Cross-Currency and Hybrid Markov-Functional Models

Christian P. Fries* — christian.fries@dresdner-bank.com
Marius G. Rott* — marius.rott@dresdner-bank.com

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*Dresdner Bank, 60301 Frankfurt, Germany

1 Introduction

In this paper we consider cross-currency Markov-functional models and their calibration under the spot measure. Hunt, Kennedy and Pelsser [12, 13, 18] introduced a single-currency Markov-functional interest rate model in the terminal measure and showed how to efficiently calibrate it to LIBOR or swaprate options. Building upon their work we will present a multi-factor cross-currency LIBOR model under different measures. We see zero correlated FX spot and LIBOR rates as a natural starting point. Under this assumption we don't need a change of numéraire drift correction. The functionals of the foreign currency rates under the domestic numéraire then are identical to the functionals of the foreign currency model under its (foreign) spot measure. This provides the motivation for first deriving a spot measure version of single-currency Markov-functional model. We will show that in the spot measure it is possible to formulate and implement a very efficient calibration procedure comparable to that provided in [12] for the terminal measure. Combining single-currency Markov-functional interest rate models with a Markov-functional FX spot model we build two and three-factor cross-currency models. Relaxing the zero correlation assumption is technically quite simple, but it entails considerable additional computational costs, mainly for the calibration of the model to FX options. To circumvent this problem we suggest a more efficient approximate procedure, which seems to work quite well for low correlations.

Lattice-type models, like trees or Markov-functional models, with numerical integration as the main computational building block are usually seen numerically too expensive in high dimensions. However, with recent advances in sparse grid integration methods and sparse grid PDE methods, see e.g. [3, 10] and references therein, high dimensional integration became computationally feasible.

The Markov-functional model captures the probability densities inferred from option prices by assigning values to the realizations of the underlying Markov process. This is similar to local volatility models [9] and implied trees [8]. The implementation of the Markov-functional models assign values (e.g. LIBORs) to nodes with given transition probabilities, while trees usually assign transition probabilities to nodes with given values (e.g. short rate).

The current standard for pricing LIBOR exotics is probably the LIBOR Market Model (cf. [4, 5, 14, 16, 17, 19, 20, 22]). We see the Markov-functional model as a complement to LIBOR Market Models. The LIBOR Market Model is Markovian in high-dimensionions, but non Markovian in low-dimensions and it is therefore usually implemented as Monte Carlo simulation.[1] Despite recent advances[2], the determination of robust and fast early exercise boundaries is still a challenge within high dimensional Monte Carlo models. The backward algorithm of Markov-functional models offer a robust and fast alternative. In the context of cross-currency models we successfully take advantage of Markov-functional models for the pricing and sensitivity calculation of products like Bermudan Callable Power Reverse Duals.

The paper is organized as follows: Section 2 reconsiders the single-currency Markov-functional model. After a review of the LIBOR Markov-functional model under the terminal measure, c.f. [12], we show how to efficiently calibrate a single-currency Markov-functional model under the spot measure (Sections 2.2). Section 2.4 briefly discusses the change of numéraire in a single-currency Markov-functional model. In Section 3 we will then consider a simplified cross-currency model with deterministic foreign interest rates and stochastic FX rates. Finally, in Section 4 we will discuss the general case of stochastic domestic interest rates, foreign interest rates and FX rates. We conclude by briefly sketching in Section 5 how to adjust the Markov-functional FX model to equity.

[1] Markovian approximation of the LIBOR Market Model exists, see, e.g., [19].

[2] See e.g. [1, 6, 7, 15, 19, 21].

2 One-factor single-currency models

We consider a time discretization of the interval $[0, T]$ into n subintervals given by $0 =: T_0 < T_1 < \ldots < T_n := T$. $P(T_i; t)$ denotes the time t value of the zero coupon bond paying 1 at time T_i and defines a $\mathcal{F}_t$-measurable random variable over the filtered probability space $(\Omega, Q, \mathcal{F}, \{\mathcal{F}_t\})$. By $P(T_i)$ we denote the corresponding $\{\mathcal{F}_t\}$-adapted stochastic process, while $P(T_i; t)$ is the $\mathcal{F}_t$-measurable random variable and $P(T_i; t, \omega)$ denotes its value at some path $\omega \in \Omega$.[3]

2.1 Markov-functional model under the terminal measure[4]

Choose the zero coupon bond which pays 1 at T_n as the numéraire $N(t) := P(T_n; t)$. Assuming the validity of the *fundamental pricing theorem*[5] (see e.g. [2, 13, 17]) the value of replicateable assets $V(t)$ is given by the expectation[6] of the N-relative value $\frac{V}{N}$ with respect to the (so called) *equivalent martingale measure* $\mathbb{Q}^N$:

$$\frac{V(t_1)}{N(t_1)} = \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(t_2)}{N(t_2)}|\mathcal{F}_{t_1}\right). \tag{1}$$

We make the following assumption:

$$\boxed{\begin{array}{c}\text{The numéraire } N(t) = P(T_n; t)\\ \text{is a (deterministic) function of } x(t),\\ \text{where } x \text{ is a Markovian process, given by}\\ \mathrm{d}x = \sigma(t)\mathrm{d}W \quad \text{under } \mathbb{Q}^N, \qquad x(0) = x_0.\end{array}} \tag{2}$$

Here W is a $\mathbb{Q}^N$-Brownian motion, adapted to the filtration $\{\mathcal{F}_t\}$. We will denote functional form of the numéraire, whose existence is postulated in (2), by N again and write $N(T_i, \xi)$. We skip $\mathbb{Q}^N$ in the expectation operator as all following expectations are with respect to the measure $\mathbb{Q}^N$. Since the distribution of the random variable $x(t)$ with respect to the measure $\mathbb{Q}^N$ is known, expectations of functions $\xi \mapsto f(\xi)$ are given by

$$\mathrm{E}(f(x(T_j))|\{x(T_i) = \xi\}) = \int_{-\infty}^{\infty} f(\eta)\phi(\eta - \xi; \bar{\sigma})\mathrm{d}\eta,$$

where $\phi(\cdot; \bar{\sigma})$ is the probability density of the normal distribution with variance $\bar{\sigma}^2 = \int_{T_i}^{T_j} \sigma^2(\tau)\mathrm{d}\tau$.

By (1) and assumption (2) all products considered in this framework are functions of x. Non-path dependent products as of time T_i are function of $x(T_i)$ alone. For example, bond prices at time $t = T_i$ ($i < j$) in state $x(T_i) = \xi$ are given by

$$P(T_j; T_i, \xi) = N(T_i, \xi) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T_j, x(T_j))}|\{x(T_i) = \xi\}\right).$$

[3] Although we sometimes consider continuous processes, we are only interested in the time T_i realizations and only the discretized processes really matter here.

[4] This section sketches the Markov-functional model under terminal measure, as introduced by Hunt, Kennedy and Pelsser, [12].

[5] For the case of discrete time and finite state space a fundamental pricing theorem holds under the assumption of no arbitrage. For continuous time, the assumption is slightly more technical, cf. [2].

[6] Here $\mathrm{E}^{\mathbb{Q}^N}(\,\cdot\,|\mathcal{F}_{t_1})$ denotes the conditional expectation with respect to the measure $\mathbb{Q}^N$, conditioned on the information available in time t_1, with $\{\mathcal{F}_t\}$ being the filtration.

To streamline notation we simply write

$$\mathrm{E}^{\mathbb{Q}^N}(\ldots|(T_i,\xi)) \quad \text{for} \quad \mathrm{E}^{\mathbb{Q}^N}(\ldots|\{\omega|x(T_i,\omega)=\xi\})$$

(note that $\{\omega|x(T_i,\omega)=\xi\}\in\mathcal{F}_{T_i}$) and we will use the same symbol for the functional $\xi\mapsto P(T_j;T_i,\xi)$ and the original random variable $\omega\mapsto P(T_j;T_i,\omega)$ (as we already did), where $P(T_j;T_i,\omega)=P(T_j;T_i,x(T_i,\omega))$. Since we will only consider functionals of x this ambiguity will not lead to confusion.

2.1.1 The LIBOR model

The above constitutes a *model framework*. It remains to specify the functional forms of the numéraire $N(T_i)$. One way to do so is to derive the numéraire from LIBOR rates and infer, i.e. calibrate, the functional forms of the LIBOR rates from market option prices. This gives the Markov-functional LIBOR model[7].

