Default Risk and Hazard Process

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1 Introduction

The so-called intensity-based approach to the modelling and valuation of defaultable securities has attracted a considerable attention of both practitioners and academics in recent years; to mention a few papers in this vein: Duffie [8], Duffie and Lando [9], Duffie et al. [10], Jarrow and Turnbull [13], Jarrow et al. [14], Jarrow and Yu [15], Lando [21], Madan and Unal [23]. In the context of financial modelling, there was also a renewed interest in the detailed analysis of the properties of random times; we refer to the recent papers by Elliott et al. [12] and Kusuoka [20] in this regard. In fact, the systematic study of stopping times and the associated enlargements of filtrations, motivated by a purely mathematical interest, was initiated in the 1970s by the French school, including: Brémaud and Yor [4], Dellacherie [5], Dellacherie and Meyer [7], Jeulin [16], and Jeulin and Yor [17]. On the other hand, the classic concept of the intensity (or the hazard rate) of a random time was also studied in some detail in the context of the theory of Cox processes, as well as in relation to the theory of martingales. The interested reader may consult, in particular, the monograph by Last and Brandt [22] for the former approach, and by Brémaud [3] for the latter. It seems to us that no single comprehensive source focused on the issues related to default risk modelling is available to financial researchers, though. Furthermore, it is worth noting that some challenging mathematical problems associated with the modelling of default risk remain still open. The aim of this text is thus to fill the gap by furnishing a relatively concise and self-contained exposition of the most relevant – from the viewpoint of financial modelling – results related to the analysis of random times and their filtrations. We also present some recent developments and we indicate the directions for a further research. Due to the limited space, the proofs of some results were omitted; a full version of the working paper [19] is available from the authors upon request.

2 Hazard Process Γ of a Random Time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, such that $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$ for any $t \geq 0$. We introduce a right-continuous process D by setting $D_t = 1\!\!1_{\{\tau \leq t\}}$, and we denote by $\mathbf{D}$ the filtration generated by D; that is, $\mathcal{D}_t = \sigma(D_u : u \leq t)$.
Setup 1. Suppose that $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ is a given (but arbitrary) filtration[1] on $(\Omega, \mathcal{G}, \mathbf{P})$. Let us consider the joint filtration $\mathbf{G} := \mathbf{D} \vee \mathbf{F}$; that is, we set $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$ for every $t \in I\!\!R_+$. Our first goal is to find a representation of the conditional expectation $\mathbf{E}\,(Y \,|\, \mathcal{G}_t)$ in terms of the random time τ and the 'hazard process' of τ with respect to $\mathbf{F}$. Subsequently, we analyse the martingale representation of the process $\mathbf{E}\,(Y \,|\, \mathcal{G}_t)$, $t \in I\!\!R_+$, and we examine the behaviour of the hazard process under an equivalent change of probability measure. The properties of the reference filtration $\mathbf{F}$ appear to play a crucial role in this study.
Setup 2. Suppose that we are given an arbitrary filtration $\mathbf{G}$ such that $\mathbf{D} \subset \mathbf{G}$. It is important to observe that the equality $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$ does not specify uniquely the 'complementary' filtration $\mathbf{F}$. For instance, when $\mathcal{G}_t = \mathcal{D}_t$, we may take the trivial filtration (as, e.g., in [9,12,18]), but also $\mathbf{F} = \mathbf{D}$ (or indeed any other sub-filtration of $\mathbf{D}$). At the intuitive level, given the random time τ and the 'enlarged' filtration $\mathbf{G}$, one might be interested in searching for a filtration which represents an additional flow of informations. Formally, one looks for a 'minimal' filtration $\widetilde{\mathbf{F}}$ such that the equality $\mathbf{G} = \mathbf{D} \vee \widetilde{\mathbf{F}}$ is valid. As soon as the minimal complementary filtration $\widetilde{\mathbf{F}}$ is determined, we are back in Setup 1.

Remark 2.1. In both cases, the σ-field $\mathcal{G}_t$ is assumed to represent all observations available at time t. Since obviously $\mathcal{D}_t \subset \mathcal{G}_t$ for any t, τ is a stopping time with respect to $\mathbf{G}$; τ is not necessarily a stopping time with respect to $\mathbf{F}$, however. In financial interpretation, the filtration $\mathbf{F}$ is assumed to model the flow of observations available to investors prior to the *default time* τ. In case of several default times (see Section 5), the filtration $\mathbf{F}$ may also include some events related to some of them. We do not pretend that the default time τ is not observed, but we are interested in the valuation of (defaultable) contingent claims only strictly prior to the default time.

For any $t \in I\!\!R_+$, we write $F_t = \mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t)$, so that $1 - F_t = \mathbf{P}(\tau > t \,|\, \mathcal{F}_t)$. It is easily seen that F is a bounded, non-negative, $\mathbf{F}$-submartingale. We may thus deal with the right-continuous modification of F. The following definition is standard.

Definition 2.2. The $\mathbf{F}$-*hazard process* of τ, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$. Equivalently, $\Gamma_t = -\ln(1 - F_t)$ for every $t \in I\!\!R_+$.

[1] In most applications, $\mathbf{F}$ is the natural filtration of a certain stochastic process. Filtrations $\mathbf{D}$ and $\mathbf{F}$ are not independent, in general.

Unless otherwise explicitly stated, it is assumed throughout that the inequality $F_t < 1$ holds for every t, and thus the $\mathbf{F}$-hazard process of τ given by Definition 2.2 exists. The special case when τ is an $\mathbf{F}$-stopping time (in other words, the case when $\mathbf{F} = \mathbf{G}$) is not analysed in detail in the present work.

2.1 Conditional Expectation with respect to G

We make throughout the technical assumption that all filtrations are $(\mathbf{P}, \mathcal{G})$-completed. In addition, we assume also that the enlarged filtration $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$ satisfies the 'usual conditions.' The first result examines the structure of the enlarged filtration $\mathbf{G}$.

Lemma 2.3. *We have $\mathcal{G}_t \subset \mathcal{G}_t^*$, where*

$$\mathcal{G}_t^* := \big\{ A \in \mathcal{G} \,|\, \exists\, B \in \mathcal{F}_t \;\; A \cap \{\tau > t\} = B \cap \{\tau > t\} \big\}.$$

PROOF: Observe that $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t = \sigma(\mathcal{D}_t, \mathcal{F}_t) = \sigma(\{\tau \le u\},\, u \le t,\, \mathcal{F}_t)$. Also, it is easily seen that the class $\mathcal{G}_t^*$ is a sub-σ-field of $\mathcal{G}$. Therefore, it is enough to check that if either $A = \{\tau \le u\}$ for some $u \le t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. Indeed, in the former case we may take $B = \emptyset$, in the latter $B = A$. $\triangle$

The next result provides the key formula (see, e.g., Dellacherie [5]) which relates the conditional expectation with respect to $\mathcal{G}_t$ to the conditional expectation with respect to $\mathcal{F}_t$.

Lemma 2.4. *For any $\mathcal{G}$-measurable[2] random variable Y we have, for any $t \in I\!R_+$,*

$$\mathbf{E}\,(1\!\!1_{\{\tau>t\}} Y | \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \frac{\mathbf{E}\,(1\!\!1_{\{\tau>t\}} Y | \mathcal{F}_t)}{\mathbf{P}(\tau > t | \mathcal{F}_t)} = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t} \mathbf{E}\,(1\!\!1_{\{\tau>t\}} Y | \mathcal{F}_t). \quad (2.1)$$

PROOF: Let us fix $t \in I\!R_+$. In view of Lemma 2.3, any $\mathcal{G}_t$-measurable random variable coincides on the set $\{\tau > t\}$ with some $\mathcal{F}_t$-measurable random variable. Therefore,

$$\mathbf{E}\,(1\!\!1_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbf{E}\,(Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} X,$$

where X is an $\mathcal{F}_t$-measurable random variable. Taking conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$\mathbf{E}\,(1\!\!1_{\{\tau>t\}} Y \,|\, \mathcal{F}_t) = \mathbf{P}(\tau > t \,|\, \mathcal{F}_t)\, X. \qquad \triangle$$

Proposition 2.5. *Let Z be a (bounded) $\mathbf{F}$-predictable process. Then for any $t < s \le \infty$*

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\le s\}} Z_\tau | \,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t}\, \mathbf{E}\,\Big(\int_{]t,s]} Z_u \, dF_u \,\Big|\, \mathcal{F}_t \Big). \quad (2.2)$$

[2] Of course, it is also assumed throughout that Y is $\mathbf{P}$-integrable.

Proof: We start by assuming that Z is a stepwise $\mathbf{F}$-predictable process, so that (we are interested only in values of Z for $u \in]t,s]$)

$$Z_u = \sum_{i=0}^{n} Z_{t_i} \mathbb{1}_{]t_i,t_{i+1}]}(u),$$

where $t_0 = t < \cdots < t_{n+1} = s$ and the random variable Z_{t_i} is $\mathcal{F}_{t_i}$-measurable. In view of (2.1), for any i we have

$$\begin{aligned}\mathbf{E}\big(\mathbb{1}_{\{t_i<\tau\le t_{i+1}\}} Z_\tau \,|\, \mathcal{G}_t\big) &= \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbf{E}\big(\mathbb{1}_{\{t_i<\tau\le t_{i+1}\}} Z_{t_i} \,|\, \mathcal{F}_t\big)\\ &= \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbf{E}\big(Z_{t_i}(F_{t_{i+1}} - F_{t_i}) \,|\, \mathcal{F}_t\big).\end{aligned}$$

In the second step we approximate an arbitrary (bounded) $\mathbf{F}$-predictable process by a sequence of stepwise $\mathbf{F}$-predictable processes. △

Let us remark that Proposition 2.5 remains valid if $\mathbf{F} = \mathbf{G}$; that is, when τ is an $\mathbf{F}$-stopping time. However, in this case, it does not provide a non-trivial formula. Indeed, the left-hand member of (2.2) is then $\mathbf{E}(\mathbb{1}_{\{t<\tau\le s\}} Z_\tau \,|\, \mathcal{F}_t)$. On the other hand, since $F_t = \mathbb{1}_{\{\tau\le t\}}$, the random variable e^{Γ_t} is equal to 1 on the set $\{\tau > t\}$, and thus the right-hand side of (2.2) is also equal to $\mathbf{E}(\mathbb{1}_{\{t<\tau\le s\}} Z_\tau \,|\, \mathcal{F}_t)$. Notice also that any $\mathbf{G}$-predictable process coincides with a (unique) $\mathbf{F}$-predictable process up to time τ; this shows that there is no point in dealing in Proposition 2.5 with $\mathbf{G}$-predictable processes.

Corollary 2.6. *Let Y be a $\mathcal{G}$-measurable random variable. Then, for $t \le s$,*

$$\mathbf{E}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbf{E}(\mathbb{1}_{\{\tau>s\}} e^{\Gamma_t} Y \,|\, \mathcal{F}_t). \tag{2.3}$$

Furthermore, for any $\mathcal{F}_s$-measurable random variable Y we have

$$\mathbf{E}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbf{E}\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{2.4}$$

If F (and thus also Γ) is a continuous increasing process then for any $\mathbf{F}$-predictable (bounded) process Z we have

$$\mathbf{E}(\mathbb{1}_{\{t<\tau\le s\}} Z_\tau \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbf{E}\Big(\int_t^s Z_u e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big). \tag{2.5}$$

Proof: In view of (2.1), to show that (2.3) holds it is enough to observe that $\mathbb{1}_{\{\tau>s\}} = \mathbb{1}_{\{\tau>t\}} \mathbb{1}_{\{\tau>s\}}$. Equality (2.4) is a straightforward consequence of (2.3). Formula (2.5) follows immediately from (2.2), since, when F is increasing, $dF_u = e^{-\Gamma_u}\, d\Gamma_u$. △

2.2 Valuation of Defaultable Claims

Let us fix $t \le T$, and let δ be a constant. In what follows, we implicitly assume that $\mathbf{P}$ is the martingale measure for a financial market model. For any $\mathcal{G}$-measurable random variable Y we have

$$\mathbf{E}(\mathbb{1}_{\{t<\tau\le T\}} \delta + \mathbb{1}_{\{\tau>T\}} Y \,|\, \mathcal{G}_t) = \delta\, \mathbf{P}(t < \tau \le T \,|\, \mathcal{G}_t) + \mathbf{E}(\mathbb{1}_{\{\tau>T\}} Y \,|\, \mathcal{G}_t).$$

In particular if Y is an $\mathcal{F}_T$-measurable random variable, using (2.4) we may rewrite the formula as follows, on the set $\{\tau > T\}$

