

OpenGamma Quantitative Research

## **Mixed Bivariate Log-Normal Model for Forex Cross**

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### **Abstract**

Options on certain currency crosses  $CCY1/CCY2$  (e.g. between two non-OECD countries) may be illiquid and therefore not have reliable market prices (if any exist at all). For a third currency  $CCY3$  (which is usually a major currency, e.g. USD, EUR, GBP), where options on the crosses  $CCY1/CCY3$  and  $CCY2/CCY3$  are liquid, we model the joint distribution of these forex rates at a particular time horizon as a mixture of bivariate log-normal distributions: From this we can infer prices on options (the volatility smile) on the cross  $CCY1/CCY2$ .

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# 1 Introduction

Implied volatilities of vanilla options in forex markets exhibit volatility smiles. Since division of two forex rates, CCY1/CCY3 and CCY2/CCY3, is again a forex rate, CCY1/CCY2, the option price of one currency pair is implicitly dependent on the other two pairs. Thus, in order to capture the forex market structure, we need to construct a multi-dimensional model that provides volatility smiles consistent with the vanilla options across the currency pairs. This problem can be addressed by an extension of a stochastic volatility model, e.g., [KG05], and alternative approaches include multi-dimensional partial differential equations [TL09], an extension of the Vanna-Volga method [Cas09] and references therein. The multi-dimensional models allow one to link the volatility smiles of two currency pairs to the third one, which is particularly useful when one currency pair is less liquid than the others.

In this note we consider a mixed bivariate log-normal model, a multi-dimensional extension of a mixed log-normal model presented in [Whi12], and discuss calibration to the currency crosses. Modelling the terminal distribution of an asset by a mixture of log-normal and a joint distribution by mixture of bivariate log-normal distributions produces a single, arbitrage-free, volatility smile.

## 2 Mixed Bivariate Log-Normal Model

While the main application of the mixed bivariate log-normal model is forex options, we shall start discussion with generic assets  $S_T^{(I)}$  ( $I = X, Y$ ) where in the forex case  $S_T^{(X)}$  and  $S_T^{(Y)}$  are the rates CCY1/CCY3 and CCY2/CCY3 at time  $T$ , respectively. Let the terminal asset prices given by

$$S_T^{(X)} = F_T^{(X)} e^{X - \omega^{(X)}}, \quad S_T^{(Y)} = F_T^{(Y)} e^{Y - \omega^{(Y)}}, \quad (1)$$

where  $F_T^{(I)}$  are the forwards and  $X, Y$  are weighted sum of bivariate normal random variables with density,

$$f_{X,Y}(x, y) = \sum_{i=1}^N w_i \phi_{X,Y} \left[ x, y; \left( \mu_i^{(X)} - \frac{\sigma_i^{(X)^2}}{2} \right) T, \sigma_i^{(X)^2} T, \left( \mu_i^{(Y)} - \frac{\sigma_i^{(Y)^2}}{2} \right) T, \sigma_i^{(Y)^2} T; \rho_i \right]. \quad (2)$$

Here the function  $\phi_{X,Y} [x, y; m_x, v_x^2, m_y, v_y^2; \rho]$  is the density of bivariate normal distribution,

$$\begin{aligned} \phi_{X,Y} [x, y; m_x, v_x^2, m_y, v_y^2; \rho] &= \frac{1}{2\pi v_x v_y \sqrt{1-\rho^2}} \exp \left( -\frac{q}{2(1-\rho^2)} \right), \\ q &= \left( \frac{x-m_x}{v_x} \right)^2 + \left( \frac{y-m_y}{v_y} \right)^2 - 2\rho \left( \frac{x-m_x}{v_x} \right) \left( \frac{y-m_y}{v_y} \right). \end{aligned} \quad (3)$$

The normalisation of the density requires

$$\sum_{i=1}^N w_i = 1, \quad w_i > 0, \quad \forall i. \quad (4)$$

While  $\omega^{(I)}$  in Eq. (1) ensure that the forward condition  $\mathbb{E} [S_T^{(I)}] = F_T^{(I)}$  is satisfied in the terminal measure, it is allowed to set  $\omega^{(I)} = 0$  keeping the forward condition held if we impose

$$\sum_{i=1}^N w_i e^{\mu_i^{(I)} T - \omega^{(I)}} = 1, \quad I = X, Y. \quad (5)$$

The marginal density for  $X$  is obtained by integrating Eq. (2) over  $y$ ,

$$f_X(x) = \sum_{i=1}^N w_i \phi_X \left[ x; \left( \mu_i^{(X)} - \frac{\sigma_i^{(X)2}}{2} \right) T, \sigma_i^{(X)2} T \right], \quad (6)$$

where  $\phi_X [x; m, v^2]$  is a normal distribution with mean  $m$  and variance  $v^2$ . Integrating Eq. (2) over  $x$  yields the marginal density for  $y$ , which can be obtained by replacement of indices,  $X \rightarrow Y$ , in the expression (6).

