Next Gen Finite Difference

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Outline

- Forward-Backward-Dupire-Monte-Carlo.
- Discrete consistency for a special case.
- A new split scheme for the general case: 4-Step.
- Drift step.
- Jump and compensator step.
- Correlated stochastic volatility step.
- Local volatility step.

- Accuracy and convergence.

- Conclusion.

References

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Forward-Backward-Dupire-Monte-Carlo

- For the local volatility model

$$ds = \sigma(t,s)dW$$

- … we have

  - Backward PDE: $0 = f_t + \frac{1}{2}\sigma^2 f_{ss} \quad , f(t,s(t)) = E_t[f(T,s(T))]$
  - Forward PDE: $0 = -p_t + \frac{1}{2}[\sigma^2 p]_{xx} \quad , p(t,x) = E[\delta(s(t)-x)]$
  - Dupire PDE: $0 = -c_t + \frac{1}{2}\sigma^2 c_{kk} \quad , c(t,k) = E[(s(t)-k)^+]$

- These equations are mutually consistent in continuous time and state, but not necessarily in any arbitrary chosen discretisation.

- It is not clear how we should choose discretisation schemes to make the methods consistent in a discrete sense.

- How can one ensure that model calibrated with a discretisation of the Dupire equation will reprice options in backward finite difference or Monte-Carlo?

Discrete Forward-Backward-Dupire

- A&H (2011) shows that there is discrete consistency between

  - Backward equation: $[1-\Delta t \frac{1}{2}\sigma^2 \delta_{ss}] f(t_h) = f(t_{h+1})$

  - Forward equation: $[1-\Delta t \frac{1}{2}\sigma^2 \delta_{ss}]' p(t_{h+1}) = p(t_h)$

  - Dupire equation: $[1-\Delta t \frac{1}{2}\sigma^2 \delta_{kk}] c(t_{h+1}) = c(t_h)$

- It is established that the transition matrix is non-negative

$$\Pr(s(t_{h+1}) = s_j \,|\, s(t_h) = s_i) = [1-\Delta t \frac{1}{2}\sigma^2 \delta_{ss}]^{-1} \geq 0.$$

- Further, A&H find that the transition matrix has a convenient decomposition

$$([1-\Delta t\frac{1}{2}\sigma^2\delta_{ss}]^{-1})_{ij}=u_i v_j$$

- … which allows consistent and efficient Monte-Carlo simulation.

- The above system generates the same arbitrage free option prices, no matter whether we price backwards, forwards, or by Monte-Carlo.

- The system can be calibrated exactly to any arbitrage free surface of European option prices through the discrete Dupire equation.

- So finite difference is the mother and brother of arbitrage free option prices.

But What About …

- Drift and discrete dividends?
- Jumps?
- Correlated stochastic volatility?
- The case when the input European option prices are not fully arbitrage consistent?

- In this talk we will show that discrete consistency and positivity can be retained if we split the single time step in four steps:

  - Drift.

  - Jump and compensator.

  - Stochastic volatility.

  - Local volatility.

SDE

- The SDE that we are considering is

$$ds = \underbrace{\mu(t,s)}_{general\,drift} dt + \underbrace{z\sigma(t,s)}_{stoch\,loc\,vol} dW + \underbrace{IdN}_{jump} + \underbrace{\alpha dt}_{\substack{compensator \\ for\ jumps}}$$

$$dz = \beta(z)dt + \varepsilon(z)dZ$$

$$dW \cdot dZ = \underset{\substack{non-zero \\ stoch\,vol\,corr}}{\rho} \quad dt$$

- So full general stochastic local volatility with general drift (dividends) and jumps.

Discrete Space/Continuous Time

- The discrete space/continuous time problem that we are working with is

$$0 = f_t + A_\mu f + \lambda[(1-A_j)^{-1}-1] f + A_\alpha f + A_y f + A_s f$$

- Drift operator $A_\mu = \mu^+ \delta_s^+ + \mu^- \delta_s^-$ uses up- and down-winding.

- Jump operator $A_j = \gamma^+ \delta_s^+ + \gamma^- \delta_s^- + \frac{1}{2}\vartheta^2 \delta_{ss}$ makes use of FD efficiency.

