Equity Variance Swap Greeks

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Abstract

We present what Greeks can (and should) be calculated for for Equity Variance Swaps with and without discrete dividends.

1 Pricing

The pricing methodology is discussed in [Whi12b]. To summarise, a dividend payment at time τ_i is assumed to be of the affine form $\alpha_i + \beta_i S_{\tau_i^-}$, where $S_{\tau_i^-}$ is the stock price immediately prior to the dividend payment. This leads to the stock price (or index value) being of the form

$$S_t = (F_t - D_t)X_t + D_t \tag{1}$$

where the forward F_t is given by

$$F_t \equiv F(0,t) = R(0,t)\left(S_t - \sum_{j:0<\tau_j\leq t} \frac{\alpha_j}{R(0,\tau_j)}\right) \tag{2}$$

the discounted future dividends D_t by

$$D_t \equiv \sum_{j:\tau_j>t} \frac{\alpha_j}{R(t,\tau_j)} \tag{3}$$

and the growth factor $R(t,T)$ by

$$R(t,T) = \frac{1}{P(t,T)} \prod_{j:t<\tau_j\leq T} (1-\beta_j) \tag{4}$$

where $P(t,T)$ is the discount factor. These are known as the *equity dividend curves.*

In this model, the positive Martingale X_t is known as the *pure* stock price. The prices of (hypothetical) European options on the pure stock price (henceforth known as pure option prices) can be derived from real (i.e. market) option prices. Then the pure implied volatility can

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be defined as the number that put into the Black formula[1] that gives the pure option price. If the pure implied volatilities from many options (on the same underlying) are interpolated in an arbitrage free way, then we have a pure implied volatility surface. Furthermore, if the pure stock price is assumed to follow the process

$$\frac{dX_t}{X_t} = \sigma^X(t, X_t)dW_t \tag{5}$$

then the pure local volatility surface , $\sigma^X(t, X_t)$ can be found by applying the Dupire formula [Dup94] to the pure implied volatility surface [Whi12b].

With the lexicon established, we highlight the steps taken in pricing an equity variance swap in the presence of discrete dividends of an affine form.

- Get market prices of european options (on the relevant underlying)
- Make (informed) assumptions on future dividends
- Obtain a discount curve
- Convert market prices to pure option prices (using the dividend assumption)
- Use these pure option prices to construct the *pure implied volatility surface* (this with implicitly depend on the dividend assumptions)
- There are three numerical procedures to price the variance swap [Whi12a]
  - using static replication for the price of a log-contact and the corrections at each dividend by deriving prices directly off the pure implied volatility surface
  - deriving the *pure local volatility surface* and either
    * solve the forward PDE for the pure option prices, then carrying out static replication as above[2]
    * solve the backwards PDE for the price of a log-contact and the corrections at each dividend

Up to the level of numerical accuracy all three methods give the same answer.

2 The Greeks

For all the Greeks discussed below, we use the data discussed in the appendix when showing concrete results. The results are given for a notional of one, so are in effect the greeks on the annualised Expected Variance (EV), or in some cases the square-root of this.

[1]as $\mathbb{E}[X_t] = 1$ the forward to put into the formula is 1.

[2]Solving the forward PDE is effectively just reversing the differentiation process that produced the local from the implied volatility surface, so is not really an independent numerical scheme to price variance swaps, unless you are given the local volatility extraniously.

2.1 Delta

This is the sensitivity of the EV to the spot. In a Black-Scholes world the delta of a variance swap is zero[DDKZ99, Bue10]. This is true for any model where the instantaneous volatility does not explicitly depend on S_t (e.g. Heston [Hes93]). We have a number of model dependent choices on the definition of delta.