The price of European options for  $S_T^{(I)}$  ( $I = X, Y$ ) is expressed as a weighted sum of the Black prices,

$$V^{(I)}(T, k) = P(0, T) \sum_{i=1}^N w_i B(F_i^{(I)*}, k, T, \sigma_i^{(I)}), \quad (7)$$

where  $F_i^{(I)*} = F_i^{(I)} e^{\mu_i^{(I)} T}$  and

$$B(F, k, T, \sigma) = c \cdot (F \Phi(c \cdot d_1) - k \Phi(c \cdot d_2)), \quad (8)$$

$$d_1 = \frac{\ln \frac{F}{k} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}, \quad c = \begin{cases} +1 & \text{(Call)} \\ -1 & \text{(Put)} \end{cases}$$

Here  $\Phi$  is the cumulative distribution function of the standard normal distribution.

We shall introduce an asset  $S_T^{(Z)} = S_T^{(X)} / S_T^{(Y)}$ . Under the terminal measure for  $Z$ , we have  $\mathbb{E}[(S_T^{(Z)})] = F_T^{(Z)} \mathbb{E}[e^{X-Y}] e^{\omega^{(Z)}}$  with  $F_T^{(Z)} = F_T^{(X)} / F_T^{(Y)}$ . Hence  $S_T^{(Z)}$  can be written (under the new measure) as

$$S_T^{(Z)} = F_T^{(Z)} e^{Z - \omega^{(Z)}}, \quad (9)$$

$Z = X - Y$  is a mixed normal random variable with density (see Appendix A),

$$f_Z(z) = \sum_{i=1}^N w_i \phi_Z \left[ z; \left( \mu_i^{(Z)} - \frac{1}{2} \sigma_i^{(Z)2} \right) T, \sigma_i^{(Z)2} T \right], \quad (10)$$

$$\mu_i^{(Z)} = \mu_i^{(X)} - \mu_i^{(Y)} + \sigma_i^{(Y)2} - \rho_i \sigma_i^{(X)} \sigma_i^{(Y)},$$

$$\sigma_i^{(Z)} = \left( \sigma_i^{(X)2} + \sigma_i^{(Y)2} - 2\rho_i \sigma_i^{(X)} \sigma_i^{(Y)} \right)^{1/2}.$$

The extra shift  $\omega^{(Z)}$  ensures the forward condition of  $Z$ , i.e.,  $\mathbb{E}[S_T^{(Z)}] = F_T^{(Z)}$ . This implies that  $\omega^{(Z)}$  should be fixed by the following equation,

$$\sum_{i=1}^N w_i e^{\mu_i^{(Z)} T - \omega^{(Z)}} = 1. \quad (11)$$

We could also consider  $S_T^{(Z)} = S_T^{(X)} * S_T^{(Y)}$  depending on the identification between  $S_T^{(I)}$  and currency pairs. In this case we have  $F_T^{(Z)} = F_T^{(X)} * F_T^{(Y)}$  and  $Z = X + Y$ . The corresponding density is shown in Appendix A.

While the mixed bivariate log-normal model contains  $6N$  parameters,  $w_i, \mu_i^{(X)}, \sigma_i^{(X)}, \mu_i^{(Y)}, \sigma_i^{(Y)}, \rho_i$ ,  $6N - 3$  of them are independent because of the 3 constraints in Eqs. (4), (5). Among these parameters, the correlations  $\rho_i$  are independent of implied volatility smiles of  $S_T^{(X)}$  and  $S_T^{(Y)}$ . Hence calibration of this model is done by finding  $5N - 3$  parameters such that the squared difference between market and model volatilities are minimised. The correlations  $\rho_i$  are fixed by the market data of  $S_T^{(Z)}$  via the relations (10) together with the constraint (11).

