A NEW CLASS OF LOCAL CORRELATION MODELS

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Abstract. Allowing correlation to be local, i.e., state-dependent, in multi-asset models allows better hedging by incorporating correlation moves in the delta. When options on a basket, be it a stock index, a cross FX rate, or an interest rate spread, are liquidly traded, one may want to calibrate a local correlation to these option prices. Only two admissible models, i.e., models that calibrate to the basket smile, have been suggested in the past. Depending on market conditions, both models may actually not be always admissible and, when they are, they impose a particular dependency of the correlation matrix on the asset values that one has no reason to undergo. In this article we introduce a new class of local correlation models that include the two existing models as special cases. Not all models in this new family are guaranteed to be always admissible but with this new family at hand, one has more chances to pick a model that not only is admissible but also has extra desirable properties, like fitting a view on correlation skew, mimicking historical correlation, or matching prices of exotic options. The models are described by nonlinear SDEs and built using the particle method. It is straightforward to generalize *at no extra cost* this construction of admissible models to the cases of (i) models that combine stochastic interest rates, stochastic dividend yield, local stochastic volatility, and local correlation; and (ii) single asset path-dependent volatility models. Our numerical tests in the case of the FX smile triangle problem show the wide variety of admissible local correlations and give insight on lower bounds/upper bounds on general multi-asset option prices given the surface of implied volatilities of a basket and those of its constituents.

1. Introduction

Many investment banks use a multi-asset version of the Dupire local volatility model to price multi-asset derivatives. Most of the time, the correlation matrix is assumed to be constant, e.g., some constant historical correlation ρ^{hist}. In the equity market, since banks usually "sell correlation," i.e., sell products that have a positive sensitivity to correlation, they tend to overprice correlation and often use a convex combination of ρ^{hist} and the matrix $\mathbf{1}$ representing full correlation of the assets, whose all entries are equal to one:

$$\rho = (1-\lambda)\rho^{\text{hist}} + \lambda\mathbf{1}, \qquad \lambda \in [0,1]$$

However, such a model is not able to reproduce the market smile of implied volatilities of stock index options: typically, when a constant correlation is picked to match the price of the at-the-money implied volatility of the index, it generates a skew which is much smaller (roughly twice smaller) than the market skew. Stated otherwise, the smile of index options contains information on how much more correlated its constituents are in a bearish market, and how less they are in a bullish one.

Local correlation models, in which the correlation matrix is allowed to be state-dependent:

$$\rho(t, S_t^1, \dots, S_t^N)$$

are able to capture this information. They are of high practical importance, not only because they include correlation variability in option prices, but mainly because they allow better hedging by incorporating correlation moves in the delta. This is crucial for short cross gamma positions, where an underestimation of correlation in periods of crises yields a daily P&L bleeding that can only be stopped by occuring a large remarking-to-market loss. Many investment banks have been affected by this effect in 2008 after the bankruptcy of Lehman Brothers. Like the local volatility model, local correlations models do not aim at describing the real world dynamics of the assets, but at helping traders risk-manage their correlation positions, especially during crises. Local correlation models are also very useful in the context of foreign exchange (FX) options. They allow to build models that are consistent with the market smiles of two FX rates, and the market smile of the cross rate. They are also used in interest rates to calibrate to spread options prices.

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	Langnau [25]	Reghai [26]	Guyon and Henry-Labordère [16]	New model
Function of basket value	Basket variance	Correlation, λ	Correlation, λ	$a + b\lambda$
Function of all underlying assets	Correlation, λ	Basket variance	Basket variance	a, b, basket variance and correlation
Possibly function of any path-dependent variable	/	/	/	a, b, basket variance and correlation
Calibration method	Closed form	Fixed point	Particle method	Particle method
Correlation candidate built explicitly	Yes	No	Yes, time step by time step	Yes, time step by time step
Avoids computing implied volatilities	Yes	No	Yes	Yes
Number of degrees of freedom	0	0	0	An infinity: all functions a and b

TABLE 1. Summary of models and methods for calibrating to basket smile. The correlation matrix $\rho = (1-\lambda)\rho^0 + \lambda\rho^1$ lies on the line defined by two fixed correlation matrices ρ^0 and ρ^1.

Let us say that a local correlation model is admissible if it calibrates to the market smile of the index. To the best of our knowledge, only two admissible models have been suggested so far in the literature. In both models the correlation matrix $\rho = (1-\lambda)\rho^0 + \lambda\rho^1$ is assumed to lie on the line defined by two fixed correlation matrices ρ^0 and ρ^1. The first model, proposed by Langnau [25], assumes that the instantaneous variance of a stock index in the multi-asset local volatility-local correlation model is "local in index," i.e., depends on the stocks only through the index value. The second model, presented in Reghai [26] and Guyon and Henry-Labordère [16], assumes that the instantaneous correlation itself (or equivalently λ) is local in index. It may seem enough to have those two models at hand. However, they both have drawbacks. First, both models may actually fail to be admissible. In [25] and [16], a unique correlation candidate is explicitly built and may fail to be positive semi-definite (PSD). Then one projects the candidate onto the set of correlation matrices, and the resulting model does not perfectly calibrate. (The correlation candidate has more chances to be PSD in the second model, see Equation (6.1)). In [26], the correlation matrix is built by solving a fixed point problem that may have no solution. (Precisely it has no solution when the correlation candidate exhibited in [16] is not PSD.) Second, even if both models are admissible, there is no reason why one would undergo either correlation structure. For instance, the resulting correlation may have a weird skew (dependence on the asset values), or its skew may be far from the one that is historically observed, or it may generate prices of other options that are far from market quotes.

In this article, we build a new family of local correlation models using the particle method. This family is parameterized by two functions a and b that depend on time and on the values of all the underlying assets. Instead of assuming that the basket variance or the correlation (or equivalently λ) is local in index, we assume that $a + b\lambda$ is. The two existing models are just two particular points in this family: they correspond to two particular choices of (a, b). We easily handle path-dependent correlation by allowing a and b to depend as well on any set of path-dependent variables. Table 1 helps compare the two existing models with the new family of models.

Using the new family, one can now design one's favorite local correlation model in order to satisfy desirable properties $\mathcal{P}$, such as matching a view on a correlation skew, reproducing some features of historical correlation, calibrating to other option prices, etc. *on top of* reproducing the market smile of the basket, be it a stock index, a cross FX rate, an interest rate spread, etc. Not all models in the family are admissible: for a given (a, b), the particle method generates an explicit local correlation candidate, and admissible models correspond to those pairs (a, b) for which the candidate is PSD at all times and for all asset values. If for some time and asset values a correlation candidate fails to be PSD, we project it onto the set of correlation matrices and carry on using the particle method. The resulting model is not perfectly admissible, but the imperfect calibration may be accurate enough for trading purposes, for instance when the correlation candidate fails to be PSD only for unlikely asset values (see examples in Section 10).

Another way of jointly calibrating to the smile of a basket and to the smiles of its components is described in [23] where Jourdain and Sbai build an incomplete stochastic volatility-stochastic correlation model by following a top-down approach in which the level of a stock index induces some feedback on the dynamics of its constituents. Attempts to approximately calibrate to a triangle of FX market smiles in a symmetric way include [7], where a multi-Heston model with constant correlations is used. Reghai [26] considers the pricing of options on worst-of in a model where the local correlation depends on the stocks only through the worst performance of the basket constituents and suggests a historical calibration procedure. Delanoe [8] addresses the question of calibrating such a model to option prices and discusses stochastic volatility extensions of local correlation models. In the context of constant correlation, Avellaneda et al. [3] are first to give the formula for the equivalent local volatility of a basket of stocks (see (9.2)), and estimate it using short term asymptotics at order zero, namely Varadhan's formula and the method of steepest descent. The expansion at order one, as well as an extension to local in index correlation models, are proved in [19]. Durrleman and El Karoui [11] price options written on a domestic asset based on implied volatilities of options on the same asset expressed in a foreign currency and the exchange rate and, given a local correlation, derive explicit formulas to compute the at-the-money implied volatility, skew, convexity, and term structure for short maturities. In [4], Cont and Deguest use a random mixture of reference models to build a multi-asset model consistent with a set of observed single- and multi-asset derivative prices. Austing [2] provides an analytic formula for a joint probability density such that all three market smiles in a FX triangle are repriced. A few stochastic correlation models have also been suggested and analyzed in the literature, including [14, 6, 1].

The paper is structured as follows. In Section 3, we introduce our new family of local correlation models in the simple context of the FX triangle smile calibration problem, briefly recalled in Section 2. In Section 4 we show how easy it is to build this new family of local correlations step by step from inception to maturity using the particle method. We highlight some key examples in Section 5. Important links between the various admissible local correlations are investigated in Section 6. In Section 7 we give an intuition of the reason why an inadequate joint extrapolation of local volatilities may lead to the non-existence of (strictly) admissible local correlation models. Section 8 deals with the impact of correlation on the prices of multi-asset options, with a reminder on implied correlation *à la* Dupire [10] and a new formula *à la* Gatheral [13]. In Section 9 we show how to build our new family of local correlation models in the context of the N-dimensional stock index smile calibration problem. In Section 10, our numerical examples in the FX context show the wide variety of admissible correlations and give insight on lower bounds and upper bounds on prices of multi-asset options when the smile of a basket and the smiles of its constituents are given. In Section 11 we generalize to models that combine stochastic interest rates, stochastic dividend yield, local stochastic volatility, and local correlation. Finally, in Section 12, we show how to easily adapt the technique presented in this article to build single asset path-dependent volatility models that calibrate to the smile, before we conclude in Section 13. The proofs are gathered in the appendix.

2. The FX triangle smile calibration problem

Let us introduce our new family of local correlation models in the simple context of the FX triangle smile calibration problem. Section 9 deals with the general N-dimensional basket case. Let S^1, S^2 be two FX rates, and $S^{12} = S^1/S^2$ be the cross rate. One can think of $S^1 = \text{EUR/USD}$, $S^2 = \text{GBP/USD}$ and $S^{12} = \text{EUR/GBP}$. Assume we know from the market the surfaces of implied volatility for S^1, S^2 and S^{12} until some maturity T, and that those surfaces are jointly arbitrage-free. They correspond to three local volatility surfaces that we denote by $\sigma_1(t, S^1)$, $\sigma_2(t, S^2)$, and $\sigma_{12}(t, S^{12})$. Assume the following model $\mathcal{M}_\rho$ for the dynamics of S^1 and S^2:

$$\begin{aligned} dS_t^1 &= \left(r_t^d - r_t^1\right) S_t^1\,dt + \sigma_1(t, S_t^1) S_t^1\,dW_t^1 \\ dS_t^2 &= \left(r_t^d - r_t^2\right) S_t^2\,dt + \sigma_2(t, S_t^2) S_t^2\,dW_t^2 \\ d\langle W^1, W^2\rangle_t &= \rho(t, S_t^1, S_t^2)\,dt \end{aligned} \tag{2.1}$$

All interest rates are deterministic; both rates S^1 and S^2 follow local volatility dynamics; the two driving processes W^1 and W^2 are Brownian motions under the risk-neutral measure $\mathbb{Q}$ associated to the anchor (domestic) currency (USD in our example); they have a local instantaneous correlation $\rho(t, S_t^1, S_t^2) \in [-1, 1]$.

Model $\mathcal{M}_\rho$ is calibrated to the market smile of the cross rate S^{12} if and only if (see proof in the appendix)

$$\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)-2\rho(t,S_t^1,S_t^2)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right) \tag{2.2}$$

for all $t\in[0,T]$, where $\mathbb{E}^{\mathbb{Q}^f}$ denotes the expectation under the risk-neutral measure $\mathbb{Q}^f$ associated to the foreign currency in S^2 (GBP in our example):

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}}=\frac{S_T^2}{S_0^2}\exp\left(\int_0^T\left(r_t^2-r_t^d\right)dt\right)$$

Equation (2.2) is equivalent to

$$\frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[S_t^2\left(\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)-2\rho(t,S_t^1,S_t^2)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}_\rho^{\mathbb{Q}}\left[S_t^2\,\middle|\,\frac{S_t^1}{S_t^2}\right]}=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right) \tag{2.3}$$

Note that the left hand side of Equation (2.2) depends on the correlation in two ways: (i) explicitly through the random variable $\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$, and (ii) implicitly through the conditional distribution of (S_t^1,S_t^2) given $\frac{S_t^1}{S_t^2}$ under $\mathbb{Q}^f$. To emphasize point (ii), we have written $\mathbb{E}_\rho^{\mathbb{Q}^f}$ instead of $\mathbb{E}^{\mathbb{Q}^f}$.

Let us denote by $\mathcal{C}$ the set of functions $\rho:[0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*\to[-1,1]$. Any $\rho\in\mathcal{C}$ satisfying (2.2) will be called an "**admissible correlation**." In this article, we are mainly interested in the following important practical question **(Q)**: How to build an almost (if not strictly) admissible correlation $\rho(t,S^1,S^2)$ having some of the desirable properties $\mathcal{P}$ listed above?

Remark 1. Two important and difficult theoretical questions are the following ones:

(1) How to verify that the three *surfaces* of implied volatility for S^1, S^2 and S^{12} are jointly arbitrage-free? How to detect joint arbitrages?
(2) Assuming no arbitrage, under which condition on $\sigma_1(t,S^1)$, $\sigma_2(t,S^2)$, and $\sigma_{12}(t,S^{12})$ does there exist an admissible correlation?

The non-existence of an admissible correlation may be due to the extrapolations of the three local volatilities (see Section 7). In practice, this means that "good" correlation candidates $\rho(t,S^1,S^2)$, such as the ones exhibited in [25, 16] may fail to be true correlations, i.e., to belong to $[-1,1]$, but only for very small or very large values of S^1 or S^2 or S^{12}. (The correlation candidate exhibited in [16] is more likely to belong to $[-1,1]$ than the one exhibited in [25], see Equation (6.1).) In practice, this may not be a problem: the "good" correlation candidates, when capped to $+1$ and floored to -1, become "almost" admissible correlations, meaning that the smile of the cross rate is correctly reproduced almost everywhere, except maybe very far from the money. One might also want to modify the extrapolations of the local volatilities that appear to be problematic. For instance, in the situation of Figure 10.5, one may want to modifiy the low strikes extrapolations of σ_1 and σ_2.

Remark 2. We may have started with a general stochastic process (ρ_t) for the correlation, that possibly depends on extra sources of randomness. In this situation, the calibration condition (2.2) still holds with $\rho(t,S_t^1,S_t^2)\equiv\mathbb{E}_{\rho_t}^{\mathbb{Q}^f}\left[\rho_t\,\middle|S_t^1,S_t^2\right]=\mathbb{E}_{\rho_t}^{\mathbb{Q}}\left[\rho_t\,\middle|S_t^1,S_t^2\right]$. This result is not completely trivial, as one needs to show that $\mathbb{E}_{\rho_t}^{\mathbb{Q}^f}$ can be replaced by $\mathbb{E}_\rho^{\mathbb{Q}^f}$. This follows from Gyöngy's theorem - see Section 8.2 for a simple derivation. As a consequence, if there exists an admissible correlation process ρ_t, then there exists an admissible local correlation $\rho(t,S^1,S^2)$: as far as calibration to the market smile of the cross rate is concerned, assuming a local correlation $\rho(t,S^1,S^2)$ is not restrictive.

3. A new representation of admissible correlations

To answer Question **(Q)**, we now introduce a new representation of admissible correlations. We consider an *admissible* correlation $\rho\in\mathcal{C}$. We say that a function is "local in X" if it is a function of (t,X) only, say $f(t,X)$. When $X=S^1/S^2$, we also say "local in cross." Let us pick two functions $a(t,S^1,S^2)$ and $b(t,S^1,S^2)$ such that b does not vanish and

$$a(t,S^1,S^2)+b(t,S^1,S^2)\rho(t,S^1,S^2)\equiv f\left(t,\frac{S^1}{S^2}\right)$$

is local in cross. We can always do so, by choosing for instance $b \equiv 1$ and $a(t, S^1, S^2) = f\left(t, \frac{S^1}{S^2}\right) - \rho(t, S^1, S^2)$ for some function f. The two already mentioned existing approaches for trying to build admissible correlations correspond to two special cases of this assumption: when $a \equiv 0$ and $b \equiv 1$, one assumes that the correlation itself is local in cross, see [16, 26]; when $a = \sigma_1^2 + \sigma_2^2$ and $b = -2\sigma_1\sigma_2$, one assumes that the instantaneous variance of the cross rate is local in cross, see [24, 25]. We will come back to both examples and introduce new ones in Sections 5 and 10. Then[1]

$$\sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]$$
$$= \mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2 + \sigma_2^2 + 2\frac{a}{b}\sigma_1\sigma_2\,\middle|\,\frac{S_t^1}{S_t^2}\right] - 2\,(a + b\rho)\left(t, \frac{S_t^1}{S_t^2}\right)\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\frac{\sigma_1\sigma_2}{b}\,\middle|\,\frac{S_t^1}{S_t^2}\right]$$

As a consequence $\rho = \rho_{(a,b)}$ satisfies $\rho_{(a,b)} \in \mathcal{C}$ and

$$\rho_{(a,b)}(t, S_t^1, S_t^2) = \frac{1}{b(t, S_t^1, S_t^2)}\left(\frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\sigma_1^2 + \sigma_2^2 + 2\frac{a}{b}\sigma_1\sigma_2\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\frac{\sigma_1\sigma_2}{b}\,\middle|\,\frac{S_t^1}{S_t^2}\right]} - a(t, S_t^1, S_t^2)\right) \tag{3.1}$$

We have thus proved that *any* admissible correlation is of the above type. Conversely, if a function $\rho_{(a,b)} \in \mathcal{C}$ satisfies (3.1), then it is an admissible correlation. We call (3.1) the "**local in cross $a + b\rho$ representation**" of admissible correlations. Question **(Q)** can now be restated as follows: Can we find a pair of functions (a, b) such that $\rho_{(a,b)}$ is (at least almost) admissible and has some of the desirable properties $\mathcal{P}$?

Note that (3.1) is a circular equation: the right hand side of (3.1) depends on $\rho_{(a,b)}$ through the two conditional expectations. To the best of our knowledge, the existence of the nonlinear stochastic differential equations (SDEs) describing the calibrated models

$$\begin{aligned} dS_t^1 &= \left(r_t^d - r_t^1\right) S_t^1\,dt + \sigma_1(t, S_t^1)S_t^1\,dW_t^1 \\ dS_t^2 &= \left(r_t^d - r_t^2\right) S_t^2\,dt + \sigma_2(t, S_t^2)S_t^2\,dW_t^2 \\ d\langle W^1, W^2\rangle_t &= \frac{dt}{b(t, S_t^1, S_t^2)}\left(\frac{\mathbb{E}^{\mathbb{Q}}\left[S_t^2\left(\sigma_1^2 + \sigma_2^2 + 2\frac{a}{b}\sigma_1\sigma_2\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)\mathbb{E}^{\mathbb{Q}}\left[S_t^2\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{2\mathbb{E}^{\mathbb{Q}}\left[S_t^2\frac{\sigma_1\sigma_2}{b}\,\middle|\,\frac{S_t^1}{S_t^2}\right]} - a(t, S_t^1, S_t^2)\right) \end{aligned}$$

is still an open mathematical question. This is common to a variety of smile calibration problems, see for instance [16, 17] for an investigation of some nonlinear SDEs describing models calibrated to a smile - including local stochastic volatility models, with or without stochastic interest rates, and local correlation models - and a discussion and numerical experiments on the existence of a solution. In practice, one may try to build a solution $\rho_{(a,b)} \in \mathcal{C}$ using the particle method [16], as we explain in the next section.

Remark 3. As stated above, one can always require that $b \equiv 1$. Consequently, *any* correlation candidate is also of the subtype $\rho_{(a,1)}$:

$$\rho_{(a,1)}(t, S_t^1, S_t^2) = \frac{\mathbb{E}_{\rho_{(a,1)}}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) + 2a(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}_{\rho_{(a,1)}}^{\mathbb{Q}^f}\left[\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]} - a(t, S_t^1, S_t^2) \tag{3.2}$$

The advantage of dealing with $a + b\rho$ instead of $a + \rho$ is that it includes the common approach where $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$ is assumed to be local in cross, with both $a = \sigma_1^2 + \sigma_2^2$ and $b = -2\sigma_1\sigma_2$ being *independent of* ρ. In general one cannot require that $a \equiv 0$, because it would impose that $\rho(t, S^1, S^2) = 0 \Rightarrow \rho(t, \lambda S^1, \lambda S^2) = 0$ for all $\lambda > 0$. Any admissible correlation satisfying the above condition - in particular any non-vanishing admissible correlation - is also of the subtype $\rho_{(0,b)}$:

$$\rho_{(0,b)}(t, S_t^1, S_t^2) = \frac{\mathbb{E}_{\rho_{(0,b)}}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2b(t, S_t^1, S_t^2)\mathbb{E}_{\rho_{(0,b)}}^{\mathbb{Q}^f}\left[\frac{\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)}{b(t, S_t^1, S_t^2)}\,\middle|\,\frac{S_t^1}{S_t^2}\right]} \tag{3.3}$$

[1] From now on, for the sake of clarity, we may omit the functions arguments t, S_t^1, S_t^2, etc. within conditional expectations in long equations.

Remark 4. Our method allows us to consider more general models where the instantaneous correlation depends on path-dependent variables as well. For instance, we can handle situations where ρ depends not only on (t, S_t^1, S_t^2) but also on the running averages $\frac{1}{t}\int_0^t S_u^1 du$ and $\frac{1}{t}\int_0^t S_u^2 du$, or on moving averages, or on the running minimums and maximums of S^1 and S^2, or on the realized correlation over the past few days, or on the realized volatilities over the past few days, etc. All one has to do is to also include the path-dependent variables in the arguments of the functions a and b.

