7 Poisson random measures

7.1 Construction and basic properties

For $\lambda \in (0, \infty)$ we say that a random variable X in $\mathbb{Z}^+$ is *Poisson of parameter* λ and write $X \sim \mathsf{Poi}(\lambda)$ if

$$\mathsf{P}(X = n) = e^{-\lambda}\lambda^n/n!\,.$$

We also write $X \sim \mathsf{Poi}(0)$ to mean $X \equiv 0$ and write $X \sim \mathsf{Poi}(\infty)$ to mean $X \equiv \infty$.

Proposition 7.1.1 (Addition property) *Let N_k, $k \in \mathbb{N}$, be independent random variables, with $N_k \sim \mathsf{Poi}(\lambda_k)$ for all k. Then*

$$\sum_k N_k \sim \mathsf{Poi}\Big(\sum_k \lambda_k\Big)\,.$$

Proposition 7.1.2 (Splitting property) *Let N, Y_n, $n \in \mathbb{N}$, be independent random variables, with $N \sim \mathsf{Poi}(\lambda)$, $\lambda < \infty$ and $\mathsf{P}(Y_n = j) = p_j$ for all $j = 1, \dots, k$ and all n. Set*

$$N_j = \sum_{n=1}^{N} 1\!\!1_{\{Y_n = j\}}\,.$$

Then $N_1, \dots, N_k$ are independent random variables with $N_j \sim \mathsf{Poi}(\lambda p_j)$ for all j.

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space. A *Poisson random measure with intensity* μ is a map

$$M : \Omega \times \mathcal{E} \to \mathbb{Z}^+$$

satisfying, for all sequences $\big(A_k : k \in \mathbb{N}\big)$ of disjoint sets in $\mathcal{E}$,

(i) $M\big(\cup_k A_k\big) = \sum_k M(A_k)$,

(ii) $M(A_k)$, $k \in \mathbb{N}$, are independent random variables,

(iii) $M(A_k) \sim \mathsf{Poi}\big(\mu(A_k)\big)$ for all k.

Denote by E^* the set of integer-valued measures on $\mathcal{E}$ and define

$$X : E^* \times \mathcal{E} \to \mathbb{Z}^+, \qquad X_A : E^* \to \mathbb{Z}^+, \qquad A \in \mathcal{E}$$

by

$$X(m, A) = X_A(m) = m(A)\,.$$

Set $\mathcal{E}^* = \sigma\left(X_A : A \in \mathcal{E}\right)$.

Theorem 7.1.3 *There exists a unique probability measure μ^* on $(E^*, \mathcal{E}^*)$ such that X is a Poisson random measure with intensity μ.*

Proof. (*Uniqueness.*) For disjoint sets $A_1, \ldots, A_k \in \mathcal{E}$ and $n_1, \ldots, n_k \in \mathbb{Z}^+$, set

$$A^* = \big\{m \in E^* : m(A_1) = n_1, \ldots, m(A_k) = n_k\big\}\,.$$

Then, for any measure μ^* making X a Poisson random measure with intensity μ,

$$\mu^*(A^*) = \prod_{j=1}^{k} e^{-\mu(A_j)} \mu(A_j)^{n_j}/n_j!\,.$$

Since the set of such sets A^* is a π-system generating $\mathcal{E}^*$, this implies that μ^* is uniquely determined on $\mathcal{E}^*$.

(*Existence.*) Consider first the case where $\lambda = \mu(E) < \infty$. There exists a probability space $\big(\Omega, \mathcal{F}, \mathsf{P}\big)$ on which are defined independent random variables N and Y_n, $n \in \mathbb{N}$, with $N \sim \mathsf{Poi}(\lambda)$ and $Y_n \sim \mu/\lambda$ for all n. Set

$$M(A) \stackrel{\mathsf{def}}{=} \sum_{n=1}^{N} \mathbb{1}_{\{Y_n \in A\}}, \qquad A \in \mathcal{E}\,. \tag{7.1}$$

It is easy to check, by the Poisson splitting property, that M is a Poisson random measure with intensity μ.

