A Multi-Factor SABR Model for Forward Inflation Rates

Fabio Mercurio
Bloomberg, New York

Nicola Moreni
Banca IMI, Milan

January 2009

Abstract

We introduce a new forward CPI model that is based on a multi-factor volatility structure and leads to SABR-like dynamics for forward inflation rates. Our approach is the first in the financial literature to reconcile zero-coupon and year-on-year quotes, granting, at the same time, a both fast and accurate calibration to market data. Explicit formulas for year-on-year caps/floors as well as for zero-coupon options are then derived in terms of the SABR volatility form. An example of calibration to market data is finally provided.

1 Introduction

The inflation rates quoted in the (interbank) derivatives market are either zero-coupon (ZC) or year-on-year (YY), and underlie ZC and YY swaps, respectively. In a ZC swap, a fixed payment, based on the annual compounding of the quoted ZC rate, is exchanged at maturity for the inflation rate corresponding to the swap application period.[1] In a YY swap, instead, payments are exchanged annually, with the floating payment that is based on the just-set annual inflation rate.

ZC and YY rates do not provide equivalent information about the forward inflation. In fact, the knowledge of all ZC rates does not imply the knowledge of YY rates, nor does the reverse hold. This is because, YY rates contain an intrinsic convexity adjustment coming from the CPI ratio that defines them. To be able to move from ZC to YY rates, or vice versa, one needs to introduce an inflation model that values such an adjustment.

Options on both ZC and YY rates are also traded in the market. Options on YY rates are quoted as inflation caps, *i.e.* strips of (yearly-spaced) options on annual rates, whereas ZC options are options on the CPI index. Inflation caps contain information on the

[1] To simplify things, we here neglect the time lags that are actually present in the contract definition. We thus assume that the CPI index in the ZC swap payoff is set at the maturity date and not two or three months earlier, as in typical market contracts.

volatility of the annual inflation rate, a fundamental economic factor and the underlying of several (inflation-indexed or hybrid) structured notes. ZC options, instead, contain information on the volatility of the CPI index, or equivalently, on the volatility of inflation rates covering periods starting today and longer than one year. ZC options are present in pension-plan payouts or in inflation bonds like TIPS that grant the payment of the initially-set nominal value at maturity, both in real and nominal terms.

When dealing with ZC and YY rates at the same time, we essentially face two choices: either we assume a market segmentation, treating the ZC and YY data (swaps and options) independently, thus using separate models for ZC and YY rates, or we look for a theoretical framework aiming to accommodate simultaneously all available market quotes.

Assuming market segmentation, inflation data can be successfully described by different market models. In the ZC case, a market model for forward consumer price indices (CPIs) coupled with stochastic volatility, see for instance Mercurio and Moreni (2005, 2006), is flexible enough to calibrate the smile, if any, of ZC options. Conversely, a market model for forward inflations, see Kenyon (2008), can well reproduce quoted prices of inflation caps. However, when jointly pricing ZC and YY basic derivatives, none of the these approaches has the desired tractability or flexibility for a satisfactory calibration of all market quotes.[2]

In this article, we propose an alternative inflation model that allows for closed-form pricing of the main inflation derivatives. The model is based on forward CPI dynamics that lead to SABR-like dynamics for forward inflation rates. The model is flexible enough to accommodate cap smiles and to retrieve ZC volatility levels as typically seen in the market.

The article is organized as follows. In the first section, we provide the main definitions and notation used in the article. In Section 3, we describe the dynamics of forward CPIs that constitute our model. In Section 4, we derive a closed-form approximation for YY caplet prices based on the SABR formula. In Section 5, we imply the dynamics of forward inflation rates and calculate the associated convexity adjustment. In Section 6, we propose an approximation procedure to price analytically ZC options, again with the SABR formula. In Section 7, we explain the difficulties one encounters when directly modeling forward inflation rates. Section 8 is devoted to numerical examples and to a case of calibration to market data. Section 9 concludes the paper.

2 Definitions and notation

Let us denote by $I(t)$ the CPI at time t.

Given the time structure $T_0 := 0, T_1, \dots, T_M$, we define, as in Kazziha (1999), the time T_i-forward CPI at time t, denoted by $\mathcal{I}_i(t)$, as the fixed amount to be exchanged at time T_i for the CPI $I(T_i)$, so that the swap has zero value at time t. By standard no-arbitrage pricing theory, we have:

$$\mathcal{I}_i(t) = E^{T_i}\big[I(T_i)|\mathcal{F}_t\big],$$

[2]Resorting to short-rate modeling, like in Jarrow and Yildirim (2003) or Kruse (2007), leads to similar inconsistencies or incompatibilities.

where E^{T_i} denotes expectation under the T_i-forward risk-adjusted measure Q^{T_i}, with numeraire the zero-coupon bond $P(t, T_i)$, and $\mathcal{F}_t$ is the sigma-algebra generated by the relevant market factors up to time t.

The value at time zero of a T_i-forward CPI can be immediately obtained from the market quote $K(T_i)$ of the ZC swap with maturity $T_i = i$ years. In fact, see . *e.g.* Brigo and Mercurio (2006),

$$\mathcal{I}_i(0) = I(0)(1 + K(T_i))^i.$$

By definition, each forward CPI is a martingale under the associated forward measure. Analogously to the market model of forward LIBOR rates, therefore, to model the evolution of forward CPIs, we just have to specify their instantaneous covariance structure. The example with lognormal dynamics has been analyzed by Kazziha (1999), Belgrade et al. (2004) and Mercurio (2005). A one-factor stochastic-volatility process as in Heston (1993) has then been introduced by Mercurio and Moreni (2005, 2006). In this article, we propose a new specification of the dynamics of forward CPIs that is based on a multi-factor stochastic volatility. Our purpose is to obtain closed-form formulas for YY options that better accommodate market smiles.

3 The model

Let us assume that the forward LIBOR rates F_i, defined by

$$F_i(t) := F_i(t; T_{i-1}, T_i) = \frac{P(t, T_{i-1}) - P(t, T_i)}{\tau_i P(t, T_i)}, \tag{1}$$

with τ_i the year fraction for the interval $(T_{i-1}, T_i]$, evolve according to a lognormal LIBOR market model

$$dF_i(t) := \sigma_i^F(t) F_i(t)\, dW_i^F(t),$$

where σ_i^F is deterministic and W_i^F is a Q^{T_i}-standard Brownian motion.

Let us consider M volatility processes $V_i(t)$ that are driftless geometric Brownian motions under their respective forward measure. Namely, we assume that under Q^{T_i},

$$dV_i(t) = \nu_i V_i(t)\, dZ_i(t) \tag{2}$$

where $V_i(0) = \alpha_i \in I\!R_0^+$ and Z_i is a Q^{T_i}-standard Brownian motion.