By definition the LIBOR L_i, seen on its fixing date T_i is given by

$$1+L_i(T_i)(T_{i+1}-T_i) := \frac{P(T_i;T_i)}{P(T_{i+1};T_i)} = \frac{1}{P(T_{i+1};T_i)}. \tag{3}$$

Since $P(T_{i+1};T_i)$ is a function of $x(T_i)$, $L_i(T_i)$ is also a function of $x(T_i)$ given by the functional

$$1+L_i(T_i,\xi)(T_{i+1}-T_i) = \frac{1}{N(T_i,\xi)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T_{i+1},x)}|(T_i,\xi)\right)}. \tag{4}$$

Rearranging we have

$$N(T_i,\xi) = \frac{1}{\mathrm{E}\left(\frac{1}{N(T_{i+1},\cdot)}|(T_i,\xi)\right)\cdot(1+L_i(T_i,\xi)(T_{i+1}-T_i))}. \tag{5}$$

Thus, given $\xi\mapsto L_i(T_i,\xi)$, this gives a backward induction step $T_{i+1}\to T_i$ to calculate $N(T_i)$ from $N(T_{i+1})$. The induction start is trivially given by $N(T_N)\equiv 1$.

The free parameters of the LIBOR model are

- the specification of the underlying process x, i.e. here $\sigma(t)$,
- the specification of the LIBOR functional $\xi\mapsto L_i(T_i;\xi)$, which is by (5) equivalent to the specification of the numéraire functional $\xi\mapsto N(T_i;\xi)$.

Note that we could reformulate assumption (2) as an assumption on the LIBOR directly:

The LIBOR $L(T_k)=L_k(T_k)=\frac{1-P(T_{k+1};T_k)}{P(T_{k+1};T_k)(T_{k+1}-T_k)}$
(seen upon its maturity) is a (deterministic) function of $x(T_k)$,
where x is a Markovian process given by
$\mathrm{d}x=\sigma(t)\mathrm{d}W \quad$ under $\mathbb{Q}^N, \qquad x(0)=x_0$. (6)

[7] As presented in [12] one might also derive the numéraire from (co-terminal) Swap rates, which then defines the Markov-functional Swap rate model.

2.1.2 Calibration of the Markov-functional model under terminal measure

The model is calibrated by deriving the functional forms $\xi \mapsto L(T_i,\xi)$ from market prices in each backward induction step.

2.1.3 Backward induction step

Assume that $\xi \mapsto L(T_k,\xi)$ for $k \geq i$ have already been calculated. Thus the numéraire functionals $N(T_k)$ are known for $k \geq i$ from (5). Let $V^{\text{market}}_{K,T_{i-1}}(T_0)$ denote the *market price* of the digital caplet with fixing date T_{i-1}, paying

$$V^{\text{market}}_{K,T_{i-1}}(T_i) = \begin{cases} 1 & \text{if } L(T_{i-1}) \geq K \\ 0 & \text{if } L(T_{i-1}) < K \end{cases} \quad \text{in } T_i.$$

Digital caplet prices can be calculated if caplet market prices for arbitrary strikes K are given (see Lemma 1 in the appendix). Although we calibrate the model to digital caplet prices, the procedure will automatically calibrate the model to the corresponding caplet market prices for all strikes K.

Assume that the functional $\xi \mapsto L(T_{i-1},\xi)$ is monotone in ξ, for a fixed $x^* \in \mathbb{R}$ the payoff of a digital caplet with strike $L(T_{i-1},x^*)$ and payment date T_i is given by

$$\xi \mapsto V^{\text{model}}_{L(T_{i-1},x^*),T_{i-1}}(T_i,\xi) = \begin{cases} 1 & \text{if } \xi \geq x^* \\ 0 & \text{if } \xi < x^* \end{cases}.$$

Thus the *model price* is[8]

$$V^{\text{model}}_{L(T_{i-1},x^*),T_{i-1}}(T_0) = \frac{1}{N(0)} \cdot \mathrm{E}\left(\frac{\mathbb{1}_{[x^*,\infty)}}{N(T_i)} | (T_0,x_0) \right).$$

Note that, since the right hand side is known and does only depend on x^* and not on $L(T_{i-1},x^*)$, the above model price can be calculated for any given x^* without knowing $L(T_{i-1},x^*)$ in the current induction step. We will thus write $V^{\text{model}}_{x^*,T_{i-1}} := V^{\text{model}}_{L(T_{i-1},x^*),T_{i-1}}$.

The requirement to have the model calibrated to market prices implies

$$V^{\text{model}}_{x^*,T_{i-1}}(T_0) = V^{\text{market}}_{K,T_{i-1}}(T_0) \tag{7}$$

for the strike $K = L(T_{i-1},x^*)$. Inverting for a given state x^* the market-price formula for the corresponding strike K we find $L(T_{i-1},x^*) = K$.

As a result this method gives functionals $\xi \mapsto L(T_{i-1},\xi)$, monotone in ξ, calibrated to caplets of all strikes.

2.2 Markov-functional model under the spot measure

In this section we will discuss the Markov-functional model under the spot measure, i.e. we choose the money market account as numéraire, and present an efficient calibration method for this model. By money market account numéraire we mean (c.f. [14, 17])

$$N(T_i) := \prod_{k=0}^{i-1} (1 + L(T_k))(T_{k+1} - T_k), \tag{8}$$

[8] Here $\mathbb{1}_{[x^*,\infty)}$ denotes the indicator function, i.e. $\mathbb{1}_{[x^*,\infty)}(\xi) = 1$ for $\xi \geq x^*$ and else 0.

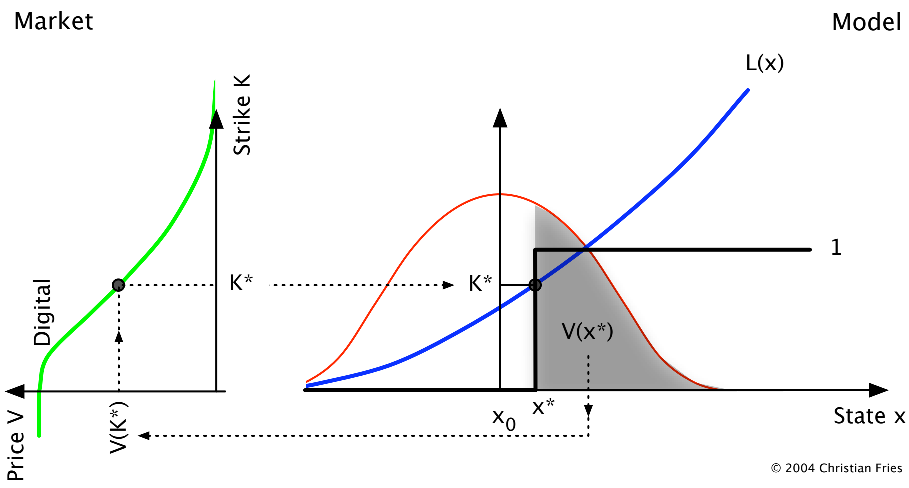

Figure 1: Calibration of the LIBOR functional of the Markov-functional model: The green curve of prices is given by / inferred from the market. For a given state x^* the value $V(x^*)$ of a calibration product (payout in black) with strike $L(x^*)$ is calculated. This can be done without knowing the functional L, knowing only the distribution of x (red). Looking up the corresponding strike K^* (on the market curve (green)) one obtains $L(x^*)$.

which is the value of repeated reinvestments of the initial value $N(0) = 1$ in the shortest bond on our time discretization $\{T_0, \ldots, T_N\}$. As in Section 2 we make the assumption:

> The LIBOR $L(T_k) = L_k(T_k) = \frac{1-P(T_{k+1};T_k)}{P(T_{k+1};T_k)(T_{k+1}-T_k)}$
> (seen upon its maturity) is a (deterministic) function of $x(T_k)$,
> where x is a Markovian process given by
> $\mathrm{d}x = \sigma(t)\mathrm{d}W \quad \text{under } \mathbb{Q}^N, \qquad x(0) = x_0.$ (6)

Note that this does imply that the numéraire $N(T_k)$ given in (8) is *not* a function of $x(T_k)$ alone. Here the numéraire $N(T_i)$ is path-dependent, i.e. it is given as a function of $x(T_0)$, $x(T_1)$,..., $x(T_{i-1})$:

$$N(T_i; x(T_0), x(T_1), \ldots, x(T_{i-1})) := \prod_{k=0}^{i-1} (1 + L(T_k; x(T_k))(T_{k+1} - T_k), \tag{9}$$

and $\mathcal{F}_{T_{i-1}}$-measurable. In contrast to this for the Markov-functional model under terminal measure we had that the numéraire $N(T_i)$ was a function of $x(T_i)$ alone (i.e. not path-dependent and $\mathcal{F}_{T_i}$-measurable, not $\mathcal{F}_{T_{i-1}}$-measurable).