More generally, if $(E, \mathcal{E}, \mu)$ is σ-finite, then there exist disjoint sets $E_k \in \mathcal{E}$, $k \in \mathbb{N}$, such that $\cup_k E_k = E$ and $\mu(E_k) < \infty$ for all k. We can construct, on some probability space, independent Poisson random measures M_k, $k \in \mathbb{N}$, with M_k having intensity $\mu|_{E_k}$. Set

$$M(A) \stackrel{\mathsf{def}}{=} \sum_{k \in \mathbb{N}} M_k(A \cap E_k), \qquad A \in \mathcal{E}\,.$$

It is easy to check, by the Poisson addition property, that M is a Poisson random measure with intensity μ. The law μ^* on E^* is then a measure with the required properties. □

7.2 Integrals with respect to a Poisson random measure

Theorem 7.2.4 *Let M be a Poisson random measure on E with intensity μ and let g be a measurable function on E. If $\mu(E)$ is finite or g is integrable, then*

$$X = \int_E g(y)\, M(dy)$$

is a well-defined random variable with

$$\mathsf{E}\big(e^{iuX}\big) = \exp\Big\{\int_E \big(e^{iug(y)} - 1\big)\, \mu(dy)\Big\}\,.$$

Moreover, if g is integrable, then so is X and

$$\mathsf{E}(X) = \int_E g(y)\, \mu(dy), \qquad \mathsf{Var}\,(X) = \int_E g(y)^2\, \mu(dy)\,.$$

Proof. Assume for now that $\lambda = \mu(E) < \infty$. Then $M(E)$ is finite a.s. so X is well defined. If $g = 1\!\!1_A$ for some $A \in \mathcal{E}$, then $X = M(A)$, so X is a random variable. This extends by linearity and by taking limits to all measurable functions g.

Since the value of $\mathsf{E}(e^{iuX})$ depends only on the law μ^* of M on E^*, we can assume that M is given as in (7.1). Then

$$\mathsf{E}\,\big(e^{iuX} \mid N = n\big) = \mathsf{E}\big(e^{iug(Y_1)}\big)^n = \Big(\int_E e^{iug(y)}\,\frac{\mu(dy)}{\lambda}\Big)^n$$

so

$$\begin{aligned}\mathsf{E}(e^{iuX}) &= \sum_{n=0}^{\infty} \mathsf{E}\,\big(e^{iuX} \mid N = n\big)\,\mathsf{P}(N = n)\\ &= \sum_{n=0}^{\infty}\Big(\int_E e^{iug(y)}\,\frac{\mu(dy)}{\lambda}\Big)^n e^{-\lambda}\lambda^n/n! = \exp\Big\{\int_E \big(e^{iug(y)} - 1\big)\,\mu(dy)\Big\}.\end{aligned}$$

If g is integrable, then formulae for $\mathsf{E}(X)$ and $\mathsf{Var}\,(X)$ may be obtained by a similar argument.

It remains to deal with the case where g is integrable and $\mu(E) = \infty$. Assume for now that $g \geq 0$, then X is obviously well defined. We can find $0 \leq g_n \uparrow g$ with $\mu(|g_n| > 0) < \infty$ for all n. The conclusions of the theorem are then valid for the corresponding integrals X_n. Note that $X_n \uparrow X$ and $\mathsf{E}(X_n) \leq \mu(g) < \infty$ for all n. It follows that X is a random variable and, by dominated convergence, $X_n \to X$ in $L^1(\mathsf{P})$. Further, using the estimate $|e^{iux} - 1| \leq |ux|$, we can obtain the desired formulae for X by passing to the limit. Finally, for a general integrable function g, we have

$$\mathsf{E}\int_E |g(y)|\,M(dy) = \int_E |g(y)|\,\mu(dy)$$

so X is well defined. Also $X = X_+ - X_-$, where

$$X_\pm = \int_{\{\pm g > 0\}} g(y)\,M(dy)$$

and X_+ and X_- are independent. Hence the formulae for X follow from those for $X_\pm$. □

We now fix a σ-finite measure space $(E, \mathcal{E}, K)$ and denote by μ the product measure on $(0, \infty) \times E$ determined by

$$\mu\big((0, t] \times A\big) = tK(A), \qquad t \geq 0, \quad A \in \mathcal{E}.$$

Let M be a Poisson random measure with intensity μ and set $\widetilde{M} = M - \mu$. Then $\widetilde{M}$ is a *compensated Poisson measure with intensity* μ.