- Compensator $A_\alpha = \alpha^+ \delta_s^+ + \alpha^- \delta_s^-$ $, \alpha \approx -\lambda[(1-A_j)^{-1}-1]s$ is chosen non-parametrically.

- Stochastic volatility operator $A_y = \varphi^+ \delta_y^+ + \varphi^- \delta_y^- + \frac{1}{2}(1-\rho^2) z^2 \delta_{yy}$ where $y = y(t,s,z)$ is a transform so that $dy \cdot ds = 0$.

- Local vol stock operator $A_s = \frac{1}{2} z^2 \sigma^2 \delta_{ss}$ is chosen for model to match input option prices.

Split Scheme

- The split in operators is used for a scheme that splits in time

$$[1-\Delta t A_s] f(t_{h+3/4}) = f(t_{h+1})$$
$$[1-\Delta t A_y] f(t_{h+1/2}) = f(t_{h+3/4})$$

$$[1-A_j] f^{j}(t_{h+1/4}) = f(t_{h+1/2})$$
$$[1-\Delta t A_\alpha] f^{c}(t_{h+1/4}) = f(t_{h+1/2})$$
$$f(t_{h+1/4}) = (1-e^{-\lambda \Delta t}) f^{j}(t_{h+1/2}) + e^{-\lambda \Delta t} f^{c}(t_{h+1/2})$$

$$[1-\Delta t A_\mu] f(t_h) = f(t_{h+1/4})$$

- At each step we choose parameters so that we hit forwards and option prices and retain positive transition probabilities.

Drift and Positivity

- Define the up- and down-winding operators:

$$\delta_x^+ f(x) = (f(x+\Delta x) - f(x))/\Delta x$$

$$\delta_x^- f(x) = (f(x) - f(x-\Delta x))/\Delta x$$

- We have

$$[1-\Delta t(\mu^+\delta_x^+ - \mu^-\delta_x^- + \frac{1}{2}\sigma^2\delta_{xx}]^{-1} \geq 0$$

- … for all vectors μ, σ^2.

- So, if we use up- and down-winding we can introduce general drift without destroying positivity.

Matching the Drift

- Consider a single drift step

$$[1-\Delta t(\mu^{+}\delta_{x}^{+}-\mu^{-}\delta_{x}^{-})]\, f(t_h)= f(t_{h+1})$$

- If we let

$$g(t_h, s(t_h)) = E[s(t_{h+1}) \mid s(t_h)]$$

- For example

$$g(t,s)=\underbrace{se^{r\Delta t}}_{\substack{grow\\by\\interest\\rate}}-\underbrace{d}_{\substack{absolute\\dividend}}$$

- Assume that $g(t_h, s)$ is increasing in s.

- For the edges of the grid we let the lower and upper bounds change over time according to

$$\begin{aligned} s_0(t_{h+1}) &= s_0(t_h) - (s_0(t_h) - g(t_h, s_0(t_h)))^+ \\ s_{n-1}(t_{h+1}) &= s_{n-1}(t_h) + (g(t_h, s_{n-1}(t_h)) - s_{n-1}(t_h))^+ \end{aligned} \tag{1}$$

- Then for the *interior* points we can set the drift to be

$$\mu(t_h, s_i) = \frac{(g(t_h, s_i) - s_i)^+}{\Delta t \delta_s^+ g(t_h, s_i)} - \frac{(s_i - g(t_h, s_i))^+}{\Delta t \delta_s^- g(t_h, s_i)} =: \tilde{\mu}(t_h, s_i) \quad , 0 < i < n-1 \tag{2}$$

- For the edge points the drift is given by (2) if the edge values are unchanged and zero otherwise, hence

$$\begin{aligned} \mu(t_h, s_0) &= \tilde{\mu}(t_h, s_0) 1_{s_0(t_{h+1}) = s_0(t_h)} \\ \mu(t_h, s_{n-1}) &= \tilde{\mu}(t_h, s_{n-1}) 1_{s_{n-1}(t_{h+1}) = s_{n-1}(t_h)} \end{aligned} \tag{3}$$

- This scheme will ensure that the drift of the finite difference grid is always matched with the continuous time case.