Let us then consider an M-dimensional Brownian motion

$$W^M := \{W_1^M, W_2^M, \dots, W_M^M\}, \tag{3}$$

with instantaneous correlation $\rho_{j,k}^W$ between W_j^M and W_k^M, and recursively define other $M - 1$ M-dimensional Brownian motions

$$W^1 := \{W_1^1, W_2^1, \dots, W_M^1\}, \ \dots \ , W^{M-1} := \{W_1^{M-1}, W_2^{M-1}, \dots, W_M^{M-1}\},$$

by the following rule:

$$dW_j^{i-1}(t) = dW_j^i(t) - \frac{\tau_i \sigma_i^F(t) F_i(t)}{1 + \tau_i F_i(t)} \rho_{i,j}^{F,W}\, dt, \tag{4}$$

where $j = 1, \dots, M$, $i = 2, \dots, M$ and

$$\rho_{i,j}^{F,W} := dW_j^i(t) dW_i^F(t)/dt = dW_j^{i-1}(t) dW_i^F(t)/dt.$$

Notice that W_j^{i-1} is indeed a $Q^{T_{i-1}}$-standard Brownian motion by the Girsanov theorem and the change-of-measure technique, since when moving from Q^{T_i} to $Q^{T_{i-1}}$, W_j^i acquires a drift term given by

$$d\langle W_j^i, \ln \frac{P(\cdot, T_{i-1})}{P(\cdot, T_i)} \rangle_t = d\langle W_j^i, \ln[1 + \tau_i F_i] \rangle_t = \frac{\tau_i \sigma_i^F(t) F_i(t)}{1 + \tau_i F_i(t)} \rho_{i,j}^{F,W}\, dt.$$

Let us finally assume that each forward CPI $\mathcal{I}_i$ evolves under the associated forward measure Q^{T_i} according to

$$d\mathcal{I}_i(t) = \mathcal{I}_i(t) \sum_{j=\beta(t)}^{i} V_j(t)\, dW_j^i(t), \tag{5}$$

where $\beta(t)$ is the index of the first tenor date T_i strictly larger than t. Since, by definition, a sum is null when its lower bound is larger that the higher one, equation (5) implicitly defines the dynamics of $\mathcal{I}_i$ after time t, too (a diffusion with zero vol). Precisely, we have that $\mathcal{I}_i(t) = \mathcal{I}_i(T_i)$ for each $t \geq T_i$.

Hereafter, with this convention at hand, we will assume that stochastic processes are defined for every time $t \in [0, T_M]$.

4 The pricing of YY caplets

A T_i-maturity caplet (floorlet) is an option on the inflation rate, paying at time T_i

$$\left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right)\right]^+,$$

where κ is the strike and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

By standard no-arbitrage pricing theory, the value at time $t \leq T_{i-1}$ of this contract is

$$\begin{aligned} \mathbf{IICplt}(t, T_{i-1}, T_i, K, \omega) &= P(t, T_i) E^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right)\right]^+ |\mathcal{F}_t \right\} \\ &= P(t, T_i) E^{T_i} \left\{ \left[\omega\big(Y_i(T_i) - \kappa\big)\right]^+ \big|\mathcal{F}_t \right\} \end{aligned}$$

where the forward inflation rate Y_i is defined by

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1, \tag{6}$$

which, remembering our convention on the definition of processes after their expiry time, reads as

$$Y_i(t) := \begin{cases} \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1, & t \le T_{i-1} \\ \frac{\mathcal{I}_i(t)}{I(T_{i-1})} - 1, & t \in [T_{i-1}, T_i) \\ \frac{I(T_i)}{I(T_{i-1})} - 1, & t \ge T_i \end{cases} \tag{7}$$

To derive the dynamics of Y_i we first need to derive the dynamics of $\mathcal{I}_{i-1}$ under Q^{T_i}, remembering that, under $Q^{T_{i-1}}$,

$$d\mathcal{I}_{i-1}(t) = \mathcal{I}_{i-1}(t) \sum_{j=\beta(t)}^{i-1} V_j(t)\, dW_j^{i-1}(t).$$

Applying the definition (4), the dynamics of $\mathcal{I}_{i-1}$ under Q^{T_i} is

$$d\mathcal{I}_{i-1}(t) = \mathcal{I}_{i-1}(t) \left[- \frac{\tau_i \sigma_i^F(t) F_i(t)}{1 + \tau_i F_i(t)} \sum_{j=\beta(t)}^{i-1} V_j(t) \rho_{i,j}^{F,W}\, dt + \sum_{j=\beta(t)}^{i-1} V_j(t)\, dW_j^i(t) \right]. \tag{8}$$

Remark 4.1. *Equation (8) can be equivalently derived by a measure-change technique. In fact, the Q^{T_i}-drift of $\mathcal{I}_{i-1}$ is*

$$\begin{aligned} d\langle \mathcal{I}_{i-1}, \ln \frac{P(\cdot, T_i)}{P(\cdot, T_{i-1})} \rangle_t &= -d\langle \mathcal{I}_{i-1}, \ln[1 + \tau_i F_i] \rangle_t \\ &= -\frac{\tau_i \sigma_i^F(t) F_i(t)}{1 + \tau_i F_i(t)} \mathcal{I}_{i-1}(t) \sum_{j=\beta(t)}^{i-1} V_j(t) \rho_{i,j}^{F,W}\, dt, \end{aligned}$$

since $dW_j^{i-1}(t) dW_i^F(t) = \rho_{i,j}^{F,W} dt$.

The Q^{T_i}-dynamics of Y_i can then be obtained by Ito's lemma, *i.e.*

$$\begin{aligned} d\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} =& \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} \Bigg[\sum_{j=\beta(t)}^{i} V_j(t)\, dW_j^i(t) \\ &+ \frac{\tau_i \sigma_i^F(t) F_i(t)}{1 + \tau_i F_i(t)} \sum_{j=\beta(t)}^{i-1} V_j(t) \rho_{i,j}^{F,W}\, dt - \sum_{j=\beta(t)}^{i-1} V_j(t)\, dW_j^i(t) \\ &+ \sum_{j=\beta(t)}^{i-1} \sum_{k=\beta(t)}^{i-1} V_j(t) V_k(t) \rho_{j,k}^{W}\, dt - \sum_{j=\beta(t)}^{i} \sum_{k=\beta(t)}^{i-1} V_j(t) V_k(t) \rho_{j,k}^{W}\, dt \Bigg]. \end{aligned}$$

After straightforward simplifications, we get:

$$
\begin{aligned}
d\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} =& \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\Bigg[\left(\frac{\tau_i\sigma_i^F(t)F_i(t)}{1+\tau_iF_i(t)}\sum_{j=\beta(t)}^{i-1}V_j(t)\rho_{i,j}^{F,W}\,dt - V_i(t)\sum_{k=\beta(t)}^{i-1}V_k(t)\rho_{i,k}^{W}\right)dt \\
&+ V_i(t)\,dW_i^i(t)\Bigg],
\end{aligned}
\tag{9}
$$

which immediately leads to

$$
dY_i(t) = [1+Y_i(t)]\left[\sum_{j=\beta(t)}^{i-1}V_j(t)\left(\frac{\tau_i\sigma_i^F(t)F_i(t)}{1+\tau_iF_i(t)}\rho_{i,j}^{F,W} - V_i(t)\rho_{i,j}^{W}\right)dt + V_i(t)\,dW_i^i(t)\right]. \tag{10}
$$