- The (market) implied volatilities are invariant to a change in the spot (so called *sticky-strike*) - the derived pure (implied and local) volatility surfaces will of course change as the spot moves.
- The pure implied volatility surface is invariant to a change in the spot. This is identical to asserting that the pure local volatility surface is invariant to a change in the spot. For the no dividends case this is just *sticky-moneyness*[3] or *sticky-delta*.
- The local volatility surface is invariant to a change in the spot. The introduction of dividends will make the pure (local and implied) volatility surfaces depend on the spot.

The results for the various "market" data and dividends scenarios are shown in table 1. Clearly for a flat volatility surface and no dividends[4] we should expect no delta in any scenario; and this is what we see (except for some numerical noise for sticky local volatility). The other expected result is that the delta is zero in the sticky pure volatility assumption with no dividends (which corresponds to a sticky-moneyness/delta assumption). In all other cases we have significant delta.

	Sticky Strike	Sticky Pure Volatility	Sticky Local Volatility
Flat surface - no dividends	0	0	1.01E-08
Flat surface - dividends corrected	2.26E-02	2.75E-02	2.75E-02
Flat surface - dividends not corrected	1.57E-02	2.06E-02	2.06E-02
Non-flat surface - no dividends	-6.07E-02	0	-2.07E-01
Non-flat surface - dividends corrected	-3.52E-02	1.52E-02	-1.26E-01
Non-flat surface - dividends not corrected	-4.21E-02	1.52E-02	-1.35E-01

Table 1: The delta of the Expected Variance (EV), scaled by the spot.

2.2 Gamma

Gamma is handled exactly like delta, except that we now scale by the square of the spot. We present the results in table 2. Zero (or very small values) appear in the same positions as for delta. Gamma for *sticky local volatility* is very high (and positive) for the non-flat volatility surface with no dividends; with dividends it becomes negative.

2.3 Vega

Vega here means the sensitivity of the variance swap to parallel moves of the various volatility surfaces. We present the vega here as the sensitivity of the square root of the expected variance to the volatility ($\frac{\partial\sqrt{EV}}{\partial\sigma}$) The four volatility surfaces to consider are:

[3] Moneyness is the strike divided by the forward

[4] for a flat volatility surface with no dividends we are in a Black-Scholes world and there is no distinction between the flavours of volatility surface - implied or local, pure or actual.

	Sticky Strike	Sticky Pure Volatility	Sticky Local Volatility
Flat surface - no dividends	0.00E+00	0.00E+00	-1.35E-03
Flat surface - dividends corrected	-4.51E-02	2.75E-02	-5.61E-02
Flat surface - dividends not corrected	-2.39E-02	2.06E-02	-3.52E-02
Non-flat surface - no dividends	3.58E-01	0.00E+00	4.29E+01
Non-flat surface - dividends corrected	3.00E-01	1.52E-02	-2.52E+01
Non-flat surface - dividends not corrected	3.24E-01	8.80E-03	-2.52E+01

Table 2: The gamma of the Expected Variance (EV), scaled by spot squared.

- The implied volatility surface
- The local volatility surface
- The pure implied volatility surface
- The pure local volatility surface

Table 3 summarises the results for the four flavours of vega[5]. That the vega for a flat surface with no dividends is exactly 1.0 should not be surprising.

The presence of dividends lowers the vega. Again this should not be a surprise; some of the variance is coming from the jumps at dividend dates (whether or not they are corrected for) so will not be affected by a change to the volatility surface.

