COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

BORIS HASSELBLATT

Tufts UNIVERSITY

CONTENTS

1. Introduction 1
2. Why complex numbers were first introduced (digression) 2
3. Complex numbers, Euler's formula 3
4. Homogeneous differential equations 8
5. Undetermined coefficients 11
6. Laplace transforms 13
7. Partial fractions 15
8. Complex eigenvalues (covered in [1, Section 3.8]) 21
References 21

1. INTRODUCTION

These notes introduce complex numbers and their use in solving differential equations. Using them, trigonometric functions can often be omitted from the *methods* even when they arise in a given problem or its solution. Still, the solution of a differential equation is always presented in a form in which it is apparent that it is real.

One one hand this approach is illustrated with the method of undetermined coefficients, where this approach offers a parallel development to the one in the text [1]. On the other hand, this is done with the Laplace transform, where the use of complex numbers replaces much otherwise needed machinery *and* at the same time covers situations that the methods in the text do not. Here a big step is to produce partial-fractions decompositions using complex numbers.

The reason the method of undetermined coefficients is revisited here in the complex context is that fluency with this method is very helpful in using the Laplace transform method reliably.

The exercises are not for credit but will help you read this text actively.

2. Why complex numbers were first introduced (digression)

Complex numbers have turned out to be tremendously useful in ways that could not possibly have been anticipated when they were introduced. They first arose in the context of finding solutions of cubic equations.

To see how finding roots of cubic polynomials leads to complex numbers, let us recapitulate the quadratic formula: The roots of a quadratic polynomial

$$x^2 + px + q$$

are given by

$$-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

This is a special case of the usual form of the quadratic formula, and this simplification foreshadows a simplification that makes a formula for cubics palatable[1].

To solve a cubic, one first simplifies it to a depressed (no x^2-term) monic (leading coefficient 1) cubic. In 16th century Italy, several people independently found that one root of a cubic

$$x^3 + px + q$$

is given by

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

1. **Example.** For $x^3 - 15x - 4 = 0$ this gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

This must have looked impossible at the time, and yet, one can also find a real solution by inspection: $x^3 - 15x - 4 = x(x^2 - 15) - 4 = 0$ for $x = 4$. And this is indeed what the formula gives, because by multiplying out, we get

$$(2 + \sqrt{-1})^3 = 2 + \underbrace{11\sqrt{-1}}_{=\sqrt{-121}} \quad \text{and} \quad (2 - \sqrt{-1})^3 = 2 - 11\sqrt{-1},$$

so the formula becomes

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4.$$

This example shows that if one is willing to allow the symbol $\sqrt{-1}$ and applies the usual rules of algebra, meaningful real solutions can be obtained that might not be otherwise obtainable.

[1] See `http://en.wikipedia.org/wiki/Cubic_function`.

3. Complex numbers, Euler's formula

2. **Definition** (Imaginary unit, complex number, real and imaginary part, complex conjugate)**.** We introduce the symbol i by the property

$$\boxed{i^2 = -1}$$

A *complex number* is an expression that can be written in the form $a + ib$ with real numbers a and b. Often z is used as the generic letter for complex numbers, just like x often stands for a generic real number.

If a and b are *real* numbers, then a is called the *real part* of $a + ib$, and b is called the *imaginary part.* (Note that both are real numbers!)

The expression $a - ib$ is called the *complex conjugate* of $a + ib$. It is sometimes denoted by a bar:

$$\overline{a + ib} = a - ib.$$

3. **Exercise.** Verify that $\overline{\overline{z}} = z$.

4. **Exercise.** Verify that any real number x satisfies $\overline{x} = x$.

5. **Exercise.** Verify that a complex number z satisfying $\overline{z} = z$ is a real number.

3.1. Adding complex numbers.

Complex numbers are added using the usual rules of algebra except that one usually brings the result into the form $a + ib$. That is,

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

Adding a complex number and its complex conjugate always gives a real number:

$$a + ib + a - ib = 2a.$$

This is twice the real part. So, if we are given a complex number $z = a + ib$ in any form, we can express the real part as

$$\Re(z) = \text{real part of } z = \frac{z + \overline{z}}{2}.$$

The imaginary part can be expressed as (check!)

$$\Im(z) = \text{imaginary part of } z = \frac{z - \overline{z}}{2i}.$$

3.2. Multiplying complex numbers.

To multiply two complex numbers just use $i^2 = -1$ and group terms:

$$(a + ib)(c + id) = ac + aid + ibc + \underbrace{ibid}_{=-bd} = ac - bd + i(ad + bc).$$

Multiplying a complex number and its complex conjugate always gives a real number:

$$(a+ib)(a-ib) = a^2 + b^2.$$

We call $\sqrt{a^2+b^2}$ the *absolute value* or *modulus* of $a+ib$:

$$\boxed{|a+ib| = \sqrt{a^2+b^2}}$$

6. **Exercise.** Verify that $|z| = \sqrt{z\bar{z}}$.

7. **Exercise.** Verify that $\overline{z_1}\,\overline{z_2} = \overline{z_1 z_2}$.

8. **Exercise.** Verify that $\overline{\overline{z_1} z_2} = z_1 \overline{z_2}$.

3.3. **Dividing complex numbers.** To divide two complex numbers and write the result as real part plus $i\times$imaginary part, multiply top and bottom of this fraction by the complex conjugate of the denominator:

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}.$$

3.4. **Factoring polynomials.** Factoring polynomials is no harder (or easier) when complex numbers are allowed *but* in this case all factors are linear. The reason is that factors $x-\alpha$ are now legal even when α is complex.

9. **Example.** The polynomial s^2+1 is irreducible over the real numbers, but it factors over the complex numbers:

$$s^2+1 = (s-i)(s+i).$$

10. **Example.** The polynomial s^2+4s+5 is irreducible over the real numbers, but by completing the square we can write

$$s^2+4s+5 = (s+2)^2+1 = [(s+2)+i][(s+2)-i] = [s+(2+i)][s+(2-i)].$$

So, the roots of this polynomial are $-2\pm i$.

