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CONSTANT ELASTICITY OF VARIANCE OPTION PRICING MODEL WITH TIME-DEPENDENT PARAMETERS

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This paper provides a method for pricing options in the constant elasticity of variance (CEV) model environment using the Lie-algebraic technique when the model parameters are time-dependent. Analytical solutions for the option values incorporating time-dependent model parameters are obtained in various CEV processes with different elasticity factors. The numerical results indicate that option values are sensitive to volatility term structures. It is also possible to generate further results using various functional forms for interest rate and dividend term structures. Furthermore, the Lie-algebraic approach is very simple and can be easily extended to other option pricing models with well-defined algebraic structures.

Keywords: Options, constant elasticity of variance, partial differential equation, Lie algebra.

1. Introduction

The Black–Scholes option pricing model [2] is a member of the class of constant elasticity of variance (CEV) option pricing models. The diffusion process of stock price S in a CEV model can be expressed as

$$dS = \mu S dt + \sigma S^{\beta/2} dZ, \qquad 0 \leq \beta < 2 \tag{1}$$

where μ is the instantaneous mean, $\sigma S^{\beta/2}$ is the instantaneous variance of the stock price, dZ is a Weiner process and β is the elasticity factor. The equation shows that the instantaneous variance of the percentage price change is equal to $\sigma^2/S^{2-\beta}$ and is a direct inverse function of the stock price. In the limiting case $\beta = 2$, the CEV model returns to the conventional Black–Scholes model in which the variance rate is independent of the stock price. In another case $\beta = 0$, it is the Ornstein–Uhlenbeck model. Several theoretical arguments imply an association between stock price and volatility. Black [3] and Christie [4] consider the effects of financial leverage on the variance of the stock. A fall in the stock price increases the debt-equity ratio of the firm; therefore, both the risk and the variance of the stock increase. Black also proposes that a downturn in the business cycle might lead to an increase in the stock price volatility and hence to a fall in the stock prices. Empirical evidence has shown that the CEV process may be a better description of stock behaviour than the more commonly used lognormal model because the CEV process allows for a non-zero elasticity of return variance with respect to prices. Schmalensee and Trippi [20] find a strong negative relationship between stock price changes and changes in implied volatility after examining over a year of weekly data on six stocks. By applying the trading profits approach on 19,000 daily warrant price observations, Hauser and Lauterbach [10] find that the CEV model roughly doubles the trading excess returns of the Black–Scholes model. The superiority of the CEV model is clearest in out-of-the-money and longer-time-to-expiration warrants. The results are consistent with the findings in Lauterbach and Schultz [11]. If the relationship between the variance and the stock price is deduced from the empirical data, an option pricing formula based on the CEV model could fit the actual market option prices better than the Black–Scholes model. Beckers [1] finds thirty-seven out of forty-seven stocks in a year daily data set to have estimated β to be less than two and concludes that the CEV diffusion process could be a better candidate for describing the actual stock price behaviour than the Black–Scholes model.

The derivation of the CEV option pricing formula with $\beta = 1$ (commonly known as the "square-root process") was first presented by Cox and Ross [6] as an alternative diffusion process for the valuation of options. Cox [5] also derived the option pricing formula for $\beta < 2$. However, the solutions of the CEV model have been considered to be computationally intensive because of its combination of computing an improper integral and an infinite sum. The difficulty in the computational aspect has made the CEV option pricing formula not used widely by practitioners [10]. Schroder [21] presented an algorithm for computing the solutions to overcome this difficulty and claimed that it was efficient but no numerical results have been demonstrated.

All the above derivations assume the model parameters such as volatility, interest rate and dividend yield are constant. However, the model parameters are actually time-dependent in the market. The time-dependent term structures of interest rates and volatility which can be implied from the money market and the option market respectively are expressed as time-dependent stepwise functions. The term structures can also be expressed as analytical functions to reflect the expectation and dynamics of market factors. In constrast to the analytical solution for the Black–Scholes model, the analytical solution for an option with the CEV process cannot be easily extended to the case where the model parameters are time-dependent. Only recently has Goldenberg [8], using an approach based upon the stochastic calculus, succeeded in introducing time-dependent volatilities into the special case of the square-root process and obtaining the closed-form option pricing formula explicitly. However, no numerical calculations have been performed to investigate the effect of the time-dependent volatilities on the option prices.