To produce tractable dynamics, we freeze in the drift of (10), $F_i(t)$ and each $V_j(t)$ to their time-0 value. Setting

$$
D_i(t) := \sum_{j=\beta(t)}^{i-1}V_j(0)\left(\frac{\tau_i\sigma_i^F(t)F_i(0)}{1+\tau_iF_i(0)}\rho_{i,j}^{F,W} - V_i(0)\rho_{i,j}^{W}\right), \tag{11}
$$

which is zero after time T_{i-1}, and

$$
\bar{Y}_i(t) := 1+Y_i(t) = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)},
$$

we obtain:

$$
\begin{aligned}
d\bar{Y}_i(t) &= \bar{Y}_i(t)\big[D_i(t)\,dt + V_i(t)\,dW_i^i(t)\big] \\
dV_i(t) &= \nu_iV_i(t)\,dZ_i(t), \quad V_i(0)=\alpha_i.
\end{aligned}
\tag{12}
$$

To finally produce SABR dynamics, we notice that

$$
\bar{Y}_i(T_i) = \tilde{Y}_i(T_i),
$$

where the process $\tilde{Y}_i$ is defined by

$$
\begin{aligned}
d\tilde{Y}_i(t) &= \tilde{Y}_i(t)V_i(t)\,dW_i^i(t), \quad \tilde{Y}_i(0) = \bar{Y}_i(0)\,e^{\int_0^{T_i}D_i(t)dt} \\
dV_i(t) &= \nu_iV_i(t)\,dZ_i(t), \quad V_i(0)=\alpha_i.
\end{aligned}
\tag{13}
$$

Therefore, setting $K := 1+\kappa$, we get:

$$
\begin{aligned}
\mathbf{IICplt}(t,T_{i-1},T_i,K,\omega) &= P(t,T_i)E^{T_i}\Big\{\big[\omega Y_i(T_i)-\omega\kappa\big]^+\big|\mathcal{F}_t\Big\} \\
&= P(t,T_i)E^{T_i}\Big\{\big[\omega\bar{Y}_i(T_i)-\omega K\big]^+\big|\mathcal{F}_t\Big\} \\
&= P(t,T_i)E^{T_i}\Big\{\big[\omega\tilde{Y}_i(T_i)-\omega K\big]^+\big|\mathcal{F}_t\Big\},
\end{aligned}
$$

so that the caplet price can be valued with a SABR lognormal formula ($\beta = 1$), see Hagan et al. (2002), assuming that the instantaneous correlation between $\tilde{Y}_i$ and V_i is given by some ρ_i:

$$\mathbf{IICplt}(t, T_{i-1}, T_i, K, \omega) = \omega P(t, T_i)\big[\tilde{Y}_i(t)\Phi(\omega d_+) - K\Phi(\omega d_-)\big]$$

where

$$d_\pm = \frac{\ln \frac{\tilde{Y}_i(t)}{K} \pm \frac{1}{2}\sigma^2(K)(T_i - t)}{\sigma(K)\sqrt{T_i - t}}$$

and

$$\sigma(K) = \alpha_i \frac{z}{x(z)} \left\{1 + \left[\frac{\rho_i \nu_i \alpha_i}{4} + \nu_i^2 \frac{2 - 3\rho_i^2}{24}\right](T_i - t)\right\}$$

with

$$z := \frac{\nu_i}{\alpha_i} \ln\left(\frac{\tilde{Y}_i(t)}{K}\right)$$

$$x(z) := \ln\left\{\frac{\sqrt{1 - 2\rho_i z + z^2} + z - \rho_i}{1 - \rho_i}\right\}.$$

5 Forward inflation rates and convexity adjustments

We have already defined a forward inflation rate in terms of the forward CPI ratio (6). A more natural definition, which refers to its homologue in the interest rate world, is the following. The time-t forward inflation rate, for the future interval $[T_{i-1}, T_i]$, is defined as the rate K that, at time t, gives zero value to the swaplet where, at time T_i, K is exchanged for $I(T_i)/I(T_{i-1}) - 1$. In formulas:

$$K = \mathcal{Y}_i(t) := E^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})} - 1\big|\mathcal{F}_t\right\} = E^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1\big|\mathcal{F}_t\right\}.$$

Under the previous assumptions and approximation, the last expectation can be calculated analytically. In fact

$$\mathcal{Y}_i(t) = E^{T_i}\left\{Y_i(T_i)|\mathcal{F}_t\right\} = E^{T_i}\left\{\bar{Y}_i(T_i) - 1|\mathcal{F}_t\right\} = \bar{Y}_i(t)e^{\int_t^{T_i} D_i(u)du} - 1. \tag{14}$$

The forward rate $\mathcal{Y}_i(t)$ is also called the time-t YY rate for the period $[T_{i-1}, T_i]$, and the exponential term in the last equality is referred to as its convexity correction.

Remark 5.1. *According to our definitions and convention, the two forward rates $Y_i(t)$ and $\mathcal{Y}_i(t)$ coincide for each $t \geq T_{i-1}$, but their values are different on times $t < T_{i-1}$. In fact, we see that $Y_i(t)$ is known as soon as the corresponding ZC swap rates (and hence forward CPIs) are known, whereas $\mathcal{Y}_i(t)$ is known as soon as the corresponding YY swap rates are known. From a mathematical point of view, the difference between the two rates is that $\mathcal{Y}_i(t)$ is a martingale under the T_i-forward measure (Q^{T_i}-conditional expectation of the random variable $Y_i(T_i)$), whereas $Y_i(t)$ is not.*

By Ito's lemma, the Q^{T_i}-dynamics of $\mathcal{Y}_i$ are given by:

$$\begin{aligned} d\mathcal{Y}_i(t) &= [1+\mathcal{Y}_i(t)]V_i(t)\,dW_i^i(t), \quad \mathcal{Y}_i(0) = \tilde{Y}_i(0) - 1 \\ dV_i(t) &= \nu_i V_i(t)\,dZ_i(t), \quad V_i(0) = \alpha_i. \end{aligned} \tag{15}$$

From equation (13), we immediately notice that $\mathcal{Y}_i(t) + 1 = \tilde{Y}_i(t)$. Therefore, the pricing formula for caplets can also be written as:

$$\mathbf{IICplt}(t, T_{i-1}, T_i, K, \omega) = \omega P(t, T_i)\big[(1+\mathcal{Y}_i(t))\Phi(\omega d_+) - (1+K)\Phi(\omega d_-)\big] \tag{16}$$

where

$$d_\pm = \frac{\ln \frac{1+\mathcal{Y}_i(t)}{1+K} \pm \frac{1}{2}\sigma^2(K)(T_i - t)}{\sigma(K)\sqrt{T_i - t}}$$
$$z := \frac{\nu_i}{\alpha_i} \ln\left(\frac{1+\mathcal{Y}_i(t)}{1+K}\right)$$

with $\sigma(K)$ and $x(z)$ defined as before.

Remark 5.2. *The sake of generality was not the only criterion that led us to assume a non-standard Brownian motion W^M in (3). In fact, both from (10) and (15) we see that*

$$\rho_{i,j}^W = Corr\big(dW_i^i(t), dW_j^i(t)\big) = Corr\big(dY_i(t), dY_j(t)\big) = Corr\big(d\mathcal{Y}_i(t), d\mathcal{Y}_j(t)\big).$$

Assuming that W^M is a standard Brownian motion, i.e. $\rho_{i,j}^W = 0$ for each $i \neq j$, implies that the instantaneous correlation between different forward inflation rates is null. This is not a desirable feature when pricing inflation derivatives. For instance, this can result in a ZC volatility that is too low compared to the values the market usually trades at. See also the example in the following section.

