



OpenGamma Quantitative Research

**Deliverable Interest Rate  
Swap Futures: Pricing in  
Gaussian HJM Model**

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### **Abstract**

CME will soon be proposing a new product: Deliverable Interest Rate Swap Futures. This note analyses the pricing of such futures in the Gaussian multi-factor HJM model and multi-curves framework. We provide also numerical example of prices and hedging with those futures.

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# 1 Introduction

CME<sup>1</sup> announced that it will launch in November 2012 a new futures contract called *Deliverable Interest Rate Swap Futures*. This futures contract has the particularity that it settles into an OTC swap cleared on CME.

In the first part of this note we first describe the most important features of the product. The futures is margined on a daily basis as like other exchange traded futures. Its specific feature is in its settlement mechanism.

In the second part we propose a pricing formula in the Gaussian HJM multi-factor model. The pricing is obtained in the nowadays standard multi-curves framework described in [Henrard \(2010\)](#) with deterministic spread. Due to the daily margining, the price of the futures is not directly in line with the one of the underlying swap; so-called *convexity adjustments* are required. In the pricing development we suppose that the underlying swap can be priced with OIS discounting applied to collateralized swaps.

The results are very similar to the one presented in [Kennedy \(2010\)](#) for SwapNote futures, but extended to multi-factor HJM and multi-curves. Both are related to results obtained for the more complex bond futures, including the delivery option, presented in [Henrard \(2006\)](#).

In the last section we present several numerical examples on hedging swaps with swap futures and short term swaps.

## 2 Swap Futures

The futures nominal is USD 100,000 per contract; the notional is denoted  $N$  in this note. The margining feature is the future-type daily margin on the quoted price<sup>2</sup>. The underlying swap has the standard conventions for USD swaps: semi-annual bond basis versus Libor 3M. The standard swap conventions and their description can be found in [Quantitative Research \(2012\)](#). The futures are quoted for swaps with tenors 2, 5, 10 and 30 years. The underlying swap has a fixed rate as decided by the exchange on the first trading date of the contract. The rate is changed in increments of 25 basis points. The rate is not fixed at a predefined value like the reference coupon of bond futures.

The *delivery dates* follow the quarterly cycle standard to interest rate futures. The *delivery date* is the third Wednesday of the quarterly months (March, June, September, December). The *last trading date* or *expiry date* is two business days prior to that date, usually on the Monday. The expiry date is denoted  $\theta$  in this note.

On the expiry date, the parties agree to enter into a swap where the party long the futures receives fixed on the swap and the party short the futures pays fixed. The *effective date* of the swap, denoted  $t_0$ , is the delivery date. The fixed rate of the swap is the one attached to the swap futures. The swap has also an up-front payment on the delivery date. The up-front payment is obtained from the futures settlement price on the last trading date, denoted  $F_\theta$ . The amount received by the long party is  $(1 - F_\theta) * N^3$ .

The 1 in the formula is cosmetic feature. One could not add it to compute the up-front amount and keep the same economic reality. The only thing that would change is that the future *price*

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<sup>0</sup>First version: September 2012; this version: September 2012.

<sup>1</sup>[www.cmegroup.com](http://www.cmegroup.com)

<sup>2</sup>Note that the price is quoted in (percentage) points and 32nd of points, like the bond futures contracts.

<sup>3</sup>If the amount is negative, it should be interpreted as the absolute value is paid by the long party.

would be around 0 and not around 1 and the range of price would be different from the other futures. Personally I would prefer it without the 1, but it is a personal taste.

The arbitrage free price of the future in  $\theta$  satisfy, for  $PV_\theta$  the present value of the underlying swap without up-front payment in  $\theta$  and with notional 1 to the long party,

$$(1 - F_\theta)NP^D(\theta, t_0) + PV_\theta N = 0$$

or

$$F_\theta = 1 + \frac{PV_\theta}{P^D(\theta, t_0)}.$$

After delivery, the swap is cleared on CME. The swap are collateralized.

Regarding the long/short choice in the contract definition (long is receiving fixed), it can be related to bond futures. The interest rate sensitivity are similar between the two types of futures. Being long the future is similar to being long a bond and short the rate/yield.

### 3 Multi-curves

We briefly describe the features of the multi-curves framework we use in this note. More details can be found in [Henrard \(2010\)](#). The discounting curve discount factors are denoted  $P^D$  and the forward curve pseudo-discount factors are denoted  $P^j$ . For the swap futures in USD, the relevant forward curve is the one associated to the Libor three months. In practice, the discounting curve is often the one linked to OIS.