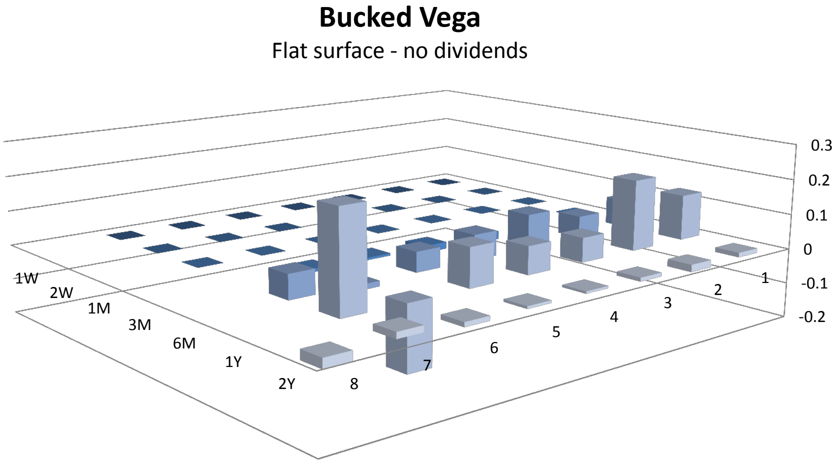
	Implied Volatility	Local Volatility	Pure Implied Volatility	Pure Local Volatility
Flat surface - no dividends	1.000	1.000	1.000	1.000
Flat surface - dividends corrected	0.967	0.991	0.919	0.919
Flat surface - dividends not corrected	0.962	0.985	0.914	0.914
Non-flat surface - no dividends	1.174	1.315	1.174	1.315
Non-flat surface - dividends corrected	1.119	1.282	1.062	1.185
Non-flat surface - dividends not corrected	1.106	1.267	1.049	1.170

Table 3: The vega of the square-root of Expected Variance (EV), defined as the sensitivity to a parallel shift of each of the four flavours of volatility surface.

2.4 Bucked vega

The sensitivity of the square-root of annualised Expected Variance to the market price of each option (expressed as implied volatility). This is done by taking finite differences - moving each market implied volatility a small amount and repricing (using any pricing method) the variance swap. Figures 1, 2 and 3 shows the result of this excise for a 9M variance swap for three different data sets. In all cases, nearly all the sensitivity is to 6M and 1Y options, as one would expect. The sensitivity across strikes appears to be very depended on the choice of smile interpolation/extrapolation. Interpolation of the implied volatility surface is discussed here [Whi12b].

[5] we have chosen to display the results as the sensitivity of the square-root of EV to parallel moves in the volatility surface, since for a Black-Scholes world this with give exactly 1.0



	1	2	3	4	5	6	7	8
1W	0	0	0	0	0	0	0	
2W	0	0	0	0	0	0	0	
1M	9.59E-05	1.61E-04	2.68E-04	2.47E-04	1.67E-04	5.62E-05	6.12E-05	
3M	-0.003213184	-1.56E-02	-0.023226351	-0.019815438	-0.006643086	-0.009317461		
6M	0.079233495	0.06193557	0.097465566	0.071355093	0.057176615	-0.01529691	0.067991407	
1Y	0.129759928	0.2006906	0.068945913	0.079046846	0.109851208	-0.185842579	0.274643788	
2Y	-0.012927597	-0.020393911	-0.010761254	-0.006071321	-0.007078312	-0.012672369	0.017452172	-0.026596342

Figure 1: The bucked vega (sensitivity of the square-root of expected volatility to implied volatility) of a 9M variance swap, where the underlying volatility surface is flat and there are no dividends.

2.5 Dividend Sensitivity

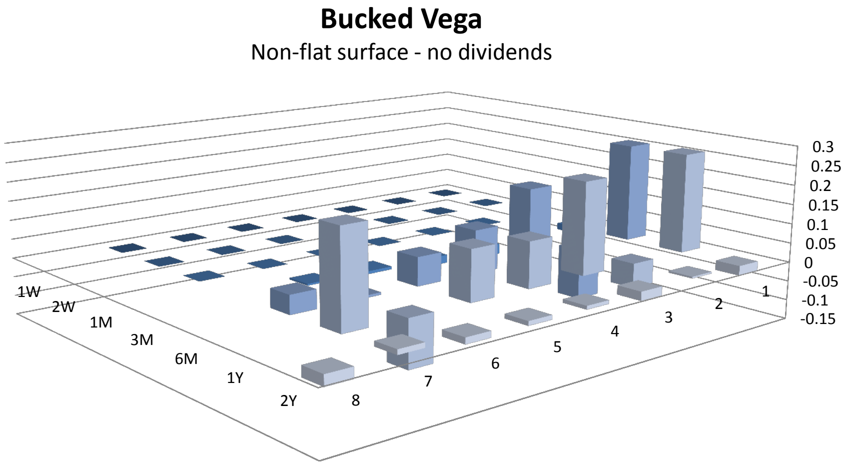
While not *standard* Greeks, the realised variance is of course sensitive to the dividends, so these must be included in a list of Greeks. The two different assumptions we make are that