This paper has two main purposes: (1) to propose a Lie-algebraic technique to value options (both call and put options) for the CEV class with time-dependent model parameters and (2) to demonstrate that the time-dependent term-structures of the model parameters have a significant effect on the option prices. The Lie-algebraic method, which has never been used in option pricing, is very simple and has been successfully applied to tackle the time-dependent Schrödinger equation associated with generalized quantum time-dependent oscillators [12, 13, 14, 18] as well as the Fokker–Plank equation [15, 16, 17]. By using this technique, the analytical solution of the CEV model with time-dependent parameters is derived and found to have the same form as the time-independent case. We also derive a generalized put-call parity relation for the time-dependent CEV model. Moreover, we show that the computation of the CEV option pricing formula with time-dependent parameters is very efficient and can be used in practice by employing a computing algorithm similar to the one proposed by Schroder [21].

The scheme of this paper is as follows. In the following section we consider the Black–Scholes equation with time-dependent parameters for a standard European option and derive its pricing formula using the Lie-algebraic approach. In Sec. 3 we apply the same approach to derive the CEV option pricing formula with time-dependent parameters. Numerical results are shown in Sec. 4. In the last section we shall summarize our investigation.

2. Black–Scholes Model for a Standard European Option

Consider the Black–Scholes equation with time-dependent model parameters for a standard European option

$$\begin{aligned}\frac{\partial P(S,\tau)}{\partial \tau} &= \frac{1}{2}\sigma(\tau)^2 S^2 \frac{\partial^2 P(S,\tau)}{\partial S^2} + [r(\tau) - d(\tau)]S\frac{\partial P(S,\tau)}{\partial S} - r(\tau)P(S,\tau) \\ &\equiv H(\tau)P(S,\tau)\,, \end{aligned} \tag{2}$$

where P is the option value, S is the underlying price, τ is the time to maturity, σ is the volatility, r is the risk-free interest rate and d is the dividend. We define the evolution operator $U(\tau, 0)$ such that

$$P(S,\tau) = U(\tau,0)P(S,0)\,. \tag{3}$$

Inserting Eq. (3) into Eq. (2) yields the evolution equation

$$H(\tau)U(\tau,0) = \frac{\partial}{\partial \tau}U(\tau,0), \qquad U(0,0) = 1\,. \tag{4}$$

It is not difficult to show that in terms of the commuting operators $\hat{e}_1 = S(\partial/\partial S)$, $\hat{e}_2 = \hat{e}_1^2$ and $\hat{e}_3 = 1$, the operator $H(\tau)$ can be rewritten as

$$H(\tau) = a_1(\tau)\hat{e}_1 + a_2(\tau)\hat{e}_2 + a_3(\tau)\hat{e}_3\,, \tag{5}$$

where

$$a_1(\tau) = r(\tau) - d(\tau) - \frac{1}{2}\sigma(\tau)^2, \qquad a_2(\tau) = \frac{1}{2}\sigma(\tau)^2, \qquad a_3(\tau) = -r(\tau). \tag{6}$$

Since the operators $\hat{e}_j$ form a solvable algebra, the Wei–Norman theorem states that the evolution operator $U(\tau, 0)$ can be expressed in the form [22]

$$U(\tau, 0) = \exp[c_1(\tau)\hat{e}_1] \exp[c_2(\tau)\hat{e}_2] \exp[c_3(\tau)\hat{e}_3] \tag{7}$$

where $c_j(\tau)$ are to be determined. Then by direct differentiation with respect to τ, we obtain

$$\frac{\partial}{\partial\tau}U(\tau, 0) = [\dot{c}_1(\tau)\hat{e}_1 + \dot{c}_2(\tau)\hat{e}_2 + \dot{c}_3(\tau)\hat{e}_3]U(\tau, 0) \tag{8}$$

Substituting Eqs. (5), (7) and (8) into Eq. (4), and comparing the two sides, we obtain after simplification

$$\begin{aligned} c_1(\tau) &= \int_0^\tau \left[r(\tau') - d(\tau') - \frac{1}{2}\sigma(\tau')^2\right] d\tau' , \\ c_2(\tau) &= \frac{1}{2}\int_0^\tau \sigma(\tau')^2 d\tau' , \\ c_3(\tau) &= -\int_0^\tau r(\tau') d\tau' . \end{aligned} \tag{9}$$

Provided that $P(S, 0) = \sum_{n=0}^\infty b_n S^n$, we can easily show that $P(S, \tau)$ is given by

$$P(S, \tau) = \sum_{n=0}^\infty b_n S^n \exp[c_2(\tau)n^2 + c_1(\tau)n + c_3(\tau)] .$$

