Calibrating Local Correlations to a Basket Smile

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Outline

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- Impact of correlation on option prices
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Motivation

- Multi-asset Dupire local volatility models with constant correlation do not **capture market skew of stock indices**
- Stocks highly correlated in bearish markets
- Local correlation (LC) models incorporate correl variability in option prices and help traders risk-manage their correl positions during crises

$$\Delta \mathrm{P\&L}_t = \frac{1}{2} \sum_{i,j=1}^{N} S_t^i S_t^j \partial^2_{S^i S^j} P(t, S_t) \left(\frac{\Delta S_t^i \Delta S_t^j}{S_t^i S_t^j} - \rho_{ij} \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) \Delta t \right)$$

- LC models also allow to build models consistent with a triangle of market smiles of FX rates
- LC also used in interest rates modeling to calibrate to spread options prices

Motivation

- Only 2 methods so far to calibrate to smile of index $I_t = \sum_{i=1}^{N} \alpha_i S_t^i$:
  - Local in index volatility of the basket (Langnau):

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j \rho_{ij}(t, S_t) \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j = f(t, I_t)$$

  - Local in index correlation matrix (Reghai, G. and Henry-Labordère):

$$\rho(t, S_t) = f(t, I_t)$$

- Both methods may lead to correlation candidates that fail to be positive semi-definite
- And anyway **why would one undergo either correlation structure?**

Motivation

- Our goal: **build all the LC models calibrated to a basket smile (stock index, cross FX rate, interest rate spread)**

- For the first time, one can design a particular calibrated model in order to
  - match a view on a correlation skew
  - reproduce some features of historical correl
  - calibrate to other option prices

- We can reconcile **static** (implied) calibration and **dynamic** (historical/statistical) calibration

The FX triangle smile calibration problem

- LC model for a triangle of FX rates (deterministic rates):

$$
\begin{aligned}
dS_t^1 &= (r_t^d - r_t^1) S_t^1\, dt + \sigma_1(t, S_t^1) S_t^1\, dW_t^1 \\
dS_t^2 &= (r_t^d - r_t^2) S_t^2\, dt + \sigma_2(t, S_t^2) S_t^2\, dW_t^2 \\
d\langle W^1, W^2 \rangle_t &= \rho(t, S_t^1, S_t^2)\, dt
\end{aligned}
$$

- Example: $S^1 = \mathrm{EUR/USD}$, $S^2 = \mathrm{GBP/USD}$ and $S^{12} = \mathrm{EUR/GBP}$
- Model calibrated to market smile of cross rate $S^{12} \equiv S^1/S^2$ iff for all t

$$
\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) \,\middle|\, \frac{S_t^1}{S_t^2}\right] = \sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right)
$$

- $\mathbb{Q}^f$ = risk-neutral measure associated to the foreign currency in S^2 (GBP):

$$
\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{S_T^2}{S_0^2} \exp\left(\int_0^T \left(r_t^2 - r_t^d\right) dt\right)
$$

The FX triangle smile calibration problem

- The calibration condition

$$\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)-2\rho(t,S_t^1,S_t^2)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right) \quad (1)$$

is equivalent to

$$\frac{\mathbb{E}_\rho^{\mathbb{Q}}\left[S_t^2\left(\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)-2\rho(t,S_t^1,S_t^2)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}_\rho^{\mathbb{Q}}\left[S_t^2\,\middle|\,\frac{S_t^1}{S_t^2}\right]}=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)$$

- Any $\rho:[0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*\to[-1,1]$ satisfying (1) is called an "**admissible correlation**." Admissible = calibrated to market smile of basket

Existing model #1: Local in cross volatility of the cross (Langnau, 2010, Kovrizhkin, 2012)

- Assume that the volatility of the cross is local in cross:

$$\sigma_1^2(t,S^1)+\sigma_2^2(t,S^2)-2\rho(t,S^1,S^2)\sigma_1(t,S^1)\sigma_2(t,S^2)\equiv f\left(t,\frac{S^1}{S^2}\right)$$

- Calibration condition reads

$$\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)-2\rho(t,S_t^1,S_t^2)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)$$

- $\Longrightarrow$

$$\rho=\frac{\sigma_1^2+\sigma_2^2-\sigma_{12}^2}{2\sigma_1\sigma_2}\equiv\rho^*$$

- Does ρ^* stay in $[-1,1]$?

Existing model #2: Local in cross correlation (Reghai, G. and Henry-Labordère, 2011)

- Assume that ρ is local in cross: $\rho\left(t, \frac{S^1}{S^2}\right)$ (Reghai, G. and Henry-Labordère, 2011)
- Calibration condition reads

$$\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]-2\rho\left(t,\frac{S_t^1}{S_t^2}\right)\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]=\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)$$

- $\Longrightarrow$

$$\begin{aligned}\rho\left(t,\frac{S_t^1}{S_t^2}\right) &= \frac{\mathbb{E}^{\mathbb{Q}^f}\left[\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)}{2\mathbb{E}^{\mathbb{Q}^f}\left[\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[S_t^2\left(\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)\mathbb{E}^{\mathbb{Q}}\left[S_t^2\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{2\mathbb{E}^{\mathbb{Q}}\left[S_t^2\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]}\end{aligned}$$

Existing model #2: Local in cross correlation (Reghai, G. and Henry-Labordère, 2011)

- Calibrated model follows the **McKean nonlinear SDE**:

$$
\begin{aligned}
dS_t^1 &= (r_t^d - r_t^1)S_t^1\,dt + \sigma_1(t, S_t^1)S_t^1\,dW_t^1 \\
dS_t^2 &= (r_t^d - r_t^2)S_t^2\,dt + \sigma_2(t, S_t^2)S_t^2\,dW_t^2 \\
d\langle W^1, W^2\rangle_t &= \frac{\mathbb{E}^{\mathbb{Q}}\left[S_t^2\left(\sigma_1^2(t,S_t^1)+\sigma_2^2(t,S_t^2)\right)\,\middle|\,\frac{S_t^1}{S_t^2}\right] - \sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)\mathbb{E}^{\mathbb{Q}}\left[S_t^2\,\middle|\,\frac{S_t^1}{S_t^2}\right]}{2\mathbb{E}^{\mathbb{Q}}\left[S_t^2\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\middle|\,\frac{S_t^1}{S_t^2}\right]}dt
\end{aligned}
$$

- $\Longrightarrow$ We can build ρ using the **particle method**
  Cf. G. and Henry-Labordère, *Being Particular About Calibration*, Risk magazine, January 2012; Jourdain and Sbai, *Coupling Index and stocks*, Quantitative Finance, October 2010.

Particle method for local in cross correl (G. and Henry-Labordère, 2011)

1. Set $k = 1$ and $\rho(t, S^1, S^2) = \frac{\sigma_1^2(0,S^1)+\sigma_2^2(0,S^2)-\sigma_{12}^2(0,\frac{S^1}{S^2})}{2\sigma_1(0,S^1)\sigma_2(0,S^2)}$ for $t \in [t_0, t_1]$
2. Simulate $(S_t^{1,i}, S_t^{2,i})_{1\leq i\leq N}$ from t_{k-1} to t_k using a discretization scheme
3. For all S^{12} in a grid G_{t_k} of cross rate values, compute non-parametric kernel estimates $E_{t_k}^{\text{num}}(S^{12})$ and $E_{t_k}^{\text{den}}(S^{12})$ of $\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2 + \sigma_2^2 \,\middle|\, \frac{S_t^1}{S_t^2}\right]$ and $\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1\sigma_2 \,\middle|\, \frac{S_t^1}{S_t^2}\right]$ at date t_k, define

   $$\rho(t_k, S^{12}) = \frac{E_{t_k}^{\text{num}}(S^{12}) - \sigma_{12}^2\left(t_k, S^{12}\right)}{2E_{t_k}^{\text{den}}(S^{12})}$$

   interpolate $\rho(t_k, \cdot)$, e.g., using cubic splines, extrapolate, and, for all $t \in [t_k, t_{k+1}]$, set $\rho(t, S^1, S^2) = \rho(t_k, \frac{S^1}{S^2})$. If $\rho(t_k, \frac{S^1}{S^2}) > 1$ (resp. < -1), cap it at $+1$ (resp. floor it at -1) $\Longrightarrow$ imperfect calibration (may be accurate enough!)
4. Set $k := k + 1$. Iterate steps 2 and 3 up to the maturity date T