3.5. **Euler's formula.** Complex numbers are useful in our context because they give Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula is easy to remember. In case you are not sure whether to attach the i to the cos or to the sin, just plug in $\theta = 0$.

Power series let us see why this is so. We have[2]

$$
\begin{aligned}
e^z = 1 + z + z^2/2 + \dots &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\cos z = 1 - z^2/2 + \dots &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\sin z = z - z^3/6 + \dots &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}
\end{aligned}
$$

With $z = i\theta$ this gives

$$
\begin{aligned}
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} &= \sum_{n=0}^{\infty} \frac{\overbrace{i^{2n}}^{=(i^2)^n=(-1)^n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\overbrace{i^{2n+1}}^{=i(i^2)^n=i(-1)^n}\theta^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} \\
&= \cos\theta + i\sin\theta.
\end{aligned}
$$

Euler's formula can be read as saying that $\cos\theta = \Re(e^{i\theta})$ and $\sin\theta = \Im(e^{i\theta})$. Writing out the definitions of $\Re$ and $\Im$ gives "backward" Euler formulas that express cos and sin in terms of complex exponentials.

In summary, we will use the "forward" and "backward" Euler formulas

$$
\boxed{e^{i\theta} = \cos\theta + i\sin\theta}
$$

$$
\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}} \quad \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}} \quad \boxed{\sin\theta = \frac{i}{2}(e^{-i\theta} - e^{i\theta})} \tag{1}
$$

11. **Exercise.** Show that $e^{\overline{z}} = \overline{e^z}$.

12. **Exercise.** Verify that if λ is complex and $ae^{\lambda t} + be^{\overline{\lambda} t}$ is real for all t, then $b = \overline{a}$ via the following steps.

(1) Rewrite the given information using Exercises 4 (page 3) and 11.
(2) The deriviative of $ae^{\lambda t} + be^{\overline{\lambda} t}$ is also real—compute it and proceed as in the previous part.
(3) Take $t = 0$ in the expressions obtained in both previous parts.

3.6. **The complex plane.** While real numbers can be geometrically represented by a number line, complex numbers can be represented by a plane, called the complex plane. Taking the real number line as a horizontal axis, one can introduce a vertical axis with i as the unit on it. Then

[2]In case you are wondering what the left-hand side of each of the equations *means*, you can take the equation to be a *definition*.

$a+ib$ can be understood as representing a point in this plane in terms of cartesian coordinates.

The power of this representation is related to the fact that Euler's formula represents the same plane but using polar coordinates.[3] To see this, look at

$$z = e^{a+ib} = e^a e^{ib} = e^a \cos b + i e^a \sin b.$$

Here e^a plays the role of r in polar coordinates, and indeed,

$$|e^{a+ib}| = |z| = \sqrt{z\bar{z}} = \sqrt{e^{a+ib}e^{a-ib}} = \sqrt{e^{2a}} = e^a.$$

So in the complex plane, e^{a+ib} has distance e^a from the origin and lies in a direction relative to the horizontal axis given by the angle b, which is called the *argument* of e^{a+ib}.

One consequence is that multiplication of complex numbers can be interpreted geometrically:

$$e^a e^{ib} \cdot e^c e^{id} = e^{a+ib} \cdot e^{c+id} = e^a e^c e^{i(b+d)}.$$

The absolute value of this product is $e^a e^c$, which is the product of the absolute values of the 2 numbers. The argument is the sum of the arguments of the 2 numbers. Geometrically: since the argument is an angle, multiplying a complex number by another rotates and scales it.

3.7. **Exponential shift.** As in [1, p. 123], the *exponential shift* works for complex exponentials (you can check that the calculation on the bottom half of that page does not use that λ is real):

$$\boxed{P(D)[e^{\lambda t} y] = e^{\lambda t} P(D+\lambda) y.}$$

13. **Example.**

$$(D+3)^5 e^{-t} = e^{-3t} D^5 e^{3t} e^{-t} = e^{-3t} D^5 e^{2t} = e^{-3t} 2^5 e^{2t} = 2^5 e^{-t}.$$

3.8. **Hyperbolic functions** (digression). While we think of t as a real variable in Euler's formula, one gets an interesting result when one plugs imaginary numbers into cos and sin:

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} =: \cosh x.$$

Here, $\cosh x$ is the hyperbolic cosine function defined by this last equality. Likewise,

$$\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\frac{e^{x} - e^{-x}}{2} = i \sinh x.$$

[3]Recall that polar coordinates in the plane are of the form $(r\cos\theta, r\sin\theta)$, where r is the distance from the origin and θ the angle with the horizontal axis.

In summary,

$$\boxed{\cosh x = \cos ix}, \quad \boxed{\sinh x = \frac{1}{i}\sin ix}, \quad \text{and} \quad \boxed{e^x = \cosh x + \sinh x}.$$

3.9. **The trigonometric-identity machine** (useful digression). Euler's formula produces many trigonometric identities by the rules of exponents.

14. **Example.** Find formulas for $\cos(a+b)$ and for $\sin(a+b)$.

We start with the law of exponents $e^{i(a+b)} = e^{ia}e^{ib}$ and rewrite every exponential using Euler's formula. On the left we have

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b).$$

On the right we have

$$\begin{aligned} e^{ia}e^{ib} &= \big(\cos a + i\sin a\big)\big(\cos b + i\sin b\big) \\ &= \big(\cos a\cos b - \sin a\sin b\big) + i\big(\sin a\cos b + \cos a\sin b\big). \end{aligned}$$

Since these two things are the same, we get

$$\cos(a+b)+i\sin(a+b) = \big(\cos a\cos b-\sin a\sin b\big)+i\big(\sin a\cos b+\cos a\sin b\big).$$

Matching real and imaginary parts, we get

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \text{ and } \sin(a+b) = \sin a\cos b + \cos a\sin b.$$

15. **Example.** To get double-angle formulas, take $a = b$:

$$\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a \quad \text{and} \quad \sin(2a) = 2\sin a\cos a.$$

16. **Example.** To obtain triple-angle formulas use the law $e^{3ia} = \big(e^{ia}\big)^3$ and rewrite every term using Euler's formula, multiply out and sort terms. (Or use Example 14 with $b = 2a$ and double-angle formulas along the way.)