4. The particle method for local correlation

The particle method for solving various smile calibration problems, including calibration of local stochastic volatility models, with or without stochastic interest rates, and of local correlation models, has been presented in [16]. It was also used in [23] to calibrate a model coupling an index and its constituents. In the context presented in Section 3, the particle algorithm can be described as follows. Let $\{t_k\}$ denote a time discretization of $[0, T]$. We simulate N processes $(S_t^{1,i}, S_t^{2,i})_{1\le i\le N}$ starting from (S_0^1, S_0^2) at time 0 using N independent Brownian motions *under the domestic measure* $\mathbb{Q}$ as follows:

(1) Initialize $k = 1$ and set $\rho_{(a,b)}(t, S^1, S^2) = \frac{\sigma_1^2(0,S^1)+\sigma_2^2(0,S^2)-\sigma_{12}^2\left(0,\frac{S^1}{S^2}\right)}{2\sigma_1(0,S^1)\sigma_2(0,S^2)}$ for all $t \in [t_0 = 0; t_1]$. (At $t = 0$, no conditional expectation is computed so $\rho_{(a,b)}$ does not depend on (a, b).)
(2) Simulate $(S_t^{1,i}, S_t^{2,i})_{1\le i\le N}$ from t_{k-1} to t_k using a discretization scheme - say a log-Euler scheme.
(3) For all S^{12} in a grid $G_{t_k}$of cross rate values, compute

$$
\begin{aligned}
E_{t_k}^{\mathrm{num}}(S^{12}) &= \frac{\sum_{i=1}^N S_{t_k}^{2,i}\left(\sigma_1^2(t_k, S_{t_k}^{1,i}) + \sigma_2^2(t_k, S_{t_k}^{2,i}) + 2\frac{a(t_k,S_{t_k}^{1,i},S_{t_k}^{2,i})}{b(t_k,S_{t_k}^{1,i},S_{t_k}^{2,i})}\sigma_1(t_k, S_{t_k}^{1,i})\sigma_2(t_k, S_{t_k}^{2,i})\right)\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)}{\sum_{i=1}^N S_{t_k}^{2,i}\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)} \\
E_{t_k}^{\mathrm{den}}(S^{12}) &= \frac{\sum_{i=1}^N S_{t_k}^{2,i}\frac{\sigma_1(t_k,S_{t_k}^{1,i})\sigma_2(t_k,S_{t_k}^{2,i})}{b(t_k,S_{t_k}^{1,i},S_{t_k}^{2,i})}\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)}{\sum_{i=1}^N S_{t_k}^{2,i}\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)} \\
f(t_k, S^{12}) &= \frac{E_{t_k}^{\mathrm{num}}(S^{12}) - \sigma_{12}^2\left(t_k, S^{12}\right)}{2E_{t_k}^{\mathrm{den}}(S^{12})}
\end{aligned}
$$

interpolate and extrapolate $f(t_k, \cdot)$, for instance using cubic splines, and, for all $t \in [t_k, t_{k+1}]$, set

$$\rho_{(a,b)}(t, S^1, S^2) = \frac{1}{b(t, S^1, S^2)}\left(f\left(t_k, \frac{S^1}{S^2}\right) - a(t, S^1, S^2)\right)$$

(4) Set $k := k + 1$. Iterate steps 2 and 3 up to the maturity date T.

Here, $\delta_{t,N}(x) = \frac{1}{h_{t,N}}K\left(\frac{x}{h_{t,N}}\right)$ is an approximation of the Delta dirac function; K is a fixed, symmetric, non-negative kernel; $h_{t,N}$ is a bandwidth that tends to zero as N grows to infinity. $E_t^{\mathrm{num}}(S^{12})$ and $E_t^{\mathrm{den}}(S^{12})$ approximate the conditional expectations $\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) + 2\frac{a(t,S_t^1,S_t^2)}{b(t,S_t^1,S_t^2)}\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2} = S^{12}\right]$ and $\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\,\middle|\,\frac{S_t^1}{S_t^2} = S^{12}\right]$ respectively. Alternative methods for estimating such conditional expectations include B-spline techniques as explained in a recent work by Corlay [5]. Implementation details can be found in Section 10.

5. Some examples of pairs of functions (a, b)

As already mentioned in Section 3, the two existing approaches for trying to build admissible correlations are special cases of the local in cross $a + b\rho$ representation:

- $a \equiv 0$ and $b \equiv 1$: In this case [16, 26], one assumes that the correlation itself is local in cross:

$$\rho_{(0,1)}(t, S_t^1, S_t^2) = \frac{\mathbb{E}_{\rho_{(0,1)}}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}_{\rho_{(0,1)}}^{\mathbb{Q}^f}\left[\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]} \tag{5.1}$$

a	b	$\rho_{(a,b)}(t, S_t^1, S_t^2)$
0	σ_1	$\frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\middle\vert\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)}{2\sigma_1(t,S_t^1)\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_2(t,S_t^2)\middle\vert\frac{S_t^1}{S_t^2}\right]}$
0	σ_2	$\frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\middle\vert\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_1(t,S_t^1)\middle\vert\frac{S_t^1}{S_t^2}\right]\sigma_2(t,S_t^2)}$
σ_1^2	$-2\sigma_1\sigma_2$	$\frac{\sigma_1^2(t,S_t^1)+\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_2^2(t,S_t^2)\middle\vert\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)}{2\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}$
σ_2^2	$-2\sigma_1\sigma_2$	$\frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_1^2(t,S_t^1)\middle\vert\frac{S_t^1}{S_t^2}\right]+\sigma_2^2(t,S_t^2)-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)}{2\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}$

TABLE 2. Examples of simple but symmetry-breaking choices of (a, b) and the corresponding correlation candidates

We then speak of the "**local in cross ρ model**" or "local in cross correlation model." If at some date $t < T$, $\rho_{(0,1)}(t, S^1, S^2) \notin [-1, 1]$ for some FX rate values S^1, S^2, then the trial is a failure: $\rho_{(0,1)}$ is not admissible.

- $a = \sigma_1^2 + \sigma_2^2$ and $b = -2\sigma_1\sigma_2$: In this case, one assumes that the instantaneous variance of the cross rate is local in cross. This is in the spirit of [25] and has been studied in this FX context in [24]. In this case, the cross rate follows a local volatility model, we speak of the "**local in cross volatility model**," and denote by ρ^* the correlation candidate:

$$\rho^*(t, S_t^1, S_t^2) = \frac{\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)} \tag{5.2}$$

Note that this is the only situation where no estimation of conditional expectation (given the value of S_t^1/S_t^2) is needed. As a consequence, ρ^* is well defined even if it exits the interval $[-1, 1]$. If at some date $t < T$, $\rho^*(t, S^1, S^2) \notin [-1, 1]$ for some S^1, S^2, then ρ^* is not admissible.

Another natural choice of (a, b) is the following:

- $a \equiv 0$ and $b = \sigma_1\sigma_2$: In this case, one assumes that the local covariance $\rho(t, S^1, S^2)\sigma_1(t, S^1)\sigma_2(t, S^2)$ of increments of S^1 and S^2 is local in cross. We then speak of the "**local in cross covariance model**." This choice defines a model calibrated to the three FX smiles if and only if

$$\frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(0,\sigma_1\sigma_2)}}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2)\middle\vert \frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)} \in [-1, 1]$$

Other simple but symmetry-breaking choices of (a, b) are given in Table 5, together with the corresponding correlation candidates.

Remark 5. Had we only considered the types $\rho_{(a,1)}$ (which span the space of all admissible correlations), by blindly applying (3.2), we would have get for the local in cross volatility assumption $a = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 - \rho$ and the following correlation candidate

$$\rho_{(a,1)}(t, S_t^1, S_t^2) = \frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,1)}}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) + 2a(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\middle\vert \frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,1)}}\left[\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)\middle\vert \frac{S_t^1}{S_t^2}\right]} - a(t, S_t^1, S_t^2)$$

Though we can recover (5.2) from the above equation, it is much more convenient to consider the $a + b\rho$ formulation, because in this case both a and b are independent of ρ.

Remark 6. In general, the instantaneous volatility of the cross $\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ in the model depends jointly on S^1 and S^2, and not only on S^{12}, whereas the instantaneous volatility of S^1 (resp. S^2) depends only on S^1 (resp. S^2). This means that in general the model is not symmetric in the three FX rates. If ρ^* takes values in $[-1, 1]$, then the local in cross volatility model is the unique local volatility-local correlation model that is symmetric in the three FX rates. Otherwise, no such model exists. The asymmetry may not necessarily be seen as a drawback: in cases where a currency is "stronger" than the other two, it is natural to choose it as the anchor rate and it might make sense to price and hedge in a model where the volatility of the cross is actually a function of the two rates against the strong currency separately, not only of the cross rate. Note however that if one takes EUR or GBP as anchor, i.e., domestic currency, instead of USD, one gets another family of admissible local correlation models, and different prices for exotic options. See for instance [7] for a symmetric way to *approximately* calibrate to a triangle of FX market smiles.

6. Some links between local correlations

Assume that $\rho_{(a,b)}$ is an *admissible* correlation. We can express the affine transform $a + b\rho_{(a,b)}$ of $\rho_{(a,b)}$ as an average of the same affine transform of the correlation candidate ρ^* (even if ρ^* takes values outside $[-1,1]$):

$$\left(a + b\rho_{(a,b)}\right)\left(t, \frac{S_t^1}{S_t^2}\right) = \frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\left(a+b\rho^*\right)\left(t, S_t^1, S_t^2\right)\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\,\middle|\,\frac{S_t^1}{S_t^2}\right]} \equiv \mathbb{E}^{\mathbb{Q}^{\frac{\sigma_1\sigma_2}{b}}}_{\rho_{(a,b)}}\left[\left(a+b\rho^*\right)\left(t, S_t^1, S_t^2\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right]$$

where

$$\frac{d\mathbb{Q}^{\frac{\sigma_1\sigma_2}{b}}}{d\mathbb{Q}^f} \equiv \frac{\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}}{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\right]}$$

In particular, if $\rho_{(0,1)}$ is an admissible correlation, $\rho_{(0,1)}$ is a weighted average of ρ^* on each line where $\frac{S^1}{S^2}$ is constant:

$$\rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) = \frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(0,1)}}\left[\rho^*\left(t, S_t^1, S_t^2\right)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(0,1)}}\left[\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]} \equiv \mathbb{E}^{\mathbb{Q}^{\sigma_1\sigma_2}}_{\rho_{(a,b)}}\left[\rho^*\left(t, S_t^1, S_t^2\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right] \tag{6.1}$$

This has two consequences:

- If $\rho_{(0,1)}$ is an admissible correlation, then its image, i.e., the range of values it takes, is included in the image of ρ^*.
- The smallest time $\tau_{\rho_{(0,1)}}$ at which $\rho_{(0,1)}$ fails to be a correlation function is larger than or equal to the smallest time τ_{ρ^*} at which ρ^* fails to be a correlation function, where

$$\tau_\rho = \inf\left\{t \in [0,T] \,|\, \exists S^1, S^2 > 0,\ \rho(t, S^1, S^2) \notin [-1,1]\right\} \tag{6.2}$$

As for the volatility of S^1/S^2, if we denote

$$\sigma^2_{(a,b)}(t, S_t^1, S_t^2) = \sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho_{(a,b)}(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)$$

we have by construction

$$\sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma^2_{(a,b)}(t, S_t^1, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right] \tag{6.3}$$

In the particular case where σ_1 and σ_2 depend only on t (no skew on S^1 nor S^2), then (6.3) simply reads

$$\sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right) = \sigma_1(t)^2 + \sigma_2(t)^2 - 2\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\rho_{(a,b)}(t, S_t^1, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]\sigma_1(t)\sigma_2(t)$$

i.e., using (5.1) and (5.2),

$$\rho^*(t, S_t^1, S_t^2) = \rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\rho_{(a,b)}(t, S_t^1, S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right] \tag{6.4}$$

Note that in this case, all seven examples of Section 5 boil down to the same correlation model, where the correlation is local in cross. In this situation, $\rho_{(0,1)} = \rho^*$ is, among all the admissible correlations $\rho(t, S^1, S^2)$ the one with the smallest image. This means that if $\rho_{(0,1)} = \rho^*$ is not admissible, then no correlation $\rho(t, S^1, S^2)$ is admissible; $\rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) = \rho^*\left(t, \frac{S_t^1}{S_t^2}\right) > 1$ corresponds to the situation where

$$|\sigma_1(t) - \sigma_2(t)| > \sigma_{12}\left(t, \frac{S_t^1}{S_t^2}\right)$$

$\rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) = \rho^*\left(t, \frac{S_t^1}{S_t^2}\right) < -1$ corresponds to the situation where

$$\sigma_1(t) + \sigma_2(t) < \sigma_{12}\left(t, \frac{S_t^1}{S_t^2}\right)$$

Equation (6.4) tells us that in the case where $S^1$and S^2 have no skew all admissible correlations have same average value under $\mathbb{Q}^f$ over each line where S^1/S^2 is constant, and this common average value is given by $\rho_{(0,1)}$.

7. Joint extrapolation of local volatilities

As stated in Section 2, failure to be a correlation fonction, i.e., the fact that $\tau_\rho < T$ (see Equation (6.2)), may be the consequence of an inadequate joint extrapolation of the three local volatilities σ_1, σ_2 and σ_{12}. To give an intuition of this, let us look at the particular case of the correlation candidate ρ^*. It takes values in $[-1, 1]$ if and only if for all t, S^1, S^2

$$\left|\sigma_1(t, S^1) - \sigma_2(t, S^2)\right| \leq \sigma_{12}\left(t, \frac{S^1}{S^2}\right) \leq \sigma_1(t, S^1) + \sigma_2(t, S^2)$$

This is equivalent to saying that

$$\underline{\sigma}_{12} \leq \sigma_{12} \leq \overline{\sigma}_{12} \tag{7.1}$$

where the functions $\underline{\sigma}_{12}$ and $\overline{\sigma}_{12}$ are defined by

$$\begin{aligned} \underline{\sigma}_{12}(t, S^{12}) &= \sup_{S^2>0} \left|\sigma_1(t, S^2 S^{12}) - \sigma_2(t, S^2)\right| \\ \overline{\sigma}_{12}(t, S^{12}) &= \inf_{S^2>0} \left\{\sigma_1(t, S^2 S^{12}) + \sigma_2(t, S^2)\right\} \end{aligned}$$

A first problem is that there is no guarantee that $\underline{\sigma}_{12} \leq \overline{\sigma}_{12}$. If this does not hold, ρ^* is guaranteed to be inadmissible, whatever the local volatility σ^{12} of the cross rate. For instance, the extrapolations of σ_1 and σ_2 may be such that $\underline{\sigma}_{12}(t, S^{12}) \equiv +\infty$, if at least one of both local volatilities is unbounded, because in the definition of $\underline{\sigma}_{12}$ we take the supremum over *all* values of S^2, even extremely unlikely values. For instance, it is common to build extrapolations where asymptotically the squared local volatility is an affine function of the log-spot. When the asymptotic slopes of σ_1^2 and σ_2^2 differ, then $\underline{\sigma}_{12}(t, S^{12}) \equiv +\infty$. In such a case ρ^* must cross the $+1$ boundary for S^1 or S^2 far enough from the money. However, this may not be a problem in practice, if ρ^* lies in $[-1, 1]$ for a broad range of likely values of S^1 and S^2.

A second problem is that, even if $\underline{\sigma}_{12} \leq \overline{\sigma}_{12}$, the market local volatility of the cross rate may fail to lie in between the two. This may indicate an inadequate joint extrapolation of the three local volatilities involved.

For general (a, b), there is no necessary and sufficient condition as simple as (7.1) for $\tau_{\rho_{(a,b)}}$ to be greater than T, i.e., for $\rho_{(a,b)}$ to be well defined over $[0, T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$. The above reasoning shows that even with "good" candidates (a, b), one may have $\tau_{\rho_{(a,b)}} = 0$ in theory, because $\rho_{(a,b)}$ exits $[-1, 1]$ for extremely small or large values of S^1, S^2, or S^{12}. However, again, this may not be a problem in practice if, in the Monte Carlo procedure, only a small proportion of paths (if any) reaches the region where $\rho_{(a,b)}$ is capped to 1 or floored to -1. See Section 10 for many numerical examples.

8. Price impact of correlation

Different choices of admissible $\rho_{(a,b)}$ will lead to different prices for exotic options on S^1 and S^2, while still producing same prices for vanilla options on S^1, vanilla options on S^2, and vanilla options on the cross rate S^1/S^2. In this section we analyze the impact of $\rho_{(a,b)}$ on the price of options on S^1 and S^2. This helps developing an intuition of the model, and choosing the right model for pricing and hedging a given option.

8.1. **The price impact formula.** Here we follow a reasoning inspired by El Karoui et al. [12] and Dupire [10]. We consider the option with payout $g(S_T^1, S_T^2)$ in domestic currency (USD in our example) at maturity T. Let us pick a correlation function $\rho_0(t, S^1, S^2)$, and denote by $P_0(t, S^1, S^2)$ the corresponding pricing function in domestic currency (Model $\mathcal{M}_{\rho_0}$). P_0 is solution to the backward PDE

$$(\partial_t + \mathcal{L})P_0 = 0 \tag{8.1}$$

$$P_0(T, S^1, S^2) = g(S^1, S^2) \tag{8.2}$$

where

$$\mathcal{L} = \frac{1}{2}\sigma_1^2(t,S^1)(S^1)^2\partial^2_{S^1} + \frac{1}{2}\sigma_2^2(t,S^2)(S^2)^2\partial^2_{S^2} + \rho_0(t,S^1,S^2)\sigma_1(t,S^1)\sigma_2(t,S^2)S^1S^2\partial^2_{S^1S^2} \\ + (r_t^d - r_t^1)S^1\partial_{S^1} + (r_t^d - r_t^2)S^2\partial_{S^2} - r_t^d.$$

We apply Itô's formula to the function P_0 and a process (S_t^1, S_t^2) whose dynamics derive, not from ρ_0, but from a general correlation process ρ_t (Model $\mathcal{M}_{\rho_t}$). Using (8.1) and (8.2), we have

$$\begin{aligned} & D^d_{0T}g(S^1_T,S^2_T) - P_0(0,S^1_0,S^2_0) \\ = & \int_0^T D^d_{0t}\Big(\partial_t P_0(t,S^1_t,S^2_t) + \frac{1}{2}\sigma_1^2(t,S^1_t)(S^1_t)^2\partial^2_{S^1}P_0(t,S^1_t,S^2_t) \\ & + \frac{1}{2}\sigma_2^2(t,S^2_t)(S^2_t)^2\partial^2_{S^2}P_0(t,S^1_t,S^2_t) + \rho_t\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_0(t,S^1_t,S^2_t) \\ & + (r^d_t - r^1_t)S^1_t\partial_{S^1}P_0(t,S^1_t,S^2_t) + (r^d_t - r^2_t)S^2_t\partial_{S^2}P_0(t,S^1_t,S^2_t) - r^d_tP_0(t,S^1_t,S^2_t)\Big)\,dt + M_t \\ = & \int_0^T D^d_{0t}(\rho_t - \rho_0(t,S^1_t,S^2_t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_0(t,S^1_t,S^2_t)dt + M_t \end{aligned}$$

where $D^d_{0T} = \exp\left(-\int_0^T r^d_t dt\right)$ is the (deterministic) discount factor and

$$M_t = \int_0^T D^d_{0t}\partial_{S^1}P_0(t,S^1_t,S^2_t)\sigma_1(t,S^1_t)S^1_t dW^1_t + \int_0^T D^d_{0t}\partial_{S^2}P_0(t,S^1_t,S^2_t)\sigma_2(t,S^2_t)S^2_t dW^2_t$$

is a local martingale under the risk-neutral measure $\mathbb{Q}$. Assuming that it is a true martingale and taking expectations, we get

$$\begin{aligned} & D^d_{0T}\mathbb{E}^{\mathbb{Q}}_{\rho_t}[g(S^1_T,S^2_T)] - P_0(0,S^1_0,S^2_0) \\ & \qquad = \mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T D^d_{0t}(\rho_t - \rho_0(t,S^1_t,S^2_t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_0(t,S^1_t,S^2_t)dt\right] \end{aligned} \tag{8.3}$$

We use the notation $\mathbb{E}^{\mathbb{Q}}_{\rho_t}$ to emphasize that the process (S^1_t, S^2_t) is simulated under Model $\mathcal{M}_{\rho_t}$. This is interpreted as follows: the price difference between Model $\mathcal{M}_{\rho_t}$ and Model $\mathcal{M}_{\rho_0}$ is the expected value of the integrated discounted tracking error, where the instantaneous tracking error at date t

$$\epsilon_t \equiv (\rho_t - \rho_0(t,S^1_t,S^2_t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_0(t,S^1_t,S^2_t)$$

consists of the spread of the two correlations times the $\mathcal{M}_{\rho_0}$-cross gamma $\partial^2_{S^1S^2}P_0$, times the product of (normal) volatilities $\sigma_1\sigma_2S^1S^2$, where all terms are evaluated at the spots (S^1_t, S^2_t) defined by Model $\mathcal{M}_{\rho_t}$. The interpretation as an error comes from the fact that $\epsilon_t dt$ is the infinitesimal P&L between t and $t+dt$ of a delta-hedged long position in one option, when one uses the pricing function P_0 and the corresponding deltas $\partial_{S^1}P_0$, $\partial_{S^2}P_0$, derived from Model $\mathcal{M}_{\rho_0}$ while the actual dynamics of the assets is given by Model $\mathcal{M}_{\rho_t}$. Equation (8.3) has several interesting consequences that we address below in Sections 8.2, 8.3, and 8.4.