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\leq T\}}\delta + 1\!\!1_{\{\tau>T\}}Y \,|\, \mathcal{G}_t) = \delta\,\mathbf{E}\,\big(1 - e^{\Gamma_t-\Gamma_T} \,|\, \mathcal{F}_t\big) + \mathbf{E}\,(e^{\Gamma_t-\Gamma_T}\,Y \,|\, \mathcal{F}_t).$$

In financial interpretation, the random variable Y represents the *promised payoff* of a defaultable claim, and δ is the *recovery payoff* – that is, the amount received if default occurs prior to the claim's maturity date T. If the constant δ is replaced by a random payoff Z_τ, where Z is an $\mathbf{F}$-predictable process, referred to as the *recovery process,* and F is an increasing continuous process, then (2.4)-(2.5) yield

$$\mathbf{E}\,(1\!\!1_{\{t<\tau\leq T\}}Z_\tau + 1\!\!1_{\{\tau>T\}}Y|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbf{E}\,\Big(\int_t^T Z_u e^{-\Gamma_u}\,d\Gamma_u + e^{-\Gamma_T}Y \,\Big|\, \mathcal{F}_t\Big).$$

In the formulae above, we have made an implicit assumption that the interest rate equals zero. In general, we introduce the savings account process B; that is, a strictly positive, $\mathbf{F}$-adapted process of finite variation. For any $t \leq T$, we set $B(t,T) := B_t\,\mathbf{E}\,(B_T^{-1} \,|\, \mathcal{F}_t)$, and we refer to $B(t,T)$ as the price of a unit default-free zero-coupon bond of maturity T. First,

$$B_t\,\mathbf{E}\,(1\!\!1_{\{\tau>T\}}B_T^{-1}Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\hat{B}_t\,\mathbf{E}\,(\hat{B}_T^{-1}Y \,|\, \mathcal{F}_t),$$

where we write $\hat{B}_t := B_t e^{\Gamma_t}$. Next, if the process F is increasing and continuous, then

$$B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\leq T\}}B_\tau^{-1}Z_\tau | \,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\hat{B}_t\,\mathbf{E}\,\Big(\int_t^T \hat{B}_u^{-1}Z_u\,d\Gamma_u \,\Big|\, \mathcal{F}_t\Big).$$

The last two formulae lead to the following result.

Corollary 2.7. *Assume that F is an increasing continuous process. Consider a defaultable contingent claim with default time τ, the promised payoff Y and the recovery process Z. The pre-default value V of this claim, defined as*

$$V_t := B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\leq T\}}B_\tau^{-1}Z_\tau + 1\!\!1_{\{\tau>T\}}B_T^{-1}Y \,|\, \mathcal{G}_t),$$

satisfies

$$V_t = 1\!\!1_{\{\tau>t\}}\,\hat{B}_t\,\mathbf{E}\,\Big(\int_t^T \hat{B}_u^{-1}Z_u\,d\Gamma_u + \hat{B}_T^{-1}Y \,\Big|\, \mathcal{F}_t\Big). \tag{2.6}$$

Example 2.8. Assume that F is an increasing continuous process. Consider a *defaultable zero-coupon bond* with the nominal value N, maturity T, and the recovery process $Z_t = h(t)$ for some function $h : [0,T] \to I\!\!R$. The *pre-default bond price* $D(t,T)$ is defined through the formula

$$D(t,T) := B_t\,\mathbf{E}\,(1\!\!1_{\{t<\tau\leq T\}}B_\tau^{-1}h(\tau) + 1\!\!1_{\{\tau>T\}}B_T^{-1}N \,|\, \mathcal{G}_t).$$

Setting $Z_t = h(t)$ and $Y = N$ in (2.6), we obtain the following representation for $D(t,T)$

$$D(t,T) = 1\!\!1_{\{\tau>t\}}\hat{B}_t \Big(\int_t^T \hat{B}_u^{-1} h(u)\, d\Gamma_u + \hat{B}_T^{-1} N \,\Big|\, \mathcal{F}_t \Big).$$

Notice that manifestly $D(t,T) = 0$ after default time τ, so that $D(t,T)$ does not give the right value of a defaultable bond after default in case of non-zero recovery (i.e., when h is nonvanishing). The correct expression for the price of a defaultable bond reads

$$1\!\!1_{\{\tau>t\}} D(t,T) + 1\!\!1_{\{\tau\le t\}} h(\tau) B^{-1}(\tau,T) B(t,T), \quad \forall\, t \in [0,T],$$

if the recovery payoff $h(\tau)$ received at default time τ is invested in default-free bonds of maturity T. Let us introduce the 'credit-risk-adjusted' bond price $\hat{B}(t,T)$ by setting $\hat{B}(t,T) := \hat{B}_t\, \mathbf{E}\,(\hat{B}_T^{-1} \,|\, \mathcal{F}_t)$. Then

$$D(t,T) = 1\!\!1_{\{\tau>t\}}\hat{B}_t\, \mathbf{E} \Big(\int_t^T \hat{B}_u^{-1} h(u)\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) + 1\!\!1_{\{\tau>t\}} N \hat{B}(t,T). \tag{2.7}$$

Assume, in addition, that $dB_t = r_t B_t\, dt$ for some process r representing the short-term interest rate, and that the process F is absolutely continuous, so that

$$1 - F_t = 1 - \int_0^t f_u\, du = e^{-\Gamma_t} = \exp\Big(- \int_0^t \gamma_u\, du \Big),$$

where the *intensity of default* γ satisfies $\gamma_u = f_u(1-F_u)^{-1}$. Then $\hat{B}_t = \exp\big(\int_0^t \hat{r}_u\, du \big)$, where $\hat{r}_u := r_u + \gamma_u$ is the 'credit-risk-adjusted' short-term interest rate. Using (2.7), we conclude that in the case of $h = 0$ (i.e., under zero recovery) to value a defaultable zero-coupon bond prior to default, it is enough to discount its nominal value N using the 'credit-risk-adjusted' rate $\hat{r}$. In the special case when the filtration $\mathbf{F}$ is trivial, and the probability law of τ under $\mathbf{P}$ admits the density function f, formula (2.7) becomes (we write $R_t = 1/B_t$ and $G(t) = 1 - F(t)$)

$$R_t G(t) D(t,T) = 1\!\!1_{\{\tau>t\}} \Big(\int_t^T R_u h(u) f(u)\, du + R_T G(T) N \Big).$$

Remark 2.9. Using the technique described above, it is possible to solve the problem studied extensively in Duffie and Lando [9] (see Elliott et al. [12] for further comments).

2.3 Semimartingale Representation of the Stopped Process

In the next auxiliary result we assume that m is an arbitrary $\mathbf{F}$-martingale, and we examine the semimartingale decomposition with respect to the enlarged filtration $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$ of the stopped process $\widetilde{m}_t = m_{t\wedge\tau}$ (we sometimes use the standard notation m^τ for the process m stopped at τ).

Lemma 2.10. *Assume that the process m is a continuous* $\mathbf{F}$*-martingale.*
(i) *If F is a continuous increasing process then the stopped process $\widetilde{m}_t = m_{t\wedge\tau}$ is a* $\mathbf{G}$*-martingale.*
(ii) *If F is a continuous submartingale then the process*

$$\hat{m}_t = \widetilde{m}_t + \int_0^{t\wedge\tau} e^{\Gamma_u}\, d\langle m, F\rangle_u = m_{t\wedge\tau} + \int_0^{t\wedge\tau} (1-F_u)^{-1}\, d\langle m, F\rangle_u \qquad (2.8)$$

is a $\mathbf{G}$*-martingale.*

Proof: We establish only the second statement, which implies the first one. For $s \geq t$,

$$\mathbf{E}\left(m_{s\wedge\tau} - m_{t\wedge\tau} \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\big(\mathbf{E}\left(m_{s\wedge\tau} \,|\, \mathcal{G}_t\right) - m_t\big).$$

The process m is $\mathbf{F}$-predictable, hence, on the set $\{\tau > t\}$, from (2.2)

$$\begin{aligned}\mathbf{E}\left(m_{s\wedge\tau} \,|\, \mathcal{G}_t\right) &= e^{\Gamma_t}\mathbf{E}\left(\int_t^\infty m_{s\wedge u}\, dF_u \,\Big|\, \mathcal{F}_t\right)\\ &= e^{\Gamma_t}\mathbf{E}\left(\int_t^s m_u\, dF_u + m_s(1-F_s) \,\Big|\, \mathcal{F}_t\right).\end{aligned}$$

Now, the integration by parts formula leads to

$$\int_t^s m_u\, dF_u = F_s m_s - F_t m_t - \int_t^s F_u\, dm_u - \langle m, F\rangle_s + \langle m, F\rangle_t.$$

From the $\mathbf{F}$-martingale property of m, it follows that

$$\mathbf{E}\left(m_{s\wedge\tau} - m_{t\wedge\tau} \,|\, \mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t}\left(\langle m, F\rangle_t - \mathbf{E}\left(\langle m, F\rangle_s \,|\, \mathcal{F}_t\right)\right).$$

Using (2.2) again, and introducing

$$\tilde{M}_t = \int_0^{t\wedge\tau} (1-F_u)^{-1}\, d\langle m, F\rangle_u = \int_0^{t\wedge\tau} e^{\Gamma_u} d\langle m, F\rangle_u,$$

we obtain $\mathbf{E}\,(\tilde{M}_s - \tilde{M}_t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} J_t^s$, where

$$J_t^s = \mathbf{E}\left(\int_t^s dF_u \int_t^u e^{\Gamma_v} d\langle m, F\rangle_v + (1-F_s)\int_t^s e^{\Gamma_v} d\langle m, F\rangle_v \,\Big|\, \mathcal{F}_t\right).$$

Using Fubini's theorem,

$$J_t^s = \mathbf{E}\left(\int_t^s e^{\Gamma_v}(F_s - F_v)\, d\langle m, F\rangle_v + (1-F_s)\int_t^s e^{\Gamma_v}\, d\langle m, F\rangle_v \,\Big|\, \mathcal{F}_t\right).$$

The result follows from $J_t^s = \mathbf{E}\left(\int_t^s d\langle m, F\rangle_v \,\Big|\, \mathcal{F}_t\right)$. △

Remark 2.11. All these results have been established in a general setting. Under some additional assumptions imposed on τ (namely, when τ is an *honest time* – i.e. the end of a predictable set). the decomposition of the semimartingale m in the filtration $\mathbf{G}$ can be given (for details, see [17,16,26]). We can not avoid the pleasure to quote from Dellacherie et Meyer (see Page 137 in [6]) "If X is an adapted continuous process, the random variable $\tau = \inf\{s \in [0,T] : X_s = \sup_{0 \le u \le T} X_u\}$ is honest. For example, if X represents the dynamics of an asset price, τ would be the best time to sell. All speculators try to obtain some knowledge on τ, but they cannot succeed; that's why this variable is named honest." Notice that formula (2.8) is similar to Girsanov's transformation; things are not so easy, though. In a typical case, when F has a nonzero martingale part, this process is equal to 1 at time $\tau-$, and the "drift" term goes to infinity.

2.4 Martingales Associated with the Hazard Process Γ

Lemma 2.12. *The process*

$$L_t := \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} = (1 - D_t) e^{\Gamma_t} = \frac{1 - D_t}{1 - F_t}$$

is a $\mathbf{G}$*-martingale. Moreover, for any* $\mathbf{F}$*-martingale* m*, the product* Lm *is a* $\mathbf{G}$*-martingale.*

PROOF: We establish the second statement which implies the first one. In view of (2.4), for $t \le s$

$$\mathbf{E}\,(\mathbb{1}_{\{\tau > s\}} e^{\Gamma_s} m_s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbf{E}\,(e^{-\Gamma_s} e^{\Gamma_s} m_s \,|\, \mathcal{F}_t) = L_t m_t.$$

△

In the next result we assume in addition that Γ is an increasing continuous process.

Proposition 2.13. *Assume that the* $\mathbf{F}$*-hazard process* Γ *of* τ *is an increasing continuous process. Then:*
(i) *the process* $M_t = D_t - \Gamma_{t \wedge \tau}$ *is a* $\mathbf{G}$*-martingale, more specifically,*

$$M_t = -\int_{]0,t]} e^{-\Gamma_u}\, dL_u. \tag{2.9}$$

Furthermore, L *satisfies*

$$L_t = 1 - \int_{]0,t]} L_{u-}\, dM_u. \tag{2.10}$$

(ii) *If an* $\mathbf{F}$*-martingale* m *is also a* $\mathbf{G}$*-martingale then the product* Mm *is a* $\mathbf{G}$*-martingale.*

PROOF: The martingale property of M and equalities (2.9)-(2.10) can be shown using Itô's lemma combined with Lemma 2.12. To establish (ii), notice that from Lemma 2.12 we know that Lm is a $\mathbf{G}$-martingale, so that $[L,m]$ is a $\mathbf{G}$-martingale. Therefore, using the equality $dM_t = -e^{-\Gamma_t} dL_t$, we get that

$$d(mM)_t = m_{t-} dM_t + M_{t-} dm_t + d[M,m]_t = m_{t-} dM_t + M_{t-} dm_t - e^{-\Gamma_t} d[L,m]_t$$

and (ii) follows. △

Remark 2.14. In general, Γ has a nonzero martingale part and we have

$$L_t = (1-D_t)e^{\Gamma_t} = 1 + \int_0^t e^{\Gamma_u} \big[(1-D_u)\,(d\Gamma_u + (1/2)\langle \Gamma \rangle_u) - dD_u\big]$$

so that M is no longer a $\mathbf{G}$-martingale (but the process $D_t - \Gamma_{t\wedge\tau} - (1/2)\langle \Gamma \rangle_{t\wedge\tau}$ is clearly a $\mathbf{G}$-martingale).