If we define $S = \exp(x)$ and $S' = \exp(x')$, then $P(S, \tau)$ can be expressed in the following form:

$$P(S, \tau) = \int_{-\infty}^\infty dx' G(x, \tau; x', 0) P(S', 0) , \tag{10}$$

where

$$G(x, \tau; x', 0) = \frac{1}{\sqrt{4\pi c_2(\tau)}} \exp = \left\{-\frac{[x - x' + c_1(\tau)]^2}{4c_2(\tau)} + c_3(\tau)\right\} . \tag{11}$$

It is obvious that $G(x, \tau; x', 0)$ is the propagator of the Black–Scholes equation in Eq. (2). Furthermore, given that $P_c(S, 0) = \max(S - S_0, 0)$ where S_0 is the strike price for a call option, we can easily perform the integration in Eq. (10) to obtain the explicit pricing formula as follows:

$$\begin{aligned} P_c(S, \tau) = {} & S \exp\left[-\int_0^\tau d(\tau')d\tau'\right] \mathrm{N}\left(\frac{x - x_0 + c_1(\tau) + 2c_2(\tau)}{\sqrt{2c_2(\tau)}}\right) \\ & - S_0 \exp\left[-\int_0^\tau r(\tau')d\tau'\right] \mathrm{N}\left(\frac{x - x_0 + c_1(\tau)}{\sqrt{2c_2(\tau)}}\right) , \end{aligned} \tag{12}$$

where N($\cdot$) is the cumulative normal distribution function. When the model parameters are time-independent, Eq. (12) is reduced to the standard Black–Scholes formula [2]. The same derivation can also be performed on a put option with $P_p(S,0) = \max(S_0 - S, 0)$.

3. CEV Model for a Standard European Option

The CEV model with time-dependent model parameters for a standard European option is described by the partial differential equation [5]

$$\frac{\partial P(S,\tau)}{\partial \tau} = \frac{1}{2}\sigma(\tau)^2 S^\beta \frac{\partial^2 P(S,\tau)}{\partial S^2} + [r(\tau) - d(\tau)]S\frac{\partial P(S,\tau)}{\partial S} - r(\tau)P(S,\tau) \tag{13}$$

for $0 \leq \beta < 2$. Introducing a simple change of variables: $x = \sqrt{S^{(2-\beta)}}$, Eq. (13) can be recast in the following form:

$$\begin{aligned}\frac{\partial u(x,\tau)}{\partial \tau} &= \frac{1}{8}\tilde{\sigma}(\tau)^2\frac{\partial^2 u(x,\tau)}{\partial x^2} + \frac{1}{2}\left[\tilde{\mu}(\tau)x - \frac{(4-\beta)\tilde{\sigma}(\tau)^2}{4(2-\beta)x}\right]\frac{\partial u(x,\tau)}{\partial x}\\ &\quad + \left[\frac{(4-\beta)\tilde{\sigma}(\tau)^2}{8(2-\beta)x^2} - r(\tau) - \frac{\tilde{\mu}(\tau)}{2}\right]u(x,\tau) \equiv H(\tau)u(x,\tau)\,,\end{aligned} \tag{14}$$

where $\tilde{\sigma}(\tau) = (2-\beta)\sigma(\tau)$, $\tilde{\mu}(\tau) = (2-\beta)[r(\tau) - d(\tau)]$ and $u(x,\tau) = xP(s,\tau)$. This equation represents a generalization of the Fokker–Planck equation associated with the well-known Rayleigh process [7]. It is not difficult to show that the operator $H(\tau)$ can be rewritten as follows:

$$H(\tau) = a_1(\tau)K_+ + a_2(\tau)K_0 + a_3(\tau)K_- + b(\tau) \tag{15}$$

where

$$\begin{aligned} K_- &= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} - \frac{4-\beta}{(2-\beta)x}\frac{\partial}{\partial x} + \frac{4-\beta}{(2-\beta)x^2}\right]\\ K_0 &= \frac{1}{2}\left(x\frac{\partial}{\partial x} - \frac{1}{2-\beta}\right), \quad K_+ = \frac{1}{2}x^2\\ a_3(\tau) &= \frac{1}{4}\tilde{\sigma}(\tau)^2, \qquad a_2(\tau) = \tilde{\mu}(\tau)\\ a_1(\tau) &= 0, \qquad b(\tau) - \frac{1-\beta}{2(2-\beta)}\tilde{\mu}(\tau) - r(\tau)\,.\end{aligned} \tag{16}$$