8.2. **Equivalent local correlation.** From (8.3), by conditioning on (S^1_t, S^2_t), we get

$$\begin{aligned} & D^d_{0T}\mathbb{E}^{\mathbb{Q}}_{\rho_t}[g(S^1_T,S^2_T)] - P_0(0,S^1_0,S^2_0) \\ & \quad = \mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T D^d_{0t}(\rho_{\mathrm{loc}}(t,S^1_t,S^2_t) - \rho_0(t,S^1_t,S^2_t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_0(t,S^1_t,S^2_t)dt\right] \end{aligned} \tag{8.4}$$

where

$$\rho_{\mathrm{loc}}(t,S^1_t,S^2_t) \equiv \mathbb{E}^{\mathbb{Q}}_{\rho_t}[\rho_t|S^1_t,S^2_t]$$

is the equivalent local correlation. From Gyöngy's theorem [18], we know that the model, say $\mathcal{M}_{\rho_{\mathrm{loc}}}$, that uses local correlation function ρ_{loc} generates the same distributions for (S^1_t, S^2_t) as Model $\mathcal{M}_{\rho_t}$, for all t. This can be easily rederived by applying (8.4) with the model $\mathcal{M}_{\rho_{\text{loc}}}$ playing the role of $\mathcal{M}_{\rho_0}$:

$$\begin{aligned} &D^d_{0T}\mathbb{E}^{\mathbb{Q}}_{\rho_t}[g(S^1_T,S^2_T)] - P_{\rho_{\text{loc}}}(0,S^1_0,S^2_0) \\ &\quad = \mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T D^d_{0t}(\rho_{\text{loc}}(t,S^1_t,S^2_t) - \rho_{\text{loc}}(t,S^1_t,S^2_t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho_{\text{loc}}}(t,S^1_t,S^2_t)dt\right] = 0 \end{aligned}$$

This proves that all vanilla payoffs $g(S^1_T,S^2_T)$ have identical prices in models $\mathcal{M}_{\rho_t}$ and $\mathcal{M}_{\rho_{\text{loc}}}$, i.e., (S^1_T,S^2_T) have identical distributions under $\mathbb{Q}$ in both models.

8.3. **Implied correlation.** Equation (8.3), or equivalently Equation (8.4), also allows us to define the implied correlation. Given a general model $\mathcal{M}_{\rho_t}$ and a payoff $g(S^1_T,S^2_T)$, we define the implied correlation $\rho(T,g)$ as the value of the constant correlation such that the option has same price in Model $\mathcal{M}_{\rho_t}$ and in the model with constant correlation function, i.e., such that

$$\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T D^d_{0t}(\rho_t - \rho(T,g))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(T,g)}(t,S^1_t,S^2_t)dt\right] = 0$$

or, equivalently,

$$\rho(T,g) = \frac{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T \rho_t D^d_{0t}\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(T,g)}(t,S^1_t,S^2_t)dt\right]}{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\int_0^T D^d_{0t}\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(T,g)}(t,S^1_t,S^2_t)dt\right]} \tag{8.5}$$

This is similar to Dupire's expression of implied volatility as a weighted average of the spot volatility [10]. Note that (8.5) is a fixed point equation, because the right hand side depends on $\rho(T,g)$ as well through the cross gamma $\partial^2_{S^1S^2}P_{\rho(T,g)}$. If (8.5) admits a unique solution, then the implied correlation exists and is uniquely defined.

Following Guyon and Henry-Labordère [15], one can estimate the implied correlation by estimating the fixed point of the mapping

$$\rho \mapsto \int_0^T\int_0^\infty\int_0^\infty \rho_{\text{loc}}(t,S^1,S^2)q_\rho(t,S^1,S^2)dS^1dS^2dt$$

where

$$q_\rho(t,S^1,S^2) = \frac{D^d_{0t}\sigma_1(t,S^1)\sigma_2(t,S^2)S^1S^2\partial^2_{S^1S^2}P_\rho(t,S^1,S^2)p(t,S^1,S^2)}{\int_0^T\int_0^\infty\int_0^\infty D^d_{0t_*}\sigma_1(t_*,S^1_*)\sigma_2(t_*,S^2_*)S^1_*S^2_*\partial^2_{S^1S^2}P_\rho(t_*,S^1_*,S^2_*)p(t_*,S^1_*,S^2_*)dS^1_*dS^2_*dt_*}$$

with $p(t,S^1,S^2)$ the probability density function of (S^1_t,S^2_t) when the correlation is ρ_t, or, equivalently, $\rho_{\text{loc}}(t,S^1_t,S^2_t)$. Of course the density $p(t,S^1,S^2)$ is unknown - otherwise we could compute exactly the price of the option. One way to estimate the implied correlation is to compute the fixed point of the approximate mapping where $p(t,S^1,S^2)$ is replaced by some explicit estimate $\hat{p}(t,S^1,S^2)$. In the particular case where the instantaneous volatilities and correlation are constant, the weight $q_\rho(t,S^1,S^2)$ is known explicitly for the payoff $g(S^1_T,S^2_T) = (S^1_T - KS^2_T)_+$. Figure 8.1 shows the graphs of $(S^1,S^2) \mapsto q_\rho(t,S^1,S^2)$ for increasing values of t, from 0 to T.

Following Gatheral [13] (see also [15]), we can get an alternative expression for the implied correlation by considering the situation where the local correlation function $\rho(t)$ is a deterministic function of time only. The option has the same price in Model $\mathcal{M}_{\rho_t}$ and in this model if and only if

$$\int_0^T D^d_{0t}\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[(\rho_t - \rho(t))\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(t)}(t,S^1_t,S^2_t)\right]dt = 0$$

There is a unique function $\rho(t) \equiv \rho(t;T,g)$ such that, not only the time integral is zero, but also the integrand vanishes for each time slice t:

$$\rho(t;T,g) = \frac{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\rho_t\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(t;T,g)}(t,S^1_t,S^2_t)\right]}{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\sigma_1(t,S^1_t)\sigma_2(t,S^2_t)S^1_tS^2_t\partial^2_{S^1S^2}P_{\rho(t;T,g)}(t,S^1_t,S^2_t)\right]}$$

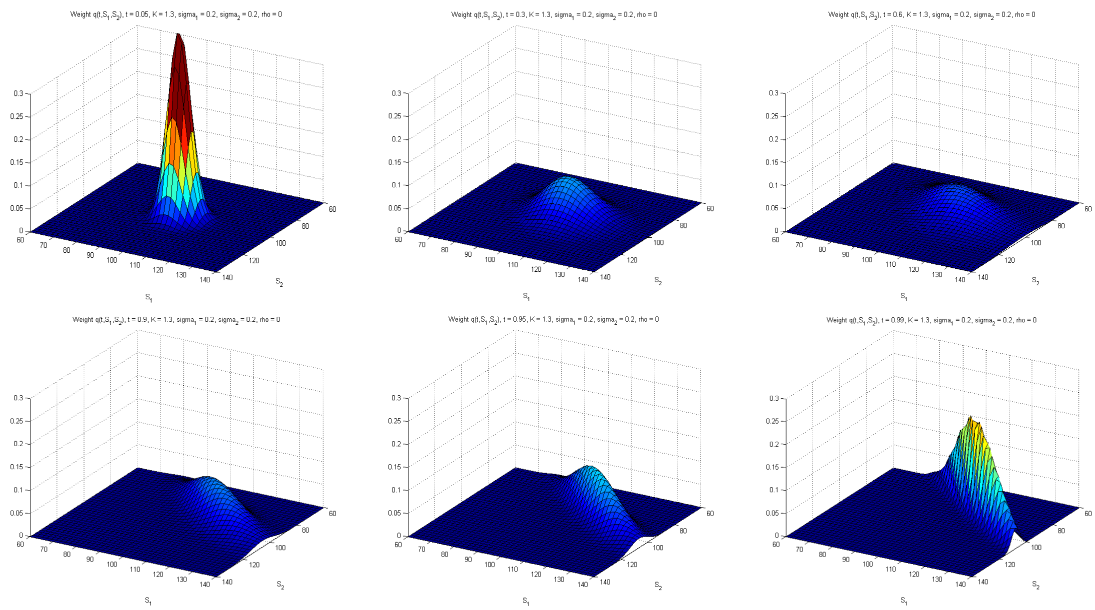

FIGURE 8.1. Graphs of $(S^1, S^2) \mapsto q_\rho(t, S^1, S^2)$ for increasing values of t. Black-Scholes model: $\sigma_1 = 20\%$, $\sigma_2 = 20\%$, $\rho = 0$, $S_0^1 = 100$, $S_0^2 = 100$. Payoff $g(S_T^1, S_T^2) = (S_T^1 - K S_T^2)_+$, $K = 1.3$, $T = 1$.

Note that this is again a fixed point equation, because the right hand side depends on $\rho(t;T,g)$ through the cross gamma $\partial^2_{S^1 S^2} P_{\rho(t;T,g)}$. In the particular case where $\sigma_1(t, S_t^1) = \sigma_1(t)$ and $\sigma_2(t, S_t^2) = \sigma_2(t)$ depend only on time (no volatility skew on S^1 and S^2), then

$$\rho(T,g) = \frac{\int_0^T \rho(t;T,g)\sigma_1(t)\sigma_2(t)dt}{\int_0^T \sigma_1(t)\sigma_2(t)dt} = \frac{\int_0^T \frac{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\rho_t S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}\sigma_1(t)\sigma_2(t)dt}{\int_0^T \sigma_1(t)\sigma_2(t)dt} \tag{8.6}$$

In the even more particular case where σ_1 and σ_2 are constant, this reads

$$\rho(T,g) = \frac{1}{T}\int_0^T \frac{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[\rho_t S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}{\mathbb{E}^{\mathbb{Q}}_{\rho_t}\left[S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}dt \tag{8.7}$$

Equations (8.6) and (8.7) are similar to Gatheral's formula for the implied volatility (see [13, 15]). Note however that we had to assume no skew on S^1 and S^2 to derive them. When S^1 or S^2 have a skew, we can no longer easily link the implied volatility $\rho(T,g)$ to $\rho(t;T,g)$. Equation (8.7) looks like Equation (8.5) but actually differs in two ways: (i) Equation (8.7) involves a time average of a space average, whereas Equation (8.5) involves a joint time and space average, and (ii) The cross gammas involved in both equations slightly differ: Equation (8.7) uses the cross gamma computed with time-dependent correlation $\rho(t;T,g)$, whereas Equation (8.5) uses the cross gamma computed with constant correlation $\rho(T,g)$.

8.4. **Impact of correlation on price.** For a given option, Equation (8.3) helps understand the impact on the option price of a particular choice of local correlation function. Basically, high option prices correspond to local correlation functions that are large in the region where the cross gamma is positive, and small (i.e., close to -1) in the region where the cross gamma is negative. Of course the cross gamma depends on the particular model picked. Equation (8.3) states that to compute the price difference between Model $\mathcal{M}_{\rho_t}$ and reference Model $\mathcal{M}_{\rho_0}$ you may compute the right hand side expectation where (S_t^1, S_t^2) is simulated under Model $\mathcal{M}_{\rho_t}$ and the cross gamma is computed under Model $\mathcal{M}_{\rho_0}$. Actually, in the case when the process ρ_t

is a local correlation process $\rho_1(t, S_t^1, S_t^2)$, the roles of $\mathcal{M}_{\rho_0}$ and $\mathcal{M}_{\rho_1}$ can be swapped:

$$\begin{aligned} & D_{0T}^d \mathbb{E}_{\rho_0}^{\mathbb{Q}}[g(S_T^1, S_T^2)] - P_1(0, S_0^1, S_0^2) \\ & \qquad = \mathbb{E}_{\rho_0}^{\mathbb{Q}}\left[\int_0^T D_{0t}^d(\rho_0(t, S_t^1, S_t^2) - \rho_1(t, S_t^1, S_t^2))\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) S_t^1 S_t^2 \partial_{S^1 S^2}^2 P_1(t, S_t^1, S_t^2) dt\right] \end{aligned}$$

or, equivalently,

$$\begin{aligned} & P_1(0, S_0^1, S_0^2) - P_0(0, S_0^1, S_0^2) \\ & \qquad = \mathbb{E}_{\rho_0}^{\mathbb{Q}}\left[\int_0^T D_{0t}^d(\rho_1(t, S_t^1, S_t^2) - \rho_0(t, S_t^1, S_t^2))\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) S_t^1 S_t^2 \partial_{S^1 S^2}^2 P_1(t, S_t^1, S_t^2) dt\right] \end{aligned}$$

Stated otherwise, to compute the price difference between two local correlation models $\mathcal{M}_{\rho_1}$ and $\mathcal{M}_{\rho_0}$, you may compute the expected value of the integrated tracking error where:

- either (S_t^1, S_t^2) is simulated under Model $\mathcal{M}_{\rho_1}$ and the cross gamma is computed under Model $\mathcal{M}_{\rho_0}$,
- or (S_t^1, S_t^2) is simulated under Model $\mathcal{M}_{\rho_0}$ and the cross gamma is computed under Model $\mathcal{M}_{\rho_1}$.

8.5. **Uncertain correlation model.** The highest possible price for payoff g, given local volatility dynamics for S^1 and S^2, is

$$D_{0T}^d \sup_{\rho_t \in \mathcal{R}} \mathbb{E}_{\rho_t}^{\mathbb{Q}}[g(S_T^1, S_T^2)]$$

where $\mathcal{R}$ denotes the set of all adapted stochastic processes taking values in $[-1, 1]$. It is given by the solution $P(0, S_0^1, S_0^2)$ to the (nonlinear) Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \partial_t P + \frac{1}{2}\sigma_1^2(t, S^1)(S^1)^2 \partial_{S^1}^2 P + \frac{1}{2}\sigma_2^2(t, S^2)(S^2)^2 \partial_{S^2}^2 P + \sup_{\rho \in [-1,1]} \left\{\rho \sigma_1(t, S^1)\sigma_2(t, S^2) S^1 S^2 \partial_{S^1 S^2}^2 P\right\} & \\ + (r_t^d - r_t^1) S^1 \partial_{S^1} P + (r_t^d - r_t^2) S^2 \partial_{S^2} P - r_t^d P &= 0 \\ P(T, S^1, S^2) &= g(S^1, S^2) \end{aligned}$$

that is,

$$\begin{aligned} \partial_t P + \frac{1}{2}\sigma_1^2(t, S^1)(S^1)^2 \partial_{S^1}^2 P + \frac{1}{2}\sigma_2^2(t, S^2)(S^2)^2 \partial_{S^2}^2 P + \rho\left(\partial_{S^1 S^2}^2 P\right) \sigma_1(t, S^1)\sigma_2(t, S^2) S^1 S^2 \partial_{S^1 S^2}^2 P & \\ + (r_t^d - r_t^1) S^1 \partial_{S^1} P + (r_t^d - r_t^2) S \partial_{S^2}^2 P - r_t^d P &= 0 \\ P(T, S^1, S^2) &= g(S^1, S^2) \end{aligned} \tag{8.8}$$

where

$$\rho(\Gamma) = \begin{cases} +1 & \text{if } \Gamma \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

As expected from (8.3), the highest option price correspond to the local correlation function that is worth $+1$ in the region where the cross gamma is positive, and -1 in the region where the cross gamma is negative. Here, for consistency, the cross gamma must be computed within this extremal model $\mathcal{M}_{\text{HJB}}$, i.e., by solving (8.8). Symmetrically, the lower bound

$$D_{0T}^d \inf_{\rho_t \in \mathcal{R}} \mathbb{E}_{\rho_t}[g(S_T^1, S_T^2)]$$

is given by $P(0, S_0^1, S_0^2)$, where P is solution to the (nonlinear) HJB equation

$$\begin{aligned} \partial_t P + \frac{1}{2}\sigma_1^2(t, S^1)(S^1)^2 \partial_{S^1}^2 P + \frac{1}{2}\sigma_2^2(t, S^2)(S^2)^2 \partial_{S^2}^2 P - \rho\left(\partial_{S^1 S^2}^2 P\right) \sigma_1(t, S^1)\sigma_2(t, S^2) S^1 S^2 \partial_{S^1 S^2}^2 P & \\ + (r_t^d - r_t^1) S^1 \partial_{S^1} P + (r_t^d - r_t^2) S \partial_{S^2}^2 P - r_t^d P &= 0 \\ P(T, S^1, S^2) &= g(S^1, S^2) \end{aligned} \tag{8.9}$$

9. The equity index smile calibration problem

Let us now see to which extent the reasoning presented above in Sections 2 and 3 for the FX smile triangle calibration problem can be extended to the N-dimensional equity index smile calibration problem. Let us consider an index $I_t = \sum_{i=1}^{N} \alpha_i S_t^i$ made of N weighted stocks, each of which modeled using its own local volatility:

$$dS_t^i = r_t S_t^i \, dt + \sigma_i(t, S_t^i) S_t^i \, dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, S_t) dt \tag{9.1}$$

The interest rate r_t is deterministic; $\{W^i\}$ denotes a multi-dimensional Brownian motion with an instantaneous correlation function of the time and the N stock values $S_t = \left(S_t^1, \ldots, S_t^N\right)$. This model is calibrated to the index smile if and only if (see proof in the appendix)

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = \mathbb{E}_\rho \left[\sum_{i,j=1}^{N} \alpha_i \alpha_j \rho_{ij}(t, S_t) \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j \,\middle|\, I_t \right] \tag{9.2}$$

where σ_{Dup}^I denotes the Dupire local volatility of the index. To ease notations, let us denote by

$$v_\rho(t, S_t) = \sum_{i,j=1}^{N} \alpha_i \alpha_j \rho_{ij}(t, S_t) \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j$$

the instantaneous (normal) variance of the basket of stocks within Model (9.1). Then Equation (9.2) simply reads

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = \mathbb{E}_\rho \left[v_\rho(t, S_t) \middle| \, I_t \right] \tag{9.3}$$

Remark 7. It is tempting but wrong to believe that in order to exclude arbitrage opportunities we must have the stronger statement $I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = v_\rho(t, S_t)$ at each point in time. This is incorrect in two ways. First, only the conditional expected value of the basket variance given the index value matters. Second, if no ρ satisfies (9.3), it does not mean that arbitrage opportunities exist, but only that prices are inconsistent with local volatilities-local correlation modeling, and that one has to consider more general models, for instance models that include stochastic volatility.

Let $\mathcal{C}$ denote the set of functions $\rho(t, S)$ taking values in the set of correlation matrices. Any function $\rho \in \mathcal{C}$ satisfying (9.3) will be called an "**admissible correlation**." We aim at identifying families of admissible correlations. Let $\rho \in \mathcal{C}$ be *admissible*. It is made of $N(N-1)/2$ parameters (the off-diagonal entries) satisfying one scalar equation, so we reduce the dimension of the problem by assuming that $\rho(t, S)$ lies on the line defined by two given correlation matrices $\rho^0(t, S)$ and $\rho^1(t, S)$ that may depend on (t, S) but are usually taken to be constant:

$$\rho(t, S) = (1 - \lambda(t, S))\rho^0(t, S) + \lambda(t, S)\rho^1(t, S), \qquad \lambda(t, S) \in \mathbb{R}$$

When $\lambda = 0$, $\rho = \rho^0$; when $\lambda = 1$, $\rho = \rho^1$. If $\lambda \in [0, 1]$, ρ is guaranteed to be a correlation matrix, because the set of correlation matrices is convex. When ρ^0 (resp. ρ^1) does not belong to the boundary of the set of correlation matrices, ρ may be a correlation matrix even if $\lambda < 0$ (resp. $\lambda > 1$). With this specification of $\rho(t, S)$, (9.3) reads

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 \quad = \quad \mathbb{E}_\rho \left[v_{\rho^0}(t, S_t) + \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \lambda(t, S_t) \middle| \, I_t \right]$$

Let us now pick two functions a and b such that b does not vanish and

$$a(t, S_t) + b(t, S_t)\lambda(t, S_t) \equiv f(t, I_t)$$

is local in index, i.e., is a function of (t, I_t) only, say $f(t, I_t)$. We can always do so, by choosing for instance $b \equiv 1$ and $a(t, S_t) = f(t, I_t) - \lambda(t, S_t)$ for some function f. Then

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = (a + b\lambda)(t, I_t) \mathbb{E}_\rho \left[\frac{v_{\rho^1} - v_{\rho^0}}{b} \middle| \, I_t \right] + \mathbb{E}_\rho \left[v_{\rho^0} - \frac{a}{b} \left(v_{\rho^1} - v_{\rho^0} \right) \middle| \, I_t \right]$$

and $\lambda = \lambda_{(a,b)}$ satisfies the self-consistency equation

$$\begin{aligned} \rho_{(a,b)} &\equiv (1 - \lambda_{(a,b)})\rho^0 + \lambda_{(a,b)}\rho^1 \in \mathcal{C} \\ \lambda_{(a,b)}(t, S_t) &= \frac{1}{b(t, S_t)} \left(\frac{ I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 - \mathbb{E}_{\rho_{(a,b)}} \left[v_{\rho^0}(t, S_t) - \frac{a(t, S_t)}{b(t, S_t)} \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \middle| \, I_t \right] }{ \mathbb{E}_{\rho_{(a,b)}} \left[\frac{1}{b(t, S_t)} \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \middle| \, I_t \right] } - a(t, S_t) \right) \end{aligned} \tag{9.4}$$

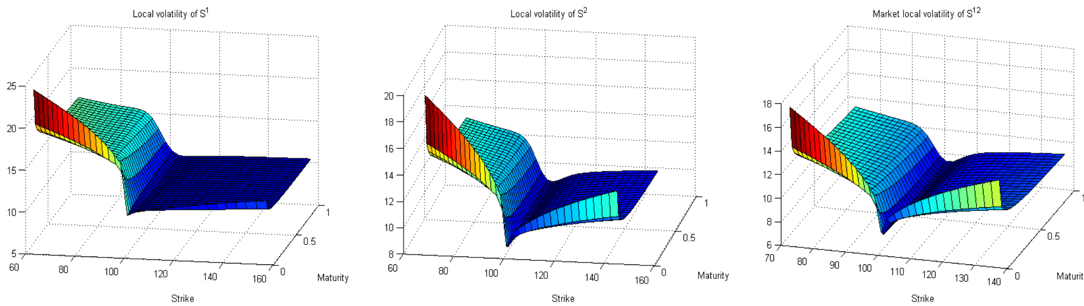

FIGURE 10.1. Surfaces of local volatilities σ_1, σ_2 and σ_{12}

Conversely, if a function $\lambda_{(a,b)}$ satisfies both conditions in (9.4), then $\rho_{(a,b)}$ is an admissible correlation. $\rho_{(a,b)}$ is guaranteed to be PSD if $\lambda_{(a,b)}$ takes values in $[0,1]$. We call the resulting model the "**local in index** $a+b\lambda$ **model**." It also depends on the choice of ρ^0 and ρ^1.