The equity index smile calibration problem

$$\begin{aligned} S_t &= (S_t^1, \ldots, S_t^N) \\ dS_t^i &= r_t S_t^i dt + \sigma_i(t, S_t^i) S_t^i dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, S_t) dt \\ v_\rho(t, S_t) &\equiv \sum_{i,j=1}^N \alpha_i \alpha_j \rho_{ij}(t, S_t) \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j \end{aligned}$$

- Index $I_t = \sum_{i=1}^N \alpha_i S_t^i$ made of N weighted stocks
- Model calibrated to index smile if and only if

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = \mathbb{E}_\rho \left[v_\rho(t, S_t) \middle| I_t \right] \tag{2}$$

- $N(N-1)/2$ parameters for 1 scalar equation. Dimension reduction (e.g., $\rho^0 = \rho^{\mathrm{hist}}$, $\rho^1 = \mathbf{1}$):

$$\rho(t, S) = (1 - \lambda(t, S))\rho^0 + \lambda(t, S)\rho^1, \qquad \lambda(t, S) \in \mathbb{R}$$

- If $\lambda \in [0, 1]$, ρ is guaranteed to be a correlation matrix
- After dimension reduction, calibration condition reads

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 \quad = \quad \mathbb{E}_\rho \left[v_{\rho^0}(t, S_t) + \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \lambda(t, S_t) \middle| I_t \right]$$

Existing model #1: Local in index volatility of the index (Langnau, 2010)

- Assume that v_ρ is local in index:

$$v_\rho(t,S) \equiv f(t,I)$$

- Then the calibration condition reads

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t,I_t)^2 = v_{\rho^0}(t,S_t) + \left(v_{\rho^1}(t,S_t) - v_{\rho^0}(t,S_t)\right)\lambda(t,S_t)$$

- $\Longrightarrow$

$$\lambda(t,S) = \frac{I^2\sigma_{\mathrm{Dup}}^I(t,I)^2 - v_{\rho^0}(t,S)}{v_{\rho^1}(t,S) - v_{\rho^0}(t,S)} \equiv \lambda^*(t,S)$$

- Does λ^* stay in $[0,1]$?

Existing model #2: Local in index correl for stock indices (G. and Henry-Labordère, 2011)

- Recall that, after dimension reduction, calibration condition reads

$$I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 \quad = \quad \mathbb{E}_\rho \left[v_{\rho^0}(t, S_t) + \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \lambda(t, S_t) \middle| I_t \right]$$

- Assume that λ is local in index: $\lambda(t, I)$
- $\Longrightarrow$

$$\lambda(t, I_t) = \frac{I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 - \mathbb{E}_\rho \left[v_{\rho^0}(t, S_t) \middle| I_t \right]}{\mathbb{E}_\rho \left[v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \middle| I_t \right]}$$

- The calibrated model then follows the **McKean nonlinear SDE**

$$\begin{aligned} dS_t^i &= r_t S_t^i dt + S_t^i \sigma_i(t, S_t^i) dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, I_t) dt \\ \lambda(t, I) &= \frac{I^2 \sigma_{\mathrm{Dup}}^I(t, I)^2 - \sum_{i,j=1}^N \alpha_i \alpha_j \rho_{ij}^0 \mathbb{E}[\sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j | I_t = I]}{\sum_{i,j=1}^N \alpha_i \alpha_j (\rho_{ij}^1 - \rho_{ij}^0) \mathbb{E}[\sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j | I_t = I]} \end{aligned}$$

- Does λ stay in $[0, 1]$?

Building all admissible local correlation models: FX (G., 2013)

- Vol or ρ local in cross/index = a particular modeling choice only guided by computational convenience. **How do we build all admissible correls?**
- Let ρ be admissible. We can always pick two functions $a(t, S^1, S^2)$ and $b(t, S^1, S^2)$ such that b does not vanish and $a + b\rho$ is local in cross:

$$a(t, S^1, S^2) + b(t, S^1, S^2)\rho(t, S^1, S^2) \equiv f\left(t, \frac{S^1}{S^2}\right)$$

e.g., $b \equiv 1$, $a(t, S^1, S^2) = f\left(t, \frac{S^1}{S^2}\right) - \rho(t, S^1, S^2)$

- Local in cross correl: $a \equiv 0$ and $b \equiv 1$ (Reghai, G. and Henry-Labordère)
- Local in cross volatility of the cross: $a = \sigma_1^2 + \sigma_2^2$ and $b = -2\sigma_1\sigma_2$ (Langnau, Kovrizhkin): $\rho^* = \frac{\sigma_1^2+\sigma_2^2-\sigma_{12}^2}{2\sigma_1\sigma_2}$

$$\sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \,\middle|\, \frac{S_t^1}{S_t^2}\right]$$
$$= \mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2 + \sigma_2^2 + 2\frac{a}{b}\sigma_1\sigma_2 \,\middle|\, \frac{S_t^1}{S_t^2}\right] - 2(a + b\rho)\left(t, \frac{S_t^1}{S_t^2}\right)\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\frac{\sigma_1\sigma_2}{b} \,\middle|\, \frac{S_t^1}{S_t^2}\right]$$

Building all admissible local correlation models: FX (G., 2013)

$$\sigma_{12}^2\left(t,\frac{S_t^1}{S_t^2}\right)=\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2+\sigma_2^2+2\frac{a}{b}\sigma_1\sigma_2\,\middle|\,\frac{S_t^1}{S_t^2}\right]-2\left(a+b\rho\right)\left(t,\frac{S_t^1}{S_t^2}\right)\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\frac{\sigma_1\sigma_2}{b}\,\middle|\,\frac{S_t^1}{S_t^2}\right]$$

$$\Longrightarrow\rho_{(a,b)}=\frac{1}{b}\left(\frac{\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\sigma_1^2+\sigma_2^2+2\frac{a}{b}\sigma_1\sigma_2\,\middle|\,\frac{S_t^1}{S_t^2}\right]-\sigma_{12}^2}{2\mathbb{E}_{\rho_{(a,b)}}^{\mathbb{Q}^f}\left[\frac{\sigma_1\sigma_2}{b}\,\middle|\,\frac{S_t^1}{S_t^2}\right]}-a\right) \qquad (3)$$

- The affine transform may seem *ad hoc* at first sight but actually *any* admissible correlation is of the above type
- Conversely, if a function $\rho_{(a,b)}:[0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*\to[-1,1]$ satisfies (3), then it is an admissible correlation
- We call (3) the "**local in cross affine transform representation**" of admissible correlations

Building all admissible local correlation models: FX (G., 2013)

- Calibrated model follows the **McKean nonlinear SDE**:

$$
\begin{aligned}
dS_t^1 &= (r_t^d - r_t^1) S_t^1 \, dt + \sigma_1(t, S_t^1) S_t^1 \, dW_t^1 \\
dS_t^2 &= (r_t^d - r_t^2) S_t^2 \, dt + \sigma_2(t, S_t^2) S_t^2 \, dW_t^2 \\
d\langle W^1, W^2 \rangle_t &= \left(\frac{\mathbb{E}^{\mathbb{Q}} \left[S_t^2 \left(\sigma_1^2 + \sigma_2^2 + 2 \frac{a}{b} \sigma_1 \sigma_2 \right) \middle| \frac{S_t^1}{S_t^2} \right] - \sigma_{12}^2 \left(t, \frac{S_t^1}{S_t^2} \right) \mathbb{E}^{\mathbb{Q}} \left[S_t^2 \middle| \frac{S_t^1}{S_t^2} \right]}{2 \mathbb{E}^{\mathbb{Q}} \left[S_t^2 \frac{\sigma_1 \sigma_2}{b} \middle| \frac{S_t^1}{S_t^2} \right]} \right. \\
&\qquad \left. - a(t, S_t^1, S_t^2) \right) dt / b(t, S_t^1, S_t^2)
\end{aligned}
$$