The operators K_+, K_0 and K_- are the generators of the Lie algebra su(1,1) [19]:

$$[K_+, K_-] = -2K_0, \qquad [K_0, K_\pm] = \pm K_\pm\,. \tag{17}$$

We may define the evolution operator $U(\tau, 0)$ such that

$$u(x,\tau) = \exp\left[\int_0^\tau d\tau' b(\tau')\right]\cdot U(\tau,0)u(x,0)\,. \tag{18}$$

Inserting Eq. (18) into Eq. (14) yields the evolution equation

$$\frac{\partial}{\partial \tau} U(\tau, 0) = H_I(\tau) U(\tau, 0), \qquad U(0,0) = 1 \tag{19}$$

with

$$H_I(\tau) = a_1(\tau) K_+ + a_2(\tau) K_0 + a_3(\tau) K_- \,. \tag{20}$$

Since the su(1,1) algebra is a real "split 3-dimensional" simple Lie algebra, the Wei-Norman theorem states that the evolution operator $U(\tau, 0)$ can be expressed in the form [22]

$$U(\tau, 0) = \exp[c_1(\tau) K_+] \cdot \exp[c_2(\tau) K_0] \cdot \exp[c_3(\tau) K_-] \tag{21}$$

where $c_i(\tau)$ are to be determined. Then by direct differentiation with respect to τ, we obtain

$$\frac{\partial}{\partial \tau} U(\tau, 0) = [h_+(\tau) K_+ + h_0(\tau) K_0 + h_-(\tau) K_-] U(\tau, 0) \tag{22}$$

with

$$\begin{aligned} h_+(\tau) &= \frac{dc_1}{d\tau} - c_1 \frac{dc_2}{d\tau} + c_1^2 \exp(-c_2) \frac{dc_3}{d\tau} \\ h_0(\tau) &= \frac{dc_2}{d\tau} - 2c_1 \exp(-c_2) \frac{dc_3}{d\tau} \\ h_-(\tau) &= \exp(-c_2) \frac{dc_3}{d\tau} \,. \end{aligned} \tag{23}$$

Substituting Eqs. (20)–(22) into Eq. (19), and comparing both sides, we obtain after simplification

$$\frac{dc_1(\tau)}{d\tau} = \frac{1}{4} \tilde{\sigma}(\tau)^2 c_1^2 + \tilde{\mu}(\tau) c_1, \quad c_1(0) = 0 \tag{24}$$

$$c_2(\tau) = \int_0^{\tau} \left[\frac{1}{2} \tilde{\sigma}(\tau')^2 c_1(\tau') + \tilde{\mu}(\tau') \right] d\tau' \tag{25}$$

$$c_3(\tau) = \frac{1}{4} \int_0^{\tau} \tilde{\sigma}(\tau')^2 \exp[c_2(\tau')] d\tau' \,. \tag{26}$$

Equation (24), which is just a Bernoulli equation, is the equation we have to solve first to determine $c_1(\tau)$, and obviously the only admissible solution is the trivial solution $c_1(\tau) = 0$. Once $c_1(\tau)$ is determined, $c_2(\tau)$ and $c_3(\tau)$ can be obtained readily by direct integration:

$$\begin{aligned} c_2(\tau) &= \int_0^{\tau} \tilde{\mu}(\tau') d\tau' \\ c_3(\tau) &= \frac{1}{4} \int_0^{\tau} \tilde{\sigma}(\tau')^2 \exp[c_2(\tau')] d\tau' \,. \end{aligned} \tag{27}$$

Hence, we have obtained an exact form of the time evolution operator $U(\tau, 0)$ of the system.

Now we apply the above results to the case of a standard European call option. Without loss of generality, we suppose that $u(x,0) = x^{(\alpha+1)/2}v(x,0)$, where $\alpha = (4-\beta)/(2-\beta)$ and

$$v(x,0) = \int_0^\infty d\nu \nu J_{(\alpha-1)/2}(x\nu) \int_0^\infty dy y J_{(\alpha-1)/2}(y\nu) v(y,0)\,. \tag{28}$$

Then it is not difficult to show that $u(x,\tau)$ is given by

$$u(x,\tau) = \int_0^\infty dy K(x,\tau;y,0) u(y,0) \tag{29}$$

with

$$\begin{aligned} K(x,\tau;y,0) = {} & \left\{ y^{1-\alpha} x^{1+\alpha} \exp\left[\frac{c_2(\tau)}{2-\beta}\right]\right\}^{1/2} \exp\left[-\int_0^\tau r(\tau')d\tau'\right] \\ & \times \int_0^\infty d\nu \nu J_{(\alpha-1)/2}(y\nu) J_{(\alpha-1)/2}(x\nu \exp[c_2(\tau)/2]) \\ & \times \exp\left[-\frac{c_3(\tau)}{2}\nu^2\right]. \end{aligned} \tag{30}$$

