Valuing Basket Options with Asset and Correlation Smiles

Peter Austing

Published in *Risk* as *Deconstructing Correlation*, June 2016

Abstract

To properly value a basket option, one should construct a joint probability density correctly repricing all asset smiles and correlation smiles. At first sight, the task seems formidable. However, by reformulating the problem, we can develop a model that is simple and fast, admitting analytic or semi-analytic valuation.

The views in this article are the author's own, and do not necessarily represent those of any employer, past or present.

1 Introduction

When markets are stressed, assets become more strongly correlated. Just as there is an implied volatility smile, there is also a correlation smile. However, though it is known to exist, the correlation smile has been largely ignored in multi-asset derivatives pricing for reasons of practicality: there may not be liquid instruments to determine the smile, and even if it is known, practical pricing algorithms taking account of the smile may not exist.

The former can be dealt with quickly. Even when multi-asset contracts are not liquid, one can pick an instrument with which to define implied correlation, and use some simple intuition to mark the wings of the smile. Indeed, this should be considered essential for trading desks with correlation risk. While the correlation smile marked may not be perfect, it allows traders to make markets in correlation, and to price and hedge conservatively. More importantly, it allows one to calculate correlation smile Greeks and so understand exposure to the high stress market events in which correlation increases towards 100%.

It is the second practicality, of correlation smile aware models, that can prove difficult. There has been exciting progress in local correlation modeling, as in Langnau (2010) and subsequent authors including Guyon (2014). However, pricing is necessarily heavy, and so local correlation is challenging when there are more than a small number of assets. More recently, a very interesting mixture dynamic approach has been proposed by Brigo, Pisani, and Rapisarda (2015). This is more tractable numerically, though less easy to calibrate to general smiles.

Baskets are the most interesting and challenging of the classic European contracts, since they are regularly traded with large numbers of assets. We will work with a basket of N fundamental assets, having N corresponding volatility smiles. In major financial institutions, it is common practice to assume constant correlations between the assets, and to use a copula or local volatility model to value contracts. Our aim is, in addition, to give each of the $N(N-1)/2$ pairwise correlations a smile.

Given an expiry and N assets, our task is to construct a joint probability density with the property that all the instruments that define the asset smiles and correlation smiles are priced correctly, and then to use that density to value basket options. A number of approaches have been proposed for valuing baskets consistent with asset smiles assuming constant correlations. These include Qu (2001, 2005), Avellaneda et al. (2002) and Hakala and Wystup (2008). A consequence of the analysis we will present is a simple way of seeing that these do indeed have underlying densities consistent with the asset smiles, even (as in Avellaneda et al. 2002) when approximations are used in going from an original underlying model. However, the meaning of the flat correlation used is dependent on the smile method, with the result that the value of the basket is dependent on that unknown parameter. In addition, there is no possibility to model correlation smile. Our method solves these issues by repricing correlation smiles whose meaning, reflecting the value of tradable assets, is unambiguous.

2 Implied Correlation Smile

In order to define implied correlation, we need to pick a derivative contract that is heavily correlation dependent. The implied correlation will then be *that correlation we plug into the Black-Scholes formula to get the true market price*. The most appropriate instrument to use depends on the asset class. In foreign exchange, cross-vanilla options are very often liquid, and even when they are not traded, they are intuitive to market makers. In other markets, two-asset baskets may be a more intuitive correlation instrument. We will begin our analysis by defining the correlation smile with respect to two-asset baskets, but we will not be limited to this case.

Let us first provide a definition of implied correlation that will be useful throughout. We must choose a contract that has a parameter – the strike – that can indicate moneyness of correlation. We choose to use 2-asset equally weighted at-the-money forward basket options. That is, given an expiry T and strike K, the implied correlation $\rho^{\text{imp}}(K,T)$, is the correlation we should plug into the Black-Scholes formula to correctly re-price the contract having payout

$$B(S_1,S_2) = (K - 0.5S_1/F_1 - 0.5S_2/F_2)_+, \tag{1}$$

where S_i are the spots at expiry and F_i are the forwards to expiry fixed at the inception date.

Before we can proceed, a number of details must be established in order to complete this definition. Firstly, there is no analytic formula for the Black-Scholes price of a 2-asset basket. Therefore, one might ask in which particular approximation to that formula? However, a 2-asset basket put option can be valued with a single numerical integration using the formula

$$E[B] = \int_0^K dU\, C\left(\frac{U}{w_1}, \frac{K-U}{w_2}\right) \tag{2}$$

where C is the joint cumulative distribution function, and in this case $w_i = 0.5/F_i$. This can be evaluated so efficiently that there is no real need for an analytic approximation. Therefore our definition for implied correlation may as well use the *true* 2-asset Black-Scholes price.

