BOND AND CDS PRICING WITH RECOVERY RISK I: THE STOCHASTIC RECOVERY MERTON MODEL

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ABSTRACT. In this work we incorporate recovery risk into Merton's original credit risk model by introducing a separate risk driver for the recovery process and rationalize this new model within a "partial information" perspective. We show that while adding the recovery risk driver has no impact on probabilities of default (PD), it does have an impact on loss given default (LGD), and on quantities that depend on LGD such as credit prices and spreads. In fact, the addition of recovery risk allows for a mechanism to increase credit spreads, and therefore may account for some of the bond mispricings typical when using Merton's model. Finally, using the new model we price both bonds and CDS, and explicitly compute the price of recovery risk.

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1. Background and Motivation

In his seminal paper [30], Merton proposed a model for assessing the credit risk of a firm by characterizing the firm's equity as a call option on its assets. Merton's model assumes that at time $t \geq 0$ the firm's capital structure consists of equity $E_{t,T}$ and a zero-coupon bond $B_{t,T}$ with maturity T and notional N. The firm's asset value A_t is then simply the sum of the equity and debt values. Under these assumptions, equity represents a call option on the firm's assets with maturity T and strike price of N, which we write as

$$E_{t,T}^{\text{Merton}} := \widetilde{\mathbb{E}}_t \left[e^{-r(T-t)} (A_T - N) \mathbb{1}_{\{A_T > N\}} \right] \tag{1.1}$$

where we have taken the numeraire to be the cash money-market account with constant deterministic interest rate.

The payoff in (1.1) describes the situation that if at maturity T the firm's asset value A_T is enough to pay back the face value of the debt N, then the firm does not default and shareholders receive equity $A_T - N \geq 0$. Otherwise $A_T < N$, the firm defaults, bondholders take control of the firm receiving fA_T, and shareholders receive nothing. The remaining assets, $(1-f)A_T$, are attributed to friction such as bankruptcy costs [1] (c.f. Section 2.3 in [11]).

We can write the default mechanism described above as a random default time τ where

$$\tau := T \mathbb{1}_{\{A_T < N\}} + \infty \mathbb{1}_{\{A_T \geq N\}}. \tag{1.2}$$

The price of a zero-coupon defaultable bond is then given by the risk-neutral expectation of the discounted general payoff Π_τ, where

$$\Pi_\tau := N \mathbb{1}_{\{\tau > T\}} + R_\tau \mathbb{1}_{\{\tau \leq T\}}. \tag{1.3}$$

Here the first term describes receiving the full payment N at maturity in the event of no default, while the second term describes receiving a recovery value R_τ in the event of default. In a one-factor structural model such as the Merton model, a simplifying assumption is made that the recovery value R_τ at default τ is simply a fixed percentage of the asset value A_τ at default time τ. Thus the Merton model assumes

$$R_\tau := f A_\tau. \tag{1.4}$$

[1] Strictly speaking, Merton's model [30] assumes no friction so $f \equiv 1$. However, here we include the generalization for $f \in [0,1]$ so as to have a framework consistent with the stochastic recovery model we present in this paper.

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Under this assumption the general payoff (1.3) reduces to

$$\Pi_\tau := N\mathbb{1}_{\{\tau>T\}} + fA_\tau\mathbb{1}_{\{\tau\le T\}} \tag{1.5}$$

and the price of a defaultable zero-coupon bond is again given by the risk-neutral expectation of Π_τ,

$$\begin{aligned} B_{t,T} &= \widetilde{\mathbb{E}}_t[e^{-r(\tau\wedge T-t)}\Pi_\tau] \\ &= Ne^{-r(T-t)}\widetilde{\mathbb{P}}[\tau>T] + \widetilde{\mathbb{E}}_t[e^{-r(\tau-t)}fA_\tau\mathbb{1}_{\{\tau\le T\}}] \end{aligned} \tag{1.6}$$

where r is the risk-free rate, assumed to be constant, and $\wedge$ is the min-operator. The first term in (1.6) is the discounted expectation of the firm not defaulting and honoring its payment obligation, while the second term is the discounted expectation of the recovery amount if the firm were to default and not honor its payment obligations. For the Merton default time (1.2), $\tau \wedge T = T$ but we prefer to keep the $\wedge$ notation in the exposition for full generality since the same framework can be extended to other credit models for which default can occur at times other than maturity T (c.f. [8]).

The typical calibration methodology for Merton's model requires estimating the unobservable asset value A and volatility σ_A from the market observable equity value E and volatility σ_E. This leads to an informational contradiction in that the inaccessible default time is estimated from accessible information which in turn leads to two main inconsistencies in structural models; predictable default and predictable recovery. More precisely, in Merton's model the assumption is that the present asset value is known exactly (perfect information) and therefore default in the short-term is a predictable process. In addition, this implies the recovery amount in the event of default is perfectly known as well. Thus the model predicts zero short-term credit spreads and recovery at default equal to the asset value. However, due to the fact that the model is calibrated with imperfect information from the market, it is empirically observed that indeed firms default despite the model predicting otherwise (non-zero short-term credit spreads) and that often the recovered amount is different than the calibrated asset value at default (recovery risk). The recovered amount at default can be higher than the calibrated asset value (as is often the case when default is triggered by liquidity issues) or lower than the calibrated asset value (as is often the case when default occurs due to misreporting of financial statements). The fact that empirically the recovery value may be different than the asset value at default amounts to recovery risk, which is typically not included in structural models of credit risk.

While several mechanisms have been proposed to address the issue of zero short-term credit spreads in structural models (including adding jumps to the asset price [40], randomizing the asset value [39] and filtering [12, 28]) few attempts have been made to address the recovery risk which arises from

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the exact same informational assumptions that leads to zero short-term credit spreads. One notable exception is in CDO modeling where the need for stochastic recovery was borne from some pricing anomalies in 2007/2008 arising from the fixed 40% recovery assumption (c.f. [3, 4, 19, 26]). However, similar modeling efforts have not been made for other credit products such as bonds and credit default swaps, and as of yet a general theory of recovery risk has not emerged.

The lack of a recovery risk in credit models such as Merton lead to several shortcomings. The first is that they treat recovery for firms with recession-proof tangible assets (say cash and physical property) the same as recovery for firms with recession-susceptible intangible assets (say patents and trademarks). And since not all firms respond to economic shocks and stresses in the same way, the parameter f should not be a constant, but depend on both the macro- and micro-economic environment. Second, is that they do not allow for separation of the risk premium into default risk and recovery risk since it is a single factor model. Indeed, the asset value serves as both the default driver (determining the probability of default) as well as the recovery driver (determining the value of recovery at default). Therefore the only type of credit risk captured in the original Merton's model is default risk.

To introduce recovery risk into the model, we introduce a separate recovery risk driver R_t correlated with A_t so that the ratio R_τ/A_τ is no longer a constant as in (1.4) but in fact stochastic. We discuss the decoupling of the default risk driver A_t and the recovery risk driver R_t in an informational perspective in Section 3.2.

This paper is organized as follows. In Section 2 we review the classic one-factor Merton model as a benchmark for our later results where we include recovery risk. In Section 3 we introduce recovery risk into the Merton model and motivate the need for such a risk driver through a partial information perspective. In Section 4 we give closed form pricing formulas for bonds and credit derivatives under this new framework. Finally, in Section 6 we analyze the effect of recovery risk on prices and spreads by comparing the original Merton model with the stochastic recovery Merton model, thereby explicitly pricing recovery risk.

2. Review of Credit Risk and Pricing in Merton's Model

In this section we review the classic Merton model of credit risk. In particular we price both bonds and CDS under this model which will serve as a benchmark for pricing bonds and CDS with recovery risk considered in Section 4.

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2.1. **Bond Pricing with Merton's Model.** Consider now the credit risk of a firm for which:

- Assets follow a geometric Brownian motion

$$dA_t = \mu_A A_t dt + \sigma_A A_t dZ_t^A \tag{2.1}$$

- Outstanding debt is of the form of a zero-coupon bond with notional N, payable at maturity T.
- Default τ is defined by (1.2).
- Payoff is given by (1.3) with recovery $R_T = fA_T$ (1.4).

In this model, default results in a turnover of the company's assets to bondholders if assets are worth less than the total debt outstanding. Consistent with these assumptions we have

$$\begin{aligned} E_{t,T}^{\text{Merton}} &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\max\{A_T - N, 0\}\right] \\ &= A_t\Phi(d_1) - Ne^{-r(T-t)}\Phi(d_0) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} B_{t,T}^{\text{Merton}} &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[N\mathbb{1}_{\{\tau>T\}} + fA_\tau\mathbb{1}_{\{\tau\leq T\}}\right] \\ &= Ne^{-r(T-t)}\Phi(d_0) + fA_t\Phi(-d_1) \end{aligned} \tag{2.3}$$

where

$$d_0 = d_0(A_t, r, \sigma_A, T-t) = \frac{\ln(A_t/N) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \tag{2.4}$$

$$d_1 = d_1(A_t, r, \sigma_A, T-t) = \frac{\ln(A_t/N) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \tag{2.5}$$

are the risk-neutral distances to default. Under the physical measure, we define the physical distances to default as

$$d_0^\mu = d_0(A_t, \mu_A, \sigma_A, T-t) = \frac{\ln(A_t/N) + (\mu_A - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \tag{2.6}$$

$$d_1^\mu = d_1(A_t, \mu_A, \sigma_A, T-t) = \frac{\ln(A_t/N) + (\mu_A + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}}. \tag{2.7}$$

A fundamental consequence is that assets can be decomposed as

$$A_t = E_{t,T}^{\text{Merton}} + B_{t,T}^{\text{Merton}} + C_{t,T}^{\text{Merton}} \tag{2.8}$$

where

$$C_{t,T}^{\text{Merton}} = (1-f)A_t\Phi(-d_1) \tag{2.9}$$

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are the friction (bankruptcy) costs which depend on f. Note that in the case that $f = 1$ the costs are zero so that $A_t = E_{t,T}^{\text{Merton}} + B_{t,T}^{\text{Merton}}$.