2.2.1 Calibration of the Markov-functional model under spot measure

The calibration procedure of the Markov-functional model under the terminal measure was presented in [12] and Section 2.1.2 gave a streamlined presentation of the idea. It seems as if the feasibility of the calibration process is tied to the choice of the terminal measure as it induced a simple *backward induction* for the LIBOR functionals. The LIBOR functionals were calculated through the pricing of digital caplets which simply involved expectations of indicator functionals (ie. half integrals over given distributions).

We will show that the calibration procedure of the Markov-functional model under the spot measure is given by a simple *forward induction* for the LIBOR functionals. They are calculated through the pricing of a portfolio of a caplet and digital caplets. This involves only a simple half integral over the given distribution and a known expectation step.

2.2.2 Forward induction step

We assume that the LIBORs $L(T_j)$ for $T_j < T_i$ and thus $N(T_i)$ have already been calculated and present the induction step $T_i \rightarrow T_{i+1}$.[9] Together with $N(0) := 1$ this gives the calibration procedure as a *forward-in-time* algorithm. Let $V_{T_i}(T_k)$ denote the time T_k value of a product with a time T_{i+1} value $V_{T_i}(T_{i+1}; L(T_i))$ depending on $L(T_i)$ only (e.g. the value of a caplet or digital caplet with fixing date T_i and payment date T_{i+1}). Then the value of this product is

$$V_{T_i}(0) = N(0)\mathrm{E}\left(\frac{V_{T_i}(T_{i+1}; L(T_i))}{(1 + L(T_i)(T_{i+1} - T_i))N(T_i)} \mid \mathcal{F}_{T_0} \right).$$

On the right hand side, we take the expectation of a function depending on $L(T_i)$ and $N(T_i)$. As the Numeraire $N(T_i)$ is known from the previous induction step the functional form of $L(T_i; x(T_i))$ is the only unknown in this equation and it may be used to calibrate the functional form $\xi \mapsto L(T_i; \xi)$ to given market prices.

2.2.3 Dealing with the path-dependency of the numéraire

The path-dependency of the numéraire (9) implies that (conditional) expectations have to be calculated time-step by time-step using

$$\mathrm{E}\Big(\frac{V_{T_i}(T_k)}{N(T_k)} \mid \mathcal{F}_{T_i} \Big) = \mathrm{E}\Big(\frac{V_{T_i}(T_{k-1})}{N(T_{k-1})} \mid \mathcal{F}_{T_i} \Big)$$

where

$$V_{T_i}(T_{k-1}) = \frac{\mathrm{E}\left(V_{T_i}(T_k) \mid \mathcal{F}_{T_{k-1}} \right)}{1 + L(T_{k-1})(T_k - T_{k-1})}.$$

The need for the time step-by-time step calculation of conditional expectations (induced by the path-dependency of the numéraire) seems to be a major computational bottleneck, when compared to the Markov-functional model in terminal measure. However, we will discuss in Section 2.3 that $\mathcal{F}_{T_0}$ conditioned expectations may be calculated fast using a single scalar product with precalculated projection vectors.

2.2.4 Efficient calculation of the LIBOR functional from given market prices

The LIBOR functional are now derived from the model pricing formula of a portfolio of a caplet and digital caplets. Consider the following payout function

$$V_{T_i,K}(T_{i+1}, L(T_i)) := \begin{cases} 1 + L(T_i)(T_{i+1} - T_i) & \text{if } L_i - K > 0 \\ 0 & \text{else} \end{cases} \qquad \text{paid in } T_{i+1}. \tag{10}$$

This is a *digital caplet in arrears* or equivalently the portfolio of 1 strike K caplet and $K + \frac{1}{(T_{i+1} - T_i)}$ strike K digital caplets. Given market prices of caplets, we have market prices for

[9] Note that $N(T_i)$ depends on $L(T_j)$ for $T_j < T_i$ only.

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the digital caplet in arrears for any strike K (see Lemma 3 in the appendix). Its model price is given by

$$\begin{aligned} V^{\mathrm{model}}_{T_i,K}(T_0) &= \mathrm{E}\Big(\frac{V_{T_i,K}(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_0}\Big) \\ &= \mathrm{E}\Big(\frac{V_{T_i,K}(T_{i+1})}{1+L(T_i,x(T_i))(T_{i+1}-T_i)\cdot N(T_i)} \mid \mathcal{F}_{T_0}\Big) \\ &= \mathrm{E}\Big(\mathbb{1}(L(T_i,x(T_i))-K) \ \cdot \ \underbrace{\frac{1}{N(T_i)}}_{\mathcal{F}_{T_{i-1}}\text{-measurable}} \mid \mathcal{F}_{T_0}\Big) \\ &= \mathrm{E}\Big(\mathrm{E}\big(\mathbb{1}(L(T_i,x(T_i))-K) \mid \mathcal{F}_{T_{i-1}}\big) \ \cdot \ \frac{1}{N(T_i)} \mid \mathcal{F}_{T_0}\Big), \end{aligned}$$

where $\mathbb{1}$ denotes the indicator function with $\mathbb{1}(R) = 0$ if $R \leq 0$ and $1(R) = 1$ if $R > 0$ and $\xi \mapsto L(T_i, \xi)$ denotes the functional form of the LIBOR, assumed to be increasing. If x^* is such that

$$L(T_i, x^*) = K \tag{11}$$

we have

$$\begin{aligned} V_{T_i,K}(T_{i-1}) = \mathrm{E}\left(\mathbb{1}(L(T_i,x(T_i))-K) \mid (T_{i-1},\xi)\right) &= \mathrm{E}\left(\mathbb{1}(x(T_i)-x^*) \mid (T_{i-1},\xi)\right) \\ &= \int_{x^*}^{\infty} \phi(\eta-\xi;\sigma(T_{i-1},T_i))\,\mathrm{d}\eta. \end{aligned}$$

This reduces the model price to an integral over the indicator function(al) and then taking the expectation $\mathrm{E}\left(\frac{V(T_{i-1})}{N(T_{i-1})} \mid \mathcal{F}_{T_0}\right)$. The latter is known from the previous calibration steps from T_{i-1} back to T_0. It is implemented efficiently as a scalar product with a pre-calculated projection vector.[10]

The calculation of the functional form $L(T_i; \xi)$ thus involves the calculation of model prices as outlined above for suitable discretization points x^* and calculating the corresponding strikes K by inverting the market-price function. This determines $L(T_i, x^*)$ through (11).

The calibration step is as simple as under the terminal measure: Model prices of calibration products are evaluated by a half-integral together with a known expectation step and matched with the market-price function. Here, the half integral only represents a slightly different product.

Often a certain measure is chosen to simplify the pricing of a given product (e.g. the Black '76 caplet pricing formula is best derived under the terminal measure associated with the caplets payment date). Here this technique is reversed by considering a certain product with a simple (model) pricing formula under a given measure. The suitable product for the terminal measure is the digital caplet while the digital caplet in arrears seems the best choice for the spot measure.

2.3 Remark on the implementation

Given some functional $\xi \mapsto f(\xi)$ and a lattice time and state discretization

$$\{x_{T_j,k} \mid k = 1,\ldots,m_i\} \subset x(T_j,\Omega) = \mathbb{R}, \qquad 0 \leq j \leq n,$$

where $m_0 = 1$, $x_{T_0,1} = x_0$. The expectation of $f(x(T_{i+1}))$ conditioned on state $x(T_i) = x_{T_i,k}$ is given by

$$\int_{-\infty}^{\infty} f(\xi)\cdot\phi(\xi - x_{T_i,k};\bar{\sigma}^2)\mathrm{d}\xi, \tag{12}$$

[10] We will discuss this aspect of the implementation in the next section.

where $\phi(\cdot;\bar{\sigma})$ is the density of the normal distribution with variance $\bar{\sigma}^2 = \int_{T_i}^{T_{i+1}} \sigma^2(\tau)\mathrm{d}\tau$. The approximation of this integral within the lattice is given by a numerical integration based on sampled values $f_k := f(x_{T_i,k})$. We represent this integration by

$$A_{T_i}^{T_{i+1}} \cdot (f_1, \ldots, f_{m_{i+1}})^{\mathsf{T}}, \tag{13}$$

where $A_{T_i}^{T_{i+1}}$ is a linear operator given by a $m_i \times m_{i+1}$-matrix. Defining

$$A_{T_0}^{T_{i+1}} := A_{T_0}^{T_i} \cdot A_{T_i}^{T_{i+1}}, \tag{14}$$

the large time expectation step

$$E(f(x(T_{i+1}) \mid \{x(T_0) = x_0\})$$

is represented numerically by $A_{T_0}^{T_{i+1}}$. The matrix multiplication with $A_{T_0}^{T_{i+1}}$ is fast as $A_{T_0}^{T_{i+1}}$ is a row vector.