- The pure (implied or local) volatility surface is invariant to changes in the dividends. This is perhaps logically inconsistent - we use the given dividends and market implied volatilities to derive the pure volatility surface, then assume this surface does **not** depend on those dividends. This may be appropriate if the pure volatility surface is given extraneously.
- The market implied volatilities are invariant to changes in the dividends.

In both of these cases we assume that the forwards do charge, that is, we assume the discount curve is invariant, therefore the forwards must be dependent on the dividends.

Tables 4 and 5 show the results for the two different assumptions. In both cases the sensitivity to the proportional part of the dividend (β) decreases with time to the dividend payment, and is zero for dividends after the expiry of the variance swap. For the case of the invariant pure implied volatility surface, the sensitivity to the cash part of the dividend (α) remains large even for dividends after the expiry of the variance swap - this is because cash dividends affect the implied volatility surface[6] for all expiries less than or equal to their own. For the invariant

[6] under the assumption of a fixed pure implied volatility surface.



	1	2	3	4	5	6	7	8
1W	0	0	0	0	0	0	0	
2W	0	0	0	0	0	0	0	
1M	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	
3M	-0.017354256	-9.45E-04	-0.026375361	-0.020686497	-0.007325342	-0.004220501		
6M	0.256043728	-0.132625378	0.188337845	0.109744605	0.072676939	0.006111976	0.04708877	
1Y	0.259008496	-0.063751059	0.236696274	0.116850601	0.1282259	-0.12234237	0.238992704	
2Y	-0.024853574	0.004153627	-0.022818804	-0.009017532	-0.011120566	-0.015838427	0.012503027	-0.026484621

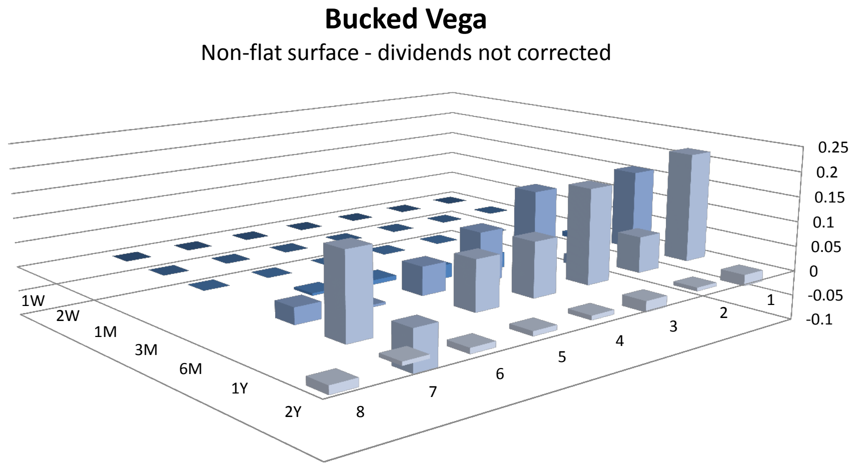
Figure 2: The bucked vega (sensitivity of the square-root of expected volatility to implied volatility) of a 9M variance swap, where the underlying volatility surface is non-flat and there are no dividends.

implied volatilities assumption, the sensitivity to the cash part of the dividend does decrease with time to dividend payment.

