On the Heston model with Stochastic Correlation

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Abstract

The degree of relationship between financial products and financial institutions, e.g., must be considered for pricing and hedging. Usually, for financial products modelled with the specification of a system of stochastic differential equations, the relationship is represented by correlated Brownian motions (BMs). For example, the BM of the asset price and the BM of the stochastic volatility in the Heston model [14] correlates with a deterministic constant.
However, market observations clearly indicate that financial quantities are correlated in a strongly nonlinear way, correlation behaves even stochastically and unpredictably. In this work we extend the Heston model by imposing a stochastic correlation given by the Ornstein-Uhlenbeck and the Jacobi processes. By approximating non-affine terms we find the characteristic function in a closed-form which can be used for pricing purposes.
Our numerical results and experiment on calibration to market data validate that incorporating stochastic correlations improves the performance of the Heston model.

Keywords *Heston model, Stochastic Correlation process, Ornstein-Uhlenbeck process, Jacobi process, Characteristic function*

1 Introduction

The Heston model [14] is one of the most widely used affine stochastic volatility models for equity prices. Heston extended the Black and Scholes model [2] by taking into account stochastic volatility given by a Cox-Ingersoll-Ross (CIR) process[7] and found the conditional characteristic function (CF) in a closed form, from which one can compute the risk-neutral exercise probabilities appearing in the option pricing formulas. Indeed, the Heston model belongs to the class of affine diffusion processes (AD), see [9, 11]. The CF of the processes in the AD class exists and can be derived as follows: Suppose a system of (stochastic differential equations) SDEs given by

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t \tag{1.1}$$

which is said to be of *affine form,* if

$$\mu(\mathbf{X}_t) = a_0 + a_1\mathbf{X}_t \qquad \text{for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \tag{1.2}$$

$$\left(\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^\top\right)_{i,j} = (b_0)_{i,j} + (b_1)_{i,j}^\top\mathbf{X}_t \qquad \text{for } (b_0, b_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n\times n}, \tag{1.3}$$

for $i, j = 1, ..., n$. Then, the CF under the risk-neutral measure $\mathbb{Q}$ takes the form

$$\phi\left(\mathbf{u}, \mathbf{X}_t, t, T\right) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\mathbf{u}^\top\mathbf{X}_T}|\mathcal{F}_t{}^1\right] = \mathrm{e}^{A(\mathbf{u},\tau)+B(\mathbf{u},\tau)\mathbf{X}_t}. \tag{1.4}$$

Now setting $\tau := T - t$, the coefficients $A(\mathbf{u}, \tau)$ and $B(\mathbf{u}, \tau)$ in (1.4) must satisfy the following complex-valued ordinary differential equations (ODEs):

$$\frac{d}{d\tau}B(\mathbf{u}, \tau) = a_1^\top B(\mathbf{u}, \tau) + \frac{1}{2}B^\top(\mathbf{u}, \tau)b_1B(\mathbf{u}, \tau), \tag{1.5}$$

$$\frac{d}{d\tau}A(\mathbf{u}, \tau) = a_0B(\mathbf{u}, \tau) + \frac{1}{2}B^\top(\mathbf{u}, \tau)b_0B(\mathbf{u}, \tau), \tag{1.6}$$

with boundary conditions $A(\mathbf{u}, 0) = 0$ and $B(\mathbf{u}, 0) = \mathrm{i}\mathbf{u}$.

Due to the fact that in many cases the Heston model can not generate enough skews or smiles in the implied volatility as market required, especially for a short maturity, a couple of ideas have been proposed to extend the Heston model: one idea is to adapt the Heston model by allowing time-dependent parameters [16, 12, 3] or time-dependent correlations [22]; Christoffersen [6] et al. specified an additional volatility process to the pure Heston model, called the double Heston model; another way is to extend the Heston model by introducing a stochastic interest rate which is the Hybrid-Heston-Hull-White model (HHW) [13]. Summarizing, we put this work's contribution into context with respect to some other extensions for the Heston model in Table 1.

[1] We fix a probability space $(\Omega, \mathcal{F}, P)$ and an information filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$ which is assumed to satisfy the usual conditions (see e.g. [18]), and $\mathbf{X}$ is assumed to be Markov relative to $(\mathcal{F}_t)$.

The Heston model [14] extended by	
allowing time-dependent parameters	Mikhailov and Nögel [16], Elices[12], Benhamou et al.[4]
specifying a two-factor volatility	Christoffersen et al. [6]
imposing a stochastic interest rate	Grzelak and Oosterlee [13]
imposing a time-dependent correlation	Teng et al. [22]
imposing a stochastic correlation	Teng et al. [23] and this work

Table 1: Literature on the extension of the Heston model.

The reason, why we incorporate a stochastic correlation into the Heston model is given as follows: firstly, due to the fact that correlation affects skew of the implied volatility, introducing a non-constant correlation can certainly improve the calibration. Teng et al. [22] have shown that calibration of the Heston model can be improved only by allowing an appropriate local time-dependent correlations; secondly, a couple of papers (e.g. [25, 19, 20, 21]) have indicated that the correlation between financial quantities can not be a constant. Thus, the correlation in the financial market must be modelled in a nonlinear way even randomly by using a mean-reverting stochastic process like modelling volatility or interest rate. This is to say that using a stochastic correlation is more realistic than using a constant correlation. Therefore we believe that introducing a stochastic correlation will improve calibration of the Heston model better compared to other extensions.

Teng et al. [22] discuss the Heston model with stochastic correlations driven by recently developed diffusion correlation processes, as the OU process [24], the Jacobi process [25, 17] and the transformed mean-reverting processes by the tangens hyperbolicus function [20, 21]. However, the speed of pricing products by numerical approximation using Monte-Carlo simulation is not acceptable for calibration purposes.

In [13], two projection techniques have been used to derive affine approximations of the Heston model incorporating stochastic interest rates driven by Hull-White (HW) [15] and CIR [7] processes, so that calibration and pricing benefits greatly from the speed of evaluating CFs. In this work, we extend the pure Heston model by imposing a stochastic correlation given by the Ornstein-Uhlenbeck (OU) and the bounded Jacobi processes. By approximating non-affine terms we bring these extended models in the class of AD processes so that the CFs in closed-form can be found. Thus, we can calibrate the model. By comparison with other models we show that the implied volatility for our model can be better than other models fitted to the market data.

The remainder of the paper is organized as follows. The next section specializes how to impose generally a stochastic correlation into the pure Heston model. In Section 3, we investigate the approximations of non-affine terms in the Heston model extended by different stochastic correlation processes and find the corresponding CFs in closed-form. Section 4 is devoted to the analysis of the approximation error and the illustration for the advantages of our models compared to some other models. Finally, Section 5 concludes this work. And an appendix stating proofs and approximations is given.

2 Stochastic correlation in the Heston model

Heston's stochastic volatility model [14] under the risk-neutral measure is specified as

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{\nu_t}S_t\,dW_t^S, & S_0 > 0, \\ d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)\,dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, & \nu_0 > 0, \end{cases} \tag{2.1}$$

where S_t is the spot price of the underlying asset, ν_t is the volatility and the Brownian motions W_t^S and W_t^ν are correlated with a constant $\rho_{S\nu}$. Under the log-transform for the asset, i.e. $x_t = \log(S_t)$, the model is represented by

$$\begin{cases} dx_t = (r - \frac{1}{2}\nu_t)\,dt + \sqrt{\nu_t}\,dW_t^x, & x_0 = \log(S_0), \\ d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)\,dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, & \nu_0 > 0, \end{cases} \tag{2.2}$$

which is in the class of AD. The discounted CF has been found by Heston [14]. We extend the model by imposing stochastic correlation between the asset price and the volatility given by an appropriate SDE system:

$$\begin{cases} dx_t = (r - \frac{1}{2}\nu_t)\,dt + \sqrt{\nu_t}\,dW_t^x, & x_0 = \log(S_0), \\ d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, & \nu_0 > 0, \\ d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)dW_t^\rho, & \rho_0 \in [-1,1], \end{cases} \tag{2.3}$$

where

$$dW_t^x dW_t^\nu = \rho_t\,dt, \quad dW_t^x dW_t^\rho = \rho_{x\rho}\,dt, \quad dW_t^\nu dW_t^\rho = \rho_{\nu\rho}\,dt, \tag{2.4}$$

i.e. the log price process and the volatility process are set to be correlated stochastically, driven by the correlation process ρ_t which is by itself correlated with the log price process by $\rho_{x\rho}$ and with the volatility by $\rho_{\nu\rho}$, respectively.

To check conveniently the affinity, we reformulate the SDE system (2.3) with respect to the independent BMs: We first rearrange the SDE system (2.3) as

$$\begin{cases} d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}\,dW_t^\nu, \\ d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)dW_t^\rho, \\ dx_t = (r - \frac{1}{2}\nu_t)\,dt + \sqrt{\nu_t}\,dW_t^x, \end{cases} \tag{2.5}$$

which has a family of correlation matrices

$$\mathcal{C}_t = \begin{pmatrix} 1 & \rho_{\nu\rho} & \rho_t \\ \rho_{\rho\nu} & 1 & \rho_{\rho x} \\ \rho_t & \rho_{x\rho} & 1 \end{pmatrix}, \quad t \geq 0, \tag{2.6}$$

which is symmetric, namely $\rho_{\nu\rho} = \rho_{\rho\nu}$ and $\rho_{x\rho} = \rho_{\rho x}$. To simplify notation we set $\rho_1 := \rho_{\nu\rho}(\rho_{\rho\nu})$ and $\rho_2 := \rho_{x\rho}(\rho_{\rho x})$. Thus, one can perform a Cholesky-decomposition $\mathcal{C}_t = \mathcal{L}_t\mathcal{L}_t^\top$, where $\mathcal{L}_t$ is a family of lower triangular matrices given by

$$\mathcal{L}_t = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_t & \frac{\rho_2 - \rho_1\rho_t}{\sqrt{1-\rho_1^2}} & \sqrt{1 - \rho_t^2 - \left(\frac{\rho_2-\rho_1\rho_t}{\sqrt{1-\rho_1^2}}\right)^2} \end{pmatrix}, \quad t \geq 0, \tag{2.7}$$

which can be employed to reformulate the SDE system (2.5) with respect to the independent BMs $\tilde{W_t^\nu}, \tilde{W_t^\rho}$ and $\tilde{W_t^x}$ as:

$$\begin{cases} d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)\,dt + \sigma_\nu\sqrt{\nu_t}\,d\tilde{W_t^\nu}, \\ d\rho_t = a(t,\rho_t)\,dt + \rho_1 b(t,\rho_t)\,d\tilde{W_t^\nu} + \sqrt{1-\rho_1^2}b(t,\rho_t)\,d\tilde{W_t^\rho}, \\ dx_t = (r - \dfrac{1}{2}\nu_t)\,dt + \rho_t\sqrt{\nu_t}\,d\tilde{W_t^\nu} + \dfrac{\rho_2 - \rho_1\rho_t}{\sqrt{1-\rho_1^2}}\sqrt{\nu_t}\,d\tilde{W_t^\rho} \\ \qquad\qquad + \sqrt{1 - \rho_t^2 - \left(\dfrac{\rho_2 - \rho_1\rho_t}{\sqrt{1-\rho_1^2}}\right)^2}\sqrt{\nu_t}\,d\tilde{W_t^x}. \end{cases} \tag{2.8}$$