The two existing approaches correspond to special cases of this formulation:

- $a \equiv 0$ and $b \equiv 1$: In this case one assumes that the linear combination parameter λ itself is local in index. Then [16]

$$\lambda_{(0,1)}(t,S_t) = \frac{I_t^2 \sigma_{\text{Dup}}^I(t,I_t)^2 - \mathbb{E}_{\rho_{(0,1)}}\left[v_{\rho^0}(t,S_t) \middle| I_t \right]}{\mathbb{E}_{\rho_{(0,1)}}\left[v_{\rho^1}(t,S_t) - v_{\rho^0}(t,S_t) \middle| I_t \right]}$$

  and we speak of the "**local in index** λ **model**." If at some date $t < T$, $\rho_{(0,1)}(t,S)$ is not a correlation matrix for some S, then the trial is a failure: $\rho_{(0,1)}$ is not admissible. In [26], the same model is investigated but no explicit formula is given for $\lambda_{(0,1)}$; instead $\lambda_{(0,1)}$ is computed as the fixed point of a mapping that requires computing basket implied volatilities at each iteration, which makes this method slower. Precisely the mapping admits no fixed point when for some t, S, the candidate $\rho_{(0,1)}$ exhibited in [16] is not PSD.
- $a = v_{\rho^0}$ and $b = v_{\rho^1} - v_{\rho^0}$: In this case one assumes that the instantaneous variance of the index within Model (9.1) is local in index, we denote $\lambda_{(a,b)} = \lambda^*$ [25]:

$$\lambda^*(t,S_t) = \frac{I_t^2 \sigma_{\text{Dup}}^I(t,I_t)^2 - v_{\rho^0}(t,S_t)}{v_{\rho^1}(t,S_t) - v_{\rho^0}(t,S_t)}$$

  and we speak of the "**local in index volatility model**." This is the only situation where no estimation of conditional expectation (given the value of I_t) is needed. Note that λ^* is well defined even if the corresponding ρ^* is not PSD. If at some date $t < T$, $\rho^*(t,S)$ is not a correlation matrix for some S, then ρ^* is not admissible.

Another choice of (a,b) that respects the symmetry of the problem is the following:

- $a \equiv 0$ and $b = v_{\rho^1} - v_{\rho^0}$: In this case

$$\lambda_{(0,v_{\rho^1}-v_{\rho^0})}(t,S_t) = \frac{I_t^2 \sigma_{\text{Dup}}^I(t,I_t)^2 - \mathbb{E}_{\rho_{(0,v_{\rho^1}-v_{\rho^0})}}\left[v_{\rho^0}(t,S_t) \middle| I_t \right]}{v_{\rho^1}(t,S_t) - v_{\rho^0}(t,S_t)}$$

Remark 8. As already mentioned in Remark 4, our method allows to handle local correlations that depend on path-dependent variables, like some running averages, moving averages, running maximums, running minimums, etc. It is enough to add those path-dependent variables to the arguments of the functions a, b and λ.

10. NUMERICAL EXPERIMENTS ON THE FX TRIANGLE PROBLEM

10.1. **Calibration.** We have tested several "local in cross $a + b\rho$ models" on March 2012 market data involving the three currencies USD, EUR and GBP, and using USD as domestic currency: $S^1 = \text{EUR/USD}$, $S^2 = \text{GBP/USD}$, $S^{12} = S^1/S^2 = \text{EUR/GBP}$. For simplicity we have assumed zero interest rates. The three surfaces of local volatilities are shown in Figure 10.1. Different pairs of functions (a,b) are tested. For

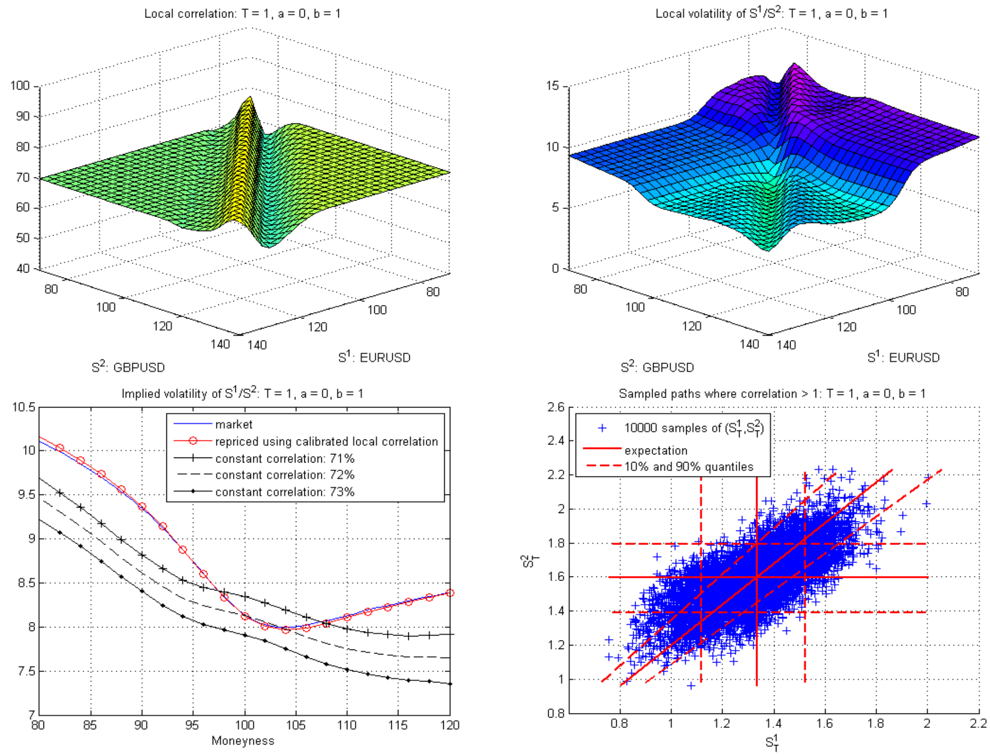

FIGURE 10.2. $a = 0$, $b = 1$

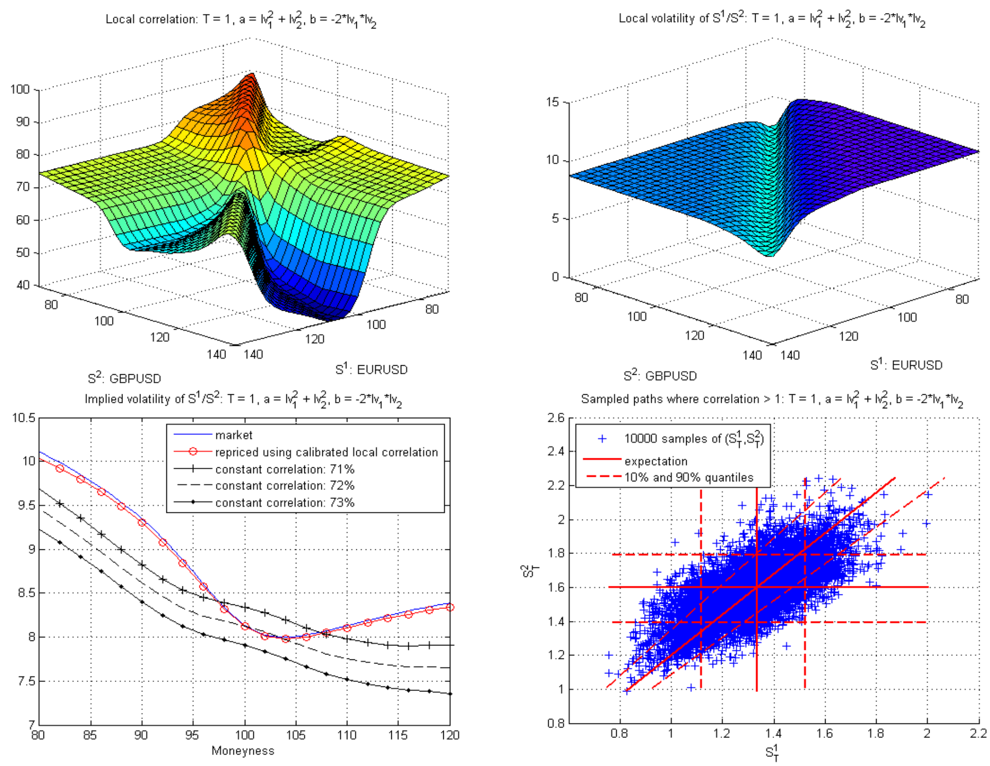

FIGURE 10.3. $a(t, S^1, S^2) = \sigma_1^2(t, S^1) + \sigma_2^2(t, S^2)$, $b(t, S^1, S^2) = -2\sigma_1(t, S^1)\sigma_2(t, S^2)$

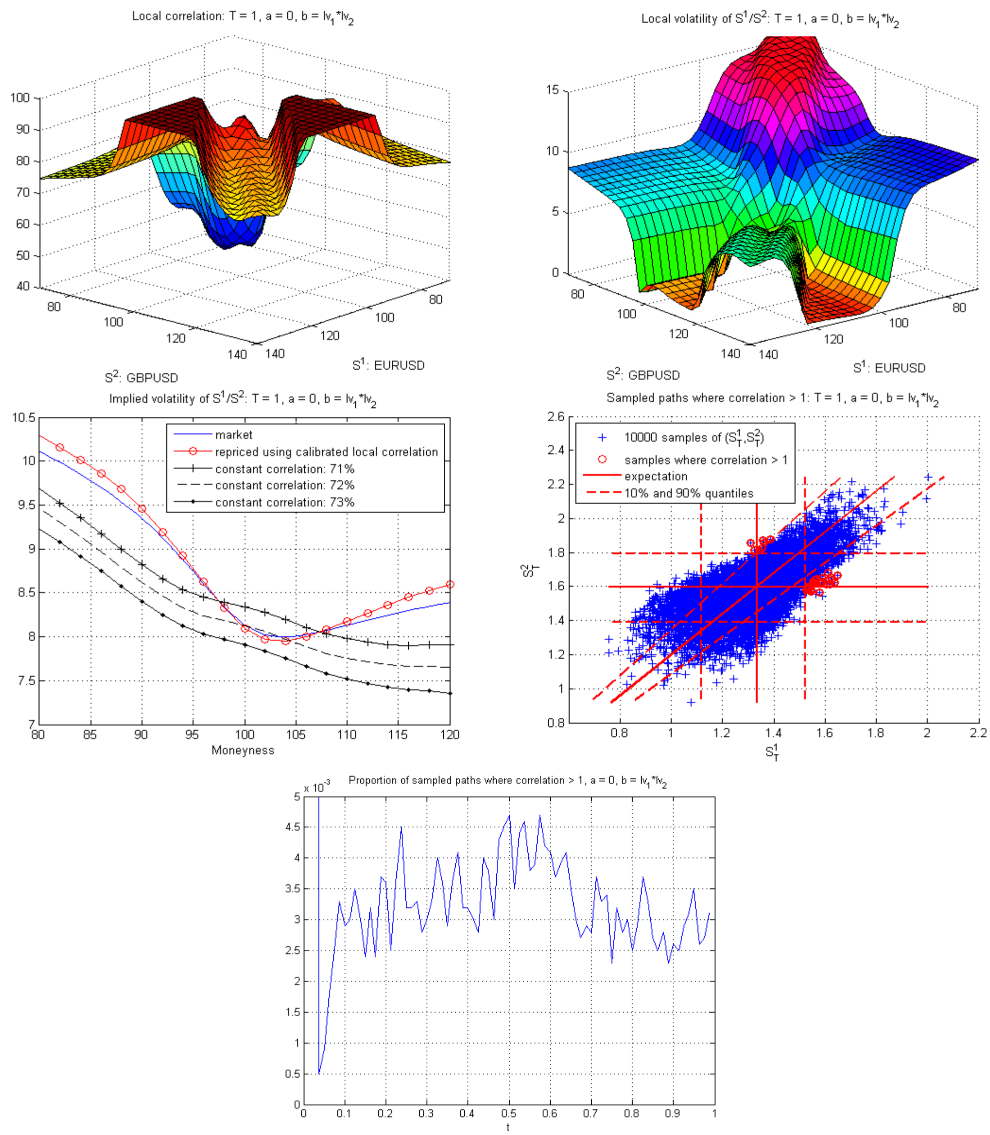

FIGURE 10.4. $a = 0$, $b(t, S^1, S^2) = \sigma_1(t, S^1)\sigma_2(t, S^2)$

each pair (a, b), the instantaneous correlation $(S^1, S^2) \mapsto \rho(T, S^1, S^2)$ at maturity (top left), and the instantaneous volatility $(S^1, S^2) \mapsto \sqrt{\sigma_1^2(T, S^1) + \sigma_2^2(T, S^2) - 2\rho_{(a,b)}(T, S^1, S^2)\sigma_1(T, S^1)\sigma_2(T, S^2)}$ of the cross rate at maturity (top right), the repriced smile of the cross rate at maturity T (bottom left), and the scatter plot of Monte Carlo sampled paths (S_T^1, S_T^2) (bottom right) are shown in Figures 10.2 through 10.16. In the case when the instantaneous correlation $\rho_{(a,b)}(t, S^1, S^2)$ takes values above $+1$, we simply cap it to $+1$, we highlight the corresponding (S_T^1, S_T^2) on the scatter plot with red circles, and we show the proportion of those paths as a function of time on a fith graph.

We picked $T = 1$, and used the particle method described in Section 4 with $N = 10,000$ Monte Carlo paths and the time step $\Delta t = \frac{1}{80}$. For the non-parametric regressions, we used the quartic kernel $K(x) = (1 - x^2)^2 1_{\{|x| \leq 1\}}$ and a bandwidth

$$h = \kappa \bar{\sigma}^{12} S_0^{12} \sqrt{\max(t, t_{\min})} N^{-\frac{1}{5}}$$

where $\bar{\sigma}^{12} = 10\%$ is a typical level for the volatility of S^{12}, $t_{\min} = 0.25$ and $\kappa = 3$. The conditional expectations are computed on a grid $G_{S,t}$ of $N_{S,t} = \max(N_S\sqrt{t}, N_S')$ values of the conditioning random

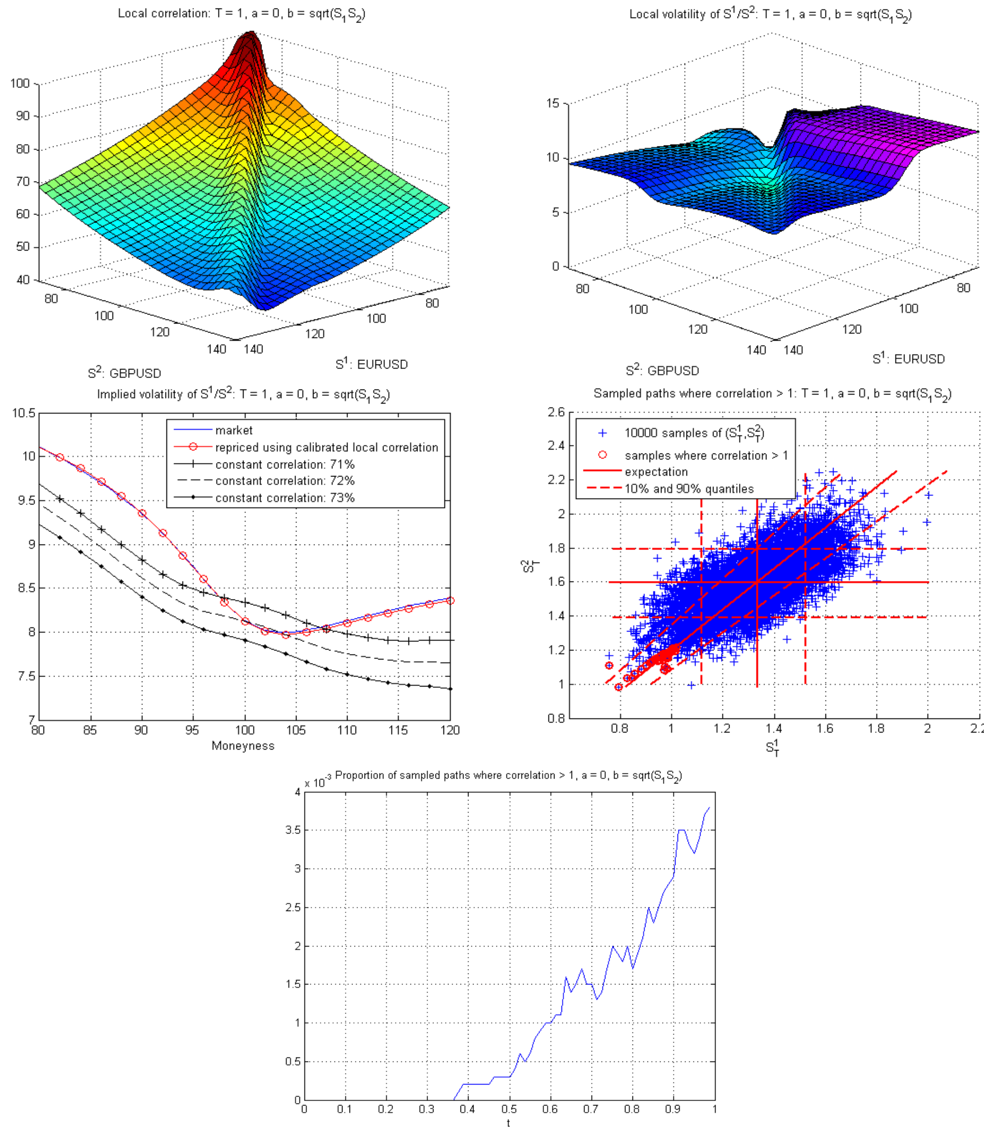

FIGURE 10.5. $a = 0$, $b(t, S^1, S^2) = \sqrt{S^1 S^2}$

variable $S^{12} = S^1/S^2$, with $N_S = 30$ and $N'_S = 15$. We use the 1% and 99% quantiles of the distribution of S_t^{12} as the minimum and maximum values of the grid $G_{S,t}$. Then the function

$$f\left(t, \frac{S^1}{S^2}\right) = a(t, S^1, S^2) + b(t, S^1, S^2)\rho(t, S^1, S^2)$$

is interpolated using cubic splines and extrapolated in a flat way.

The bottom left graphs also show the market implied volatilities of the cross rate at maturity, as well as the smiles produced by the constant correlation model for three values of constant correlation: 71%, 72% and 73%. This allows to translate the calibration error in terms of correlation points. 72% is the value of the constant correlation that fits the market value of the ATM implied volatility of the cross rate at maturity.

