MULTI-FACTOR CROSS CURRENCY LIBOR MARKET MODELS: IMPLEMENTATION, CALIBRATION AND EXAMPLES

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Cross currency derivatives are increasingly more popular in OTC markets. Some versions of these derivatives have a very high degree of risk associated with changes in either domestic or foreign currency yield curve and foreign exchange rate. One example of these derivatives is Bermudan callable power reverse dual note. They are extremely sensitive to spot FX rate as well as changes in foreign or domestic currency yield curves. It is very appropriate that a model that takes into account stochastic yield curve of domestic and foreign economy is used for their pricing. We hereby extend the cross currency LIBOR Market Model presented by Schlogl (1999) and Mikkelsen (1999) to a multifactor case which is much more realistic. The model presented here takes into account the possibility of correlations between the stochastic domestic and foreign interest rates and FX rates. It is usually said that calibration of cross currency LIBOR Market Models is not easy. Particularly a notion that is oft heard is that the spectrum of foreign exchange calls for different maturities cannot be easily calibrated in this model. We show that the calibration of this model though numerical, is actually very straightforward and the whole spectrum of FX option prices can be fitted while simultaneously calibrating the model to interest rate option prices in both economies. Finally we show how very complex derivatives like Bermudan callable PRDCs can easily be priced in this model using a version of Least Squares Monte Carlo method that I present here.

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Cross-Currency LIBOR Market Model

Notation

$L_i(t)$: Domestic Forward rate observed at time t for the period (t_i, t_{i+1}), expressed with a compounding period of δ_i

$B(t,T)$: Price at time t of a zero coupon bond that provides a payoff of one unit of domestic currency at time T.

$m(t)$: Index for next reset date at time t.

p : Number of orthogonal factors

$\lambda_{i,q}(t)$: qth component of volatility of $L_i(t)$

$\Lambda_i(t)$: total volatility of $L_i(t)$. This equals in terms of p orthogonal components as $\sqrt{\sum_{q=1}^{p} \lambda_{i,q}{}^2(t)} = \Lambda_i(t)$

$\widetilde{L}_i(t)$: Foreign Forward rate observed at time t for the period (t_i, t_{i+1}), expressed with a compounding period of δ_i

$\widetilde{B}(t,T)$: Price in foreign currency, at time t ,of a zero coupon bond that provides a payoff of one unit of foreign currency at time T.

$\widetilde{\lambda}_{i,q}(t)$: qth component of volatility of Foreign forward rate $\widetilde{L}_i(t)$

$\widetilde{\Lambda}_i(t)$: Total volatility of $\widetilde{L}_i(t)$. This equals in terms of p orthogonal components as $\sqrt{\sum_{q=1}^{p} \widetilde{\lambda}_{i,q}{}^2(t)} = \widetilde{\Lambda}_i(t)$

$Q_T(t)$: Forward exchange rate observed at time t for maturity at time T. So $Q_t(t)$ is the spot exchange rate and $Q_{t+1}(t)$ is forward exchange rate for one period ahead observed at time t. The exchange rate is denominated in domestic currency per unit of foreign currency.

$\hat{\lambda}_q(t)$: qth component of volatility of spot exchange rate $Q_t(t)$

$\hat{\Lambda}(t)$: total volatility of $Q_t(t)$. This equals in terms of p orthogonal components as $\sqrt{\sum_{q=1}^{p} \hat{\lambda}_q^{\,2}(t)} = \hat{\Lambda}(t)$

Single Factor Set UP

Here we follow Schlogl and present a single factor cross-currency LIBOR Market Model on Forward and Spot LIBOR measures.

First of all, we take the case of Forward LIBOR measure. Forward rate process of domestic interest rates is given as on the Domestic Spot LIBOR measure

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=m(t)}^{i} \frac{\delta_j L_j(t)\lambda_j(t)\lambda_i(t)}{1+\delta_j L_j(t)} dt + \lambda_i(t)dz$$

Foreign interest rates are assumed to be lognormal martingales under their respective forward measures in foreign economy and on the foreign spot LIBOR measure are given as

$$\frac{d\widetilde{L}_i(t)}{\widetilde{L}_i(t)} = \sum_{j=m(t)}^{i} \frac{\delta_j \widetilde{L}_j(t)\widetilde{\lambda}_j(t)\widetilde{\lambda}_i(t)}{1+\delta_j \widetilde{L}_j(t)} dt + \widetilde{\lambda}_i(t)d\widetilde{z}$$