2.3.1 Fast calculation of price functionals

In the model calibration and the application of the model to derivative pricing expectations of numéraire relative prices have to be calculated. For a given time T_{i+1} functional V we have to calculate

$$I_{T_i}^{T_{i+1}}[V](x_{T_i,k}) := \int_{-\infty}^{\infty} \frac{V(\xi)}{N(T_{i+1},\xi)} \cdot \phi(\xi - x_{T_i,k};\bar{\sigma}^2)\mathrm{d}\xi. \tag{15}$$

It is advantageous to view $\xi \mapsto \frac{1}{N(T_{i+1},\xi)} \cdot \phi(\xi - x_{T_i,k};\bar{\sigma}^2)$ as convolution kernel and directly prelculate the numerical approximation of the (linear) operator $V \mapsto I[V]$.

Redefining the $A_{T_i}^{T_{i+1}}$ in this sense, we are able to numerically calculate large time-step expectations

$$\mathrm{E}\Big(\frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_0}\Big)$$

even for the path-dependent numéraire (9) by a single scalar product of the projection vector $A_{T_0}^{T_{i+1}}$ with the sample vector $(V(x_{T_{i+1},1}),\ldots,V(x_{T_{i+1},n_{i+1}}))$. The vectors $A_{T_0}^{T_{i+1}}$ may be precalculated iteratively in each forward induction step.

The elements of projection vector $A_{T_0}^{T_{i+1}}$ are Arrow-Debreu like prices.

2.3.2 Discussion on the implementation of the Markov-functional model under terminal and spot measure

It appears that the precalculation of the large time expectation step is only necessary to cope with the path-dependent numéraire in the spot measure Markov-functional model. However, in our experience the precalculation of projection vectors through the iteration (14) is advantageous even for the terminal measure variant as it will prevent numerically inconsistent ways of calculating the large-time expectation. Numerical approximation errors will lead to significant differences between iterated expectation and single, large time-step expectations, thus violating the tower law[11]. By enforcing the calculation of large time step expectations by iterated expectations the tower law will be valid in the model implementation by definition. It might seem as if the iteration (14) will then lead to a propagation of numerical errors. Indeed the terminal distributions are much less closer to a normal distribution, but exact sampling of the terminal distribution is not crucial and the calibration quality of the discrete model will not suffer.

[11] The tower law is the equation of iterated expectation, i.e. $\mathrm{E}\Big(\mathrm{E}(Z \mid \mathcal{F}_{T_j}) \mid \mathcal{F}_{T_i}\Big) = \mathrm{E}(Z \mid \mathcal{F}_{T_i})$ for $T_i < T_j$.

2.4 Change of numéraire in a Markov-Functional Model

Having presented Markov-functional models under different measures it is natural to ask how the functionals relate, i.e. under which condition a functional calibrated in one measure may be reused in the other.

Let N, M be two numéraires. Then for any traded asset V we have:

$$\frac{V(T_i)}{N(T_i)} = \mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i}\right)$$
$$\frac{V(T_i)}{M(T_i)} = \mathrm{E}^{Q^M}\left(\frac{V(T_{i+1})}{M(T_{i+1})} \mid \mathcal{F}_{T_i}\right) = \mathrm{E}^{Q^M}\big(\frac{V(T_{i+1})}{N(T_{i+1})}\frac{N(T_{i+1})}{M(T_{i+1})} \mid \mathcal{F}_{T_i}\big).$$

Thus

$$\mathrm{E}^{Q^M}\big(\frac{V(T_{i+1})}{N(T_{i+1})}\underbrace{\frac{N(T_{i+1})}{M(T_{i+1})}\frac{M(T_i)}{N(T_i)}}_{=:C(T_i,T_{i+1})} \mid \mathcal{F}_{T_i}\big) \;=\; \mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i}\right)$$

i.e.

$$\mathrm{E}^{Q^M}\left(\frac{V(T_{i+1})}{N(T_{i+1})}\cdot C(T_i,T_{i+1}) \mid \mathcal{F}_{T_i}\right) \;=\; \mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i}\right). \tag{16}$$

We want to see this in the light of a Markov-functional model and thus impose that all three quantities V, N and M are functions of a (scalar) time-discrete Markovian stochastic process $x(t)$. For illustration purposes we additionally impose that the functional of V under Q^M is the same as V under Q^N and that

$$\begin{aligned} x(T_{i+1}) &= x(T_i) + \sigma(T_i)\Delta W(T_i,T_{i+1}) && \text{under } Q^N \\ x(T_{i+1}) &= x(T_i) + \mu(T_i,x(T_i))\Delta T_i + \sigma(T_i)\Delta W(T_i,T_{i+1}) && \text{under } Q^M, \end{aligned}$$

where $\Delta T_i := (T_{i+1} - T_i)$ - it will become clear below, that this assumption cannot hold in general. Under these assumptions we have from (16) that[12]

$$\mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})}\cdot C(T_i,T_{i+1}) \mid x(T_i)+\mu(x(T_i))\Delta T_i\right) = \mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid x(T_i)\right),$$

i.e.

$$\mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})}\cdot C(T_i,T_{i+1}) \mid x(T_i)\right) \;=\; \mathrm{E}^{Q^N}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid x(T_i)+\mu(x(T_i))\Delta T_i\right). \tag{17}$$

The Equation (17) is valid for all traded assets V.[13] Choosing $V(T_2) \equiv 1$ (a bond) we see that Equation (17) determines $\mu(x(T_1))$ from the change of numéraire integration kernel C[14]. With μ fixed we see that (17) cannot hold for general functionals V. This is clear from Girsanov's Theorem: Over a discrete time step a change of numéraire will introduce a change in the conditional probability density, which can not be just a shift of the mean as in general $\int_t^{t+\Delta t}\mu(t)\mathrm{d}t$ is not $\mathcal{F}_t$ previsible. (Girsanov's Theorem [2] states that the conditional probability density changes by an infinitesimal shift of the mean (the drift adjustment) over an *infinitesimal* time-step.

Therefore, in a discrete time model it is usually not possible to perform a change of numéraire via an adapted change of the drift (if done, it is an approximation).

Thus we have to relax our assumptions to either

12 We assume here that the two measures are identical on $\mathcal{F}_{T_i}$, i.e. $[T_i, T_{i+1}]$ is the first time interval where the change of numéraire applies. This is no restriction, for example the argument applies to the first time step $[T_0, T_1]$.
13 Equation (17) is just a discrete version of Girsanov's theorem.
14 Note that $C(T_i, T_{i+1})$ is $\mathcal{F}_{T_{i+1}}$-measurable but not $\mathcal{F}_{T_i}$-measurable.

- the drift μ is path-dependent, i.e. we consider

$$\mathrm{d}x = \mu(t,x)\mathrm{d}t + \sigma(t)\mathrm{d}W \text{resulting in}$$
$$x(T_i) = x(T_{i-1}) + \int_{T_{i-1}}^{T_i} \mu(t,x(t))\mathrm{d}t + \sigma(T_{i-1})\Delta W(T_{i-1},T_i) \quad \text{under } Q^M\text{, or}$$

- the functional V is different under Q^N and Q^M.

The first will work, because it is just the proposition of Girsanov's Theorem.

As we do not want such a path-dependent drift in the driving process, we choose the second approach. Then we can fit *terminal* (!) distributions of V through a change of the functional form. Under the changed numéraire one has to recalculate the functional form V^M.