The function J_μ is the Bessel function of the first kind of order μ. Here we have made use of the fact that $x^{(\alpha+1)/2} J_{(\alpha-1)/2}(x\nu)$ is an eigenfunction of the operator K_- with the eigenvalue $-\nu^2/2$ as well as the well-known relation

$$\exp\left(\eta x \frac{\partial}{\partial x}\right) f(x) = f(x \exp(\eta))\,. \tag{31}$$

The integral over ν can be evaluated to give [9]

$$\frac{1}{c_3(\tau)} \exp\left\{-\frac{y^2 + x^2 \exp[c_2(\tau)]}{2c_3(\tau)}\right\} I_{(\alpha-1)/2}\left(\frac{yx \exp[c_2(\tau)/2]}{c_3(\tau)}\right) \tag{32}$$

for $(\alpha-1)/2 > -1$, $y > 0$, $x \exp[c_2(\tau)/2] > 0$ and $|\arg[c_3(\tau)/2]^{1/2}| < \pi/4$. The function I_μ is the modified Bessel function of the first kind of order μ. As a result, the desired kernel $K(x,\tau;y,0)$ is found to be

$$\begin{aligned} K(x,\tau;y,0) = {} & \frac{1}{c_3(\tau)} \left\{ y^{1-\alpha} x^{1+\alpha} \exp\left[\frac{c_2(\tau)}{2-\beta}\right]\right\}^{1/2} \exp\left\{-\frac{y^2 + x^2\exp[c_2(\tau)]}{2c_3(\tau)}\right\} \\ & \times \exp\left[-\int_0^\tau r(\tau')d\tau'\right] I_{(\alpha-1)/2}\left(\frac{yx\exp[c_2(\tau)/2]}{c_3(\tau)}\right). \end{aligned} \tag{33}$$

Since $S = x^{\alpha-1}$ and $P_c(S,\tau) = u(x,\tau)/x$, we can readily obtain $P_c(S,\tau)$ as follows:

$$P_c(S,\tau) = \int_0^\infty dR \mathcal{K}(S,\tau;R,0) P_c(R,0) \tag{34}$$

with

$$\mathcal{K}(S,\tau;R,0) = \frac{2-\beta}{2c_3(\tau)}\left\{\frac{S}{R^{2\beta-1}}\exp\left[\frac{c_2(\tau)}{2-\beta}\right]\right\}^{1/2}\exp\left\{-\frac{R^{2-\beta}+S^{2-\beta}\exp[c_2(\tau)]}{2c_3(\tau)}\right\}$$
$$\times\exp\left[-\int_0^\tau r(\tau')d\tau'\right] I_{1/(2-\beta)}\left(\frac{\{R^{2-\beta}S^{2-\beta}\exp[c_2(\tau)]\}^{1/2}}{c_3(\tau)}\right). \tag{35}$$

The propagator $\mathcal{K}(S,\tau;R,0)$ is for the most general CEV model with time-dependent parameters and thus results for any special case can be easily deduced from it. For instance, in the case of constant model parameters the corresponding propagator is simply given by

$$\mathcal{K}(S,\tau;R,0) = \frac{2(r-d)\exp(-r\tau)}{\sigma^2\{\exp[(2-\beta)(r-d)\tau]-1\}}\left\{\frac{S}{R^{2\beta-1}}\exp[(r-d)\tau]\right\}^{1/2}$$
$$\times\exp\left\{-\frac{2(r-d)\{R^{2-\beta}+S^{2-\beta}\exp[(2-\beta)(r-d)\tau]\}}{(2-\beta)\sigma^2\{\exp[(2-\beta)(r-d)\tau]-1\}}\right\}$$
$$\times I_{1/(2-\beta)}\left(\frac{4(r-d)\{R^{2-\beta}S^{2-\beta}\exp[(2-\beta)(r-d)\tau]\}^{1/2}}{(2-\beta)\sigma^2\{\exp[(2-\beta)(r-d)\tau]-1\}}\right), \tag{36}$$