	$i = 1: w_1 = 0.7$		$i = 2: w_2 = 0.3$	
	$\mu_1$	$\sigma_1$	$\mu_2$	$\sigma_2$
X: $F_T^{(X)} = 1$	0	0.25	0	0.7
Y: $F_T^{(Y)} = 1$	-0.2	0.3	0.1531931731	0.5

Table 1: Parameters for calibration test

In the absence of any option prices on  $S_T^{(Z)}$ ,  $\rho_i$  can be treated as extraneous parameters. If a single option is available (usually this will be the At-The-Money (ATM), or Delta Neutral Straddle (DNS), for forex), one can set  $\rho_i$  such that the single option price is recovered. Otherwise one could use historical time series to estimate the correlations.

### 3 Calibration Test

To examine the calibration stability, we shall generate fake “market” data from a known set of parameters, then treat this data set as a real one, and check the the calibration fits (in the sense of zero  $\chi^2$ ) and that the original parameters are recovered.

Under the parameter choice for the  $N = 2$  model with  $T = 1$  given in Table 1, the implied volatility smiles are depicted for  $S_T^{(X)}$  (blue line) and  $S_T^{(Y)}$  (red line) in Figure 1. Given 7+7 data points on these implied volatility smiles corresponding to strikes  $k = 0.5, 0.7, 0.9, 1.0, 1.2, 1.5, 1.8$ , the least square fitting determines the parameters of the mixed bivariate log-normal model. Then the implied volatility smiles of  $S_T^{(Z)}$  are computed once correlations  $\rho_i$  are chosen. We show the comparison between the volatility smile of  $S_T^{(Z)}$  derived by this data fitting (green line) and data points ( $k = 0.5, 0.7, 0.9, 1.0, 1.2, 1.5, 1.8$ ) of the “market” volatility smile derived from the model parameters in Table 1 (triangles) for  $\rho_1 = \rho_2 = 0$  in Figure 2,  $\rho_1 = \rho_2 = 0.4$  in Figure 3 and  $\rho_1 = 0.2, \rho_2 = 0.9$  in Figure 4. We also test the case where the “market” data of  $S_T^{(X)}$  and  $S_T^{(Y)}$  contain normal random noise  $N(0, 10^{-3})$ . The implied volatility smile of  $S_T^{(Z)}$  in this case is depicted with a black dashed line in each figure.

### 4 Calibration to Market Volatility Smile

To test with real market data we shall pick up three currencies, EUR, USD and GBP, where options on all three crosses are liquid across various expiries. Under the identification,  $(S_T^{(X)}, S_T^{(Y)}, S_T^{(Z)}) = (\text{EUR/USD}, \text{GBP/USD}, \text{EUR/GBP})$ , the parameters of the mixed bivariate log-normal model are fixed such that implied volatilities of  $S_T^{(X)}$  and  $S_T^{(Y)}$  fit market volatility smiles of EUR/USD and GBP/USD, respectively. Then implied volatilities of EUR/GBP are derived once the correlations  $\rho_i$  are given.

Figures 5 - 13 show the market volatility smiles (triangles) and the implied volatilities of EUR/USD (blue line), GBP/USD (red line) and EUR/GBP (green line) by the calibration of the  $N = 2$  mixed bivariate log-normal model for various expiries,  $T = 1$  week in Figures 5, 6, 7,  $T = 1$  month in Figures 8, 9, 10 and  $T = 6$  months in Figures 11, 12, 13. The optimal correlations  $\rho_i$  depend on the expiries.

## 5 Conclusion

This note provides a brief description on a mixed bivariate log-normal model and its application to foreign exchange markets. In the absence of liquid prices for options on a particular currency cross, this model offers a fast, arbitrage free way of inferring prices from other quoted options in the market.

We have discussed the only  $N = 2$  case in this note. One may consider that a mixed bivariate log-normal model with  $N > 2$  would allow for a better fit. However, it is observed that calibration of the  $N > 2$  models always ends up with weights only two of which are not sufficiently small. This is a strong indication that the  $N = 2$  model is complicated enough for our present purpose.