The bond credit spread $S_{t,T}$ is defined as the spread over the constant risk-free rate r which reprices the bond. Thus $S_{t,T}$ satisfies

$$B_{t,T} = Ne^{-(r+S_{t,T})(T-t)} \tag{2.10}$$

which for Merton's model yields

$$S_{t,T}^{\text{Merton}} = -\frac{1}{T-t} \ln \left(\Phi(d_0) + \frac{fA_t}{e^{-r(T-t)}N} \Phi(-d_1) \right). \tag{2.11}$$

Finally, defining $\widetilde{\text{PD}}_{t,T}^{\text{Merton}}$ to be the risk-neutral probability at time t of defaulting at time T, and $\widetilde{\text{LGD}}_{t,T}^{\text{Merton}}$ to be the risk-neutral expected loss at time t given default at time T, we have

$$\widetilde{\text{PD}}_{t,T}^{\text{Merton}} = \widetilde{\mathbb{P}}_t[\tau \leq T] = \widetilde{\mathbb{P}}_t[A_T \leq N] = \Phi(-d_0) \tag{2.12}$$

$$\widetilde{\text{LGD}}_{t,T}^{\text{Merton}} = \widetilde{\mathbb{P}}_t[1 - \frac{A_T}{N} | \tau \leq T] = 1 - e^{r(T-t)} \frac{fA_t}{N} \frac{\Phi(-d_1)}{\Phi(-d_0)}. \tag{2.13}$$

In Section 6 we analyze these quantities when Recovery Risk is added to the model.

2.2. **CDS Pricing with Merton's Model.** A Credit Derivative Swap (CDS) is an OTC contract in which one party (the buyer) pays premiums to another party (the seller) to insure against default on a bond (c.f. [22, 33]). Pricing consists of separately modeling the present value of the fixed premiums paid by the protection buyer, and the present value of the contingent default payment leg received by the buyer. The difference between the two is then the value of the CDS. If there is no upfront fee at initiation of the contract, then the premium P is given as the value that makes the contract worthless at initiation.

To be more precise, let T be the expiry of the CDS contract and let $\mathbb{T}_n := \{0 = t_0, t_1, t_2, ...t_n = T\}$ be the premium payment dates. For $i = 1...n$ we define $\Delta t_i = t_i - t_{i-1}$ to be the time between payments. The premium leg of the transaction is then given by the risky prevent value of the premium payments $P_{t,T}$ that the buyer pays (and seller receives), namely

$$V_{t,T}^{\text{Premium}} = P_{t,T} \left[N \sum_{i=1}^{n} D(t, t_i) \widetilde{\mathbb{P}}[\tau > t_i] \Delta t_i + \mathcal{A}_p \right] \tag{2.14}$$

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where $\mathcal{A}_p$ is the accrual payment in case default occurs between two payment dates. We consider the continuous setting so that (2.14) becomes

$$V_{t,T}^{\text{Premium}} = P_{t,T}\left[N\int_t^T D(t,s)\widetilde{\mathbb{P}}[\tau > s]ds + \mathcal{A}_p\right]. \tag{2.15}$$

Similarly, in the continuous setting the protection (default) leg is then given as

$$V_{t,T}^{\text{Protection}} = N\widetilde{\mathbb{E}}_t[D(t,\tau)(1-\bar{R}_\tau)\mathbb{1}_{\tau\leq T}] \tag{2.16}$$

where $\bar{R}_\tau$ is the *recovery rate*[2] at time τ. Using usual no arbitrage principles, the CDS premium $P_{t,T}$ is given as the value that balances these two equations, namely

$$P_{t,T} = \frac{\widetilde{\mathbb{E}}_t[D(t,\tau)(1-\bar{R}_\tau)\mathbb{1}_{\{\tau\leq T\}}]}{\int_t^T D(t,s)\widetilde{\mathbb{P}}[\tau > s]ds + \frac{\mathcal{A}_p}{N}}. \tag{2.17}$$

To evaluate (2.17) we need a model for the recovery rate $\bar{R}_t$ and default τ. A standard assumption [22, 33] in a hazard rate framework is that recovery is a constant (i.e. $\bar{R}_t = \bar{R}$ for all t) so that the CDS premium (2.17) reduces to

$$P_{t,T} = (1-\bar{R})\cdot\frac{\widetilde{\mathbb{E}}_t\left[D(t,s)\mathbb{1}_{\{\tau\leq T\}}\right]}{\int_t^T e^{-r(s-t)}\widetilde{\mathbb{P}}_t[\tau > s]ds + \frac{\mathcal{A}_p}{N}}. \tag{2.18}$$

However, in a structural formulation of the CDS premium, one could use (2.17) directly instead of (2.18) of a process for R_τ is defined. In the classical Merton model where there is no recovery risk driver, recovery is not modeled explicity but instead (2.17) is evaluated using $\bar{R}_\tau = A_\tau/N$. In Section 4.3 we will relax this assumption further and price using the stochastic recovery dynamics for R_τ described in Section 3. In particular we will use $\bar{R}_\tau = R_\tau/N$ where R_τ is given by (3.8).

Despite the restrictive assumption in Merton's model that default is possible only at maturity T, closed form prices and premiums for CDS can nevertheless be computed under the model assumptions. The model assumes interest rates are constant so the discount factor D is simply $D(t,s) = e^{-r(s-t)}$. Thus using the Merton default time (1.2) and assuming assets follow (2.1)the value of the premium leg (2.14) is explicitly computed as

[2]In our framework the recovery rate $\bar{R}_\tau$ is related to recovery R_τ by $\bar{R}_\tau = R_\tau$/N.

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$$
\begin{aligned}
V_{t,T}^{\text{Premium}} &= P_{t,T}\left[N\widetilde{\mathbb{E}}_t\left[\int_t^T e^{-r(s-t)}\mathbb{1}_{\{\tau>s\}}ds\right]\right]\\
&= P_{t,T}\left[N\int_t^T e^{-r(s-t)}\widetilde{\mathbb{P}}_t[\tau>s]ds\right]\\
&= P_{t,T}^{\text{Merton}}N\int_t^T e^{-r(s-t)}ds\\
&= P_{t,T}^{\text{Merton}}N\left[\tfrac{1-e^{-r(T-t)}}{r}\right]
\end{aligned} \tag{2.19}
$$

Similarly, the value of the protection leg (2.16) is computed as

$$
\begin{aligned}
V_{t,T}^{\text{Protection}} &= N\widetilde{\mathbb{E}}_t\left[e^{-r(\tau-t)}\left(1-\frac{R_\tau}{N}\right)\mathbb{1}_{\{\tau\leq T\}}\right]\\
&= Ne^{-r(T-t)}\cdot\widetilde{\mathbb{E}}_t\left[\left(1-\frac{A_T}{N}\right)\mathbb{1}_{\{\tau_{\text{Merton}}\leq T\}}\right]\\
&= Ne^{-r(T-t)}\cdot\widetilde{\mathbb{E}}_t\left[\left(\frac{N-A_T}{N}\right)\mathbb{1}_{\{A_T<N\}}\right]\\
&= Ne^{-r(T-t)}\cdot\widetilde{\mathbb{E}}_t\left[\left(\frac{N-A_T}{N}\right)\mid A_T<N\right]\cdot\widetilde{\mathbb{P}}_t[A_T<N]\\
&= Ne^{-r(T-t)}\left[\Phi(-d_0)-e^{r(T-t)}\frac{A_t}{N}\Phi(-d_1)\right]\\
&= Ne^{-r(T-t)}\cdot\widetilde{\text{LGD}}_{t,T}^{\text{Merton}}\cdot\widetilde{\text{PD}}_{t,T}^{\text{Merton}}
\end{aligned} \tag{2.20}
$$

where $\widetilde{\text{PD}}_{t,T}^{\text{Merton}}$ and $\widetilde{\text{LGD}}_{t,T}^{\text{Merton}}$ are the risk-neutral Merton PD and LGD given by (2.12) and (2.13). Thus in the Merton model, the CDS premium $P_{t,T}$ (2.17) is computed as

$$
\begin{aligned}
P_{t,T}^{\text{Merton}} &= \frac{re^{-r(T-t)}}{1-e^{-r(T-t)}}\left[\Phi(-d_0)-e^{r(T-t)}\frac{A_t}{N}\Phi(-d_1)\right]\\
&= r\frac{e^{-r(T-t)}}{1-e^{-r(T-t)}}\cdot\widetilde{\text{PD}}_{t,T}^{\text{Merton}}\cdot\widetilde{\text{LGD}}_{t,T}^{\text{Merton}}.
\end{aligned} \tag{2.21}
$$

Note using the Merton model to price Bonds and CDS gives consistent spreads in that the CDS premium (2.21) is approximately equal to the Merton bond spread (2.11). The difference is primarily due to the model dependence in defining the Premium-leg (2.14). We record this observation in the following Lemma.

Lemma 2.1. (Consistency of Bond and CDS Spreads in Merton's Model). *Consider a parameter regime* $(r, \sigma_A, T-t, \frac{A_t}{N})$ *where both* $r(T-t)$ *and* $\widetilde{\text{LGD}}_{t,T}^{\text{Merton}}\cdot\widetilde{\text{PD}}_{t,T}^{\text{Merton}} << 1$. *Then, in this regime, the resulting Merton CDS premium per unit face and Merton bond credit spread are approximately equal, i.e.*

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$$P^{\text{Merton}}_{t,T} \approx S^{\text{Merton}}_{t,T}. \tag{2.22}$$

Proof. Define $z = r(T-t)$. Then it follows that

$$r(T-t)\frac{e^{-r(T-t)}}{1-e^{-r(T-t)}} = z\frac{e^{-z}}{1-e^{-z}} = \frac{1-z+\frac{z^2}{2!}}{1-\frac{z}{2!}+\frac{z^2}{3!}} \approx 1 \tag{2.23}$$

and so

$$\begin{aligned} P^{\text{Merton}}_{t,T} &= r\frac{e^{-r(T-t)}}{1-e^{-r(T-t)}} \cdot \widetilde{\text{LGD}}^{\text{Merton}}_{t,T} \cdot \widetilde{\text{PD}}^{\text{Merton}}_{t,T} \\ &\approx \frac{1}{T-t}\left(\widetilde{\text{LGD}}^{\text{Merton}}_{t,T} \cdot \widetilde{\text{PD}}^{\text{Merton}}_{t,T}\right) \end{aligned} \tag{2.24}$$

Furthermore, by employing a Taylor expansion around $x = 0$, we obtain the approximation $\ln\left(\frac{1}{1-x}\right) \approx x$ for small x. Recalling the formula for credit spread under the Merton model, we return the approximation for credit spread

$$\begin{aligned} S^{\text{Merton}}_{t,T} &= \frac{1}{T-t}\ln\left(\frac{1}{1-\widetilde{\text{LGD}}^{\text{Merton}}_{t,T} \cdot \widetilde{\text{PD}}^{\text{Merton}}_{t,T}}\right) \\ &\approx \frac{1}{T-t}\left(\widetilde{\text{LGD}}^{\text{Merton}}_{t,T} \cdot \widetilde{\text{PD}}^{\text{Merton}}_{t,T}\right). \end{aligned} \tag{2.25}$$

It follows that in this parameter regime, $P^{\text{Merton}}_{t,T} \approx S^{\text{Merton}}_{t,T}$.