Note, that a recalculation of the functional forms *is not* a change of numéraire in the strict sense. The functional forms may be used to match the relevant terminal, but not the transition, distributions under Q^N and Q^M. As a result prices of some products, e.g. Bermudans, may differ. The two models are not "equivalent".[15]

This can already be seen for the Markov-functional model under the terminal measure. Two such models with different time horizons $T_m < T_n$ are not equivalent over the common time interval $[0, T_m]$.[16]

[15] This problem exists also in Monte Carlo simulations. For example, the Euler discretization of the LIBOR Market Model's SDE $\mathrm{d}L_i(t) = \mu_i(t)L_i(t)\mathrm{d}t + \sigma_i(t)L_i(t)\mathrm{d}W(t)$ exhibits different discretization errors for different measures. For the Monte Carlo simulation this problem may be solved by arbitrage-free discretization techniques [11] or by reducing the size of the discrete time step Δt.

[16] It is a charming aspect of the spot measure Markov-functional model that it does not exhibit this dependence on the time horizon (since there is no time horizon at all).

3 The two-factor cross-currency model (stochastic FX rates)

Consider

$$\begin{aligned} \mathrm{d}x &= \sigma_x(t)\mathrm{d}W_1 & x(0) &= x_0 \\ \mathrm{d}y &= \mu(t,x,y)\mathrm{d}t + \sigma_y(t)\mathrm{d}W_2 & y(0) &= y_0 \end{aligned} \tag{18}$$

with independent increments[17], i.e. $< \mathrm{d}W_1, \mathrm{d}W_2 >= 0$. Note, that we allow for a state-dependent drift in the y process. Furthermore, let the domestic LIBOR rate $L(T_i)$ at its fixing date T_i be a function of $x(T_i)$ within a (calibrated) one dimensional Markov-functional model with numéraire $N(T_i)$.

3.1 Discrete approximations of the driving processes

Since we are only interested in functionals of (x,y) at the T_i's, we consider a discretization of (18). Using a Euler discretization gives

$$\begin{aligned} \Delta x(T_{i-1}) &= \sigma_{x,i-1}\sqrt{\Delta T_{i-1}}\Delta W_1 \\ \Delta y(T_{i-1}) &= \mu(T_{i-1}, x(T_{i-1}), y(T_{i-1}))\Delta T_{i-1} + \sigma_{y,i-1}\sqrt{\Delta T_{i-1}}\Delta W_2 \end{aligned} \tag{19}$$

with $\Delta T_{i-1} = (T_i - T_{i-1})$ and

$$\begin{pmatrix} \Delta W_1 \\ \Delta W_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \qquad \begin{aligned} \sigma_{x,i-1} &:= \sqrt{\tfrac{1}{\Delta T_{i-1}} \int_{T_{i-1}}^{T_i} \sigma_x^2(t)\mathrm{d}t}, \\ \sigma_{y,i-1} &:= \sqrt{\tfrac{1}{\Delta T_{i-1}} \int_{T_{i-1}}^{T_i} \sigma_y^2(t)\mathrm{d}t}, \end{aligned}$$

where in (19). From now on $x(T_i)$, $y(T_i)$ denote the Euler scheme approximations of $x_0 + \int_0^{T_i} \mathrm{d}x$, $y_0 + \int_0^{T_i} \mathrm{d}y$, given by

$$\begin{aligned} x(T_0) &= x_0 \\ y(T_0) &= y_0 \\ x(T_i) &= x(T_{i-1}) + \Delta x(T_{i-1}) \\ y(T_i) &= y(T_{i-1}) + \Delta y(T_{i-1}). \end{aligned}$$

and we will consider functionals of the time-discrete processes (19) and not functionals of the time-continuous processes (18).

In the one dimensional case, there was actually no difference between the approximation $x(T_i)$ given through the Euler scheme and the random variable $x_0 + \int_0^{T_i} \mathrm{d}x(t)$ since - trivially - the approximation was exact due to the absence of a (nonlinear) drift. For the y process, however, we allow time and state-dependent drifts and there is a potential approximation error in the Euler scheme. This will make it important to consider functionals of the Euler scheme approximations $x(T_i)$ and $y(T_i)$ and not of their continuous processes $x_0 + \int_0^{T_i} \mathrm{d}x$, $y_0 + \int_0^{T_i} \mathrm{d}y$.

3.1.1 A note on the importance of 'early' discretization

In the Section 3.2 we will derive the drift of the *discrete version of the underlying process*, i.e. μ in (19) (the Euler-Scheme) from no-arbitrage considerations. This will make the *discrete model* an *arbitrage free model*.

Another possible approach would be to derive the drift of the time-continuous process from no-arbitrage considerations and then use a discretization scheme with an approximation of the time-continuous drift. This, however, only guarantees that the discretized model satisfies the no-arbitrage condition up to a discretization error.

[17] This requirement may be relaxed later.

Note on the notation

In order to calculate conditional expectations of functionals of x and y it is sufficient to know the distribution of x and y. The distribution of $x(T_i)$, $y(T_i)$ is known from (19). For the calculation of the expectation over one time step $T_{i+1} \to T_i$ of a functional G of some random variable $U = U(T_i) + \Delta U(T_i)$ conditioned on state $U(T_i) = \xi$, we need to know the distribution of $\Delta U(T_i)(\xi)$ only. To stress that the conditional expectation of G, conditioned on state (T_i, ξ) depends only on the increment $\Delta U(T_i)$ we will write

$$\mathrm{E}^{\Delta U(T_i)}\left(G|(T_i,\xi)\right) := \int_{-\infty}^{\infty} G(\eta) \cdot \phi^{\Delta U(T_i)}(\eta|\xi)\mathrm{d}\eta,$$

where $\phi^{\Delta U(T_i)}(\cdot|\xi)$ is the conditional probability density[18] of the random variable $\Delta U(T_i)$ under the condition $U(T_i) = \xi$.

3.2 The functional form and the derivation of the drift

Let $\tilde{P}(T_{i+1};T_i)$ denote the foreign bond guaranteeing the payment of 1 unit of foreign currency in T_{i+1} observed in T_i. From the fundamental pricing formula (1) we have that the process $FX \cdot \frac{\tilde{P}(T_{i+1})}{N}$ is a Q^N-martingale and thus

$$FX(T_i) \cdot \frac{\tilde{P}(T_{i+1};T_i)}{N(T_i)} = \mathrm{E}^{Q^N}\left(\frac{FX(T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}\right).$$

Assuming independent changes in interest rates and FX spot over the time step $T_i \to T_{i+1}$ (this is the discrete version on the assumption of instantaneously uncorrelated rates) we have

$$FX(T_i) \cdot \frac{\tilde{P}(T_{i+1};T_i)}{N(T_i)} = \mathrm{E}^{Q^N}\left(FX(T_{i+1})|\mathcal{F}_{T_i}\right) \cdot \mathrm{E}^{Q^N}\left(\frac{1}{N(T_{i+1})}|\mathcal{F}_{T_i}\right),$$

i.e.

$$FX(T_i) \cdot \frac{\tilde{P}(T_{i+1};T_i)}{P(T_{i+1};T_i)} = \mathrm{E}^{Q^N}\left(FX(T_{i+1})|\mathcal{F}_{T_i}\right). \tag{20}$$

We model the spot FX rate as instantaneously uncorrelated to the interest rates L and assume

$$\boxed{FX(T_k) \text{ is a function of } y(T_k) \text{ only.}} \tag{21}$$

Rewriting (20) using functional forms, and further assuming that the foreign bond is deterministic, (stochastic foreign interest rates will be considered later) we have (with (19))

$$\begin{aligned} FX(T_i, y(T_i)) = \; & \frac{P(T_{i+1};T_i,x(T_i))}{\tilde{P}(T_{i+1};T_i)} \\ & \cdot \mathrm{E}^{\Delta y(T_i)}\left(FX(T_{i+1}, y(T_{i+1}))|(T_i, x(T_i), y(T_i))\right). \end{aligned} \tag{22}$$

The no-arbitrage condition (20) – and (22) – gives a relation between the functional form $\eta \mapsto FX(T_i, \eta)$ and the specification of the transition probability $\Delta y(T_i)$, i.e. the drift μ and σ_y.

Thus, the free parameters of the FX model are

- the specification of the underlying process, here σ_y,
- the specification of the FX functional $FX(T_i; y)$.

and setting the drift according to (22) will ensure that the model is arbitrage free.

[18] The conditional probability densities, which form the basis of the model, are normal densities.