6 The pricing of ZC options

A T_i-maturity inflation ZC option is an option on the CPI, paying at time T_i the positive part of the difference between the fixed and floating payoffs of the corresponding ZC swap, namely

$$\left[\omega\left(\frac{I(T_i)}{I(0)} - K\right)\right]^+,$$

where K is the strike and $\omega = 1$ for a call and $\omega = -1$ for a put.

By standard no-arbitrage pricing theory, the value at time $t \leq T_{i-1}$ of this contract is

$$\begin{aligned} \mathbf{IIZCO}(t, T_i, K, \omega) &= P(t, T_i) E^{T_i}\left\{\left[\omega\left(\frac{I(T_i)}{I(0)} - K\right)\right]^+ |\mathcal{F}_t\right\} \\ &= \frac{P(t, T_i)}{I(0)} E^{T_i}\Big\{\big[\omega\big(\mathcal{I}_i(T_i) - K I(0)\big)\big]^+ \big|\mathcal{F}_t\Big\}. \end{aligned} \tag{17}$$

The forward CPI $\mathcal{I}_i$ evolves under the associated measure Q^{T_i} according to (5). Apart from the trivial case where $\beta(t) = i$, the forward CPI dynamics

$$d\mathcal{I}_i(t) = \mathcal{I}_i(t) \sum_{j=\beta(t)}^{i} V_j(t)\, dW_j^i(t) \tag{18}$$

depends on more than one volatility process. Our objective is to derive an approximated SDE

$$\begin{aligned} d\mathcal{I}_i(t) &= \mathcal{I}_i(t) V(t)\, dW(t) \\ dV(t) &= \nu V(t)\, dZ(t), \quad dW(t)\, dZ(t) = \rho\, dt \end{aligned} \tag{19}$$

that still allows us to use the SABR option pricing formula of Hagan et al. (2002). Assuming $t = 0$, by matching (instantaneous) quadratic variations, and noticing that $j \geq \beta(t)$ iff $t < T_j$, we get

$$V^2(t) = \sum_{j,k=\beta(t)}^{i} V_j(t)V_k(t)\rho_{j,k}^W = \sum_{j,k=1}^{i} V_j(t)V_k(t)1_{\{t<T_j \wedge T_k\}}\rho_{j,k}^W, \tag{20}$$

with $a \wedge b := \min(a, b)$, so that:

$$V(0) = \sqrt{\sum_{j,k=1}^{i} V_j(0)V_k(0)\rho_{j,k}^W}.$$

This formula, however, does not consider the fact that $V(0)$ has to account for the drift correction of the process $V(t)$ defined by (20), which, contrary to (19), is not a martingale. Moreover, this value of $V(0)$ does not satisfy the limit case where all the "vol-of-vols" ν_j go to zero. We thus resort to a better approximation by noticing that

$$E^{T_i}\left[\frac{1}{T_i}\int_0^{T_i} V^2(t)\, dt\right] = V^2(0)\frac{e^{\nu^2 T_i} - 1}{\nu^2 T_i},$$

and

$$\lim_{\nu \to 0} V^2(0)\frac{e^{\nu^2 T_i} - 1}{\nu^2 T_i} = V^2(0).$$

To this end, we first calculate the expected mean-square volatility

$$\begin{aligned} E^{T_i}\left[\int_0^{T_i} \sum_{j,k=1}^{i} V_j(t)V_k(t)1_{\{t<T_j\wedge T_k\}}\rho_{j,k}^W\, dt\right] &= \sum_{j,k=1}^{i} \int_0^{T_i} E^{T_i}\left[V_j(t)V_k(t)\right]1_{\{t<T_j\wedge T_k\}}\rho_{j,k}^W\, dt \\ &= \sum_{j,k=1}^{i} \rho_{j,k}^W \int_0^{T_j\wedge T_k} V_j(0)V_k(0)e^{\rho_{j,k}^V \nu_j \nu_k t}\, dt \\ &= \sum_{j,k=1}^{i} V_j(0)V_k(0)\frac{\rho_{j,k}^W}{\rho_{j,k}^V \nu_j \nu_k}\left(e^{\rho_{j,k}^V \nu_j \nu_k (T_j \wedge T_k)} - 1\right), \end{aligned}$$

where $\rho^V_{j,k}$ is the instantaneous correlation between $Z_j(t)$ and $Z_k(t)$, and then match the two expected mean-square volatilities (of $\mathcal{I}_i$ and its approximation) in the limit of all "vol-of-vols" going to zero:

$$\lim_{\nu\to 0} V^2(0)\frac{e^{\nu^2 T_i}-1}{\nu^2 T_i} = \lim_{\nu_1,\dots,\nu_i\to 0}\sum_{j,k=1}^{i} V_j(0)V_k(0)\frac{\rho^W_{j,k}}{\rho^V_{j,k}\nu_j\nu_k T_i}\big(e^{\rho^V_{j,k}\nu_j\nu_k(T_j\wedge T_k)}-1\big).$$

We thus set:

$$V(0) := \sqrt{\sum_{j,k=1}^{i} V_j(0)V_k(0)\rho^W_{j,k}\frac{T_j\wedge T_k}{T_i}}.$$

Notice that this value coincides with that obtained by matching mean-square volatilities after freezing the $V_j(t)$ (but not $\beta(t)$) to the their time-0 value.

The "vol-of-vol" ν can be found in a similar fashion by matching the mean-square volatilities, see also Rebonato and White (2007):

$$V^2(0)\frac{e^{\nu^2 T_i}-1}{\nu^2 T_i} = \sum_{j,k=1}^{i} V_j(0)V_k(0)\frac{\rho^W_{j,k}}{\rho^V_{j,k}\nu_j\nu_k T_i}\big(e^{\rho^V_{j,k}\nu_j\nu_k(T_j\wedge T_k)}-1\big).$$

Expanding both sides in the 'vol-of-vols" and matching second-order terms (the first ones have been already matched through $V(0)$), we get:

$$\nu^2 = \sum_{j,k=1}^{i}\frac{V_j(0)V_k(0)}{V^2(0)}\rho^W_{j,k}\rho^V_{j,k}\nu_j\nu_k\left(\frac{T_j\wedge T_k}{T_i}\right)^2 = \sum_{j,k=1}^{i}\bar{V}_j(0)\bar{V}_k(0)\rho^W_{j,k}\rho^V_{j,k}\nu_j\nu_k\left(\frac{T_j\wedge T_k}{T_i}\right)^2$$

where, for each t and j, we set

$$\bar{V}_j(t) := \frac{V_j(t)}{V(t)}.$$

The last parameter to determine is the (instantaneous) correlation ρ between $V(t)$ and $W(t)$, whose values are derived from (18), (19) and (20):

$$V(t) = \sqrt{\sum_{j,k=1}^{i} V_j(t)V_k(t)1_{\{t<T_j\wedge T_k\}}\rho^W_{j,k}}$$

$$dW(t) = \sum_{j=\beta(t)}^{i}\frac{V_j(t)}{V(t)}\,dW^i_j(t) = \sum_{j=1}^{i}\bar{V}_j(t)1_{\{t<T_j\}}\,dW^i_j(t).$$