2.5 Intensity of a Random Time

Let us consider the classic case of an absolutely continuous, increasing, $\mathbf{F}$-hazard process Γ. We assume that $\Gamma_t = \int_0^t \gamma_u\, du$ for some $\mathbf{F}$-progressively measurable process γ, referred to as the $\mathbf{F}$-*intensity* of a random time τ. By virtue of Proposition 2.13, the process M, which is given by the formula

$$M_t = D_t - \int_0^{t\wedge\tau} \gamma_u\, du = D_t - \int_0^t 1\!\!1_{\{\tau>u\}} \gamma_u\, du,$$

follows a $\mathbf{G}$-martingale. The property above is frequently used in the financial literature as the definition of the $\mathbf{F}$-intensity of a random time. The intuitive meaning of the $\mathbf{F}$-intensity γ as the "intensity of survival given the information flow $\mathbf{F}$" becomes apparent from the following corollary.

Corollary 2.15. *If the* $\mathbf{F}$*-hazard process* Γ *of* τ *is absolutely continuous then for any* $t \le s$

$$\mathbf{P}(t < \tau \le s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbf{E}\big(1 - e^{-\int_t^s \gamma_u du} \,\big|\, \mathcal{F}_t\big).$$

Remark 2.16. The $\mathbf{F}$-hazard function Γ is not well defined when τ is a $\mathbf{F}$-stopping time (that is, when $\mathbf{D} \subset \mathbf{F}$ so that $\mathbf{G} = \mathbf{F}$), and thus Corollary 2.15 cannot be directly applied in this case. It appears, however, that for a certain class of a $\mathbf{G}$-stopping times we can find an increasing $\mathbf{G}$-predictable process Λ such that for any $t \le s$

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbf{E}\big(e^{\Lambda_t - \Lambda_s} \,\big|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau>t\}} \mathbf{E}\big(e^{-\int_t^s \lambda_u du} \,\big|\, \mathcal{G}_t\big),$$

where the second equality holds provided that the process Λ is absolutely continuous. It seems natural to conjecture that the *martingale hazard* process Λ, which is formally defined in Section 4 below, coincides with the $\hat{\mathbf{F}}$-hazard process of τ for some filtration $\hat{\mathbf{F}}$, such that τ is not an $\hat{\mathbf{F}}$-stopping time. Let us again emphasise that the existence and the value of the intensity depend strongly on the choice of $\mathbf{F}$. If τ is an $\mathbf{F}$-predictable stopping time (so that, in particular, $\mathbf{G} = \mathbf{F}$), it does not admit an intensity with respect to $\mathbf{F}$. The intensity of τ with respect to the trivial filtration exists, however, provided that $F(t) = \mathbf{P}(\tau \le t) < 1$ for every t and F admits a density function.

2.6 Tradable Contingent Claims in a Defaultable Market

Blanchet-Scalliet and Jeanblanc [2] give the following suggestion to study a defaultable market. Assume that every $\mathcal{F}_T$-square-integrable contingent claim X is hedgeable, i.e., there exists a real number x and a square-integrable $\mathbf{F}$-predictable process θ such that[3]

$$R_T X = x + \int_0^T \theta_u \, d\widetilde{S}_u,$$

where $\widetilde{S} = RS$; we shall refer to this property as the completeness of the $\mathcal{F}_T$-market. The t-time price of X is equal to V_t, where

$$R_t V_t = \mathbf{E}_{\mathbf{Q}}(R_T X \,|\, \mathcal{F}_t) = x + \int_0^t \theta_u \, d\widetilde{S}_u$$

and $\mathbf{Q}$ the unique equivalent martingale measure (e.m.m.) for $\widetilde{S}$ with respect to the filtration $\mathbf{F}$.

Assume that the $\mathcal{F}_T$-measurable claims X are available in the defaultable market – that is, it is possible to get a payoff equal to X, no matter whether the default has occurred prior to T or not. Then these claims are obviously also hedgeable in the $\mathcal{G}_T$-market (with the same hedging portfolio θ), and thus the discounted price of X must be equal to $\mathbf{E}_{\mathbf{R}}(XR_T \,|\, \mathcal{G}_t)$, where $\mathbf{R}$ is any e.m.m. with respect to $\mathbf{G}$ (we assume that the $\mathbf{G}$-market is arbitrage-free; more precisely, that there exists at least one e.m.m.) The uniqueness of the price of a hedgeable claim yields

$$\mathbf{E}_{\mathbf{Q}}(XR_T \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{R}}(XR_T \,|\, \mathcal{G}_t) = x + \int_0^t \theta_u \, d\widetilde{S}_u.$$

This means, in particular, that $\mathbf{E}_{\mathbf{Q}}(Z) = \mathbf{E}_{\mathbf{R}}(Z)$ for any $Z \in \mathcal{F}_T$ (take $t = 0$ and $X = ZR_T^{-1}$), and thus the restriction of any e.m.m. $\mathbf{R}$ to the σ-field $\mathcal{F}_T$ coincides with $\mathbf{Q}$. Moreover, since any square-integrable $\mathbf{F}$-martingale can be written as $\mathbf{E}_{\mathbf{R}}(X \,|\, \mathcal{F}_t)$ under any e.m.m. $\mathbf{R}$, we conclude that any square-integrable $\mathbf{F}$-martingale is also a $\mathbf{G}$-martingale.

[3] Recall that the process R equals $1/B$, where B is assumed to represent the savings account.

2.7 Hypothesis (H)

Let us introduce the following hypothesis.

(H) Every **F** square-integrable martingale is a **G** square-integrable martingale.

Hypothesis (H) (which implies, in particular, that any **F**-Brownian motion remains a Brownian motion in the enlarged filtration) was studied by Brémaud and Yor [4], Mazziotto and Szpirglas [24], and in the financial context by Kusuoka [20]. Hypothesis (H) can also be expressed directly in terms of filtrations **F** and **G** ; namely, the following condition (H*) is equivalent to (H):

(H*) For any t, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.

Lemma 2.17. *If* $\mathbf{G} = \mathbf{F} \vee \mathbf{D}$ *then* (H) *is equivalent to any of the two equivalent conditions* (H′)

$$\forall\, s \le t, \quad \mathbf{P}(\tau \le s \,|\, \mathcal{F}_\infty) = \mathbf{P}(\tau \le s \,|\, \mathcal{F}_t),$$

$$\forall\, t \in I\!\!R_+, \quad \mathbf{P}(\tau \le t \,|\, \mathcal{F}_\infty) = \mathbf{P}(\tau \le t \,|\, \mathcal{F}_t). \tag{2.11}$$

Proof: The proof of this result can be found in [7] (see also [19]). △

Remark 2.18. (i) Equality (2.11) appears in several papers devoted to default risk, usually with no explicit reference to condition (H). For instance, the main theorem in Madan and Unal [23] follows from the fact that (2.11) holds (see the proof of B9 in the appendix of this paper). This is also the case for Wong's [25] setup.
(ii) If τ is $\mathcal{F}_\infty$-measurable and (2.11) holds, then τ is an **F**-stopping time. If τ is an **F**-stopping time, equality (2.11) holds.
(iii) Though condition (H) is not always satisfied, it holds when τ is constructed through a standard approach (see Section 4.4 below). This hypothesis is quite natural under the historical probability, and it is stable under some changes of the underlying probability measure. However, Kusuoka [20] provides a simple example in which (H) is satisfied under the historical probability, but fails to hold after an equivalent change of a probability measure. This counter-example is linked to the *default correlations* across various firms (see Section 5.4 below).

3 Martingale Representation Theorems

It is well known that the concept of replication of contingent claims is closely related to martingale representation theorems. In the case of defaultable markets this problem of hedging of contingent claims becomes more delicate than in the classic case of default-free markets.

3.1 Martingale Representation Theorem: General Case

We shall first consider a general setup; that is, we do not assume that the filtration $\mathbf{F}$ supports only continuous martingales. We shall postulate that the process F is increasing and continuous, however. We reproduce here without proof the result due to Blanchet-Scalliet and Jeanblanc [2], a useful tool for hedging purposes.

Proposition 3.1. *Assume that the* $\mathbf{F}$*-hazard process* Γ *of* τ *is an increasing continuous process. Let* Z *be an* $\mathbf{F}$*-predictable process such that the random variable* Z_τ *is integrable. Then the* $\mathbf{G}$*-martingale* $M_t^Z := \mathbf{E}\,(Z_\tau \,|\, \mathcal{G}_t)$ *admits the following decomposition*

$$M_t^Z = m_0 + \int_{]0,t\wedge\tau]} L_{u-} dm_u + \int_{]0,t\wedge\tau]} (Z_u - M_{u-}^Z)\, dM_u,$$

where m *is an* $\mathbf{F}$*-martingale*

$$m_t := \mathbf{E}\left(\int_0^\infty Z_u e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t \right) = \mathbf{E}\left(\int_0^\infty Z_u \, dF_u \,\Big|\, \mathcal{F}_t \right).$$

As usual $M_t = D_t - \Gamma_{t\wedge\tau}$. Note also that $m_0 = M_0^Z$.

3.2 Martingale Representation Theorem: Case of a Brownian Filtration

We shall now consider the case of the Brownian filtration; that is, we assume that $\mathbf{F} = \mathbf{F}^W$ for some Brownian motion W. We postulate that W remains a martingale (and thus a Brownian motion) with respect to the enlarged filtration $\mathbf{G}$ (see Section 2.7). The next result is borrowed from Kusuoka [20].

Proposition 3.2. *Assume that the hazard process* Γ *is an increasing continuous process. Then for any* $\mathbf{G}$*-martingale* N *we have*

$$N_t = N_0 + \int_0^t \xi_u dW_u + \int_{]0,t]} \zeta_u dM_u = N_0 + \widetilde{N}_t + \hat{N}_t,$$

where ξ *and* ζ *are* $\mathbf{G}$*-predictable stochastic processes, and* M *is given by* (2.9). *Moreover, the continuous* $\mathbf{G}$*-martingale* $\widetilde{N}$ *and the purely discontinuous* $\mathbf{G}$*-martingale* $\hat{N}$ *are mutually orthogonal.*

The proposition above shows that in order to get a complete market with defaultable securities, a default-free asset and a defaultable asset (for instance, a defaultable zero-coupon bond) should be taken as hedging instruments to mimic the price process of a defaultable claim (i.e., to generate an arbitrary discontinuous $\mathbf{G}$-martingale).

4 Martingale Hazard Process Λ of a Random Time

In this section, the case of $\mathbf{D} \subset \mathbf{F}$ (i.e., the case when $\mathbf{F} = \mathbf{G}$) is not excluded. Put another way, the case when τ is an $\mathbf{F}$-stopping time is also covered by the foregoing results.

Definition 4.1. An $\mathbf{F}$-predictable right-continuous increasing process Λ is called an $(\mathbf{F},\mathbf{G})$-*martingale hazard process* of a random time τ if and only if the process $\widetilde{M}_t := D_t - \Lambda_{t\wedge\tau}$ is a $\mathbf{G}$-martingale. In addition, $\Lambda_0 = 0$.

For the sake of brevity, we shall frequently refer to the $(\mathbf{F},\mathbf{G})$-martingale hazard process as the $\mathbf{F}$-*martingale hazard process,* if the meaning of $\mathbf{G}$ is clear from the context.

Remark 4.2. We assume here that the reference filtration $\mathbf{F}$ is given a priori. It seems natural, however, to search for the 'minimal' filtration such that $\hat{\mathbf{F}} \subset \mathbf{F}$ and the $(\hat{\mathbf{F}},\mathbf{G})$-martingale hazard process is actually $\hat{\mathbf{F}}$-adapted (we do not insist that the equality $\mathbf{G} = \mathbf{D} \vee \hat{\mathbf{F}}$ needs to hold). Though this problem is rather difficult to solve in general, in particular cases the filtration $\hat{\mathbf{F}}$ emerges in the calculation of the $\mathbf{F}$-martingale hazard process (see, e.g., Example 4.1 in [19]).