- The particle method allows us to estimate the conditional expectations

Building $\rho_{(a,b)}$ using the particle method

1. Set $k = 1$ and $\rho_{(a,b)} = \frac{\sigma_1^2(0,S^1)+\sigma_2^2(0,S^2)-\sigma_{12}^2\left(0,\frac{S^1}{S^2}\right)}{2\sigma_1(0,S^1)\sigma_2(0,S^2)}$ for $t \in [t_0 = 0; t_1]$.
2. Simulate $(S_t^{1,i}, S_t^{2,i})_{1\leq i\leq N}$ from t_{k-1} to t_k using a discretization scheme.
3. For all S^{12} in a grid G_{t_k} of cross rate values, compute

$$E_{t_k}^{\text{num}}(S^{12}) = \frac{\sum_{i=1}^N S_{t_k}^{2,i}\left(\sigma_1^2(t_k,S_{t_k}^{1,i}) + \sigma_2^2(t_k,S_{t_k}^{2,i}) + 2\frac{a(t_k,S_{t_k}^{1,i},S_{t_k}^{2,i})}{b(t_k,S_{t_k}^{1,i},S_{t_k}^{2,i})}\sigma_1(t_k,S_{t_k}^{1,i})\sigma_2(t_k,S_{t_k}^{2,i})\right)\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)}{\sum_{i=1}^N S_{t_k}^{2,i}\delta_{t_k,N}\left(\frac{S_{t_k}^{1,i}}{S_{t_k}^{2,i}} - S^{12}\right)}$$

Building $\rho_{(a,b)}$ using the particle method

$$E^{\text{den}}_{t_k}(s^{12}) = \frac{\sum_{i=1}^N S^{2,i}_{t_k} \frac{\sigma_1(t_k, S^{1,i}_{t_k})\sigma_2(t_k, S^{2,i}_{t_k})}{b(t_k, S^{1,i}_{t_k}, S^{2,i}_{t_k})} \delta_{t_k,N}\left(\frac{S^{1,i}_{t_k}}{S^{2,i}_{t_k}} - s^{12}\right)}{\sum_{i=1}^N S^{2,i}_{t_k} \delta_{t_k,N}\left(\frac{S^{1,i}_{t_k}}{S^{2,i}_{t_k}} - s^{12}\right)}$$

$$f(t_k, s^{12}) = \frac{E^{\text{num}}_{t_k}(s^{12}) - \sigma^2_{12}\left(t_k, s^{12}\right)}{2E^{\text{den}}_{t_k}(s^{12})}$$

3 cont'd interpolate and extrapolate $f(t_k, \cdot)$, for instance using cubic splines, and, for all $t \in [t_k, t_{k+1}]$, set

$$\rho_{(a,b)}(t, S^1, S^2) = \frac{1}{b(t, S^1, S^2)}\left(f\left(t_k, \frac{S^1}{S^2}\right) - a(t, S^1, S^2)\right)$$

If $\rho_{(a,b)}(t, S^1, S^2) > 1$ (resp. < -1), cap it at $+1$ (resp. floor it at -1) $\Longrightarrow$ imperfect calibration (may still be very accurate!)

4 Set $k := k + 1$. Iterate steps 2 and 3 up to the maturity date T.

Links between local correlations

$$\left(a+b\rho_{(a,b)}\right)\left(t,\frac{S_t^1}{S_t^2}\right) = \frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\left(a+b\rho^*\right)\left(t,S_t^1,S_t^2\right)\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\,\Big|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\frac{\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)}{b(t,S_t^1,S_t^2)}\,\Big|\,\frac{S_t^1}{S_t^2}\right]}$$

$$\rho_{(0,1)}\left(t,\frac{S_t^1}{S_t^2}\right) = \frac{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(0,1)}}\left[\rho^*\left(t,S_t^1,S_t^2\right)\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\Big|\,\frac{S_t^1}{S_t^2}\right]}{\mathbb{E}^{\mathbb{Q}^f}_{\rho_{(0,1)}}\left[\sigma_1(t,S_t^1)\sigma_2(t,S_t^2)\,\Big|\,\frac{S_t^1}{S_t^2}\right]}$$

- $\rho_{(0,1)}$ = an average of ρ^*
- If $\rho_{(0,1)}$ is admissible then its image is included in the image of ρ^*
- $\tau_{\rho^*} \leq \tau_{\rho_{(0,1)}}$ with τ_ρ the smallest time at which ρ fails to be a correlation function:

$$\tau_\rho = \inf\left\{t\in[0,T]\,|\,\exists S^1,S^2>0,\ \rho(t,S^1,S^2)\notin[-1,1]\right\}$$

Links between local correlations

$$\sigma_{12}^2\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho_{(a,b)}(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) \,\middle|\, \frac{S_t^1}{S_t^2}\right]$$

If no skew on S^1 and S^2:

$$\rho^*(t, S_t^1, S_t^2) = \rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) = \mathbb{E}^{\mathbb{Q}^f}_{\rho_{(a,b)}}\left[\rho_{(a,b)}(t, S_t^1, S_t^2) \,\middle|\, \frac{S_t^1}{S_t^2}\right]$$

- All admissible correls have same average value over constant cross lines
- $\rho_{(0,1)} = \rho^*$ is among all the admissible correls the one with smallest image
- $\rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) > 1 \iff |\sigma_1(t) - \sigma_2(t)| > \sigma_{12}\left(t, \frac{S_t^1}{S_t^2}\right)$
- $\rho_{(0,1)}\left(t, \frac{S_t^1}{S_t^2}\right) < -1 \iff \sigma_1(t) + \sigma_2(t) < \sigma_{12}\left(t, \frac{S_t^1}{S_t^2}\right)$

Building whole families of admissible local correlation models: equity (G., 2013)

$$
\begin{aligned}
dS_t^i &= S_t^i \sigma_i(t, S_t^i) dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, S_t) dt \\
v_\rho(t, S_t) &\equiv \sum_{i,j=1}^{N} \alpha_i \alpha_j \rho_{ij}(t, S_t) \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) S_t^i S_t^j
\end{aligned}
$$

- After dimension reduction, calibration condition reads

$$
I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 \quad = \quad \mathbb{E}_\rho \left[v_{\rho^0}(t, S_t) + \left(v_{\rho^1}(t, S_t) - v_{\rho^0}(t, S_t) \right) \lambda(t, S_t) \middle| I_t \right]
$$

- We can always pick two functions a and b such that b does not vanish and $a(t, S_t) + b(t, S_t)\lambda(t, S_t) \equiv f(t, I_t)$ is local in index. Then

$$
I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 = (a + b\lambda)(t, I) \mathbb{E}_\rho \left[\frac{1}{b} \left(v_{\rho^1} - v_{\rho^0} \right) \middle| I_t \right] + \mathbb{E}_\rho \left[v_{\rho^0} - \frac{a}{b} \left(v_{\rho^1} - v_{\rho^0} \right) \middle| I_t \right]
$$

$$
\Longrightarrow \lambda_{(a,b)} = \frac{1}{b} \left(\frac{I_t^2 \sigma_{\mathrm{Dup}}^I(t, I_t)^2 - \mathbb{E}_{\rho_{(a,b)}} \left[v_{\rho^0} - \frac{a}{b} \left(v_{\rho^1} - v_{\rho^0} \right) \middle| I_t \right]}{\mathbb{E}_{\rho_{(a,b)}} \left[\frac{1}{b} \left(v_{\rho^1} - v_{\rho^0} \right) \middle| I_t \right]} - a \right)
$$