A The Data

We generate four sets of option implied volatilities, two with dividends and two without. These four data sets are treated as the ”market”. The steps are as follows:

- Start with two pure implied volatility surfaces: one flat at $\sigma = 0.45$ and the other generated from a mixed log-normal model[Whi12c], the parameters of which are detailed below
- Chose a spot level and discount curve - we use 65.4 and a constant short rate of 7%.
- Chose a dividend structure - this is detailed in table 6
- Build two sets equity dividend curves, R_t, D_t and F_t, one for and one without dividends
- Chose the option expiries and strikes - we use the fairly standard 1W, 2W, 1M, 3M 6M, 1Y and 2Y. The strikes are detailed in table 7
- Price options (by first pricing pure options, then converting these to actual option prices) using the two pure implied volatility surfaces with and without dividends (giving four sets)



	1	2	3	4	5	6	7	8
1W	0	0	0	0	0	0	0	
2W	0	0	0	0	0	0	0	
1M	2.79E-04	-3.70E-05	3.43E-04	3.51E-04	2.05E-04	4.71E-05	3.97E-05	
3M	-0.010491222	-0.011086861	-0.031527104	-0.027657854	-0.00868219	-0.004759834		
6M	0.160836121	-0.019209828	0.158911192	0.099196324	0.056206039	0.004510024	0.03203213	
1Y	0.219097466	0.071736518	0.187907173	0.108242288	0.098957209	-0.083544724	0.16365163	
2Y	-0.022362074	-0.007133007	-0.021421026	-0.008647881	-0.009321774	-0.010901842	0.007259911	-0.015666958

Figure 3: The bucked vega (sensitivity of the square-root of expected volatility to implied volatility) of a 9M variance swap, where the underlying volatility surface is non-flat and there are dividends.

- Find implied volatilities for the four option price sets - these are then treated as the market data

Our representative variance swap has an expiry of nine months, so there is neither options or dividends at that time. We consider the expected variance with and without the correction made for dividend payments.

Clearly for the flat pure volatility surface at 0.45 without dividends, the ”market” implied volatilities will all be at 0.45. The other implied volatility sets are shown in tables 8, 9 and 10.

A.1 Mixed Log-Normal parameters

Full details of the model are given here[Whi12c]. We use three normals with weights of 15%, 80% and 5%, volatilities of 0.15, 0.3 and 0.8, and drifts of 0.04, 0.02 and -0.2. This produces an arbitrage free pure implied volatility surface.

References

[Bue10] Hans Buehler. Volatility and dividends. Working paper, 2010.

[DDKZ99] Demeterfi, Derman, Kamal, and Zou. More than you ever wanted to know about

	Flat surface dividends corrected	Flat surface dividends not corrected	Non-flat surface dividends corrected	Non-flat surface dividends not corrected
α_1	-0.159	-0.045	-0.086	0.019
β_1	-0.021	0.083	-0.011	0.088
α_2	-0.291	-0.292	-0.162	-0.163
β_2	-0.014	-0.014	-0.008	-0.008
α_3	-0.289	-0.291	-0.161	-0.162
β_3	-0.005	-0.005	-0.003	-0.003
α_4	-0.291	-0.292	-0.162	-0.163
β_4	0.000	0.000	0.000	0.000
α_5	-0.296	-0.297	-0.165	-0.166
β_5	0.000	0.000	0.000	0.000

Table 4: The dividend sensitivity under the assumption that the pure volatility surface is invariant to the dividends.

	Flat surface dividends corrected	Flat surface dividends not corrected	Non-flat surface dividends corrected	Non-flat surface dividends not corrected
α_1	-0.178	-0.063	-0.056	0.049
β_1	-0.018	0.085	0.031	0.131
α_2	-0.185	-0.185	-0.082	-0.082
β_2	-0.007	-0.007	0.019	0.019
α_3	-0.026	0.026	0.010	0.010
β_3	0.000	0.000	-0.002	-0.002
α_4	0.026	0.026	0.010	0.010
β_4	0.000	0.000	-0.002	-0.002
α_5	-0.001	-0.001	-0.003	-0.003
β_5	0.000	0.000	0.000	0.000

Table 5: The dividend sensitivity under the assumption that the market implied volatilities are invariant to the dividends.

