustrate how the Lognormal Variance model introduces arbitrage in the implied volatility surface, the calibrated/estimated levels of σ_t, ρ_t, w_t, η_t, κ_t and θ_t have been collected from Carr and Wu [2010] for the last available date in their dataset, the 26th of December 2007, and is now used in an implementation of the model to calculate the price of a bull spread for different levels of moneyness. The collected values are $\sigma_t = 20\%$, $\rho_t = -0.95$, $w_t = 0.8$, $\eta_t = 0.05$, $\kappa_t = 0.6$ and $\theta_t = 0.09$.

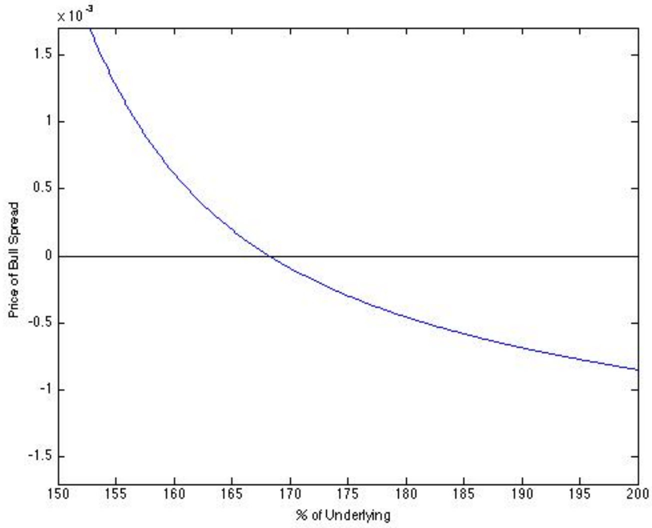

Figure 3.1: Price of a bull spread with one year maturity for the Lognormal Variance model.

When the maturity of the option is one year, we see that the bull spread is priced negatively already for a strike about 68% in the money, which of course constitutes an arbitrage opportunity. The first negatively priced bull spread appears for a strike less in the money for options with shorter maturities, ceteris paribus. If σ_t, ρ_t, η_t, κ_t, or θ_t is decreased, ceteris paribus, so is the strike for when the first negatively priced bull spread appears in the money. Only if w_t is decreased, ceteris paribus, the strike for which the first negatively priced bull spread appears is increased.

3.4 The Dynamics Proposed in Carr and Wu [2013]

In Carr and Wu [2013] a new model called the Proportional Volatility Model has been proposed. The dynamics is described by

$$d\hat{\sigma}_t(K,T) = e^{-\eta_t(T-t)}\hat{\sigma}_t(K,T)(m_t dt + w_t dW_1), \quad w_t, \eta_t > 0, \tag{3.5}$$

where m_t, w_t and η_t are stochastic processes that do not depend on K, T or $\hat{\sigma}(K,T)$. However, we will also now show that if the processes m_t, w_t and η_t do not dependent on K and T, the implied volatility surface in the model cannot be free of arbitrage.

Theorem 7 (Arbitrage in the Proportional Volatility Model of Carr and Wu)**.** *Assume $\hat{\sigma}_t$ is an implied volatility surface following the proportional volatility model where the dynamics are described by equations (3.1) and (3.5) with generic assumptions on the processes m_t, w_t and η_t. Then for any $\hat{\sigma}_t$ that fulfills the risk-neutral drift condition in Theorem 5, the necessary condition on the large moneyness behavior of $\hat{\sigma}_t$ to exclude static arbitrage in Theorem 4 cannot be fulfilled. In other words, m_t, w_t and η_t cannot all be chosen independent of K and T.*

Proof. For the Proportional Volatility model the drift condition from Theorem 5 is

$$\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2\hat{\sigma}_t^4+\left[1-2m_te^{-\eta_t(T-t)}(T-t)-\rho_t w_t\sigma_t e^{-\eta_t(T-t)}(T-t)\right]\hat{\sigma}_t^2$$
$$-\left[\sigma_t^2+2\rho_t w_t\sigma_t e^{-\eta_t(T-t)}x+w_t^2e^{-2\eta_t(T-t)}x^2\right]=0.$$