## A Probability Density Function of $S_T^{(Z)}$

In this appendix we show the derivation of Eq. (10). We shall pick up one of the bivariate normal distributions in Eq. (2),  $\phi_Z \left[ z; \left( \mu_i^{(Z)} - \frac{1}{2} \sigma_i^{(Z)2} \right) T, \sigma_i^{(Z)2} T \right]$ , and employ the following simple notation,

$$\phi_{X,Y} [x, y; m_x, v_x^2, m_y, v_y^2; \rho] = \frac{1}{2\pi v_x v_y \sqrt{1-\rho^2}} \exp \left( -\frac{q}{2(1-\rho^2)} \right), \quad (12)$$

$$q = \left( \frac{x-m_x}{v_x} \right)^2 + \left( \frac{y-m_y}{v_y} \right)^2 - 2\rho \left( \frac{x-m_x}{v_x} \right) \left( \frac{y-m_y}{v_y} \right),$$

where the model parameters  $\mu_i^{(I)}$ ,  $\sigma_i^{(I)}$ ,  $\rho_i$  are encoded as

$$\begin{aligned} m_x &= \left( \mu_i^{(X)} - \frac{\sigma_i^{(X)2}}{2} \right) T, & v_x^2 &= \sigma_i^{(X)2} T, \\ m_y &= \left( \mu_i^{(Y)} - \frac{\sigma_i^{(Y)2}}{2} \right) T, & v_y^2 &= \sigma_i^{(Y)2} T. \end{aligned} \quad (13)$$

$$\rho = \rho_i.$$

Corresponding to the two cases,  $S_T^{(Z)} = S_T^{(X)} * S_T^{(Y)}$  and  $S_T^{(Z)} = S_T^{(X)} / S_T^{(Y)}$ , the densities for  $z = x \pm y$  are obtained by the integrals,

$$\begin{aligned} f(z) &= \int \phi_{Z \pm Y, Y} [z \pm y, y; m_x, v_x^2, m_y, v_y^2; \rho] dy \\ &= \phi_Z [z; m_x \pm m_y, v_x^2 + v_y^2 \pm 2\rho v_x v_y] \end{aligned} \quad (14)$$

If we introduce new parameters by  $m_z = m_x \pm m_y$  and  $v_z^2 = v_x^2 + v_y^2 \pm 2\rho v_x v_y$ , the last line is rewritten as

$$f(z) = \phi_Z \left[ z; \left( m_z + \frac{1}{2} v_z^2 \right) - \frac{1}{2} v_z^2, v_z^2 \right]. \quad (15)$$

Plugging Eqs. (13) into this yields

$$\begin{aligned} \left( m_z + \frac{1}{2} v_z^2 \right) &= \begin{cases} \mu_i^{(X)} + \mu_i^{(Y)} + \rho_i \sigma_i^{(X)} \sigma_i^{(Y)} & (z = x + y) \\ \mu_i^{(X)} - \mu_i^{(Y)} + \sigma_i^{(Y)2} - \rho_i \sigma_i^{(X)} \sigma_i^{(Y)} & (z = x - y) \end{cases} \\ v_z &= \left( \sigma_i^{(X)2} + \sigma_i^{(Y)2} \pm 2\rho_i \sigma_i^{(X)} \sigma_i^{(Y)} \right)^{1/2} \end{aligned} \quad (16)$$

Taking the weighted sum of the derived distribution functions, one ends up with the density for  $Z$  in Eq. (10).



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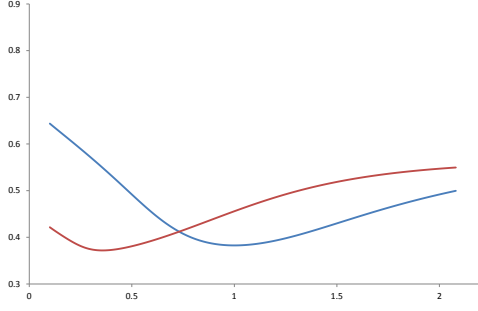


Figure 1: Volatility smiles of  $S_T^{(X)}$  (blue line) and  $S_T^{(Y)}$  (red line).

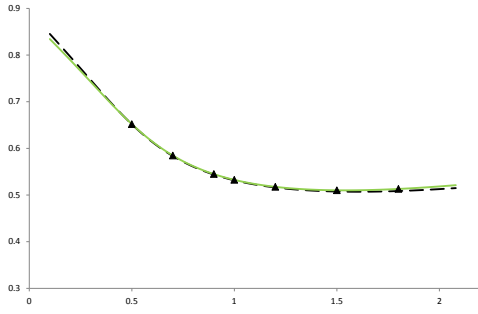


Figure 2: Volatility smiles of  $S_T^{(Z)}$  derived by “data fitting” without random noise (green line) and with random noise (black dashed line) and computed with the parameters given in Table 1 (triangles) for  $\rho_1 = \rho_2 = 0$ .