□

We remark here that recently a framework for pricing CDS using the Merton model was proposed in [17] and obtain a spread that is fundamentally different from (2.21) (c.f. equation 14 in [17]). The main difference is that they use (2.18) instead of (2.17) despite using a structural framework to price the CDS. This, of course, is in contrast with with our formulation where we use (2.17) to compute the Merton model CDS spread. We argue that our formulation of the Merton CDS spread is correct in that it is internally consistent. That is, if one uses a structural framework, then it is not correct to pull the recovery out of the integral in (2.17) since it is given by the asset value at default A_τ.

3. Modeling Recovery Risk within A Structural Framework

The Merton model considered in Section 2 has been extended in several directions, including adding stochastic interest rates, bankruptcy costs, taxes, debt subordination, strategic default, time dependent stochastic default barrier, jumps in the asset value process, etc. However, none of these

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extensions consider recovery as a risk factor, and therefore the only credit risk they account for is default risk.

Recovery risk, however, is an important factor to model for several reasons. One reason is that including recovery risk in a structural framework allows for larger credit spreads and a decomposition of the spread into a default risk and recovery risk premium. This is important because empirical literature suggests that structural models tend to underestimate observed credit spreads by $10 - 15\%$ on average (c.f. [14, 15, 21, 25]) and it is often argued that this extra observed spread is explained by liquidity risk. However in [6] Chen, Colin-Dufresne & Goldstein suggest that recovery risk could be a mechanism to reconcile the discrepancies between spreads generated by structural models with those observed in the market and in a recent empirical study, Schläfer & Uhrig-Homburg [36] show that recovery risk is mispriced in the market in that there is pronounced systemic component in recovery rates for which investors should receive a risk premium. *By adding a recovery risk driver to the classical Merton model through a stochastic recovery process, we show theoretically that recovery risk can lead to an increase in the credit spread and are able to give formulas for the additional recovery risk premium.*

Recovery risk has been has been investigated by several researchers in the context of credit capital (c.f. [16, 32]. Recently, Levy & Hu [29] introduced a model which attempts to account for this correlation by explicitly modeling the correlation between the asset and recovery risk drivers. The model begins by introducing an additional shadow recovery process R_t which attempts to capture the empirical PD-LGD correlation, and they rationalize the model via economic arguments (c.f. [1, 20, 34, 38]). In Section 3.1 we present this model and in Section 3.2 we recast recovery risk within the context of a partial information perspective. Note that the partial information perspective of recovery risk we present here gives an alternative explanation for PD-LGD correlation beyond the usual economic-cycle explanation in the literature. We then use this new model to price bonds and CDS and give prices in closed form in Section 4. We discuss calibration to market data in Section 5. Finally in Section 6 we explicitly obtain the market price of recovery risk.

3.1. **The Correlated Asset-Recovery Model.** We model the recovery R_t as a geometric Brownian motion similar to A_t. The justification for R_t having similar dynamics as A_t comes from the observation that typically what is recovered in the event of default are parts of the firm's assets, and therefore it is natural for the recovery price process to have the same behavior as the asset price process.

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To be more precise, let A_t denote the asset price at time $t > 0$ and let R_t denote the recovery amount at time $t > 0$. The unobservable process R_t is interpreted as the amount of the asset that would be recovered if default were to occur at t. The asset and recovery processes are modeled as two correlated geometric Brownian motions on $(\Omega, \mathcal{F}_t, \mathbb{P})$ given by

$$
\begin{aligned}
dA_t &= \mu_A A_t dt + \sigma_A A_t dZ_t^A \\
dR_r &= \mu_R R_t dt + \sigma_R R_t dZ_t^R \\
\langle dZ_t^A, dZ_t^R \rangle &= \rho_{A,R} dt.
\end{aligned} \tag{3.1}
$$

3.2. **Connection between Recovery Risk and Partial Information.** There is growing literature on credit modeling under partial or incomplete information. In this perspective, initiated by Duffie & Lando [12] and further elaborated upon in [5, 7, 9, 18, 24, 28] among others, the asset value in a structural model is interpreted as the asset value as seen by the *firm's Manager*. This leads to a firm bankruptcy being predictable for the manager (accessible default time) but unpredictable for the market (inaccessible default time), and has been used to explain zero short-term credit spreads in Structural models versus non-zero short-term credit spreads for hazard-rate models,

The correlated asset-recovery process (3.1) can be interpreted within this partial information context as follows. The variable A_t is the firm's assets as seen by the Manager and hence is the default driver. The variable R_t on the other hand are the actual assets recovered in the event of default and only seen by the market at the time of default. This explains the need for different variables and the possibility that $R_t > A_t$ so that equity $E_{t,T}$ may be non-zero at default. That is, default occurs because the Manager has a view that there are insufficient funds to pay bondholders, but the actual value of the assets is not known until liquidation, at which time the market may discover that the value was more than sufficient to pay bondholders, and in addition there is some remaining capital left over for equity holders. In fact, this frequently occurs in situations where default is driven by liquidity issues. This decoupling of a variable A_t that is estimated by the firm's Manager and a variable R_t that is estimated by the market manifests itself in the pricing formulas for bonds and CDS. In particular, it gives a deeper understanding of the original Merton model.

3.3. **Some Preliminary Results.**

Lemma 3.1. *(Existence of Risk Neutral Measure) Let (A_t, R_t) be the coupled measurable stochastic processes on $(\Omega, \mathcal{F}_t, \mathbb{P})$ given by (3.1). Then there exists a risk-neutral measure $\widetilde{\mathbb{P}}_t$ such that the process $X_t := e^{-r(T-t)}(A_t, R_t)$*

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is martingale under $\widetilde{\mathbb{P}}_t$. Furthermore, there exists a 2-d $\mathcal{F}_s$-adapted Brownian motion $(\widetilde{Z}^A, \widetilde{Z}^R)$ under the risk-neutral measure $\widetilde{\mathbb{P}}$ such that (A_t, R_t) satisfies

$$\begin{aligned} dA_t &= rA_t dt + \sigma_A A_t d\widetilde{Z}_t^A \\ dR_t &= rR_t dt + \sigma_R R_t d\widetilde{Z}_t^R \\ \langle d\widetilde{Z}, d\widetilde{Z}_t^R \rangle &= \rho_{A,R} dt. \end{aligned} \tag{3.2}$$

Proof. The proof follows from the results in [27] as the pair (A_t, R_t) is a two-dimensional diffusion.

□

As a direct corollary, we have the following result:

Lemma 3.2. *(Solution to the PD-LGD Equations) Let (A_t, R_t) be given by* (3.1). *Then, under the physical measure, (A_t, R_t) is given by*

$$A_t = A_0 \exp\Big((\mu_A - \frac{1}{2}\sigma_A^2)t + \sigma_A Z_t^A \Big) \tag{3.3}$$

$$R_t = R_0 \exp\Big((\mu_R - \frac{1}{2}\sigma_R^2)t + \sigma_R Z_t^R \Big) \tag{3.4}$$

and under the risk-neutral measure,

$$A_t = A_0 \exp\Big((r - \frac{1}{2}\sigma_A^2)t + \sigma_A \widetilde{Z}_t^A \Big) \tag{3.5}$$

$$R_t = R_0 \exp\Big((r - \frac{1}{2}\sigma_R^2)t + \sigma_R \widetilde{Z}_t^R \Big) \tag{3.6}$$

Note that the recovery process R_t in (3.2) is not the recovery rate at default, but rather a shadow process which when stopped at default time τ gives us the value of recovered assets R_τ. Thus it is only observed at time τ. The recovery rate $\bar{R}$ of a zero-coupon bond, for instance, would then be the ratio of recovered assets over the notional amount (i.e. $\bar{R} = \frac{R_\tau}{N}$).

In order to express R as a function of A as well as a stochastic process independent of A, we introduce another Brownian motion $\widetilde{W}$ on our probability space where we can express the solution to (3.2) as

$$\begin{aligned} A_t &= A_0 \exp\Big((r - \frac{1}{2}\sigma_A^2)t + \sigma_A \widetilde{Z}_t^A) \Big) \\ R_t &= R_0 \exp\Big((r - \frac{1}{2}\sigma_R^2)t + \sigma_R \widetilde{Z}_t^R \Big) \\ \widetilde{Z}_t^R &= \rho_{A,R} \widetilde{Z}_t^A + \sqrt{1 - \rho_{A,R}^2}\, \widetilde{W}_t \\ \langle d\widetilde{Z}_t^A, d\widetilde{W}_t \rangle &= 0. \end{aligned} \tag{3.7}$$

Solving explicitly for the recovery process we have

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$$R_t = R_0\Big(\frac{A_t}{A_0}\Big)^{\gamma} \exp\Big(\delta t + \sigma_R\sqrt{1-\rho_{A,R}^2}\widetilde{W}_t\Big) \tag{3.8}$$

where

$$\begin{aligned} \gamma &:= \rho_{A,R}\frac{\sigma_R}{\sigma_A} \\ \delta &:= (r - \frac{1}{2}\sigma_R^2) - \gamma(r - \frac{1}{2}\sigma_A^2) \end{aligned} \tag{3.9}$$

Given the correlated asset-recovery process (3.2), it is natural to ask what the ratio of recoverable assets to total assets is at any given time. In particular, since the recoverable assets R_t also are a geometric Brownian motion (3.8), it is theoretically possible for recovery at default, R_T, to be larger than the value of assets at default, A_T. We calculate below the probability that recovery at default is greater than asset value at default.