3.2.1 A note on the functional form

In the single-currency Markov-functional model we could arbitrarily choose an underlying process without imposing any restriction on the functional form of the LIBOR. The model would always remain arbitrage free and we naturally opted for a zero-drift process. In the cross-currency model the freedom to choose a y drift is gone, as for a given FX functional we now need a corresponding drift for the model to remain arbitrage free. If we would a priory insist on a specific drift, e.g. a zero drift, specifying the functional form of $FX(T_N)$ would automatically determine all other functional forms $FX(T_i)$ by (22).

The reason for the "freedom" to choose the drift for the process x lies in the ability to (still) choose the numéraire functional. Because of

$$P(T_{i+1};T_i,x(T_i)) = N(T_i,x(T_i)) \cdot E^{\Delta x(T_i)}\left(\frac{1}{N(T_{i+1},x(T_{i+1}))}|(T_i,x(T_i))\right)$$

$T_i \mapsto P(T_{i+1};T_i,x(T_i))$ is a $\mathbb{Q}^N$-martingale for any functional form $\xi \mapsto N(T_i,\xi)$. Conversely, for any functional form $\xi \mapsto P(T_{i+1};T_i,\xi)$ we may choose the numéraire $\xi \mapsto N(T_i,\xi)$ such that $T_i \mapsto P(T_{i+1};T_i,x(T_i))$ is a $\mathbb{Q}^N$-martingale. This freedom is gone once the numéraire has been chosen.

3.3 Examples for analytical drift calculation for special functional forms

In this section we will derive analytic formulas for the drift for some specific (analytical) functionals for $FX(T_i)$.

3.3.1 Linear functional form

Choose $FX(T_i,y) = y$ and $\sigma_y(T_i,y) = \sigma_{y,i} \cdot y$ for all times T_i, where $\sigma_{y,i}$ denotes the *constant* instantaneous volatility. For this functional form y is not just some driving process of FX, it *is* the FX and due to the choice of $\sigma_y(T_i,y)$ it is modelled as an Euler approximation of a log-normal process. In this case (22) results in

$$\mu(T_i,\xi,\eta)\Delta T_i = \eta \cdot \left(\frac{\tilde{P}(T_{i+1};T_i)}{P(T_i,T_{i+1},\xi)} - 1\right) = FX(T_i,\eta) \cdot \left(\frac{\tilde{P}(T_{i+1};T_i)}{P(T_i,T_{i+1},\xi)} - 1\right).$$

This is the Euler scheme of a log-normal process which converges to a log-normal process for $\Delta T \to 0$. A similar log-normal process is given by discretizing the log of the process through an Euler scheme and applying the exponential. This is our next example.

3.3.2 Exponential functional form

Fixing the functional form to $FX(T_i,y) = \exp(ay)$ $(a > 0)$ and the diffusion to $\sigma(T_i,y) = \sigma_i = \text{const.}$ for all i results in a simple (instantaneously) log-normal model. By

$$\begin{aligned}
&\mathrm{E}^{\Delta y(T_i)}(FX(T_{i+1},y(T_{i+1}))|(T_i,\xi,\eta) \\
&= \mathrm{E}^{\Delta y(T_i)}(\exp(ay(T_{i+1}))|(T_i,\xi,\eta)) \\
&= \exp\left(a(\eta + \mu_i(\xi,\eta)\Delta T_i) + \frac{a^2}{2} \cdot \sigma_{y,i}^2 \Delta T_i\right) \\
&= FX(T_i,\eta) \cdot \exp\left(a\mu_i(\xi,\eta)\Delta T_i + \frac{a^2}{2} \cdot \sigma_{y,i}^2 \Delta T_i\right)
\end{aligned}$$

(see Lemma 2 in the appendix) and by (22) we get

$$\frac{\tilde{P}(T_{i+1};T_i)}{P(T_i,T_{i+1},\xi)} = \exp\left(a\mu_i(\xi,\eta)\Delta T_i + \frac{a^2}{2} \cdot \sigma_{y,i}^2 \Delta T_i\right)$$

and thus

$$\mu_i(\xi, \eta)\Delta T_i = \frac{1}{a} \log\left(\frac{\tilde{P}(T_{i+1}; T_i)}{P(T_i, T_{i+1}, \xi)}\right) - \frac{a}{2}\sigma_{y,i}^2 \Delta T_i. \tag{23}$$

For this model the free parameters $\sigma_{y,i}$'s and a already provide enough freedom to replicate ATM FX option prices, but not enough flexibility to replicate the FX smile.
Note, that in contrast to the log-normal model in the previous section the drift μ_i does not depend on η here.

3.3.3 Other functional forms

It is possible to use other functional forms for the FX. This freedom allows to generate FX skews and smiles. For some parametrized functional forms it is possible to calculate expectations analytically deriving analytical formulas for the drift[19]. Of course it is also possible to solve for the drift numerically using a fast one dimensional root finder, but this will somewhat slow down the calibration process.

3.4 Calibration to FX option prices

In this section we will describe the forward in time induction step to calibrate the model to FX option prices. Two different routes are possible:

1. Fix a reasonable functional form, e.g. as in Section 3.3.2 and calibrate to ATM FX options by choosing the $\sigma_{y,i}$.
2. Freely choose the functional form for each time step, e.g. a parametrized functional; this allows for a full FX option smile calibration.

In this section we will consider the steps necessary for both alternatives.

Model price of an FX option with maturity T_i and strike K are given by

$$V_{\mathrm{FXOption}}^{T_i,K}(T_0) = N(0)\mathrm{E}^{\Delta x_{T_0}^{T_i}, \Delta y_{T_0}^{T_i}}\left(\frac{[FX(T_i, y(T_i)) - K]^+}{N(T_i, x(T_i))} | (T_0, x_0, y_0)\right)$$

Assuming that $FX(T_{i-1})$ and $\sigma(T_j, y)$ for $T_j < T_{i-1}$ are known from previous calibration steps, the forward in time induction $T_{i-1} \to T_i$ is completed by calibrating $\sigma(T_{i-1})$ and/or $FX(T_i)$. The objective to fit FX option model prices to given market prices is either achieved by using some minimization / root finding algorithm or by methods similar to those presented for the single-currency model. Since we have already fixed $\sigma(T_j, y)$ for $T_j < T_{i-1}$ we know the (backward) transition probabilities from T_{i-1} to T_0. Using the tower law for the conditional expectation operator, we have

$$\begin{aligned} V_{\mathrm{FXOption}}^{T_i,K}(T_0) &= N(0)\mathrm{E}\left(\frac{[FX(T_i, y(T_i)) - K]^+}{N(T_i, x(T_i))} | (T_0, x_0, y_0)\right) \\ &= N(0)\mathrm{E}\left(\hat{V}_{\mathrm{FXOption}}^{T_i,K}(T_{i-1}, x(T_{i-1}), y(T_{i-1})) | (T_0, x_0, y_0)\right) \end{aligned}$$

where

$$\hat{V}_{\mathrm{FXOption}}^{T_i,K}(T_{i-1}, \xi, \eta) := \mathrm{E}\left(\frac{[FX(T_i, y(T_i)) - K]^+}{N(T_i, x(T_i))} | (T_{i-1}, \xi, \eta)\right). \tag{24}$$

[19] As a hint we refer to the parametrized functional forms for the single-currency Markov-functional model proposed Pelsser, see [18].

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The calculation of the expectation

$$\mathrm{E}\left(\hat{V}^{T_i,K}_{\mathrm{FXOption}}(T_{i-1}, x(T_{i-1}), y(T_{i-1})) | (T_0, x_0, y_0)\right)$$

entails a single convolution with known two-dimensional transition probabilities $T_0 \mapsto T_{i-1}$. Since we only condition to a single point (T_0, x_0, y_0) the computational cost is bearable in general and quite low within a lattice implementation using precalculated projection vectors as discussed in Section 2.3.

Apart from this large time expectation step we have to calculate in each optimization step $\hat{V}^{T_i,K}_{FXOption}(T_{i-1}, \xi, \eta)$ for all sample points ξ, η. An optimization step consists of

1. adjusting the functional form of $FX(T_i)$ and/or $\sigma(T_{i-1})$,
2. solving for the drift using the drift condition (22),
3. calculating model prices as described above.