Solving this biquadratic equation for the implied volatility by taking the positive root gives

$$\begin{aligned}\hat{\sigma}_t=\Bigg(&-\frac{1-2m_te^{-\eta_t(T-t)}(T-t)-\rho_t w_t\sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2}\\&+\Bigg[\left(\frac{1-2m_te^{-\eta_t(T-t)}(T-t)-\rho_t w_t\sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2}\right)^2\\&+\frac{\sigma_t^2}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}+\frac{8\rho_t\sigma_t x}{w_te^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\Bigg]^{1/2}\Bigg)^{1/2}.\end{aligned}\tag{3.6}$$

In order to simplify, lets introduce the following notation

$$a=\frac{1-2m_te^{-\eta_t(T-t)}(T-t)-\rho_t w_t\sigma_t e^{-\eta_t(T-t)}(T-t)}{\frac{w_t^2}{2}e^{-2\eta_t(T-t)}(T-t)^2},$$
$$b=\frac{\sigma_t^2}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2},$$

where a and b do not depend on the log-moneyness x. Using this notation the implied volatility can now be written as

$$\hat{\sigma}_t=\left(-a+\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_te^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}\right)^{1/2}.$$

If it first can be asserted that the second derivative of the implied volatility surface exists for every time to maturity larger than zero, $\tau=T-t>0$, and that the implied volatility surface is strictly positive for every $x\in\mathbb{R}$ and $\tau>0$ then Theorem 4 applies and the condition on the large moneyness behavior of $\hat{\sigma}_t$ is a necessary condition for $\hat{\sigma}_t$ to be free of static arbitrage. By deriving the first and second derivative with respect to x as

$$\frac{\partial}{\partial x}\hat{\sigma}_t=\frac{2}{\hat{\sigma}_t}\frac{\frac{\rho_t\sigma_t}{w_te^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_te^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}},$$

and

$$\frac{\partial^2}{\partial x^2}\hat{\sigma}_t = -\frac{4}{\hat{\sigma}_t^3}\frac{\left(\frac{\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}\right)^2}{a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}}$$
$$+\frac{2}{\hat{\sigma}_t}\frac{\frac{1}{(T-t)^2}}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{1/2}}$$
$$-\frac{8}{\hat{\sigma}_t}\frac{\left(\frac{\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{x}{(T-t)^2}\right)^2}{\left[a^2+b+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}\right]^{3/2}}.$$

it is clear that the second derivative exists for all $\tau > 0$. Second, it is seen from equation (3.6) that the implied volatility will be strictly positive if

$$0 < \frac{\sigma_t^2}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}+\frac{8\rho_t\sigma_t x}{w_t e^{-\eta_t(T-t)}(T-t)^2}+\frac{4x^2}{(T-t)^2}.$$

But this expression's minimum is equal to $\frac{\sigma_t^2}{\frac{w_t^2}{4}e^{-2\eta_t(T-t)}(T-t)^2}$ for $x = -\frac{2\rho_t\sigma_t}{w_t e^{-\eta_t(T-t)}}$. Therefore the implied volatility must be strictly positive for all $x \in \mathbb{R}$ and $\tau > 0$ since σ_t, w_t and η_t are strictly positive. Now the implied volatility surface satisfy the requirements of Theorem 4 and therefore all the necessary conditions from the theorem must hold for the surface. In particular the condition

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty}\frac{-x+\frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}} = -\infty,$$

must hold. But for $t = 0$ the limit behavior when $x \to \infty$ of the implied volatility surface in equation (3.6) follows

$$\hat{\sigma}_0 = \sqrt{\frac{2x}{T}} + \mathcal{O}(x^{-1/4}) \quad \text{as } x \to \infty,$$

and hence

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty}\frac{-x+\frac{1}{2}\hat{\sigma}_0^2 T}{\hat{\sigma}_0\sqrt{T}} = \lim_{x\to\infty}\frac{-x+x}{\sqrt{2x}} = 0.$$