Building families of admissible local correlation models: equity (G., 2013)

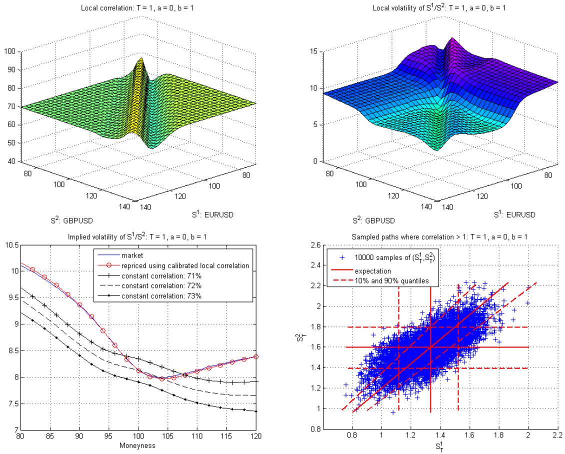
$$\lambda_{(a,b)} = \frac{1}{b}\left(\frac{I_t^2\sigma_{\mathrm{Dup}}^I(t,I_t)^2 - \mathbb{E}_{\rho_{(a,b)}}\left[v_{\rho^0} - \frac{a}{b}\left(v_{\rho^1} - v_{\rho^0}\right)\middle| I_t\right]}{\mathbb{E}_{\rho_{(a,b)}}\left[\frac{1}{b}\left(v_{\rho^1} - v_{\rho^0}\right)\middle| I_t\right]} - a\right) \qquad (4)$$

- If a function $\lambda_{(a,b)}$ satisfies (4) and is s.t. $\rho_{(a,b)} \equiv (1-\lambda_{(a,b)})\rho^0 + \lambda_{(a,b)}\rho^1$ is positive semi-definite, then $\rho_{(a,b)}$ is an admissible correlation
- We call this procedure the **local in index $a + b\lambda$ method**.
- Local in index λ method: $a \equiv 0$ and $b \equiv 1$ (Reghai, G. and Henry-Labordère)
- Local in index volatility method: $a = v_{\rho^0}$ and $b = v_{\rho^1} - v_{\rho^0}$ (Langnau, Kovrizhkin)
- Particle method $\Longrightarrow \lambda_{(a,b)}$

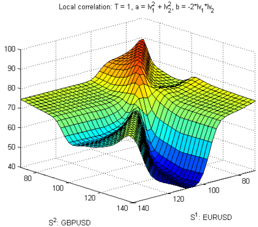
Numerical examples: FX

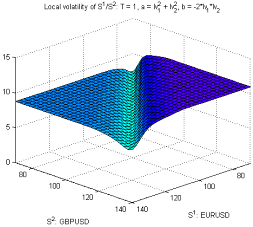
- $S^1 = \mathrm{EURUSD}$, $S^2 = \mathrm{GBPUSD}$, $S^{12} = S^1/S^2 = \mathrm{EURGBP}$ (March 2012)
- $T = 1$
- $N = 10,000$ particles
- $\Delta t = \frac{1}{80}$
- $K(x) = (1 - x^2)^2 1_{\{|x| \leq 1\}}$
- Bandwidth $h = \kappa \bar{\sigma}^{12} S_0^{12} \sqrt{\max(t, t_{\min})} N^{-\frac{1}{5}}$, $\bar{\sigma}^{12} = 10\%$, $t_{\min} = 0.25$ and $\kappa \approx 3$.
- The constant correlation that fits ATM implied volatility of cross rate at maturity $= 72\%$

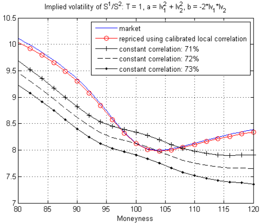
$a = 0$, $b = 1$ (local in cross correlation)

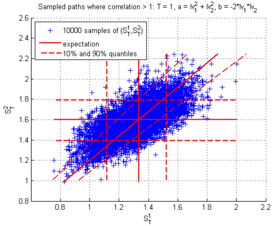


$a = \sigma_1^2 + \sigma_2^2$, $b = -2\sigma_1\sigma_2$ (local in cross volatility of the cross)

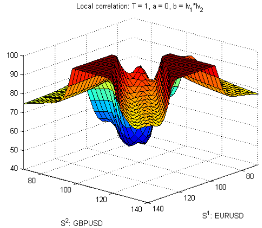


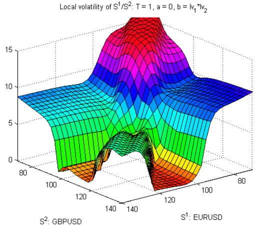


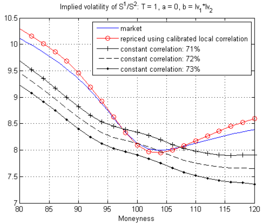


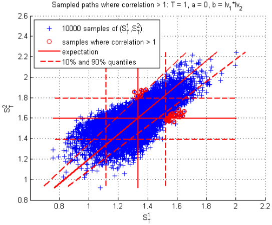


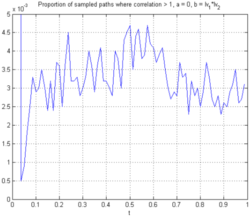
$a = 0,\ b = \sigma_1 \sigma_2$ (local in cross covariance)



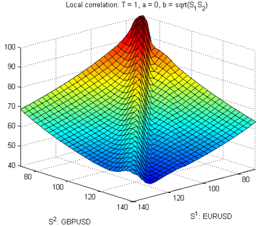


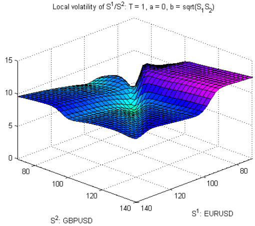


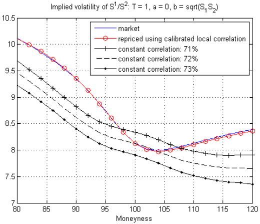


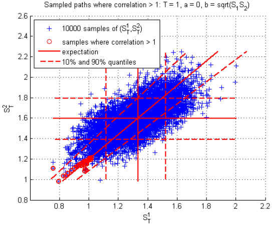


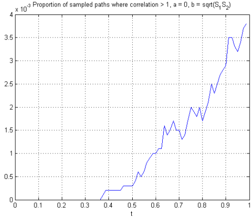
$a = 0,\ b = \sqrt{S^1 S^2}$



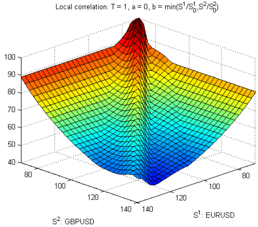


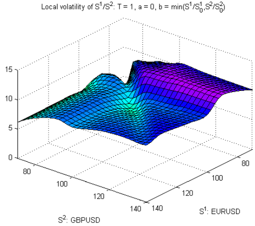


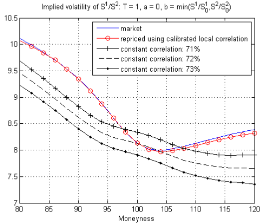


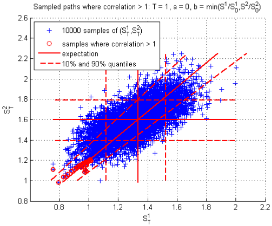


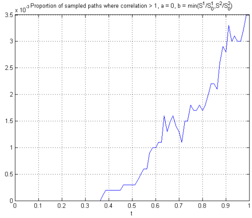
$a = 0,\ b = \min(S^1, S^2)$


Local correlation: T = 1, a = 0, b = min(S1/S10,S2/S20)
S2: GBPUSD
S1: EURUSD

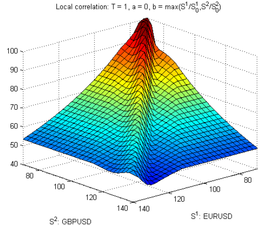

Local volatility of S1/S2: T = 1, a = 0, b = min(S1/S10,S2/S20)
S2: GBPUSD
S1: EURUSD