4.1 Evaluation of Λ: Special Case

Our first goal will be to examine a special case when the $\mathbf{F}$-martingale hazard process Λ can be expressed by means of Γ. We find it convenient to introduce the following condition (notice that (H) implies (G)):
(G) F is an increasing process.

Proposition 4.3. *Assume that* (G) *holds. If the process Λ given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1 - F_{u-}} = \int_{]0,t]} \frac{d\mathbf{P}(\tau \le u \,|\, \mathcal{F}_u)}{1 - \mathbf{P}(\tau < u \,|\, \mathcal{F}_u)} \tag{4.12}$$

is $\mathbf{F}$-predictable, then Λ is the $\mathbf{F}$-martingale hazard process of a random time τ.

PROOF: It suffices to check that $D_t - \Lambda_{t\wedge\tau}$ is a $\mathbf{G}$-martingale, where $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$. Using (2.1), we obtain for $t < s$

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbf{P}(t < \tau \le s \,|\, \mathcal{F}_t)}{\mathbf{P}(\tau > t \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} \frac{\mathbf{E}\,(F_s \,|\, \mathcal{F}_t) - F_t}{1 - F_t}.$$

On the other hand,

$$J_t^s := \mathbf{E}\left(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\right) = \mathbf{E}\left(\mathbb{1}_{\{\tau > s\}}(\Lambda_s - \Lambda_t) \,|\, \mathcal{G}_t\right) + \mathbf{E}\left(\mathbb{1}_{\{t < \tau \le s\}}\, \widetilde{\Lambda}_\tau \,|\, \mathcal{G}_t\right),$$

where, for a fixed t, we write $\widetilde{\Lambda}_u = (\Lambda_u - \Lambda_t)1\!\!1_{]t,\infty[}(u)$ (so that $\widetilde{\Lambda}$ is an **F**-predictable process). Therefore, an application of formula (2.2) gives

$$\mathbf{E}\left(1\!\!1_{\{t<\tau\le s\}}\,\widetilde{\Lambda}_\tau \,|\, \mathcal{G}_t\right) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbf{E}\left(\int_{]t,s]}(\Lambda_u-\Lambda_t)\,dF_u \,\Big|\, \mathcal{F}_t\right).$$

Furthermore, (2.3) yields

$$\mathbf{E}\left(1\!\!1_{\{\tau>s\}}(\Lambda_s-\Lambda_t)\,|\,\mathcal{G}_t\right) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\,\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,(\Lambda_s-\Lambda_t)\,|\,\mathcal{F}_t).$$

Combining the formulae above, we get, on the set $\{\tau > t\}$

$$\begin{aligned} J_t^s &= e^{\Gamma_t}\,\mathbf{E}\left(1\!\!1_{\{\tau>s\}}\,(\Lambda_s-\Lambda_t) + \int_{]t,s]}(\Lambda_u-\Lambda_t)\,dF_u \,\Big|\, \mathcal{F}_t\right) \\ &= e^{\Gamma_t}\,\mathbf{E}\left((1-F_s)(\Lambda_s-\Lambda_t) + \int_{]t,s]}(\Lambda_u-\Lambda_t)\,dF_u \,\Big|\, \mathcal{F}_t\right), \end{aligned}$$

where the last equality follows from the conditioning with respect to the σ-field $\mathcal{F}_s$. To conclude, it is enough to apply Itô's integration by parts formula (see [19] for details). △

4.2 Evaluation of Λ: General Case

Assume now that either (G) is not satisfied (so that F is not an increasing process), or (G) holds, but the increasing process F is not **F**-predictable.[4] The **F**-martingale hazard process Λ can nevertheless be found through a suitable modification of formula (4.12).

We write $\widetilde{F}$ to denote the **F**-compensator of the bounded **F**-submartingale F. This means that $\widetilde{F}$ is the unique **F**-predictable, increasing process, with $\widetilde{F}_0 = 0$, and such that the process $U = F - \widetilde{F}$ is an **F**-martingale (the existence and uniqueness of $\widetilde{F}$ is a consequence of the Doob-Meyer decomposition theorem). In the next result it is not assumed that (G) holds.

Proposition 4.4. (i) *The* **F***-martingale hazard process of a random time* τ *is given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{d\widetilde{F}_u}{1-F_{u-}}. \tag{4.13}$$

(ii) *If* $\widetilde{F}_t = \widetilde{F}_{t\wedge\tau}$ *for every* $t \in I\!\!R_+$ *(that is, the process* $\widetilde{F}$ *is stopped at* τ*) then* $\Lambda = \widetilde{F}$.

[4] For instance, τ can be an **F**-stopping time, which is not **F**-predictable. If τ is an **F**-stopping time, we have simply $F = D$, and the process D is not **F**-predictable, unless the stopping time τ is **F**-predictable.

Proof: It is clear that the process Λ given by (4.13) is predictable. Therefore, one needs only to verify that the process $\widetilde{M}_t = D_t - \Lambda_{t\wedge\tau}$ is a $\mathbf{G}$-martingale. To this end, it is enough to proceed along the same lines as in the proof of Proposition 4.3 (see, e.g., [19]).

We shall now prove part (ii). We assume that $\widetilde{F}_{t\wedge\tau} = \widetilde{F}_t$ for every $t \in I\!R_+$. This means, in particular, that the process $F_t - \widetilde{F}_{t\wedge\tau}$ is an $\mathbf{F}$-martingale. We wish to show that the process $D_t - \widetilde{F}_{t\wedge\tau}$ is a $\mathbf{G}$-martingale – that is, for any $t \le s$

$$\mathbf{E}\,(D_s - \widetilde{F}_{s\wedge\tau}\,|\,\mathcal{G}_t) = D_t - \widetilde{F}_{t\wedge\tau},$$

or equivalently,

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = \mathbf{E}\,(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau}\,|\,\mathcal{G}_t).$$

By virtue of (2.1), we have

$$\mathbf{E}\,(D_s - D_t\,|\,\mathcal{G}_t) = (1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)}. \tag{4.14}$$

On the other hand,

$$\begin{aligned}\mathbf{E}\,(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau}\,|\,\mathcal{G}_t) &= (1 - D_t)\,\frac{\mathbf{E}\,(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau}\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)}\\ &= (1 - D_t)\,\frac{\mathbf{E}\,(F_s - F_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)} = (1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t\,|\,\mathcal{F}_t)}{\mathbf{E}\,(1 - D_t\,|\,\mathcal{F}_t)},\end{aligned}$$

where the second equality follows from (2.1), and the third is a consequence of our assumption that the process $F_t - \widetilde{F}_{t\wedge\tau}$ is an $\mathbf{F}$-martingale. $\triangle$

Remark 4.5. In standard examples, τ is a totally inaccessible $\mathbf{F}$-stopping time, and $\widetilde{F}$ is an $\mathbf{F}$-adapted process with continuous increasing sample paths. For example, if τ is the first jump time of a Poisson process N, and $\mathbf{F} = \mathbf{F}^N$ is the natural filtration of this process, then clearly $F_t = D_t$ and $\widetilde{F}_t = \lambda t$, where λ is the (constant) intensity of N. Let us mention that if $\mathbf{F}$ is the Brownian filtration, the process $\widetilde{F}$ is continuous if and only if for any $\mathbf{F}$-stopping time U we have: $\mathbf{P}(\tau = U) = 0$.

Remark 4.6. Under Hypothesis (H), the process $\widetilde{F}$ is never stopped at τ, unless τ is an $\mathbf{F}$-stopping time. To show this assume, on the contrary, that $\widetilde{F}_t = \widetilde{F}_{t\wedge\tau}$. Let us stress that if (H) holds, the process $F_t - \widetilde{F}_{t\wedge\tau}$ is not only an $\mathbf{F}$-martingale, but also a $\mathbf{G}$-martingale. Since by virtue of part (ii) in Proposition 4.4 the process $D_t - \widetilde{F}_{t\wedge\tau}$ is a $\mathbf{G}$-martingale, we see that $D - F$ also is a $\mathbf{G}$-martingale. In view of the definition of F, the last property reads

$$\mathbf{E}\,\big(D_s - \mathbf{E}\,(D_s\,|\,\mathcal{F}_s)\,\big|\,\mathcal{G}_t\big) = D_t - \mathbf{E}\,(D_t\,|\,\mathcal{F}_t),$$

for $t \leq s$, or equivalently

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{G}_t) = \mathbf{E}\big(\mathbf{E}\,(D_s \,|\, \mathcal{F}_s) \,|\, \mathcal{G}_t\big) - \mathbf{E}\,(D_t \,|\, \mathcal{F}_t) = I_1 - I_2. \tag{4.15}$$

Under (H), we have

$$I_1 = \mathbf{E}\,(\mathbf{P}(\tau \leq s \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t \vee \mathcal{D}_t) = \mathbf{E}\,(\mathbf{P}(\tau \leq s \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t)$$

since the random variable $\mathbf{P}(\tau \leq s \,|\, \mathcal{F}_\infty)$ is obviously $\mathcal{F}_\infty$-measurable, and the σ-fields $\mathcal{F}_\infty$ and $\mathcal{D}_t$ are conditionally independent given $\mathcal{F}_t$. Consequently, $I_1 = \mathbf{E}\,(\mathbf{E}\,(D_s \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t) = \mathbf{E}\,(D_s \,|\, \mathcal{F}_t)$. Therefore, (4.15) can be rewritten as follows

$$\mathbf{E}\,(D_s - D_t \,|\, \mathcal{G}_t) = \mathbf{E}\,(D_s \,|\, \mathcal{F}_t) - \mathbf{E}\,(D_t \,|\, \mathcal{F}_t).$$

Furthermore, applying (4.14) to the left-hand side of the last equality, we obtain

$$(1 - D_t)\,\frac{\mathbf{E}\,(D_s - D_t \,|\, \mathcal{F}_t)}{\mathbf{E}\,(1 - D_t \,|\, \mathcal{F}_t)} = \mathbf{E}\,(D_s - D_t \,|\, \mathcal{F}_t).$$

By letting s tend to ∞, we obtain $D_t = \mathbf{E}\,(D_t \,|\, \mathcal{F}_t)$ or more explicitly, $\mathbf{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau \leq t\}}$ for every $t \in \mathbb{R}_+$. We conclude that τ is a $\mathbf{F}$-stopping time.

The theory of the compensator proves that the process $\widetilde{F}$ enjoys the property that for any $\mathbf{F}$-predictable bounded process Z we have

$$\mathbf{E}\,(Z_\tau \,|\, \mathcal{G}_t) = Z_\tau \mathbb{1}_{\{\tau \leq t\}} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbf{E}\,\Big(\int_t^\infty Z_u \, d\widetilde{F}_u \,\Big|\, \mathcal{F}_t \Big).$$

This property appears to be useful, for instance, in the computation of the value of a rebate (recovery payoff at default)

$$\mathbf{E}\,\big(\mathbb{1}_{\{\tau \leq T\}} Z_\tau\big) = \mathbf{E}\,\Big(\int_0^T Z_u \, d\widetilde{F}_u \Big).$$

Let us examine the relationship between the concept of an $\mathbf{F}$-martingale hazard process Λ of τ and the classic concept of a compensator.

Definition 4.7. The *compensator* of a $\mathbf{G}$-stopping time τ is a process A which satisfies: (i) A is a $\mathbf{G}$-predictable right-continuous increasing process, with $A_0 = 0$, (ii) the process $D - A$ is a $\mathbf{G}$-martingale.

It is well known that for any random time τ and any filtration $\mathbf{G}$ such that τ is a $\mathbf{G}$-stopping time there exists a unique $\mathbf{G}$-compensator A of τ. Moreover, $A_t = A_{t \wedge \tau}$, that is, A is stopped at τ. In the next proposition, $\mathbf{F}$ is an arbitrary filtration such that $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$.

Proposition 4.8. (i) *Let Λ be an $\mathbf{F}$-martingale hazard process of τ. Then the process $A_t = \Lambda_{t \wedge \tau}$ is the $\mathbf{G}$-compensator of τ.*
(ii) *Let A be the $\mathbf{G}$-compensator of τ. Then there exists an $\mathbf{F}$-martingale hazard process Λ such that $A_t = \Lambda_{t \wedge \tau}$.*

Remark 4.9. For a given filtration $\mathbf{G}$ and a given $\mathbf{G}$-stopping time τ, the condition $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$ does not specify uniquely the filtration $\mathbf{F}$, in general. Assume that $\mathbf{G} = \mathbf{D} \vee \mathbf{F}^1 = \mathbf{D} \vee \mathbf{F}^2$, and denote by Λ^i the $\mathbf{F}^i$-martingale hazard process of τ. Then obviously $\Lambda^1_{t\wedge\tau} = A_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$ so that the stopped hazard processes coincide.