The dynamics of FX rate are given as

$$dQ_t(t) = Q_t(t)((r_t(t) - \widetilde{r}_t(t))dt + \hat{\lambda}(t)dz)$$

A log-Euler version of the above equation especially suited to simulation discretization with discrete LIBOR rates is

$$Q_{t+1}(t+1) = Q_t(t) \times \left(\frac{1+\delta L_t(t)}{1+\delta \widetilde{L}_t(t)} \right) \exp(-.5\hat{\lambda}\ (t)^2 + \hat{\lambda}\ dz\ )$$

In other words

$$Q_{t+1}(t+1) = Q_{t+1}(t) \exp(-.5\hat{\lambda}\ (t)^2 + \hat{\lambda}\ dz\ )$$

And $Q_{t+n}(t)$ is calculated at time t as

$$Q_{t+n}(t) = Q_t(t) \times \frac{\widetilde{B}(t, t_n)}{B(t, t_n)}$$

$$Q_{t+n}(t) = Q_t(t) \times \prod_{j=0}^{n-1} \frac{1+\delta L_{t+j}(t)}{1+\delta \widetilde{L}_{t+j}(t)}$$

But in the domestic economy foreign rates follow a Brownian motion with a drift dictated by their correlation with FX rate. Since in our one factor model foreign interest rate forward measure and Fx rates are perfectly correlated we make the adjustment as

$$d\widetilde{z}(t) = dz(t) - \hat{\lambda}(t)dt$$

This adjustment results in Foreign interest rates on the domestic spot LIBOR measure.

$$\frac{d\widetilde{L}_i(t)}{\widetilde{L}_i(t)} = \sum_{j=m(t)}^{i} \frac{\delta_j \widetilde{L}_j(t) \widetilde{\lambda}_j(t) \widetilde{\lambda}_i(t)}{1+\delta_j \widetilde{L}_j(t)} dt - \widetilde{\lambda}_i(t)\hat{\lambda}(t)dt + \widetilde{\lambda}_i(t)dz(t)$$

Three Factor Setup

In the three-factor set up, we allow the three processes involved to be partially uncorrelated and possibly be totally uncorrelated with each other. Unlike the extreme of single factor model where everything was perfectly correlated, this model allows for a realistic evolution of the three stochastic processes. Even after calibration, this change from single factor to three factor results in drastic differences in the prices of exotic products.

In the three-factor setup, for simplicity we assume that the correlations between interest rates in domestic and foreign economy and FX rates are constant. Though our analysis can be extended to more factors and time changing correlations.

The Domestic interest rates follow the process given below on the domestic spot LIBOR measure. Please note that all Domestic forward rates are perfectly correlated but this assumption can be relaxed by adding more factors to the model.

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=m(t)}^{i} \frac{\delta_j L_j(t) \sum_{q=1}^{p} \lambda_{j,q}(t) \lambda_{i,q}(t)}{1+\delta_j L_j(t)} dt + \sum_{q=1}^{p} \lambda_{i,q}(t) dz_q$$

The dynamics of the exchange rate are given as

$$Q_{t+1}(t+1) = Q_t(t) \left(\frac{1+\delta L_t(t)}{1+\delta \widetilde{L}_t(t)} \right) \exp(-\sum_{q=1}^{p} .5 \hat{\lambda}_q(t)^2 dt + \sum_{q=1}^{p} \hat{\lambda}_q dz_q)$$

The dynamics of foreign forward rates on the domestic spot rate are given below. Please note that we have made the Girsanov adjustment required to convert the foreign forward rates from foreign spot measure to domestic spot measure. This is just multi factor version of the single factor correction mentioned in the previous section.

$$\frac{d\widetilde{L}_i(t)}{\widetilde{L}_i(t)} = \sum_{j=m(t)}^{i} \frac{\delta_j \widetilde{L}_j(t) \sum_{q=1}^{p} \widetilde{\lambda}_{j,q}(t) \widetilde{\lambda}_{i,q}(t)}{1+\delta_j \widetilde{L}_j(t)} dt - \sum_{q=1}^{p} \widetilde{\lambda}_{i,q}(t) \hat{\lambda}_q(t) dt + \sum_{q=1}^{p} \widetilde{\lambda}_{i,q}(t) dz_q$$

Calibration of Three Factor Model

we calibrate to both interest rate processes in usual manner that is well know.