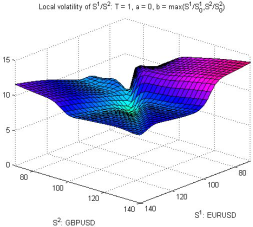

Implied volatility of S1/S2: T = 1, a = 0, b = min(S1/S10,S2/S20)
market
repriced using calibrated local correlation
constant correlation: 71%
constant correlation: 72%
constant correlation: 73%
Moneyness

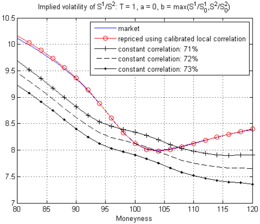

Sampled paths where correlation > 1: T = 1, a = 0, b = min(S1/S10,S2/S20)
10000 samples of (S1T,S2T)
samples where correlation > 1
expectation
10% and 90% quantiles
S2T
S1T

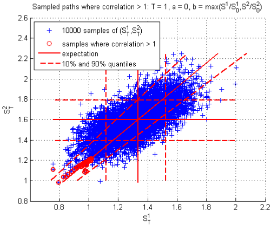

Proportion of sampled paths where correlation > 1, a = 0, b = min(S1/S10,S2/S20)
x 10-3
t

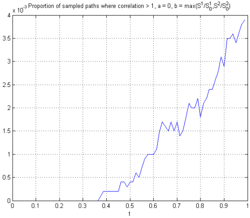
$a = 0,\ b = \max(S^1, S^2)$


Local correlation: T = 1, a = 0, b = $\max(S^1/S^1_0, S^2/S^2_0)$
S^2: GBPUSD
S^1: EURUSD

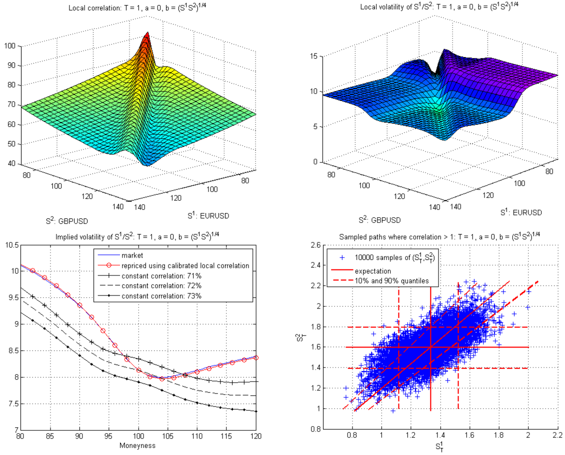

Local volatility of S^1/S^2: T = 1, a = 0, b = $\max(S^1/S^1_0, S^2/S^2_0)$
S^2: GBPUSD
S^1: EURUSD


Implied volatility of S^1/S^2: T = 1, a = 0, b = $\max(S^1/S^1_0, S^2/S^2_0)$
market
repriced using calibrated local correlation
constant correlation: 71%
constant correlation: 72%
constant correlation: 73%
Moneyness

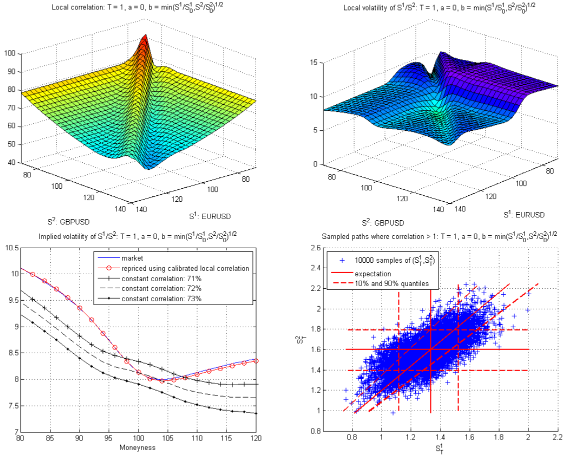

Sampled paths where correlation > 1: T = 1, a = 0, b = $\max(S^1/S^1_0, S^2/S^2_0)$
10000 samples of (S^1_T, S^2_T)
samples where correlation > 1
expectation
10% and 90% quantiles
S^1_T
S^2_T


$\times 10^{-3}$ Proportion of sampled paths where correlation > 1, a = 0, b = $\max(S^1/S^1_0, S^2/S^2_0)$
t

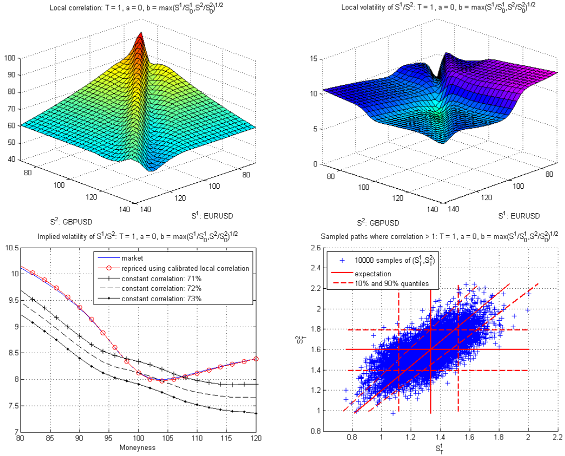
$a = 0,\ b = (S^1 S^2)^{1/4}$


Local correlation: T = 1, a = 0, b = (S1S2)1/4
S2: GBPUSD
S1: EURUSD
Local volatility of S1/S2: T = 1, a = 0, b = (S1S2)1/4
S2: GBPUSD
S1: EURUSD
Implied volatility of S1/S2: T = 1, a = 0, b = (S1S2)1/4
market
repriced using calibrated local correlation
constant correlation: 71%
constant correlation: 72%
constant correlation: 73%
Moneyness
Sampled paths where correlation > 1: T = 1, a = 0, b = (S1S2)1/4
10000 samples of (S1T,S2T)
expectation
10% and 90% quantiles
S2T
S1T