The spread hypothesis we use are based on the following definition of spread:

**Definition 1** *The spread between a forward curve and the discounting curve is*

$$\beta_t^j(u, u + j) = \frac{P^j(t, u)}{P^j(t, u + j)} \frac{P^D(t, u + j)}{P^D(t, u)}. \quad (1)$$

The main spread hypothesis itself reads as:

**SI** The spreads  $\beta_t^j(u, u + j)$ , defined in Equation (1), are independent of the discount factors  $P^D(t, v)$  for all maturities  $v$ .

As the swaps are collateralized, we suppose that their valuation can be done in the multi-curves framework with the OIS curve as discounting curve.

An IRS is described by a set of fixed coupons or cash flows  $c_i$  at dates  $\tilde{t}_i$  ( $1 \leq i \leq \tilde{n}$ ). For those flows, the discounting is used. It also contains a set of floating coupons or cash flows over the periods  $[t_{i-1}, t_i]$  with  $t_i = t_{i-1} + j$  ( $1 \leq i \leq n$ )<sup>4</sup>.

The value of a (fixed rate) receiver swap is

$$PV_t = \sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^n P^D(t, t_i) \left( \frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right). \quad (2)$$

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<sup>4</sup>In practice, due to weekends and holidays, the periods used for the fixings can be slightly different from the payment dates. We will not make that distinction here.

With Definition 1, the floating coupons can be rewritten as fixed coupon with amounts function of  $\beta$ :

$$\begin{aligned} P^D(t, t_i) \left( \frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right) &= P_X^D(t, t_i) \left( \beta_t^j(t_{i-1}, t_i) \frac{P^D(t, t_{i-1})}{P^D(t, t_i)} - 1 \right) \\ &= \beta_t^j(t_{i-1}, t_i) P^D(t, t_{i-1}) - P^D(t, t_i). \end{aligned}$$

The swap value becomes

$$PV_t = \sum_{i=0}^n d_t^i P^D(t, t_i)$$

with  $d_t^i$  the appropriate quantities, dependent linearly on  $\beta_t^i(t_i, t_{i+1})$ .

## 4 Model and hypothesis

A term structure model describes the behaviour of  $P^D(t, u)$ . When the discount curve  $P^D(t, \cdot)$  is absolutely continuous, which is something that is the case in practice as the curve is constructed by some kind of interpolation, there exists  $f(t, u)$  such that

$$P^D(t, u) = \exp \left( - \int_t^u f(t, s) ds \right). \quad (3)$$

The short rate associated to the curve is  $(r_t)_{0 \leq t \leq T}$  with  $r_t = f(t, t)$ . The cash-account numeraire is  $N_t = \exp(\int_0^t r_s ds)$ .

The idea of [Heath et al. \(1992\)](#) was to model  $f$  with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u) \cdot dW_t$$

for some suitable  $\mu$  and  $\sigma$  and deducing the behavior of  $P^D$  from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of [Hunt and Kennedy \(2004\)](#).

To simplify the writing, the notation

$$\nu(t, u) = \int_t^u \sigma(t, s) ds$$

is used.

The equations of the model in the cash-account numeraire measure associated to  $N_t$  are

$$df(t, u) = \sigma(t, u) \cdot \nu(t, u)dt + \sigma(t, u) \cdot dW_t.$$

The HJM model is called Gaussian when the volatility  $\sigma$  is deterministic. The Hull and White one-factor volatility model [Hull and White \(1990\)](#) satisfies the above description with  $\nu(s, t) = (1 - \exp(-a(t - s)))\eta(s)/a$  and  $\sigma(s, t) = \eta(s) \exp(-a(t - s))$ . It is also the case for the G2++ model described in [Brigo and Mercurio \(2006\)](#).

The following technical lemma was presented in [Henrard \(2003\)](#) for the Gaussian one-factor HJM. Similar formulas can be found in ([Brody and Hughston, 2004](#), (3.3),(3.4)) in the framework of coherent interest-rate models and in [Nunes and de Oliveira \(2004\)](#) for multi-factor Gaussian HJM.

**Lemma 1** *Let  $0 \leq t \leq u \leq v$ . In HJM framework the price of the zero coupon bond is, in the cash-account numeraire,*

$$P(u, v) = \frac{P(t, v)}{P(t, u)} \exp \left( - \int_t^u (\nu(s, v) - \nu(s, u)) \cdot dW_s - \frac{1}{2} \int_t^u (|\nu(s, v)|^2 - |\nu(s, u)|^2) ds \right).$$

Using the lemma twice, the discount factor ratios in the futures price can be written as

$$\frac{P(\theta, t_i)}{P(\theta, t_0)} = \frac{P(0, t_i)}{P(0, t_0)} \exp \left( -\alpha_i \cdot X_i - \frac{\alpha_i^2}{2} \right) \gamma(\theta, t_i)$$

with

$$\gamma(\theta, t_i) = \exp \left( \int_0^\theta \nu(s, t_0) \cdot (\nu(s, t_0) - \nu(s, t_i)) ds \right),$$

$$\alpha_i^2 = \int_0^\theta |\nu(s, t_i) - \nu(s, t_0)|^2 ds,$$

and  $X_i$  N-standard normal random variables. Note that as  $\nu$  is increasing in the second variable and  $t_i \geq t_0$ , the adjustment factor  $\gamma$  is less than 1.