We first calculate the differential of $V(t)$:

$$
\begin{aligned}
dV(t) &= \frac{1}{2V(t)} \sum_{j,k=1}^{i} d\big[V_j(t)V_k(t)1_{\{t<T_j\wedge T_k\}}\big]\rho^W_{j,k} \\
&= \frac{1}{2V(t)} \sum_{j,k=1}^{i} d\big[V_j(t)V_k(t)\big]1_{\{t<T_j\wedge T_k\}}\rho^W_{j,k} + \cdots dt \\
&= \frac{1}{2V(t)} \sum_{j,k=1}^{i} V_j(t)V_k(t)\big[\nu_j\, dZ_j(t) + \nu_k\, dZ_k(t)\big]1_{\{t<T_j\wedge T_k\}}\rho^W_{j,k} + \cdots dt \\
&= V(t) \sum_{j,k=1}^{i} \bar{V}_j(t)\bar{V}_k(t)1_{\{t<T_j\wedge T_k\}}\rho^W_{j,k}\nu_j\, dZ_j(t) + \cdots dt,
\end{aligned}
$$

implying that

$$
dV(t)\, dV(t) = V^2(t) \sum_{j,k,l,h=1}^{i} \bar{V}_j(t)\bar{V}_k(t)\bar{V}_l(t)\bar{V}_h(t)1_{\{t<T_j\wedge T_k\wedge T_l\wedge T_h\}}\rho^W_{j,k}\rho^W_{l,h}\nu_j\nu_l\rho^V_{j,l}\, dt. \tag{21}
$$

Then, the instantaneous covariation:

$$
\begin{aligned}
dV(t)\, dW(t) &= V(t) \sum_{j,k=1}^{i} \bar{V}_j(t)\bar{V}_k(t)1_{\{t<T_j\wedge T_k\}}\rho^W_{j,k}\nu_j\, dZ_j(t) \sum_{h=1}^{i} \bar{V}_h(t)1_{\{t<T_h\}}\, dW^i_h(t) \\
&= V(t) \sum_{j,k,h=1}^{i} \bar{V}_j(t)\bar{V}_k(t)\bar{V}_h(t)1_{\{t<T_j\wedge T_k\wedge T_h\}}\rho^W_{j,k}\nu_j\rho^{V,W}_{j,h}\, dt,
\end{aligned}
$$

where $\rho^{V,W}_{j,h}$ is the instantaneous correlation between $Z_j(t)$ and $W^i_h(t)$.

Taking values at $t = 0$, we finally set:

$$
\rho := \frac{\sum_{j,k,h=1}^{i} \bar{V}_j(0)\bar{V}_k(0)\bar{V}_h(0)\rho^W_{j,k}\nu_j\rho^{V,W}_{j,h}}{\sqrt{\sum_{j,k,l,h=1}^{i} \bar{V}_j(0)\bar{V}_k(0)\bar{V}_l(0)\bar{V}_h(0)\rho^W_{j,k}\rho^W_{l,h}\nu_j\nu_l\rho^V_{j,l}}}.
$$

The ZC option price is then given by the SABR formula with parameters $V(0)$, ν, ρ and time to maturity $T_i - t$.

Remark 6.1. *A more involved, but maybe better, approximated value for ν can be derived, from (21), by freezing the values of the $\bar{V}$ to time zero and taking the mean-square "vol-of-vol":*

$$
\nu := \sqrt{\sum_{j,k,l,h=1}^{i} \bar{V}_j(0)\bar{V}_k(0)\bar{V}_l(0)\bar{V}_h(0)\frac{T_j\wedge T_k\wedge T_l\wedge T_h}{T_i}\rho^W_{j,k}\rho^W_{l,h}\nu_j\nu_l\rho^V_{j,l}}.
$$

Another possibility is to use Piterbarg's (2006) Markovian projection method.

In case we are willing to use Hagan et al. (2002) formula for time-dependent parameters, the "vol-of-vol" ν can become a function $\nu(t)$. Again, from (21), we would obtain:

$$\nu(t) := \sqrt{\sum_{j,k,l,h=1}^{i} \bar{V}_j(0)\bar{V}_k(0)\bar{V}_l(0)\bar{V}_h(0)1_{\{t<T_j\wedge T_k\wedge T_l\wedge T_h\}}\rho^W_{j,k}\rho^W_{l,h}\nu_j\nu_l\rho^V_{j,l}}.$$

7 A market model for YY rates

In perfect analogy with the market model of forward LIBOR rates, one could model the evolution of each YY rate $\mathcal{Y}_i(t)$ under the associated measure Q^{T_i},[3] directly starting from dynamics (15) instead of deriving them from different assumptions and arguments. To this end, one would assume that

$$\begin{aligned} d\mathcal{Y}_i(t) &= [1+\mathcal{Y}_i(t)]V_i(t)\,dW_i^i(t), \\ dV_i(t) &= \nu_i V_i(t)\,dZ_i(t), \quad V_i(0) = \alpha_i, \end{aligned} \tag{22}$$

where the initial YY rates $\mathcal{Y}_i(0)$ would be model inputs. The resulting caplet prices would then coincide with (16).

The problem with this formulation is that, currently, the YY swap market is not as liquid as the ZC one. If only ZC swap rates are quoted, one can calculate the initial values $Y_i(0)$ (by stripping the forward CPI for the quoted maturities), but not the $\mathcal{Y}_i(0)$, which require the knowledge of YY swap rates.

The direct modeling of YY rates does not seem to be the right approach to follow in absence of a reliable YY swap market. The advantage of an exact formula for caplet prices does not sufficiently compensate for the loss of an automatic calibration to the inflation linear instruments. In fact, to accommodate the market ZC swap quotes, by calibration to the initial forward CPIs, one has to derive a formula for the generic $\mathcal{I}_i(0)$ as a function of the model parameters $\mathcal{Y}_j(0)$, α_j, ν_j and ρ_j, $j = 1,\dots,M$. To this end, we notice that, under (22), the forward CPIs can be defined only at their expiry time through

$$\mathcal{I}_i(T_i) = I(0)\prod_{j=1}^{i}[1+\mathcal{Y}_j(T_j)]$$

so that $\mathcal{I}_i(0)$ can be calculated by taking expectation under the T_i-forward measure:

$$\begin{aligned} \mathcal{I}_i(0) = E^{T_i}\{\mathcal{I}_i(T_i)\} &= I(0)E^{T_i}\left\{\prod_{j=1}^{i}[1+\mathcal{Y}_j(T_j)]\right\} \\ &=: I(0)f\big(\mathcal{Y}_1(0),\dots,\mathcal{Y}_i(0),\alpha_1,\dots,\alpha_i,\nu_1,\dots,\nu_i,\rho_1,\dots,\rho_i\big). \end{aligned}$$

This function f can be calculated either by Monte Carlo simulation or in closed form by some non-standard approximation. This is the major complication one has to deal with

[3]Clearly, the correlation between different processes needs to be modeled, too.

when modeling YY rates. When a joint calibration to the ZC ad YY markets is needed, it is therefore advisable to model the inflation rates Y_i, through the modeling of forward CPIs, and obtain some approximate dynamics for the rates $\mathcal{Y}_i$, than doing the reverse.

8 Numerical examples and calibration to market data

In order to provide examples of calibration of our model to real market data, let us consider as underlying the Euro-zone harmonized index of consumer prices excluding tobacco (HICP-XT).