4.3 Relationships Between Hazard Processes Γ and Λ

Let us assume that the $\mathbf{F}$-hazard process Γ is well defined (in particular, τ is not an $\mathbf{F}$-stopping time). Recall that for any $\mathcal{F}_s$-measurable random variable Y we have (cf. (2.4))

$$\mathbf{E}\,(1\!\!1_{\{\tau>s\}}\,Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbf{E}\,\big(Ye^{\Gamma_t-\Gamma_s} \,|\, \mathcal{F}_t\big). \tag{4.16}$$

The natural question which arises in this context is: can we substitute Γ with the $\mathbf{F}$-martingale hazard function Λ in the formula above? Of course, the answer is trivial when it is known that the equality $\Lambda = \Gamma$ is valid, for instance, when condition (G) are satisfied and F is a continuous process. More precisely, we have the following result (see [19] for the proof).

Proposition 4.10. *Under assumption* (G) *the following assertions are valid.*
(i) *If the* $\mathbf{F}$*-hazard process* Γ *is continuous, then the* $\mathbf{F}$*-martingale hazard process* Λ *is also continuous, and both processes coincide, namely,*

$$\Gamma_t = \Lambda_t = -\ln(1-F_t), \quad \forall\, t \in I\!\!R_+.$$

(ii) *If the* $\mathbf{F}$*-hazard process* Γ *is a discontinuous process then the equality* $\Lambda = \Gamma$ *is never satisfied. More precisely, we have*

$$e^{-\Gamma_t} = e^{-\Lambda^c_t} \prod_{0<u\le t} (1-\Delta\Lambda_u),$$

where Λ^c *is the continuous component of* Λ *– that is,* $\Lambda^c_t = \Lambda_t - \sum_{0\le u\le t} \Delta\Lambda_u$.

We shall now examine the following question: does the continuity of the $\mathbf{F}$-martingale hazard process Λ imply the equality $\Lambda = \Gamma$? The next result provides only a partial solution to this problem.

Proposition 4.11. *Under* (G), *assume that any* $\mathbf{F}$*-martingale is continuous. If the* $\mathbf{F}$*-martingale hazard process* Λ *is a continuous process then for arbitrary* $t \le s$ *and any bounded* $\mathcal{F}_s$*-measurable random variable* Y *we have*

$$1\!\!1_{\{\tau>t\}}\mathbf{E}\,(Ye^{\Gamma_t-\Gamma_s} \,|\, \mathcal{F}_t) = 1\!\!1_{\{\tau>t\}}\mathbf{E}\,(Ye^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t). \tag{4.17}$$

Proof: We shall show that for any $t \le s$

$$\mathbf{E}\,(1\!\!1_{\{\tau>s\}}Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\mathbf{E}\,(Ye^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t). \tag{4.18}$$

Let us introduce the **F**-martingale $m_t = \mathbf{E}\,(Ye^{-\Lambda_s} \,|\, \mathcal{F}_t)$, where Y is a bounded $\mathcal{F}_s$-measurable random variable. Also let $\widetilde{L}_t = 1\!\!1_{\{\tau>t\}}\, e^{\Lambda_t}$. The **G**-martingale property of $\widetilde{L}$ follows easily from Itô's lemma and the continuity of Λ. Indeed,

$$d\widetilde{L}_t = (1 - D_{t-})e^{\Lambda_t}\, d\Lambda_t - e^{\Lambda_t}\, dD_t = -e^{\Lambda_t}\, d\widetilde{M}_t = -\widetilde{L}_{t-}\, d\widetilde{M}_t. \tag{4.19}$$

By virtue of part (i) in Lemma 2.10 the stopped process $\widetilde{m}_t = m_{t\wedge\tau}$ is a continuous **G**-martingale, so that it is orthogonal to the **G**-martingale $Z_t = \widetilde{L}_{t\wedge s}$ (which is obviously of finite variation). Therefore the product $\widetilde{m}Z$ is a **G**-martingale, and thus

$$\mathbf{E}\,(\widetilde{m}_s Z_s \,|\, \mathcal{G}_t) = \widetilde{m}_t Z_t = 1\!\!1_{\{\tau>t\}}\, m_t e^{\Lambda_t} = 1\!\!1_{\{\tau>t\}} \mathbf{E}\,(Ye^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t).$$

Furthermore,

$$\widetilde{m}_s Z_s = 1\!\!1_{\{\tau>s\}}\, m_{s\wedge\tau} e^{\Lambda_s} = 1\!\!1_{\{\tau>s\}}\, Y m_s e^{\Lambda_s} = Y 1\!\!1_{\{\tau>s\}}.$$

This shows that (4.18) is indeed satisfied. Combining (4.18) with (4.16), we get (4.17). △

It appears that under the assumptions of Proposition 4.11 we can establish the equality $\Gamma = \Lambda$, as the following result shows.

Proposition 4.12. *Under* (G), *assume that any* **F***-martingale is continuous. Then:*
(i) *if Λ is a continuous process, then Γ is also continuous and $\Lambda = \Gamma$,*
(ii) *if Λ is a discontinuous process, then Γ is also a discontinuous process, and $\Lambda \neq \Gamma$.*

PROOF: We know that the **F**-martingale hazard process Λ is given by (4.13). Therefore, if Λ is continuous then also $\widetilde{F}$ is continuous, and thus also $F = \widetilde{M} + \widetilde{F}$ is an increasing continuous process. Consequently, Λ is given by (4.12) and thus $\Lambda_t = -\ln(1 - F_t) = \Gamma_t$. The second statement follows by similar arguments. △

To the best of our knowledge, the problem whether the continuity of Λ implies the continuity of Γ remains open in general (under (G), say). The following conjecture seems to be natural.

Conjecture (A). If (G) holds and the **F**-martingale hazard process Λ is continuous, then $\Gamma = \Lambda$.

In view of Proposition 4.3, it would be enough to show that Γ is a continuous process, and the equality $\Gamma = \Lambda$ would then follow. The following example[5] shows that Conjecture (A) is false, in general, when hypothesis (G) fails to hold.

[5] More examples of this kind can be found in [18].

Example 4.13. Let $(W_t,\, t \geq 0)$ be a standard Brownian motion on $(\Omega, \mathbf{F}, \mathbf{P})$, where $\mathbf{F} = \mathbf{F}^W$ is the natural filtration of W. Let S_t denote the maximum of W on $[0, t]$; that is, $S_t = \sup\{W_s : s \leq t\}$. A random time $\widetilde{\tau}$ is defined by setting $\widetilde{\tau} = \inf\{t \leq 1 : W_t = S_1\}$ (notice that $\widetilde{\tau}$ is the honest time of Meyer mentioned in Remark 2.11). We set $\mathbf{G} = \mathbf{D} \vee \mathbf{F}$. Notice that

$$\{\widetilde{\tau} \leq t\} = \{\sup_{s \leq t} W_s \geq \sup_{t \leq u \leq 1} W_u\} = \{S_t - W_t \geq \sup_{0 \leq s \leq 1-t} \hat{W}_s\},$$

where $(\hat{W}_s := W_{s+t} - W_t\,, s \geq 0)$ is a standard Brownian motion independent of $\mathcal{F}_t$. Therefore,

$$\mathbf{P}(\widetilde{\tau} \leq t | \mathcal{F}_t) = \mathbf{P}\big(\sup_{0 \leq s \leq 1-t} \hat{W}_s \leq S_t - W_t \,\big|\, \mathcal{F}_t\big) = \mathbf{P}\big(\sup_{0 \leq s \leq 1-t} \hat{W}_s \leq x\big)_{|x = S_t - W_t}.$$

Using the standard relationships $\mathbf{P}\big(\sup_{0 \leq s \leq 1-t} \hat{W}_s \leq x\big) = 1 - 2\mathbf{P}(\hat{W}_{1-t} \geq x) = \mathbf{P}(|\hat{W}_{1-t}| \leq x)$ which are valid for any $x \geq 0$, we finally obtain, for $t \in [0, 1)$,

$$F_t = \mathbf{P}(\tilde{\tau} \leq t \,|\, \mathcal{F}_t) = \widetilde{\Phi}(t, S_t - W_t) = \Phi\left(\frac{S_t - W_t}{\sqrt{1-t}}\right),$$

where (G stands here for a random variable with the standard Gaussian law under $\mathbf{P}$)

$$\widetilde{\Phi}(t, x) := \mathbf{P}(|\hat{W}_{1-t}| \leq x) = \mathbf{P}(\sqrt{1-t}\,|G| \leq x) = \sqrt{\frac{2}{\pi}} \int_0^{x/\sqrt{1-t}} e^{-\frac{y^2}{2}}\, dy$$

and

$$\Phi(x) := \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{y^2}{2}} dy, \quad \forall\, x \in \mathbb{R}_+.$$

Since the $\mathbf{F}$-hazard process of $\widetilde{\tau}$ equals $\Gamma_t = -\ln(1 - F_t)$, it is apparent that it follows a process of infinite variation. The next goal is to find the canonical decomposition of the submartingale F. Let us denote by Z the non-negative continuous semimartingale, for $t \in [0, 1)$,

$$Z_t = \frac{S_t - W_t}{\sqrt{1-t}}.$$

Since clearly $F_t = \Phi(Z_t)$, using Itô's formula, we obtain

$$\begin{aligned} F_t &= \int_0^t \Phi'(Z_u)\, dZ_u + \frac{1}{2} \int_0^t \Phi''(Z_u)\, \frac{du}{1-u} \\ &= -\int_0^t \Phi'(Z_u) \frac{dW_u}{\sqrt{1-u}} + \sqrt{\frac{2}{\pi}} \int_0^t \frac{dS_u}{\sqrt{1-u}} - \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^t Z_u e^{-Z_u^2/2} \frac{du}{1-u}, \end{aligned}$$

where in the second equality we have used the fact that $\Phi'(0) = \sqrt{2/\pi}$, and that the process S_t, $t < 1$, increases only on the set $\{t \in [0, 1) : S_t = W_t\} =$

$\{t \in [0,1) : Z_t = 0\}$. By virtue of Proposition 4.4, the martingale hazard process of $\widetilde{\tau}$ equals, for $t \leq 1$ (cf. (4.13))

$$\Lambda_t = \int_0^t \frac{d\widetilde{F}_u}{1-F_u} = \int_0^t \frac{d\widetilde{F}_u}{1-\Phi(Z_u)}.$$

Of course, processes Γ and Λ do not coincide (recall that Γ is not of finite variation).

As noted in [1], such a model would admit arbitrage opportunities if the occurrence of the random time $\widetilde{\tau}$ could be observed by an investor. From a financial point of view, they are obvious. Consider any date t. If $\widetilde{\tau}$ is smaller than t, the investor should not enter the market. If $\widetilde{\tau} > t$, the investor knows that the supremum is not yet attained. Therefore, he should buy the asset and wait until the asset price attains some higher value (this will happen with probability one). Finally, he should sell the asset at some time when his profit[6] is strictly positive (for example, at time $\widetilde{\tau}$).

Let us now take a purely mathematical point of view. Suppose that there exists a probability $\mathbf{Q}$, equivalent to the historical probability $\mathbf{P}$, and such that the stopped process $W_{t\wedge\widetilde{\tau}}$ is a $\mathbf{G}$-martingale. Let a be a constant such that the event $A = \{\widetilde{\tau} > \frac{1}{2}, W_{1/2} < a\}$ has a positive probability under $\mathbf{P}$. Since $A \in \mathcal{G}_{1/2}$, we have

$$0 = \mathbf{E}_{\mathbf{Q}}\big(\mathbb{1}_A W_{\widetilde{\tau}}\big) - \mathbf{E}_{\mathbf{Q}}\big(\mathbb{1}_A W_{1/2}\big) = \mathbf{E}_{\mathbf{Q}}\big(\mathbb{1}_A (W_{\widetilde{\tau}} - W_{1/2})\big).$$

We conclude that $\mathbf{Q}(A) = 0$, and thus $\mathbf{Q}$ and $\mathbf{P}$ are not mutually equivalent.

Remark 4.14. The example above shows that the "stochastic intensity" does not characterize the default time, in general. It is well known (see the next section) that it is possible to construct a random time τ with a given hazard processes $\Gamma = \Lambda$. Unlike $\widetilde{\tau}$, the random time τ defined below satisfies (H).