Figures 10.2 to 10.14 illustrate the variety of (at least almost) admissible correlations. Before we introduce the local in cross $a + b\rho$ representation, one would only choose between two local correlations: the local in cross correlation $\rho_{(0,1)}$ (Figure 10.2) or the correlation ρ^* corresponding to a local in cross volatility of the cross (Figure 10.3). Thanks to our local in cross $a + b\rho$ representation, among the variety of admissible correlations that it produces, one can now pick one's favourite depending on one's criterion:

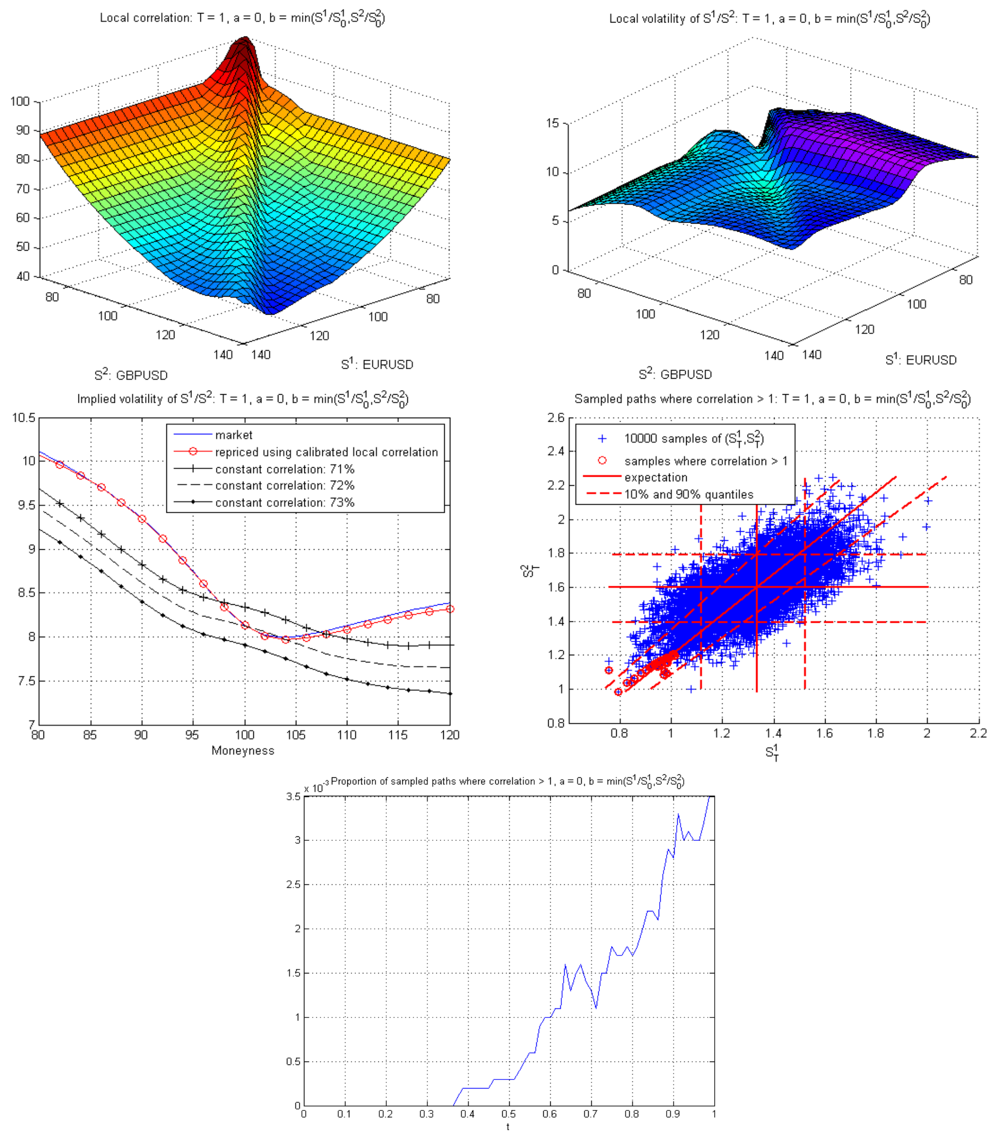

FIGURE 10.6. $a = 0$, $b(t, S^1, S^2) = \min\left(\frac{S^1}{S^1_0}, \frac{S^2}{S^2_0}\right)$

- Match a view on the correlation skew, for instance match a value of

$$\Delta\rho \equiv \rho(T, 1.05S^1_0, 1.05S^2_0) - \rho(T, 0.95S^1_0, 0.95S^2_0)$$

- Reproduce some features of the historical shape of the correlation between the returns in $S^1$and S^2, as a function of the values of $S^1$and S^2. Figure 10.17 shows what this shape looks like for EUR/USD and GBP/USD over the periods January 2007-June 2013 (left) and January 2011-June 2013 (right).
- Fit the price of options on S^1 and S^2 (other than the payoffs $(S^1 - KS^2)_+$ which are automatically fitted to the market since the correlation is admissible). See Table 3 below.

For instance, one may fit a negative skew by using $a = 0$ and $b = (S^1S^2)^\alpha$ with $\alpha > 0$, see Figures 10.5 and 10.8. The larger α, the more negative the skew $\Delta\rho$. However, too large values of α produce correlation candidates that are not admissible. For instance, in Figure 10.5, we observe that we had to cap the correlation to 1 for some small values of $S^1$and S^2. However, this is still acceptable in practice because only 0.3% of the simulated spots undergo this capped correlation. Conversely, one may fit a positive skew by using a negative value for α, see Figure 10.11.

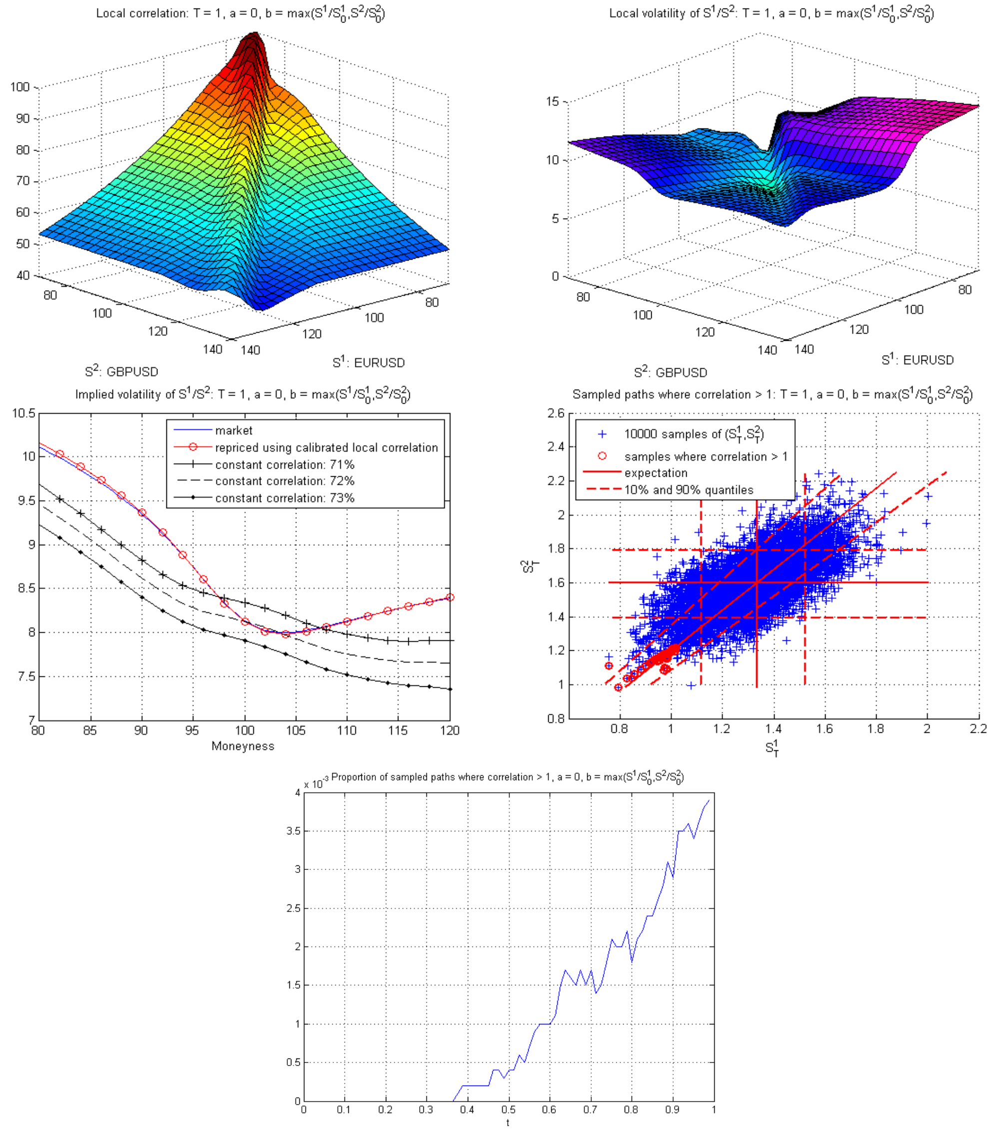

FIGURE 10.7. $a = 0$, $b(t, S^1, S^2) = \max\left(\frac{S^1}{S^1_0}, \frac{S^2}{S^2_0}\right)$

If one observes that the historical correlation is large when the spots are low - which is typical in equity markets - then one may slightly transform function b and use for instance $b = \min(S^1, S^2)^{2\alpha}$, see Figures 10.6 and 10.9. This has almost no impact on the price of many products (see Table 3), but allows to incorporate correlation impact in the delta hedge and avoids posting large remarking-to-market loss in case of crisis. If, on the contrary, one wants to decrease correlation for low spot values, one may use for instance $b = \max(S^1, S^2)^{2\alpha}$, see Figures 10.7 and 10.10.

Choosing $\rho_{(0,1)}$ implies pricing vanishing correlation skew across lines where S^1/S^2 is constant, and may not be desirable. Choosing ρ^* may imply pricing and hedging with a correlation that varies strongly with the spot values and which is highly asymmetric (see Figure 10.3). As expected from (6.1), the image of $\rho_{(0,1)}$ is much narrower than the image of ρ^*: $\rho_{(0,1)}$ varies much less than ρ^*.

Note that, in this numerical example, the third financially natural correlation, namely the local in cross covariance correlation, varies a lot (and the volatility of S^1/S^2 as well) and has to be capped to 1 for S^1 large and S^2 around the money, and for S^2 large and S^1around the money. However, only around 0.5% of the simulated paths are affected by the cap (see Figure 10.4).

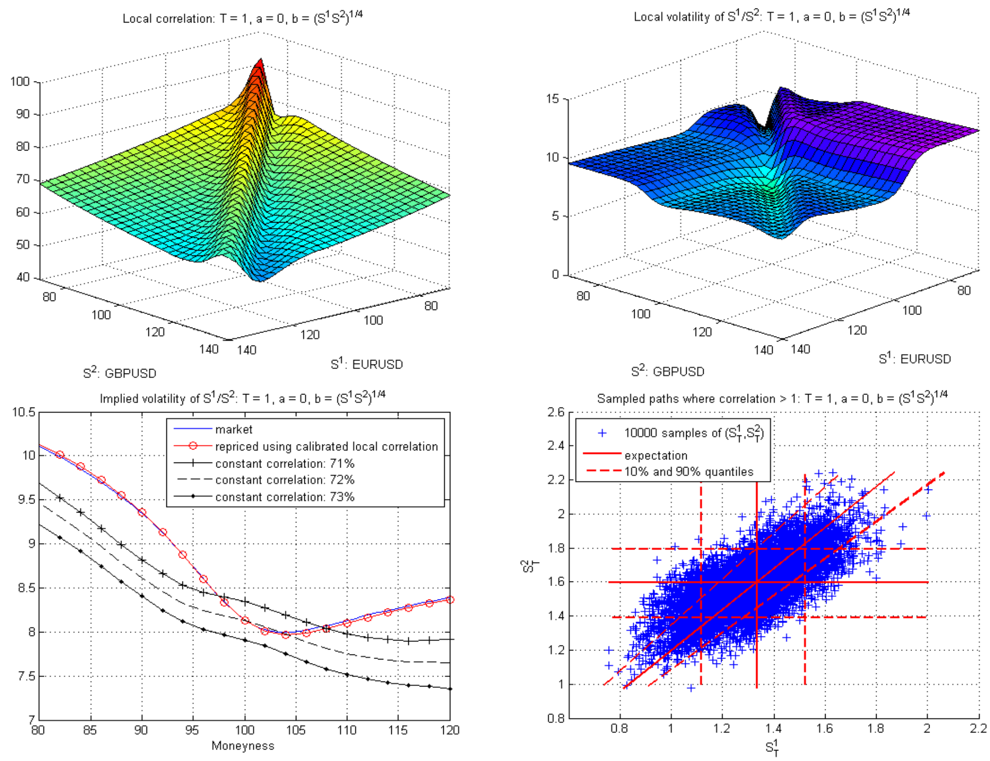

FIGURE 10.8. $a = 0$, $b(t, S^1, S^2) = (S^1 S^2)^{1/4}$

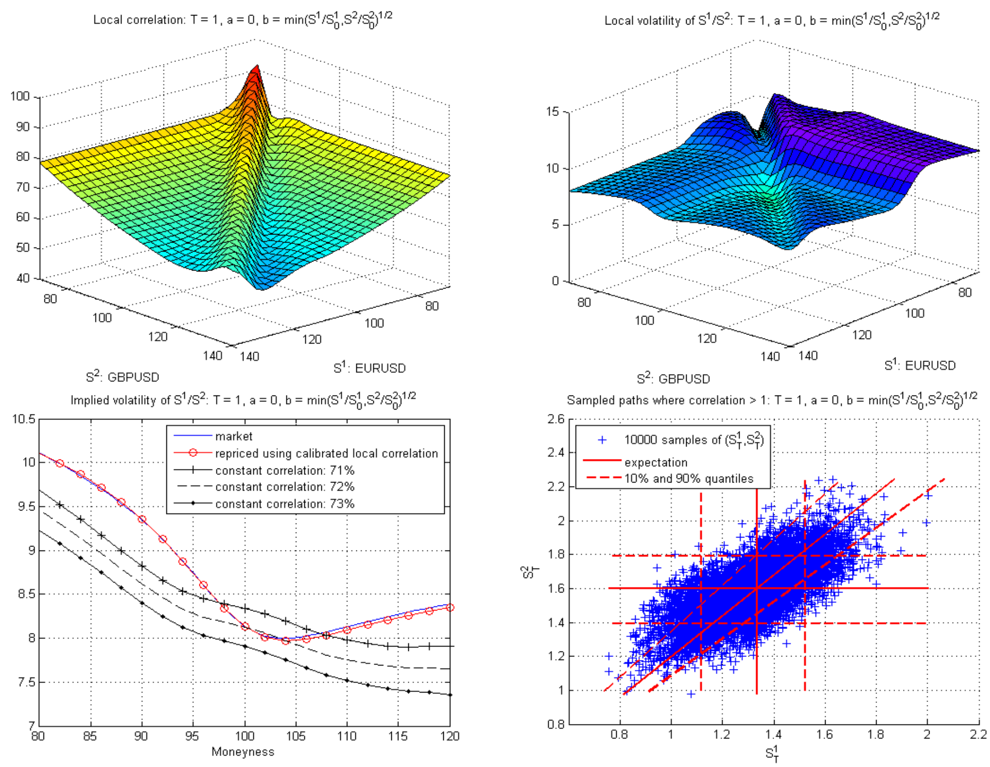

FIGURE 10.9. $a = 0$, $b(t, S^1, S^2) = \sqrt{\min(S^1, S^2)}$

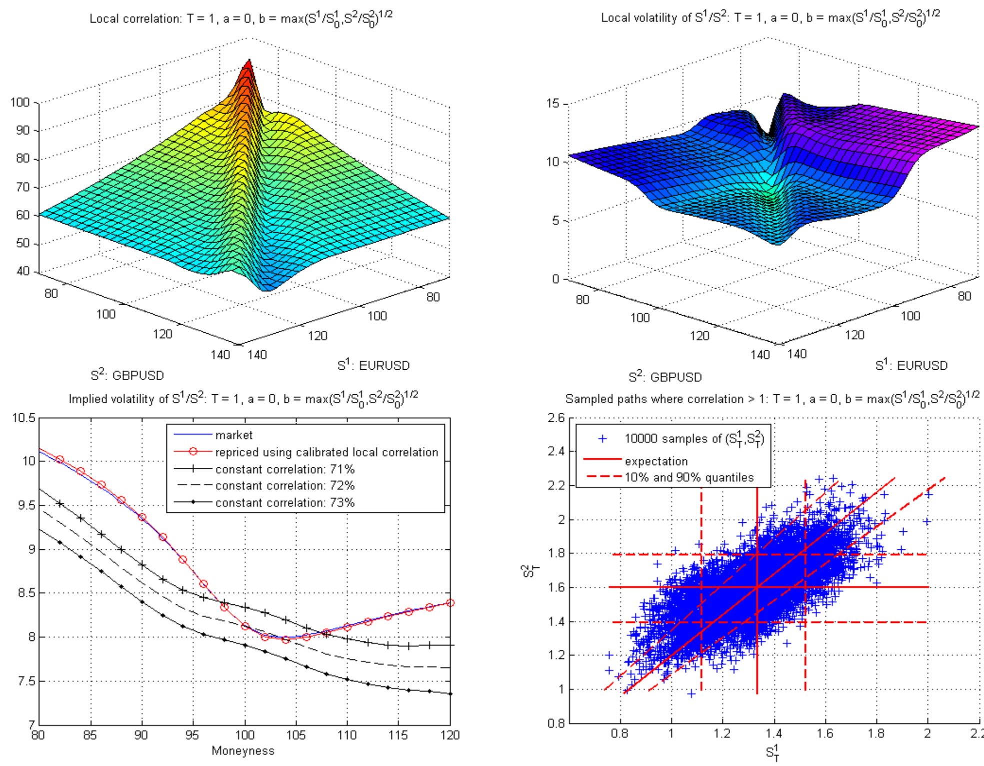

FIGURE 10.10. $a = 0$, $b(t, S^1, S^2) = \sqrt{\max(S^1, S^2)}$

An extreme admissible correlation is shown in Figure 10.12. From (6.4), we know that all admissible correlations approximately share the same average value on each line where S^1/S^2 is constant. (This is exact only when the two rates have no skew, and when the average is taken under $\mathbb{Q}^f$.) The correlation in Figure 10.12 was built so that the local correlation is very high and roughly constant when $\frac{S^1}{S^1_0} + \frac{S^2}{S^2_0}$ is lesser than 2, and very low and roughly constant when $\frac{S^1}{S^1_0} + \frac{S^2}{S^2_0}$ is greater than 2, and has the correct average value on those lines where the cross is constant. A smoothed version using the tanh function is shown in Figure 10.13.

Eventually, Figure 10.14 shows that our new method allows to build very diverse admissible correlations, here for instance a local correlation which is peaked around (S^1_0, S^2_0). As for Figures 10.15 and 10.16, they illustrate that a wrong choice of functions (a, b) can lead to inadmissible correlations, with a high proportion (resp. 22% and 16%) of correlations that have to be capped, resulting in a poor calibration of the smile of the cross rate.

10.2. **Pricing.** To illustrate the impact of the local correlation model on the price of options, we have considered the following three derivative products:

$$\begin{aligned}
\text{Min of calls}: \quad g(S^1_T, S^2_T) &= \min\left(\left(\frac{S^1_T}{K^1} - 1\right)_+, \left(\frac{S^2_T}{K^2} - 1\right)_+\right), \quad K^1 = S^1_0, \quad K^2 = S^2_0 \\
\text{Put on worst}: \quad g(S^1_T, S^2_T) &= \left(K - \min\left(\frac{S^1_T}{S^1_0}, \frac{S^2_T}{S^2_0}\right)\right)_+, \quad K = 0.95 \\
\text{Put on basket}: \quad g(S^1_T, S^2_T) &= \left(K - \left(\frac{S^1_T}{S^1_0} + \frac{S^2_T}{S^2_0}\right)\right)_+, \quad K = 1.8
\end{aligned}$$

The prices are shown in Table 3 for different admissible correlations. For each of these products, we can build an intuition of the impact of the local correlation model on the price by looking at the instantaneous correlation surfaces in Figures 10.2-10.13 and using the price impact formula (8.3). To this end, note that

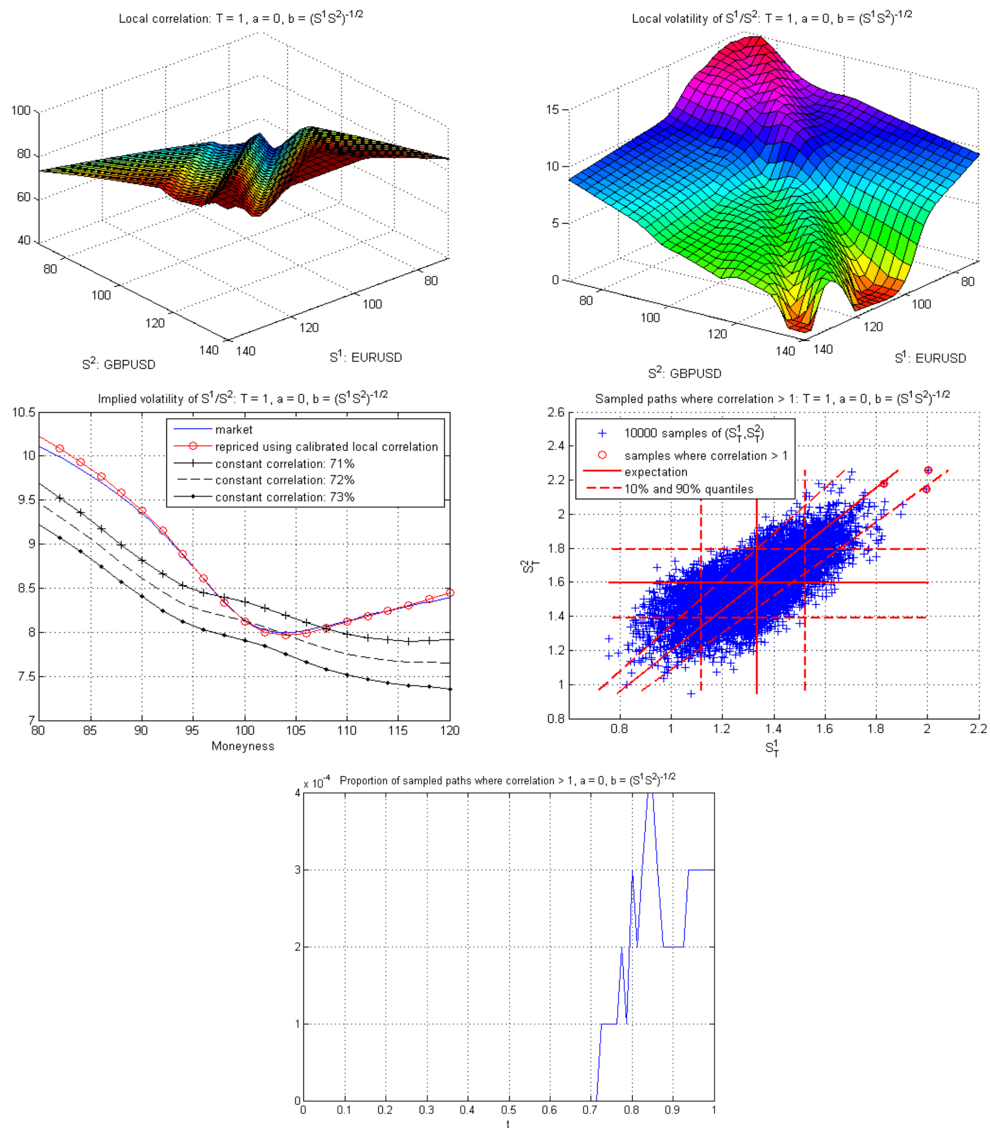

FIGURE 10.11. $a = 0$, $b(t, S^1, S^2) = \frac{1}{\sqrt{S^1 S^2}}$

the cross gammas of these options at maturity are simply proportional to the following Dirac masses:

$$\begin{aligned}
\text{Min of calls}: \quad & \delta\left(\frac{S^2}{K^2} - \frac{S^1}{K^1}\right) 1_{\left\{\frac{S^1}{K^1} \geq 1\right\}} \\
\text{Put on worst}: \quad & -\delta\left(\frac{S^2}{S_0^2} - \frac{S^1}{S_0^1}\right) 1_{\left\{\frac{S_T^1}{S_0^1} \leq K\right\}} \\
\text{Put on basket}: \quad & \delta\left(\frac{S^1}{S_0^1} + \frac{S^2}{S_0^2} - K\right)
\end{aligned}$$

From (8.3), we expect the higher prices to correspond to local correlation surfaces that are:

- larger in the neighborhood of the half-line $\frac{S^1}{S^2} = \frac{K^1}{K^2}$, $S^1 \geq K^1$, for the min of calls,
- smaller in the neighborhood of the half-line $\frac{S^1}{S^2} = \frac{S_0^1}{S_0^2}$, $S^1 \leq K S_0^1$, for the put on worst,
- larger in the neighborhood of the segment $\frac{S^1}{S_0^1} + \frac{S^2}{S_0^2} = K$, $S^1, S^2 > 0$, for the put on basket.