Therefore, when m_t, w_t and η_t are non-negative stochastic processes that do not depend on K, T and $\hat{\sigma}$ in the dynamics given by equations (3.1) and (3.5) the necessary condition on the large moneyness behavior of $\hat{\sigma}$ cannot be fulfilled and the initial implied volatility surface is not free of arbitrage. □

3.4.1 Negatively Priced Bull Spread

Carr and Wu [2013] used long time series of option prices together with the unscented Kalman filter to find the levels of all the stochastic processes σ_t, ρ_t, m_t, w_t and η_t at the same point in time. To illustrate how the Proportional Volatility model introduces arbitrage in the implied volatility surface, the calibrated/estimated levels of σ_t, ρ_t, m_t, w_t and η_t have been collected from Carr and Wu [2013] for the last available date in their dataset, the 26th of December 2007, and is now used in an implementation of the model to calculate the price of a bull spread for different levels of moneyness. The collected values are $\sigma_t = 20\%$, $\rho_t = -0.87$, $m_t = 0.2$, $w_t = 0.55$ and $\eta_t = 0.4$.

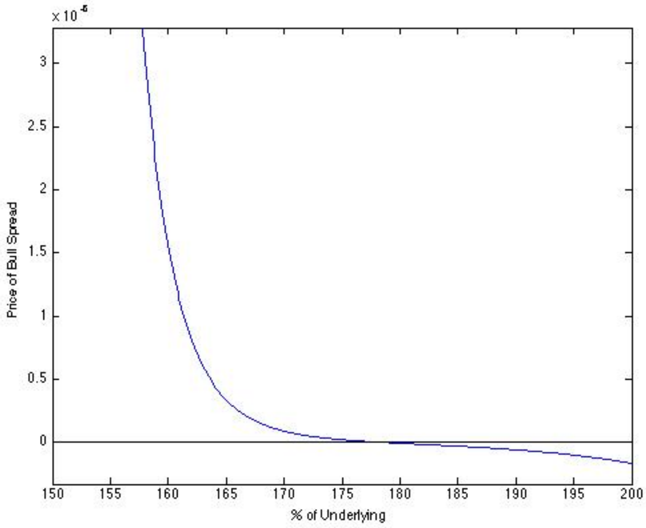

Figure 3.2: Price of a bull spread with one year maturity for the Proportional Volatility model.

When the maturity of the option is one year, we see that the bull spread is priced negatively for a strike about 78% in the money, which of course constitutes an arbitrage opportunity. The first negatively priced bull spread appears for a strike less in the money for options with shorter maturities, ceteris paribus. If σ_t, ρ_t, or η_t, is decreased, ceteris paribus, so is the strike for when the first negatively priced bull spread appears in the money. Contrary, if m_t or w_t is decreased, ceteris paribus, the strike for which the first negatively priced bull spread appears is increased.

3.5 A General Dynamics

In this section we prove a general statement for a range of models of the form

$$\begin{aligned}
d\hat{\sigma}_t(K,T) &= u_t \hat{\sigma}_t^p(K,T)dt + w_t \hat{\sigma}_t^q(K,T)dW_t^{(1)}, \\
dS_t &= \sigma_t S_t dW_t^{(2)}, \\
d\langle W^{(1)}, W^{(2)}\rangle_t &= \rho_t dt,
\end{aligned} \tag{3.7}$$

where $p \in [0,1]$, $q \in [1/2, 1]$ and u_t, w_t do not depend on K, T or $\hat{\sigma}(K,T)$.

Theorem 8. *Consider the general model in equation (3.7) with generic assumptions on the processes u_t and w_t. For an implied volatility surface $\hat{\sigma}_t$ that follows this dynamics and*

- *is twice differentiable with respect to log-moneyness x for every time to maturity $\tau > 0$,*
- *is strictly positive for every $x \in \mathbb{R}$ and $\tau > 0$,*
- *fulfills the risk neutral drift condition in Theorem 5,*

the necessary condition on the large moneyness behavior of $\hat{\sigma}_t$ to exclude static arbitrage in Theorem 4 cannot be fulfilled.