Only FX rate of one maturity can be modeled as lognormal and rest have stochastic volatility but we show how to calibrate the model to all FX call prices.

We calibrate to FX calls of all maturities using the following technique. Since, at any given time we can take only one FX rate volatility to be lognormal, we choose the rate with the shortest maturity to have lognormal volatility. This calibrates to FX call with the shortest maturity. We call this shortest maturity as time one. When the call with the shortest maturity expires, we choose the volatility of the FX rate for next maturity (time two) to have lognormal volatility during the time after first FX rate matured i.e between time one and time two. This lognormal volatility is chosen so that the FX call for this maturity (time two) is perfectly calibrated. Let us reiterate that the evolution of FX rate for maturity at time two was stochastic between time zero and time one but this has a lognormal volatility between time one and time two and this volatility is chosen so that FX call price for maturity at time two is perfectly priced. So we continue to fit all FX calls in this manner.

Please note that the procedure we follow here is closely related to Rebonato (1999). We only describe an approach that is numerical version of his procedure.

We embed simulation with optimization for the calibration of foreign exchange options. We start with a 3 X 3 correlation matrix for time t that has been pre-specified representing the correlation between domestic and foreign interest rates and FX process. We orthogonalize this symmetric positive definite matrix using eigenvalue decomposition or singular value decomposition. Suppose $\hat{\alpha}_q(t)$ is the factor loading for the exchange rate process and the *q*th factor (eigenvector), and $s_q(t)$ is the eigenvalue or singular value of the *q*th factor (factor score). Please note that both *i* and *q* range from 1 to 3 since there are three factors and three processes. Please note that Forward rates for different maturity in both local and foreign economy are considered to be perfectly correlated though this assumption can be relaxed by adding more factors. Since in our current setup, the number of processes and factors are the same, we can set the volatility coefficients of FX process with respect to factor q as

$$\hat{\lambda}_q(t) = \hat{\Lambda}(t) s_q(t) \hat{\alpha}_q(t)$$

where $\hat{\Lambda}(t)$ indicates the total volatility of foreign exchange process which is as yet unknown and has to be backed out from FX option prices. We, however, do ask the reader to make note that in our three factor setup the following identity is satisfied by the exchange rate process.

$$\sqrt{\sum_{q=1}^{p} \hat{\lambda}_q{}^2(t)} = \hat{\Lambda}(t)$$

Our intention, in this section, is to show how to choose this volatility $\hat{\Lambda}(t)$ at each time step such that Foreign Exchange options are properly calibrated.

As we have stated before, we simulate the Foreign Exchange rate for maturity at the next date in tenor structure and rest of the rates are retrieved by the deterministic relationship which we give below again between this "spot" Exchange rate and the Forward Exchange rates. Note that by each advance to next date in the tenor structure, the forward exchange rate for that particular date becomes spot rate. Specifically, we have that

$$Q_{t+1}(t+1) = Q_{t+1}(t) \exp(-\sum_{j=1}^{3} .5 \hat{\lambda}_j(t)^2 dt + \sum_{j=1}^{3} \hat{\lambda}_j dz^j)$$

Where $Q_{t+1}(t)$ has already been calculated at time t as

$$Q_{t+1}(t) = Q_t(t) \times \frac{1 + \delta L_t(t)}{1 + \delta \widetilde{L}_t(t)}$$

So it is clear that evolution of FX rate for maturity t+1 has stochastic volatility till time t and it has a lognormal volatility from time t to t+1 which we find using the procedure given below such the prices of FX call option for maturity t+1 are properly calibrated.

So the calibration procedure now can be summarized in a step by step manner as

1. At every time step t start with a decent volatility approximation. This can be any reasonable number. Let us choose $\hat{\Lambda}(t) = .2$ for illustration purpose. Next we shock this value up and down by a very small amount say $\Delta\hat{\Lambda} = .0005$. New volatility values are given as $\hat{\Lambda}(t) \pm \Delta\hat{\Lambda}$

2. For each of these three values of volatility, determine the volatility coefficients based on the equation $\hat{\lambda}_q(t) = \hat{\Lambda}(t) s_q(t) \hat{\alpha}_q(t)$. Please note that $s_q(t)$ and $\hat{\alpha}_q(t)$ are constant at any time depending only upon the correlation matrix and represent corresponding eigenvalue and relevant member of the eigenvector associated with the correlation matrix. We now have three sets of volatility coefficients for original and perturbed volatility values.