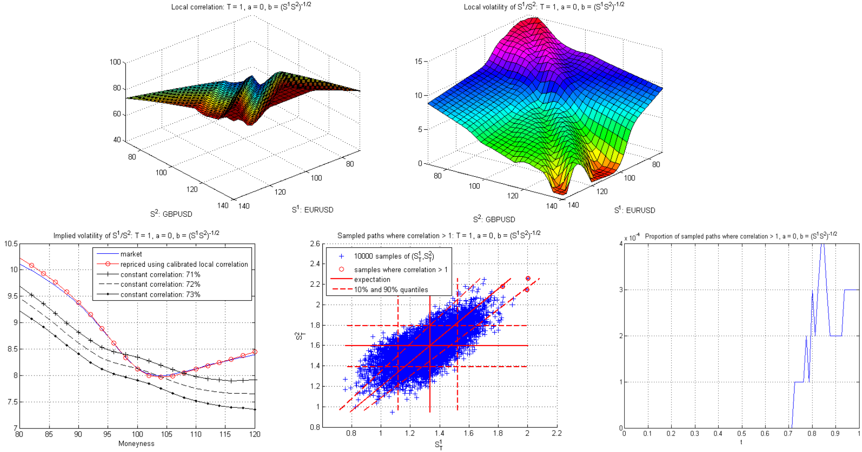
$a = 0,\ b = \sqrt{\min(S^1, S^2)}$



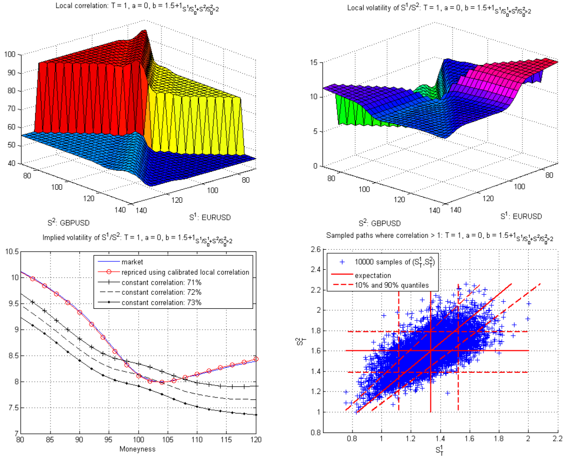
$a = 0,\ b = \sqrt{\max(S^1, S^2)}$



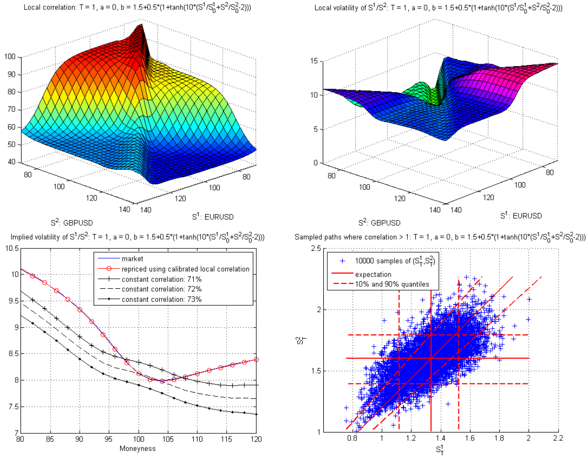
$a = 0,\ b = \frac{1}{\sqrt{S^1 S^2}}$



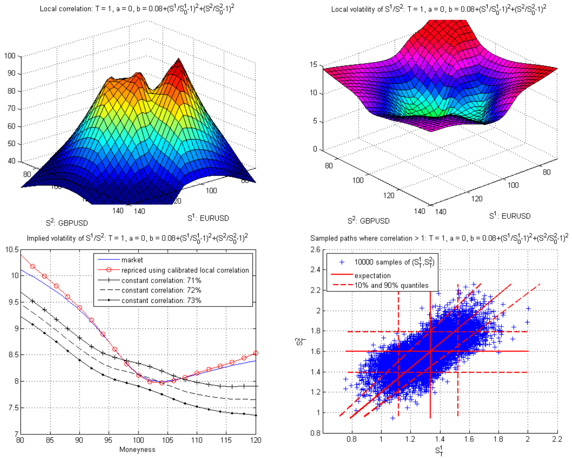
$a = 0,\ b = 1.5 + 1_{\left\{\frac{S^1}{S^1_0} + \frac{S^2}{S^2_0} > 2\right\}}$



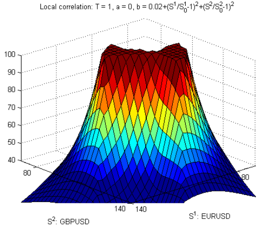
$a = 0,\ b = 1.5 + \frac{1}{2}\left(1 + \tanh\left(10\left(\frac{S^1}{S^1_0} + \frac{S^2}{S^2_0} - 2\right)\right)\right)$

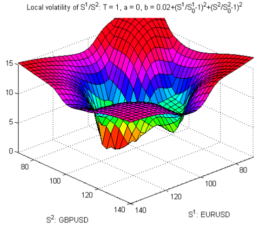


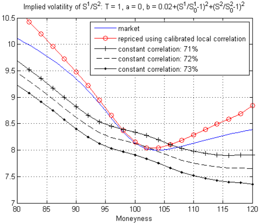
$a = 0,\ b = 0.08 + \left(\frac{S^1}{S^1_0} - 1\right)^2 + \left(\frac{S^2}{S^2_0} - 1\right)^2$

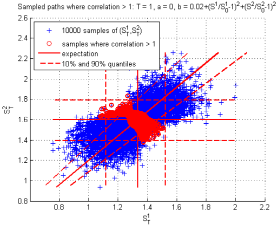


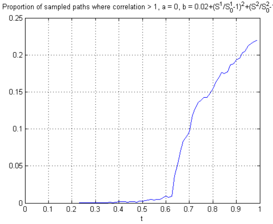
$$a = 0,\ b = 0.02 + \left(\frac{S^1}{S^1_0} - 1\right)^2 + \left(\frac{S^2}{S^2_0} - 1\right)^2$$



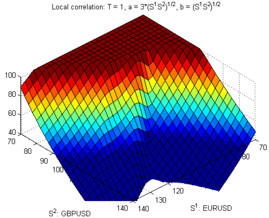


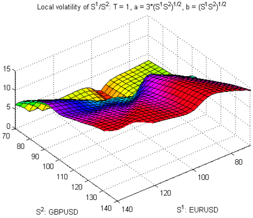


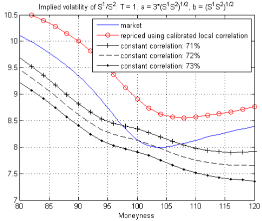


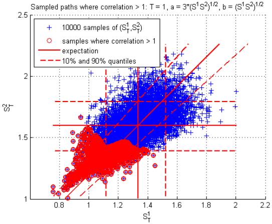


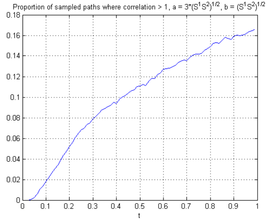
$a = 3\sqrt{S^1 S^2}$, $b = \sqrt{S^1 S^2}$



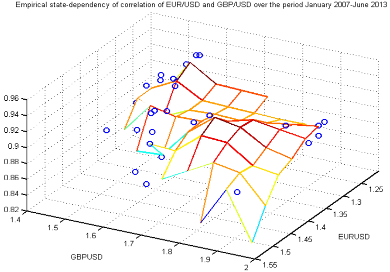


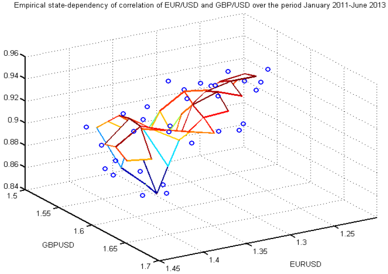






Historical local correlation EUR/USD & GBP/USD, 2007-13 vs 2011-13





Impact of correlation on option prices

- Price impact formula (similar to El Karoui, Dupire for vol):

$$\mathbb{E}_{\rho_t}[g(S_T^1, S_T^2)] - P_0(0, S_0^1, S_0^2)$$
$$= \mathbb{E}_{\rho_t}\left[\int_0^T (\rho_t - \rho_0(t, S_t^1, S_t^2))\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) S_t^1 S_t^2 \partial^2_{S^1 S^2} P_0(t, S_t^1, S_t^2)dt\right]$$

- Implied correlation (cf Dupire for implied vol):

$$\rho(T,g) = \frac{\mathbb{E}_{\rho_t}\left[\int_0^T \rho_t \sigma_1(t, S_t^1)\sigma_2(t, S_t^2) S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(T,g)}(t, S_t^1, S_t^2)dt\right]}{\mathbb{E}_{\rho_t}\left[\int_0^T \sigma_1(t, S_t^1)\sigma_2(t, S_t^2) S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(T,g)}(t, S_t^1, S_t^2)dt\right]}$$