17. **Example.** Should you ever need a formula for $\cos(a+b+c)$, you can start with the law of exponents $e^{i(a+b+c)} = e^{ia}e^{ib}e^{ic}$ and rewrite every exponential using Euler's formula, then multiply out and sort terms.

18. **Example** (Compare with [1, p. 158])**.** Second-order differential equations often have solutions that can be written as

$$x(t) = \Re(ce^{\lambda t}) = \frac{c}{2}e^{\lambda t} + \frac{\bar{c}}{2}e^{\bar{\lambda} t}$$

(see Exercise 12). If we are given initial conditions $x(0) = x_0$ and $x'(0) = v_0$ and write $\lambda = -\sigma + i\omega$, then we find that $x_0 = x(0) = \Re(ce^{\lambda 0}) = \Re c$ and

$$v_0 = x'(0) = \Re(c\lambda e^{\lambda 0}) = \Re(c\lambda) = \Re((x_0 + i\Im c)\cdot(-\sigma + i\omega)) = -\sigma x_0 - \omega\Im c,$$

so $c = \Re c + i\Im c = x_0 - i\frac{v_0+\sigma x_0}{\omega}$, and the solution $x(t) = \Re(ce^{\lambda t})$ can be written as

$$e^{-\sigma t}\Re\Big((x_0 - i\frac{v_0+\sigma x_0}{\omega})(\cos\omega t + i\sin\omega t)\Big) = e^{-\sigma t}[x_0\cos\omega t + \frac{v_0+\sigma x_0}{\omega}\sin\omega t].$$

This can be rewritten in a way that involves only a sine or a cosine, but not both. To see how, write $c = e^{a-i\alpha}$ to get

$$\frac{c}{2}e^{\lambda t} + \frac{\bar{c}}{2}e^{\bar{\lambda}t} = \frac{1}{2}e^{a-i\alpha}e^{(-\sigma+i\omega)t} + e^{a+i\alpha}e^{(-\sigma-i\omega)t} = e^a e^{-\sigma t}\underbrace{\frac{e^{i(\omega t-\alpha)}+e^{-i(\omega t-\alpha)}}{2}}_{=\cos(\omega t-\alpha)}.$$

If we set $\quad A = e^a = |c| = \sqrt{x_0^2 + \left(\frac{v_0+\sigma x_0}{\omega}\right)^2}, \quad$ then this means that

$$x(t) = \Re(ce^{\lambda t}) = Ae^{-\sigma t}\cos(\omega t - \alpha),$$

where

$$\cos\alpha + i\sin\alpha = e^{i\alpha} = \overline{e^{-i\alpha}} = \frac{1}{e^a}\overline{e^{a-i\alpha}} = \frac{1}{A}\bar{c} = \frac{x_0}{A} + i\frac{v_0+\sigma x_0}{\omega A},$$

that is,

$$\cos\alpha = \frac{x_0}{A} \quad \text{and} \quad \sin\alpha = \frac{v_0+\sigma x_0}{\omega A}.$$

4. Homogeneous differential equations

If one uses complex numbers, then the method described in [1, Sections 2.5, 2.6] looks slightly different, and there are a few choices one can make.

The point of [1, p. 131] is that the method of [1, Section 2.5] works even for complex roots of the characteristic polynomial, but that some care should be taken to make sure one ultimately writes down real solutions only.

19. **Example** ([1, Example 2.6.1])**.** Solve $(D^2+4D+5)x = 0$.

The characteristic polynomial has roots $\lambda_\pm = -2 \pm i$ (see Example 10), which gives solutions $e^{\lambda_\pm t}$, or $e^{\lambda_+ t}$ and $e^{\lambda_- t}$, or

$$e^{(-2+i)t} \text{ and } e^{(-2-i)t}.$$

However, neither of these is a real solution[4]. There are different ways of obtaining real solutions. We can write the general solution as

$$ce^{(-2+i)t} + \bar{c}e^{(-2-i)t},$$

and for $c = \frac{1}{2}$ this becomes

$$\frac{1}{2}e^{(-2+i)t} + \frac{1}{2}e^{(-2-i)t} = \Re(e^{(-2+i)t}),$$

[4] i.e., a real-valued function that solves the differential equation.

while for $c = \frac{1}{2i}$ this becomes

$$\frac{1}{2i}e^{(-2+i)t} - \frac{1}{2i}e^{(-2-i)t} = \Im(e^{(-2+i)t}).$$

So, for the purpose of finding the general solution, we can choose, as in [1, Section 2.6] but with different notation, the solutions

$$\Re(e^{(-2+i)t}) = e^{-2t}\cos t \quad \text{and} \quad \Im(e^{(-2+i)t}) = e^{-2t}\sin t.$$

This gives the formulas at the end of [1, Section 2.6].

20. **Example** ([1, Exercise 2.6.17])**.** Solve the initial-value problem

$$(5D^2 + 2D + 1)x = 0; \quad x(0) = 0,\ x'(0) = 1.$$

The roots of the characteristic polynomial are $\lambda_\pm = -\frac{1}{5} \pm \frac{2}{5}i$, giving the "general solution" $c_+e^{\lambda_+ t} + c_-e^{\lambda_- t}$, or

$$c_+e^{(-\frac{1}{5}+\frac{2}{5}i)t} + c_-e^{(-\frac{1}{5}-\frac{2}{5}i)t}.$$

This is not quite "proper" since it involves complex functions, but unlike in the previous example, this is also not the desired final answer anyway since we wish to solve an initial-value problem. While we could proceed as before and write out the general solution in real terms and then determine the correct coefficients from the initial values, we can instead directly do so with the complex solution above.