- …and it ensures that the transition probabilities are non-negative.

- If g is linear in s, then the scheme will automatically hit the forward exactly per construction.

- If not, we may need to adjust g to hit the forward.

Jumps

- Suppose we wish to approximate the model

$$ds = IdN + \alpha dt \qquad (4)$$

- … where N is a Poison process with intensity λ, and α is set to ensure that s is a martingale, i.e. $\alpha = -\lambda E_t[I]$.

- A convenient way of modelling the jumps in an FD setting is to set

$$f(t_h) = \underbrace{(1-e^{-\lambda\Delta t})}_{\substack{jump\\prob}} \underbrace{[1-(\gamma^+\delta_s^+ - \gamma^-\delta_s^- + \frac{1}{2}\vartheta^2\delta_{ss})]^{-1} f(t_{h+1})}_{jump\ \ with\ \ mean\ \ \gamma\ \ and\ \ std\ \ \vartheta}$$

$$+ \underbrace{e^{-\lambda\Delta t}}_{\substack{no\ jump\\prob}} \underbrace{[1-\Delta t(\alpha^+\delta_s^+ - \alpha^-\delta_s^-)]^{-1} f(t_{h+1})}_{drift\ \ with\ \ compensator\ \alpha} \tag{5}$$

- For the stock to be a martingale we can solve for α by setting $f = s$ in (5).

- We get

$$s_i = \underbrace{(1-e^{-\lambda\Delta t})}_{\substack{jump\\ prob}} \underbrace{s_i^j}_{\substack{\text{expected}\\ stock\ price\\ conditional\\ on\ jump}} + \underbrace{e^{-\lambda\Delta t}}_{\substack{no\\ jump\\ prob}} \underbrace{s_i^c}_{\substack{\text{expected}\\ stock\ price\\ conditional\\ on\,no\ jump}}$$

$$[1-(\gamma^+\delta_s^+ - \gamma^-\delta_s^- + \frac{1}{2}\vartheta^2\delta_{ss})]s^j = s$$

$$[1-\Delta t(\alpha^+\delta_s^+ - \alpha^-\delta_s^-)]s^c = s$$

- If we assume ***no jumps*** on the grid edges, the solution is well defined and given by

$$\alpha_i = \frac{1}{\Delta t}\left(\frac{(s_i^c - s_i)^+}{\delta_s^+ s_i^c} - \frac{(s_i^c - s_i)^-}{\delta_s^- s_i^c}\right)$$

$$s_i^c = e^{\lambda\Delta t}(s_i - (1-e^{-\lambda\Delta t})s_i^j) \tag{6}$$

- We note that the computational cost of jumps using this methodology is $O(n)$ rather than $O(n^2)$.

Stochastic Volatility

- Consider the stochastic differential equations

$$\begin{aligned} ds &= \mu dt + z\sigma(t,s)dW \\ dz &= \beta(z)dt + \varepsilon(z)dZ \\ dW \cdot dZ &= \rho dt \end{aligned} \tag{7}$$

- We'll look for a transform $y = y(t,s,z)$ so that $dy \cdot ds = 0$, i.e.

$$\begin{aligned} 0 &= (y_s ds + y_z dz) \cdot ds \\ &= (y_s z^2 \sigma(s)^2 + y_z \sigma(s)\rho\varepsilon(z))dt \end{aligned} \tag{8}$$

- It can be verified that

$$y(t,s,z) = -\rho \int_{s_0}^{s} \frac{1}{\sigma(t,u)} du + \int_{z_0}^{z} \frac{v}{\varepsilon(v)} dv \tag{9}$$

- … solves (8) with

$$dy = \sqrt{1-\rho^2} z dB + \varphi dt$$

$$dB \cdot dW = 0 \tag{10}$$

$$\varphi = y_t + y_s \mu + y_z \beta + \frac{1}{2} y_{ss} z^2 \sigma^2 + \frac{1}{2} y_{zz} \varepsilon^2$$

- With this we have established the PDE

$$0 = f_t + A_\mu f + \lambda[(1 - A_j)^{-1} - 1] f + A_\alpha f + A_y f + A_s f \tag{11}$$

- Hence, there is no cross term between s and y.