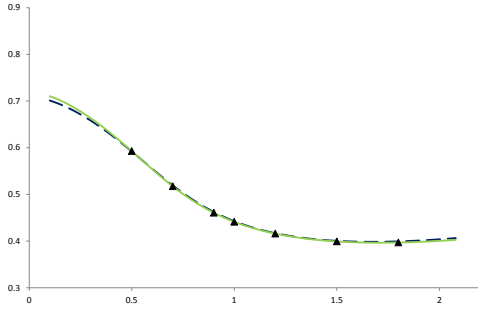


Figure 3: Volatility smiles of  $S_T^{(Z)}$  derived by “data fitting” without random noise (green line) and with random noise (black dashed line) and computed with the parameters given in Table 1 (triangles) for  $\rho_1 = \rho_2 = 0.4$ .

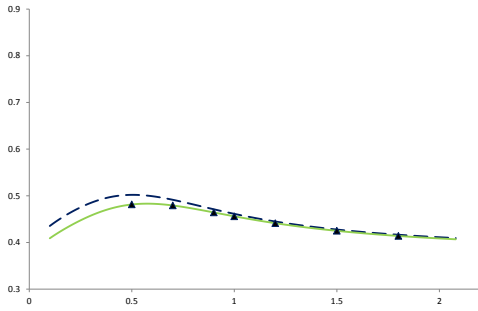


Figure 4: Volatility smiles of  $S_T^{(Z)}$  derived by “data fitting” without random noise (green line) and with random noise (black dashed line) and computed with the parameters given in Table 1 (triangles) for  $\rho_1 = 0.2, \rho_2 = 0.9$ .

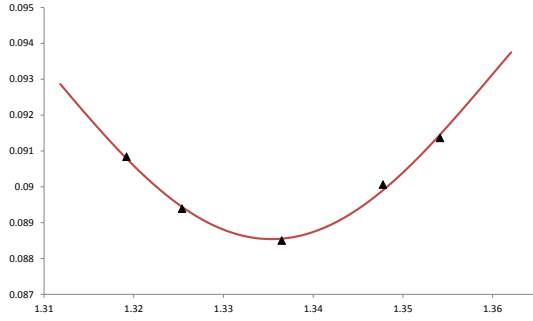


Figure 5: Market data of EUR/USD (triangles) and implied volatility smile by the calibration (red line) for  $T = 1$  week.

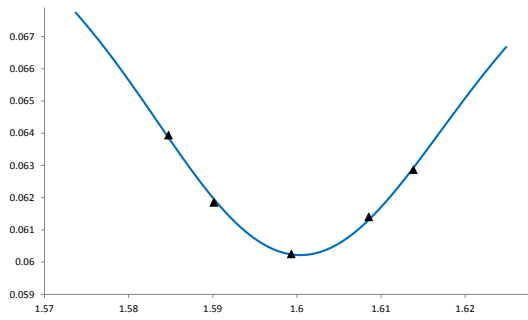


Figure 6: Market data of GBP/USD (triangles) and implied volatility smile by the calibration (blue line) for  $T = 1$  week.

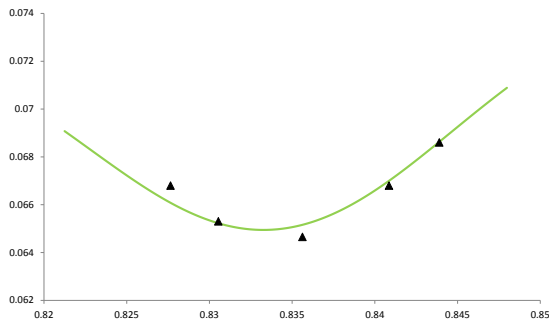


Figure 7: Market data of EUR/GBP (triangles) and implied volatility smile derived from market data of EUR/USD, GBP/USD (green line) for  $T = 1$  week. Here the optimal  $\rho_i$  are (0.709851512, 0.671188015).