The family of symmetric instantaneous covariance matrices for $\mathbf{X_t} := [\nu_t, \rho_t, x_t]^\top$ reads

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^\top = \begin{pmatrix} \nu_t\sigma_\nu^2 & \rho_1\sigma_\nu\sqrt{\nu_t}b(t,\rho_t) & \sigma_\nu\nu_t\rho_t \\ * & b^2(t,\rho_t) & \rho_2 b(t,\rho_t)\sqrt{\nu_t} \\ * & * & \nu_t \end{pmatrix}, \quad t \geq 0. \tag{2.9}$$

Since our main aim is to impose a stochastic correlation between the asset process and the stochastic volatility process, we first assume $\rho_1 = 0$ so that the latter SDE system becomes

$$\begin{cases} d\nu_t = \kappa_\nu(\mu_\nu - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}d\tilde{W}_t^\nu, \\ d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)d\tilde{W}_t^\rho, \\ dx_t = \left(r - \frac{1}{2}\nu_t\right)dt + \rho_t\sqrt{\nu_t}d\tilde{W}_t^\nu + \rho_2\sqrt{\nu_t}d\tilde{W}_t^\rho + \sqrt{1-\rho_t^2-\rho_2^2}\sqrt{\nu_t}d\tilde{W}_t^x, \end{cases} \tag{2.10}$$

and the family of symmetric instantaneous covariance matrices reads

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^\top = \begin{pmatrix} \nu_t\sigma_\nu^2 & 0 & \sigma_\nu\nu_t\rho_t \\ * & b^2(t,\rho_t) & \rho_2 b(t,\rho_t)\sqrt{\nu_t} \\ * & * & \nu_t \end{pmatrix}, \quad t \geq 0. \tag{2.11}$$

We define the discounted characteristic function $\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)+\mathrm{i}\mathbf{u}^\top\mathbf{X}_T}|\mathcal{F}_t\right]$, whose Kolmogorov's backward equation is given by

$$\begin{aligned} \frac{\partial\phi}{\partial t} + (r - \frac{1}{2}\nu)\frac{\partial\phi}{\partial x} + \kappa_\nu(\mu_\nu - \nu)\frac{\partial\phi}{\partial\nu} + a(t,\rho_t)\frac{\partial\phi}{\partial\rho} + \frac{1}{2}\nu\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}\nu\sigma_\nu^2\frac{\partial^2\phi}{\partial\nu^2} \\ + \frac{1}{2}b^2(t,\rho)\frac{\partial^2\phi}{\partial\rho^2} + \sigma_\nu\nu_t\rho_t\frac{\partial^2\phi}{\partial\nu\partial x} + \rho_2 b(t,\rho_t)\sqrt{\nu_t}\frac{\partial^2\phi}{\partial\rho\partial x} - r\phi = 0 \end{aligned} \tag{2.12}$$

subject to the terminal condition $\phi(\mathbf{u}, \mathbf{X}_T, T, T) = \mathrm{e}^{\mathrm{i}ux_t}$. Obviously, the system (2.10) is not in the affine form. We can use appropriate approximations in order to generate an affine form. We first consider $\sigma_\nu\nu_t\rho_t$: assuming independence between ρ_t and ν_t we can straightforwardly take the following approximation

$$\sigma_\nu\nu_t\rho_t \approx \mathbb{E}\left[\sigma_\nu\nu_t\rho_t\right] = \sigma_\nu\mathbb{E}\left[\nu_t\right]\mathbb{E}\left[\rho_t\right]. \tag{2.13}$$

A better approximation could be

$$\sigma_\nu\nu_t\rho_t \approx \sigma_\nu\mathbb{E}\left[\nu_t\right]\rho_t, \tag{2.14}$$

which is justified due to the assumption $\rho_1 = 0$, because the stochasticity of the correlation process is kept. We discuss the affinity of the terms including $a(t,\rho_t)$ and $b(t,\rho_t)$ in the next section, as it will depend on the chosen stochastic correlation process.

3 Stochastic correlation processes

In this section, we apply an OU process and a Jacobi process to model stochastic correlation and discuss their merits to be a correlation process. By employing appropriate approximations for non-affine terms we find the CF in closed-form for the extended Heston model by imposing stochastic correlation.

3.1 The Ornstein-Uhlenbeck process

We first use an OU process [24] to be a stochastic process which is defined by the SDE

$$d\rho_t = \kappa_\rho(\mu_\rho - \rho_t)\, dt + \sigma_\rho\, d\tilde{W}_t^\rho. \tag{3.1}$$

Therefore, the functions $a(t, \rho_t)$ and $b(t, \rho_t)$ defined in (2.10) and (2.11) are known as $\kappa_\rho(\mu_\rho - \rho_t)$ and σ_ρ, respectively. The major drawback of using an OU process for stochastic correlation is that the process is not bounded. This is to say the generated correlations can be out of the correlation interval $[-1, 1]$; this specially occurs for a small value of κ_ρ and a large value of σ_ρ. However, due to its analytical tractability, one would like to use it for modelling correlation; e.g., Düllmann et al. [10] estimated asset correlations from stock prices or default rates by assuming correlation following an OU process.

We employ it for modelling stochastic correlations while we limit the mean value μ_ρ to be in $(-1, 1)$ and choose a relative large value of κ_ρ, a small value of σ_ρ. We name this extended Heston model as "HO" model. In the HO model, the remaining non-affine term is only $\rho_2\sigma_\rho\sqrt{\nu_t}$, see (2.11). For its approximation we use the following result [13]:

$$\rho_2\sigma_\rho\sqrt{\nu_t} \approx \rho_2\sigma_\rho\mathbb{E}\left[\sqrt{\nu_t}\right], \tag{3.2}$$

where $\mathbb{E}\left[\sqrt{\nu_t}\right]$ is given in the next proposition.

Proposition 3.1. $\mathbb{E}\left[\sqrt{\nu_t}\right]$ *can be approximated by*

$$\mathbb{E}\left[\sqrt{\nu_t}\right] \approx m + ne^{-lt}, \tag{3.3}$$

where

$$m := \sqrt{\mu_\nu - \frac{\sigma_\nu^2}{8\kappa_\nu}}, \quad n := \sqrt{\nu_0} - m, \quad l := -\log\left(n^{-1}\left(\hat{d} - m\right)\right), \tag{3.4}$$

$$\hat{d} := \sqrt{\left(\nu_0 e^{-\kappa_\nu} - \frac{\sigma_\nu^2(1-e^{-\kappa_\nu})}{4\kappa_\nu}\right) + \mu_\nu(1-e^{-\kappa_\nu}) + \frac{\sigma_\nu^2\mu_\nu(1-e^{-\kappa_\nu})^2}{8\kappa_\nu\mu_\nu + 8\kappa_\nu e^{-\kappa_\nu}(\nu_0-\mu_\nu)}}. \tag{3.5}$$

The detailed derivation and the test of quality of the approximation can be found in [13].

We start to derive the CF for the HO model, according to [11]. We first assume that the discounted CF for the HO model is of the following form:

$$\phi_{HO}\left(\mathbf{u}, \mathbf{X_t}, \tau\right) = \mathrm{e}^{-r\tau + A(u,\tau) + B(u,\tau)x_t + C(u,\tau)\rho_t + D(u,\tau)\nu_t} \tag{3.6}$$

with final conditions $A(u,0) = 0$, $B(u,0) = iu$, $C(u,0) = 0$, $D(u,0) = 0$ and $\tau := T-t$. By substituting (3.6) into (2.12) we obtain the ODEs related to the HO model given in the following lemma.

Lemma 3.1. *The functions in* (3.6) $A(u,\tau), B(u,\tau), C(u,\tau)$ *and* $D(u,\tau)$ *for the HO model satisfy the following ODE system:*

$$B'(u,\tau) = 0, \quad B(u,0) = iu, \tag{3.7}$$

$$C'(u,\tau) = \sigma_\nu \mathbb{E}[\nu_t]B(u,\tau)D(u,\tau) - \kappa_\rho C(u,\tau), \quad C(u,0) = 0, \tag{3.8}$$

$$D'(u,\tau) = \frac{1}{2}B^2(u,\tau) + \frac{1}{2}\sigma_\nu^2 D(u,\tau) - \frac{1}{2}B(u,\tau) - \kappa_\nu D(u,\tau), \quad D(u,0) = 0, \tag{3.9}$$

$$\begin{aligned} A'(u,\tau) =& (B(u,\tau)-1)r + \kappa_\nu\mu_\nu D(u,\tau) + \kappa_\rho\mu_\rho C(u,\tau) \\ &+ \frac{1}{2}\sigma_\rho^2 C^2(u,\tau) + \sigma_\rho\rho_2\mathbb{E}[\sqrt{\nu_t}]B(u,\tau)C(u,\tau), \quad A(u,0) = 0, \end{aligned} \tag{3.10}$$

Obviously, the discounted CF can be obtained as long as the closed-form solution of the latter ODE system is available.

Lemma 3.2. *The solution of the ODE system in Lemma 3.1 is given by*

$$B(u,\tau) = iu, \tag{3.11}$$

$$D(u,\tau) = \frac{\kappa_\nu - D_1}{\sigma_\nu^2} \cdot \frac{1-e^{-D_1\tau}}{1-D_2 e^{-D_1\tau}}, \tag{3.12}$$

$$A(u,\tau) = H_1(u,\tau) + \alpha H_2(u,\tau) + \beta H_3(u,\tau) + \frac{\sigma_\rho^2}{2}H_4(u,\tau), \tag{3.13}$$

$$C(u,\tau) = \frac{C_1(\mu_\nu - \nu_0)}{\kappa_\nu + \kappa_\rho - l_1} e^{(\kappa_\nu - l_1)\tau - \kappa_\nu T} + \frac{C_1(\nu_0 - \mu_\nu)}{\kappa_\nu + \kappa_\rho} e^{\kappa_\nu(\tau - T)} + \frac{C_1\mu_\nu}{\kappa_\rho} - \frac{C_1\mu_\nu}{\kappa_\rho - l_1} e^{-l_1} + C_1C_2 e^{-\kappa_\rho\tau}, \tag{3.14}$$

where m, n, *and* l *defined in* (3.4) - (3.5) *and*

$$D_1 = \sqrt{\kappa_\nu^2 + \sigma_\nu^2(u^2 + iu)}, \quad D_2 = \frac{\kappa_\nu - D_1}{\kappa_\nu + D_1}, \quad C_1 = iu\frac{\kappa_\nu - D_1}{\sigma_\nu^2}, \tag{3.15}$$

$$l_1 = -\ln\left(\frac{e^{-D_1} - D_2 e^{-D_1}}{1 - D_2 e^{-D_1}}\right), \quad \alpha = \kappa_\rho\mu_\rho + m\sigma_\rho\rho_2 ui, \quad \beta = n\sigma_\rho\rho_2 ui, \tag{3.16}$$