Lemma 3.3. (Dynamics of the Recovery Fraction). *Let $f_t := R_t/A_t$ be the recovery fraction at time $t \in [0,T]$ where A_t and R_t are given by* (3.2). *Then f_t satisfies the stochastic differential equation*

$$\begin{aligned} \frac{df_t}{f_t} &= \big(\mu_R - \mu_A + \sigma_A^2 - \rho_{A,R}\sigma_A\sigma_R\big)\,dt + (\sigma_R dZ_t^R - \sigma_A dZ_t^A) \\ &= \big(\sigma_A^2 - \rho_{A,R}\sigma_A\sigma_R\big)\,dt + (\sigma_R d\widetilde{Z}_t^R - \sigma_A d\widetilde{Z}_t^A) \\ &= \big(\sigma_A^2 - \rho_{A,R}\sigma_A\sigma_R\big)\,dt + \sigma_d d\overline{Z}_t \\ &= (1-\gamma)\sigma_A^2 dt + \sigma_d d\bar{Z}_t \\ \sigma_d^2 &:= \sigma_A^2 + \sigma_R^2 - 2\rho_{A,R}\sigma_A\sigma_R = (1-2\gamma)\sigma_A^2 + \sigma_R^2 \\ \overline{Z}_s &:= \frac{\sigma_R}{\sigma_d}\widetilde{Z}_s^R - \frac{\sigma_A}{\sigma_d}\widetilde{Z}_s^A \\ f_0 &= R_0/A_0 \end{aligned} \tag{3.10}$$

and has solution

$$\begin{aligned} f_t &= f_0 \exp\left[\frac{1}{2}\left(\sigma_A^2 - \sigma_R^2\right)t + \sigma_R\widetilde{Z}_t^R - \sigma_A\widetilde{Z}_t^A\right] \\ &= f_0 \exp\left[\frac{1}{2}\left(\sigma_A^2 - \sigma_R^2\right)t + \sigma_d\bar{Z}_t\right]. \end{aligned} \tag{3.11}$$

Proof. The proof follows from applying Ito's Lemma to the function $h(A,R) = \frac{R}{A}$, or alternatively simply computing the division of the solutions for R_t, A_t directly. □

Remark 3.4. *Notice that this fraction f_t approaches f_0 in the appropriate sense as $\rho_{A,R}, \gamma \to 1$ (i.e. $R_t \to A_t$). See the following Lemma for a measure*

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of this convergence, as well as a measure of the likelihood that recovered values surpass that of the asset.

Lemma 3.5. (Probability that Recovery is Greater Than Assets).

The probability that the recovery R_T is larger than assets A_T at maturity T is

$$\widetilde{\mathbb{P}}_t[R_T > A_T] = \Phi(-d) \tag{3.12}$$

where $\rho \in (-1, 1)$, $\sigma_A \neq \sigma_R$, and $A_0 \neq R_0$, and

$$d := \frac{\ln\left(\frac{A_t}{R_t}\right) + \frac{1}{2}\left(\sigma_R^2 - \sigma_A^2\right)(T-t)}{\sqrt{(\sigma_A^2 + \sigma_R^2 - 2\rho_{A,R}\sigma_A\sigma_R)(T-t)}}. \tag{3.13}$$

If $\rho = 1$, $\sigma_A = \sigma_R$, and $A_0 = R_0$, then $\widetilde{\mathbb{P}}_t[R_T > A_T] = 0$.

Proof. Without loss of generality, we set $t = 0$. Defining $\sigma_d^2 := \sigma_A^2 + \sigma_R^2 - \rho_{A,R}\sigma_A\sigma_R$, it follows that for a standard normal random variable $Z \sim N(0, \sigma_d^2 T)$ under $\tilde{\mathbb{P}}$,

$$\begin{aligned}
\widetilde{\mathbb{P}}_0[R_T > A_T] &= \widetilde{\mathbb{P}}_0\left[\frac{R_T}{A_T} > 1\right] = \widetilde{\mathbb{P}}_0[f_T > 1] \\
&= \widetilde{\mathbb{P}}\left[\sigma_d \bar{Z}_T > \ln\left(\frac{A_0}{R_0}\right) - \frac{1}{2}\left(\sigma_A^2 - \sigma_R^2\right)T\right] \\
&= \widetilde{\mathbb{P}}\left[\sqrt{\sigma_d^2 T} Z > \ln\left(\frac{A_0}{R_0}\right) + \frac{1}{2}\left(\sigma_R^2 - \sigma_A^2\right)T\right] \\
&= \widetilde{\mathbb{P}}\left[\sqrt{\sigma_d^2 T} Z < -\Big(\ln\left(\frac{A_0}{R_0}\right) + \frac{1}{2}\left(\sigma_R^2 - \sigma_A^2\right)T\Big)\right].
\end{aligned} \tag{3.14}$$

□

Remark 3.6. *The result that recovery R_T may be larger than the asset value A_T at default has a clear justification within the context of the "partial information" interpretation described in Section 3.2. Since in a structural model, the assets and default risk are as seen by the firm's Manager, the result shows how default may be decided by the firm's manager upon assessing the firms assets and liabilities and goes into default, only to find out during bankruptcy workout that the firm value is larger than what the manager had assessed.*

Lemma 3.7. (Expected Recovery Given Default). *The expected recovery at default in the Stochastic Recovery Merton model is*

$$\widetilde{\mathbb{E}}_t\left[R_\tau | \tau \leq T\right] = R_t e^{r(T-t)} \frac{\Phi(-d_\gamma)}{\Phi(-d_0)} \tag{3.15}$$

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where in analogy with d_0 and d_1 (2.4)

$$\begin{aligned} d_\gamma &= d_0 + \gamma\sigma_A\sqrt{T-t} \\ d_\gamma^\mu &= d_0^\mu + \gamma\sigma_A\sqrt{T-t} \\ \gamma &= \rho_{A,R}\frac{\sigma_R}{\sigma_A} \end{aligned} \tag{3.16}$$

Furthermore, the relationship between the risk-neutral and real-world expected recoveries is then

$$\frac{\widetilde{\mathbb{E}}_t\left[R_\tau|\tau \le T\right]}{\mathbb{E}_t\left[R_\tau|\tau \le T\right]} = e^{(r-\mu_R)(T-t)}\frac{\Phi(-d_\gamma)}{\Phi(-d_0)}\frac{\Phi(-d_0^\mu)}{\Phi(-d_\gamma^\mu)}. \tag{3.17}$$

Proof. By employing the solution for (A_T, R_T) outlined in (3.7) and by setting $t = 0$ without loss of generality, we obtain

(3.18)

$$\begin{aligned} \widetilde{\mathbb{E}}_0\left[R_\tau|\tau \le T\right] &= \widetilde{\mathbb{E}}_0\left[R_T|A_T < N\right] \\ &= \widetilde{\mathbb{E}}_0\left[R_T|\widetilde{Z}_T^A < -d_0\sqrt{T}\right] \\ &= \widetilde{\mathbb{E}}\left[R_0 e^{(r-\frac{1}{2}\sigma_R^2)T+\sigma_R\left(\rho_{A,R}\widetilde{Z}_T^A+\sqrt{1-\rho_{A,R}^2}\widetilde{W}_T\right)} \mid \widetilde{Z}_T^A < -d_0\sqrt{T}\right] \\ &= R_0 e^{(r-\frac{1}{2}\sigma_R^2)T}\widetilde{\mathbb{E}}\left[e^{\sigma_R\sqrt{1-\rho_{A,R}^2}\widetilde{W}_T}\right]\widetilde{\mathbb{E}}\left[e^{\rho_{A,R}\sigma_R Z_T^A} \mid \widetilde{Z}_T^A < -d_0\sqrt{T}\right] \\ &= R_0 e^{(r-\frac{1}{2}\sigma_R^2)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}\widetilde{\mathbb{E}}\left[e^{\rho_{A,R}\sigma_R Z_T^A} \mid \widetilde{Z}_T^A < -d_0\sqrt{T}\right] \\ &= R_0 e^{(r-\frac{1}{2}\rho_{A,R}^2\sigma_R^2)T}\frac{\int_{-\infty}^{-d_0}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}e^{\rho_{A,R}\sigma_R\sqrt{T}z}dz}{\int_{-\infty}^{-d_0}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz} \\ &= R_0 e^{rT}\frac{\int_{-\infty}^{-d_0}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\gamma\sigma_A\sqrt{T})^2}{2}}dz}{\Phi(-d_0)} \\ &= R_0 e^{rT}\frac{\Phi(-d_\gamma)}{\Phi(-d_0)}. \end{aligned}$$

A similar calculation using (2.6) and the solution for A under the real-world measure in Lemma 3.2 returns

$$\mathbb{E}_t\left[R_\tau|\tau \le T\right] = R_t e^{\mu_R(T-t)}\frac{\Phi(-d_\gamma^\mu)}{\Phi(-d_0^\mu)}. \tag{3.19}$$

This then yields (3.17).

□

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In Section 4 below, we use the previous Lemma to compute the bond and CDS prices when recovery is assumed to follow the correlated Asset-Recovery model (3.2).

4. Merton's Model with Recovery Risk

Let us now return to the pricing of zero-coupon bonds when both default and recovery risk are modeled. Recall the generalized payoff Π_τ is given by

$$\Pi_\tau = N\mathbb{1}_{\{\tau>T\}} + R_\tau \mathbb{1}_{\{\tau\leq T\}}. \tag{4.1}$$

and the bond price is given as the discounted risk-neutral expectation of Π_τ

$$\begin{aligned} B_{t,T} &= \widetilde{\mathbb{E}}_t[e^{-r((\tau\wedge T)-t)}\Pi_\tau] \\ &= Ne^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau > T] + \widetilde{\mathbb{E}}_t[e^{-r(\tau-t)}R_\tau \mathbb{1}_{\{\tau\leq T\}}]. \end{aligned} \tag{4.2}$$

If we include only default risk as in the original Merton model, then both the default time τ as well as recovery R_τ are defined directly through the asset value (c.f. (1.2) and (1.4)). In the Stochastic Recovery Merton model, default τ is still defined by a condition on the assets but we introduce an additional risk driver for the recovery value R_τ. The recovery risk driver is correlated to the asset risk driver so as to model the empirically observed PD-LGD correlation.

4.1. **Equity as a Call on Recovery.** Adding recovery risk to the Merton model gives rise to a new paradigm for the capital structure related to the decoupling of the default driver and the recovery driver.

Recall that under the partial information interpretation of recovery risk, default is driven by the Manager's estimation of the firm's assets A_t while the true value of the assets are given by R_t. Hence A_t determines the default rate while R_t determines the recovery rate. To derive a formulation of equity within this partial information framework of recovery risk, consider two cases:

i. Default. If $A_T < N$, then default occurs and the market estimated value of the assets R_T become known through liquidation so that $E_{T,T} = \max\{R_T - N, 0\}$.

ii. No Default. If $A_T \geq N$ then default does not occur and since equity is observed by the market then again it should be given through the market estimated value of the assets R_T so that again $E_{T,T} = \max\{R_T - N, 0\}$.