Assuming zero instantaneous correlation $< \Delta W_1, \Delta W_2 >= 0$ the two-dimensional expectation (24) can be simplified to allow for a much faster calibration. The transition probability $T_{i-1} \mapsto T_i$ given by the Euler approximation then is

$$\Delta x(T_{i-1}) \sim \mathcal{N}(0, \sigma^2_{x,i-1} \Delta T_{i-1}) \tag{25}$$

$$\Delta y(T_{i-1}) \sim \mathcal{N}(\mu_{i-1}(y(T_{i-1}))\Delta T_{i-1}, \sigma^2_{y,i-1} \Delta T_{i-1}). \tag{26}$$

For the case of the exponential functional form of Section 3.3.2 μ is given by an analytical formula

$$\mu_{i-1}(\xi)\Delta T_{i-1} = \frac{1}{a} \log\left(\frac{P(T_{i-1}, T_i, \xi)}{\tilde{P}(T_{i-1}, T_i)}\right) - \frac{a}{2}\sigma^2_{y,i-1}\Delta T_{i-1},$$

while in other cases one may be forced to solve for μ numerically using equation (22). Thus

$$\begin{aligned}
&\hat{V}^{T_i,K}_{FXOption}(T_{i-1}, \xi, \eta) \\
&= \mathrm{E}^{\Delta x, \Delta y}\left(\frac{[FX(T_i, y(T_i)) - K]^+}{N(T_i, x(T_i))} | (T_{i-1}, \xi, \eta)\right) \\
&= \mathrm{E}^{\Delta y}\left([FX(T_i, y(T_i)) - K]^+ | (T_{i-1}, \xi, \eta)\right) \cdot \mathrm{E}^{\Delta x}\left(\frac{1}{N(T_i, x(T_i))} | (T_{i-1}, \xi)\right) \\
&= \mathrm{E}^{\sigma_{y,i-1}\sqrt{\Delta T_{i-1}}\Delta W_2}\left([FX(T_i, y(T_i)) - K]^+ | (T_{i-1}, \eta + \mu_{i-1}(\xi)\Delta T_{i-1})\right) \\
&\quad \cdot \mathrm{E}^{\sigma_{x,i-1}\sqrt{\Delta T_{i-1}}\Delta W_1}\left(\frac{1}{N(T_i, x(T_i))} | (T_{i-1}, \xi)\right),
\end{aligned} \tag{27}$$

where the latter involves two, only one dimensional, expectations providing a major improvement in terms of calculation complexity. Note also that the one dimensional expectation $\mathrm{E}^{\sigma_{x,i-1}\sqrt{\Delta T_{i-1}}\Delta W_1}$ is independent of $\sigma_{y,i-1}$ and η and it can therefore be calculated outside the optimization for $\sigma_{y,i-1}$ and/or the optimization/calibration of the functional form $\eta \mapsto FX(T_i, \eta)$.

4 The three-factor cross-currency model

We consider a Markov process under a measure Q^N given by

$$\begin{aligned} \mathrm{d}x &= \sigma_x(t)\mathrm{d}W_1 \\ \mathrm{d}y &= \mu(t,x,y,z)\mathrm{d}t + \sigma_y(t)\mathrm{d}W_2 \\ \mathrm{d}z &= \sigma_z(t)\mathrm{d}W_3 \end{aligned} \tag{28}$$

and as before we choose Euler approximations of the the processes x, y, z, given as

$$\begin{aligned} \Delta x(T_{i-1}) &= \sigma_{x,i-1}\Delta W_1 \\ \Delta y(T_{i-1}) &= \mu(T_{i-1}, x(T_{i-1}), y(T_{i-1}), z(T_{i-1}))(T_i - T_{i-1}) + \sigma_{y,i-1}\Delta W_2 \\ \Delta z(T_{i-1}) &= \sigma_{z,i-1}\Delta W_3 \end{aligned} \tag{29}$$

and consider functionals of the Euler discretisation (which are denoted by x, y, z from now on).

Let L denote the domestic LIBOR depending on x only, as given by assumption (6) and FX the FX rate depending on y only, as given by assumption (21). Let $N(T_i)$ denote the numéraire of the single currency domestic model, which should also serve as numéraire for the three-factor model. We make the following assumption on the foreign interest rates functional

$$\boxed{\begin{array}{c} \text{The foreign LIBOR } \tilde{L}(T_i) := \tilde{L}(T_i, T_{i+1}; T_i) := \frac{1-\tilde{P}(T_{k+1};T_k)}{\tilde{P}(T_{k+1};T_k)(T_{k+1}-T_k)} \\ \text{(evaluated at its maturity } T_i\text{) is a (deterministic) function of } z(T_i), \\ \text{where } z \text{ is a Markovian process, given by} \\ \mathrm{d}z = \sigma_z(t)\mathrm{d}W_3 \quad \text{under } \mathbb{Q}^N, \qquad z(0) = z_0. \end{array}} \tag{30}$$

As a start for our analysis we assume that interest rates and FX rates are instantaneously independent in the following sense that their conditional transition probabilities are independent over the smallest discrete time step T_{i-1}, T_i. In other words, we assume that ΔW_i's are independent. This assumption may be relaxed.

Consider a foreign currency product $\tilde{V}$ depending at time T_i on the foreign LIBOR $\tilde{L}(T_i)$ paying $\tilde{V}(\tilde{L}(T_i)) \cdot FX(T_{i+1})$ at T_{i+1} to a domestic investor (e.g. a foreign caplet or a foreign digital caplet). This is a traded asset for the domestic investor and the price of this product is given by

$$\begin{aligned} \tilde{V}(T_0) \cdot \frac{FX(T_0)}{N(T_0)} &= \mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i)) \cdot FX(T_{i+1})}{N(T_{i+1})} | \mathcal{F}_{T_0} \right) \\ &= \mathrm{E}\left(\tilde{V}(\tilde{L}(T_i)) \cdot \mathrm{E}\left(\frac{FX(T_{i+1})}{N(T_{i+1})} | \mathcal{F}_{T_i} \right) | \mathcal{F}_{T_0} \right). \end{aligned}$$

We have $\mathrm{E}\left(\frac{FX(T_{i+1})}{N(T_{i+1})} | \mathcal{F}_{T_i} \right) = \frac{\tilde{P}(T_{i+1};T_i)}{N(T_i)} FX(T_i)$ and thus

$$\tilde{V}(T_0) \cdot \frac{FX(T_0)}{N(T_0)} = \mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i))}{1 + \tilde{L}(T_i)(T_{i+1} - T_i)} \cdot \frac{FX(T_i)}{N(T_i)} | \mathcal{F}_{T_0} \right).$$

Since the conditional transition probabilities from T_{i+1} to T_i of z and x, y are independent we have

$$
\begin{aligned}
&= \mathrm{E}\left(\mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i))}{1+\tilde{L}(T_i)(T_{i+1}-T_i)}|\mathcal{F}_{T_{i-1}}\right)\cdot \mathrm{E}\left(\frac{FX(T_i)}{N(T_i)}|\mathcal{F}_{T_{i-1}}\right)|\mathcal{F}_{T_0}\right)\\
&= \mathrm{E}\left(\mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i))}{1+\tilde{L}(T_i)(T_{i+1}-T_i)}|\mathcal{F}_{T_{i-1}}\right)\cdot \frac{FX(T_{i-1})}{(1+\tilde{L}(T_{i-1})(T_i-T_{i-1}))\cdot N(T_{i-1})}|\mathcal{F}_{T_0}\right)\\
&= \mathrm{E}\left(\mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i))}{1+\tilde{L}(T_i)(T_{i+1}-T_i)\cdot(1+\tilde{L}(T_{i-1})(T_i-T_{i-1}))}|\mathcal{F}_{T_{i-1}}\right)\cdot \frac{FX(T_{i-1})}{N(T_{i-1})}|\mathcal{F}_{T_0}\right)
\end{aligned}
$$

and by induction we have

$$
= \mathrm{E}\left(\frac{\tilde{V}(\tilde{L}(T_i))}{\prod_{k=0}^{i}(1+\tilde{L}(T_k)(T_{k+1}-T_k))}|\mathcal{F}_{T_0}\right)\cdot\frac{FX(T_0)}{N(T_0)}.
$$

This shows that in our time-discrete model, under the assumption of independent increments for the processes x, y and z, the pricing in the foreign currency behaves as if one considers $\prod_{k=0}^{i}(1+\tilde{L}(T_k)(T_{k+1}-T_k))$ as numéraire, i.e. as if the foreign currency functionals would be calibrated to the *foreign* currencies spot measure.

Thus the consideration of Section 2.4 apply and the foreign currency interest rate Makrov functionals may be calibrated as presented there. In addition we may draw the following conclusions:

- For model consistency it is preferable to chose the spot measure for the domestic Markov-functional model.
- For efficiency, it is preferable to calibrate the single-currency model(s) in spot measure, because then, the functionals may be used without any change in the three-factor model with instantaneously uncorrelated driving processes.