4.4 Random Time with a Given Hazard Process

We shall now examine the 'standard' construction of a random time for a given 'hazard process' Ψ. In the 'standard' construction of τ, the following properties hold:
(i) Ψ coincides with the $\mathbf{F}$-hazard process Γ of τ,
(ii) Ψ is the $\mathbf{F}$-martingale hazard process of a random time τ,
(iii) Ψ is a $\mathbf{G}$-martingale hazard process of τ considered as a $\mathbf{G}$-stopping time.
Let us notice that the random time τ defined below is not a stopping time with respect to the filtration $\mathbf{F}$, but it is a totally inaccessible stopping time with respect to the enlarged filtration $\mathbf{G}$. Let Ψ be an $\mathbf{F}$-adapted, continuous,

[6] For simplicity, we assume that the interest rate is zero. Otherwise, we should take into account the cost of borrowed money.

increasing process, defined on a filtered probability space $(\widetilde{\Omega}, \mathbf{F}, \widetilde{\mathbf{P}})$ such that $\Psi_0 = 0$ and $\Psi_\infty = +\infty$. For instance, it can be given by the formula

$$\Psi_t = \int_0^t \psi_u \, du, \quad \forall\, t \in I\!R_+, \tag{4.20}$$

where ψ is a non-negative $\mathbf{F}$-progressively measurable process. Our goal is to construct a random time τ, on an enlarged probability space $(\Omega, \mathcal{G}, \mathbf{P})$, in such a way that Ψ is an $\mathbf{F}$-(martingale) hazard process of τ. To this end, we assume that ξ is a random variable on some probability space[7] $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}})$, with the uniform probability law on $[0,1]$. We may take the product space $\Omega = \widetilde{\Omega} \times \hat{\Omega}$, $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and $\mathbf{P} = \widetilde{\mathbf{P}} \otimes \hat{\mathbf{P}}$. We introduce the random time τ by setting

$$\tau = \inf \{\, t \in I\!R_+ : e^{-\Psi_t} \leq \xi \,\} = \inf \{\, t \in I\!R_+ : \Psi_t \geq -\ln \xi \,\}.$$

As usual, we set $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$ for every t. We shall now check that properties (i)-(iii) also hold.
Indeed, since clearly $\{\tau > t\} = \{e^{-\Psi_t} > \xi\}$, we get $\mathbf{P}(\tau > t \,|\, \mathcal{F}_\infty) = e^{-\Psi_t}$. Consequently,

$$1 - F_t = \mathbf{P}(\tau > t | \mathcal{F}_t) = \mathbf{E}\left(\mathbf{P}(\tau > t | \mathcal{F}_\infty) | \mathcal{F}_t\right) = e^{-\Psi_t} = \mathbf{P}(\tau > t | \mathcal{F}_\infty) \tag{4.21}$$

and thus F is an $\mathbf{F}$-adapted continuous increasing process. We conclude that Ψ coincides with the $\mathbf{F}$-hazard process Γ. Since (H) is valid (cf. (4.21) and (2.11)), using Proposition 4.3, we conclude that the $\mathbf{F}$-martingale hazard process Λ of τ coincides with Γ. To be more specific, we have $\Psi_t = \Lambda_t = \Gamma_t = -\ln(1 - F_t)$ and thus (ii) is valid. Furthermore, the process $D_t - \Psi_{t \wedge \tau}$ is indeed a $\mathbf{G}$-martingale so that (iii) holds.

Remark 4.15. If, in addition, Ψ satisfies (4.20) then

$$\mathbf{P}(t < \tau \leq s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \, \mathbf{E}\left(1 - e^{-\int_t^s \psi_u \, du} \,\Big|\, \mathcal{F}_t\right).$$

In particular, the cumulative distribution function of τ equals (we write γ^0 to denote the unique intensity function of τ with respect to the trivial filtration)

$$\mathbf{P}(\tau \leq t) = 1 - \mathbf{E}\left(e^{-\int_0^t \psi_u \, du}\right) = 1 - e^{-\int_0^t \gamma^0(u)\, du}.$$

Remark 4.16. If Hypothesis (H) is satisfied, there exists an $\mathbf{F}$-adapted increasing process ζ such that $\mathbf{P}(\tau > t \,|\, \mathcal{F}_\infty) = e^{-\zeta_t}$. The variable $\xi := e^{-\zeta_\tau}$ is independent of $\mathcal{F}_\infty$, it is uniformly distributed on $[0,1]$ and $\tau = \inf\{t : \zeta_t \geq -\ln \xi\}$. See [11].

[7] In principle, it is enough to assume that there exists a random variable ξ on $(\Omega, \mathcal{G}, \mathbf{P})$ such that ξ is uniformly distributed on $[0,1]$, and it is independent of the process Ψ (we then set $\hat{\mathcal{F}} = \sigma(\xi)$).

5 Analysis of Several Random Times

Assume that we are given random times $\tau_1, \ldots, \tau_n$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a filtration $\mathbf{F}$. For $i = 1, \ldots, n$ we set $D^i_t = 1\!\!1_{\{\tau_i \le t\}}$, and we denote by $\mathbf{D}^i$ the filtration generated by the process D^i. We introduce the enlarged filtration $\mathbf{G} := \mathbf{D}^1 \vee \cdots \vee \mathbf{D}^n \vee \mathbf{F}$. It is thus evident that $\tau_1, \ldots, \tau_n$ are stopping times with respect to the filtration $\mathbf{G}$.

5.1 Ordered Random Times

Consider the two $\mathbf{F}$-adapted increasing continuous processes, Ψ^1 and Ψ^2, which satisfy $\Psi^2_0 = \Psi^1_0 = 0$ and $\Psi^1_t > \Psi^2_t$ for every $t \in I\!\!R_+$. Let ξ be a random variable uniformly distributed on $[0,1]$, independent of the processes $\Psi^i, i = 1,2$. We introduce random times satisfying $\tau_1 < \tau_2$ with probability 1 by setting

$$\tau_i = \inf\{\, t \in I\!\!R_+ : e^{-\Psi^i_t} \le \xi \,\} = \inf\{\, t \in I\!\!R_+ : \Psi^i_t \ge -\ln \xi \,\}.$$

We shall write $\mathbf{G}^i = \mathbf{D}^i \vee \mathbf{F}$, for $i = 1,2$, and $\mathbf{G} = \mathbf{D}^1 \vee \mathbf{D}^2 \vee \mathbf{F}$. An analysis of each random time τ_i with respect to its 'natural' enlarged filtration $\mathbf{G}^i$ can be done along the same lines as in the previous section.

From results of Section 4.4, it follows that for each i the process Ψ^i represents: (i) the $(\mathbf{F}, \mathbf{G}^i)$-hazard process Γ^i of τ_i, (ii) the $(\mathbf{F}, \mathbf{G}^i)$-martingale hazard process Λ^i of τ_i, and (iii) the $\mathbf{G}^i$-martingale hazard process of τ_i when τ_i is considered as a $\mathbf{G}^i$-stopping time.

We find it convenient to introduce the following notation:[8] $\mathbf{F}^i = \mathbf{D}^i \vee \mathbf{F}$, so that $\mathbf{G} = \mathbf{D}^1 \vee \mathbf{F}^2$ and $\mathbf{G} = \mathbf{D}^2 \vee \mathbf{F}^1$. Let us start by an analysis of τ_1. We search for the $(\mathbf{F}^2, \mathbf{G})$-hazard process $\widetilde{\Gamma}^1$ of τ_1 and for the $(\mathbf{F}^2, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^1$ of τ_1. We shall first check that $\widetilde{\Gamma}^1 \neq \Gamma^1$. Indeed, by virtue of the definition of a hazard process we have, for $t \in I\!\!R_+$,

$$e^{-\Gamma^1_t} = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t) = e^{-\Psi^1_t}.$$

and

$$e^{-\widetilde{\Gamma}^1_t} = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t).$$

Equality $\widetilde{\Gamma}^1 = \Gamma^1$ would thus imply the following equality, for every $t \in I\!\!R_+$,

$$\mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t) = \mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t). \tag{5.22}$$

The equality above is not valid, however. Indeed, the condition $\tau_2 \le t$ implies $\tau_1 \le t$, an thus on the set $\{\tau_2 \le t\} \in \mathcal{D}^2_t$ we obtain $\mathbf{P}(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^2_t) = 0$. The last equality contradicts (5.22) since the right-hand side in (5.22) is non-zero.

[8] Though $\mathbf{F}^i = \mathbf{G}^i$ in the present setup, this double notation will prove useful in what follows.

Notice also that the $(\mathbf{F}^2, \mathbf{G})$-hazard process $\widetilde{\Gamma}^1$ is well defined only strictly before τ_2.

On the other hand, one can check that the process $D^1_t - \Psi^1_{t\wedge\tau_1}$, which is obviously stopped at τ_1, is not only a $\mathbf{G}^1$-martingale, but also a $\mathbf{G}$-martingale. To check this, let us consider an arbitrary $\mathbf{G}^1$-martingale M stopped at τ_1. To show that M is a $\mathbf{G}$-martingale, it is enough to check that for any bounded $\mathbf{G}$-stopping time τ we have $\mathbf{E}\,(M_\tau) = M_0$. Since M is stopped at τ_1, it is clear that $\mathbf{E}\,(M_\tau) = \mathbf{E}\,(M_{\tau\wedge\tau_1})$. Furthermore, for any bounded $\mathbf{G}$-stopping time τ, the random time $\tilde{\tau} = \tau \wedge \tau_1$ is a bounded $\mathbf{G}^1$-stopping time. To see that, take an arbitrary t, and consider the event $\{\tilde{\tau} \le t\}$. We have

$$\{\tilde{\tau} \le t\} = \{\tau \wedge \tau_1 \le t\} = \{\tau_1 \le t\} \cup (\{\tau \le t\} \cap \{\tau_1 > t\}) = A \cup B\,.$$

Clearly $A = \{\tau_1 \le t\} \in \mathcal{D}^1_t \subset \mathcal{F}_t \vee \mathcal{D}^1_t = \mathcal{G}^1_t$. Since $\tau_1 \le \tau_2$, we have

$$B = \{\tau \le t\} \cap \{\tau_1 > t\} = \{\tau \le t\} \cap \{\tau_1 > t\} \cap \{\tau_2 > t\}.$$

Since $\mathcal{G}_t = \mathcal{G}^1_t \vee \mathcal{D}^2_t$, there exists a set $C \in \mathcal{G}^1_t$ such that $B = C \cap \{\tau_2 > t\}$. Thus

$$B = C \cap \{\tau_2 > t\} = C \cap \{\tau_1 > t\} \cap \{\tau_2 > t\} = C \cap \{\tau_1 > t\} \in \mathcal{G}^1_t\,.$$

By assumption, M is a $\mathbf{G}^1$-martingale, and thus $\mathbf{E}\,(M_{\tilde{\tau}}) = M_0$ for any bounded $\mathbf{G}^1$-stopping time $\tilde{\tau}$. Combining the properties above, we get the equality $\mathbf{E}\,(M_\tau) = M_0$ for any bounded $\mathbf{G}$-stopping time τ. Notice that since $D^1_t - \Psi^1_{t\wedge\tau_1}$ is a $\mathbf{G}$-martingale, Ψ^1 coincides with the $(\mathbf{F}^2, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^1$ of τ_1. Furthermore, Ψ^1 represents also the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}^1$ of τ_1.

As expected, the properties of τ_2 with respect to the filtration $\mathbf{F}^1$ are different. First, by definition of the $(\mathbf{F}^1, \mathbf{G})$-martingale hazard process of τ_2 we have

$$e^{-\widetilde{\Gamma}^2_t} = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}^1_t) = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t).$$

We claim that $\widetilde{\Gamma}^2 \ne \Gamma^2$; that is, that the equality

$$\mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t) = \mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t) \tag{5.23}$$

does not hold, in general. Indeed, the inequality $\tau_1 > t$ implies $\tau_2 > t$, and thus on set $\{\tau_1 > t\} \in \mathcal{D}^1_t$ we have $\mathbf{P}(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{D}^1_t) = 1$, in contradiction with (5.23). Notice that the process $\widetilde{\Gamma}^2$ is not well defined after time τ_1.

Furthermore, the process $D^2_t - \Psi^2_{t\wedge\tau_2}$ is a $\mathbf{G}^2$-martingale; it does not follow a $\mathbf{G}$-martingale, however (otherwise, the equality $\widetilde{\Gamma}^2_t = \Gamma^2_t = \Psi^2_t$ would be true for $t < \tau_2$, but this is not the case). The explicit formula for the $(\mathbf{F}^1, \mathbf{G})$-martingale hazard process $\widetilde{\Lambda}^2$ of τ_2 is not easily available (it seems plausible that $\widetilde{\Lambda}^2$ has discontinuity at τ_1).

Let us observe that τ_1 is a totally inaccessible stopping time not only with respect to $\mathbf{G}^1$, but also with respect to $\mathbf{G}$. On the other hand, τ_2 is a totally inaccessible stopping time with respect to $\mathbf{G}^1$, but it is a predictable stopping time with respect to $\mathbf{G}$. Indeed, we may easily find an announcing sequence τ_2^n of $\mathbf{G}$-stopping times, for instance,

$$\tau_2^n = \inf \{ t \geq \tau_1 : \Psi_t^2 \geq -\ln \xi - \tfrac{1}{n} \}.$$

Therefore, the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}^2$ of τ_2 coincides with the $\mathbf{G}$-predictable process D^2.