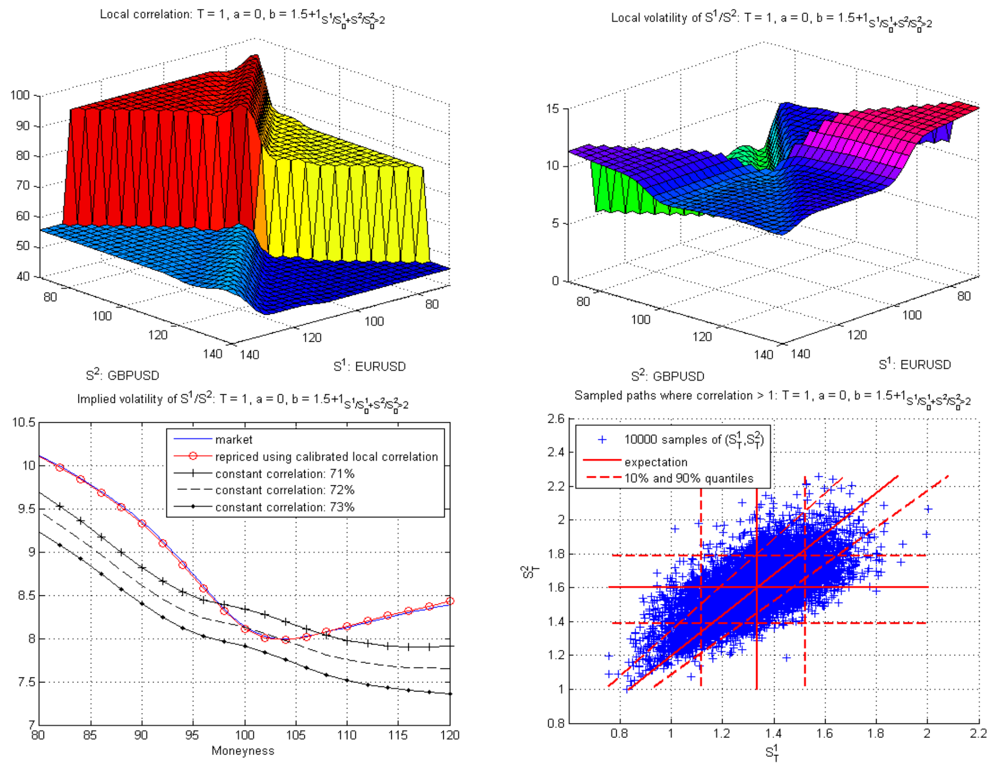

FIGURE 10.12. $a = 0$, $b(t, S^1, S^2) = 1.5 + 1_{\left\{\frac{S^1}{S^1_0}+\frac{S^2}{S^2_0}>2\right\}}$

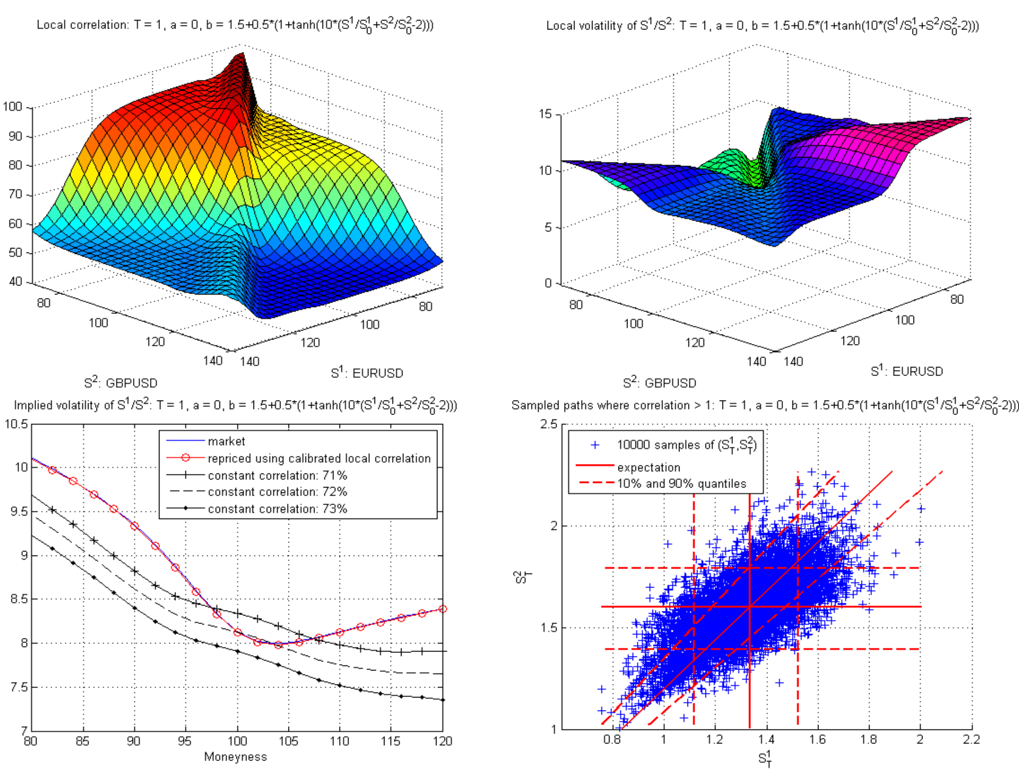

FIGURE 10.13. $a = 0$, $b(t, S^1, S^2) = 1.5 + \frac{1}{2}\left(1 + \tanh\left(10\left(\frac{S^1}{S^1_0} + \frac{S^2}{S^2_0} - 2\right)\right)\right)$

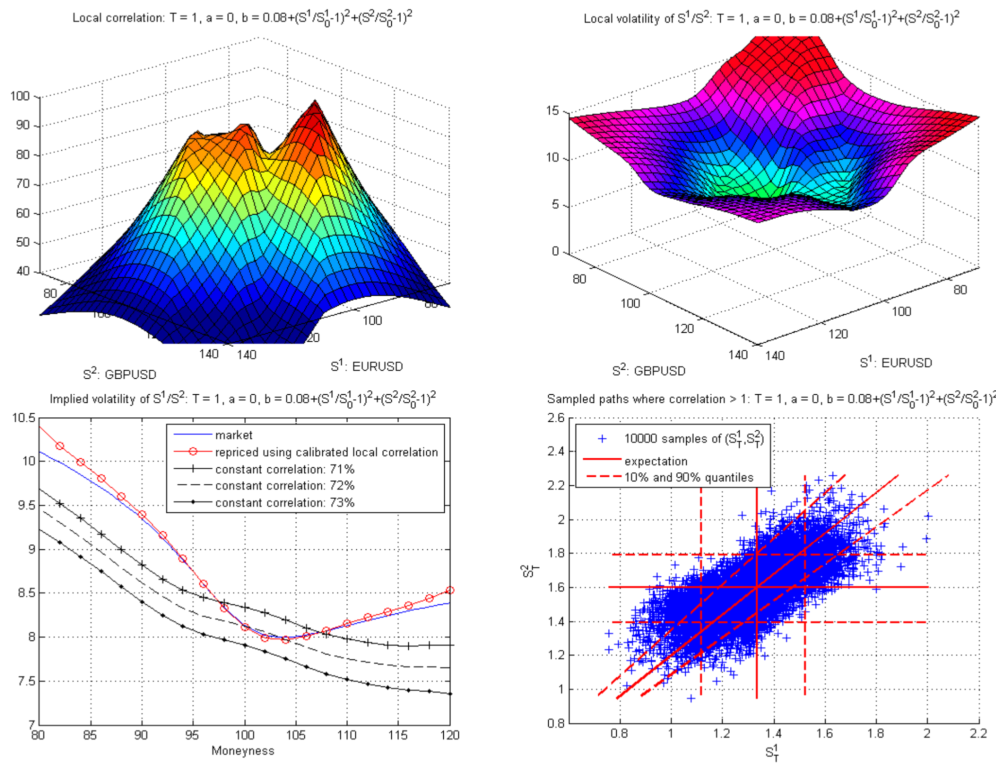

FIGURE 10.14. $a = 0$, $b(t, S^1, S^2) = 0.08 + \left(\frac{S^1}{S^1_0} - 1\right)^2 + \left(\frac{S^2}{S^2_0} - 1\right)^2$

This is indeed verified. For these products, the highest and lowest prices always correspond to the local in cross covariance correlation (Figure 10.4), and to the extreme correlation of Figure 10.12. This way, we have provided a numerical partial answer to the difficult problem of determining the lower and upper bounds of prices of options on (S^1, S^2) given the three surfaces of implied volatilities on S^1, S^2, and S^1/S^2, and the corresponding models. The answer is only partial because here we have only considered local volatility models (see Section 11 for a generalization to stochastic local volatility models) and a small number of admissible local correlations $\rho_{(a,b)}$. Note that the range of prices is already quite large, from 2.37 to 2.91 for instance for the min of calls, despite the fact that the three surfaces of implied volatilities are calibrated. In Table 3 we have also reported prices of a digital call on the cross rate with strike $K = 1.1\frac{S^1_0}{S^2_0}$ and a double-no-touch on the cross rate with barriers $K_1 = 0.9\frac{S^1_0}{S^2_0}$ and $K_2 = 1.1\frac{S^1_0}{S^2_0}$. The derivation of lower and upper bounds of prices of calls on the cross rate S^1/S^2 of maturity T given the two smiles at maturity T of $S^1$and S^2 and the at-the-money implied volatility and skew of S^1/S^2 can be found in [20].

11. Generalization to stochastic volatility, stochastic interest rates, and stochastic dividend yield

The FX triangle smile calibration problem. Let us show how to generalize the construction of families of local correlation models for the FX triangle smile calibration problem in the presence of local stochastic volatility, stochastic interest rates, and local correlation. For the sake of simplicity (see Remark 9 for a more general case), let us assume that the extra Brownian motions W^3, W^4, $W^5 \ldots$ that drive the dynamics of the stochastic volatilities and the stochastic interest rates are *independent* of the two Brownian motions W^1 and W^2 that drive the dynamics of (S^1, S^2). The correlation matrix of W^3, W^4, $W^5 \ldots$ is assumed to be known and constant. Only the correlation between W^1 and W^2 is unknown; it is assumed to

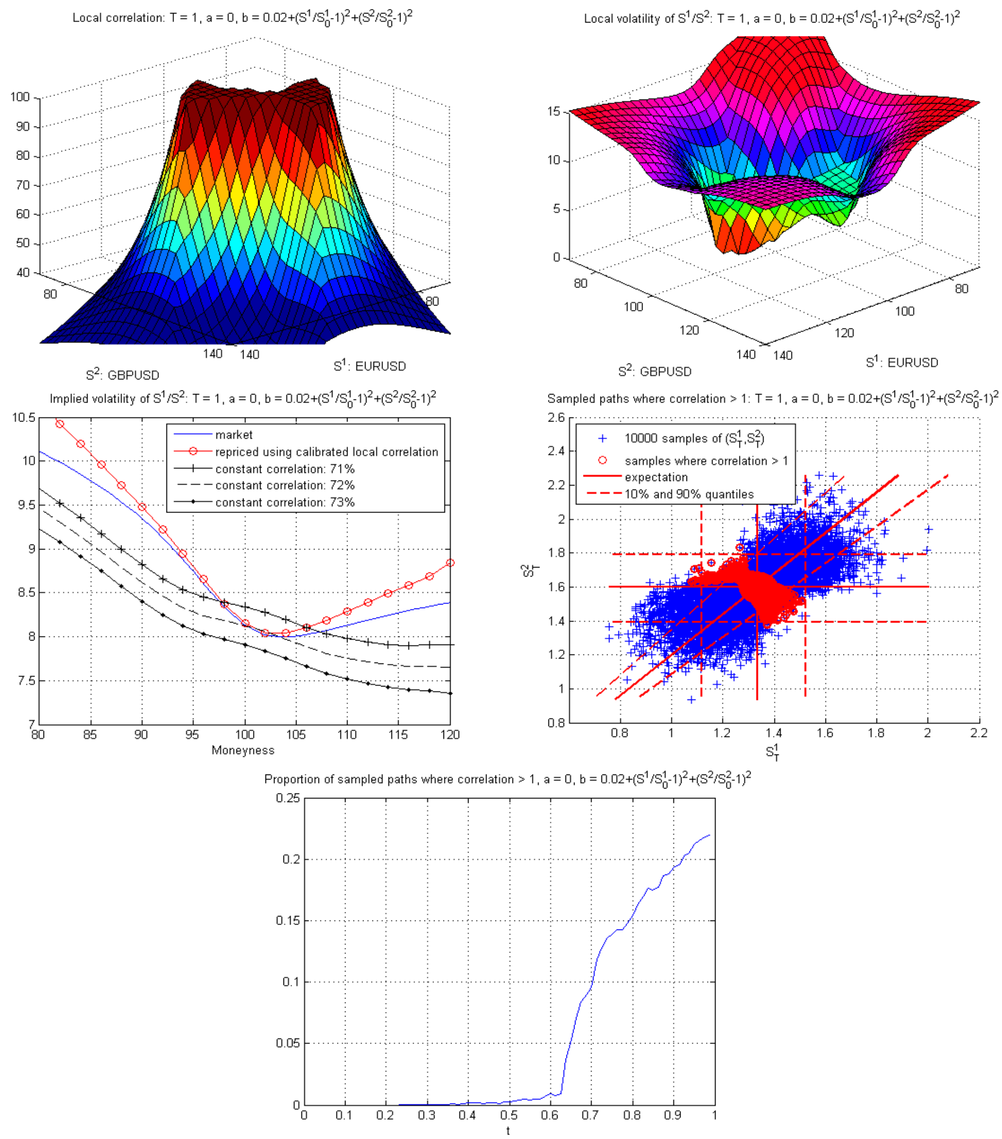

FIGURE 10.15. $a = 0$, $b(t, S^1, S^2) = 0.02 + \left(\frac{S^1}{S^1_0} - 1\right)^2 + \left(\frac{S^2}{S^2_0} - 1\right)^2$

be local:

$$\begin{aligned} dS_t^1 &= \left(r_t^d - r_t^1\right) S_t^1\,dt + \sigma_1(t, S_t^1) a_t^1 S_t^1\,dW_t^1 \\ dS_t^2 &= \left(r_t^d - r_t^2\right) S_t^2\,dt + \sigma_2(t, S_t^2) a_t^2 S_t^2\,dW_t^2 \\ d\langle W^1, W^2\rangle_t &= \rho(t, S_t^1, S_t^2, a_t^1, a_t^2, D_{0t}^d, D_{0t}^1, D_{0t}^2)\,dt \end{aligned} \tag{11.1}$$

a_t^1, a_t^2, r_t^d, r_t^1 and r_t^2 are Itô processes driven by the extra Brownian motions W^3, W^4, $W^5 \ldots$ The instantaneous correlation between W^1 and W^2 is assumed to depend not only on the FX rates S_t^1 and S_t^2 but also on the stochastic volatilities a_t^1 and a_t^2 and on the (stochastic) discount factors D_{0t}^d, D_{0t}^1 and D_{0t}^2:[2] $D_{0t}^i = \exp\left(-\int_0^t r_s^i ds\right)$. To keep notations short, we write $\rho(t, X_t)$ with $X_t = (S_t^1, S_t^2, a_t^1, a_t^2, D_{0t}^d, D_{0t}^1, D_{0t}^2)$.

[2]The instantaneous correlation may also depend on r_t^d, r_t^1, r_t^2. It may actually depend on any $\mathcal{F}_t$-measurable random variable, including path-dependent variables (see Remark 4).

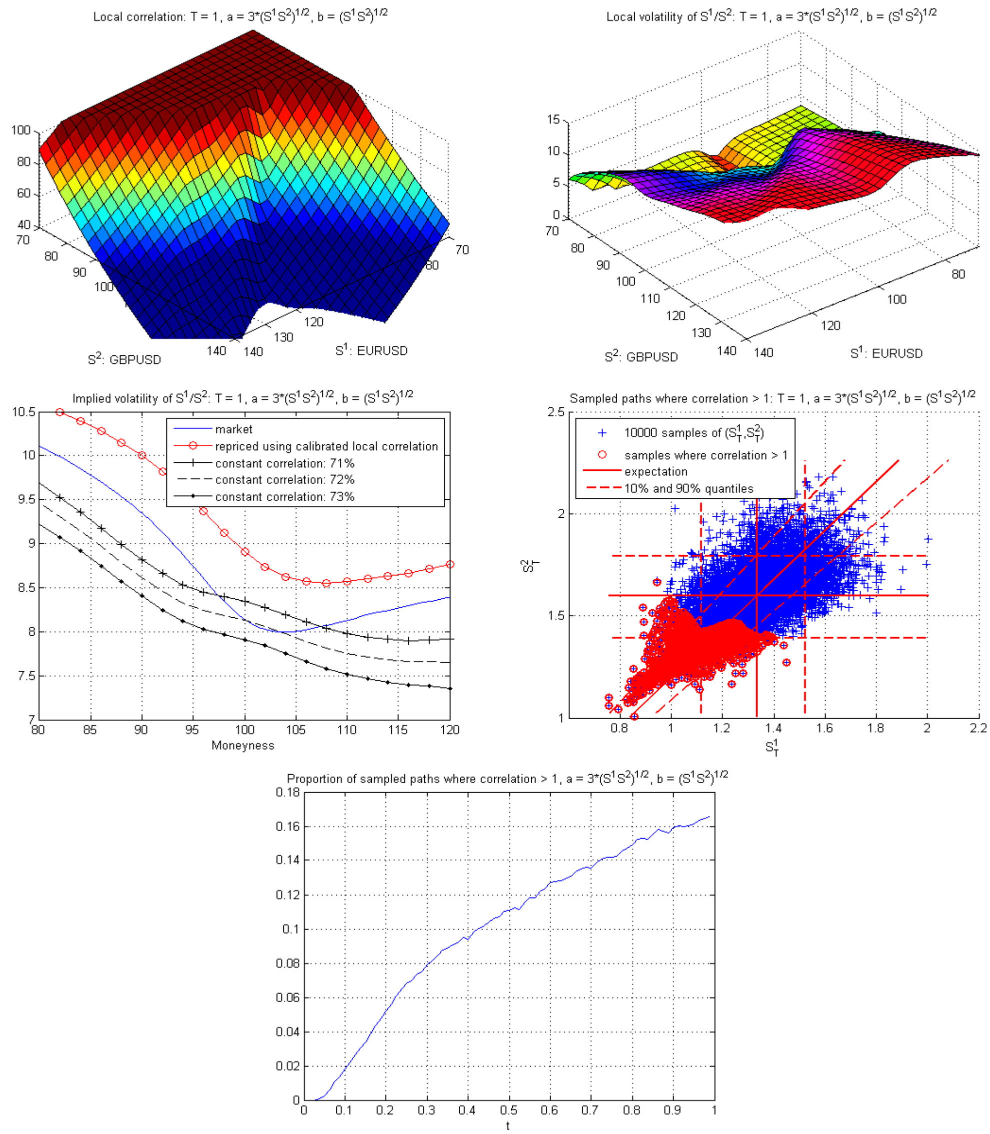

FIGURE 10.16. $a(t,S^1,S^2) = 3\sqrt{S^1S^2}$, $b(t,S^1,S^2) = \sqrt{S^1S^2}$

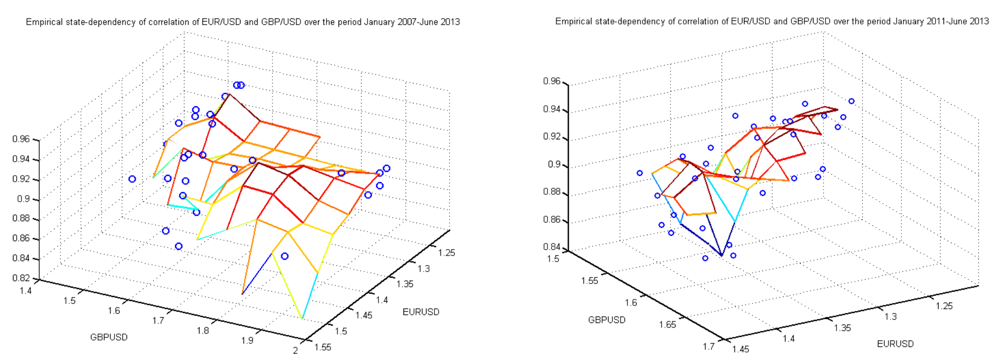

FIGURE 10.17. Empirical state-dependency of correlation of EUR/USD and GBP/USD over the period January 2007-June 2013 (left) and over the period January 2011-June 2013 (right).