$$C_2 = \frac{\mu_\nu - \nu_0}{\kappa_\nu + \kappa_\rho - l_1} e^{-\kappa_\nu T} + \frac{\nu_0 - \mu_\nu}{\kappa_\nu + \kappa_\rho} e^{-\kappa_\nu T} - \frac{\mu_\nu}{\kappa_\rho} + \frac{1}{\kappa_\rho - l_1}, \tag{3.17}$$

$$H_1(u,\tau) = (iu - 1)r\tau + \frac{\kappa_\nu\mu_\nu}{\sigma_\nu^2}\left((\kappa_\nu - D_1)\tau - 2\ln\left(\frac{1 - D_2 e^{-D_1\tau}}{1 - D_2}\right)\right), \tag{3.18}$$

$$H_2(u,\tau) = \frac{C_1(\mu_\nu - \nu_0)e^{\kappa_\nu(\tau - T) - l_1\tau}}{(\kappa_\nu + \kappa_\rho - l_1)(\kappa_\nu - l_1)} + \frac{C_1(\nu_0 - \mu_\nu)e^{\kappa_\nu(\tau - T)}}{\kappa_\nu(\kappa_\nu + \kappa_\rho)} + \frac{\mu_\nu\tau C_1}{\kappa_\rho} + \frac{\mu_\nu C_1 e^{-l_1\tau}}{(\kappa_\rho - l_1)l_1} - \frac{C_1C_2 e^{-\kappa_\rho\tau}}{\kappa_\rho} + H_{2c}, \tag{3.19}$$

$$H_3(u,\tau) = \frac{C_1(\mu_\nu - \nu_0)e^{\tau(\kappa_\nu + l - l_1) - T(\kappa_\nu + l)}}{(\kappa_\nu + \kappa_\rho - l_1)(\kappa_\nu + l - l_1)} + \frac{C_1(\nu_0 - \mu_\nu)e^{(\tau - T)(l + \kappa_\nu)}}{(l + \kappa_\nu)(\kappa_\nu + \kappa_\rho)} + \frac{\mu_\nu C_1 e^{l(\tau - T)}}{\kappa_\rho l} - \frac{\mu_\nu C_1 e^{\tau(l - l_1) - lT}}{(\kappa_\rho - l_1)(l - l_1)} + \frac{C_1C_2 e^{\tau(l - \kappa_\rho) - lT}}{l - \kappa_\rho} + H_{3c}, \tag{3.20}$$

$$H_{2c} = \frac{C_1(\nu_0 - \mu_\nu)e^{-\kappa_\nu T}}{(\kappa_\nu + \kappa_\rho - l_1)(\kappa_\nu - l_1)} - \frac{C_1(\nu_0 - \mu_\nu)e^{-\kappa_\nu T}}{\kappa_\nu(\kappa_\nu + \kappa_\rho)} - \frac{\mu_\nu C_1}{(\kappa_\rho - l_1)l_1} + \frac{C_1C_2}{\kappa_\rho}, \tag{3.21}$$

$$H_{3c} = \frac{C_1(\mu_\nu - \nu_0)e^{-T(\kappa_\nu + l)}}{(\kappa_\nu + \kappa_\rho - l_1)(\kappa_\nu + l - l_1)} + \frac{C_1(\nu_0 - \mu_\nu)e^{-T(l + \kappa_\nu)}}{(l + \kappa_\nu)(\kappa_\nu + \kappa_\rho)} + \frac{\mu_\nu C_1 e^{-lT}}{\kappa_\rho l} - \frac{\mu_\nu C_1 e^{-lT}}{(\kappa_\rho - l_1)(l - l_1)} + \frac{C_1C_2 e^{-lT}}{l - \kappa_\rho}, \tag{3.22}$$

$$
\begin{aligned}
H_4(u,\tau) = {} & H_{4c_1} e^{2\kappa_\nu(\tau-T)} + H_{4c_4} e^{-\tau l_1} + H_{4c_5} e^{(-l_1-\kappa_\rho)\tau} + H_{4c_9} e^{\tau(\kappa_\nu-\kappa_\rho)-\kappa_\nu T} \\
& + H_{4c_2} e^{2\tau(\kappa_\nu-l_1)-2\kappa_\nu T} + H_{4c_3} e^{\tau(2\kappa_\nu-l_1)-2\kappa_\nu T} + H_{4c_{11}} e^{\tau(\kappa_\nu-\kappa_\rho-l_1)-\kappa_\nu T} \\
& + H_{4c_{12}} e^{\tau(\kappa_\nu-l_1)-\kappa_\nu T} + H_{4c_{13}} e^{\tau(\kappa_\nu-2l_1)-\kappa_\nu T} + H_{4c_6} e^{\tau(-2\kappa_\rho\tau} + H_{4c_7} e^{-\kappa_\rho\tau} \\
& + H_{4c_8} e^{-2l_1\tau} + H_{4c_{10}} e^{\tau(\kappa_\nu-l_1)-\kappa_\nu T} + H_{4c_{14}} e^{\kappa_\rho(\tau-T)} + \frac{C_1^2\mu_\nu^3\tau}{\kappa_\rho^2} + H_{4c},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
H_{4c} = {} & (H_{4c_1} + H_{4c_2} + H_{4c_3}) e^{-2\kappa_\nu T} + H_{4c_4} + H_{4c_5} + H_{4c_6} + H_{4c_7} + H_{4c_8} \\
& + (H_{4c_9} + H_{4c_{10}} + H_{4c_{11}} + H_{4c_{12}} + H_{4c_{13}} + H_{4c_{14}}) e^{-\kappa_\nu T},
\end{aligned}
\tag{3.24}
$$

with

$$
H_{4c_1} := \frac{C_1^2(\nu_0-\mu_\nu)^2}{2\kappa_\nu(\kappa_\nu+\kappa_\rho)^2}, \quad H_{4c_2} := \frac{C_1^2(\nu_0-\mu_\nu)^2}{2(2\kappa_\nu+\kappa_\rho-l_1)^2(\kappa_\nu-l_1)}, \tag{3.25}
$$

$$
H_{4c_3} := \frac{2C_1^2(\nu_0-\mu_\nu)^2}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu+\kappa_\rho)(l_1-2\kappa_\nu)}, \quad H_{4c_4} := \frac{2C_1^2\mu_\nu^2}{\kappa_\nu l_1(\kappa_\rho-l_1)}, \tag{3.26}
$$

$$
H_{4c_5} := \frac{2\mu_\nu C_1^2 C_2}{\kappa_\rho^2-l_1^2}, \quad H_{4c_6} := -\frac{1}{2}\frac{C_1^2C_2^2}{\kappa_\rho}, \quad H_{4c_7} := -\frac{2\mu_\nu C_1^2 C_2}{\kappa_\rho^2}, \tag{3.27}
$$

$$
H_{4c_8} := -\frac{1}{2}\frac{\mu_\nu^2 C_1^2}{l_1(\kappa_\rho-l_1)^2}, \quad H_{4c_9} := \frac{2(\nu_0-\mu_\nu)C_1^2C_2}{(\kappa_\nu+\kappa_\rho)(\kappa_\nu-\kappa_\rho)}, \quad H_{4c_{14}} := \frac{2C_1^2(\nu_0\mu_\nu-\mu_\nu^2)}{\kappa_\nu\kappa_\rho(\kappa_\nu+\kappa_\rho)^2}, \tag{3.28}
$$

$$
H_{4c_{10}} := \frac{2C_1^2(\mu_\nu^2-\nu_0\mu_\nu)}{(\kappa_\nu+\kappa_\rho)(\kappa_\nu-l_1)(\kappa_\rho-l_1)}, \quad H_{4c_{11}} := \frac{2(\mu_\nu-\nu_0)C_1^2C_2}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu-\kappa_\rho-l_1)}, \tag{3.29}
$$

$$
H_{4c_{12}} := \frac{2C_1^2(\mu_\nu^2-\nu_0\mu_\nu)}{\kappa_\rho(\kappa_\nu-l_1)(\kappa_\nu+\kappa_\rho-l_1)}, \quad H_{4c_{13}} := \frac{2C_1^2(\nu_0\mu_\nu-\mu_\nu^2)}{(\kappa_\rho-l_1)(\kappa_\nu-2l)(\kappa_\nu+\kappa_\rho-l_1)}. \tag{3.30}
$$

The proof can be found in appendix A.1.

3.2 The bounded Jacobi process

In this section, we consider modelling stochastic correlation using the bounded Jacobi process

$$
d\rho_t = \kappa_\rho(\mu_\rho-\rho_t)\,dt + \sigma_\rho\sqrt{1-\rho_t^2}\,d\tilde{W}_t^\rho, \tag{3.31}
$$

where the functions $a(t, \rho_t)$ and $b(t, \rho_t)$ defined in (2.10) and (2.11) are $\kappa_\rho(\mu_\rho - \rho_t)$ and $\sigma_\rho\sqrt{1-\rho_t^2}$, respectively. Van Emmerich [25] proved that the boundaries -1 and 1 of (3.31) are not attractive and unattainable with the following restriction of the parameter range

$$\kappa_\rho > \frac{\sigma_\rho^2}{1 \pm \mu_\rho}. \tag{3.32}$$

For the detailed derivation we refer to [25]. We call this extended Heston model as "HJ" model. Similar to the HO model, from (2.11) we observe that the non-affine terms in the HJ model are $b^2(t, \rho_t)$ and $\rho_2 b(t, \rho_t)\sqrt{\nu_t}$, as

$$b^2(t, \rho_t) = \sigma_\rho^2(1 - \rho_t^2), \tag{3.33}$$

$$\rho_2 b(t, \rho_t)\sqrt{\nu_t} = \rho_2\sigma_\rho\sqrt{1-\rho_t^2}\sqrt{\nu_t}. \tag{3.34}$$

We try to find appropriate approximations for (3.33) and (3.34) which are affine. We consider first (3.33) which could be approximated by

$$\sigma_\rho^2(1 - \mathbb{E}[\rho_t^2]), \tag{3.35}$$

where $\mathbb{E}[\rho_t^2])$ is given by [26]

$$\begin{aligned}\mathbb{E}[\rho_t^2] =& \frac{1}{\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2} \mathrm{e}^{-t(\sigma_\rho^3 + 2\kappa_\rho)} \Big((\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2)\rho_0^2 \\ &+ 2\mu_\rho\kappa_\rho\rho_0(\sigma_\rho^2 + 2\kappa_\rho)(\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)} - 1) + \sigma_\rho^2(\sigma_\rho^2 + \kappa_\rho)(\mathrm{e}^{t(\sigma_\rho^2+2\kappa_\rho)} - 1) \\ &- 2\mu_\rho^2\kappa_\rho\big(\kappa_\rho(2\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)} - \mathrm{e}^{t(\sigma_\rho^2+2\kappa_\rho)} - 1) - \sigma_\rho^2\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)}(\mathrm{e}^{t\kappa_\rho} - 1)\big)\Big).\end{aligned} \tag{3.36}$$

We see that the latter equation is rather complicated and not convenient for further calculation. Therefore, we introduce the following approximation.