## 5 Swap Futures pricing

The results presented here are similar to the one presented in [Kennedy \(2010\)](#) for SwapNote futures and in [Henrard \(2006\)](#) for bond futures. They are more general as they are for multi-factor models and in the multi-curves framework.

Using the generic pricing future price process theorem ([Hunt and Kennedy, 2004](#), Theorem 12.6),

$$F_0 = \mathbb{E}_{\mathbb{N}} [F_\theta].$$

Using the swap pricing formula,

$$F_\theta = 1 + \sum_{i=0}^n d_\theta^i \frac{P^D(\theta, t_i)}{P^D(\theta, t_0)}.$$

With the above developments, the futures can be written as function of random variable  $X_i$ :

$$1 + \sum_{i=0}^n d_t^i \frac{P^D(0, t_i)}{P^D(0, t_0)} \exp \left( -\alpha_i \cdot X_i - \frac{1}{2} \alpha_i^2 \right) \gamma(\theta, t_i).$$

The exponential terms have a expectation of 1. We thus obtain the following result:

**Theorem 1** *In the multi-curves framework with independent spread hypothesis **SI** in the Gaussian HJM model, the price of a swap futures with expiry date  $\theta$  is given by*

$$F_0 = 1 + \frac{1}{P^D(0, t_0)} \sum_{i=0}^n \mathbb{E}_{\mathbb{N}} [d_\theta^i] P^D(0, t_i) \gamma_\theta(t_i).$$

A simplifying hypothesis often used in the multi-curves framework is a deterministic hypothesis



**S0** The multiplicative coefficients between discount factor ratios,  $\beta_t^j(u, u+j)$ , as defined in Equation (1), are constant through time:  $\beta_t^j(u, u+j) = \beta_0^j(u, u+j)$  for all  $t$  and  $u$ .

With hypothesis **S0**, the price is

$$F_0 = 1 + \frac{1}{P^D(0, t_0)} \sum_{i=0}^n d_0^i P^D(0, t_i) \gamma_\theta(t_i). \quad (4)$$

## 6 Numerical examples

In this section we give some numerical<sup>5</sup> examples of the impact of the futures features on the price and risks. The analysis is done in a one-factor Hull-White model. We compare the futures to the underlying swap. The comparison is on the price (for market making) and the curve risk (for hedging). the comparisons are done with a flat curve with a 3% (continuously compounded) rate and a 2% Hull-White constant volatility.

From Formula (4), it is clear that the impact on the value is two fold. The first one is the multiplication by the common factor  $1/P^D(0, t_0)$ . This factor is always larger than 1 (when interest are positive). This represent the fact that the profit is paid immediately (through the margin) and not a settlement. As the futures have usually an expiry up to three or six month in the future, the impact is up to 0.5 times the short term rate. With the current very low rate this impact is almost negligible (below 0.10%). For higher level of rates, the impact can be non-negligible. When interest are 3%, the impact on a six month future is around 1.5%.

The differences between the futures price (with the 1 subtracted) and the underlying swap present value are displayed in Table 1. The differences clearly depend on all three item: expiry, tenor and moneyness.

Expiry	Moneyness	2Y	5Y	10Y	30Y
3M	-100 bps	-1.56	-3.73	-6.94	-15.85
	0 bps	-0.26	-0.62	-1.14	-2.47
	+100 bps	1.04	2.49	4.66	10.91
9M	-100 bps	-6.11	-14.51	-26.72	-59.16
	0 bps	-2.22	-5.25	-9.59	-20.73
	+100 bps	1.67	4.02	7.54	17.69
Price differences in basis points.					

Table 1: Difference in future price and swap present value for different expiries, tenor and moneyness.

On the curve risk side, the derivative with respect to the rate will be roughly  $t_n - t_0$  for the future and  $t_n$  for the swap. The ratio swap notional/future notional will be impacted by this. Using this very crude approximation, for a 3 months futures, the ratio is around 1.125 for a 2 years tenor and 1.01 for a 30 year tenor. Actual ratios for different scenarios are given in Table 2. The ratios strongly depend on the future expiry and underlying swap tenor but not on the moneyness.