Implied correl is the fixed point of

$$\rho \mapsto \int_0^T \int_0^\infty \int_0^\infty \rho_{\text{loc}}(t, S^1, S^2) q_\rho(t, S^1, S^2) dS^1 dS^2 dt$$

$$q_\rho(t, S^1, S^2) = \frac{\sigma_1(t, S^1)\sigma_2(t, S^2) S^1 S^2 \partial^2_{S^1 S^2} P_\rho(t, S^1, S^2) p(t, S^1, S^2)}{\int_0^T \int_0^\infty \int_0^\infty \sigma_1(t_*, S_*^1)\sigma_2(t_*, S_*^2) S_*^1 S_*^2 \partial^2_{S^1 S^2} P_\rho(t_*, S_*^1, S_*^2) p(t_*, S_*^1, S_*^2) dS_*^1 dS_*^2}$$

$p \to \hat{p} \Longrightarrow \rho(T,g) \to \hat{\rho}(T,g)$ (cf G. and Henry-Labordère, 2011)

Graphs of $(S^1, S^2) \mapsto q_\rho(t, S^1, S^2)$ for different values of t

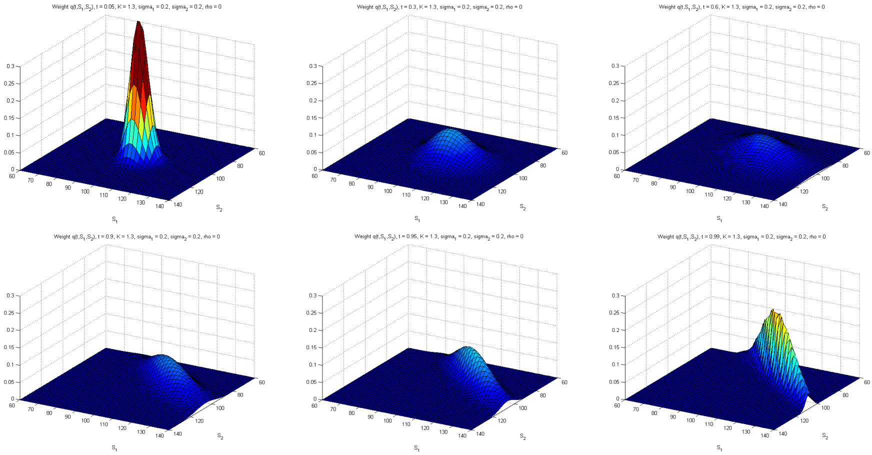


Figure : Black-Scholes model: $\sigma_1 = 20\%$, $\sigma_2 = 20\%$, $\rho = 0$, $S_0^1 = 100$, $S_0^2 = 100$. Payoff $g(S_T^1, S_T^2) = (S_T^1 - KS_T^2)_+$, $K = 1.3$, $T = 1$.

- Following Gatheral, we get another expression for implied correl by considering a time-dependent $\rho(t)$
- Models with ρ_t and $\rho(t)$ give same price to the option iff

$$\int_0^T \mathbb{E}_{\rho_t}\left[(\rho_t - \rho(t))\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t)}(t, S_t^1, S_t^2)\right] dt = 0$$

- The integrand vanishes for each time slice $t \Longrightarrow$

$$\rho(t; T, g) = \frac{\mathbb{E}_{\rho_t}\left[\rho_t \sigma_1(t, S_t^1)\sigma_2(t, S_t^2)S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}{\mathbb{E}_{\rho_t}\left[\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}$$

- When S^1 and S^2 have no skew,

$$\rho(T, g) = \frac{\int_0^T \rho(t; T, g)\sigma_1(t)\sigma_2(t)dt}{\int_0^T \sigma_1(t)\sigma_2(t)dt} = \frac{\int_0^T \frac{\mathbb{E}_{\rho_t}\left[\rho_t S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}{\mathbb{E}_{\rho_t}\left[S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]} \sigma_1(t)\sigma_2(t)dt}{\int_0^T \sigma_1(t)\sigma_2(t)dt}$$

- When σ_1 and σ_2 are constant, this reads (cf Gatheral for implied vol)

$$\rho(T, g) = \frac{1}{T}\int_0^T \frac{\mathbb{E}_{\rho_t}\left[\rho_t S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]}{\mathbb{E}_{\rho_t}\left[S_t^1 S_t^2 \partial^2_{S^1 S^2} P_{\rho(t;T,g)}(t, S_t^1, S_t^2)\right]} dt$$

Impact of admissible $\rho_{(a,b)}$ on option prices

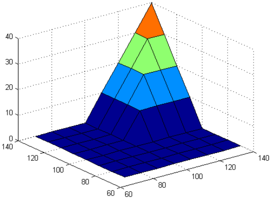
- Options considered:

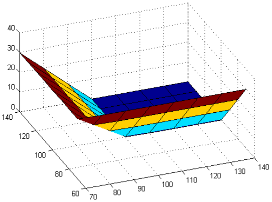
$$
\begin{aligned}
\text{Min of calls}: \quad g(S_T^1, S_T^2) &= \min\left(\left(\frac{S_T^1}{K^1}-1\right)_+, \left(\frac{S_T^2}{K^2}-1\right)_+\right), \quad K^i = S_0^i \\
\text{Put on worst}: \quad g(S_T^1, S_T^2) &= \left(K - \min\left(\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}\right)\right)_+, \quad K = 0.95 \\
\text{Put on basket}: \quad g(S_T^1, S_T^2) &= \left(K - \left(\frac{S_T^1}{S_0^1} + \frac{S_T^2}{S_0^2}\right)\right)_+, \quad K = 1.8
\end{aligned}
$$

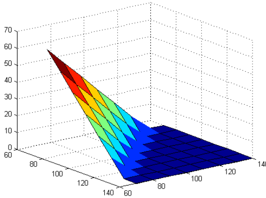
- Their cross gamma at maturity is proportional to

$$
\begin{aligned}
\text{Min of calls} &: \quad \delta\left(\frac{S^2}{K^2} - \frac{S^1}{K^1}\right) 1_{\left\{\frac{S^1}{K^1} \geq 1\right\}} \\
\text{Put on worst} &: \quad -\delta\left(\frac{S^2}{S_0^2} - \frac{S^1}{S_0^1}\right) 1_{\left\{\frac{S_T^1}{S_0^1} \leq K\right\}} \\
\text{Put on basket} &: \quad \delta\left(\frac{S^1}{S_0^1} + \frac{S^2}{S_0^2} - K\right)
\end{aligned}
$$

Impact of admissible $\rho_{(a,b)}$ on option prices







a	b	Min of calls	Put on worst	Put on basket
Standard deviation		≈ 0.020	≈ 0.027	≈ 0.027
Constant correlation 72%		2.59	3.47	1.88
0	1	2.65	3.49	1.91
$\sigma_1^2 + \sigma_2^2$	$-2\sigma_1\sigma_2$	2.53	3.37	1.99
0	$\sigma_1\sigma_2$	**2.91**	**3.70**	**1.78**
0	$\sqrt{S^1 S^2}$	2.56	3.41	1.95
0	$\max(S^1, S^2)$	2.56	3.40	1.95
0	$\min(S^1, S^2)$	2.56	3.41	1.95
0	$(S^1 S^2)^{1/4}$	2.61	3.45	1.93
0	$\sqrt{\max(S^1, S^2)}$	2.61	3.45	1.93
0	$\sqrt{\min(S^1, S^2)}$	2.61	3.45	1.93
0	$\frac{1}{\sqrt{S^1 S^2}}$	2.74	3.56	1.87
0	$1.5 + 1_{\left\{\frac{S^1}{S_0^1} + \frac{S^2}{S_0^2} > 2\right\}}$	**2.37**	**3.25**	**2.06**
0	$2 + \frac{1}{2}\mathrm{th}\left(10\left(\frac{S^1}{S_0^1} + \frac{S^2}{S_0^2} - 2\right)\right)$	2.42	3.28	2.04