It is now straight forward to calibrate FX Options, which is done exactly as presented in Section 3.4.

The case of non-zero correlation

We have restricted the presentation to the case of zero instantaneous correlation of the driving processes. It is straight forward to derive and implement the model in the case of non-zero correlations. While zero correlation simplified the calibration of the models (see (22), (27)), the expectation operator will not decouple in the case of non-zero correlation and the calculation cost are significantly increased. In addition one has to consider a change of numéraire drift for the foreign currency's driving process z. Considering only low correlations one may retain the ability of fast calibration by assuming zero correlation over the calibration time step $[T_{i-1}, T_i]$ while using the correlated transition probabilities for the expectation over the time interval $[0, T_{i-1}]$ (we refer here to the notation in Section 3.4).

5 Other hybrid Markov-functional models

In the same way as we introduced a two factor Markov-functional model to model the FX along a single-factor Markov-functional model (see Section 3) we may model an equity Markov-functional model.

Given the driving processes

$$\begin{aligned} \mathrm{d}x &= \sigma_x(t)\mathrm{d}W_1 & x(0) &= x_0 \\ \mathrm{d}y &= \mu(t,x,y)\mathrm{d}t + \sigma_y(t)\mathrm{d}W_2 & y(0) &= y_0 \end{aligned} \tag{31}$$

let $L(T_i)$ denote the LIBOR rate seen upon its fixing date T_i given as a functional of a (calibrated) one dimensional Markov-functional model, i.e. depending on $x(T_i)$ only and let $N(T_i)$ denote the numéraire of this Markov-functional model.

Let S denote some traded asset (equity) for which we assume that

$$\boxed{S(T_k) \text{ is a function of } y(T_k) \text{ only.}} \tag{32}$$

Assuming that the discrete driving process have independent increments over a time step $[T_i, T_{i+1}]$ (see Section 3) we get (corresponding to (22))

$$\begin{aligned} S(T_i, y(T_i)) = \ & P(T_{i+1}; T_i, x(T_i)) \\ & \cdot \mathrm{E}^{\Delta y(T_i)}\left(S(T_{i+1}, y(T_{i+1})) | (T_i, x(T_i), y(T_i))\right). \end{aligned} \tag{33}$$

from which we may solve for the drift. So all aspects are as discussed in Section 3 (setting $FX = S$ and $\tilde{P} \equiv 1$).

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A Auxiliary calculations

For convenience we provide here some of the (well known) results used during our discussion.

Lemma 1 (Call-spread approximation of a digital caplet): Let $V^{T_i}_{\mathrm{caplet}}(K)$ denote the price of a caplet with strike K (and maturity T_i) given in some (arbitrary) arbitrage free pricing model. Assume that a differentiable market-price curve $K \mapsto V^{T_i}_{\mathrm{caplet}}(K)$ is given. Then the prices of digital caplets $V^{T_i}_{\mathrm{digital}}(K)$ under the same model are given by

$$V^{T_i}_{\mathrm{digital}}(K) = -\frac{1}{T_{i+1}-T_i}\frac{\partial}{\partial K}V^{T_i}_{\mathrm{caplet}}(K).$$

The approximation of the digital caplet prices by finite differences

$$\tilde{V}_{\mathrm{digital}}(K;0) = -\frac{V_{\mathrm{caplet}}(K+\epsilon;0) - V_{\mathrm{caplet}}(K-\epsilon;0)}{2\epsilon(T_{i+1}-T_i)}$$

is called *call-spread* approximation.

Lemma 2 (Expectation of $\exp(a\cdot X)$, X normally distributed): Let $\phi(\xi;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(\xi-\mu)^2}{2\sigma^2}}$ denote density of the normal distribution and $\Phi(x) := \int_{-\infty}^{x}\phi(\xi;0,1)\mathrm{d}\xi$ the distribution function of the standard normal distribution. Then

$$\int_{h_1}^{h_2} e^{a\cdot x}\cdot\phi(x;y,\sigma)\,\mathrm{d}x \;=\; \left[\Phi(\frac{h_2-(y+a\sigma^2)}{\sigma}) - \Phi(\frac{h_1-(y+a\sigma^2)}{\sigma})\right]e^{ay+\frac{a^2\sigma^2}{2}},$$

and thus we have for the expectation $\mathrm{E}(\exp(a\cdot X))$ for a random variable X with density $\phi(\cdot;y,\sigma)$:

$$\int_{-\infty}^{\infty} e^{a\cdot x}\cdot\phi(x;y,\sigma)\,\mathrm{d}x \;=\; e^{ay+\frac{a^2\sigma^2}{2}}.$$

Lemma 3 (Price of a digital caplet in arrears, given implied Black volatility): Let $\sigma(T_i,K)$ denote the implied Black volatility of a strike K caplet with fixing date T_i and payment date T_{i+1}, i.e. the price of such a caplet is given by

$$V^{\mathrm{Caplet}}_{T_i,K}(0) = P(T_{i+1})\cdot(T_{i+1}-T_i)\cdot\big(L_i(0)\Phi(d_+) - K\Phi(d_-)\big), \tag{34}$$

where $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(-\frac{y^2}{2})\mathrm{d}y$ and $d_\pm = \frac{\ln(\frac{L_i(0)}{K})\pm\frac{1}{2}\sigma^2(T_i,K)T_n}{\sigma(T_i,K)\sqrt{T_i}}$.

Given a digital caplet in arrears[20] paying in T_i

$$V^{\mathrm{Dig.Arrears}}_{T_i,K}(T_i) = \begin{cases} 1 & \text{if } L_i(T_i) > K \\ 0 & \text{else.}\end{cases}$$

the value is given by

$$\begin{aligned} V^{\mathrm{Dig.Arrears}}_{T_i,K}(0) =& P(T_{i+1})\cdot(T_{i+1}-T_i)\cdot L_i(0)\Phi(d_+) + P(T_{i+1})\cdot\Phi(d_-) \\ &+ (1+(T_{i+1}-T_i)K)P(T_{i+1})L_i(0)(T_{i+1}-T_i)\sqrt{T_i}\Phi'(d_+)\frac{\partial\sigma(T_i,K)}{\partial K} \end{aligned}$$

[20] The word *arrears* here relates to the fixing, which is behind the period if compared to a digital with fixing date T_{i-1} and payment date T_i.

Proof: The payout of the digital in arrears is equivalent in value to a payout of

$$V^{\text{Dig.Arrears}}_{T_i,K}(T_{i+1}) = \begin{cases} 1 + L_i(T_i) \cdot (T_{i+1} - T_i) & \text{if } L_i(T_i) > K \\ 0 & \text{else} \end{cases}$$

made in T_{i+1}, which is equivalent to the payout of $(1 + (T_{i+1} - T_i)K)$ times a digital caplet and 1 caplet (both with strike K). Since the value of a digital caplet is given by[21]

$$V^{\text{Digital}}_{T_i,K}(0) = P(T_{i+1})\big(\Phi(d_-) + L_i(0)(T_{i+1} - T_i)\sqrt{T_i}\Phi'(d_+)\frac{\partial\sigma(T_i,K)}{\partial K}\big) \qquad (35)$$

we have

$$\begin{aligned} &V^{\text{Dig.Arrears}}_{T_i,K}(0) \\ &= (1 + (T_{i+1} - T_i)K)P(T_{i+1})\big(\Phi(d_-) + L_i(0)(T_{i+1} - T_i)\sqrt{T_i}\Phi'(d_+)\frac{\partial\sigma(T_i,K)}{\partial K}\big) \\ &\quad + P(T_{i+1}) \cdot (T_{i+1} - T_i) \cdot \big(L_i(0)\Phi(d_+) - K\Phi(d_-)\big) \\ &= P(T_{i+1}) \cdot (T_{i+1} - T_i) \cdot L_i(0)\Phi(d_+) + P(T_{i+1}) \cdot \Phi(d_-) \\ &\quad + (1 + (T_{i+1} - T_i)K)P(T_{i+1})L_i(0)(T_{i+1} - T_i)\sqrt{T_i}\Phi'(d_+)\frac{\partial\sigma(T_i,K)}{\partial K}. \end{aligned}$$

[21] Equation (35) follows from differentiating (34) with respect to K. The first term relates to the K derivative, while the second term relates to the Vega followed by the $\frac{\partial\sigma(T_i,K)}{\partial K}$.

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