The condition

$$\begin{aligned} x(0) &= 0, \\ x'(0) &= 1 \end{aligned}$$

applied to $x(t) = c_+e^{(-\frac{1}{5}+\frac{2}{5}i)t} + c_-e^{(-\frac{1}{5}-\frac{2}{5}i)t}$ gives

$$\begin{aligned} c_+ + c_- &= 0, \\ c_+\lambda_+ + c_-\lambda_- &= 1. \end{aligned}$$

Cramer's rule gives

$$c_+ = \frac{\det\begin{pmatrix} 0 & 1 \\ 1 & \lambda_- \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}} = -\frac{1}{\lambda_- - \lambda_+} = \frac{1}{2\cdot\frac{2}{5}i} = \frac{5}{2}\frac{1}{2i}.$$

Now, Exercise 12 (on page 5) or an analogous computation or the first equation tells us that $c_- = -c_+ = -\frac{5}{2}\frac{1}{2i}$. Therefore the solution of the initial-value problem is

$$x(t) = \underbrace{\frac{5}{2}\frac{1}{2i}}_{=c_+} e^{(-\frac{1}{5}+\frac{2}{5}i)t} \underbrace{-\frac{5}{2}\frac{1}{2i}}_{=c_-} e^{(-\frac{1}{5}-\frac{2}{5}i)t} = \frac{5}{2}e^{-t/5}\cdot \underbrace{\frac{e^{\frac{2}{5}it}-e^{-\frac{2}{5}it}}{2i}}_{=\sin\frac{2}{5}t} = \frac{5}{2}e^{-t/5}\sin\frac{2}{5}t.$$

Which version works better is a matter of taste; the one presented here leads directly to the coefficients from the complex form with a computation involving 2 equations in 2 unknowns, and it is reassuring (basic sanity check) that the end result is real.

21. **Example** ([1, Exercise 2.6.17] with real functions)**.** Solve the initial-value problem

$$(5D^2+2D+1)x = 0; \quad x(0) = 0,\ x'(0) = 1.$$

The roots of the characteristic polynomial are $\lambda_\pm = -\frac{1}{5} \pm \frac{2}{5}i$, giving by Exercise 12 the general solution $ce^{\lambda_+ t} + \bar{c}e^{\lambda_- t}$, or

$$ce^{(-\frac{1}{5}+\frac{2}{5}i)t} + \bar{c}e^{(-\frac{1}{5}-\frac{2}{5}i)t}.$$

Since this is a real function by Exercise 12, it is quite proper, even though the coefficients are complex. There is only one (complex) coefficient to determine.

The condition

$$x(0) = 0,$$
$$x'(0) = 1$$

applied to $x(t) = ce^{(-\frac{1}{5}+\frac{2}{5}i)t} + \bar{c}e^{(-\frac{1}{5}-\frac{2}{5}i)t}$ gives

$$c + \bar{c} = 0,$$
$$c\lambda_+ + \bar{c}\bar{\lambda}_+ = 1.$$

The first equation (which says that c is imaginary) gives $\bar{c} = -c$, so the second equation gives

$$1 = c\cdot(\lambda_+ - \bar{\lambda}_+) = c\cdot 2i\Im\lambda_+ = c\cdot 2i\frac{2}{5},$$

so

$$c = \frac{5}{2}\frac{1}{2i}.$$

As above, one then finds that the solution of the initial-value problem is

$$x(t) = \frac{5}{2}e^{-t/5}\sin\frac{2}{5}t.$$

5. UNDETERMINED COEFFICIENTS

The method of undetermined coefficients works much like in [1, Section 2.7], but we use complex exponentials and complex factoring of polynomials where appropriate.

22. **Example** ([1, Example 2.7.5]). Solve

$$(D^2-4)x = 1+65e^t\cos 2t. \tag{N}$$

Since $D^2-4=(D-2)(D+2)$, the general solution of the associated homogeneous equation

$$(D^2-4)x = 0 \tag{H}$$

is

$$x = H(t) = c_1e^{2t}+c_2e^{-2t}.$$

The forcing term of (N) can be rewritten as $1+\frac{65}{2}e^{(1+2i)t}+\frac{65}{2}e^{(1-2i)t}$ and is therefore annihilated by

$$A(D) = D(D-(1+2i))(D-(1-2i)).$$

Therefore a particular solution $x=p(t)$ of (N) will also satisfy

$$\underbrace{D(D-(1+2i))(D-(1-2i))}_{A(D)}\underbrace{(D-2)(D+2)}_{D^2-4}x = 0. \tag{H*}$$

This has characteristic polynomial $r(r-(1+2i))(r-(1-2i))(r-2)(r+2)$, so $p(t)$ has to be of the form

$$p(t) = k_1+k_2e^{(1+2i)t}+k_3e^{(1-2i)t}\underbrace{+k_4e^{2t}+k_5e^{-2t}}_{\text{omit for simplified guess}}.$$

23. **Remark.** When we use this method later to predict what terms to expect in applying the Laplace transform method, then we can stop at this point.

We get a simplified guess by omitting the terms that solve (H):

$$p(t) = k_1+k_2e^{(1+2i)t}+k_3e^{(1-2i)t}.$$

To determine the (as yet undetermined) coefficients k_1, k_2 and k_3, insert this simplified guess into (N), written with the forcing term on the left:

$$\begin{aligned}
1+\frac{65}{2}e^{(1+2i)t}+\frac{65}{2}e^{(1-2i)t} &\overset{!}{=} (D^2-4)p(t)\\
&= (D^2-4)[k_1+k_2e^{(1+2i)t}+k_3e^{(1-2i)t}]\\
&= (D^2-4)k_1+(D^2-4)k_2e^{(1+2i)t}\\
&\qquad\qquad +(D^2-4)k_3e^{(1-2i)t}\\
\text{(complex exponential shift)}\rightsquigarrow &= -4k_1+e^{(1+2i)t}([D+(1+2i)]^2-4)k_2\\
&\qquad +e^{(1-2i)t}([D+(1-2i)]^2-4)k_3\\
\text{(derivatives of } k_2, k_3 \text{ are zero)}\rightsquigarrow &= -4k_1+e^{(1+2i)t}((1+2i)^2-4)k_2\\
&\qquad +e^{(1-2i)t}((1-2i)^2-4)k_3
\end{aligned}$$