a	b	Min of calls	Put on worst	Put on basket	DC on S^{12}	DNT on S^{12}
Standard deviation		≈ 0.020	≈ 0.027	≈ 0.027	≈ 0.18	≈ 0.22
Constant correlation 72%		2.59	3.47	1.88	21.18	57.97
0	1	2.65	3.49	1.91	**20.53**	**58.02**
$\sigma_1^2+\sigma_2^2$	$-2\sigma_1\sigma_2$	2.53	3.37	1.99	20.46	58.17
0	$\sigma_1\sigma_2$	**2.91**	**3.70**	**1.78**	19.75	59.41
0	σ_1	2.81	3.62	1.83	20.22	58.67
0	σ_2	2.78	3.60	1.85	20.27	58.39
σ_1^2	$-2\sigma_1\sigma_2$	2.67	3.51	1.91	20.47	58.12
σ_2^2	$-2\sigma_1\sigma_2$	2.80	3.60	1.84	20.38	58.50
0	$\sqrt{S^1S^2}$	2.56	3.41	1.95	20.48	58.11
0	$\max(S^1,S^2)$	2.56	3.40	1.95	20.50	58.10
0	$\min(S^1,S^2)$	2.56	3.41	1.95	20.46	58.15
0	$(S^1S^2)^{1/4}$	2.61	3.45	1.93	20.45	58.09
0	$\sqrt{\max(S^1,S^2)}$	2.61	3.45	1.93	20.46	58.08
0	$\sqrt{\min(S^1,S^2)}$	2.61	3.45	1.93	20.44	58.10
0	$\frac{1}{\sqrt{S^1S^2}}$	2.74	3.56	1.87	20.41	58.21
0	$1.5+1_{\left\{\frac{S^1}{S^1_0}+\frac{S^2}{S^2_0}>2\right\}}$	**2.37**	**3.25**	**2.06**	**19.71**	**59.26**
0	$2+\frac{1}{2}\tanh\left(10\left(\frac{S^1}{S^1_0}+\frac{S^2}{S^2_0}-2\right)\right)$	2.42	3.28	2.04	20.14	58.65

TABLE 3. Price in pct of the min of calls, put on worst, put on basket, digital call (DC) on S^{12}, and double-no-touch (DNT) on S^{12} for different admissible (or almost admissible) correlations described by the pair of functions (a,b). We used the same $50,000 = 10,000 + 40,000$ Brownian paths for all choices of (a,b). The first 10,000 paths are those used for calibration of the correlation.

First, the local volatility $\sigma_1(t,S^1)$ is calibrated to the market smile of S^1 using Propostion 10 in the appendix with $r_t = r_t^d$, $q_t = r_t^1$ and $a_t = \sigma_1(t,S_t^1)a_t^1$:

$$\sigma_1(t,K)^2\frac{\mathbb{E}^{\mathbb{Q}}[D_{0t}^d(a_t^1)^2|S_t^1=K]}{\mathbb{E}^{\mathbb{Q}}[D_{0t}^d|S_t^1=K]} = \sigma_{\mathrm{Dup}}^1(t,K)^2 - \frac{\mathbb{E}^{\mathbb{Q}}\left[D_{0t}^d\left(r_t^d-r_t^1-(r_t^{d,0}-r_t^{1,0})\right)1_{S_t^1>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}^{\mathbb{Q}}\left[D_{0t}^d\left(r_t^1-r_t^{1,0}\right)\left(S_t^1-K\right)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}$$

where $r_t^{d,0} = -\partial_t \ln P_{0t}^d$, $r_t^{1,0} = -\partial_t \ln P_{0t}^1$, and

$$\sigma_{\mathrm{Dup}}^1(t,K)^2 = \frac{\partial_t\mathcal{C}_1(t,K) + (r_t^{d,0}-r_t^{1,0})K\partial_K\mathcal{C}_1(t,K) + r_t^{1,0}\mathcal{C}_1(t,K)}{\frac{1}{2}K^2\partial_K^2\mathcal{C}_1(t,K)}$$

with $\mathcal{C}_1(t,K)$ the market price of the call option on S^1 with strike K and maturity t; and likewise for $\sigma_2(t,S^2)$. This is achieved using the particle algorithm (see [16, 17]). The knowledge of the local correlation $\rho(t,X)$ between the two FX rates is not required at this step. Then $\rho(t,X)$ is calibrated to the market smile of the cross rate S^{12} by requiring that (see proof in the appendix)

$$\frac{\mathbb{E}_\rho^{\mathbb{Q}^f}\left[D_{0t}^2\left(\left(\sigma_1(t,S_t^1)a_t^1\right)^2+\left(\sigma_2(t,S_t^2)a_t^2\right)^2-2\rho(t,X_t)\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2\right)|S_t^{12}=K\right]}{\mathbb{E}_\rho^{\mathbb{Q}^f}\left[D_{0t}^2|S_t^{12}=K\right]}$$
$$= \sigma_{\mathrm{Dup}}^{12}(t,K)^2 - \frac{\mathbb{E}_\rho^{\mathbb{Q}^f}\left[D_{0t}^2\left(r_t^2-r_t^1-(r_t^{2,0}-r_t^{1,0})\right)1_{S_t^{12}>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho^{\mathbb{Q}^f}\left[D_{0t}^2(r_t^1-r_t^{1,0})(S_t^{12}-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)} \quad (11.2)$$

for all (t, K), where $D_{0t}^2 = \exp\left(-\int_0^t r_s^2 ds\right)$, $r_t^{1,0}$ and $r_t^{2,0}$ are deterministic interest rates, and

$$\sigma_{\rm Dup}^{12}(t,K)^2 = \frac{\partial_t \mathcal{C}(t,K) + (r_t^{2,0} - r_t^{1,0}) K \partial_K \mathcal{C}(t,K) + r_t^{1,0}\mathcal{C}(t,K)}{\frac{1}{2}K^2 \partial_K^2 \mathcal{C}(t,K)}$$

is the market local volatility of the cross rate S^{12} computed using the determimistic interest rates $r_t^{1,0}$ and $r_t^{2,0}$. $\mathcal{C}(t,K)$ is the market price of the call option on S^{12} with strike K and maturity t. Equation (11.2) is equivalent to

$$\begin{aligned}
&\frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left(\left(\sigma_1(t,S_t^1)a_t^1\right)^2 + \left(\sigma_2(t,S_t^2)a_t^2\right)^2 - 2\rho(t,X_t)\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2\right)\left|S_t^{12}=K\right.\right]}{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left|S_t^{12}=K\right.\right]}\\
&= \sigma_{\rm Dup}^{12}(t,K)^2 - \frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}\left(r_t^2 - r_t^1 - (r_t^{2,0}-r_t^{1,0})\right)1_{S_t^{12}>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}(r_t^1 - r_t^{1,0})(S_t^{12}-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)} \quad (11.3)
\end{aligned}$$

and to

$$\begin{aligned}
&\mathbb{E}_\rho^{\mathbb{Q}^{f,t}}\left[\left(\sigma_1(t,S_t^1)a_t^1\right)^2 + \left(\sigma_2(t,S_t^2)a_t^2\right)^2 - 2\rho(t,X_t)\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2\left|S_t^{12}=K\right.\right]\\
&= \sigma_{\rm Dup}^{12}(t,K)^2 - P_{0t}^2\frac{\mathbb{E}_\rho^{\mathbb{Q}^{f,t}}\left[\left(r_t^2 - r_t^1 - (r_t^{2,0}-r_t^{1,0})\right)1_{S_t^{12}>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + P_{0t}^2\frac{\mathbb{E}_\rho^{\mathbb{Q}^{f,t}}\left[(r_t^1 - r_t^{1,0})(S_t^{12}-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)} \quad (11.4)
\end{aligned}$$

where $\mathbb{Q}^{f,t}$ denotes the foreign t-forward measure: $\frac{d\mathbb{Q}^{f,t}}{d\mathbb{Q}^f} = \frac{D_{0t}^2}{P_{0t}^2}$.

We say that a correlation ρ is admissible if Equation (11.2), or equivalently (11.3) or (11.4), holds. To build the set of all admissible correlations, we easily extend the local in cross $a + b\rho$ representation that was presented in Section 3 in the following way: for an admissible ρ, pick two functions $a(t,X)$ and $b(t,X)$ such that b does not vanish and

$$a(t,X) + b(t,X)\rho(t,X)$$

is local in cross, i.e., depends on X only through $S^{12} \equiv S^1/S^2$, where $X = (S^1, S^2, a^1, a^2, D_{0\cdot}^d, D_{0\cdot}^1, D_{0\cdot}^2)$. One can always do so by choosing $b \equiv 1$ and $a(t,X) = f\left(t, S^{12}\right) - \rho(t,X)$ for some function f. Then (11.3) is equivalent to

$$\begin{aligned}
&\frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left(\left(\sigma_1(t,S_t^1)a_t^1\right)^2 + \left(\sigma_2(t,S_t^2)a_t^2\right)^2 + 2\frac{a(t,X_t)}{b(t,X_t)}\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2\right)\left|S_t^{12}=K\right.\right]}{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left|S_t^{12}=K\right.\right]}\\
&\qquad - 2\left(a + b\rho\right)(t,K)\frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \frac{\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2}{b(t,X_t)}\left|S_t^{12}=K\right.\right]}{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left|S_t^{12}=K\right.\right]}\\
&= \sigma_{\rm Dup}^{12}(t,K)^2 - \frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}\left(r_t^2 - r_t^1 - (r_t^{2,0}-r_t^{1,0})\right)1_{S_t^{12}>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}(r_t^1 - r_t^{1,0})(S_t^{12}-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}
\end{aligned}$$

from which one gets $\rho(t,X) = \rho_{(a,b)}(t,X) \equiv \frac{f\left(t,S^{12}\right) - a(t,X)}{b(t,X)}$ with $f\left(t,S^{12}\right) \equiv \frac{N_f(t,S^{12})}{D_f(t,S^{12})}$ defined by

$$\begin{aligned}
N_f(t,K) &= \frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left(\left(\sigma_1(t,S_t^1)a_t^1\right)^2 + \left(\sigma_2(t,S_t^2)a_t^2\right)^2 + 2\frac{a(t,X_t)}{b(t,X_t)}\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2\right)\left|S_t^{12}=K\right.\right]}{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left|S_t^{12}=K\right.\right]}\\
&\quad -\sigma_{\rm Dup}^{12}(t,K)^2 + \frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}\left(r_t^2 - r_t^1 - (r_t^{2,0}-r_t^{1,0})\right)1_{S_t^{12}>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)}\\
&\quad - \frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d \frac{S_t^2}{S_0^2}(r_t^1 - r_t^{1,0})(S_t^{12}-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}\\
D_f(t,K) &= 2\frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \frac{\sigma_1(t,S_t^1)a_t^1\sigma_2(t,S_t^2)a_t^2}{b(t,X_t)}\left|S_t^{12}=K\right.\right]}{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}}\left[D_{0t}^d S_t^2 \left|S_t^{12}=K\right.\right]}
\end{aligned}$$

One can then compute $\rho_{(a,b)}$ using the particle method. Eventually one has to verify that $\rho_{(a,b)}(t,X) \in [-1,1]$. If this is not the case, one may cap and floor $\rho_{(a,b)}$ when needed and check how large is the resulting smile calibration error.

Remark 9. One may wish to correlate the extra Brownian motions that drive the dynamics of the stochastic volatilities and the stochastic interest rates, and the two Brownian motions that drive the dynamics of the two FX rates. Here is one way to adapt the above method. Assume for simplicity that each of the extra processes a_t^1, a_t^2, r_t^d, r_t^1 and r_t^2 is driven by exactly one extra Brownian motion. Pick a set C^* of admissible constant values for the 12 correlations

$$C(\rho) = \{\rho_{S^1a^1}, \rho_{S^1r^d}, \rho_{S^1r^1}, \rho_{a^1r^d}, \rho_{a^1r^1}, \rho_{r^dr^1}, \rho_{S^2a^2}, \rho_{S^2r^d}, \rho_{S^2r^2}, \rho_{a^2r^d}, \rho_{a^2r^2}, \rho_{r^dr^2}\}$$

The first six correlations are used to calibrate σ_1; the last six to calibrate σ_2. Then one builds two full correlation matrices ρ^0 and ρ^1 as follows: first one picks constant values for all the unspecified correlations in the matrix except $\rho_{S^1S^2}$. Those values can be arbitrary or inferred from historical data, and may make the matrix fail to be PSD. Then one chooses the extremal value $\rho_{S^1S^2} = -1$ (resp. 1) and projects the resulting matrix onto the space of correlation matrices to get ρ^0 (resp. ρ^1). The projection method *must leave $C(\rho)$ unchanged*. This can be done by using weighted norms on matrices (see for instance [21]). Then one assumes that the entire 7×7 correlation matrix $\rho(t,X)$ lies on the line defined by ρ^0 and ρ^1: $\rho(t,X) = (1-\lambda(t,X))\rho^0 + \lambda(t,X)\rho^1$, picks two functions $a(t,X)$ and $b(t,X)$ (with b non-vanishing) and, using the particle method, builds $\lambda_{(a,b)}(t,X)$ such that $a + b\lambda_{(a,b)}$ is local in cross and the calibration condition (11.2) is satisfied, with $\rho(t,X_t)$ replaced by $(1-\lambda_{(a,b)}(t,X_t))\rho^0_{12} + \lambda_{(a,b)}(t,X_t)\rho^1_{12}$. Then one has to verify that $\lambda_{(a,b)}$ takes values in $[0,1]$. Actually, any ρ^0 and ρ^1 for which $C(\rho^0) = C(\rho^1) = C^*$ will do the job: this guarantees that the knowledge of $\rho(t,X)$ is not needed during the first step of the calibration procedure, i.e., the calibration of σ_1 and σ_2, so that indeed the calibration procedure can be cut in two consecutive steps. It is indeed desirable to calibrate σ_1 and σ_2 independently of "cross-correlations" such as $\rho_{S^1S^2}$, $\rho_{S^1a^2}$, $\rho_{S^2a^1}$...

The equity index smile calibration problem. Let us consider a model that combines local stochastic volatility, stochastic interest rate, stochastic repo (inclusive of the dividend yield), and local correlation. Here again let us assume for simplicity that the extra Brownian motions that drive the dynamics of the stochastic volatilities, the stochastic interest rate, and the stochastic repos are *independent* of the Brownian motions $(W^1,\dots,W^N)$ that drive the dynamics of the N stocks $(S^1,\dots,S^N)$. The correlation matrix of the extra Brownian motions is assumed to be known and constant. Only the correlation of $(W^1,\dots,W^N)$ is unknown; it is assumed to be local:

$$dS_t^i = (r_t - q_t^i)S_t^i\,dt + \sigma_i(t,S_t^i)a_t^iS_t^i\,dW_t^i, \qquad d\langle W^i,W^j\rangle_t = \rho_{ij}(t,X_t)dt \tag{11.5}$$

where r_t, q_t^i, a_t^i are stochastic processes, $X_t = (S_t^1,\dots,S_t^N,a_t^1,\dots,a_t^N,D_{0t})$,[3] and $D_{0t} = \exp\left(-\int_0^t r_s ds\right)$.

First, the local volatilities $\sigma_i(t,S^i)$ are calibrated to the market smiles of the S^i's using Propostion 10 in the appendix:

$$\sigma_i(t,K)^2\frac{\mathbb{E}[D_{0t}(a_t^i)^2|S_t^i=K]}{\mathbb{E}[D_{0t}|S_t^i=K]} = \sigma^i_{\mathrm{Dup}}(t,K)^2 - \frac{\mathbb{E}\left[D_{0t}\left(r_t - q_t^i - (r_t^0 - q_t^{i,0})\right)1_{S_t^i>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}_i(t,K)} + \frac{\mathbb{E}\left[D_{0t}\left(q_t^i - q_t^{i,0}\right)\left(S_t^i-K\right)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}_i(t,K)}$$

where $r_t^0 = -\partial_t \ln P_{0t}$, $q_t^{i,0} = r_t^0 - \partial_t \ln \frac{f_0^{i,t}}{S_0^i}$ (with $f_0^{i,t}$ the forward of maturity t), and

$$\sigma^i_{\mathrm{Dup}}(t,K)^2 = \frac{\partial_t\mathcal{C}_i(t,K) + (r_t^0 - q_t^{i,0})K\partial_K\mathcal{C}_i(t,K) + q_t^{i,0}\mathcal{C}_i(t,K)}{\frac{1}{2}K^2\partial_K^2\mathcal{C}_i(t,K)}$$

where $\mathcal{C}_i(t,K)$ is the market price of the call option on S^i with strike K and maturity t. This is achieved in practice thanks to the particle algorithm [16, 17]. At this step the local correlation $\rho(t,X)$ does not need to

[3] X_t may actually include any $\mathcal{F}_t$-measurable random variable (see Remark 4).

be known. Then $\rho(t,X)$ is calibrated to the market smile of the index $I_t = \sum_{i=1}^N \alpha_i S_t^i$ by requiring that (see proof in the appendix)

$$\frac{\mathbb{E}_\rho[D_{0t} v_\rho(t,X_t)|I_t=K]}{\mathbb{E}_\rho[D_{0t}|I_t=K]} = K^2 \sigma^I_{\text{Dup}}(t,K)^2 - K\frac{\mathbb{E}_\rho\left[D_{0t}\left(r_t - q_t - (r_t^0 - q_t^0)\right)1_{I_t>K}\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho\left[D_{0t}\left(q_t - q_t^0\right)(I_t-K)^+\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} \quad (11.6)$$

for all (t,K), where

$$\begin{aligned} v_\rho(t,X_t) &= \sum_{i,j=1}^N \alpha_i\alpha_j\rho_{ij}(t,X_t)\sigma_i(t,S_t^i)a_t^i\sigma_j(t,S_t^j)a_t^jS_t^iS_t^j \\ q_t &= \frac{\sum_{i=1}^N \alpha_i S_t^i q_t^i}{\sum_{i=1}^N \alpha_i S_t^i} \end{aligned} \quad (11.7)$$

r_t^0 and q_t^0 are deterministic interest rate and repo, and

$$\sigma^I_{\text{Dup}}(t,K)^2 = \frac{\partial_t\mathcal{C}(t,K) + (r_t^0 - q_t^0)K\partial_K\mathcal{C}(t,K) + q_t^0\mathcal{C}(t,K)}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}$$

with $\mathcal{C}(t,K)$ the market price of the call option on I with strike K and maturity t.