Proof. The implied volatility surface fulfills the risk neutral drift condition

$$\begin{aligned}0 = \hat{\sigma}_t^2(K,T) - \sigma_t^2 - 2(T-t)u_t\hat{\sigma}_t^{1+p}(K,T) - \frac{2\rho_t\sigma_t x w_t \hat{\sigma}_t^q(K,T)}{\hat{\sigma}_t(K,T)} \\ -\rho_t\sigma_t(T-t)w_t\hat{\sigma}_t^{1+q}(K,T) - \frac{x^2w^2\hat{\sigma}_t^{2q}(K,T)}{\hat{\sigma}_t^2(K,T)} + \frac{1}{4}(T-t)^2w^2\hat{\sigma}_t^{2(1+q)}(K,T),\end{aligned}$$

which can be rewrite in the following form

$$\begin{aligned}0 = \hat{\sigma}_t^4(K,T) + \frac{4\hat{\sigma}_t^{4-2q}(K,T)}{(T-t)^2w_t^2} - \frac{8u_t\hat{\sigma}_t^{3+p-2q}(K,T)}{(T-t)w_t} - \frac{4\rho_t\sigma_t\hat{\sigma}_t^{3-q}(K,T)}{(T-t)w_t} \\ -\frac{4\sigma_t^2\hat{\sigma}_t^{2-2q}(K,T)}{(T-t)^2w_t^2} - \frac{8\rho_t\sigma_t x\hat{\sigma}_t^{1-q}(K,T)}{(T-t)^2w_t} - \frac{4x^2}{(T-t)^2}.\end{aligned} \tag{3.8}$$

Since the implied volatility surface satisfy the requirements from Theorem 4 all the necessary conditions must hold for the surface to be free of arbitrage. In particular the following necessary condition

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty} \frac{-x + \frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}} = -\infty,$$

must hold. However, when $x \to \infty$

$$\hat{\sigma}_t^4(K,T) + \mathcal{O}\left(\hat{\sigma}_t^{4-2q}(K,T) + \hat{\sigma}_t^{3+p-2q}(K,T)\right) = \frac{4x^2}{(T-t)^2}.$$

since $\hat{\sigma}_t$ satisfy equation (3.8) and $p \in [0,1]$ and $q \in [1/2, 1]$. But this means that

$$\lim_{x\to\infty} d_+ = \lim_{x\to\infty} \frac{-x + \frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}} = \lim_{x\to\infty} \frac{-x + \frac{1}{2}\sqrt{\frac{4x^2}{(T-t)^2}}(T-t)}{\left(\frac{4x^2}{(T-t)^2}\right)^{1/4}\sqrt{T-t}} = \lim_{x\to\infty} \frac{-x+x}{\sqrt{2x}} = 0.$$

Hence, for any implied volatility surface that follows the dynamics in equation (3.7) and fulfills the risk neutral drift condition in Theorem 5, the condition on the large moneyness behavior of $\hat{\sigma}$ cannot be fulfilled. □

3.6 Summary

In this chapter we have derived a risk neutral drift condition for the option price surface given a model with a joint dynamics between the implied volatility surface and the underlying of the options. Further, we have shown that for an implied volatility surface that follows the Lognormal Variance model proposed in Carr and Wu [2010] and fulfills a risk-neutral drift condition, the necessary condition on the large moneyness behavior of the surface to exclude static arbitrage cannot be fulfilled. A similar statement has also been shown for the Proportional Volatility model proposed in Carr and Wu [2013], where an implied volatility surface that follows this model and fulfills a risk-neutral drift condition cannot fulfill the necessary condition on the large moneyness behavior of the surface to exclude static arbitrage. Finally, when the dynamics of the implied volatility surface follow $d\hat{\sigma}_t(K,T) = u_t\hat{\sigma}_t^p(K,T)dt + w_t\hat{\sigma}_t^q(K,T)dW_t^{(1)}$, where $p \in [0,1]$, $q \in [1/2, 1]$, we have showed that any implied volatility surface that is strictly positive, twice differentiable with respect to log-moneyness x, and satisfy the risk neutral drift condition in Theorem 5, will not be an arbitrage free implied volatility surface.