3. Suppose we have N paths in simulation. We run three sets, one for each volatility scenario, of N path simulation where for each of these paths, based on the values of FX rates at time t, $Q_{t+1}(t)$ in each path, we determine the value of $Q_{t+1}(t+1)$ by using the same random numbers for each of the three set of volatility coefficients. We use same random numbers so that we can differentiate the price with respect to volatility $\hat{\Lambda}(t)$.

4. Calculate the price of foreign exchange ATM call option for each of the three different volatility scenarios. Let us denote this call price at $\hat{\Lambda}(t)$ volatility as

$C(t+1,\hat{\Lambda}(t))$. Differentiate the price with respect to volatility using the formula

$$\frac{dC(t+1,\hat{\Lambda}(t))}{d\hat{\Lambda}(t)}=\frac{C(t+1,\hat{\Lambda}(t)+\Delta\hat{\Lambda})-C(t+1,\hat{\Lambda}(t)-\Delta\hat{\Lambda})}{\Delta\hat{\Lambda}\times 2}$$

5. Having calculated the derivative of option price with respect to volatility, we use Newton-Raphson to update the volatility to a new value. This update is done as

$$\hat{\Lambda}(t)=\hat{\Lambda}(t)-\frac{C(t+1,\hat{\Lambda}(t))-FXCall\_marketprice(t+1)}{\dfrac{dC(t+1,\hat{\Lambda}(t))}{d\hat{\Lambda}(t)}}$$

6. We repeat steps 1 to 5 with this newly calculated volatility until the calibrated call price is within a tolerable limit. In our experience only two to three iterations of Newton-Raphson are enough to reach effective convergence.

BERMUDAN CALLABLE PRDC'S IN THREE-FACTOR BGM MARKET MODEL

A power reverse dual note has the following cash flows: At each of the dates T_i, $i=1,..,n$ in the tenor structure holder of the note gets, at each date in the tenor structure, a coupon equal to

$$A\%\times\frac{\text{FX Rate}}{\text{Strike}}-B\%>\mathbf{0}$$

Issuer of the note earns domestic floating LIBOR at each date T_i reset at previous date.

$\delta L(T_{i-1})$

Cash flows at each date to the issuer at date T_i thus become

$$\delta L(T_{i-1}) - \max((A\% / Strike \times FX(T_i) - B\%), 0)$$

So the value of non-callable PRDC, to the issuer, becomes

$$E_P\left\{\sum_{i=1}^{n} \frac{B_t}{B_{T_i}} (\delta L(T_{i-1}) - \max((A\% / 120 \times FX(T_i) - B\%), 0)) \middle| \Im(t)\right\}$$

here B_{T_i} represents the value of money market account associated with the spot LIBOR measure at time T_i and expectation is taken with respect to spot LIBOR measure.

EARLY EXERCISE FRONTIER DETERMINATION FOR BERMUDAN CALLABLE PRDC

Since the issuer can redeem the note at par on each of the dates in the tenor structure, the decision by the issuer depends upon the expected future value of the cash flows discounted to the exercise time. This is exactly captured by the regression in Longstaff and Schwartz's Least Square Monte Carlo method. We choose some basis functions that we consider are closely related to the present value of the callable PRDC. Please note that basis functions represent the information which is known to us on the exercise date and strictly do not include any knowledge of future information.

Starting from last date at which the swap can be called, we regress the future cash flows on these basis functions and find regression coefficients associated with each of these basis functions. Using these regression coefficients, for each of the path, we determine whether discounted future cash flows of the callable PRDC will have a positive or negative value. If the expected discounted value of continuing the swap to the issuer is positive, note is not called and vice versa. Once we have determined whether it is optimal to call the swap at last callable date, we continue backward to the next date on which the swap can be called and repeat the procedure. Only this time, we do not regress on the cash flows till PRDC maturity but we regress on the discounted cash flows until the date when the note was called in the future date (which has previously been determined at future date using the same procedure.) We continue the above procedure till we reach the first date when the swap can be called.