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[Whi12b] Richard White. Local volatility. OG notes, OpenGamma, 2012. `http://developers.opengamma.com/quantitative-research/Local-Volatility-OpenGamma.pdf`.

[Whi12c] Richard White. Mixed log-normal volatility model. Technical report, OpenGamma, 2012. `http://developers.opengamma.com/quantitative-research/Mixed-Log-Normal-Volatility-Model-OpenGamma.pdf`.

τ	5M	11M	17M	23M	29M
α	3.0	2.0	1.0	0.0	0.0
β	0.00	0.02	0.03	0.04	0.05

Table 6: The dividend structure. Dividend payments are every six months, with the first payment in five months being purely cash (assume the dividend has been announced). The cash dividends fall off to zero and are replaced by proportional dividends - there is no point specifying these beyond the expiry of the variance swap as they have no affect on its price.

Expiry \index	1	2	3	4	5	6	7	8
1W	50	55	60	65	70	75	80	
2W	50	55	60	65	70	75	80	
1M	50	55	60	65	70	75	80	
3M	40	50	60	70	80	90		
6M	40	50	60	70	80	90	100	
1Y	40	50	60	70	80	90	100	
2Y	20	40	55	65	75	90	105	125

Table 7: The strikes at each expiry.

Expiry\index	1	2	3	4	5	6	7	8
1W	0.404	0.407	0.408	0.410	0.412	0.413	0.414	
2W	0.404	0.406	0.408	0.410	0.411	0.413	0.414	
1M	0.404	0.406	0.408	0.410	0.411	0.413	0.414	
3M	0.398	0.404	0.408	0.411	0.414	0.416		
6M	0.423	0.426	0.428	0.430	0.431	0.432	0.433	
1Y	0.439	0.442	0.443	0.443	0.444	0.444	0.444	
2Y	0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450

Table 8: The market implied volatilities by expiry and strike-index when the pure implied volatility surface is flat at 0.45 and the dividends are as given in table 6. We see that the implied volatility of low strike and expiry options are dragged down.

Expiry\index	1	2	3	4	5	6	7	8
1W	0.607	0.507	0.371	0.303	0.339	0.440	0.529	
2W	0.531	0.432	0.340	0.303	0.320	0.373	0.446	
1M	0.447	0.372	0.324	0.303	0.309	0.333	0.369	
3M	0.469	0.366	0.314	0.302	0.319	0.350		
6M	0.408	0.343	0.311	0.301	0.307	0.319	0.337	
1Y	0.438	0.331	0.310	0.301	0.300	0.304	0.311	
2Y	0.465	0.349	0.317	0.305	0.299	0.297	0.299	0.305

Table 9: The market implied volatilities by expiry and strike-index when the pure implied volatility surface is given by a mixed log-normal model and there are no dividends.

Expiry\index	1	2	3	4	5	6	7	8
1W	0.566	0.480	0.349	0.277	0.316	0.421	0.505	
2W	0.500	0.410	0.315	0.276	0.296	0.354	0.427	
1M	0.423	0.348	0.297	0.276	0.284	0.310	0.351	
3M	0.440	0.340	0.286	0.276	0.297	0.331		
6M	0.379	0.318	0.292	0.288	0.299	0.315	0.337	
1Y	0.409	0.314	0.298	0.295	0.299	0.306	0.315	
2Y	0.432	0.329	0.305	0.298	0.297	0.300	0.306	0.315

Table 10: The market implied volatilities by expiry and strike-index when the pure implied volatility surface is given by a mixed log-normal model and the dividends re as given in table 6.