Since the functions 1, $e^{(1+2i)t}$ and $e^{(1-2i)t}$ are linearly independent, we can equate coefficients of like terms on left and right:

$$-4k_1=1,\qquad ((1+2i)^2-4)k_2=\frac{65}{2},\qquad ((1-2i)^2-4)k_3=\frac{65}{2}.$$

Thus, $k_1=-1/4$ and

$$k_2=\frac{65}{2((1+2i)^2-4)}=\frac{65}{2(1+4i-4-4)}=\frac{65}{2(4i-7)}=\frac{1}{2}(-4i-7)$$

and (either by a like computation or because it must be the complex conjugate[5])

$$k_3=\frac{1}{2}(4i-7).$$

This gives

$$\begin{aligned}
p(t)&=-\frac{1}{4}+\frac{1}{2}(-4i-7)e^{(1+2i)t}+\frac{1}{2}(4i-7)e^{(1-2i)t}\\
&=-\frac{1}{4}-\frac{4i}{2}e^t(e^{2it}-e^{-2it})-\frac{7}{2}e^t(e^{2it}+e^{-2it})\\
&=-\frac{1}{4}+4e^t\sin 2t-7e^t\cos 2t.
\end{aligned}$$

The general solution of (N) then is the sum of $H(t)$ and $p(t)$:

$$x=H(t)+p(t)=c_1e^{2t}+c_2e^{-2t}-\frac{1}{4}+4e^t\sin 2t-7e^t\cos 2t.$$

This completes the example. We note that when using trigonometric functions as in the text, one has to solve a 2×2 system of equations to

[5]Don't use the "must be the complex conjugate"-shortcut unless you know why it's true!

find k_2 and k_3, while with complex exponentials one instead computes the reciprocal of a complex number.

6. Laplace transforms

The definition of the Laplace transform in the textbook [1, page 412] (see also [1, Example 5.2.1] and [1, Example 5.2.4]) gives

$$\begin{aligned}
\mathscr{L}[t^n e^{\lambda t}](s) &= \int_0^\infty e^{-st} t^n e^{\lambda t}\, dt \\
&= \int_0^\infty t^n e^{-(s-\lambda)t}\, dt \\
\left(\substack{\text{integration by parts:} \\ u=t^n,\ dv=e^{-(s-\lambda)t}\,dt}\right) \rightsquigarrow &= \begin{cases} \dfrac{1}{s-\lambda} & \text{if } n=0 \\ 0 + \dfrac{n}{s-\lambda}\displaystyle\int_0^\infty t^{n-1} e^{-(s-\lambda)t}\, dt & \text{if } n \geq 1 \end{cases} \\
&= \begin{cases} \dfrac{1}{s-\lambda} & \text{if } n=0 \\ \dfrac{n}{s-\lambda}\mathscr{L}[\underbrace{t^{n-1}e^{\lambda t}}_{\text{next lower power of } t}](s) & \text{if } n \geq 1. \end{cases}
\end{aligned}$$

Applying this recursively, we find

$$\boxed{\mathscr{L}[t^n e^{\lambda t}](s) = \frac{n!}{(s-\lambda)^{n+1}} \qquad \mathscr{L}^{-1}\left[\frac{1}{(s-\lambda)^n}\right](t) = \frac{1}{(n-1)!} t^{n-1} e^{\lambda t}} \tag{2}$$

This will go quite far when combined with partial fractions. Note that

- for $n = 0$ this gives

$$\boxed{\mathscr{L}[e^{\lambda t}](s) = \frac{1}{s-\lambda} \qquad \mathscr{L}^{-1}\left[\frac{1}{s-\lambda}\right](t) = e^{\lambda t}}$$

- for $\lambda = 0$ this gives

$$\boxed{\mathscr{L}[t^n](s) = \frac{n!}{s^{n+1}} \qquad \mathscr{L}^{-1}\left[\frac{1}{s^n}\right](t) = \frac{1}{(n-1)!} t^{n-1}}$$

- and for $n = \lambda = 0$ this gives

$$\boxed{\mathscr{L}[1](s) = \frac{1}{s} \qquad \mathscr{L}^{-1}\left[\frac{1}{s}\right](t) = 1}$$

We can go further without more calculus. Taking $n = 0$ and replacing λ by $\lambda \pm i\beta$ we get from Euler's formula (1)

$$\mathscr{L}[\underbrace{\cos\beta t}_{\frac{e^{i\beta t}+e^{-i\beta t}}{2}} e^{\lambda t}](s) = \frac{1}{2}\mathscr{L}[e^{(\lambda+i\beta)t} + e^{(\lambda-i\beta)t}](s)$$

$$= \frac{1/2}{s-(\lambda+i\beta)} + \frac{1/2}{s-(\lambda-i\beta)} = \frac{1/2}{(s-\lambda)+i\beta} + \frac{1/2}{(s-\lambda)-i\beta)}$$

$$= \frac{s}{(s-\lambda)^2+\beta^2}$$

and

$$\mathscr{L}[\underbrace{\sin\beta t}_{\frac{e^{i\beta t}-e^{-i\beta t}}{2i}} e^{\lambda t}](s) = \frac{1}{2i}\mathscr{L}[e^{(\lambda-i\beta)t} - e^{(\lambda-i\beta)t}](s)$$

$$= \frac{1/2i}{s-(\lambda+i\beta)} - \frac{1/2i}{s-(\lambda-i\beta)} = \frac{1/2i}{(s-\lambda)+i\beta} - \frac{1/2i}{(s-\lambda)-i\beta}$$

$$= \frac{\beta}{(s-\lambda)^2+\beta^2}.$$

The case $\lambda = 0$ is worth remembering[6]:

$$\boxed{\mathscr{L}[\cos\beta t] = \frac{s}{s^2+\beta^2} \qquad \mathscr{L}[\sin\beta t] = \frac{\beta}{s^2+\beta^2}}$$

Here is a slightly different way of getting at these.

$$\mathscr{L}[\overbrace{\underbrace{\cos\beta t}_{\Re(e^{i\beta t})} e^{\lambda t}}^{\Re(e^{(\lambda+i\beta)t})}](s) = \Re(\mathscr{L}[e^{(\lambda+i\beta)t}](s)) = \Re\Big(\frac{1}{s-\lambda-i\beta}\Big)$$

Indeed, more generally,

$$\mathscr{L}[\overbrace{t^n \underbrace{\cos\beta t}_{\Re(e^{i\beta t})} e^{\lambda t}}^{\Re(t^n e^{(\lambda+i\beta)t})}](s) = \Re\Big(\frac{n!}{(s-\lambda-i\beta)^{n+1}}\Big)$$

This is not in itself very useful, however. In any one problem, decomposing everything into exponentials is the way to go. Example 30 below shows this in an instance where other methods are a lot more work,

[6] To remember which is which, remember that sin is an odd function, so $\mathscr{L}[\sin\beta t](s)$ must be an odd function of β, while $\mathscr{L}[\cos\beta t](s)$ must be an even function of β.