Let us set $\tau = \tau_1 \wedge \tau_2$. In the present setup, it is evident that $\tau = \tau_1$, and thus the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}$ of τ is equal to Ψ^1. It is also equal to the sum of $\mathbf{G}$-martingale hazard processes $\hat{\Lambda}^i$ of τ_i, $i = 1, 2$, stopped at τ. Indeed, we have

$$\hat{\Lambda}_{t\wedge\tau} = \Psi^1_{t\wedge\tau} = \Psi^1_{t\wedge\tau} + D^2_{t\wedge\tau} = \hat{\Lambda}^1_{t\wedge\tau} + \hat{\Lambda}^2_{t\wedge\tau}.$$

We shall see in the next section that this property is universal (notice that it is of limited use though).

5.2 Properties of the Minimum of Several Random Times

The exposition here is partially based on Duffie [8] and Kusuoka [20]. We shall examine the following problem: given a finite family of random times τ_i, $i = 1, \ldots, n$, and the associated hazard processes, find the hazard process of the random time $\tau = \tau_1 \wedge \cdots \wedge \tau_n$. The problem above cannot be solved in such a generality; that is, without the knowledge of the joint law of $(\tau_1, \ldots, \tau_n)$. Indeed, the solution depends on specific assumptions on random times and the choice of filtrations.

When the reference filtration $\mathbf{F}$ is trivial, the hazard process is a deterministic function, known as the hazard function: $\Gamma(t) = -\ln \mathbf{P}(\tau > t)$. The next simple result deals with the hazard function of the minimum of mutually independent random times.

Lemma 5.1. *Let τ_i, $i = 1, \ldots, n$, be n random times defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Assume that τ_i admits the hazard function Γ^i. If τ_i, $i = 1, \ldots, n$, are mutually independent random variables, the hazard function Γ of τ is equal to the sum of hazard functions Γ^i, $i = 1, \ldots, n$.*

Proof: For any $t \in \mathbb{R}_+$ we have

$$e^{-\Gamma(t)} = \mathbf{P}(\tau_1 \wedge \cdots \wedge \tau_n > t) = \prod_{i=1}^n (1 - F_i(t)) = e^{-\sum_{i=1}^n \Gamma^i(t)}. \qquad \triangle$$

Conversely, if the hazard function of $\tau = \tau_1 \wedge \cdots \wedge \tau_n$ satisfies $\Lambda(t) = \Gamma(t) = \sum_{i=1}^n \Gamma^i(t) = \sum_{i=1}^n \Lambda^i(t)$ for every t then we obtain

$$\mathbf{P}(\tau_1 > t, \ldots, \tau_n > t) = \prod_{i=1}^n \mathbf{P}(\tau_i > t), \quad \forall\, t \in \mathbb{R}_+.$$

Lemma 5.1 admits a rather trivial extension to the general case. Let τ_i, $i = 1, \dots, n$, be n random times defined on a common probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Assume that τ_i admits the $\mathbf{F}^i$-hazard process Γ^i. If, for every $t \in I\!R_+$, the events $\{\tau_i < t\}$, $i = 1, \dots, n$, are conditionally independent with respect to $\mathcal{F}_t$, then the hazard process Γ of τ is equal to the sum of hazard processes Γ^i, $i = 1, \dots, n$.

We borrow from Duffie [8] the following simple result (see Lemma 1 in [8]).

Lemma 5.2. *Let τ_i, $i = 1, \dots, n$, be $\mathbf{G}$-stopping times such that $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Then the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ of $\tau = \tau_1 \wedge \dots \wedge \tau_n$ is equal to the sum of $(\mathbf{F}, \mathbf{G})$-martingale hazard processes Λ^i; that is, $\Lambda_t = \sum_{i=1}^n \Lambda_t^i$ for $t \in I\!R_+$. If Λ is a continuous process then the process $\widetilde{L}$ given by the formula $\widetilde{L}_t = (1 - D_t)e^{\Lambda_t}$ is a $\mathbf{G}$-martingale.*

We consider once again the case of the Brownian filtration; that is, we assume that $\mathbf{F} = \mathbf{F}^W$ for some Brownian motion W. We postulate that W remains a martingale (and thus a Brownian motion) with respect to the enlarged filtration $\mathbf{G} = \mathbf{D}^1 \vee \dots \vee \mathbf{D}^n \vee \mathbf{F}$. In view of the martingale representation property of the Brownian filtration this means, of course, that any $\mathbf{F}$-local martingale is also a local martingale with respect to $\mathbf{G}$ (or indeed with respect to any enlargement of the filtration $\mathbf{F}$), so that (H) holds. It is worthwhile to stress that the case of a trivial filtration $\mathbf{F}$ is also covered by the results of this section.

Our next goal is to generalize the martingale representation property established in Proposition 3.2. Recall that in Proposition 3.2 we have assumed that the $\mathbf{F}$-hazard process Γ of a random time τ is an increasing continuous process. Also, by virtue of results of Section 4.3 (see Proposition 4.10) under the assumptions of Corollary 3.2 we have $\Gamma = \Lambda$; that is, the $\mathbf{F}$-hazard process Γ and the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ coincide.

In the present setup, we prefer to make assumptions directly about the $(\mathbf{F}, \mathbf{G})$-martingale hazard processes Λ^i of random times τ_i, $i = 1, \dots, n$. We assume throughout that the processes Λ^i, $i = 1, \dots, n$ are continuous. As before, we assume that $\mathbf{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. Recall that by virtue of the definition of the $(\mathbf{F}, \mathbf{G})$-martingale hazard process Λ^i of a random time τ_i the process $\widetilde{M}_t^i = D_t^i - \Lambda_{t \wedge \tau_i}^i$ is a $\mathbf{G}$-martingale. Notice that the process $\widetilde{L}_t^i = (1 - D_t^i)e^{\Lambda_t^i}$ is also a $\mathbf{G}$-martingale, since clearly (cf. (4.19))

$$\widetilde{L}_t^i = 1 - \int_{]0,t]} \widetilde{L}_{u-}^i \, d\widetilde{M}_u^i.$$

It is easily seen that $\widetilde{L}^i$ and $\widetilde{L}^j$ are mutually orthogonal $\mathbf{G}$-martingales for any $i \neq j$ (a similar remark applies to $\widetilde{M}^i$ and $\widetilde{M}^j$).

For a fixed k with $0 \le k \le n$, we introduce the filtration $\widetilde{\mathbf{G}} = \mathbf{D}^1 \vee \dots \vee \mathbf{D}^k \vee \mathbf{F}$. Then obviously $\widetilde{\mathbf{G}} = \mathbf{G}$ if $k = n$, and by convention $\widetilde{\mathbf{G}} = \mathbf{F}$ for $k = 0$. It is clear that for any fixed k and arbitrary $i \le k$ processes $\widetilde{L}^i$

and $\widetilde{M}^i$ are $\widetilde{\mathbf{G}}$-adapted. More specifically, $\widetilde{L}^i$ and $\widetilde{L}^j$ are mutually orthogonal $\widetilde{\mathbf{G}}$-martingales for $i, j \leq k$ provided that $i \neq j$. A trivial modification of Lemma 5.2 shows that the $(\mathbf{F}, \widetilde{\mathbf{G}})$-martingale hazard process of the random time $\widetilde{\tau} := \tau_1 \wedge \cdots \wedge \tau_k$ equals $\widetilde{\Lambda} = \sum_{i=1}^k \Lambda^i$. In other words, the process $\widetilde{D}_t - \sum_{i=1}^k \Lambda^i_{t\wedge\tilde{\tau}}$ is a $\widetilde{\mathbf{G}}$-martingale, where we set $\widetilde{D}_t = \mathbb{1}_{\{\tilde{\tau}\leq t\}}$. For a fixed k with $0 \leq k \leq n$, we set $\widetilde{\mathbf{F}} := \mathbf{D}^{k+1} \vee \cdots \vee \mathbf{D}^n \vee \mathbf{F}$. The next two results are due to Kusuoka [20].

Proposition 5.3. *Assume that the* $\mathbf{F}$*-Brownian motion* W *remains a Brownian motion with respect to the enlarged filtration* $\mathbf{G}$*. Let* Y *be a bounded* $\mathcal{F}_T$*-measurable random variable, and let* $\widetilde{\tau} = \tau_1 \wedge \cdots \wedge \tau_k$*. Then for any* $t \leq s \leq T$ *we have*

$$\mathbf{E}\,(\mathbb{1}_{\{\tilde{\tau}>s\}}Y \,|\, \mathcal{G}_t) = \mathbf{E}\,(\mathbb{1}_{\{\tilde{\tau}>s\}}Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\mathbf{E}\,\big(Ye^{\widetilde{\Lambda}_t-\widetilde{\Lambda}_s} \,|\, \mathcal{F}_t\big).$$

In particular,

$$\mathbf{P}(\widetilde{\tau} > s \,|\, \mathcal{G}_t) = \mathbf{P}(\widetilde{\tau} > s \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\mathbf{E}\,(e^{\widetilde{\Lambda}_t-\widetilde{\Lambda}_s} \,|\, \mathcal{F}_t).$$

Let us set $\Lambda = \sum_{i=1}^n \Lambda^i$*. Then for* $\tau = \tau_1 \wedge \cdots \wedge \tau_n$ *we have*

$$\mathbf{P}(\tau > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}\,(e^{\Lambda_t-\Lambda_s} \,|\, \mathcal{F}_t).$$

Proposition 5.4. *Assume that the* $\mathbf{F}$*-Brownian motion* W *remains a Brownian motion with respect to* $\mathbf{G}$*, and that for each* $i = 1, \ldots, n$ *the* $\mathbf{F}$*-martingale hazard process* Λ^i *is continuous. Then any* $\widetilde{\mathbf{F}}$*-martingale* N *admits the integral representation*

$$N_t = N_0 + \int_0^t \xi_u\, dW_u + \sum_{i=k+1}^n \int_{]0,t]} \zeta^i_u\, d\widetilde{M}^i_u, \tag{5.24}$$

where ξ *and* ζ^i*,* $i = k+1, \ldots, n$ *are* $\widetilde{\mathbf{F}}$*-predictable processes.*

It is interesting to observe that the $\mathcal{F}_T$-measurability of Y can be replaced by the $\widetilde{\mathcal{F}}_T$-measurability of Y in Proposition 5.3. Indeed, it is clear that $\widetilde{\Lambda} = \sum_{i=1}^k \Lambda^i$ is also the $(\widetilde{\mathbf{F}}, \mathbf{G})$-martingale hazard process of τ. Furthermore, Proposition 5.4 shows that the process $\hat{Y}$, given by the formula

$$\hat{Y}_t = \mathbf{E}\,(Ye^{-\widetilde{\Lambda}_s} \,|\, \widetilde{\mathcal{F}}_t), \quad \forall\, t \in [0, T],$$

where Y is an $\widetilde{\mathcal{F}}_T$-measurable random variable, admits the following integral representation

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \xi_u\, dW_u + \sum_{i=k+1}^n \int_{]0,t]} \zeta^i_u\, d\widetilde{M}^i_u,$$

where ξ and ζ^i, $i = k+1, \dots, n$ are $\widetilde{\mathbf{F}}$-predictable processes. We conclude that $\hat{Y}$ follows a $\mathbf{G}$-martingale orthogonal to the $\mathbf{G}$-martingale U, which is given by

$$U_t = (1 - \widetilde{D}_{t \wedge s}) e^{\widetilde{\Lambda}_{t \wedge s}} = \prod_{i=1}^{k} \widetilde{L}^i_{t \wedge s}.$$

Arguing in a much the same way as in the proof of Proposition 5.3, we obtain the following result (see [19] for the details).

Corollary 5.5. *Let Y be a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable. Let $\widetilde{\tau} = \tau_1 \wedge \dots \wedge \tau_k$. Then for any $t \le s \le T$ we have*

$$\mathbf{E}\,(1\!\!1_{\{\tilde{\tau} > s\}} Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tilde{\tau} > t\}} \mathbf{E}\,\big(Y e^{\widetilde{\Lambda}_t - \widetilde{\Lambda}_s} \,|\, \widetilde{\mathcal{F}}_t\big).$$

5.3 Change of a Probability Measure

In this section – in which we follow Kusuoka [20] – it is assumed throughout that the filtration $\mathbf{F}$ is generated by a Brownian motion W, which is also a $\mathbf{G}$-martingale (the case of a trivial filtration $\mathbf{F}$ is also covered by the results of this section, though). For a fixed $T > 0$, we shall examine the properties of $\widetilde{\tau}$ under a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{G}_t)$, for every $t \in I\!R_+$. We introduce the $\mathbf{G}$-martingale η by setting

$$\eta_t := \frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|_{\mathcal{G}_t}, \quad \mathbf{P}\text{-a.s.},$$

By virtue of Proposition 5.4 (with $k = 0$), the Radon-Nikodým density process η admits the integral representation

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \sum_{i=1}^{n} \int_{]0,t]} \zeta^i_u \, d\widetilde{M}^i_u, \tag{5.25}$$

where ξ and ζ^i, $i = 1, \dots, n$ are $\mathbf{G}$-predictable stochastic processes. It can be shown that η is a strictly positive process, so that we may rewrite (5.25) as follows

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\beta_u \, dW_u + \sum_{i=1}^{n} \kappa^i_u \, d\widetilde{M}^i_u\big), \tag{5.26}$$

where β and $\kappa^i > -1$, $i = 1, \dots, n$ are $\mathbf{G}$-predictable processes. For the proof of the next result, we refer to [20] or [19].