The second impact is the so-called convexity adjustment  $\gamma_\theta(t_i)$ . As mentioned earlier, those adjustments are always less than 1 and this will decrease the price. The factors depend on the

<sup>5</sup>All the figures are computed using the OG-Analytics library. The library is open source and available at [www.opengamma.com](http://www.opengamma.com).

Expiry	Moneyness	2Y	5Y	10Y	30Y
3M	-100 bps	1.13	1.05	1.02	1.01
	+100 bps	1.14	1.05	1.02	1.01
6M	-100 bps	1.32	1.10	1.04	1.01
	+100 bps	1.32	1.10	1.05	1.01
9M	-100 bps	1.58	1.16	1.07	1.02
	+100 bps	1.58	1.17	1.07	1.02
Hedging ratios: the notional of futures required to hedge the same amount of swap.					

Table 2: Swap/futures nominal total sensitivity hedging.

level of volatility (the adjustment is larger for higher volatilities) and on the time to maturity (the adjustment is larger for longer maturities). As the underlying tenor are between 2 and 30 years, the impact will be considerably different between the different futures. Also the adjustment factor impact will depend on the *moneyness* (difference between the current rate level and the coupon of the underlying swap). As the coupon is fixed on the first trading date and start to trade several quarter before the delivery date, the difference can be important.

We analyse also the bucketed sensitivities. The difference appears mainly at delivery. The futures has an extra sensitivity on the discounting curve, as expected from the  $1/P^D(0, t_0)$  term. In Table 3 we display the sensitivities of a futures on the two years swap with a coupon close to the money and an expiry in three months time (futures with delivery on 19-Dec-12 valued on 20-Sep-12). The sensitivities are the derivative with respect to the rates (continuously compounded) at the node point indicated and for a notional of 1. We work in an OIS discounting framework and use OIS to hedge part of the difference between swaps and futures. The OIS we use is a swap with end of its fixing period on the futures delivery date. We hedge the underlying swap with a notional of 100m using the futures and the OIS. The quantity of futures is obtain by hedging the total sensitivity to the forward curve. The quantity of OIS is obtained by hedging the total sensitivity of the swap and its futures hedge to the discounting curve. The residual of hedging 100m receiver swap with the above described portfolio. The portfolio consists of selling 993<sup>6</sup> contracts and receiving OIS for 102.4m is also displayed in the last columns. For those two column the figures represent the sensitivity to a one basis point move for the total portfolio.

Table 4 displays the same information for a 30 years tenor futures with expiry in six months. Note that the hedging quality on the forward curve is not uniform through the tenors. This is expected as in Equation 4 the cash-flows are multiplied by convexity adjustments that are different for each cash-flow date.

## 7 Conclusion

We propose a pricing formula for swap futures in the Gaussian multi-factor HJM and in a multi-curves framework. We also provide some numerical examples showing the difference between the futures and the underlying swap prices and curves sensitivities.

<sup>6</sup>The figure provided are for a not rounded hedge. In practice it is not possible to trade fractions of contracts. The figures achievable in practice will be slightly different.

Tenor	Futures		Swap		OIS		Residual	
	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd
3m	0.24	0.25	0.00	0.24	-0.23	0.00	0.37	-0.02
6m	0.00	-0.01	0.00	-0.01	0.00	0.00	-0.08	0.03
1y	0.01	-0.02	0.01	-0.02	0.00	0.00	-0.10	0.10
2y	0.01	-1.99	0.01	-1.98	0.00	0.00	-0.18	-0.08
5y	0.00	-0.18	0.00	-0.17	0.00	0.00	-0.01	-0.02
10y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of 100m receiver swap, selling 993 futures contracts and paying 102.4m OIS up to expiry. The sensitivities of the portfolio are in USD for a one basis point move.

Table 3: Swap/futures/OIS sensitivities and hedging efficiency.

Tenor	Futures		Swap		OIS		Residual	
	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd	Dsc	Fwd
3m	0.01	0.01	0.00	0.01	0.00	0.00	-107.98	-0.05
6m	0.49	0.48	0.00	0.48	-0.48	0.00	219.91	-2.68
1y	0.00	-0.02	0.00	-0.02	0.00	0.00	-0.47	0.54
2y	0.00	-0.14	0.00	-0.13	0.00	0.00	-2.36	2.98
5y	0.00	-0.54	0.00	-0.53	0.00	0.00	-8.91	10.56
10y	0.02	-3.33	0.01	-3.29	0.00	0.00	-50.02	45.97
30y	-0.03	-16.96	-0.04	-16.75	0.00	0.00	-50.17	-57.31

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of 100m receiver swap, selling 987 futures contracts and paying 103.7m OIS up to expiry. The sensitivities of the portfolio are in USD for a one basis point move.

Table 4: Swap/futures/OIS sensitivities and hedging efficiency.

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