Therefore equity E_T at maturity T is given by

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$$E^{\text{SRM}}_{T,T} := \max\{R_T - N, 0\} \underbrace{\mathbb{1}_{\{A_T < N\}}}_{\text{Default}} + \max\{R_T - N, 0\} \underbrace{\mathbb{1}_{\{A_T \geq N\}}}_{\text{No Default}} \tag{4.3}$$
$$= \max\{R_T - N, 0\}$$

Lemma 4.1. (Equity in the Stochastic Recovery Merton Model). *Let equity at maturity T be given by (4.3). Then for any time $t \in [0, T]$ we have*

$$E^{\text{SRM}}_{t,T} = R_t \Phi(d^R_1) - N e^{-r(T-t)} \Phi(d^R_0) \tag{4.4}$$

where d^R_0 and d^R_1 are given by (2.4) with A replaced with R, namely

$$d^R_0 = \frac{\ln(R_t/N) + (r - \frac{1}{2}\sigma^2_R)(T-t)}{\sigma_R \sqrt{T-t}} \tag{4.5}$$

$$d^R_1 = \frac{\ln(R_t/N) + (r + \frac{1}{2}\sigma^2_R)(T-t)}{\sigma_R \sqrt{T-t}}. \tag{4.6}$$

Proof. The proof follows from the Fundamental Theorem of Asset Pricing [37] on the discounted payoff $E^{\text{SRM}}_{T,T}$. □

Lemma 4.2. (Capital Structure in the Stochastic Recovery Merton Model). *Let $C^{\text{SRM}}_{t,T}$ be given by*

$$C^{\text{SRM}}_{t,T} = e^{-r(T-t)} \widetilde{E}_t \left[(N - R_T) \mathbb{1}_{\{A_T \geq N, R_T < N\}} + (R_T - N) \mathbb{1}_{\{A_t < N, R_T \geq N\}} \right] \tag{4.7}$$

Then for any $t \in [0, T]$ we have

$$R_t = B^{\text{SRM}}_{t,T} + E^{\text{SRM}}_{t,T} - C^{\text{SRM}}_{t,T} \tag{4.8}$$

$$A_t = B^{\text{SRM}}_{t,T} + E^{\text{Merton}}_{t,T} + e^{-r(T-t)} \widetilde{\mathbb{E}}_t \left[(A_T - R_T) \mathbb{1}_{\{A_T < N\}} \right] \tag{4.9}$$

$$= B^{\text{SRM}}_{t,T} + E^{\text{Merton}}_{t,T} + A_t \Phi(-d_1) - R_t \Phi(-d_\gamma) \tag{4.10}$$

with the computed form of $C^{\text{SRM}}_{t,T}$ given via

(4.11)
$$e^{-r(T-t)} \widetilde{E}_t \left[(N - R_T) \mathbb{1}_{\{A_T \geq N, R_T < N\}} \right] = N e^{-r(T-t)} P_{\rho_{A,R}}(d_0, d^R_0) - R_t P_{\rho_{A,R}} \left(d_0 + \rho_{A,R} \sigma_R \sqrt{T-t}, d^R_1 \right)$$
$$e^{-r(T-t)} \widetilde{E}_t \left[(R_T - N) \mathbb{1}_{\{A_t < N, R_T \geq N\}} \right] = R_t P_{\rho_{A,R}} \left(d^R_1, d_0 + \rho_{A,R} \sigma_R \sqrt{T-t} \right) - N e^{-r(T-t)} P_{\rho_{A,R}}(d^R_0, d_0)$$
$$P_{\rho_{A,R}}(x, y) := \int_{z_1=-x}^{\infty} \int_{z_2=-\infty}^{y} \frac{1}{2\pi \sqrt{1 - \rho^2_{A,R}}} e^{-\frac{z_1^2 + z_2^2 - 2\rho_{A,R} z_1 z_2}{2(1-\rho^2_{A,R})}} dz_1 dz_2.$$

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Proof. Define a pair (Z_1, Z_2) of standard normal random variables under $\widetilde{\mathbb{P}}$ that have correlation $\rho_{A,R}$, and another pair of normal random variables under $\widetilde{\mathbb{P}}$ with correlation $\rho_{A,R}$ via the linear transformation

$$(U_1, U_2) = (Z_1 - \rho_{A,R}\sigma_R\sqrt{T-t}, Z_2 - \sigma_R\sqrt{T-t}). \tag{4.12}$$

We can express

$$\begin{aligned} A_T &\sim A_t e^{(r-\frac{1}{2}\sigma_A^2)(T-t)+\sigma_A\sqrt{T-t}Z_1} \\ R_T &\sim R_t e^{(r-\frac{1}{2}\sigma_R^2)(T-t)+\sigma_R\sqrt{T-t}Z_2} \end{aligned} \tag{4.13}$$

and so

(4.14)

$$\begin{aligned}
&e^{-r(T-t)}\widetilde{E}_t\left[(N-R_T)\mathbb{1}_{\{A_T\geq N, R_T<N\}}\right] \\
&= Ne^{-r(T-t)}\widetilde{\mathbb{P}}[Z_1 \geq -d_0, Z_2 < -d_0^R] - R_t e^{-\frac{1}{2}\sigma_R^2(T-t)}\widetilde{\mathbb{E}}\left[e^{\sigma_R\sqrt{T-t}Z_2}\mathbb{1}_{\{Z_1\geq -d_0, Z_2<-d_0^R\}}\right] \\
&= Ne^{-r(T-t)}\widetilde{\mathbb{P}}[Z_1 \geq -d_0, Z_2 < -d_0^R] \\
&- R_t e^{-\frac{1}{2}\sigma_R^2(T-t)}\int_{z_1=-d_0}^{\infty}\int_{z_2=-\infty}^{-d_0^R}\frac{1}{2\pi\sqrt{1-\rho_{A,R}^2}}e^{\sigma_R\sqrt{T-t}z_2}e^{-\frac{z_1^2+z_2^2-2\rho z_1 z_2}{2(1-\rho_{A,R}^2)}}dz_1dz_2 \\
&= Ne^{-r(T-t)}P_{\rho_{A,R}}(d_0, d_0^R) \\
&- R_t e^{-\frac{1}{2}\sigma_R^2(T-t)}\int_{z_1=-d_0}^{\infty}\int_{z_2=-\infty}^{-d_0^R}\frac{1}{2\pi\sqrt{1-\rho_{A,R}^2}}e^{\sigma_R\sqrt{T-t}z_2}e^{-\frac{z_1^2+z_2^2-2\rho_{A,R} z_1 z_2}{2(1-\rho_{A,R}^2)}}dz_1dz_2 \\
&= Ne^{-r(T-t)}P_{\rho_{A,R}}(d_0, d_0^R) - R_t\int_{u_1=-d_0-\rho_{A,R}\sigma_R\sqrt{T-t}}^{\infty}\int_{z_2=-\infty}^{-d_1^R}\frac{1}{2\pi\sqrt{1-\rho_{A,R}^2}}e^{-\frac{u_1^2+u_2^2-2\rho u_1 u_2}{2(1-\rho^2)}}du_1du_2 \\
&= Ne^{-r(T-t)}P_{\rho_{A,R}}(d_0, d_0^R) - R_t P_{\rho_{A,R}}\left(d_0 + \rho_{A,R}\sigma_R\sqrt{T-t}, d_1^R\right).
\end{aligned}$$

The computation for $e^{-r(T-t)}\widetilde{E}_t\left[(R_T-N)\mathbb{1}_{\{A_t<N, R_T\geq N\}}\right]$ is similar. □

4.2. **Bond Pricing with Recovery Risk.** We now are in a position to write the problem definition and solution for the price of a zero-coupon bond with both default risk and recovery risk. In our new model, default risk is driven by the assets A_t while the recovery risk is driven by the recovery R_t.

Proposition 4.3. (Bond Pricing PDE for SRM Model). *The price of a zero-coupon bond under the Stochastic Recovery Merton (SRM) model, denoted by $B_{t,T}^{\mathrm{SRM}}$, solves the terminal value problem*

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$$
\begin{aligned}
&\frac{\partial}{\partial t} B_{t,T}^{\mathrm{SRM}} + \mathcal{L}^{\mathrm{SRM}} B_{t,T}^{\mathrm{SRM}} = 0 \\
&B_{T,T}^{\mathrm{SRM}} = G(A, R)
\end{aligned}
\tag{4.15}
$$

where $\mathcal{L}^{\mathrm{SRM}}$ is the partial differential operator

$$
\mathcal{L}^{\mathrm{SRM}} := rA\frac{\partial}{\partial A} + rR\frac{\partial}{\partial R} + \frac{1}{2}\sigma_A^2 A^2 \frac{\partial^2}{\partial A^2} + \frac{1}{2}\sigma_R^2 R^2 \frac{\partial^2}{\partial R^2} + \rho_{A,R}\sigma_A\sigma_R AR\frac{\partial^2}{\partial A \partial R} - r
\tag{4.16}
$$

and

$$
G(A_T, R_T) = R_T 1_{\{A_T < N\}} + N 1_{\{A_T \geq N\}}.
\tag{4.17}
$$

Theorem 4.4. (Defaultable Zero-Coupon Bond Price Under Stochastic Recovery Merton Model). *Suppose the asset and recovery process follow* (**??**) *under the risk neutral measure $\widetilde{\mathbb{P}}$. Furthermore, suppose the assumptions of the debt structure* (4.17) *hold, where default is (still) defined in* (1.2). *Then the defaultable zero-coupon bond price is given by*

$$
B_{t,T}^{\mathrm{SRM}} = Ne^{-r(T-t)}\Phi(d_0) + R_t\Phi(-d_\gamma)
\tag{4.18}
$$

where $d_\gamma := d_0 + \gamma\sigma_A\sqrt{T-t}$ and $\gamma \in [-\frac{\sigma_R}{\sigma_A}, \frac{\sigma_R}{\sigma_A}]$ is defined in (3.9).