Proposition 5.6. *Let $\mathbf{P}^*$ be a probability measure which is equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{G}_t)$, for every $t \in I\!R_+$. If the Radon-Nikodým density of $\mathbf{P}^*$ with respect to $\mathbf{P}$ on $(\Omega, \mathcal{G}_t)$ is given by* (5.26), *then the process*

$$W^*_t = W_t - \int_0^t \beta_u \, du,$$

is a $\mathbf{G}$*-Brownian motion under* $\mathbf{P}^*$*, and for each* $i = 1, \ldots, n$ *the process*

$$M_t^{i*} := \widetilde{M}_t^i - \int_{]0,t\wedge\tau_i]} \kappa_u^i \, d\Lambda_u^i = D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^i) \, d\Lambda_u^i,$$

is a $\mathbf{G}$*-martingale orthogonal to* W^* *under* $\mathbf{P}^*$*. Moreover, processes* M^{i*} *and* M^{j*} *follow mutually orthogonal* $\mathbf{G}$*-martingales under* $\mathbf{P}^*$ *for any* $i \neq j$*.*

Though the process M^{i*} is a $\mathbf{G}$-martingale under $\mathbf{P}^*$, it should be stressed that the process $\int_{]0,t]}(1 + \kappa_u^i) \, d\Lambda_u^i$ is not necessarily the $(\mathbf{F}, \mathbf{G})$-martingale hazard process of τ_i under $\mathbf{P}^*$, since it is not $\mathbf{F}$-adapted, but merely $\mathbf{G}$-adapted, in general. To circumvent this, we choose, for any fixed i, a suitable version of the process κ^i. Specifically, we take a process κ^{i*}, which coincides with κ^i on a random interval $[0, \tau_i]$, and which is predictable with respect to the enlarged filtration $\mathbf{F}^{i*} := \mathbf{D}^1 \vee \cdots \vee \mathbf{D}^{i-1} \vee \mathbf{D}^{i+1} \vee \ldots \mathbf{D}^n \vee \mathbf{F}$. Since

$$D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^{i*}) \, d\Lambda_u^i = D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \kappa_u^i) \, d\Lambda_u^i,$$

we conclude that for each fixed i the process

$$\Lambda_t^{i*} = \int_{]0,t]} (1 + \kappa_u^{i*}) \, d\Lambda_u^i$$

represents the $(\mathbf{F}^{i*}, \mathbf{G})$-martingale hazard process of τ_i under $\mathbf{P}^*$. This does not mean, however, that the equality

$$\mathbf{P}^*(\tau_i > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau_i > t\}} \mathbf{E}_{\mathbf{P}^*}(e^{\Lambda_t^{i*} - \Lambda_s^{i*}} \,|\, \mathcal{F}_t^{i*})$$

is valid for every $s \leq t$. We prefer to examine the validity of the last formula in a slightly more general setting. For a fixed $k \leq n$, we set $\widetilde{\tau} = \tau_1 \wedge \cdots \wedge \tau_k$, and we write $\widetilde{\mathbf{F}} = \mathbf{D}^{k+1} \vee \cdots \vee \mathbf{D}^n \vee \mathbf{F}$. For any $i = 1, \ldots, n$ we denote by $\widetilde{\kappa}^i$ ($\widetilde{\beta}$, resp.) the $\widetilde{\mathbf{F}}$-predictable process such that $\widetilde{\kappa}^i = \kappa^i$ ($\widetilde{\beta} = \beta$, resp.) on the random set $[0, \widetilde{\tau}]$.

Lemma 5.7. *The* $(\widetilde{\mathbf{F}}, \mathbf{G})$*-martingale hazard process of the random time* $\widetilde{\tau}$ *under* $\mathbf{P}^*$ *is given by the formula*

$$\Lambda_t^* = \sum_{i=1}^k \int_{]0,t]} (1 + \widetilde{\kappa}_u^i) \, d\Lambda_u^i. \tag{5.27}$$

Proof: Let us set

$$\widetilde{W}_t^* = W_t - \int_0^t \widetilde{\beta}_u \, du,$$

and

$$\widetilde{M}_t^{i*} = D_t^i - \int_{]0,t\wedge\tau_i]} (1 + \widetilde{\kappa}_u^i) \, d\Lambda_u^i$$

for $i = 1, \dots, n$. The processes $\widetilde{W}^*$ and $\widetilde{M}^{i*}$ follow **G**-martingales under $\mathbf{P}^*$, provided that they are stopped at $\widetilde{\tau}$ (since $\widetilde{W}^*_{t\wedge\tilde{\tau}} = W^*_{t\wedge\tilde{\tau}}$ and $\widetilde{M}^{i*}_{t\wedge\tilde{\tau}} = M^{i*}_{t\wedge\tilde{\tau}}$). Consequently, the process

$$\widetilde{D}_t - \sum_{i=1}^{k} \int_{]0,t\wedge\tilde{\tau}]} (1 + \widetilde{\kappa}^i_u)\, d\Lambda^i_u = \sum_{i=1}^{k} (\widetilde{M}^{i*}_t)^{\tilde{\tau}}$$

is a **G**-martingale. △

In view of Corollary 5.5 and Lemma 5.7, it would be natural to conjecture that, for any bounded $\widetilde{\mathcal{F}}_T$-measurable random variable Y, and every $t \le s \le T$, we have

$$\mathbf{E}_{\mathbf{P}^*}(\mathbb{1}_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\, \mathbf{E}_{\mathbf{P}^*}\big(Y e^{\Lambda^*_t - \Lambda^*_s} \,|\, \widetilde{\mathcal{F}}_t\big). \tag{5.28}$$

It appears, however, that the last formula is not valid, in general, unless we substitute the probability measure $\mathbf{P}^*$ in the right-hand side of (5.28) with some related probability measure. To this end, we introduce the following auxiliary processes $\hat{\eta}^\ell$, for $\ell = 1, 2, 3$,

$$\hat{\eta}^1_t = 1 + \int_{]0,t]} \hat{\eta}^1_{u-}\Big(\widetilde{\beta}_u\, dW_u + \sum_{i=k+1}^{n} \widetilde{\kappa}^i_u\, d\widetilde{M}^i_u\Big), \tag{5.29}$$

$$\hat{\eta}^2_t = 1 + \int_{]0,t]} \hat{\eta}^2_{u-}\Big(\widetilde{\beta}_u\, dW_u + \sum_{i=1}^{n} \widetilde{\kappa}^i_u\, d\widetilde{M}^i_u\Big),$$

and

$$\hat{\eta}^3_t = 1 + \int_{]0,t]} \hat{\eta}^3_{u-}\Big(\widetilde{\beta}_u\, dW_u + \sum_{i=1}^{k} \kappa^i_u\, d\widetilde{M}^i_u + \sum_{i=k+1}^{n} \widetilde{\kappa}^i_u\, d\widetilde{M}^i_u\Big).$$

It is worth noting that the process $\hat{\eta}^1$ is $\widetilde{\mathbf{F}}$-adapted (since, in particular, each process $\widetilde{M}^i$ is adapted to the filtration $\mathbf{D}^i \vee \mathbf{F}$). On the other hand, processes $\hat{\eta}^2$ and $\hat{\eta}^3$ are merely **G**-adapted, but not necessarily $\widetilde{\mathbf{F}}$-adapted, in general.

For $\ell = 1, 2, 3$, we define a probability measure $\widetilde{\mathbf{P}}_\ell$, equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{G}_t)$, by setting

$$\hat{\eta}^\ell_t := \frac{d\widetilde{\mathbf{P}}_\ell}{d\mathbf{P}}\Big|_{\mathcal{G}_t}, \quad \mathbf{P}\text{-a.s.}$$

The following proposition, which generalizes a result of Kusuoka [20], is a counterpart of Corollary 5.5. We refer to [19] for the proof.

Proposition 5.8. *Let Y be a bounded $\widetilde{\mathcal{F}}_T$-measurable random variable, and let Λ^* be given by* (5.27)*. Then for any $t \le s \le T$ and any $\ell = 1, 2, 3$, we have*

$$\mathbf{E}_{\mathbf{P}^*}(\mathbb{1}_{\{\tilde{\tau}>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tilde{\tau}>t\}}\, \mathbf{E}_{\widetilde{\mathbf{P}}_\ell}\big(Y e^{\Lambda^*_t - \Lambda^*_s} \,|\, \widetilde{\mathcal{F}}_t\big).$$

5.4 Kusuoka's Example

We shall now examine a purely mathematical example, due to Kusuoka [20]; the calculations presented in [20] are not complete, though (see [19] for details). In this example, the hazard process is not increasing, but the market is still arbitrage-free. Under the real-world probability $\mathbf{P}$, the random times $\tau_i,\ i=1,2$ are mutually independent random variables with the exponential law with constant parameters λ_1 and λ_2, respectively. The joint law of (τ_1,τ_2) under $\mathbf{P}$ has thus the density $f(x,y)=\lambda_1\lambda_2 e^{-(\lambda_1 x+\lambda_2 y)}$ for $(x,y)\in I\!\!R^2_+$. We denote by $M^i_t=D^i_t-\lambda_i t$ the martingales associated with these random times. Let α_1 and α_2 be strictly positive real numbers, and let the probability measure $\mathbf{Q}$ on $(\Omega,\mathcal{G})$ be given on $\mathcal{G}_t$ by the formula

$$\frac{d\mathbf{Q}}{d\mathbf{P}}=\eta_t,\quad \mathbf{P}\text{-a.s.},$$

with

$$\eta_t=1+\sum_{i=1}^{2}\int_{]0,t]}\eta_{u-}\kappa^i_u\,dM^i_u,$$

where in turn

$$\kappa^1_t=1\!\!1_{\{\tau_2<t\}}\Big(\frac{\alpha_1}{\lambda_1}-1\Big),\quad \kappa^2_t=1\!\!1_{\{\tau_1<t\}}\Big(\frac{\alpha_2}{\lambda_2}-1\Big).$$

Assume that the filtration in the default-free market is $\mathcal{D}^2_t$, and that τ_1 represents the default time. We have

$$\mathbf{Q}(\tau_1>t\,|\,\mathcal{D}^2_t)=(1-D^2_t)\,\frac{\mathbf{Q}(\tau_1>t,\tau_2>t)}{\mathbf{Q}(\tau_2>t)}+D^2_t\mathbf{Q}(\tau_1>t\,|\,\tau_2).$$

Rather tedious calculations show that (see [20] or [19])

$$\mathbf{Q}(\tau_1>t,\tau_2>t)=e^{-(\lambda_1+\lambda_2)t},$$

$$\mathbf{Q}(\tau_2>t)=\frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\lambda_1 e^{-\alpha_2 t}+(\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)t}\Big),$$

and

$$\mathbf{Q}(\tau_1>t\,|\,\tau_2=u)=\frac{(\lambda_1+\lambda_2-\alpha_2)\lambda_2 e^{-(\lambda_1+\lambda_2)ut}e^{-\alpha_1(t-u)}}{\lambda_1\alpha_2 e^{-\alpha_2 u}+(\lambda_2-\alpha_2)(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)u}}.$$

Consequently, we obtain

$$1-F_t=1\!\!1_{\{t<\tau_2\}}\,\frac{c}{\lambda_1 e^{ct}+\lambda_2-\alpha_2}+1\!\!1_{\{\tau_2\le t\}}\,\frac{c\lambda_2 e^{-\alpha_1(t-\tau_2)}}{\lambda_1\alpha_2 e^{c\tau_2}+(\lambda_2-\alpha_2)(\lambda_1+\lambda_2)},$$

where $c=\lambda_1+\lambda_2-\alpha_2$. The two terms in the right-hand side are decreasing functions, the jump in τ_2 equals

$$\Delta=\frac{c}{\lambda_1 e^{ct}+\lambda_2-\alpha_2}-\frac{c\lambda_2}{\lambda_1\alpha_2 e^{c\tau_2}+(\lambda_2-\alpha_2)(\lambda_1+\lambda_2)}$$

and thus it is negative if and only if $\lambda_2\le\alpha_2$. We conclude that Hypothesis (G) is not satisfied.

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