Proposition 3.2. $\mathbb{E}[\rho_t^2]$ *can be approximated by*

$$f_2(t) := \mathbb{E}[\rho_t^2] \approx e^{-m_2 t} + b_2 e^{-n_2 t} + a_2, \tag{3.37}$$

where

$$a_2 = \frac{(\sigma_\rho^2 + \kappa_\rho)(\sigma_\rho^2 + 2\kappa_\rho\mu_\rho^2)}{\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2}, \quad b_2 = \rho_0^2 - a_2 - 1, \tag{3.38}$$

$$m_2 = -2\log\left(\gamma_1 - b_2 e^{-\frac{n_2}{2}}\right), \quad n_2 = -2\log\left(\frac{b_2\gamma_1 - \sqrt{b_1^2\gamma_1^2 - \gamma_2\gamma_3}}{\gamma_2}\right), \tag{3.39}$$

with

$$\gamma_1 := f_2(0.5) - a_2, \quad \gamma_2 := b_2 + b_1^2, \quad \gamma_3 := \gamma_1^2 + a_2 - f_2(1). \tag{3.40}$$

The proof and the test of quality of the approximation can be found in Appendix B.2.

Next, we investigate the approximation for the other non-affine term (3.34). Due to $\rho_1 = 0$ we propose to approximate (3.34) using its expectation

$$\rho_2 \sigma_\rho \sqrt{\nu_t} \approx \rho_2 \sigma_\rho \mathbb{E}[\sqrt{1-\rho_t^2}]\mathbb{E}[\sqrt{\nu_t}]. \tag{3.41}$$

$\mathbb{E}[\sqrt{\nu_t}]$ is already known, see Prop. 3.1. The remaining task is to find a formula for $\mathbb{E}[\sqrt{1-\rho_t^2}]$; for this we apply the delta method which has also been used to find the approximation for $\mathbb{E}[\sqrt{\nu_t}]$ in [13]. Say that $\psi(X)$ is sufficiently smooth where the first two moments of X exist, then with the aid of Taylor expansion we have

$$\psi(X) \approx \psi(\mathbb{E}[X]) + (X - \mathbb{E}[X])\frac{\partial \psi}{\partial X}\mathbb{E}[X], \tag{3.42}$$

such that the variance of $\psi(X)$ can be given by

$$\mathbb{V}[\psi(X)] \approx \mathbb{V}\left[\psi(\mathbb{E}[X]) + (X - \mathbb{E}[X])\frac{\partial \psi}{\partial X}\mathbb{E}[X]\right] = \left(\frac{\partial \psi}{\partial X}\mathbb{E}[X]\right)^2 \mathbb{V}[X]. \tag{3.43}$$

Now, setting $\psi(\rho_t) = \sqrt{1-\rho_t^2}$ we obtain

$$\mathbb{V}\left[\sqrt{1-\rho_t^2}\right] = \frac{\mathbb{E}[\rho_t]^2}{1-\mathbb{E}[\rho_t]^2}\mathbb{V}[\rho_t]. \tag{3.44}$$

On the other hand, from the definition of the variance we also get

$$\mathbb{V}\left[\sqrt{1-\rho_t^2}\right] = \mathbb{E}[1-\rho_t^2] - \mathbb{E}\left[\sqrt{1-\rho_t^2}\right]^2. \tag{3.45}$$

Directly, from the latter two equations we obtain finally

$$\mathbb{E}\left[\sqrt{1-\rho_t^2}\right] = \sqrt{\mathbb{E}[1-\rho_t^2] - \frac{\mathbb{E}[\rho_t]^2}{1-\mathbb{E}[\rho_t]^2}\mathbb{V}[\rho_t]} = \sqrt{1 - \frac{\mathbb{E}[\rho_t^2] - \mathbb{E}[\rho_t]^4}{1-\mathbb{E}[\rho_t]^2}}, \tag{3.46}$$

where $\mathbb{E}[\rho_t^2]$ is given in (3.36) and its approximation in (3.37). Besides, we know $\mathbb{E}[\rho_t] = \mu_\rho + (\rho_0 - \mu_\rho)\mathrm{e}^{-\kappa_\rho t}$ for the correlation process ρ_t defined in (3.31). In the same way as above we try to find a suitable approximation for (3.46) which has a more convenient form.

Proposition 3.3. $\mathbb{E}[\sqrt{1-\rho_t^2}]$ *can be approximated by*

$$f_3(t) := \mathbb{E}[\sqrt{1-\rho_t^2}] \approx e^{-m_3 t} + b_3 e^{-n_3 t} + a_3, \tag{3.47}$$

with

$$a_3 = \sqrt{1 - \frac{(\sigma_\rho^2+\kappa_\rho)(\sigma_\rho^2+2\kappa_\rho\mu_\rho^2) - \mu_\rho^4(\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2)}{(1-\mu_\rho^2)(\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2)}}, \quad b_3 = \sqrt{1-\rho_0^2} - a_3 - 1 \tag{3.48}$$

$$m_3 = -2\log\left(\eta_1 - b_3 e^{-\frac{n_3}{2}}\right), \quad n_3 = -2\log\left(\frac{b_3\eta_1 - \sqrt{b_3^2\eta_1^2 - \eta_2\eta_3}}{\eta_2}\right), \tag{3.49}$$

with

$$\eta_1 := f_3(0.5) - a_3, \quad \eta_2 := b_3 + b_3^2, \quad \eta_3 := \eta_1^2 + a_3 - f_3(1). \tag{3.50}$$

We show the proof and measure quality of the approximation in Appendix B.3. Again, we assume that the discounted CF for the HJ model to be of the following form:

$$\phi_{HJ}(\mathbf{u}, \mathbf{X_t}, \tau) = \mathrm{e}^{-r\tau + \tilde{A}(u,\tau) + \tilde{B}(u,\tau)x_t + \tilde{C}(u,\tau)\rho_t + \tilde{D}(u,\tau)\nu_t} \tag{3.51}$$

with final conditions $\tilde{A}(u,0) = 0$, $\tilde{B}(u,0) = iu$, $\tilde{C}(u,0) = 0$, $\tilde{D}(u,0) = 0$ and $\tau := T-t$. By substituting (3.51) into (2.12) we can obtain a similar ODEs as in Lemma 3.1.

Lemma 3.3. *The functions in* (3.51) $\tilde{A}(u,\tau), \tilde{B}(u,\tau), \tilde{C}(u,\tau)$ *and* $\tilde{D}(u,\tau)$ *for the HJ model satisfy the following ODE system:*

$$\tilde{B}'(u,\tau) = 0, \quad \tilde{B}(u,0) = iu, \tag{3.52}$$

$$\tilde{C}'(u,\tau) = \sigma_\nu \mathbb{E}[\nu_t]\tilde{B}(u,\tau)\tilde{D}(u,\tau) - \kappa_\rho \tilde{C}(u,\tau), \quad \tilde{C}(u,0) = 0, \tag{3.53}$$

$$\tilde{D}'(u,\tau) = \frac{1}{2}\tilde{B}^2(u,\tau) + \frac{1}{2}\sigma_\nu^2\tilde{D}(u,\tau) - \frac{1}{2}\tilde{B}(u,\tau) - \kappa_\nu\tilde{D}(u,\tau), \quad \tilde{D}(u,0) = 0, \tag{3.54}$$

$$\begin{aligned}\tilde{A}'(u,\tau) =& (\tilde{B}(u,\tau)-1)r + \kappa_\nu\mu_\nu\tilde{D}(u,\tau) + \kappa_\rho\mu_\rho\tilde{C}(u,\tau) + \frac{1}{2}\sigma_\rho^2\mathbb{E}[1-\rho_t^2]\tilde{C}^2(u,\tau) \\ &+ \sigma_\rho\rho_2\mathbb{E}[\sqrt{\nu_t}]\mathbb{E}[\sqrt{1-\rho_t^2}]\tilde{B}(u,\tau)\tilde{C}(u,\tau), \quad A(u,0) = 0.\end{aligned} \tag{3.55}$$

We observe that there is only a difference between the ODEs in Lemma 3.1 and 3.3 in $A(u,\tau)$ because of the distinct correlation processes used. This also means that the solutions of $\tilde{B}(u,\tau)$, $\tilde{C}(u,\tau)$ and $\tilde{D}(u,\tau)$ coincide with $B(u,\tau)$, $C(u,\tau)$ and $D(u,\tau)$ in the HO model. Therefore we only need to calculate (3.55) to gain the discounted CF for the HJ model (3.51). We state our result in the following lemma.

Lemma 3.4. *The solutions of* $\tilde{B}(u,\tau)$, $\tilde{C}(u,\tau)$, $\tilde{D}(u,\tau)$ *are respectively equal to* (3.11), (3.14), (3.12), *and*

$$\begin{aligned} A(u,\tau) =& \tilde{H}_1(u,\tau) + (\kappa_\rho\mu_\rho + a_3 m\zeta)\tilde{H}_2(u,\tau) + a_3 n\zeta\tilde{H}_3(u,\tau,l) + b_3 m\zeta\tilde{H}_3(u,\tau,n_3) \\ &+ m\zeta\tilde{H}_3(u,\tau,m_3) + b_3 n\zeta\tilde{H}_3(u,\tau,(l+n_3)) + n\zeta\tilde{H}_3(u,\tau,(l+m_3)) \\ &+ \frac{\sigma_\rho^2}{2}(1-a_2)\tilde{H}_4(u,\tau) - \frac{\sigma_\rho^2}{2}\tilde{H}(u,\tau,m_2) - \frac{b_2\sigma_\rho^2}{2}\tilde{H}(u,\tau,n_2), \end{aligned} \tag{3.56}$$

where $\zeta = \sigma_\rho\rho_2 ui$, $\tilde{H}_1(u,\tau) = H_1(u,\tau)$ (3.18), $\tilde{H}_2(u,\tau) = H_2(u,\tau)$ (3.19), $\tilde{H}_4(u,\tau) = H_4(u,\tau)$ (3.23),

$$\begin{aligned} \tilde{H}_3(u,\tau,y) =& \frac{C_1(\mu_\nu-\nu_0)e^{\tau(\kappa_\nu+y-l_1)-T(\kappa_\nu+y)}}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu+y-l_1)} + \frac{C_1(\nu_0-\mu_\nu)e^{(\tau-T)(y+\kappa_\nu)}}{(y+\kappa_\nu)(\kappa_\nu+\kappa_\rho)} \\ &+ \frac{\mu_\nu C_1 e^{y(\tau-T)}}{\kappa_\rho y} - \frac{\mu_\nu C_1 e^{\tau(y-l_1)-yT}}{(\kappa_\rho-l_1)(y-l_1)} + \frac{C_1 C_2 e^{\tau(y-\kappa_\rho)-yT}}{y-\kappa_\rho} + H_{3c}, \end{aligned} \tag{3.57}$$