Following the lines of Section 9, if one assumes that the correlation matrix lies on the line defined by two correlation matrices ρ_0 and ρ_1, which may depend on (t,X_t),

$$\rho(t,X_t) = (1-\lambda(t,X_t))\rho^0(t,X_t) + \lambda(t,X_t)\rho^1(t,X_t), \qquad \lambda(t,X_t)\in\mathbb{R}$$

then (11.6) reads

$$\begin{aligned} &\frac{\mathbb{E}_\rho\left[D_{0t}\left(v_{\rho^0}(t,X_t) + (v_{\rho^1} - v_{\rho^0})(t,X_t)\lambda(t,X_t)\right)|I_t=K\right]}{\mathbb{E}_\rho[D_{0t}|I_t=K]} \\ &\quad = K^2\sigma^I_{\text{Dup}}(t,K)^2 - K\frac{\mathbb{E}_\rho\left[D_{0t}\left(r_t - q_t - (r_t^0-q_t^0)\right)1_{I_t>K}\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho\left[D_{0t}\left(q_t-q_t^0\right)(I_t-K)^+\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} \end{aligned}$$

When one further assumes that there exist two functions $a(t,X)$ and $b(t,X)$ such that b does not vanish and $a+b\lambda$ is local in index, then one has

$$\begin{aligned} &\frac{\mathbb{E}_\rho\left[D_{0t}\left(v_{\rho^0} - \frac{a}{b}(v_{\rho^1}-v_{\rho^0})\right)|I_t=K\right]}{\mathbb{E}_\rho[D_{0t}|I_t=K]} + (a+b\lambda)(t,K)\frac{\mathbb{E}_\rho\left[D_{0t}\frac{v_{\rho^1}-v_{\rho^0}}{b}|I_t=K\right]}{\mathbb{E}_\rho[D_{0t}|I_t=K]} \\ &\quad = K^2\sigma^I_{\text{Dup}}(t,K)^2 - K\frac{\mathbb{E}_\rho\left[D_{0t}\left(r_t - q_t - (r_t^0-q_t^0)\right)1_{I_t>K}\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_\rho\left[D_{0t}\left(q_t-q_t^0\right)(I_t-K)^+\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} \end{aligned}$$

from which one gets $\lambda(t,X) = \lambda_{(a,b)}(t,X) \equiv \frac{f(t,I)-a(t,X)}{b(t,X)}$ with $f(t,I) \equiv \frac{N_f(t,I)}{D_f(t,I)}$ defined by

$$\begin{aligned} N_f(t,K) &= K^2\sigma^I_{\text{Dup}}(t,K)^2 - K\frac{\mathbb{E}_{\rho_{(a,b)}}\left[D_{0t}\left(r_t-q_t-(r_t^0-q_t^0)\right)1_{I_t>K}\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}_{\rho_{(a,b)}}\left[D_{0t}\left(q_t-q_t^0\right)(I_t-K)^+\right]}{\frac{1}{2}\partial_K^2\mathcal{C}(t,K)} \\ &\quad - \frac{\mathbb{E}_{\rho_{(a,b)}}\left[D_{0t}\left(v_{\rho^0}(t,X_t) - \frac{a(t,X_t)}{b(t,X_t)}(v_{\rho^1}(t,X_t) - v_{\rho^0}(t,X_t))\right)|I_t=K\right]}{\mathbb{E}_{\rho_{(a,b)}}[D_{0t}|I_t=K]} \\ D_f(t,K) &= \frac{\mathbb{E}_{\rho_{(a,b)}}\left[D_{0t}\frac{v_{\rho^1}(t,X_t)-v_{\rho^0}(t,X_t)}{b(t,X_t)}|I_t=K\right]}{\mathbb{E}_{\rho_{(a,b)}}[D_{0t}|I_t=K]} \end{aligned}$$

where $\rho_{(a,b)} = (1-\lambda_{(a,b)})\rho^0 + \lambda_{(a,b)}\rho^1$. One can then compute $\rho_{(a,b)}$ using the particle method. Eventually, one has to verify that $\rho_{(a,b)}(t,X)$ is a true correlation matrix. If this is not the case, one may "cap" and "floor"

$\rho_{(a,b)}$ (to ρ^0 or ρ^1) when needed and check how large is the resulting smile calibration error. It is very easy to adapt Remark 9 to extend to cases where the extra Brownian motions are correlated with $(W^1, \ldots, W^N)$.

12. Path-dependent volatility

In Remark 4 we noticed that the particle method easily accomodates path-dependent correlation. It is easy to adapt this remark to build single asset path-dependent volatility models that calibrate to the smile. In such a model,

$$\frac{dS_t}{S_t} = (r_t - q_t)\,dt + \sigma(t, S_t, X_t)\,dW_t \tag{12.1}$$

where X_t stands for a set of path-dependent variables. The interest and repo rates r_t and q_t are assumed deterministic. The vector X_t may for instance include the running average, a moving average, the running minimum, the running maximum, the realized volatility on the past few days, etc. Hobson and Rogers [22] suggested a model where X_t is a collection of exponentially weighted moments of past returns.

Path-dependent volatility models of type (12.1) have the very nice property of being complete so that, unlike stochastic volatility models, prices are uniquely defined, independently of utility or preferences. The path-dependency allows for dynamics for the spot and the implied volatility that are richer than those produced by the local volatility model.

Assume that the market smile of S is arbitrage-free. How to calibrate σ to it? If X_t is void, the answer is well known [9]: there is a unique solution $\sigma(t, S) = \sigma_{\mathrm{Dup}}(t, S)$, called the local volatility of S, given by (14.3) with $r_t^0 = r_t$ and $q_t^0 = q_t$. This is the famous local volatility model. What if X_t includes some information on the past of S? Assume that σ calibrates to the market smile of S. From Proposition 10 in the appendix, this is equivalent to saying that

$$\mathbb{E}_\sigma[\sigma(t, S_t, X_t)^2 | S_t] = \sigma^2_{\mathrm{Dup}}(t, S_t) \tag{12.2}$$

We then say that σ is admissible. Choose two functions $a(t, S, X)$ and $b(t, S, X)$ such that b does not vanish and

$$a(t, S, X) + b(t, S, X)\sigma^2(t, S, X)$$

is local in spot, i.e., depends on (S, X) only through S. One can always do so by picking $b \equiv 1$ and $a(t, S, X) = f(t, S) - \sigma^2(t, S, X)$ for some local in spot function f. If $\sigma(t, S, X)$ does not vanish, one can also pick $a \equiv 0$ and $b(t, S, X) = \frac{f(t,S)}{\sigma^2(t,S,X)}$. Then from (12.2)

$$\left(a + b\sigma^2\right)(t, S_t)\mathbb{E}_\sigma\left[\left.\frac{1}{b(t, S_t, X_t)}\right| S_t\right] - \mathbb{E}_\sigma\left[\left.\frac{a(t, S_t, X_t)}{b(t, S_t, X_t)}\right| S_t\right] = \sigma^2_{\mathrm{Dup}}(t, S_t)$$

from which we get $\sigma = \sigma_{(a,b)}$ solution to

$$\sigma^2_{(a,b)}(t, S_t, X_t) = \frac{1}{b(t, S_t, X_t)}\left(\frac{\sigma^2_{\mathrm{Dup}}(t, S_t) + \mathbb{E}_{\sigma_{(a,b)}}\left[\left.\frac{a(t,S_t,X_t)}{b(t,S_t,X_t)}\right| S_t\right]}{\mathbb{E}_{\sigma_{(a,b)}}\left[\left.\frac{1}{b(t,S_t,X_t)}\right| S_t\right]} - a(t, S_t, X_t)\right) \tag{12.3}$$

We have thus proved that any admissible path-dependent σ is of the above type. Conversely, if a function $\sigma_{(a,b)}$ satisfies (12.3), then it is an admissible path-dependent volatility. We call (12.3), the "**local in cross $a + b\sigma^2$ representation**" of admissible path-dependent volatilities.

Note that, like (3.1), (12.3) is a circular equation: the two conditional expectations on the right hand side depend on $\sigma_{(a,b)}$. To the best of our knowledge, the existence of the nonlinear stochastic differential equations (SDEs) describing the calibrated models

$$\frac{dS_t}{S_t} = (r_t - q_t)\,dt + \sqrt{\frac{1}{b(t, S_t, X_t)}\left(\frac{\sigma^2_{\mathrm{Dup}}(t, S_t) + \mathbb{E}\left[\left.\frac{a(t,S_t,X_t)}{b(t,S_t,X_t)}\right| S_t\right]}{\mathbb{E}\left[\left.\frac{1}{b(t,S_t,X_t)}\right| S_t\right]} - a(t, S_t, X_t)\right)}\,dW_t$$

is still an open mathematical question.

In practice, one may try to build a solution $\sigma_{(a,b)}$ using the particle method (see Section 4 and [16, 17]):

(1) Initialize $k = 1$ and set $\sigma_{(a,b)}(t, S, X) = \sigma_{\mathrm{Dup}}(0, S)$ for all $t \in [t_0 = 0; t_1]$.
(2) Simulate $(S_t^i)_{1 \le i \le N}$ from t_{k-1} to t_k using a discretization scheme - say a log-Euler scheme.

(3) For all S in a grid G_{t_k} of spot values, compute non-parametric estimations $E^{\mathrm{num}}_{t_k}(S)$ and $E^{\mathrm{den}}_{t_k}(S)$ of $\mathbb{E}\left[\left.\frac{a(t_k,S_{t_k},X_{t_k})}{b(t_k,S_{t_k},X_{t_k})}\right| S_{t_k}=S\right]$ and $\mathbb{E}\left[\left.\frac{1}{b(t_k,S_{t_k},X_{t_k})}\right| S_{t_k}=S\right]$, set $f(t_k,S) = \frac{\sigma^2_{\mathrm{Dup}}(t_k,S)+E^{\mathrm{num}}_{t_k}(S)}{E^{\mathrm{den}}_{t_k}(S)}$, interpolate and extrapolate $f(t_k,\cdot)$, for instance using cubic splines, and, for all $t\in[t_k,t_{k+1}]$, set $\sigma_{(a,b)}(t,S,X)=\sqrt{\frac{f(t_k,S)-a(t,S,X)}{b(t,S,X)}}$.

(4) Set $k:=k+1$. Iterate steps 2 and 3 up to the maturity date T.

For a given pair (a,b), if at some point in time and for some path $\frac{f(t_k,S)-a(t,S,X)}{b(t,S,X)}$ is negative, i.e., $\sigma^2_{(a,b)}$ is negative, this means that there is no admissible path-dependent volatility such that $a+b\sigma^2$ is local in spot. However one can then floor $\sigma^2_{(a,b)}(t,S,X)$ to zero and carry on using the particle method until maturity. Then one must check how bad the smile calibration is. It may happen that the path-dependent volatility has to be floored on only a few paths, in which case the calibration error may be acceptable. We then say that the path-dependent volatility is almost admissible.

Given a set X of path-dependent variables, the method offers a huge number of degrees of freedom, namely the functions a and b, that can be used to build a path-dependent volatility that not only is (at least almost) admissible, but also is better than the local volatility model at reproducing some historical features of volatility, or calibrates to extra option prices, etc.

The generalization to stochastic volatility, stochastic interest rates, and stochastic dividend yield is straightforward. Assume that

$$\frac{dS_t}{S_t}=(r_t-q_t)\,dt+\sigma(t,S_t,X_t)\alpha_t\,dW_t$$

where now r_t and q_t are Itô processes, as well as the stochastic volatility α_t. From Proposition 10 in the appendix, this model is calibrated to the market smile of S if and only if for all t,K

$$\frac{\mathbb{E}[D_{0t}\sigma^2(t,S_t,X_t)\alpha_t^2|S_t=K]}{\mathbb{E}[D_{0t}|S_t=K]}=\sigma^2_{\mathrm{Dup}}(t,K)-\frac{\mathbb{E}\left[D_{0t}\left(r_t-q_t-(r_t^0-q_t^0)\right)1_{S_t>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)}+\frac{\mathbb{E}\left[D_{0t}\left(q_t-q_t^0\right)(S_t-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}$$

where σ_{Dup} is the Dupire local volatility computed using the deterministic rate r_t^0 and the derministic dividend yield q_t^0. Following the same reasoning as above, we have that in this model a path-dependent volatility $\sigma(t,X)$ calibrates to the smile of S (we say, is admissible) if and only if there exists two functions $a(t,S,X)$ and $b(t,S,X)$ such that b does not vanish and σ satisfies the self-consistency equation $\sigma^2(t,S,X)=\frac{f(t,S)-a(t,S,X)}{b(t,S,X)}$ with $f(t,K)\equiv\frac{N_f(t,K)}{D_f(t,K)}$ defined by

$$\begin{aligned}
N_f(t,K) &= \frac{\mathbb{E}_\sigma\left[\left.\frac{a(t,S_t,X_t)}{b(t,S_t,X_t)}D_{0t}\alpha_t^2\right| S_t=K\right]}{\mathbb{E}_\sigma[D_{0t}|S_t=K]}\\
&\quad+\sigma^2_{\mathrm{Dup}}(t,K)-\frac{\mathbb{E}_\sigma\left[D_{0t}\left(r_t-q_t-(r_t^0-q_t^0)\right)1_{S_t>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)}+\frac{\mathbb{E}_\sigma\left[D_{0t}\left(q_t-q_t^0\right)(S_t-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}\\
D_f(t,K) &= \frac{\mathbb{E}_\sigma\left[\left.\frac{D_{0t}\alpha_t^2}{b(t,S_t,X_t)}\right| S_t=K\right]}{\mathbb{E}_\sigma[D_{0t}|S_t=K]}
\end{aligned}$$

Again one can use the particle method to check if a pair (a,b) gives rise to an admissible path-dependent volatility. Some pairs (a,b) may be such that the self-consistency equation has no solution. Within the particle method, this is reflected in the quantity $\frac{f(t,S)-a(t,S,X)}{b(t,S,X)}$ being negative at some point in time and for some simulated path.

13. Conclusion

Only two local correlation models have been proposed in the past in order to exactly calibrate to the smile of a basket, be it a stock index, a cross FX rate, an interest rate spread, etc. Both models may actually fail to calibrate the basket smile, and, even if they do not, they impose a particular shape of the correlation matrix that one has no reason to undergo. In this article we have suggested a general procedure that produces a whole family of local correlation models among which many calibrate to the basket smile. The two existing models are just special points in the new family of models. We have also shown how to build admissible models that combine stochastic interest rates, stochastic dividend yield, local stochastic volatility, and local correlation. This generality is reached *at no cost*: the usual particle method does the job. Our procedure also easily adapts to build single asset path-dependent volatility models that calibrate to the market smile. The huge number of degrees of freedom, represented by the two functions a and b, allows one to pick one's favorite correlation with desirable properties among the new family of admissible correlations. This way we reconcile static calibration, i.e., calibration from snapshot of prices of options on basket, and dynamic calibration, i.e., calibration from historical study of state-dependency of correlation. Our numerical tests show the wide variety of admissible correlations and give insight on lower bounds/upper bounds on general multi-asset option prices given the smile of a basket and the smiles of its constituents. The derivation of the exact bounds; the derivation of conditions under which a triangle of *surfaces* of FX implied volatilities is jointly arbitrage-free; and, when so, the derivation of conditions under which an admissible local correlation does exist in theory, are three examples of important open questions that we leave for future work.

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14. Appendix: proofs

The following proposition gives a necessary and sufficient condition for a model to be calibrated to a given smile, in the presence of stochastic volatility (possibly including some local volatility component, or even some path-dependent volatility component), stochastic interest rates and stochastic dividend yield.

Proposition 10. *Let us consider the following dynamics for an asset S, where the volatility a_t, the interest rate r_t, and the repo q_t, inclusive of the dividend yield, are all stochastic processes:*

$$\frac{dS_t}{S_t} = (r_t - q_t)\,dt + a_t\,dW_t \tag{14.1}$$

Model (14.1) is exactly calibrated to the market smile of S if and only if

$$\frac{\mathbb{E}[D_{0t}a_t^2|S_t = K]}{\mathbb{E}[D_{0t}|S_t = K]} = \sigma^2_{\mathrm{Dup}}(t,K) - \frac{\mathbb{E}\left[D_{0t}\left(r_t - q_t - (r_t^0 - q_t^0)\right)1_{S_t>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}\left[D_{0t}\left(q_t - q_t^0\right)(S_t-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)} \tag{14.2}$$

for all (t,K), where $D_{0t} = \exp\left(-\int_0^t r_s ds\right)$ is the discount factor, r_t^0 and q_t^0 are deterministic rates and repos, and

$$\sigma^2_{\mathrm{Dup}}(t,K) = \frac{\partial_t\mathcal{C}(t,K) + (r_t^0 - q_t^0)K\partial_K\mathcal{C}(t,K) + q_t^0\mathcal{C}(t,K)}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)} \tag{14.3}$$

with $\mathcal{C}(t,K)$ the market price of the call option on S with strike K and maturity t.

Remark 11. The deterministic rate r_t^0 is typically taken to be equal to $-\partial_t \ln P_{0t}$, with P_{0t} the price at time 0 of a zero-coupon bond maturing at time t. Then one can infer a deterministic repo rate q_t^0 from the forward price f_0^t:

$$q_t^0 = r_t^0 - \partial_t \ln\frac{f_0^t}{S_0}$$

Proof. By applying Itô-Tanaka's formula on a discounted vanilla call payoff with maturity t and strike K, $\mathcal{P}_t \equiv D_{0t}(S_t-K)^+$, we have:

$$\begin{aligned} d\mathcal{P}_t &= -D_{0t}(S_t-K)^+r_tdt + D_{0t}1_{S_t>K}S_t\left((r_t-q_t)dt + a_tdW_t\right) + \frac{1}{2}S_t^2a_t^2D_{0t}\delta(S_t-K)dt \\ &= D_{0t}1_{S_t>K}(r_t-q_t)Kdt - D_{0t}q_t(S_t-K)^+dt + D_{0t}1_{S_t>K}a_tS_tdW_t + \frac{1}{2}K^2a_t^2D_{0t}\delta(S_t-K)dt \end{aligned}$$

By taking the expectation $\mathbb{E}[\cdot]$ on both sides of the above equation and assuming that $M_t = \int_0^t D_{0s}1_{S_s>K}a_sS_sdW_s$ is a true martingale, we get

$$\partial_t\mathcal{C}_{\mathrm{m}}(t,K) = K\mathbb{E}[D_{0t}(r_t-q_t)1_{S_t>K}] - \mathbb{E}[D_{0t}q_t(S_t-K)^+] + \frac{1}{2}K^2\sigma(t,K)^2\mathbb{E}[D_{0t}a_t^2\delta(S_t-K)]$$

where $C_{\mathrm{m}}(t,K) = \mathbb{E}[\mathcal{P}_t]$ denotes the price of the call option in the model. Then, by using that $\partial_K\mathcal{C}_{\mathrm{m}}(t,K) = -\mathbb{E}[D_{0t}1_{S_t>K}]$ and $\partial_K^2\mathcal{C}_{\mathrm{m}}(t,K) = \mathbb{E}[D_{0t}\delta(S_t-K)]$, we deduce that

$$\begin{aligned} \partial_t\mathcal{C}_{\mathrm{m}}(t,K) = {} & K\mathbb{E}[D_{0t}(r_t-q_t-(r_t^0-q_t^0))1_{S_t>K}] - (r_t^0-q_t^0)K\partial_K\mathcal{C}_{\mathrm{m}}(t,K) \\ & - \mathbb{E}[D_{0t}(q_t-q_t^0)(S_t-K)^+] - q_t^0\mathcal{C}_{\mathrm{m}}(t,K) + \frac{1}{2}K^2\partial_K^2\mathcal{C}_{\mathrm{m}}(t,K)\frac{\mathbb{E}[D_{0t}a_t^2|S_t=K]}{\mathbb{E}[D_{0t}|S_t=K]} \end{aligned}$$

with the initial condition $\mathcal{C}_{\mathrm{m}}(0,K) = (S_0-K)^+$ so by uniqueness of the solution of this PDE the model is calibrated to the market smile of S if and only if

$$\begin{aligned} \partial_t\mathcal{C}(t,K) = {} & K\mathbb{E}[D_{0t}(r_t-q_t-(r_t^0-q_t^0))1_{S_t>K}] - (r_t^0-q_t^0)K\partial_K\mathcal{C}(t,K) \\ & - \mathbb{E}[D_{0t}(q_t-q_t^0)(S_t-K)^+] - q_t^0\mathcal{C}(t,K) + \frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)\frac{\mathbb{E}[D_{0t}a_t^2|S_t=K]}{\mathbb{E}[D_{0t}|S_t=K]} \end{aligned}$$

From the definition of $\sigma_{\mathrm{Dup}}(t,K)$, this is equivalent to (14.2), which completes the proof. $\square$

The FX triangle smile calibration problem. Under Model (11.1), the dynamics of the cross rate $S^{12} = S^1/S^2$ reads

$$\frac{dS_t^{12}}{S_t^{12}} = (r_t^2 - r_t^1)\,dt + \sigma_1(t, S_t^1)a_t^1\,dW_t^{1,f} - \sigma_2(t, S_t^2)a_t^2\,dW_t^{2,f}$$

where

$$\begin{aligned} W_t^{1,f} &= W_t^1 - \int_0^t \rho(s, X_s)\sigma_2(s, S_s^2)a_s^2\,ds \\ W_t^{2,f} &= W_t^2 - \int_0^t \sigma_2(s, S_s^2)a_s^2\,ds \end{aligned}$$

are two $\mathbb{Q}^f$-Brownian motions, where $\mathbb{Q}^f$ is the risk-neutral measure associated to the foreign currency in S^2 (GBP in our example):

$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{S_T^2}{S_0^2}\exp\left(\int_0^T \left(r_t^2 - r_t^d\right)dt\right) \equiv \frac{S_T^2}{S_0^2}\frac{D_{0T}^d}{D_{0T}^2}$$

From Proposition 10, Model (11.1) is calibrated to the smile of the cross rate if and only if (11.2) holds. In the particular case of Model (2.1), where the interest rates are deterministic ($r_t^1 = r_t^{1,0}$ and $r_t^2 = r_t^{2,0}$), and the volatilities are purely local ($a_t^1 = a_t^2 \equiv 1$), (11.2) boils down to (2.2).

The equity index smile calibration problem. Under Model (11.5), the dynamics of the index $I_t = \sum_{i=1}^N \alpha_i S_t^i$ reads

$$dI_t = (r_t - q_t)I_t\,dt + \sqrt{v_\rho(t, X_t)}\,dW_t$$

where q_t and v_ρ are defined by (11.7) and W is a Brownian motion. From Proposition 10, the model is exactly calibrated to the market smile of the index I if and only if (11.6) holds. In the particular case where the interest rate is deterministic ($r_t = r_t^0$), and the repos are deterministic and equal ($q_t^1 = \cdots = q_t^N = q_t = q_t^0$; in particular when this common value is zero), (11.6) boils down to (9.2).

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