7. Partial fractions

Partial-fractions decompositions are *easier* with complex numbers, because irreducible quadratics are no longer a difficulty: everything reduces to linear factors. This is of interest wherever partial fractions might be useful: They first appeared in the context of the integration of rational functions, and if only linear factors occur, then all integrals are logarithms. Likewise, linear denominators are most convenient for inverse Laplace transforms.

One usually has to patiently work the partial-fractions decompositions that arise and put up with complex coefficients, but partial-fractions decompositions can often be found "by inspection" rather than plodding through an algorithm. For instance, they are always easy when the denominator consists of 2 linear factors and the numerator is constant:

$$\frac{1}{(s-a)(s-b)} = \frac{(s-a)-(s-b)}{(s-a)(s-b)}\frac{1}{b-a} = \frac{\frac{1}{b-a}}{s-b} - \frac{\frac{1}{b-a}}{s-a}.$$

24. **Example** ([1, Example 5.3.3]: Distinct roots)**.** Solve the initial-value problem

$$x' - x = 2\sin t \qquad x(0) = 0.$$

Note first that the method of undetermined coefficients tells us to expect no terms other than e^t, $\cos t$ and $\sin t$ in the solution (or: no terms other than e^t, e^{it} and e^{-it}). This was the purpose of Remark 23.

Applying $\mathscr{L}$ we get

$$(s-1)\mathscr{L}[x] = 2\mathscr{L}\Big[\frac{i}{2}(e^{-it}-e^{it})\Big] = \frac{i}{s+i} - \frac{i}{s-i}$$

and hence

$$x(t) = \mathscr{L}^{-1}\left[\frac{i}{(s-1)(s+i)}\right] + \mathscr{L}^{-1}\left[\frac{-i}{(s-1)(s-i)}\right] = 2\Re\mathscr{L}^{-1}\left[\frac{i}{(s-1)(s+i)}\right]$$

since the 2 terms are complex conjugates. This partial-fractions decomposition is easy:

$$\frac{i}{(s-1)(s+i)} = \underbrace{\frac{i}{1+i}}_{=\frac{i}{1+i}\frac{1-i}{1-i}=\frac{1+i}{2}}\frac{\overbrace{(s+i)-(s-1)}^{=1+i}}{(s-1)(s+i)} = \frac{1+i}{2}\Big(\frac{1}{s-1} - \frac{1}{s+i}\Big),$$

so

$$\begin{aligned} x(t) = 2\Re\mathscr{L}^{-1}\left[\frac{i}{(s-1)(s+i)}\right] &= \Re[(1+i)(e^t - e^{-it})] \\ &= \Re[(1+i)(e^t - \cos t + i\sin t)] \\ &= \Re[e^t - \cos t - \sin t + i(\text{something})] \\ &= e^t - \cos t - \sin t. \end{aligned}$$

We conclude that the initial-value problem

$$Dx - x = 2\sin t \qquad x(0) = 0.$$

has the unique solution

$$x = e^t - \cos t - \sin t.$$

The form matches with our expectations from the method of undetermined coefficients (Remark 23), so one only needs to check the initial condition to verify that this is correct: $x(0) = e^0 - \cos 0 - \sin 0 = 1 - 1 = 0$, as required.

25. **Example** (Previous example on autopilot)**.** Even plodding through the partial-fractions algorithm is not too bad. We found

$$(s-1)\mathscr{L}[x] = 2\mathscr{L}\left[\frac{i}{2}(e^{-it} - e^{it})\right] = \frac{i}{s+i} - \frac{i}{s-i} = \frac{2}{(s-i)(s+i)}$$

To find x we look for the partial-fractions decomposition of $\mathscr{L}[x]$:

$$\frac{2}{(s-1)(s+i)(s-i)} = \frac{A}{s-1} + \frac{B}{s-i} + \frac{C}{s+i}$$

To determine A, B, and C, clear fractions

$$2 = A(s+i)(s-i) + B(s-1)(s+i) + C(s-1)(s-i)$$

and then insert the values of s for which the original denominator is zero, that is, $s = 1, \pm i$.

For $s = 1$ this gives

$$2 = A(1+i)(1-i) = A(1-(i^2)) = 2A,$$

so $A = 1$.

For $s = i$ this gives

$$2 = B(i-1)(i+i) = 2B(i-1)i = 2B(i^2 - i) = 2B(-1-i),$$

so $B = \dfrac{1}{-1-i} = -\dfrac{1}{2} + \dfrac{i}{2}$.

Finally, for $s = -i$ we get

$$2 = C(-i-1)(-i-i) = 2C(-i-1)(-i) = 2C((-i)^2 + i) = 2C(-1+i),$$

so $C = \dfrac{1}{-1+i} = \dfrac{1}{-1+i}\dfrac{-1-i}{-1-i} = \dfrac{-1-i}{(-1)^2-i^2} = -\dfrac{1}{2} - \dfrac{i}{2}$.

26. **Remark.** Time-saver: B and C are complex conjugates. This is no accident!

We thus obtain the partial-fractions decomposition

$$\frac{2}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{-\frac{1}{2}+\frac{i}{2}}{s-i} + \frac{-\frac{1}{2}-\frac{i}{2}}{s+i}$$

Note that the two summands involving i are complex conjugates of each other, so they sum to a real number. This is why we *expect* their numerators B and C to be complex conjugates.