$$\begin{aligned} \tilde{H}_{3c} =& \frac{C_1(\mu_\nu-\nu_0)e^{-T(\kappa_\nu+y)}}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu+y-l_1)} + \frac{C_1(\nu_0-\mu_\nu)e^{-T(y+\kappa_\nu)}}{(y+\kappa_\nu)(\kappa_\nu+\kappa_\rho)} + \frac{\mu_\nu C_1 e^{-yT}}{\kappa_\rho y} \\ &- \frac{\mu_\nu C_1 e^{-yT}}{(\kappa_\rho-l_1)(y-l_1)} + \frac{C_1 C_2 e^{-yT}}{y-\kappa_\rho}, \end{aligned} \tag{3.58}$$

$$\begin{aligned} \tilde{H}(u,\tau,y) =& I_1 e^{(y+2\kappa_\nu-l_1)\tau-(y+2\kappa_\nu)T} + I_2 e^{(y+\kappa_\nu-l_1)\tau-(y+\kappa_\nu)T} + I_3 e^{(y+\kappa_\nu-2l_1)\tau-(y+\kappa_\nu)T} \\ &+ I_4 e^{(y+\kappa_\nu)(\tau-T)} + I_5 e^{(y+\kappa_\nu-l_1)\tau-(y+\kappa_\nu)T} + I_6 e^{(y-2l_1)\tau-yT} + I_7 e^{(y+2\kappa_\nu-2l_1)\tau-(y+2\kappa_\nu)T} \\ &+ I_8 e^{(y+2\kappa_\nu)(\tau-T)} + I_9 e^{(\kappa_\nu-\kappa_\rho+y-l_1)\tau-(y+\kappa_\nu)T} + I_{10} e^{(\kappa_\nu-\kappa_\rho+y)\tau-(y+\kappa_\nu)T} \\ &+ I_{11} e^{(y-2\kappa_\rho)\tau-yT} + I_{12} e^{(y-\kappa_\rho)\tau-yT} + I_{13} e^{y(\tau-T)} + I_{14} e^{(y-l_1)\tau-yT} + I_{15} e^{(y-l_1-\kappa_\rho)\tau-yT} + \tilde{H}_c, \end{aligned}$$

$$\begin{aligned} \tilde{H}_c =& (I_1+I_7+I_8)e^{-(y+2\kappa_\nu)T} + (I_2+I_3+I_4+I_5+I_9+I_{10})e^{-(y+\kappa_\nu)T} \\ &+ (I_6+I_{11}+I_{12}+I_{13}+I_{14}+I_{15})e^{-yT}, \end{aligned} \tag{3.59}$$

with

$$I_1 = \frac{-2C_1^2(\nu_0-\mu_\nu)^2}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu+\kappa_\rho)(y+2\kappa_\nu-l_1)}, \quad I_2 = \frac{2C_1^2(\mu_\nu^2-\mu_\nu\nu_0)}{\kappa_\rho(\kappa_\nu+\kappa_\rho-l_1)(y+\kappa_\nu-l_1)},$$
$$I_3 = \frac{2C_1^2(\mu_\nu\nu_0-\mu_\nu^2)}{(\kappa_\rho-l_1)(\kappa_\nu+\kappa_\rho-l_1)(y+\kappa_\nu-2l_1)}, \quad I_4 = \frac{2C_1^2(\mu_\nu\nu_0-\mu_\nu^2)}{\kappa_\rho(\kappa_\rho+\kappa_\nu)(\kappa_\nu+y)},$$
$$I_5 = \frac{2C_1^2(\mu_\nu^2-\mu_\nu\nu_0)}{(\kappa_\rho-l_1)(\kappa_\nu+\kappa_\rho)(y+\kappa_\nu-l_1)}, \quad I_6 = \frac{C_1^2\mu_\nu^2}{(\kappa_\rho-l_1)^2(y-2l_1)},$$
$$I_7 = \frac{C_1^2(\nu_0-\mu_\nu)^2}{(\kappa_\nu+\kappa_\rho-l_1)^2(y+2\kappa_\nu-2l_1)}, \quad I_8 = \frac{C_1^2(\nu_0+\mu_\nu)^2}{(\kappa_\rho+\kappa_\nu)^2(2\kappa_\nu+y)},$$
$$I_9 = \frac{2C_1^2C_2(\mu_\nu-\nu_0)}{(\kappa_\nu+\kappa_\rho-l_1)(\kappa_\nu-\kappa_\rho+y-l_1)}, \quad I_{10} = \frac{2C_1^2C_2(\nu_0-\mu_\nu)}{(\kappa_\nu+\kappa_\rho)(\kappa_\nu-\kappa_\rho+y)},$$
$$I_{11} = \frac{C_1^2C_2^2}{y-2\kappa_\rho}, \quad I_{12} = \frac{2C_1^2C_2\mu_\nu}{y\kappa_\rho-\kappa_\rho^2}, \quad I_{13} = \frac{C_1^2\mu_\nu^2}{y\kappa_\rho^2},$$
$$I_{14} = \frac{-2C_1^2\mu_\nu^2}{\kappa_\rho(y\kappa_\rho-l_1\kappa_\rho-yl_1+l_1^2)}, \quad I_{15} = \frac{-2C_1^2C_2\mu_\nu}{(\kappa_\rho-l_1)(y-\kappa_\rho-l_1)}.$$

C_1, l_1 *and* C_2 *are respectively located in* (3.15), (3.16) *and* (3.17).

The proof can be found in appendix A.2.

We have now obtained the explicit CFs for both models. One can thus do fast pricing by inverting the CFs directly using numerical integration, e.g., see [5, 8] for Fourier methods.

4 Numerical experiments

In this section, we conduct some numerical experiments to firstly justify the proposed approximations of non-affine terms and secondly to confirm our statement that imposing a stochastic correlation into the Heston model can improve the performance of the Heston model.

4.1 Approximation error

We compare the implied volatilities for the extended Heston model (HO and HJ) to the volatilities implied by performing a Monte-Carlo simulation as the benchmark. We define the approximation error as absolute difference between them. For a Monte-Carlo simulation of the extended Heston with stochastic correlation we use the method proposed in [23]. The basic idea is: first to discretize the volatility process ν_t by employing quadratic-exponential scheme [1]; however, instead of using the way proposed by Andersen [1] to discretize the log-price process, one can directly use the Euler or Milstein scheme for the log-price process in (2.8) or (2.10). For the motivation and the merits of this simple discretization we refer to [23].

For the Monte-Carlo simulation using the OU process, in order to ensure that the generated correlations lie in the interval $(-1, 1)$, as mentioned before, we choose values of μ_ρ and ρ_0 from $(-1, 1)$ and a large value of κ_ρ, a small value of σ_ρ. For using the bounded Jacobi process we only need to take care of the condition (3.32). We consider a Call-option ($S_0 = 100$) for the maturity of 5 years and present our results in Table 2 and 3, where $20T$ steps and 10^5 paths are used for the Monte-Carlo simulation; the implied volatilities and errors are expressed as percentage. We consider first Table 2 where σ_ρ is set to be 0.1. From the values of error we see that the approximations in both models give highly accurate results. Besides, we observe that the values of implied volatilities are the same for the HO and HJ model; there is no significant difference by varying ρ_2. This observation can be explained as follows: the OU process and the bounded Jacobi process are both mean-reverting processes; more exactly, they have a same structure for the drift. When the value of σ_ρ is so small that the random part in correlation process will play a minor role, one obtains thus same implied volatilities for using the OU and the bounded Jacobi process. Similarly, if the correlation process is not so random for a small value σ_ρ, the effect of ρ_2 will be rather small. In Table 3, we increase the value of σ_ρ to be 0.18, the mentioned differences between using the HO and HJ model, or for varying ρ_2 can be seen. The error values in this table showed again that the approximations give a rather accurate result. For the Monte-Carlo simulation we remark: while choosing the parameters one needs to pay attention to keep the value inside of the square root to be positive, see (2.10).

Model		HO			HJ		
ρ_2	Strike	MC Imp. vol.	Approx	Err.	MC Imp. vol.	Approx.	Err.
-0.2	40	19.45 (0.16)	19.12	0.33	19.38 (0.16)	19.12	0.26
	80	17.26 (0.20)	17.46	0.20	17.22 (0.20)	17.46	0.24
	100	16.58 (0.23)	16.83	0.25	16.61 (0.23)	16.83	0.22
	120	16.28 (0.25)	16.26	0.02	16.27 (0.25)	16.26	0.01
	160	15.08 (0.30)	15.17	0.09	15.33 (0.30)	15.17	0.16
0	40	19.18 (0.16)	19.13	0.05	19.38 (0.16)	19.13	0.25
	80	17.32 (0.20)	17.46	0.14	17.27 (0.20)	17.46	0.19
	100	16.65 (0.23)	16.83	0.18	16.71 (0.23)	16.83	0.12
	120	16.16 (0.25)	16.26	0.10	16.06 (0.25)	16.26	0.20
	160	15.23 (0.30)	15.17	0.06	15.22 (0.30)	15.17	0.05
0.4	40	19.45 (0.16)	19.14	0.31	19.31 (0.16)	19.14	0.17
	80	17.30 (0.20)	17.46	0.16	17.25 (0.20)	17.46	0.20
	100	16.59 (0.23)	16.82	0.24	16.59 (0.23)	16.82	0.24
	120	16.22 (0.25)	16.25	0.03	16.10 (0.25)	16.25	0.15
	160	15.57 (0.30)	15.18	0.39	15.52 (0.30)	15.18	0.35

Table 2: The other parameters are assumed as: $\nu_0 = 0.02$, $\kappa_\nu = 2.1$, $\mu_\nu = 0.03$, $\sigma_\nu = 0.2$, $\rho_0 = -0.4$, $\kappa_\rho = 3.4$, $\mu_\rho = -0.6, \sigma_\rho = 0.1$, the numbers in round brackets represent the standard deviations.