Extension to stochastic interest rates, stochastic dividend yield, and stochastic volatility (G., 2013)

- Our method is **robust** and easily handles **stoch rates**, **stoch div yield**, and **stoch vol**. For example in FX:

$$
\begin{aligned}
dS_t^1 &= (r_t^d - r_t^1) S_t^1 \, dt + \sigma_1(t, S_t^1) a_t^1 S_t^1 \, dW_t^1 \\
dS_t^2 &= (r_t^d - r_t^2) S_t^2 \, dt + \sigma_2(t, S_t^2) a_t^2 S_t^2 \, dW_t^2 \\
d\langle W^1, W^2 \rangle_t &= \rho(t, S_t^1, S_t^2, a_t^1, a_t^2, D_{0t}^d, D_{0t}^1, D_{0t}^2) \, dt \equiv \rho(t, X_t) \, dt
\end{aligned}
$$

- Easy case: (W^1, W^2) indepdt of $(W^3, W^4, \ldots)$. **First calibrate** σ_1 **and** σ_2 using the particle method (cf. calibration condition in the article). **Then calibrate** ρ by picking a and b such that b does not vanish and $a(t, X) + b(t, X)\rho(t, X)$ is local in cross.
- If (W^1, W^2) **not** indepdt of $(W^3, W^4, \ldots)$, be careful! Fix all the correls that are needed to first calibrate σ_1 and σ_2. Then both ρ^0 and ρ^1 **must have those fixed correl values**, so that calibration of ρ does not destroy calibration of σ_1 and σ_2.

Path-dependent volatility (G., 2013)

- Our method is **robust** and also works for **path-dep correlation models**:

$$a(t,S^1,S^2,X)+b(t,S^1,S^2,X)\rho(t,S^1,S^2,X)\equiv f\left(t,\frac{S^1}{S^2}\right)$$

- X can be any path-dep variable: running averages, moving averages, running maximums/mimimums, moving maximums/minimums, realized correlations/variances over the last month (cf GARCH), etc.
- What works for correl works for vol! **Path-dep vol model**:

$$\frac{dS_t}{S_t}=(r_t-q_t)\,dt+\sigma(t,S_t,X_t)\,dW_t$$

- Determ. rates and div yield: model calibrated to the smile iff for all t

$$\mathbb{E}[\sigma(t,S_t,X_t)^2|S_t]=\sigma^2_{\mathrm{Dup}}(t,S_t)$$

- All calibrated models can be built by picking a and b such that

$$a(t,S,X)+b(t,S,X)\sigma^2(t,S,X)\equiv f(S)$$

and using the particle method

Path-dependent volatility (G., 2013)

- Calibrated model follows the **McKean nonlinear SDE**

$$\frac{dS_t}{S_t} = (r_t - q_t)\,dt + \sqrt{\frac{1}{b(t,S_t,X_t)}\left(\frac{\sigma^2_{\mathrm{Dup}}(t,S_t) + \mathbb{E}\left[\left.\frac{a}{b}\right| S_t\right]}{\mathbb{E}\left[\left.\frac{1}{b}\right| S_t\right]} - a(t,S_t,X_t)\right)}\,dW_t$$

- Complete model: prices are uniquely defined. Asset and implied vol dynamics richer than in the local vol model. Possibly better fit to exotic option prices (work in progress)
- Extension to stochastic vol, stochastic rates, stochastic div yield is easy:

$$\frac{dS_t}{S_t} = (r_t - q_t)\,dt + \sigma(t,S_t,X_t)\alpha_t\,dW_t$$

Model calibrated to the smile iff for all t, K

$$\frac{\mathbb{E}[D_{0t}\sigma^2(t,S_t,X_t)\alpha_t^2|S_t=K]}{\mathbb{E}[D_{0t}|S_t=K]} = \sigma^2_{\mathrm{Dup}}(t,K)$$

$$-\frac{\mathbb{E}\left[D_{0t}\left(r_t - q_t - (r_t^0 - q_t^0)\right)1_{S_t>K}\right]}{\frac{1}{2}K\partial_K^2\mathcal{C}(t,K)} + \frac{\mathbb{E}\left[D_{0t}\left(q_t - q_t^0\right)(S_t-K)^+\right]}{\frac{1}{2}K^2\partial_K^2\mathcal{C}(t,K)}$$

where σ_{Dup} is the Dupire local volatility computed using r_t^0 and q_t^0

Conclusion

- **Only two methods proposed in the past** to calibrate a LC to smile of a basket (stock index, cross FX rate, interest rate spread...). Both may fail to generate a true correl + they impose particular shape of correl matrix
- We suggest a **general method that produces a whole family of admissible local correlations**. It spans **all the admissible local correlations such that** $\rho \in (\rho^0, \rho^1)$. The two existing methods are just particular points.
- **No added complexity: the particle method does the job!**
- The huge number of degrees of freedom (represented by two functions a and b) allows one to pick one's favorite correl with desirable properties among this family of admissible correls. **It reconciles static calibration (calibration from snapshot of prices of options on basket) and dynamic calibration (calibration from historical state-dependency of correlation)**
- Numerical tests show the **wide variety of admissible correlations** and give **insight on lower bounds/upper bounds** on option prices given smile of basket and smiles of its constituents

Open questions

- Under which condition are the 3 surfaces of implied volatility of a triangle of FX rates jointly arbitrage-free? How to detect a arbitrage?
- Under which necessary and sufficient conditions on σ_1, σ_2 and σ_{12} does there exist an admissible local correlation?
- What are the lower bound and upper bound on the price of $g(S_T^1, S_T^2)$ (or more complex payoffs) given the 3 surfaces of implied volatility of a triangle of FX rates?

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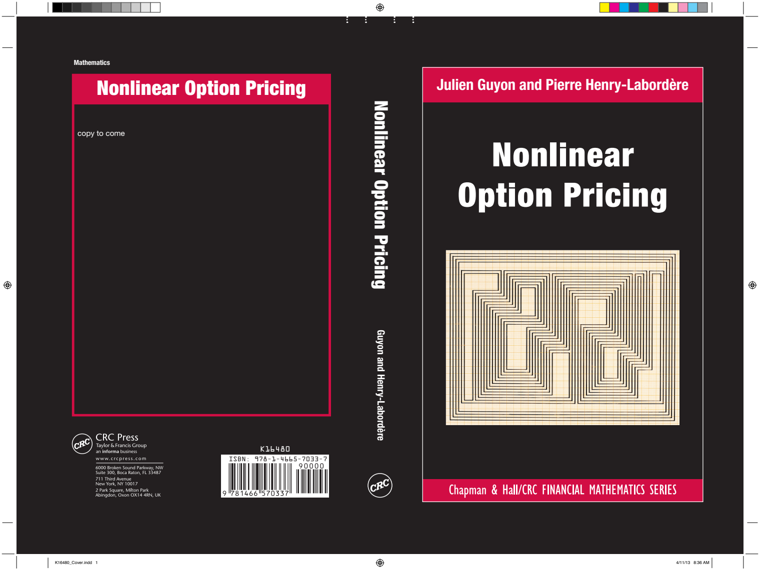
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The field of mathematics is very wide and it is not easy to predict what happens next, but I can tell you it is alive and well. Two general trends are obvious and will surely persist. In its pure aspect, the subject has changed, much for the better I think, by moving to more concrete problems. In both its pure and applied aspects, an equally beneficial shift to nonlinear problems can be seen. Most mathematical questions suggested by Nature are genuinely nonlinear, meaning very roughly that the result is not proportional to the cause, but varies with it as the square or the cube, or in some more complicated way. The study of such questions is still, after two or three hundred years, in its infancy. Only a few of the simplest examples are understood in any really satisfactory way. I believe this direction will be a principal theme in the future.

— **Henry P. McKean**, *Some Mathematical Coincidences (May 2003)*