which is in agreement with the results of previous studies [5, 6, 21]. Also, for the square-root case, i.e. $\beta = 1$, we recover the transition probability density derived by Goldenberg [8] using the stochastic calculus approach. Furthermore, provided a call option with $P_c(S,0) = \max(S-S_0, 0)$, the integration in Eq. (34) can be easily carried out to yield the explicit pricing formula

$$P_c(S,\tau) = S\exp\left[-\int_0^\tau d(\tau')d\tau'\right]\sum_{n=0}^{\infty}\frac{z^n\exp(-z)}{\Gamma(n+1)}G(n+1+1/(2-\beta),\omega)$$
$$- S_0\exp\left[-\int_0^\tau r(\tau')d\tau'\right]\sum_{n=0}^{\infty}\frac{z^{n+1/(2-\beta)}\exp(-z)}{\Gamma(n+1+1/(2-\beta))}G(n+1,\omega) \tag{37}$$

where

$$z = \frac{S^{2-\beta}\exp[c_2(\tau)]}{2c_3(\tau)}$$
$$\omega = \frac{S_0^{2-\beta}}{2c_3(\tau)} \tag{38}$$
$$G(\xi,\omega) = \frac{1}{\Gamma(\xi)}\int_\omega^\infty \zeta^{\xi-1}\exp(-\zeta)d\zeta\,.$$

The function $G(\xi,\omega)$ is the complementary incomplete gamma function [9]. It is straightforward to show that for the case of constant model parameters this valuation formula is reduced to the standard CEV valuation formula [5, 6, 21].

Next, the case of a standard European put option with $P_p(S,0) = \max(S_0 - S, 0) = \max(S - S_0, 0) - (S - S_0) = P_c(S,0) - (S - S_0)$ is considered. It is obvious that the put option price $P_p(S,\tau)$ at any time $\tau > 0$ is given by

$$P_p(S,\tau) = P_c(S,\tau) - \frac{1}{x}\exp\left[\int_0^\tau d\tau' b(\tau')\right] U(\tau,0)(x^\alpha - S_0 x)\,. \tag{39}$$

Making use of the relations:

$$\begin{aligned} K_-(x^\alpha - S_0 x) &= 0 \\ \exp[c_2(\tau)K_0](x^\alpha - S_0 x) &= \left\{x^\alpha \exp\left[\frac{\alpha-1}{2}c_2(\tau)\right] - S_0 x\right\} \\ &\quad \times \exp\left[\frac{1-\beta}{2(2-\beta)}c_2(\tau)\right], \end{aligned} \tag{40}$$

Eq. (39) can be re-written as

$$P_p(S,\tau) = P_c(S,\tau) - S\exp\left[-\int_0^\tau d(\tau')d\tau'\right] + S_0 \exp\left[-\int_0^\tau r(\tau')d\tau'\right], \tag{41}$$

which beyond question satisfies the desired boundary conditions [23]. It is interesting to note that Eq. (41) closely resembles the conventional put-call parity relation in the case of constant model parameters [23]. In fact, it is the generalized put-call parity relation for the time-dependent CEV model. As a consequence, once the call option price $P_c(S,\tau)$ is evaluated, the put option price $P_p(S,0)$ can be obtained readily.

In the following section we shall present numerical results of the time-dependent CEV model, which are obtained by employing a very efficient computing algorithm similar to the one proposed by Schroder [21]. The numerical data shows that the time-dependent term-structures of the model parameters have a significant effect on the option prices.

4. Numerical Results

The pricing formula Eq. (37) is used to evaluate call option values with time-dependent volatility $\sigma(\tau)$ for different CEV processes. The risk-free interest rate and dividend are assumed to be constant. As examples, we consider the following Gaussian type term structures for the volatility variance $\sigma(\tau)$:

(1) Term structure A

$$\sigma(\tau)^2 = \sigma_0^2\left\{1 + a_0\exp\left[-\frac{(\tau-\tau_0)^2}{b_0}\right]\right\}$$

(2) Term structure B

$$\sigma(\tau)^2 = \sigma_0^2\left\{1 + a_0\exp\left[-\frac{(\tau-\tau_0)^2}{b_0}\right] + a_1\exp\left[-\frac{(\tau-\tau_1)^2}{b_1}\right]\right\}$$

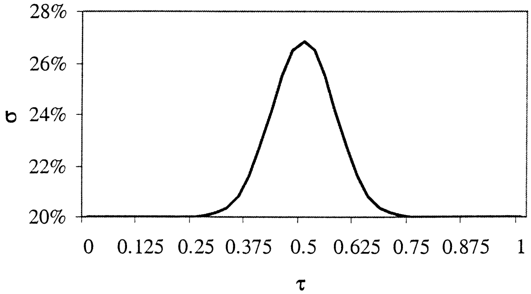

Fig. 1. Volatility term structure A with $\sigma_0 = 20\%$, $a_0 = 1$, $b_0 = 0.01$ and $\tau_0 = 0.5$.