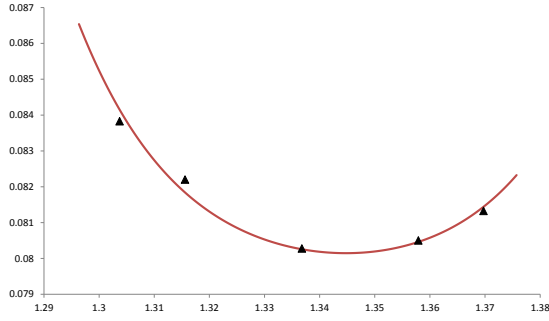


Figure 8: Market data of EUR/USD (triangles) and implied volatility smile by the calibration (red line) for  $T = 1$  month.

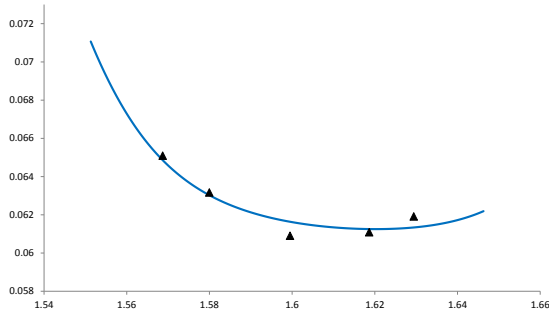


Figure 9: Market data of GBP/USD (triangles) and implied volatility smile by the calibration (blue line) for  $T = 1$  month.

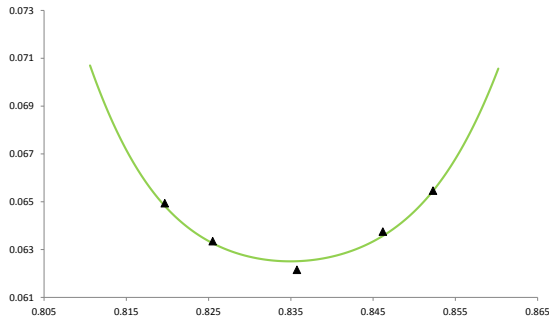


Figure 10: Market data of EUR/GBP (triangles) and implied volatility smile derived from market data of EUR/USD, GBP/USD (green line) for  $T = 1$  month. Here the optimal  $\rho_i$  are (0.650905705, 0.489285135).

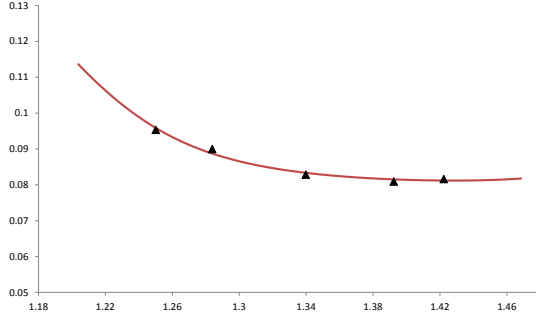


Figure 11: Market data of EUR/USD (triangles) and implied volatility smile by the calibration (red line) for  $T = 6$  months.

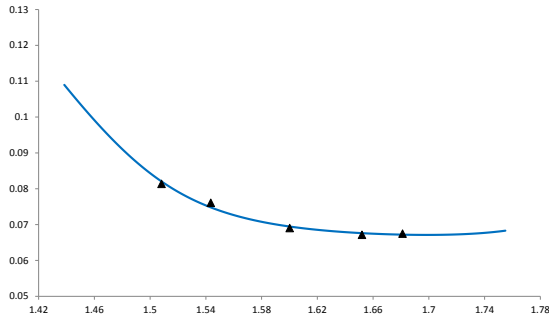


Figure 12: Market data of GBP/USD (triangles) and implied volatility smile by the calibration (blue line) for  $T = 6$  months.

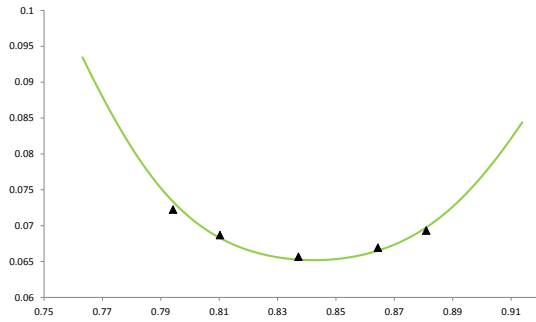


Figure 13: Market data of EUR/GBP (triangles) and implied volatility smile derived from market data of EUR/USD, GBP/USD (green line) for  $T = 6$  months. Here the optimal  $\rho_i$  are (0.677494528, 0.689476212).



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