4.2 Calibration to the market data

In order to recognize the performance of our models in a calibration setting, we compare the calibration using the Heston model extended with stochastic correlation to the calibrations using the pure Heston model and the double Heston model. For the market data, we choose Put-options on the Nikk300 index on December 31, 2012, which is representative for the skew and patterns observed. Since our aim is a comparison of our models to the pure Heston model [14] and the double Heston model [6], we thus just use the standard optimization methods: we fit the prices computed by the different models to the market observed prices for several strikes T_i and maturities K_j; one can obtain the parameter estimates by minimizing, e.g., mean squared error (MSE)

$$\frac{1}{N}\sum_{i,j} w_{ij}(P^{Mkt}(T_i, K_j) - P^{Mod}(T_i, K_j))^2, \tag{4.1}$$

Model		HO			HJ		
ρ_2	Strike	MC Imp. vol.	Approx	Err.	MC Imp. vol.	Approx.	Err.
-0.2	40	19.24 (0.16)	19.51	0.27	19.27 (0.16)	19.02	0.25
	80	17.38 (0.20)	17.39	0.01	17.37 (0.20)	17.42	0.05
	100	16.82 (0.23)	16.86	0.04	16.75 (0.23)	16.84	0.08
	120	16.05 (0.25)	16.27	0.22	16.18 (0.25)	16.31	0.13
	160	15.31 (0.30)	15.35	0.04	15.16 (0.30)	15.35	0.19
0	40	19.29 (0.16)	19.72	0.43	19.25 (0.16)	19.03	0.22
	80	17.34 (0.20)	17.38	0.04	17.24 (0.20)	17.42	0.18
	100	16.70 (0.22)	16.86	0.16	16.71 (0.22)	16.83	0.12
	120	16.26 (0.25)	16.25	0.01	16.14 (0.25)	16.30	0.16
	160	15.22 (0.30)	15.37	0.15	15.41 (0.30)	15.36	0.05
0.2	40	19.36 (0.16)	20.00	0.64	19.33 (0.16)	19.04	0.29
	80	17.35 (0.20)	17.37	0.02	17.31 (0.20)	17.42	0.11
	100	16.61 (0.23)	16.86	0.25	16.79 (0.23)	16.82	0.03
	120	16.36 (0.25)	16.22	0.14	16.07 (0.25)	16.30	0.22
	160	15.63 (0.30)	15.39	0.24	15.46 (0.30)	15.36	0.10

Table 3: The other parameters are assumed as: $\nu_0 = 0.02$, $\kappa_\nu = 2.1$, $\mu_\nu = 0.03$, $\sigma_\nu = 0.2$, $\rho_0 = -0.4$, $\kappa_\rho = 3.5$, $\mu_\rho = -0.55, \sigma_\rho = 0.18$, the numbers in round brackets represent the standard deviations.

with the market price $P^{Mkt}(T_i, K_j)$ and the corresponding model price $P^{Mod}(T_i, K_j)$; w_{ij} is an optional weight.

We report our results in Table 4, where ν_0^k, κ_ν^k, μ_ν^k, σ_ν^k, σ_ν^k are parameters for the two stochastic volatilities in the double heston model, $k = 1, 2$. We see that the

Pure	ν_0	κ_ν	μ_ν	σ_ν	ρ			MSE			
Heston	0.05	4.13	0.05	0.39	-0.47			2.3×10^{-2}			
Double	ν_0^1	κ_ν^1	μ_ν^1	σ_ν^1	ρ^1	ν_0^1	κ_ρ^2	μ_ρ^2	σ_ρ^2	ρ^2	MSE
Heston	0.05	6.36	0.02	0.49	-0.23	0.01	5.69	0.03	0.62	-0.44	1.3×10^{-2}
Heston	ν_0	κ_ν	μ_ν	σ_ν	ρ_0	κ_ρ	μ_ρ	σ_ρ	$\rho_{x\rho}$	MSE	
OU	0.06	3.32	0.07	2.02	-0.01	2.12	-0.31	0.33	-0.88	6.7×10^{-3}	
Heston	ν_0	κ_ν	μ_ν	σ_ν	ρ_0	κ_ρ	μ_ρ	σ_ρ	$\rho_{x\rho}$	MSE	
Jacobi	0.05	0.75	0.07	0.50	-0	2.72	-0.17	0.02	-0.91	5.4×10^{-3}	

Table 4: Estimated model parameters for the Nikk300 index on December 31, 2012.

MSE values for the Heston model with stochastic correlation is smaller than the pure Heston model and the double Heston model.

To illustrate more clearly, we define the error as absolute value of the difference between the implied market volatilities and the model implied volatility, namely

$$Error := |Vol^{Mkt}(T_i, K_j) - Vol^{Mod}(T_i, K_j)|. \tag{4.2}$$

Then, we compare the errors for these models in Figure 1 for relatively short maturities $T = 30, 90, 180, 360$ days and in Figure 2 for relative long maturities $T = 2, 3, 4, 5$ years.

We observe, for all maturities, that the Heston model extended by incorporating stochastic correlation (in the both cases HO and HJ) can be better fitted to real market data not only than the pure Heston model but also than the double Heston model, although the extended Heston model with stochastic correlation has one parameter less than the double Heston model. This proves that introducing a stochastic correlation can significantly improve calibration. About how each parameter of the stochastic correlation process effect the implied volatilities we refer to [23].

5 Conclusion

In this article we have presented how to impose a stochastic correlation generally into the pure Heston model. We have considered the HO and HJ model by using the OU process and the bounded Jacobi process to model stochastic correlation. By approximating appropriately non-affine terms we bring the extended model into the class of AD processes. Thus we find the CF in closed-form, which can be employed for fast fast pricing and calibration purposes. The error of approximations has been analyzed by comparison to the Monte-Carlo simulation. The experiments on the calibration to real market data has shown that introducing a stochastic correlation can not only improve significantly the performance of the pure Heston model, but also be better than the double Heston model. The great importance of modelling financial correlation as a stochastic process has thus been validated.

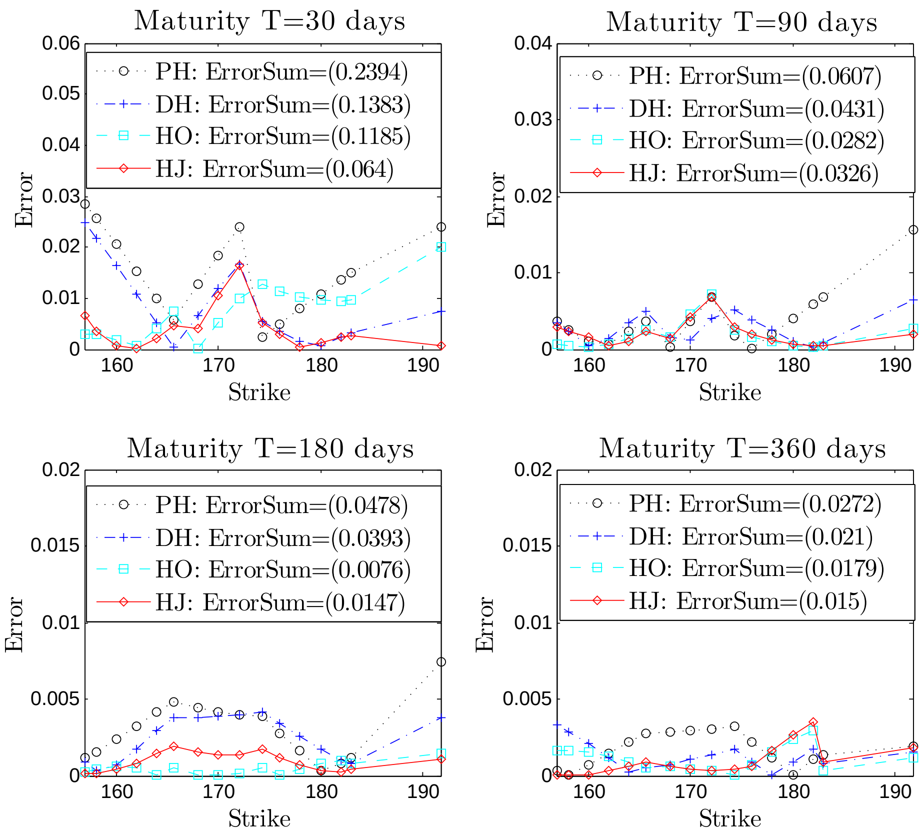

Figure 1: Using the Nikk300 index on December 31, 2012 where spot price is 174.3, and for some short maturities $T = 30, 90, 180, 360$ days, the errors (defined as absolute value of the differences between the implied market volatilities and the implied volatilities for the models) are compared for the pure Heston model ('PH'), the double Heston model ('DH'), the HO model and the HJ model. ErrorSum denotes the sum of errors for each maturity with different strikes.

Appendices

A Proofs

A.1 The proof of Lemma 3.2

Proof. Recall the ODE system in Lemma 3.1

$$B'(u,\tau) = 0, \quad B(u,0) = \mathrm{i}u, \tag{A.1}$$

$$C'(u,\tau) = \sigma_\nu \mathbb{E}[\nu_t] B(u,\tau) D(u,\tau) - \kappa_\rho C(u,\tau), \quad C(u,0) = 0, \tag{A.2}$$

$$D'(u,\tau) = \frac{1}{2} B^2(u,\tau) + \frac{1}{2}\sigma_\nu^2 D(u,\tau) - \frac{1}{2} B(u,\tau) - \kappa_\nu D(u,\tau), \quad D(u,0) = 0, \tag{A.3}$$

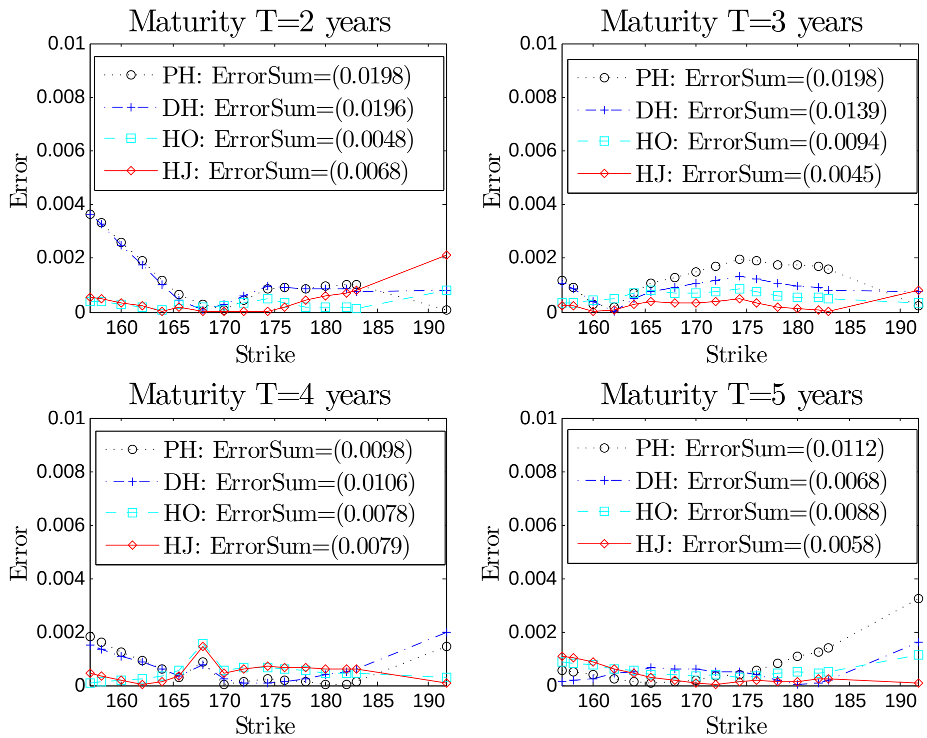

Figure 2: Using the Nikk300 index on December 31, 2012 where spot price is 174.3, and for some long maturities $T = 2,\ 3,\ 4,\ 5$ years, the errors (defined as absolute value of the differences between the implied market volatilities and the implied volatilities for the models) are compared for the pure Heston model ('PH'), the double Heston model ('DH'), the HO model and the HJ model. ErrorSum denotes the sum of errors for each maturity with different strikes.