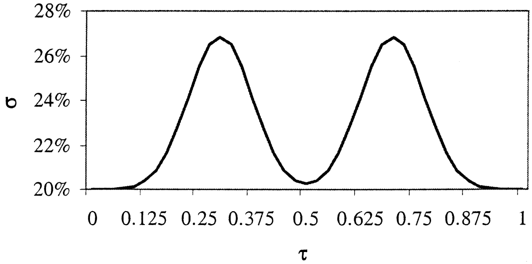

Fig. 2. Volatility term structure B with $\sigma_0 = 20\%$ $a_0 = a_1 = 1$, $b_0 = b_1 = 0.01$ and $\tau_0 = \tau_1 = 0.5$.

The above term structures can be interpreted as pulses of surge or drop (depending on the sign of a_0 and a_1) in market volatility. The centres of the pulses are at time τ_0 and τ_1. The width of the pulses is determined by b_0 and b_1. Volatility term structures with one pulse and two pulses of surge, i.e. positive a_0 and a_1, are chosen to perform numerical calculations. The term structures A and B with parameters $\sigma_0 = 20\%$, $a_0 = a_1 = 1$, $b_0 = b_1 = 0.01$, $\tau_0 = 0.5$ for term structure A, and $\tau_0 = 0.25$ and $\tau_1 = 0.75$ for term structure B are illustrated in Figs. 1 and 2. Since the term structures are Gaussian functions, $c_3(\tau)$ in Eq. (27) can be determined analytically.

To illustrate the effects of the volatility term structures on option valuation, we choose a call option with underlying price $S = 20$, risk-free interest rate $r = 5\%$ and dividend $d = 0$. We consider three CEV processes for $\beta = 0$ (Uhlenbeck), $\beta = 1$ (square root) and $\beta = 2$ (Black–Scholes) in the numerical calculations. The option maturity τ is up to 1 year. The strike prices S_0 are 18, 20 and 22 respectively. The numerical results are summarised in Tables 1 and 2 for term structure A and term structure B, respectively. The results for 20% flat volatility are shown in Table 3.

The volatility pulse causes the increases in the option values in general, especially for options with a longer maturity period. For a shorter maturity period, say $\tau = 0.25$, the impact of the pulse does not result in much difference to the option values among different volatility term structures. By comparing the option

Table 1. Call values with volatility term structure A.

	$\beta = 0$			$\beta = 1$			$\beta = 2$		
T-t	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 18$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$
0.25	2.3537	0.9209	0.2096	2.3442	0.9231	0.2245	2.3340	0.9231	0.2383
0.5	2.8004	1.4663	0.6126	2.7825	1.4720	0.6440	2.7603	1.4709	0.6697
0.75	3.1891	1.8926	0.9754	3.1683	1.9039	1.0211	3.1383	1.9024	1.0551
1	3.4870	2.2006	1.2451	3.4679	2.2188	1.3031	3.4344	2.2180	1.3428

Table 2. Call values with volatility term structure B.

	$\beta = 0$			$\beta = 1$			$\beta = 2$		
T-t	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$
0.25	2.3827	0.9729	0.2463	2.3717	0.9750	0.2627	2.3598	0.9747	0.2779
0.5	2.8384	1.5177	0.6591	2.8195	1.5241	0.6926	2.7961	1.5235	0.7202
0.75	3.2339	1.9486	1.0298	3.2118	1.9603	1.0774	3.1801	1.9589	1.1129
1	3.5457	2.2725	1.3170	3.5252	2.2912	1.3775	3.4895	2.2903	1.4190

Table 3. Call values with flat volatility 20%.