Now we "untransform" and sort terms:

$$\begin{aligned} x &= \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{-\frac{1}{2}+\frac{i}{2}}{s-i} + \frac{-\frac{1}{2}-\frac{i}{2}}{s+i}\right] \\ &= e^t + \big(-\frac{1}{2}+\frac{i}{2}\big)e^{it} + \big(-\frac{1}{2}-\frac{i}{2}\big)e^{-it} \\ &= e^t - \frac{1}{2}(e^{it}+e^{-it}) + \frac{i}{2}(e^{it}-e^{-it}) \\ &= e^t - \cos t - \sin t \end{aligned}$$

27. **Example** ([1, Example 5.6.3]: Pair of double complex roots)**.**
Find $\mathscr{L}^{-1}\Big[\dfrac{s}{(s^2+1)^2}\Big]$.

We rewrite

$$\frac{s}{(s^2+1)^2} = \frac{s}{(s-i)^2(s+i)^2}$$

and look for the partial-fractions decomposition. Seeking a free ride, note that

$$(s+i)^2 - (s-i)^2 = 4is,$$

so

$$\frac{s}{(s^2+1)^2} = \frac{s}{(s-i)^2(s+i)^2} = \frac{1}{4i}\frac{(s+i)^2-(s-i)^2}{(s-i)^2(s+i)^2} = \frac{1}{4i}\Big(\frac{1}{(s-i)^2} - \frac{1}{(s+i)^2}\Big)$$

Therefore,

$$\mathscr{L}^{-1}\Big[\frac{s}{(s-i)^2(s+i)^2}\Big] = \mathscr{L}^{-1}\Big[\frac{1/4i}{(s-i)^2} - \frac{1/4i}{(s+i)^2}\Big] = \frac{t}{2}\Big[\frac{e^{it}-e^{-it}}{2i}\Big] = \frac{t}{2}\sin t.$$

This concludes the example, but let us also show that even on autopilot we can produce the partial-fractions decomposition without undue exertions:

$$\frac{s}{(s-i)^2(s+i)^2} = \frac{A}{s-i} + \frac{B}{(s-i)^2} + \frac{C}{s+i} + \frac{D}{(s+i)^2}$$

becomes (by clearing fractions)

$$s = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2.$$

Set $s = i$ to get

$$i = B(2i)^2 = -4B, \text{ hence } B = -\frac{i}{4} = \frac{1}{4i}.$$

Set $s = -i$ to get

$$-i = D(-2i)^2 = -4D, \text{ hence } D = \frac{i}{4} = -\frac{1}{4i}.$$

Before going on we consolidate the corresponding terms:

$$\begin{aligned} B(s+i)^2 + D(s-i)^2 &= -\frac{i}{4}(s+i)^2 + \frac{i}{4}(s-i)^2 \\ &= \frac{i}{4}\big(-(s^2+2si-1) + (s^2-2si-1)\big) = -\frac{i}{4}(4si) = s. \end{aligned}$$

Therefore we can rewrite

$$s = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2$$

as

$$0 = A(s-i)(s+i)^2 + C(s+i)(s-i)^2.$$

This means that we can (and hence must!) take $A = C = 0$. Thus,

$$\frac{s}{(s^2+1)^2} = \frac{1}{4i}\Big(\frac{1}{(s-i)^2} - \frac{1}{(s+i)^2}\Big)$$

28. **Example** ([1, Example 5.6.5]: Application to initial-value problem)**.** Solve

$$(D^2+1)x = \cos t, \qquad x(0) = x'(0) = 0.$$

Apply $\mathscr{L}$ to both sides to get

$$\underbrace{(s^2+1)}_{(s-i)(s+i)}\mathscr{L}[x] = \mathscr{L}[\cos t] = \frac{1}{2}\mathscr{L}[e^{it}+e^{-it}] = \frac{1}{2}\Big(\frac{1}{s-i} + \frac{1}{s+i}\Big) = \frac{s}{(s-i)(s+i)}$$

Thus, $\quad x(t) = \mathscr{L}^{-1}\Big[\frac{s}{(s-i)^2(s+i)^2}\Big] = \frac{t}{2}\sin t \quad$ by the previous example.

29. **Example** ([1, Example 5.6.4]: Same pair of double complex roots)**.** Find $\mathscr{L}^{-1}\Big[\frac{1}{(s^2+1)^2}\Big]$.

We can repeatedly use here that $(s+i)-(s-i)$ is a constant, but this will need to be done in 2 stages. Still useful, but let's see how this plays out on autopilot.

We write $\dfrac{1}{(s^2+1)^2} = \dfrac{1}{(s-i)^2(s+i)^2}$ and produce the partial-fractions decomposition:

$$\frac{1}{(s-i)^2(s+i)^2} = \frac{A}{s-i} + \frac{B}{(s-i)^2} + \frac{C}{s+i} + \frac{D}{(s+i)^2}$$

becomes (by clearing fractions)

$$1 = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2. \tag{3}$$

Algorithm: Solve for A, B, C and D as follows

- multiply out the right-hand side,
- sort by powers of s,
- compare coefficients to get 4 linear equations in these 4 unknowns.

Shortcut: Sample useful s-values first to break the problem up.