$$\begin{aligned} A'(u,\tau) &= (B(u,\tau)-1)r + \kappa_\nu\mu_\nu D(u,\tau) + \kappa_\rho\mu_\rho C(u,\tau) \\ &\quad + \frac{1}{2}\sigma_\rho^2 C^2(u,\tau) + \sigma_\rho\rho_2\mathbb{E}[\sqrt{\nu_t}]B(u,\tau)C(u,\tau), \quad A(u,0)=0. \end{aligned} \tag{A.4}$$

Straightforwardly, due to the final condition $B(u,0) = \mathrm{i}u$ we obtain $B(u,\tau) = \mathrm{i}u$. We consider first the following Riccati-type equation:

$$\begin{aligned} \frac{\partial D(u,\tau)}{\partial\tau} &= \frac{1}{2}B^2(u,\tau) + \frac{1}{2}\sigma_\nu^2 D(u,\tau) - \frac{1}{2}B(u,\tau) - \kappa_\nu D(u,\tau), \quad D(u,0)=0, \\ H_1(u,\tau) &= (\mathrm{i}u-1)r\tau + \kappa_\nu\mu_\nu\int_0^\tau D(u,s)\,ds, \quad H_1(u,0)=0, \end{aligned}$$

which has the same form as those in [14] so that we can gain the solution given by

$$D(u,\tau) = \frac{\kappa_\nu - D_1}{\sigma_\nu^2} \cdot \frac{1-\mathrm{e}^{-D_1\tau}}{1-D_2\mathrm{e}^{-D_1\tau}}, \tag{A.5}$$

$$H_1(u,\tau) = (\mathrm{i}u-1)r\tau + \frac{\kappa_\nu\mu_\nu}{\sigma_\nu^2}\left((\kappa_\nu - D_1)\tau - 2\ln\left(\frac{1-D_2\mathrm{e}^{-D_1\tau}}{1-D_2}\right)\right), \tag{A.6}$$

where $D_1 = \sqrt{\kappa_\nu^2 + \sigma_\nu^2(u^2+\mathrm{i}u)}$ and $D_2 = \frac{\kappa_\nu - D_1}{\kappa_\nu + D_1}$.
We turn to (A.2) where

$$\mathbb{E}[\nu_t] = (\nu_0 - \mu_\nu)\mathrm{e}^{-\kappa_\nu(T-\tau)} + \mu_\nu. \tag{A.7}$$

To find its analytical solution we use the approximation

$$1 - \mathrm{e}^{-l_1\tau} \approx \frac{1-\mathrm{e}^{-D_1\tau}}{1-D_2\mathrm{e}^{-D_1\tau}}, \tag{A.8}$$

where l_1 defined in (B.4). The detailed information and the measure of the quality of this approximation can be found in Appendix B.1. We can thus rewrite (A.5) as

$$D(u,\tau) = \frac{\kappa_\nu - D_1}{\sigma_\nu^2} \cdot (1-\mathrm{e}^{-l_1\tau}). \tag{A.9}$$

and set

$$C_1 := \mathrm{i}u\frac{\kappa_\nu - D_1}{\sigma_\nu^2}, \tag{A.10}$$

sequentially, (A.2) can be rewritten as

$$C'(u,\tau) = \sigma_\nu C_1\left((\nu_0-\mu_\nu)\mathrm{e}^{-\kappa_\nu(T-\tau)} + \mu_\nu\right)\cdot(1-\mathrm{e}^{-l_1\tau}) - \kappa_\rho C(u,\tau), \quad C(u,0)=0, \tag{A.11}$$

which has a analytical solution, although its calculation is a bit tedious but straightforward. We obtain

$$\begin{aligned} C(u,\tau) = {} & \frac{C_1(\mu_\nu-\nu_0)}{\kappa_\nu+\kappa_\rho-l_1}\mathrm{e}^{(\kappa_\nu-l_1)\tau-\kappa_\nu T} + \frac{C_1(\nu_0-\mu_\nu)}{\kappa_\nu+\kappa_\rho}\mathrm{e}^{\kappa_\nu(\tau-T)} + \frac{C_1\mu_\nu}{\kappa_\rho} \\ & - \frac{C_1\mu_\nu}{\kappa_\rho - l_1}\mathrm{e}^{-l_1} + C_1C_2\mathrm{e}^{-\kappa_\rho\tau}, \end{aligned} \tag{A.12}$$

where l_1 defined in (B.4), C_1 defined in (A.10) and C_2 is given by

$$C_2 := \frac{\mu_\nu - \nu_0}{\kappa_\nu+\kappa_\rho-l_1}\mathrm{e}^{-\kappa_\nu T} + \frac{\nu_0-\mu_\nu}{\kappa_\nu+\kappa_\rho}\mathrm{e}^{-\kappa_\nu T} - \frac{\mu_\nu}{\kappa_\rho} + \frac{1}{\kappa_\rho - l_1}. \tag{A.13}$$

Finally, we rewrite (A.4) with approximations as

$$A(u,\tau) = H_1(u,\tau) + \underbrace{(\kappa_\rho\mu_\rho + m\sigma_\rho\rho_2 u\mathrm{i})}_{:=\alpha} H_2(u,\tau) + \underbrace{n\sigma_\rho\rho_2 u\mathrm{i}}_{:=\beta} H_3(u,\tau) + \frac{\sigma_\rho^2}{2} H_4(u,\tau), \tag{A.14}$$

for whose solutions we only need to calculate the following integrals

$$H_2(u,\tau) = \int_0^\tau C(u,s)\,ds, \quad H_2(u,0) = 0, \tag{A.15}$$

$$H_3(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)l} C(u,s)\,ds, \quad H_3(u,0) = 0, \tag{A.16}$$

$$H_4(u,\tau) = \int_0^\tau C^2(u,s)\,ds, \quad H_4(u,0) = 0, \tag{A.17}$$

where the constants m, n, and l are defined in (3.4) - (3.5). The calculation of the integrals above is straightforward but rather tedious. □

A.2 The proof of Lemma 3.4

Proof. As indicated before, the solutions of $\tilde{B}(u,\tau)$, $\tilde{C}(u,\tau)$ and $\tilde{D}(u,\tau)$ are the same as B, C and D in the HO model. We consider now only

$$\begin{aligned} A'(u,\tau) =& (\tilde{B}(u,\tau) - 1)r + \kappa_\nu\mu_\nu\tilde{D}(u,\tau) + \kappa_\rho\mu_\rho\tilde{C}(u,\tau) + \frac{1}{2}\sigma_\rho^2\mathbb{E}[1-\rho_t^2]\tilde{C}^2(u,\tau) \\ &+ \sigma_\rho\rho_2\mathbb{E}[\sqrt{\nu_t}]\mathbb{E}[\sqrt{1-\rho_t^2}]\tilde{B}(u,\tau)\tilde{C}(u,\tau), \quad \tilde{A}(u,0) = 0. \end{aligned} \tag{A.18}$$

By substituting the approximations of $\mathbb{E}[\rho_t^2]$, $\mathbb{E}[\sqrt{1-\rho_t^2}]$ and $\mathbb{E}[\sqrt{\nu_t}]$ into (A.18) we obtain

$$\begin{aligned} \tilde{A}'(u,\tau) =& (\tilde{B}(u,\tau) - 1)r + \kappa_\nu\mu_\nu\tilde{D}(u,\tau) + \kappa_\rho\mu_\rho\tilde{C}(u,\tau) \\ &+ \sigma_\rho\rho_2(m + n\mathrm{e}^{-l(T-\tau)})(\mathrm{e}^{-m_3(T-\tau)} + b_3\mathrm{e}^{-n_3(T-\tau)} + a_3)\tilde{B}(u,\tau)\tilde{C}(u,\tau) \\ &+ \frac{1}{2}\sigma_\rho^2\tilde{C}^2(u,\tau)(1 - \mathrm{e}^{-m_2(T-\tau)} - b_2\mathrm{e}^{-n_2(T-\tau)} - a_2), \quad \tilde{A}(u,0) = 0, \end{aligned} \tag{A.19}$$

which can be reformulated as

$$\begin{aligned}\tilde{A}(u,\tau) =& \tilde{H}_1(u,\tau) + (\kappa_\rho\mu_\rho + a_3 m\sigma_\rho\rho_2 u\mathrm{i})\tilde{H}_2(u,\tau) + a_3 n\sigma_\rho\rho_2 u\mathrm{i}\tilde{H}_3(u,\tau)\\ &+ \frac{\sigma_\rho^2}{2}(1-a_2)\tilde{H}_4(u,\tau) + b_3 m\sigma_\rho\rho_2 u\mathrm{i}\tilde{H}_5(u,\tau) + m\sigma_\rho\rho_2 u\mathrm{i}\tilde{H}_6(u,\tau)\\ &+ b_3 n\sigma_\rho\rho_2 u\mathrm{i}\tilde{H}_7(u,\tau) + n\sigma_\rho\rho_2 u\mathrm{i}\tilde{H}_8(u,\tau) - \frac{\sigma_\rho^2}{2}\tilde{H}_9(u,\tau) - \frac{b_2\sigma_\rho^2}{2}\tilde{H}_{10}(u,\tau)\end{aligned}$$

with the following integrals

$$\begin{aligned}&\tilde{H}_1(u,\tau) = (\mathrm{i}u-1)r\tau + \kappa_\nu\mu_\nu\int_0^\tau \tilde{D}(u,s)\,ds,\ \tilde{H}_2(u,\tau) = \int_0^\tau \tilde{C}(u,s)\,ds,\\ &\tilde{H}_3(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)l}\tilde{C}(u,s)\,ds,\ \tilde{H}_4(u,\tau) = \int_0^\tau \tilde{C}^2(u,s)\,ds,\\ &\tilde{H}_5(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)n_3}\tilde{C}(u,s)\,ds,\ \tilde{H}_6(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)m_3}\tilde{C}(u,s)\,ds,\\ &\tilde{H}_7(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)(l+n_3)}\tilde{C}(u,s)\,ds,\ \tilde{H}_8(u,\tau) = \int_0^\tau \mathrm{e}^{-(T-\tau)(l+m_3)}\tilde{C}(u,s)\,ds,\\ &\tilde{H}_9(u,\tau) = \int_0^\tau \mathrm{e}^{-m_2(T-\tau)}\tilde{C}^2(u,s)\,ds,\ \tilde{H}_{10}(u,\tau) = \int_0^\tau \mathrm{e}^{-n_2(T-\tau)}\tilde{C}^2(u,s)\,ds,\\ &\tilde{H}_i(u,0) = 0 \text{ for } i = 1\cdots 10.\end{aligned}$$