Proof. By direct definition, or by applying the Feynman-Kac Lemma (c.f. [37]) to the PDE in Proposition 4.3, we have

$$
B_{t,T}^{\mathrm{SRM}} = \underbrace{\widetilde{\mathbb{E}}_t[Ne^{-r(T-t)}\mathbb{1}_{\{\tau > T\}}]}_{\mathcal{I}_1} + \underbrace{\widetilde{\mathbb{E}}_t[e^{-r(T-t)}R_\tau \mathbb{1}_{\{\tau \leq T\}}]}_{\mathcal{I}_2}
\tag{4.19}
$$

Without loss of generality, we may set $t = 0$ and compute the first expectation as

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$$
\begin{aligned}
\mathcal{I}_1 &= \widetilde{\mathbb{E}}_0[Ne^{-r(T-t)}\mathbb{1}_{\{\tau>T\}}] \\
&= Ne^{-r(T-t)}\widetilde{\mathbb{P}}_0[\tau > T] \\
&= Ne^{-rT}\widetilde{\mathbb{P}}_0[A_T \geq N] \\
&= Ne^{-rT}\widetilde{\mathbb{P}}_0\Big[A_0 \exp\Big((r-\frac{1}{2}\sigma_A^2)t + \sigma_A \widetilde{Z}_T^A\Big) \geq N\Big] \\
&= Ne^{-rT}\widetilde{\mathbb{P}}_0\left[\widetilde{Z}_T^A \geq \frac{\ln\frac{N}{A_0} - (r-\frac{1}{2}\sigma_A^2)T}{\sigma_A}\right] \\
&= Ne^{-rT}\cdot\left(1-\Phi\left(\frac{\ln\frac{N}{A_0} - (r-\frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}}\right)\right) \\
&= Ne^{-rT}\Phi(d_0).
\end{aligned} \tag{4.20}
$$

by definition of d_0 (2.4).

Similarly

(4.21)

$$
\begin{aligned}
&\mathcal{I}_2 := \widetilde{\mathbb{E}}_0[e^{-rT}R_\tau \mathbb{1}_{\{\tau\leq T\}}] = \widetilde{\mathbb{E}}_0\Big[e^{-rT}R_T 1_{\{A_T<N\}}\Big] \\
&= R_0 e^{(\delta-r)T}\widetilde{\mathbb{E}}\Big[e^{\sigma_R\sqrt{1-\rho_{A,R}^2}\widetilde{W}_T}\Big]\tilde{\mathbb{E}}\Big[\Big(\frac{A_T}{A_0}\Big)^\gamma 1_{\{A_T<N\}}\Big] \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}\tilde{\mathbb{E}}\Big[\Big(\frac{A_T}{A_0}\Big)^\gamma 1_{\{A_T<N\}}\Big] \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}e^{\gamma(r-\frac{1}{2}\sigma_A^2)T}\tilde{\mathbb{E}}\left[e^{\gamma\sigma_A\widetilde{Z}_T^A}1_{\left\{\widetilde{Z}_T^A<\frac{\ln\frac{N}{A_0}-(r-\frac{1}{2}\sigma_A^2)T}{\sigma_A}\right\}}\right] \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}e^{\gamma(r-\frac{1}{2}\sigma_A^2)T}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\frac{\ln\frac{N}{A_0}-(r-\frac{1}{2}\sigma_A^2)T}{\sigma_A}} e^{\gamma\sigma_A x}e^{-\frac{x^2}{2T}}dx \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}e^{\gamma(r-\frac{1}{2}\sigma_A^2)T}e^{\frac{\gamma^2\sigma_A^2T}{2}}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\frac{\ln\frac{N}{A_0}-(r-\frac{1}{2}\sigma_A^2)T}{\sigma_A}-\gamma\sigma_A T} e^{-\frac{x^2}{2T}}dx \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}e^{\gamma(r-\frac{1}{2}\sigma_A^2)T}e^{\frac{\gamma^2\sigma_A^2T}{2}}\Phi\left(\frac{\ln\frac{N}{A_0}-(r-\frac{1}{2}\sigma_A^2)T-\gamma\sigma_A^2T}{\sigma_A\sqrt{T}}\right) \\
&= R_0 e^{(\delta-r)T}e^{\frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)T}e^{\gamma(r-\frac{1}{2}\sigma_A^2)T}e^{\frac{\gamma^2\sigma_A^2T}{2}}\Phi(-d_\gamma) \\
&= R_0 e^{(r-r)T}\Phi(-d_\gamma) \\
&= R_0\Phi(-d_\gamma)
\end{aligned}
$$

□

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Remark 4.5. *Some observations on the bond price* (4.18) *are in order. The first is that recovery R only occurs linearly in the price through the second term. Thus the Manager's estimate of the firms asset value A_t affects the distances to default d_0, d_γ, while the actual market value of the assets R_t affect the recovery value conditioned on default. This is a benefit of decoupling the default driver and the recovery driver and adds insight into the classical one-factor Merton model.*

Using the bond price under Stochastic Recovery Merton (SRM) model we obtain

$$\widetilde{\mathrm{PD}}_{t,T}^{\mathrm{SRM}} = \widetilde{\mathrm{PD}}_{t,T}^{\mathrm{Merton}} = \Phi(-d_0) \tag{4.22}$$

and

$$\begin{aligned} \widetilde{\mathrm{LGD}}_{t,T}^{\mathrm{SRM}} &= \frac{N - \tilde{\mathbb{E}}_t[R_T \mid A_T < N]}{N} \\ &= 1 - \frac{R_t}{N} e^{r(T-t)} \frac{\Phi(-d_\gamma)}{\Phi(-d_0)} \end{aligned} \tag{4.23}$$

and we obtain the following direct corollary.

Corollary 4.6. (Credit Spread under Stochastic Recovery Merton Model). *Under the assumptions of the Stochastic Recovery Merton model, the credit spread of a zero-coupon bond is*

$$\begin{aligned} S_{t,T}^{\mathrm{SRM}} &:= \frac{1}{T-t} \ln\left(\frac{N e^{-r(T-t)}}{B_{t,T}^{\mathrm{SRM}}}\right) \\ &= -\frac{1}{T-t} \ln\left(\Phi(d_0) + e^{r(T-t)} \frac{R_t}{N} \Phi(-d_\gamma)\right) \end{aligned} \tag{4.24}$$

Proof. The proof is given by direct substitution of the bond price (4.18) into the definition of the credit spread.

□

Just as in the bond price, the Manager's estimate of the assets affect only the distances to default d_0, d_γ, while the market value of the assets affects the recovery term. This decoupling suggests another interpretation of this model is a randomization of the recovery value, similar to how the asset value is randomized in the Randomized Merton Model [39].

Lemma 4.7. (Greeks for Stochastic Recovery Merton Model). *Let $B_{t,T}^{\mathrm{SRM}}$ be the zero-coupon bond price from the Merton model with stochastic recovery* (4.18). *Then the Greeks are given by*

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(4.25)
$$\frac{\partial}{\partial R}B^{\text{SRM}}_{t,T} = \Phi(-d_\gamma)$$
(4.26)
$$\frac{\partial}{\partial \rho}B^{\text{SRM}}_{t,T} = -\sigma_R R\varphi(-d_\gamma)\sqrt{T-t}$$
(4.27)
$$\frac{\partial}{\partial \sigma_R}B^{\text{SRM}}_{t,T} = -\rho R\varphi(-d_\gamma)\sqrt{T-t}$$
(4.28)
$$\frac{\partial}{\partial \sigma_A}B^{\text{SRM}}_{t,T} = -\frac{d_1}{\sigma_A}[Ne^{-r(T-t)}\varphi(d_0) - R\varphi(-d_\gamma)]$$
(4.29)
$$\frac{\partial}{\partial r}B^{\text{SRM}}_{t,T} = \frac{\sqrt{T-t}}{\sigma_A}[Ne^{-r(T-t)}\varphi(d_0) - R\varphi(-d_\gamma)] - (T-t)[Ne^{-r(T-t)}\Phi(d_0)]$$
(4.30)
$$\frac{\partial}{\partial A}B^{\text{SRM}}_{t,T} = \frac{Ne^{-r(T-t)}\varphi(d_0) - R\varphi(-d_\gamma)}{A\sigma_A\sqrt{T-t}}$$
where as usual $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ *and* $\Phi(x) = \int_{-\infty}^{x}\varphi(\xi)$.

Proof. The proof is given simply by taking the relevant partial differentials of (4.18). □

4.3. **CDS Pricing with Stochastic Recovery Merton Model.** In standard hazard-rate models [22, 33], recoveries are assumed to be constant and hence typically (2.21) is used to price the CDS spread. However, as mentioned in Section 2.2 the structural framework allows us to remove the constant recovery rate assumption and incorporate recovery-risk though a PD-LGD correlation model [29]. By our results for Loss given Default and Probability of Default in the Stochastic Recovery Model, it follows that

$$\begin{aligned} P^{\text{SRM}}_{t,T} &= \frac{re^{-r(T-t)}}{1-e^{-r(T-t)}}\widetilde{\text{PD}}^{\text{SRM}}_{t,T} \cdot \widetilde{\text{LGD}}^{\text{SRM}}_{t,T} \\ &= \frac{re^{-r(T-t)}}{1-e^{-r(T-t)}}\Big[\Phi(-d_0) - e^{r(T-t)}\frac{R_t}{N}\Phi(-d_\gamma)\Big]. \end{aligned} \tag{4.31}$$

5. Calibration and the Implied Recovery

In the classical Merton model, there are only two parameters to be calibrated, the unobservable asset value A_t and unobservable asset volatility σ_A. These are estimated through the observable equity value E_t and observable equity volatility σ_E, leading to an informational discrepancy. In the Stochastic Recovery Merton Model, there are five parameters to be calibrated,

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namely, A_t, R_t, σ_A, σ_R and $\rho_{A,R}$. Recall that A_t and σ_A are determined by the firm's Manager while R_t and σ_R are determined by the market. The parameter $\rho_{A,R}$ is a measure of the market's visibility of the firm's assets. Here we propose a calibration procedure for the model that respects the informational asymmetry.

5.1. **Calibrating to the Bond Market.** We now proceed to describe a calibration of the model under the assumption that equity and bond prices are available.

I. Calibration of R_t and σ_R. The first step is to calibrate R_t and σ_R. Since in the model these are market defined quantities, they must be calibrated using market data. Hence, we use the equity and equity volatility to set up two equations for the two unknowns.

$$\begin{aligned} E_t^{\text{Market}} &= E_t^{\text{SRM}}(R, \sigma_R) \\ \sigma_E^{\text{Market}} E_t^{\text{Market}} &= \sigma_R R_t \Phi(d_1^R) \end{aligned} \tag{5.1}$$

where σ_E^{Market} and E_t^{Market} are the equity and volatility observed directly from the market.

This procedure yields calibrated values $\widehat{R}_t$ and $\widehat{\sigma}_R$.