4 Conclusion

In this thesis we have focused on models where the joint dynamics between the implied volatility surface, denoted $\hat{\sigma}$, and the underlying of the options, denoted S_t, is given by

$$\begin{aligned} d\hat{\sigma}_t(K,T) &= \mu_t(K,T)dt + \omega_t(K,T)dW_t^{(1)}, \\ dS_t &= \sigma_t S_t dW_t^{(2)}, \\ d\langle W^{(1)}, W^{(2)}\rangle_t &= \rho_t dt. \end{aligned} \tag{4.1}$$

We have showed that for these types of models the risk neutral drift condition of the option price surface is given by

$$\begin{aligned} 0 = \hat{\sigma}^2(K,T) - \sigma^2 - 2(T-t)\mu(K,T)\hat{\sigma}(K,T) - \frac{2\rho\sigma x\omega(K,T)}{\hat{\sigma}(K,T)} \\ - \rho\sigma(T-t)\omega(K,T)\hat{\sigma}(K,T) - \frac{x^2\omega^2(K,T)}{\hat{\sigma}^2(K,T)} + \frac{1}{4}(T-t)^2\omega^2(K,T)\hat{\sigma}^2(K,T), \end{aligned} \tag{4.2}$$

where x denotes the log-moneyness. However, for these type of models the exact choice of the dynamics for the implied volatility surface is non-trivial which we have seen when we carefully examined the models proposed in Carr and Wu [2010] and in Carr and Wu [2013]. For both the Lognormal Variance model with a dynamics of the form

$$d\hat{\sigma}_t^2(K,T) = \kappa_t\left[\theta_t - \hat{\sigma}_t^2(K,T)\right]dt + 2w_t e^{-\eta_t(T-t)}\hat{\sigma}_t^2(K,T)dW_1, \tag{4.3}$$

where w_t, η_t, κ_t and θ_t are non-negative stochastic processes that do not depend on K, T and $\hat{\sigma}$; and for the Proportional Volatility model with a dynamics of the form

$$d\hat{\sigma}_t(K,T) = e^{-\eta_t(T-t)}\hat{\sigma}_t(K,T)(m_t dt + w_t dW_1), \quad w_t, \eta_t > 0, \tag{4.4}$$

where m_t, w_t and η_t are stochastic processes that do not depend on K, T or $\hat{\sigma}(K,T)$; we have showed that for any implied volatility surface which follows one of these models and fulfills a risk-neutral drift condition, the necessary condition on the large moneyness behavior of the surface to exclude static arbitrage cannot be fulfilled.

Further, we have also been able to extend our result from the two models of Carr and Wu to a result that applies to all models following the dynamics in equation (4.1) when the dynamics of the implied volatility surface has the form

$$d\hat{\sigma}_t(K,T) = u_t\hat{\sigma}_t^p(K,T)dt + w_t\hat{\sigma}_t^q(K,T)dW_t^{(1)}, \tag{4.5}$$

where $p \in [0,1]$, $q \in [1/2, 1]$ and u_t, w_t do not depend on K, T or $\hat{\sigma}(K,T)$. For an implied volatility surface that follows a model in this range we have showed that if an implied volatility surface fulfills a risk-neutral drift condition, then the condition on the large moneyness behavior of the surface to exclude static arbitrage cannot be fulfilled.

4.1 Future Research

We have eliminated some of the choices of the dynamics for the implied volatility surface in equation (4.1), however there are still choices for the μ_t and ω_t that can be evaluated. It is however important to note that this is a complex task and that it is not sufficient to only find a surface that satisfy the risk neutral drift condition in equation (4.2). For an implied volatility surface to be free of dynamic arbitrage it must both be a solution to the SDE system in equation (4.1) and satisfy the condition in equation (4.2). Indeed not every implied volatility surface that satisfies the risk neutral drift condition is free of arbitrage, but only the evolutions that starts from an arbitrage-free surface and satisfy the risk neutral drift condition is free of arbitrage.

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