	$\beta = 0$			$\beta = 1$			$\beta = 2$		
T-t	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$
0.25	2.3537	0.9209	0.2096	2.3442	0.9231	0.2245	2.3340	0.9230	0.2382
0.5	2.7333	1.3718	0.5281	2.7184	1.3780	0.5573	2.6997	1.3777	0.5813
0.75	3.0742	1.7437	0.8316	3.0574	1.7549	0.8733	3.0327	1.7545	0.9044
1	3.3859	2.0738	1.1179	3.3694	2.0908	1.1714	3.3399	2.0901	1.2080

values of different strikes, we see that the volatility term structures have a greater influence on the out-of-the-money (OTM) call values in absolute numerical terms and in percentage terms. This is shown from the in-the-money (ITM) (strike = 18) Black–Scholes model value with term structure A, which is 3% higher than that with the flat volatility, while the OTM (strike = 22) Black–Scholes model value with term structure A is 11% higher than that with the flat volatility. Similarly, the ITM (strike = 18) Uhlenbeck model value with term structure A is 3% higher than that with the flat volatility, while the OTM (strike = 22) Uhlenbeck model value with term structure A is 11% higher than that with the flat volatility. The same observation can also be found in numerical results using term structure B. The reason for this observation is that an OTM option value is mainly due to time value which is affected by volatility. Therefore OTM options are more sensitive to volatility term structures.

Table 4. Call values with volatility term structure A of different t_0.

	$\beta = 0$			$\beta = 1$			$\beta = 2$		
t_0	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$	$S_0 = 18$	$S_0 = 20$	$S_0 = 22$
0.25	3.9538	3.9291	3.8841	2.7467	2.7754	2.7802	1.7922	1.8748	1.9346
0.5	3.9959	3.9644	3.9126	2.7940	2.8163	2.8142	1.8396	1.9168	1.9703
0.75	3.9777	3.9407	3.8841	2.7736	2.7889	2.7802	1.8191	1.8887	1.9346

The numerical results show that the OTM option model values decrease with the decrease in the elasticity factor β while the ITM option model values increase slightly with the increase in β. They are consistent with the calculations performed by Beckers [1]. Since OTM options with a longer maturity period are more sensitive to volatility term structures, if the superiority of the CEV model vis-a-vis the OTM and warrants with longer maturity periods [10] is clear, the CEV option pricing model incorporating time-dependent model parameters could be important in making the model a more attractive alternative formulation of equity option pricing.

In order to study the impact of the location of surges in volatility term structures on option model values, term structure A with parameters $\sigma_0 = 20\%$, $a_0 = 3$, $b_0 = 0.005$ is used to perform calculations by putting the surges at $t_0 = 0.25$, 0.5 and 0.75 respectively. The option maturity τ is 1 year. The same strike prices and β values in the previous calculations are used in this study. The numerical results are shown in Table 3. The results show that in general the surge in the middle of the option life gives the highest option values. However the impact is not considered to be significant.

In summary, we have shown that the volatility term structure effects have a significant impact on the valuation of options, especially OTM options with long maturity periods. The CEV option pricing model incorporating time-dependent model parameters could provide a better formulation of equity option pricing. In the above illustrations, the volatility term structures being used can be analytically integrated to obtain option model values. Using numerical integration, it is easy to apply the formalism in the previous section to other term structures which cannot be integrated analytically. It is also possible to generate further analytical results using various functional forms for interest rate and dividend term structures.

5. Conclusion

This paper provides an easy-to-use method for pricing options (both call and put options) in the CEV model environment using the Lie-algebraic technique when the model parameters are time-dependent. It provides analytical solutions for the option values incorporating time-dependent model parameters in various stock dynamic processes such as the Ornstein–Uhlenbeck, square root and lognormal. The numerical results indicate that option values are sensitive to volatility term structures. The results are computed by employing a very efficient algorithm similar to the one proposed by Schroder [21]. It is also possible to generate further numerical results using various functional forms for interest rate and dividend term structures.

It is often important to determine the hedge parameters such as delta, gamma, vega and theta risks of equity options in order to hedge option positions. From the analytical option pricing formulae for the CEV model with time-dependent model parameters, the hedge parameters can be computed easily. One can achieve more accuracy to compute the hedge parameters from the CEV model using more realistic term-structures in volatility, interest rate and dividend yield. In view of the CEV model being empirically considered to be a better candidate in equity option pricing than the traditional Black–Scholes model, more comparative pricing and precise risk management in equity options can be achieved by incorporating term-structures of interest rates, volatility and dividend into the CEV option valuation model.

Finally, we would like to make a couple of remarks: (1) The results presented in this paper can be straightforwardly applied to those non-CEV processess which possess the same dynamic symmetry of su(1,1). Since there are many different possible realizations of the generators of the Lie algebra su(1,1), we have basically solved a very large class of option pricing problems. (2) The Lie-algebraic approach is indeed very simple and can be easily extended to other models of derivatives with well-defined algebraic structures. For instance, both the risky bond and the commodity-linked bond can be solved by this method. Detailed results of these two bond models will be presented elsewhere.

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