A more pressing issue is the choice of implied volatilities to use when backing out the implied correlation. As correlation varies, a tight upper bound on the basket price is obtained when the assets are perfectly correlated. This is intuitively sensible, and can be obtained with an application of the Lagrangian method as in Laurence and Wang (2004). In that case, the basket price is equal to the value of a portfolio of vanillas, giving the bound

$$E[(K - 0.5S_1/F_1 - 0.5S_2/F_2)_+] \le 0.5E[(K_1 - S_1/F_1)_+] + 0.5E[(K_2 - S_2/F_2)_+] \tag{3}$$

where K_1 and K_2 are the strikes having the same risk-neutral probability as each other and $0.5K_1 + 0.5K_2 = K$.

For the purpose of our definition of implied correlation, we choose to simplify this prescription, ignoring the skew term from the calculation of the risk-neutral probability, and approximating with the Black-Scholes delta. This will give a prescription that is more intuitive to market makers. Let $d_1(K)$ be the standard parameter from Black-Scholes theory and define $\Delta_i(K_i) = N[d_1(K_i)]$ where N is the cumulative normal function. Then we choose our strikes to be the solution of

$$\Delta_1(K_1) = \Delta_2(K_2) \tag{4}$$
$$0.5K_1/F_1 + 0.5K_2/F_2 = K, \tag{5}$$

and call these the *optimal strikes*. The implied volatilities that we use in our definition of implied correlation are taken from the volatility surfaces at these strikes

$$\sigma_i = \sigma_i^{\mathrm{imp}}(K_i, T). \tag{6}$$

Similarly, we call these the *optimal volatilities*.

More generally, if we had not restricted ourselves to an equally weighted 2-asset basket, but instead considered the payout

$$(\sum_{i=1}^{N} w_i S_i - K)_+ \tag{7}$$

the same argument would provide us with optimal strikes given by solving

$$\Delta_1(K_1) \;=\; \cdots = \Delta_N(K_N) \tag{8}$$
$$\sum_i w_i K_i \;=\; K \tag{9}$$

and corresponding optimal volatilities.

To summarise, we define implied correlation $\rho^{\mathrm{imp}}(K,T)$ to be the correlation to plug into the Black-Scholes formula for a 2-asset basket option with payout (1) and implied volatilities given by (6) to get the true market price. As we remarked earlier, if the market price does not exist liquidly, it is up to market makers to mark an implied correlation smile in the same way that vanilla traders do for the implied volatility of single-asset options.

3 Basket prices versus probability density

We are going to use the observation that knowledge of all N-asset basket prices implies knowledge of all N-asset European option values. This was discovered independently in the three ground breaking works of Henkin and Shananin (1990), Baxter (1998) and Lipton (2001). The former approach was motivated by problems in economics, and its importance to finance was recognised in d'Aspremont and El Ghaoui (2006).

While Baxter (1998) used the Fourier transform, and Lipton (2001) the Radon transform, we follow Henkin and Shananin (1990) in using the Laplace transform to explain the result.

Our aim is to start with the set of basket prices and to deduce the joint probability density function. The undiscounted basket price is

$$B(w,K) = \int (K - S\cdot w)_+ f_T(S)\, dS_1 \cdots dS_N \tag{10}$$

where $f_T(S)$ is the joint probability density function. Then differentiating twice with respect to K gives

$$\frac{\partial^2}{\partial K^2} B = E[\delta(w \cdot S - K)], \tag{11}$$

which is the probability density function of $w \cdot S$.

On the other hand, the Laplace transform of $f_T(S)$ is the expectation of a function of $w \cdot S$, and so we can compute it

$$\begin{aligned} \phi(w) &= E[e^{-w \cdot S}] && (12) \\ &= \int_0^\infty dk\, e^{-k} \frac{\partial^2}{\partial K^2} B(w, K). && (13) \end{aligned}$$

Uniqueness of the Laplace transform on $\mathbb{R}^n_+$ tells us that knowledge of the function $\phi(w)$ for positive weights w is enough to determine the joint probability density $f(S)$. Then regarding the basket pricing formula (10) as an integral transform on the probability density function, we denote the inverse by $\mathcal{T}^{-1}$

$$f_T(S) = (\mathcal{T}^{-1} B)(S). \tag{14}$$

The uniqueness theorems for the Fourier or Radon transforms require knowledge of the basket prices for weights in all of $\mathbb{R}^n$ and therefore we would need the prices of all basket *spreads* in addition to pure baskets to pin down the joint probability density. By using the uniqueness property of the Laplace transform, Henkin and Shananin (1990) show that the prices of baskets with positive weights only are enough to determine the joint density. Furthermore, they are able to provide conditions on the basket prices for the density to be everywhere positive and therefore a true arbitrage-free probability density.

To apply (14) in practice, it would be necessary to know the values of basket options for all positive weights and strikes. Although major banks will provide quotes on such options, they are certainly not liquid enough to deduce the joint density. Fortunately, we will never need to perform the inversion numerically.[1] Rather, it is the mathematical equivalence between knowledge of basket prices and knowledge of the joint probability density that is important to us in what follows.