Set $s = i$ to get

$$1 = B(2i)^2 = -4B, \text{ hence } B = -\frac{1}{4}.$$

Set $s = -i$ to get

$$1 = D(-2i)^2 = -4D, \text{ hence } D = -\frac{1}{4}.$$

Consolidate the corresponding terms in (3):

$$B(s+i)^2 + D(s-i)^2 = -\frac{1}{4}\big((s^2+2si-1) + (s^2-2si-1)\big) = -\frac{1}{2}(s^2-1).$$

Therefore, (3) becomes

$$\begin{aligned} 1 &= A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2 \\ &= A(s-i)(s+i)^2 + C(s+i)(s-i)^2 - \frac{1}{2}(s^2-1) \end{aligned}$$

and, adding $\frac{1}{2}(s^2-1)$ to both sides,

$$\begin{aligned} \frac{1}{2}(s^2+1) = 1 + \frac{1}{2}(s^2-1) &= A(s-i)(s+i)^2 + C(s+i)(s-i)^2 \\ &= A(s^2+1)(s+i) + C(s^2+1)(s-i). \end{aligned}$$

Dividing by the common factor s^2+1, this becomes

$$\frac{1}{2} = A(s+i) + C(s-i) = \underbrace{\overbrace{(A+C)}^{\text{must be } 0} s + i \overbrace{(A-C)}^{\text{must be } 1/2}}_{\text{sorted by powers of } s}.$$

Comparing coefficients of like powers of s we find that

$$A + C = 0 \quad \text{and} \quad \frac{1}{2} = i(A-C) = 2iA,$$

so

$$A=\frac{1}{4i}\quad\text{and}\quad C=-\frac{1}{4i}.$$

Inserting these into the partial-fractions decomposition we find

$$\begin{aligned}\frac{1}{(s-i)^2(s+i)^2}&=\frac{A}{s-i}+\frac{B}{(s-i)^2}+\frac{C}{s+i}+\frac{D}{(s+i)^2}\\&=\frac{1/4i}{s-i}-\frac{1/4}{(s-i)^2}-\frac{1/4i}{s+i}-\frac{1/4}{(s+i)^2}\end{aligned}$$

This gives us

$$\begin{aligned}\mathscr{L}^{-1}\Big[\frac{1}{(s^2+1)^2}\Big]&=\mathscr{L}^{-1}\Big[\frac{1/4i}{s-i}-\frac{1/4}{(s-i)^2}-\frac{1/4i}{s+i}-\frac{1/4}{(s+i)^2}\Big]\\&=\frac{1}{4i}e^{it}-\frac{t}{4}e^{it}-\frac{1}{4i}e^{-it}-\frac{t}{4}e^{-it}\\&=\frac{1}{2}\frac{e^{it}-e^{-it}}{2i}-\frac{t}{2}\frac{e^{it}+e^{-it}}{2}\\&=\frac{1}{2}\sin t-\frac{t}{2}\cos t.\end{aligned}$$

30. **Example** (Initial-value problem producing triple complex roots)**.** Solve

$$(D^2+9)^2x=-4\sin 3t,\qquad x(0)=x'(0)=x''(0)=0,\ x'''(0)=1.$$

We first note that we should expect only the functions $\sin 3t$, $\cos 3t$, $t\sin 3t$, $t\cos 3t$, $t^2\sin 3t$, $t^2\cos 3t$ in the ultimate solution (Remark 23).[7]

Apply $\mathscr{L}$ to get

$$(s^2+9)^2\mathscr{L}[x]-1=-4\frac{3}{s^2+9},$$

so

$$\mathscr{L}[x]=\frac{1}{(s^2+9)^2}-\frac{12}{(s^2+9)^3}=\frac{\overbrace{s^2-3}^{=(s^2+9)-12}}{(\underbrace{s^2+9}_{=(s-3i)(s+3i)})^3}\tag{4}$$

We need to decompose the right-hand side as follows:

$$\frac{s^2-3}{(s-3i)^3(s+3i)^3}=\frac{A}{s-3i}+\frac{B}{s+3i}+\frac{C}{(s-3i)^2}+\frac{D}{(s+3i)^2}+\frac{E}{(s-3i)^3}+\frac{F}{(s+3i)^3}.$$

[7] Spoiler alert: keeping this in mind *and* looking at the initial values one might spot that of these functions only $t\sin 3t$ has first and second derivative equal to 0 for $t=0$, so we might guess that maybe a multiple of $t^2\sin 3t$ alone constitutes the desired solution.

Clear fractions to get

$$\begin{aligned} s^2-3 =& A(s-3i)^2(s+3i)^3 + B(s-3i)^3(s+3i)^2 \\ &+ C(s-3i)(s+3i)^3 + D(s-3i)^3(s+3i) \\ &+ E(s+3i)^3 + F(s-3i)^2. \end{aligned}$$

For $s=3i$ this becomes $E = \dfrac{-9-3}{(6i)^3} = \dfrac{2\cdot 6}{6\cdot 3\cdot 2\cdot 6i} = \dfrac{1}{18i}$, and F must be the complex conjugate: $F = -\dfrac{1}{18i}$. We consolidate these 2 terms first:

$$\frac{E}{(s-3i)^3} + \frac{F}{(s+3i)^3} = \frac{1/18i}{(s-3i)^3} - \frac{1/18i}{(s+3i)^3} = \frac{\overbrace{s^2-3}^{=\frac{(s+3i)^3}{18i}-\frac{(s-3i)^3}{18i}}}{(\underbrace{s^2+9}_{=(s-3i)(s+3i)})^3} = \text{RHS of (4)!}$$

This is the right-hand side of equation (4), so we just stumbled upon the partial-fractions decomposition we need, that is, (using (2)) we have

$$x(t) = \mathscr{L}^{-1}\left[\frac{s^2-3}{(s^2+9)^3}\right](t) = \frac{1}{18i}\mathscr{L}^{-1}\left[\frac{1}{(s-3i)^3} - \frac{1}{(s+3i)^3}\right] = \frac{t^2}{18}\underbrace{\frac{e^{3it}-e^{-3it}}{2i}}_{=\sin 3t}.$$

In short, the solution to the given initial-value problem is

$$x = \frac{1}{18}t^2\sin 3t.$$

By inspection (really!) one can see that this satisfies the initial condition.

This illustrates that we can handle complex roots of any multiplicity, not just double complex roots (as is the case with convolutions).

8. Complex eigenvalues (covered in [1, Section 3.8])

One context in which the use of complex numbers is definitely not optional is that of homogeneous systems of linear differential equations with constant coefficients in which the coefficient matrix has complex eigenvalues. Since this is not an optional subject, all that is needed is covered in [1, Section 3.8].

References

[1] Martin M. Guterman, Zbigniew H. Nitecki, *Differential Equations – A First Course*, 3^{rd} ed., Saunders (1992).