Proposition 7.2.5 *Let g be an integrable function on E. Set*

$$X_t \overset{\mathsf{def}}{=} \int_{(0,t] \times E} g(y)\,\widetilde{M}(ds, dy).$$

Then $\big(X_t\big)_{t\geq 0}$ *is a cadlag martingale with stationary independent increments. Moreover,*

$$\mathsf{E}(e^{iuX_t}) = \exp\Big\{t\int_E \big(e^{iug(y)} - 1 - iug(y)\big)\,K(dy)\Big\},$$

$$\mathsf{E}(X_t^2) = t\int_E g(y)^2\,K(dy).$$

Theorem 7.2.6 *Let* $g \in L^2(K)$ *and let* $\big(g_n : n \in \mathbb{N}\big)$ *be a sequence of integrable functions such that* $g_n \to g$ *in* $L^2(K)$*. Set*

$$X_t^n \stackrel{\mathsf{def}}{=} \int_{(0,t]\times E} g_n(y)\,\widetilde{M}(ds,dy).$$

Then there exists a cadlag martingale $\big(X_t\big)_{t\geq 0}$ *such that*

$$\mathsf{E}\Big(\sup_{s\leq t}|X_s^n - X_s|^2\Big) \to 0$$

for all $t \geq 0$*. Moreover,* $\big(X_t\big)_{t\geq 0}$ *has stationary independent increments and*

$$\mathsf{E}\big(e^{iuX_t}\big) = \exp\Big\{t\int_E \big(e^{iug(y)} - 1 - iug(y)\big)\,K(dy)\Big\}.$$

The notation $\int_{(0,t]\times E} g(y)\,\widetilde{M}(ds,dy)$ is used for X_t even when g is not integrable with respect to K. Of course $\big(X_t\big)_{t\geq 0}$ does not depend on the choice of approximating sequence (g_n). This is a simple example of a *stochastic integral*.

Proof. Fix $t > 0$. By Doob's L^2-inequality and Proposition 7.2.5,

$$\mathsf{E}\Big(\sup_{s\leq t}|X_s^n - X_s^m|^2\Big) \leq 4\mathsf{E}\big((X_t^n - X_t^m)^2\big) = 4t\int_E (g_n - g_m)^2\,K(dy) \to 0$$

as n, $m \to \infty$. Hence X_s^n converges in L^2 for all $s \leq t$. For some subsequence we have

$$\sup_{s\leq t}|X_s^{n_k} - X_s^{n_j}| \to 0 \qquad \text{a.s.}$$

as j, $k \to \infty$. The uniform limit of cadlag functions is cadlag, so there is a cadlag process $(X_s)_{s\leq t}$ such that

$$\sup_{s\leq t}|X_s^{n_k} - X_s| \to 0 \qquad \text{a.s.}$$

Since X_s^n converges in L^2 for all $s \leq t$, $(X_s)_{s\leq t}$ is a martingale and so by Doob's L^2-inequality

$$\mathsf{E}\Big(\sup_{s\leq t}|X_s^n - X_s|^2\Big) \leq 4\mathsf{E}\big((X_t^n - X_t)^2\big) \to 0.$$

Note that $|e^{iug} - 1 - iug| \leq u^2 g^2/2$. Hence, for $s < t$ we have

$$\begin{aligned}
\mathsf{E}\left(e^{iu(X_t - X_s)} \mid \mathcal{F}_s^M\right) &= \lim_n \mathsf{E}\left(e^{iu(X_t^n - X_s^n)} \mid \mathcal{F}_s^M\right) \\
&= \lim_n \exp\Big\{(t-s)\int_E \big(e^{iug_n(y)} - 1 - iug_n(y)\big)\, K(dy)\Big\} \\
&= \exp\Big\{(t-s)\int_E \big(e^{iug(y)} - 1 - iug(y)\big)\, K(dy)\Big\}
\end{aligned}$$

which shows that $\big(X_t\big)_{t\geq 0}$ has stationary independent increments with the claimed characteristic function. $\square$