We report in Tables 1 and 2 data as of September 4th 2008, which refers, due to the three month market quotation lag, to the inflation of the month of June. In particular, we report in Table 1 the ZC swap rates, the corresponding forward CPI's $\mathcal{I}_i(0)$, the forward CPI ratios $Y_i(0)$, and the YY swap rates computed under the assumption of no-drift adjustment ($D_i \approx 0$ in Eq.(11)). In the last two columns, we report YY swap rates quoted by a broker and the corresponding YY forward inflation rates $\mathcal{Y}_i(0)$, respectively. At first glance, we see that the rough no-drift approximation may indeed be consistent with quoted YY swap rates, whose bid-ask spread is typically of 1 basis point for maturities up to 10 years, and of 2-3 bps for greater expiries.

Maturity (Y)	Mkt ZC swap	$\mathcal{I}_i(0)$	$Y_i(0)$	implied YY swap ($D \approx 0$)	Mkt YY swap ($\pm$1bp)	Mkt implied $\mathcal{Y}_i(0)$,
1	1.865%	110.6	1.865%	1.865%	1.865%	1.865%
2	2.190%	113.3	2.516%	2.197%	2.190%	2.528%
3	2.280%	116.1	2.460%	2.284%	2.275%	2.458%
4	2.335%	119.0	2.500%	2.336%	2.330%	2.510%
5	2.370%	122.0	2.510%	2.368%	2.364%	2.516%
6	2.400%	125.1	2.550%	2.396%	2.393%	2.557%
7	2.433%	128.4	2.628%	2.425%	2.425%	2.653%
8	2.460%	131.8	2.653%	2.450%	2.453%	2.684%
9	2.485%	135.4	2.685%	2.472%	2.478%	2.725%
10	2.513%	139.1	2.760%	2.495%	2.505%	2.812%
11	2.528%	142.8	2.679%	2.508%	2.519%	2.645%
12	2.535%	146.6	2.616%	2.515%	2.525%	2.674%
13	2.544%	150.5	2.649%	2.523%	2.533%	2.701%
14	2.554%	154.5	2.692%	2.532%	2.544%	2.729%
15	2.565%	158.7	2.714%	2.540%	2.555%	2.761%

Table 1: HICP-XT market data as of September 4th, 2008 (June inflation). June 2008 fixing was 108.54 .

In Table 2 we show a set of bid/ask prices of caps and floors for given maturities and strikes.[4] We notice that these quotes, displaying wide bid/ask spreads,[5] witness the poor liquidity of such options.

[4]For the sake of simplicity, we only report here a subset of a wider set of data used for calibration, in which a finer strike grid was actually considered.

[5]Actually we have half bid/ask spreads ranging from 6% to 55% of the corresponding mid prices.

opt type	F	F	F	F	C	C	C	C
T / K	-1%	0%	1%	2%	2.5%	3%	4%	5%
3	3/6	7/14	26/40	98/118	109/128	67/84	26/37	11/18
5	5/17	14/34	51/73	165/189	218/238	143/161	62/79	29/45
7	14/22	39/49	73/104	220/256	321/348	212/241	97/119	49/67
10	23/38	45/78	102/152	293/351	467/510	315/362	152/189	82/114
15	40/70	72/130	150/233	405/499	676/746	465/543	237/299	136/191
20	56/100	96/181	191/306	499/629	834/930	578/684	302/389	179/253
30	82/154	136/268	256/432	650/841	1047/1186	730/885	390/517	238/348

Table 2: HICP-XT cap (C) and floor (F) bid/ask quotes for different maturities T (in years) and strikes K. Prices are expressed in basis points.

In order to calibrate our model and show its flexibility, we first bootstrapped from the cap/floor strips,[6] sequences of caplets and floorlets for maturities from 1Y to 15Y and then made trials with exogenously fixed correlation scenarios. The optimization was done on the set of SABR parameters $\{\alpha_i, \rho_i, \nu_i\}$, $i = 1, \ldots, 15$, by minimizing the squared relative differences between market mids and model prices of the -1%, 0%, 1%, 1.5%, 2%-strike floorlets and of the 2.5%, 3%, 3.5%, 4%, 4.5%, 5%-strike caplets. As for the time grid, we assume, for simplicity, that the June CPI is published on the first business day in July, thus setting T_0=04-Sept-08, T_1=01-July-09,…, T_{15}=01-July-23.

Our first calibration test, consistent with the zero-drift approximation, is done by setting $\rho^{F,W}_{i,j} = 0$ for all i, j and $\rho^{W}_{i,j} = 0$ for all $i \neq j$. Within this simplified framework, all inflation rates are independent from each other and from the interest rates evolution. We can see in Figure 1, in which we report market and model implied volatilities,[7] that the fit is accurate. Let us stress that market data (prices as well as implied volatilities) display small non-smooth behaviors either where the cap and floor branches meet or on single strikes. We consider these discrepancies as being essentially bound to liquidity reasons.[8] Our model also provides a useful smoothing tool of such volatilities.

The main drawback of this simplified framework is that the unrealistic assumption of independent inflation rates implies an underestimation of ZC options variances, as we may check even in the case where the V_i are not stochastic.[9] In order to test our model under

[6]We took market YY forward rates $\mathcal{Y}_i(0)$ (Table 1) and inverted the 3Y, 5Y, 7Y, 10Y, 15Y cap and floor prices finding the corresponding flat implied volatilities as it is commonly done in the interest rate market. These flat volatilities were then linearly interpolated to obtain cap and floor prices for missing intermediate maturities.

[7]Implied volatilities are here obtained by inversion of shifted-lognormal Black-Scholes formulae like that of Eq. (16) with $\mathcal{Y}_i(0)$ equal to $Y_i(0)$ and $\nu_i = 0$.

[8]Let us also remind the wide bid/ask spreads of Table 2.

[9]Actually, we have $I(T_i) = I(T_0)\prod_{j=1}^{i}(1 + Y_j(T_j))$, so that

$$\mathrm{Var}[\ln I(T_i)] = \sum_{j=1}^{i} \mathrm{Var}[\ln(1 + Y_j(T_j))] + \sum_{\substack{j,k=1\\k\neq j}}^{i} \mathrm{Cov}[\ln(1 + Y_j(T_j)), \ln(1 + Y_k(T_k))],$$

where the cross terms are likely to give non negative contributions.