It is easy to see that $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$ and $\tilde{H}_4$ are respectively equal to H_1, H_2, H_3 and H_4 which have been given before. Besides, the solutions of $\tilde{H}_5$, $\tilde{H}_6$, $\tilde{H}_7$, $\tilde{H}_8$ can be directly obtained by adopting the solution of $\tilde{H}_3$, as they have only different constant coefficient in the exponential function. For simplicity of notation, we let this coefficient to be a variable of $\tilde{H}_3$, namely $\tilde{H}_3(u,\tau,l)$. The solutions of $\tilde{H}_5$, $\tilde{H}_6$, $\tilde{H}_7$ and $\tilde{H}_8$ can thus be immediately given by $\tilde{H}_3(u,\tau,n_3)$, $\tilde{H}_3(u,\tau,m_3)$, $\tilde{H}_3(u,\tau,(l+n_3))$ and $\tilde{H}_3(u,\tau,(l+m_3))$, respectively. Now, only the integral in the following form

$$\tilde{H}(u,\tau,y) = \int_0^\tau \mathrm{e}^{-y(T-\tau)}C^2(u,s)\,ds, \quad \tilde{H}(u,0) = 0$$

need to be calculated. The calculation is straightforward, however, rather tedious. It is Obvious that $\tilde{H}(u,\tau,m_2) = \tilde{H}_9(u,\tau)$ and $\tilde{H}(u,\tau,n_2) = \tilde{H}_{10}(u,\tau)$. Finally, by defining $\zeta := \sigma_\rho\rho_2 u\mathrm{i}$, $A(u,\tau)$ can be rewritten as

$$\begin{aligned}A(u,\tau) =& \tilde{H}_1(u,\tau) + (\kappa_\rho\mu_\rho + a_3 m\zeta)\tilde{H}_2(u,\tau) + a_3 n\zeta\tilde{H}_3(u,\tau,l) + b_3 m\zeta\tilde{H}_3(u,\tau,n_3)\\ &+ m\zeta\tilde{H}_3(u,\tau,m_3) + b_3 n\zeta\tilde{H}_3(u,\tau,(l+n_3)) + n\zeta\tilde{H}_3(u,\tau,(l+m_3))\\ &+ \frac{\sigma_\rho^2}{2}(1-a_2)\tilde{H}_4(u,\tau) - \frac{\sigma_\rho^2}{2}\tilde{H}(u,\tau,m_2) - \frac{b_2\sigma_\rho^2}{2}\tilde{H}(u,\tau,n_2).\end{aligned}$$

$\square$

B Approximations

B.1 Approximation I

We match $f_1(\tau) := \frac{1-\mathrm{e}^{-D_1\tau}}{1-D_2\mathrm{e}^{-D_1\tau}} \approx m_1 + n_1\mathrm{e}^{-l_1\tau} := \tilde{f}_1(\tau)$ for $\tau \to 0$, $\tau \to \infty$, $\tau \to 1$:

$$\lim_{\tau\to 0} f_1(\tau) = 0 = m_1 + n_1 = \lim_{\tau\to 0} \tilde{f}_1(\tau), \tag{B.1}$$

$$\lim_{\tau\to \infty} f_1(\tau) = 1 = m_1 = \lim_{\tau\to \infty} \tilde{f}_1(\tau), \tag{B.2}$$

$$\lim_{\tau\to 1} f_1(\tau) = \frac{1-\mathrm{e}^{-D_1}}{1-D_2\mathrm{e}^{-D_1}} = 1-\mathrm{e}^{-l_1} = \lim_{\tau\to 1} \tilde{f}_1(\tau), \tag{B.3}$$

which give us that

$$m_1 = 1, \quad n_1 = -1, \quad l_1 = -\ln\left(\frac{\mathrm{e}^{-D_1} - D_2\mathrm{e}^{-D_1}}{1-D_2\mathrm{e}^{-D_1}}\right). \tag{B.4}$$

In order to measure the quality of this approximation we compare $f_1(\tau)$ to $\tilde{f}_1(\tau)$ for different randomly chosen parameters in Figure 3.

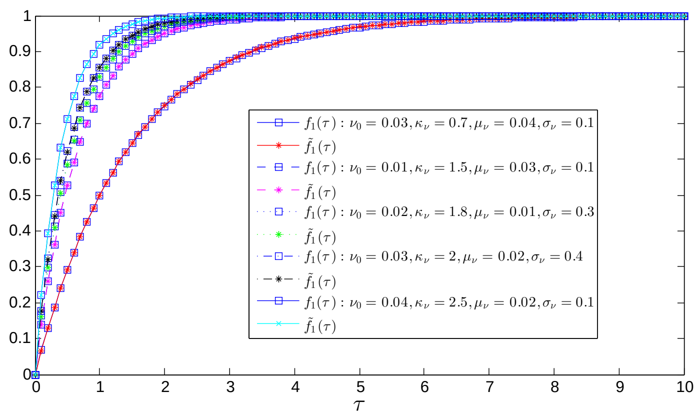

Figure 3: The quality of the approximation $\tilde{f}_1(\tau)$ versus the original $f_1(\tau)$ for randomly chosen parameters.

B.2 Approximation II

We match $f_2(t) := \mathbb{E}[\rho_t^2] \approx \mathrm{e}^{-m_2 t} + b_2 \mathrm{e}^{-n_2 t} + a_2 := \tilde{f}_2(t)$ for $t \to 0$, $t \to \frac{1}{2}$, $t \to 1$, $t \to \infty$ as follows:

$$\lim_{t\to\infty} f_2(t) = \frac{(\sigma_\rho^2 + \kappa_\rho)(\sigma_\rho^2 + 2\kappa_\rho \mu_\rho^2)}{\sigma_\rho^4 + 3\kappa_\rho \sigma_\rho^2 + 2\kappa_\rho^2} = a_2 = \lim_{t\to\infty} \tilde{f}_2(t), \tag{B.5}$$

$$\lim_{t\to 0} f_2(t) = \rho_0^2 = 1 + b_2 + a_2 = \lim_{t\to 0} \tilde{f}_2(t), \tag{B.6}$$

$$\lim_{t\to \frac{1}{2}} f_2(t) = f_2(0.5) = \mathrm{e}^{-\frac{m_2}{2}} + b_2 \mathrm{e}^{-\frac{n_2}{2}} + a_2 = \lim_{t\to \frac{1}{2}} \tilde{f}_2(t), \tag{B.7}$$

$$\lim_{t\to 1} f_2(t) = f_2(1) = \mathrm{e}^{-m_2} + b_2 \mathrm{e}^{-n_2} + a_2 = \lim_{t\to 1} \tilde{f}_2(t). \tag{B.8}$$

From (B.5) and (B.6) one obtains directly $a_2 = \frac{(\sigma_\rho^2+\kappa_\rho)(\sigma_\rho^2+2\kappa_\rho\mu_\rho^2)}{\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2}$ and $b_2 = \rho_0^2 - a_2 - 1$. Then one needs to solve the system of equations (B.7) and (B.8) to find m_2 and n_2 which has been given in (3.39). Like in the last section, we compare $f_2(\tau)$ to $\tilde{f}_2(\tau)$ for different randomly chosen parameters to measure the quality of the proposed approximation.

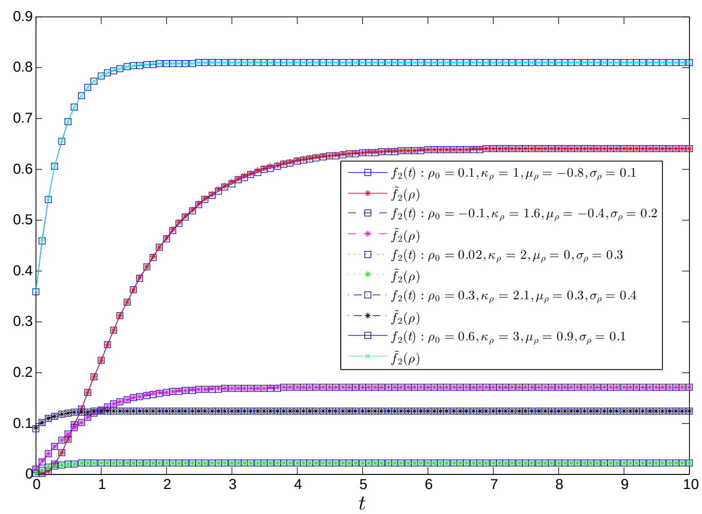

Figure 4: The quality of the approximation $\tilde{f}_2(t)$ versus the original $f_2(t)$ for randomly chosen parameters.

B.3 Approximation III

We match $f_3(t) := \mathbb{E}[\sqrt{1-\rho_t^2}] \approx \mathrm{e}^{-m_3 t} + b_3 \mathrm{e}^{-n_3 t} + a_3 := \tilde{f}_3(t)$ for $t \to 0$, $t \to \frac{1}{2}$, $t \to 1$, $t \to \infty$ as follows:

$$\lim_{t\to\infty} f_3(t) = \sqrt{1 - \frac{a_2 - \mu_\rho^4}{1-\mu_\rho^2}} = a_3 = \lim_{t\to\infty} \tilde{f}_2(t), \tag{B.9}$$

$$\lim_{t\to 0} f_3(t) = \sqrt{1-\rho_0^2} = 1 + b_3 + a_3 = \lim_{t\to 0} \tilde{f}_3(t), \tag{B.10}$$

$$\lim_{t\to\frac{1}{2}} f_3(t) = f_3(0.5) = \mathrm{e}^{-\frac{m_3}{2}} + b_3 \mathrm{e}^{-\frac{n_3}{3}} + a_3 = \lim_{t\to\frac{1}{2}} \tilde{f}_3(t), \tag{B.11}$$

$$\lim_{t\to 1} f_3(t) = f_3(1) = \mathrm{e}^{-m_3} + b_3 \mathrm{e}^{-n_3} + a_3 = \lim_{t\to 1} \tilde{f}_3(t). \tag{B.12}$$

From (B.9) and (B.10) one obtains directly

$$a_3 = \sqrt{1 - \frac{(\sigma_\rho^2+\kappa_\rho)(\sigma_\rho^2+2\kappa_\rho\mu_\rho^2) - \mu_\rho^4(\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2)}{(1-\mu_\rho^2)(\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2)}}$$

and $b_3 = \sqrt{1-\rho_0^2} - a_3 - 1$. Further, we solve the system of equations (B.11) and (B.12) to find m_3 and n_3 which has been given in (3.49). The comparison of $f_3(\tau)$ with $\tilde{f}_3(\tau)$ and the measure of quality of the approximation for different randomly chosen parameters is exhibited in Figure 5.

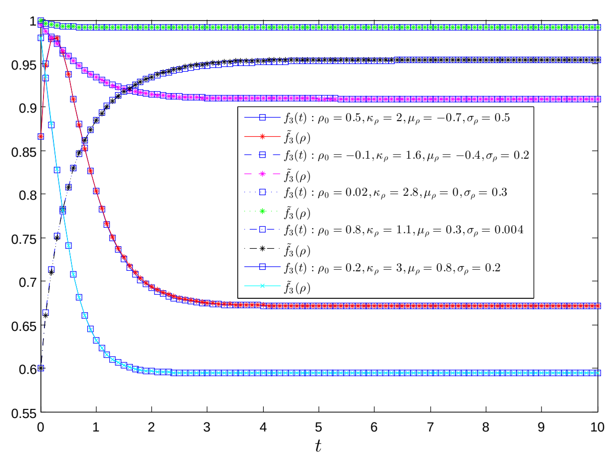

Figure 5: The quality of the approximation $\tilde{f}_3(t)$ versus the original $f_3(t)$ for randomly chosen parameters.

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