So we can describe the process in a step-by-step manner as:

1. Start from the last date when the PRDC can be called. For each path calculate the payoffs from this date to the maturity of PRDC.
2. Regress the discounted payoff on value of chosen basis functions (given below) at this date.
3. Find using these basis functions, expected value of future payoff for each path.
4. If the payoff is negative, call the swap and stop the process and note this date.
5. Now advance back to the date before the last date and regress the discounted payoff until the process was stopped, again, on the value of basis functions at that time. This stopping time is maturity of the PRDC if the PRDC was never called during the path and last callable date if the PRDC was stopped there. Find using these basis functions, expected value of future payoff for each path. If the payoff is negative, call the swap and stop the process and note this date.
6. Continue until we reach the first date when the PRDC can be called.
7. Now find the value of PRDC payoffs for each path until the stopping time, discount using money market process and find the average over all paths to find the value of Bermudan callable PRDC deal.

Please note that it is also possible to model callable notes and derivatives as sum of a plain swap etc and a Bermudan option to get into the other side of the deal. The method presented above gives the same results and is easier to implement since in the pricing of Bermudan options on swaps, we have to compare exercise value and continuation value. For many derivatives it is non trivial to find the exercise value, especially in Monte Carlo. Please note that in the method given above, we do not find the present exercise value of the PRDC. This will not be easily possible in our setup since, it will require an estimate of string of call option values for each path that is not possible, as we do not have any closed form formulas for the option prices. We just find the expected continuation value and that is enough for our purpose.

CHOICE OF BASIS FUNCTIONS

The basis functions that we choose are the following

1. The two powers of expected value of redemption cost for the PRDC.
2. The three FX forward rates for the maturity at next date, maturity at the terminal date, and maturity at a date mid between the last and terminal date of PRDC.
3. The two powers of the expected value of underlying swap till maturity of PRDC. This swap consists of a leg receiving float and the other leg paying terms of PRDC.

CONCLUSIONS

In this research note we have described the dynamics of multi-factor cross currency LIBOR BGM market model. We show how to calibrate the model. We also show how to price Bermudan exotic products in this model.

REFERENCES

Andersen, L. and J. Andreasen(1998), "Volatility skews and Extensions of the Libor Market Model." Working Paper, General Re Financial Products.

Andersen, L. (1998), "A Simple Approach to Pricing of Bermudan Swaptions in the Multi-Factor Libor Market Model," Working Paper, General Re Financial Products.

Brace, A., D. Gatarek, and M. Musiela (1997), "The Market Model of Interest Rate Dynamics," Mathematical Finance, 7, 127-55.

Carr, P. and G.Yang (1997), "Simulating Bermudan Interest Rate Derivatives," Working Paper, Bank of America / Numerix.

Hull, J., and A. White (1999), "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the Libor Market Model," Working Paper, University of Toronto.

Jamshidian, F. (1997), "Libor and Swap Market Models and Measures," Finance and Stochastics, 1, 293-330.

Longstaff, F. and E. Schwartz (1998), "Valuing American options by simulation: A simple least squares approach," Working Paper Andersen School of Business, UCLA.

Mikkelsen, P. (2001),"Cross-Currency LIBOR Market Models," Working Paper Aarhus School of Business.

Miltersen K., K. Sandmann, and D. Sondermann (1997), " Closed Form solutions for Term Structure Derivatives with Lognormal Interest Rates," Journal of Finance 409-430.

Musiela, M., and M. Rutkowski (1998), Martingale Methods in Financial Modelling, Springer Verlag Inc., New York, NY.

Pedersen, M. (1999a), “Bermudan Swaptions in the Libor Market Model,”Working Paper, SimCorp A/S.

Pedersen, M. (1999b), “On the Libor Market Models,” Working Paper, SimCorp A/S.

Press W., S. Teukolsky, W. Vetterling, and B. Flannery (1992). Numerical Recipes in C, Cambridge University Press.

Rebonato, R. (1999), “On the Simultaneous Calibration of Multi-Factor Lognormal Interest Rate Models to Black Volatilities and to the Correlation Matrix,” Journal of Computational Finance 2, 5-27.

Schlogl, E. (1999),“ A Multi-Currency Extension of the Lognormal Interest Rate Market Models,” Working Paper, University of Technology, Sydney.