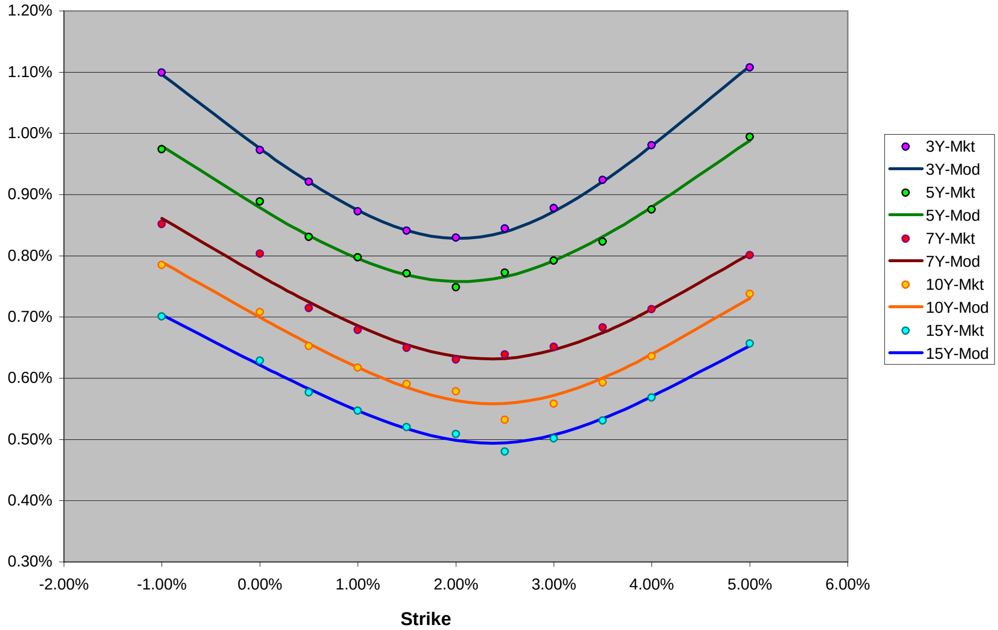

Figure 1: Calibration results: market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10, 15 years, uncorrelated case.

more realistic correlation assumptions, we set

$$\rho_{i,j}^{W} = \mathrm{e}^{-\lambda|i-j|} \qquad \text{and} \qquad \rho_{i,j}^{F,W} = \frac{\sum_{k=1}^{15} c_{i,k}\rho_{k,j}^{W}}{\sqrt{1+\sum_{k,k'=1}^{15} c_{i,k}c_{i,k'}\rho_{k,k'}^{W}}}\,, \tag{23}$$

with

$$c_{i,j} = c\,\mathrm{e}^{-\lambda_c|i-j-1|}.$$

Within this framework, the $\rho_{i,j}^{W}$ are simply derived from a two-parameters form with zero asymptotic correlation, while the interdependencies between inflation and interest rates are expressed as a function of the $\rho_{i,j}^{W}$ and of the set of (idiosyncratic) coupling coefficients $\{c_{i,j}\}$.[10] The way we obtained the expression for the $\{\rho_{i,j}^{F,W}\}$ in Eq.(23) is detailed in appendix A. By chosing reasonable parameter values as $\lambda = 3/2$, $\lambda_c = 1/10$, $c = 0.1$, we obtain, for $i > 4$, positive drift corrections D_i, leading to forward inflation rates and YY swap rates closer to broker quotations, which we report in Table 3 .

Calibration results in terms of market and model implied volatilities are reported in Figure 2.[11] We notice that the volatilities are well recovered, hence confirming that our model is flexible enough to fit prices under different correlation assumptions.

[10]Maximum coupling is between F_{i+1} and Y_i. We thought of the monetary policy over $[T_i, T_{i+1}]$, summarized by F_{i+1}, as a response to inflation behavior over $[T_{i-1}, T_i]$.

[11]Inversion of market and model prices is made with model post-calibration forwards $\mathcal{Y}_i(0)$ of Table 3.

i	YY swap	$\mathcal{Y}_i(0)$
1	1.865%	1.865%
2	2.197%	2.515%
3	2.283%	2.459%
4	2.335%	2.500%
5	2.368%	2.511%
6	2.396%	2.554%
7	2.426%	2.634%
8	2.451%	2.662%
9	2.474%	2.697%
10	2.499%	2.775%
11	2.513%	2.696%
12	2.521%	2.636%
13	2.530%	2.672%
14	2.540%	2.718%
15	2.549%	2.743%

Table 3: Model (post-calibration) YY swap rates and forward inflation rates $\mathcal{Y}_i(0)$ (see Eq.(14)), correlated case.

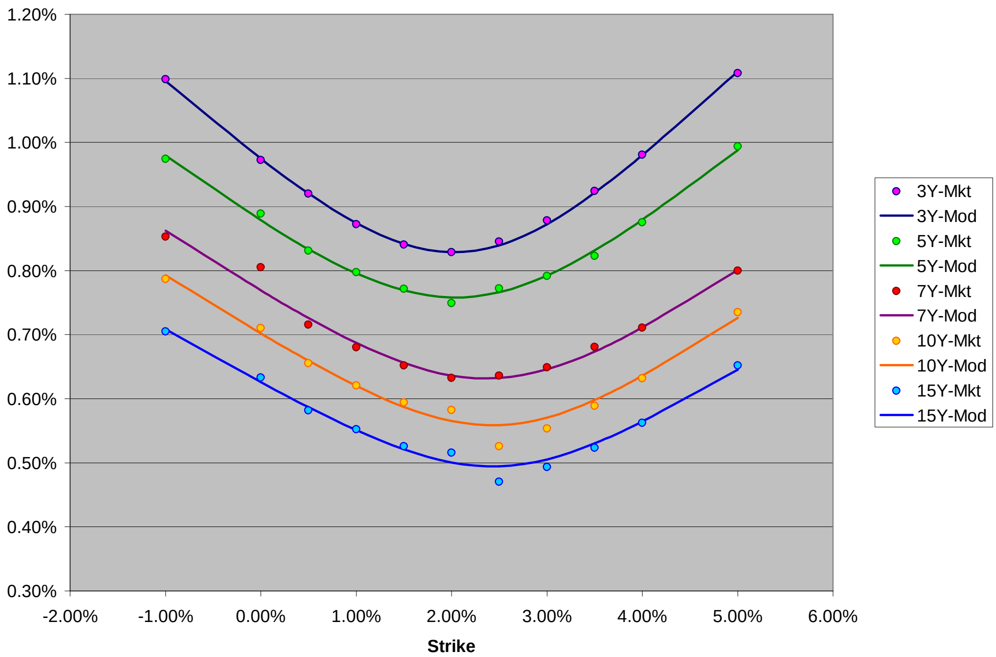

Figure 2: Calibration results: market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10,15 years, correlated case, model implied forwards.

9 Conclusions

We have derived a new inflation model with the main advantage of analytical tractability combined with the accuracy of calibration to both ZC and YY market data. Compared to other approaches in the financial literature, our model has also the advantage of an exogenous correlation structure between forward inflation rates. This allows us to price

ZC options closer to market quotes.

We have then considered an example of calibration to real market data with two different correlation assumptions. The purpose of our example is to show the flexibility of our model and its accuracy in the fitting.

The forward CPI model we propose represents a major attempt to reconcile ZC and YY (linear and option) data. We believe that modeling forward indices is preferable to modeling forward rates since only the former approach allows for the exact calibration to ZC swap rates, which are the most liquid quotes currently available in the inflation (interbank) market.

We finally stress that the assumption of a stochastic volatility of SABR type is not essential in the derivation of explicit caplet/floorlet prices. For instance, a square-root volatility process, as in Heston (1993), could also be considered. Besides the simplicity of the corresponding option pricing formula, the main reason for assuming SABR dynamics lies in the possibility to derive an analytical approximation for a ZC option price, which is a less straightforward task under other volatility dynamics.

A Parsimonious correlation structure

As usual, we can tackle the task of setting exogenous correlation patterns by analysing historical variance-covariance matrices. This will lock in all the degrees of freedom bound to the correlations themselves. Conversely, our choice is of letting some parameters free to be able to tune in levels of integrated (co)variances.