Local Volatility Step

- For the final local volatility step we have the backward equation

$$[1-\Delta t\frac{1}{2}z^2\sigma(t_h,s)^2\delta_{ss}]f(t_{h+3/4})=f(t_{h+1}) \tag{12}$$

- The associated forward equation is

$$[1-\Delta t\frac{1}{2}z^2\sigma(t_h,s)^2\delta_{ss}]'p(t_{h+1})=p(t_{h+3/4}) \tag{13}$$

- This leads to the discrete Dupire equation

$$[1-\Delta t\frac{1}{2}E[z(t_{h+1})^2|s(t_{h+1})=k]\sigma(t_h,k)^2\delta_{kk}]c(t_{h+1})=c(t_{h+3/4}) \tag{14}$$

- Here we have

$$E[z(t_{h+1})^2|s(t_{h+1})=k]=\frac{\sum_y z(t_{h+1},k,y)^2 p(t_{h+1},k,y)}{\sum_y p(t_{h+1},k,y)}$$

$$c(t_{h+3/4},k)=\sum_s \sum_y (s-k)^+ p(t_{h+3/4},s,y) \tag{15}$$

- … as in A&H (2011).

- So to calibrate the model to observed option prices we insert ***model*** values for $c(t_{h+3/4})$ and ***market*** values for $c(t_{h+1})$ in (14) and solve for the local volatility function $\sigma(t_h,s)$.

- If input option prices are not consistent with absence of arbitrage or the jump and stochastic volatility parameters, it can be a useful tactic to bound the local volatility function

$$\sigma := \min(\max(\underline{\sigma}, \sigma), \bar{\sigma})$$

- If this is a local phenomenon we can, to some extent, rely on the model to ***catch up*** at the next expiry.

Calibration by Forward Scheme

- Move edges of grid outwards, compute drift and roll fwd:

$$[1-\Delta t A_\mu]' \, p(t_{h+1/4}) = p(t_h)$$

- Calculate expected jumps and compensator and roll fwd:

$$[1-A_j]' \, p^j(t_{h+1/2}) = p(t_{h+1/4})$$

$$[1-\Delta t A_\alpha]' \, p^c(t_{h+1/2}) = p(t_{h+1/4})$$

$$p(t_{h+1/2}) = (1-e^{-\lambda\Delta t}) \, p^j(t_{h+1/2}) + e^{-\lambda\Delta t} \, p^c(t_{h+1/2})$$

- Solve ODE to find $z = z(t,s,y)$, roll fwd, compute $E[z(t_{h+1})^2 | s(t_{h+1}) = k]$, solve for local volatility using discrete Dupire equation, iterate (repeat):

$$[1 - \Delta t A_y]' p(t_{h+3/4})$$

$$[1 - \Delta t A_s]' p(t_{h+1}) = p(t_{h+3/4})$$

- For the case of non-stochastic volatility, iteration is not necessary.

Discussion

- As the solution is always locked to the input option prices the scheme is useful for finding the effect on exotics of changing esoteric parameters such as jump and stochastic volatility.

- For example American options, barriers, and volatility derivatives.

- Due to upwinding, the convergence and accuracy of the scheme is $O(\Delta t + \Delta s + \Delta y)$.

- However, as we're continuously adjusting the drift to hit the one-period forward, we can argue that the convergence is actually $O(\Delta t + \Delta s^2 + \Delta y)$.

- Due to the positivity of all transition probabilities, the convergence profile will be smooth.

- So will risk reports.

- Due to discrete space we will generally need to use likelihood ratio tricks for the Monte-Carlo.

- Adjoint differentiation can be used for this.

Conclusion

- We have managed to expand the discrete duality results of A&H (2011).
- … we can incorporate:
  - Arbitrary grid spacing and drift.
  - Jumps.
  - Correlated stochastic volatility.
  - Local volatility.
- Backward, forward, Dupire and Monte-Carlo is fully consistent.