II. Calibration of A_t, σ_A and $\rho_{A,R}$. Next, we must set up three equations for the three remaining unknowns. The three coupled equations are,

$$B_{t,T}^{\text{Market}} = B_{t,T}^{\text{SRM}}(A, \sigma_A, \rho_{A,R}; \widehat{R}, \widehat{\sigma}_R) \tag{5.2}$$

$$\frac{\mu_R^{\text{Market}} B_t^{\text{Market}}}{\mu_E^{\text{Market}} E_t^{\text{Market}}} = \frac{\Phi(-d_\gamma)}{\Phi(d_1^R)} \left(1 + \widehat{\sigma}_R \widehat{R}_t \frac{\Phi(-d_1^R)}{\phi(d_1^R)} \sqrt{T-t} \mu_E^{\text{Market}} \right) \tag{5.3}$$

$$\sigma_B^{\text{Market}} = \sigma_B^{\text{SRM}}(A, \sigma_A, \rho_{A,R}; \widehat{R}, \widehat{\sigma}_R) \tag{5.4}$$

where the second equation is simply a rewriting of the hedge ratio given by (7.1) and σ_B^{SRM} is the volatility of bond returns given by (7.9).

Calibration of the Probability of Default. Notice that the if the goal is simply to obtain the probability of default $\widetilde{\mathbb{P}}_t[\tau \leq T] = \Phi(-d_0)$ we need not calibrate the full model. Once we have gotten R_t and σ_R from (5.1) we can isolate the probability of default directly from (5.2) via

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$$
\begin{aligned}
\widetilde{\mathbb{P}}_t[\tau \le T] &= \Phi(-d_0) \\
&= 1 - \left(\frac{B_{t,T}^{\text{Market}} - R_t \Phi(-d_\gamma)}{N e^{-r(T-t)}} \right) \\
&= 1 - \frac{B_{t,T}^{\text{Market}}}{N e^{-r(T-t)}} \left(\frac{\mu_E^{\text{market}} E_t^{\text{Market}} - \mu_B^{\text{market}} \widehat{R}_{t,T} \widehat{\Phi}(d_0^R)}{\mu_E^{\text{market}} E_t^{\text{Market}}} \right).
\end{aligned} \tag{5.5}
$$

Note that the right-hand-side is all either directly observable from the market or already calibrated via the equity markets.

5.2. **Calibrating to the CDS Market.** We define the recovery rate $\bar{R}_{t,T} = \bar{R}(t,T) := \frac{R_t}{N}$, and solve the Stochastic Recovery Merton CDS premium $P_{t,T}^{\text{SRM}}$ in (4.31) for the recovery rate to yield the *Implied Recovery Rate*

$$
\bar{R}_{t,T}^{\text{Imp}} = e^{-r(T-t)} \left[\frac{\Phi(-d_0^T) - \frac{1-e^{-r(T-t)}}{r \cdot e^{-r(T-t)}} P_{t,T}^{\text{Mkt}}}{\Phi(-d_\gamma^T)} \right]. \tag{5.6}
$$

Suppose now there are two CDS on the same obligor, except one references a senior issue with maturity T_{Sr} while the other references a junior issue with maturity T_{Jr}. Denote the premiums associated to these two CDS as $P_{\text{Sr}}^{\text{Mkt}}$ and $P_{\text{Jr}}^{\text{Mkt}}$ respectively. Then the market implied recovery ratio is

(5.7)
$$
\frac{\bar{R}_{\text{Jr}}^{\text{Imp}}(t, T_{\text{Jr}})}{\bar{R}_{\text{Sr}}^{\text{Imp}}(t, T_{\text{Sr}})} = \frac{e^{-r(T_{\text{Jr}}-t)}}{e^{-r(T_{\text{Sr}}-t)}} \left[\frac{\Phi(-d_0^{\text{Jr}}) - \frac{1-e^{-r(T_{\text{Jr}}-t)}}{r \cdot e^{-r(T_{\text{Jr}}-t)}} P_{\text{Jr}}^{\text{Mkt}}(t, T_{\text{Jr}})}{\Phi(-d_0^{\text{Sr}}) - \frac{1-e^{-r(T_{\text{Sr}}-t)}}{r \cdot e^{-r(T_{\text{Sr}}-t)}} P_{\text{Sr}}^{\text{Mkt}}(t, T_{\text{Sr}})} \right] \cdot \frac{\Phi(-d_\gamma^{\text{Sr}})}{\Phi(-d_\gamma^{\text{Jr}})}
$$

where the superscript Jr and Sr on d_0 and d_γ denote evaluation at T_{Jr} and T_{Sr} respectively. If the CDSs have the same maturity, i.e. $T_{\text{Jr}} = T_{\text{Sr}} = T$ then (5.7) reduces to

$$
\frac{\bar{R}_{\text{Jr}}^{\text{Imp}}(t, T)}{\bar{R}_{\text{Sr}}^{\text{Imp}}(t, T)} = \frac{\Phi(-d_0) - \frac{1-e^{-r(T-t)}}{r \cdot e^{-r(T-t)}} P_{\text{Jr}}^{\text{Mkt}}(t, T)}{\Phi(-d_0) - \frac{1-e^{-r(T-t)}}{r \cdot e^{-r(T-t)}} P_{\text{Sr}}^{\text{Mkt}}(t, T)} \tag{5.8}
$$

where as before $\bar{R}_{\text{Sr}}^{\text{Imp}}(t,T)$ and $\bar{R}_{\text{Jr}}^{\text{Imp}}(t,T)$ are the market implied term structures for recovery rates at time t. Notice that the right hand side of (5.8) does not require any knowledge of the recovery process, but just observed CDS premiums and calibrated parameters of the original Merton model, namely (σ_A, A) which are the same regardless of the seniority of the issue. Inverting recovery to obtain implied Premiums for junior and senior issues with the same maturity leads to

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$$(5.9) \qquad \frac{P_{\mathrm{Jr}}^{\mathrm{Imp}}(t,T)}{P_{\mathrm{Sr}}^{\mathrm{Imp}}(t,T)} = \frac{\widetilde{\mathrm{LGD}}_{\mathrm{Jr}}^{\mathrm{SRM}}(t,T)}{\widetilde{\mathrm{LGD}}_{\mathrm{Sr}}^{\mathrm{SRM}}(t,T)} = \frac{e^{-r(T-t)} - \bar{R}_{t,T}^{\mathrm{Jr}} \frac{\Phi(-d_\gamma)}{\Phi(-d_0)}}{e^{-r(T-t)} - \bar{R}_{t,T}^{\mathrm{Sr}} \frac{\Phi(-d_\gamma)}{\Phi(-d_0)}}$$

which does require knowledge of the recovery process.

Remark 5.1. *The spread-ratio* (5.9) *is the Stochastic Recovery Merton model implementation of equation (6) in* [36] *used to extract recovery risk premiums from empirical data.*

6. The Recovery Risk Premium

In this Section we compare bond prices, spreads, loss-given-default and costs between the classical Merton model and our Stochastic Recovery Merton model. In particular we compute the recovery risk premium for Bonds and CDS spreads.

We begin by recording the difference in credit prices and other metrics in the Merton model when recovery risk is added to the model.

Lemma 6.1. (Comparison of the Merton and Stochastic Recovery Merton Model).

Let

$$(6.1) \qquad \Theta_t := \frac{1}{\sqrt{2\pi}} \int_{-d_1}^{-d_1+(1-\gamma)\sigma_A\sqrt{T-t}} e^{-x^2/2} dx$$

then

(6.2)

$$B_{t,T}^{\mathrm{Merton}} - B_{t,T}^{\mathrm{SRM}} = (fA_t - R_t)\Phi(-d_1) - \Theta_t R_t$$

$$\widetilde{PD}_t^{\mathrm{Merton}} - \widetilde{PD}_t^{\mathrm{SRM}} = 0$$

$$\widetilde{LGD}_t^{\mathrm{Merton}} - \widetilde{LGD}_t^{\mathrm{SRM}} = \frac{1}{Ne^{-r(T-t)}\Phi(-d_0)} \left[(R_t - fA_t)\Phi(-d_1) + R_t\Theta_t\right]$$

$$S_{t,T}^{\mathrm{Merton}} - S_{t,T}^{\mathrm{SRM}} = -\frac{1}{T-t} \ln\left(\frac{N\Phi(d_0) + fA_t\Phi(-d_1)}{N\Phi(d_0) + R_t\Phi(-d_\gamma)}\right)$$

$$P_{t,T}^{\mathrm{Merton}} - P_{t,T}^{\mathrm{SRM}} = \frac{r}{1-e^{-r(T-t)}} \left(\frac{R_t - fA_t}{N}\right) \Phi(-d_1) + R_t\Theta_t$$

$$C_t^{\mathrm{Merton}} - C_t^{\mathrm{SRM}} = (R_t - fA_t)\Phi(-d_1) + R_t\Theta_t = B_{t,T}^{\mathrm{SRM}} - B_{t,T}^{\mathrm{Merton}}.$$

Proof. By (2.3) and (4.18) we have

$$(6.3) \qquad B_{t,T}^{\mathrm{Merton}} - B_{t,T}^{\mathrm{SRM}} = (fA_t - R_t)\Phi(-d_1) - R_t\left[\Phi(-d_\gamma) - \Phi(-d_1)\right].$$

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What is left to do is compare the difference $\Phi(-d_\gamma) - \Phi(-d_1)$. By definition $d_\gamma = d_0 + \gamma\sigma_A\sqrt{T-t}$ so we can write $d_\gamma = d_1 - (1-\gamma)\sigma_A\sqrt{T-t}$ and therefore $\Phi(-d_\gamma) = \Phi(-d_1 + (1-\gamma)\sigma_A\sqrt{T-t})$. Thus,

$$
\begin{aligned}
\Phi(-d_\gamma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_\gamma} e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_1} e^{-x^2/2} dx + \frac{1}{\sqrt{2\pi}} \int_{-d_1}^{-d_1+(1-\gamma)\sigma_A\sqrt{T-t}} e^{-x^2/2} dx \\
&= \Phi(-d_1) + \Theta_t
\end{aligned}
\tag{6.4}
$$

Substituting (6.4) into (6.3) gives (6.2).

The other calculations are similar. □

A simple Corollary to Lemma (6.1) is that the Stochastic Recovery Merton Model converges to the classic Merton model as the Recovery process R_t converges A_t. That is, if there is no informational difference between the A and R and therefore no recovery risk, then the prices and risks generated by the two models should be the same.