In order to tie together interest and inflation rates shocks, we choose idiosyncratic Brownian motions $\{\hat{W}_1^F, \ldots, \hat{W}_M^F\}$, $\{\hat{W}_1, \ldots, \hat{W}_M\}$ endowed with the independent correlation structures:

$$\hat{\rho}_{i,j}^F := \rho_\infty^F + (1-\rho_\infty^F)\mathrm{e}^{-\lambda^F|i-j|}\,, \qquad \hat{\rho}_{i,j} := \rho_\infty + (1-\rho_\infty)\mathrm{e}^{-\lambda|i-j|}\,, \qquad \hat{\rho}_{i,j}^{F,W} = 0.$$

As we are not interested in pricing interest rates derivatives, we give hierarchical priority to inflation rates[12], by setting $W_i \equiv \hat{W}_i$ and $\rho_{i,j}^W \equiv \hat{\rho}_{i,j}$. On the other side interest rates are driven, in Eq.(1), by Brownian motions $\{W_i^F\}$ that we want to be function of both the $\{W_i\}$'s and the $\{\hat{W}_i^F\}$'s. Hence we introduce a set of coupling coefficients $c_{i,k}$ and define

$$\begin{aligned} dW_k^F(t) &:= \mathcal{N}_k \left[\sum_{l=1}^{M} c_{k,l} dW_l(t) + d\hat{W}_k^F(t)\right] \\ \mathcal{N}_k &:= \left(1 + \sum_{l,l'=1}^{M} c_{k,l} c_{k,l'} \rho_{l,l'}^W\right)^{-1/2}, \end{aligned} \tag{24}$$

where $\mathcal{N}_k$ is a normalizing factor ensuring $dW_k^F(t) dW_k^F(t) = dt$.

[12] Conversely, one could choose to modify inflation rates patterns and set forward rates to be driven only by their idiosyncratic component.

It follows that the $2M$-dimensional Brownian motion $\{W_1^F, \ldots, W_M^F, W_1, \ldots, W_M\}$ has the following correlation structure:

$$
\begin{aligned}
&dW_k^F(t)dW_{k'}^F(t) = \mathcal{N}_k\mathcal{N}_{k'}\left(\sum_{l,l'=1}^{M} c_{k,l}c_{k'l'}\rho_{l,l'}^W + \hat{\rho}_{k,k'}^F\right) := \rho_{k,k'}^F dt \\
&dW_k^F(t)dW_l(t) = \mathcal{N}_k\left(\sum_{l'=1}^{M} c_{k,l'}\rho_{l,l'}^W\right) dt := \rho_{k,l}^{F,W} dt \\
&dW_l(t)dW_{l'}(t) = \rho_{l,l'}^W dt.
\end{aligned}
\tag{25}
$$

This approach consists in a perturbation of the idiosyncratic LIBOR-LIBOR correlation structure $\hat{\rho}_{i,j}^F$ to take into account the extra correlation contribution coming from the fact that interest rates are also correlated to inflation rates.

Let us moreover introduce a $2M$-dimensional standard Brownian motion $\tilde{W} := \{\tilde{W}_1^F, \ldots, \tilde{W}_M^F, \tilde{W}_1, \ldots, \tilde{W}_M\}$. It is easy to interprete our approach as the change of coordinates

$$
\begin{pmatrix} dW_1^F \\ \cdots \\ dW_M^F \\ dW_1 \\ \cdots \\ dW_M \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{N}\cdot C^F & \mathcal{N}\cdot c \cdot C^Y \\ \hline 0 & C^Y \end{array}\right) \cdot \begin{pmatrix} d\tilde{W}_1^F \\ \cdots \\ d\tilde{W}_M^F \\ d\tilde{W}_1 \\ \cdots \\ d\tilde{W}_M \end{pmatrix} := \mathcal{C}\cdot \begin{pmatrix} d\tilde{W}_1^F \\ \cdots \\ d\tilde{W}_M^F \\ d\tilde{W}_1 \\ \cdots \\ d\tilde{W}_M \end{pmatrix}
\tag{26}
$$

where C^F and C^Y are pseudo-roots[13] of the idiosyncratic correlation matrices $\{\hat{\rho}_{k,k'}^F\}$ and $\{\rho_{l,l'}^W\}$, respectively and $\mathcal{N}$ is a diagonal matrix with $\mathcal{N}_{kk} := \mathcal{N}_k$.

As a consequence, we may see the variance covariance matrix $\mathcal{V}$ of the set $\{W_1^F, \ldots, W_M^F, W_1, \ldots W_M\}$ as the square of the $2M \times 2M$ matrix $\mathcal{C}$ of Eq.(26), that is $\mathcal{V} = \mathcal{C}\mathcal{C}^T$. Hence, $\mathcal{V}$ is semidefinite-positive by construction and, because of the normalization[14] of the rows of $\mathcal{C}$, has unit entries on the diagonal and all off-diagonal terms have absolute value less or equal to one. In other words, $\mathcal{V}$ is a valid correlation matrix and we have achieved our goal of perturbing the idiosyncratic LIBOR-inflation rates correlation.

References

[1] Belgrade, N., Benhamou, E., and Koehler E. (2004). A Market Model for Inflation , *SSRN Working Paper*, available online at `http://ssrn.com/abstract=576081`

[2] Brigo, D., and F. Mercurio. *Interest-Rate Models: Theory and Practice. With Smile, Inflation and Credit.* Springer Finance, 2006.

[13] Whose rows are unitary-norm vectors.

[14] Actually, we have $\sum_{i=1}^{2M}\mathcal{C}_{ji}^2 = 1$ for any $j = 1, \ldots, 2M$, that is $dW_k^F dW_k^F = dt$, $dW_l dWl = dt$.

[3] Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002) Managing Smile Risk. *Wilmott magazine*, September, 84-108.

[4] Heston, S.L. (1993) A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies* 6, 327-343.

[5] Jarrow, R., and Yildirim Y. (2003). Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM Model. *Journal of Financial and Quantitative Analysis* 38(2), 409-430.

[6] Kazziha, S. (1999). Interest Rate Models, Inflation-based Derivatives, Trigger Notes And Cross-Currency Swaptions. PhD Thesis, Imperial College of Science, Technology and Medicine. London.

[7] Kenyon, C. (2008) Inflation is normal. *Risk*, 21(7).

[8] Kruse, S. (2007). Pricing of Inflation-Indexed Options Under the Assumption of a Lognormal Inflation Index as Well as Under Stochastic Volatility. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=948399

[9] Mercurio, F. (2005). Pricing Inflation-Indexed Derivatives, *Quantitative Finance*, 5(3), 289-302.

[10] Mercurio, F., and Moreni, N. (2005). Pricing Inflation-Indexed Options with Stochastic Volatility. Internal report. Banca IMI, Milan. Available online at: www.fabiomercurio.it/stochinf.pdf

[11] Mercurio, F. and Moreni, N. (2006) Inflation with a smile. *Risk* March, Vol. 19(3), 70-75.

[12] Rebonato, R. and White, R. (2007) Linking Caplets and Swaptions Prices in the LMM-SABR Model. Available online at: http://www.riccardorebonato.co.uk/papers/LMMSABRSwaption.pdf

[13] Piterbarg, V. (2006) Markovian Projection Method for Volatility Calibration. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=906473