Corollary 6.2. (Convergence of Stochastic Recovery Merton Model).

$$
\begin{aligned}
\lim_{\gamma\to 1, R_0 \to fA_0} B_{t,T}^{\text{SRM}} &= B_{t,T}^{\text{Merton}} \\
\lim_{\gamma\to 1, R_0 \to fA_0} \widetilde{\text{PD}}_{t,T}^{\text{SRM}} &= \widetilde{\text{PD}}_{t,T}^{\text{Merton}} \\
\lim_{\gamma\to 1, R_0 \to fA_0} \widetilde{\text{LGD}}_{t,T}^{\text{SRM}} &= \widetilde{\text{LGD}}_{t,T}^{\text{Merton}} \\
\lim_{\gamma\to 1, R_0 \to fA_0} \widetilde{\text{S}}_{t,T}^{\text{SRM}} &= \widetilde{\text{S}}_{t,T}^{\text{Merton}}. \\
\lim_{\gamma\to 1, R_0 \to fA_0} \widetilde{\text{P}}_{t,T}^{\text{SRM}} &= \widetilde{\text{P}}_{t,T}^{\text{Merton}}.
\end{aligned}
\tag{6.5}
$$

Proof. The proof follows directly from Lemma 6.1, Lemma 3.5, and the fact that

$$
\lim_{\gamma\to 1} d_\gamma = d_1. \tag{6.6}
$$

□

Finally, we have our main Theorem.

Theorem 6.3. (Recovery Risk Premium in Credit Spreads). *The Bond and CDS spreads in the Stochastic Recovery Merton model can be given as*

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$$S_{t,T}^{\mathrm{SRM}} = S_{t,T}^{\mathrm{Merton}} + RR(S_{t,T}) \tag{6.7}$$

$$P_{t,T}^{\mathrm{SRM}} = P_{t,T}^{\mathrm{Merton}} + RR(P_{t,T}) \tag{6.8}$$

where $RR(\cdot)$ is the Recovery Risk premium in the spread given by

$$RR(S_{t,T}) = \frac{1}{T-t} \ln\left(\frac{N\Phi(d_0) + R_t\Phi(-d_\gamma)}{N\Phi(d_0) + fA_t\Phi(-d_1)}\right) \tag{6.9}$$

and

$$RR(P_{t,T}) = \frac{r}{1 - e^{-r(T-t)}} \left(\frac{R_t - fA_t}{N}\right) \Phi(-d_1) + R_t\Theta_t \tag{6.10}$$

Remark 6.4. *Since the Merton bond and CDS spread only contain a default risk premium, Theorem 6.3 is essentially a decomposition of the Stochastic Merton bond and CDS spread into a premium due to default risk, and a premium due to recovery risk. Notice again that the recovery risk premium vanishes as $\gamma \to 1, R_t \to fA_t$.*

7. Conclusions

In this work, we presented a two-factor Merton model with recovery risk as the extra risk factor. The extra risk factor can be interpreted as arising from the information available about the firms asset value and thus posed within a partial information perspective. We then obtain closed-form and internally consistent prices for bonds and CDS. The addition of the extra recovery risk factor allows us to price the recovery risk premium, which is not possible in the original Merton model. In addition, recovery risk generally increases spreads and may account for some of the empirical mispricing observed when using the original Merton model.

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Appendix A. Bond-Equity Hedge Ratio with Recovery Risk

Lemma 7.1. (Bond-Equity Hedge Ratio with Recovery Risk) *Let μ_B and μ_E be the instantaneous bond and equity returns respectively. Then we have*

$$\mu_B = h_1^{\mathrm{SRM}}\mu_E + h_2^{\mathrm{SRM}}\mu_E^2 + \mathcal{O}(\mu_E^3) \tag{7.1}$$

where h_1^{SRM} and h_2^{SRM} are the the first- and second-order hedge ratios for the Stochastic Recovery Merton model given by

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$$h_1^{\mathrm{SRM}}(A, R, \sigma_A, \sigma_R, \rho_{A,R}) := \frac{E_t^{\mathrm{SRM}}}{B_{t,T}^{\mathrm{SRM}}} \frac{\Phi(-d_\gamma)}{\Phi(d_1^R)} \tag{7.2}$$

and

$$h_2^{\mathrm{SRM}}(A, R, \sigma_A, \sigma_R, \rho_{A,R}) = \frac{\sigma_R R(E_t^{\mathrm{SRM}})^2}{B_{t,T}^{\mathrm{SRM}}} \frac{\Phi(-d_\gamma)}{\phi(d_1^R)} \sqrt{T-t} \tag{7.3}$$

Proof. Taylor expanding the bond price $B = B_{t,T}^{\mathrm{SRM}}$ with respect to equity yields

$$dB = \left(\frac{\partial B}{\partial E}\right) dE + \left(\frac{\partial^2 B}{dE^2}\right) dE^2 + \mathcal{O}(dE^3). \tag{7.4}$$

Dividing through by B and introducing bond returns $\mu_B := dB/B$ and equity returns $\mu_E := dE/E$ we can write this as

$$\mu_B = h_1^{\mathrm{SRM}} \mu_E + h_2^{\mathrm{SRM}} \mu_E^2 + \mathcal{O}(\mu_E^3) \tag{7.5}$$

where recalling (4.4) and (4.25) we have

$$\begin{aligned} h_1^{\mathrm{SRM}} &:= \frac{E}{B}\left(\frac{\partial B}{\partial E}\right) \\ &= \frac{E}{B}\left(\frac{\partial B}{\partial R}\right)\left(\frac{\partial E}{\partial R}\right)^{-1} \\ &= \frac{E}{B}\frac{\Phi(-d_\gamma)}{\Phi(d_1^R)} \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} h_2^{\mathrm{SRM}} &:= \frac{E^2}{B}\left(\frac{\partial^2 B}{\partial E^2}\right) \\ &= \frac{E^2}{B}\left[\underbrace{\left(\frac{\partial^2 B}{\partial R^2}\right)}_{=0}\left(\frac{\partial E}{\partial R}\right)^{-2} + \left(\frac{\partial B}{\partial R}\right)\left(\frac{\partial^2 E}{\partial R^2}\right)^{-1}\right] \\ &= \frac{E^2}{B}\Phi(-d_\gamma)\frac{\sigma_R R\sqrt{T-t}}{\phi(d_1^R)} \end{aligned} \tag{7.7}$$

□

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Appendix B: Volatility of Bond Returns with Recovery Risk

The volatility σ_X of an asset is computed via

$$\sigma_X^2 dt = \widetilde{\mathbb{E}}\left[\left(\frac{dX_t}{X_t} - \widetilde{\mathbb{E}}\left[\frac{dX_t}{X_t}\right]\right)^2\right] = \mathbb{E}\left[\left(\frac{dX_t}{X_t} - \mathbb{E}\left[\frac{dX_t}{X_t}\right]\right)^2\right] \tag{7.8}$$

That is, σ_X is invariant under whether we use the risk-neutral or real-world measure.

Lemma 7.2. *The volatility σ_B^{SRM} of bond returns in the Stochastic Recovery Merton model is given by*

$$\sigma_B^{\mathrm{SRM}} = \sqrt{\Omega_A^2\sigma_A^2 + \Omega_R^2\sigma_R^2 + 2\Omega_A\Omega_R\rho_{A,R}\sigma_A\sigma_R} \tag{7.9}$$

where

$$\begin{aligned} \Omega_A &= \frac{A}{B_{t,T}^{\mathrm{SRM}}}\frac{\partial B_{t,T}^{SRM}}{\partial A} \\ &= \frac{1}{\sigma_A\sqrt{T-t}}\left(\frac{Ne^{-r(T-t)}\phi(d_0) - R\phi(-d_\gamma)}{Ne^{-r(T-t)}\Phi(d_0) + R\Phi(-d_\gamma)}\right) \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} \Omega_R &= \frac{R}{B_{t,T}^{\mathrm{SRM}}}\frac{\partial B_{t,T}^{SRM}}{\partial R} \\ &= \frac{R\Phi(-d_\gamma)}{Ne^{-r(T-t)}\Phi(d_0) + R\Phi(-d_\gamma)} \end{aligned} \tag{7.11}$$

Proof. Applying the usual Ito Calculus to $B_{t,T}^{\mathrm{SRM}} = B(A_t, R_t)$ we have

$$dB = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial A}dA_t + \frac{\partial B}{\partial R}dR_t + \frac{1}{2}\frac{\partial^2 B}{\partial A^2}\langle dA_t, dA_t\rangle + \frac{1}{2}\frac{\partial^2 B}{\partial R^2}\langle dR_t, dR_t\rangle + \frac{\partial^2 B}{\partial A\partial R}\langle dA_t, dR_t\rangle \tag{7.12}$$

Recalling that A_t and R_t satisfy (3.1) we have

$$\begin{aligned} dB = &\left(\frac{\partial B}{\partial t} + rA\frac{\partial B}{\partial A} + rR\frac{\partial B}{\partial R} + \frac{1}{2}\sigma_A^2A^2\frac{\partial^2 B}{\partial A^2} + \frac{1}{2}\sigma_R^2R^2\frac{\partial^2 B}{\partial R^2} + \rho_{A,R}\sigma_A\sigma_R AR\frac{\partial^2 B}{\partial A\partial R}\right)dt \\ &+ \sigma_A A\frac{\partial B}{\partial A}dW_t^A + \sigma_R R\frac{\partial B}{\partial R}dW_t^R \end{aligned} \tag{7.13}$$

Therefore by Proposition 4.3 we have

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(7.14)
$$dB - rBdt = \underbrace{\left(\frac{\partial B}{\partial t} + \mathcal{L}^{\mathrm{SRM}}B\right)}_{=0 \text{ from Proposition 4.3}} dt + \left(\sigma_A A \frac{\partial B}{\partial A} dW_t^A + \sigma_R R \frac{\partial B}{\partial R} dW_t^R\right)$$

Finally, dividing both sides by B and taking the expectation yields

(7.15)
$$\begin{aligned}\widetilde{\mathbb{E}}\left[\left(\frac{dB}{B} - rdt\right)^2\right] &= \widetilde{\mathbb{E}}\left[\left(\sigma_A \frac{A}{B}\frac{\partial B}{\partial A} dW_t^A + \sigma_R \frac{R}{B}\frac{\partial B}{\partial R} dW_t^R\right)^2\right] \\ &= \left(\frac{A}{B}\frac{\partial B}{\partial A}\right)^2 \sigma_A^2 + \left(\frac{R}{B}\frac{\partial B}{\partial R}\right)^2 \sigma_R^2 + 2\rho_{A,R}\left(\frac{A}{B}\frac{\partial B}{\partial A}\right)\left(\frac{R}{B}\frac{\partial B}{\partial R}\right)\sigma_A\sigma_R\end{aligned}$$

and the result follows.

□

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