4 Valuing basket options

We are now ready to value the N asset basket payout

$$B = (\sum_{i=1}^{N} w_i S_i - K)_+. \tag{15}$$

As we have emphasized, in order to do this we need to construct a joint probability density function. That density function must have the property that when it is used to value any

[1] If we did wish to extract the density function from basket prices, we certainly would not attempt to invert a Laplace transform numerically (a notoriously difficult problem).

vanilla or correlation instrument (the 2-asset baskets that define the correlation smiles) it must give the correct price. Therefore, we need to construct something resembling a copula, except that in addition to the N single-asset marginals, a set of $N(N-1)/2$ cross marginals must also be matched. Once this density function has been constructed, we'll need to integrate it against the payout to obtain the price.

On the face of it, this would seem to be a difficult task. But we can use the equivalence between knowledge of basket prices and knowledge of the density function. Instead of attempting to construct the joint probability density directly, we simply postulate the value of basket options with all possible weights. As long as the prices we postulate are smooth as the weights vary, and have the property that the price is correct in the limits when the basket becomes an asset vanilla or one of the correlation instruments, then the underlying density function must re-price all asset smiles and correlation smiles.

There are many ways to do this, but it is important to provide a prescription giving as smooth a deformation from Black-Scholes as possible. We begin with the Black-Scholes formula for an N asset basket

$$P_{\mathrm{BS}}(w, K; \sigma, \rho) \tag{16}$$

where w are the weights, K the strike, σ the vector of asset volatilities and ρ the matrix of correlations. As there is no analytic representation of (16), we must use an approximation. Our results were obtained by matching three moments to a shifted log-normal distribution.

The next step is to postulate a set of basket prices by plugging a suitable functional form for the volatilities and correlations into the Black-Scholes formula

$$P(w, K) = P_{\mathrm{BS}}(w, K; \sigma(w, K), \rho(w, K)). \tag{17}$$

For the volatilities, we use the optimal volatilities (8), (9), that is, the implied vols at the optimal strikes

$$\sigma_i(w, K) = \sigma_{\mathrm{imp}}(K_i). \tag{18}$$

These have the desired property that when all weights but one (say the ith weight w_i) go to zero, the corresponding strike converges to the basket strike

$$K_i \to K\,, \quad w_j \to 0 \;\; j \neq i \tag{19}$$

so that, indeed, vanilla options are correctly priced.

Finally we need to choose the functional form of the correlations $\rho(w, K)$ so that the correlation instruments (2-asset baskets) are correctly priced. To do so, we construct strikes for each possible 2-asset sub-basket from the optimal strikes

$$K_{ij} = w_i K_i + w_j K_j \tag{20}$$

and then read correlations off from the implied correlation smile

$$\rho_{ij}(w, K) = \rho^{\mathrm{imp}}(K_{ij}). \tag{21}$$

Again, this choice has the desired property that the price is guaranteed correct when all weights but two go to zero so that the basket becomes an equally weighted 2-asset basket.

Our postulated basket prices are

$$P(w, K) = P_{\text{BS}}(w, K; \sigma_i^{\text{imp}}(K_i), \rho_{ij}^{\text{imp}}(K_{ij})). \tag{22}$$

where K_i are the solution of (8), (9). It follows that our joint density function is given by applying the basket transform (14)

$$f_T(S) = \mathcal{T}^{-1}\left[P(w, K)\right]. \tag{23}$$

By construction, this density function has the properties that it correctly re-prices the asset smiles and correlation smiles. Furthermore, it reduces to (a good approximation of) the Black-Scholes density when all volatility and correlation smiles are flat since in that case it values basket options according to Black-Scholes.

For theoretical reasons we have presented the algorithm as construction of a density function. However, for practical pricing, an attractive aspect of the approach is that it amounts to careful selection of volatilities and correlations to plug into the Black–Scholes formula. This approach is an extension of the usual smile pricing of vanilla options. In addition to payoff dependent volatilities we have payoff dependent correlations.

In practice, the correlation matrix constructed may sometimes go negative definite. In this case one approach is to regularise the matrix by flooring eigenvalues at zero and re-scaling. Alternatively, as the moment matching formula does not rely on positive definiteness, one can simply go ahead and use the matrix without modification. When the number of assets is larger, it is harder to have a positive definite correlation matrix, but it also becomes harder to monetize any theoretical arbitrage. Therefore our preference is for the latter "calculate anyway" regularisation approach as a means of adding a cost to such market uncertainty. Assuming the at-the-money correlations are positive definite, this will only happen away from the money where the price impact is small.

5 The foreign exchange case

Two-asset baskets are not necessarily the most convenient correlation instruments. In foreign exchange, cross vanillas are more appropriate. They have payout, expressed in a common domestic currency,

$$(S_1 - KS_2)_+ \tag{24}$$

but when expressed in units of the second asset, the payout is an ordinary vanilla

$$(S_1/S_2 - K)_+. \tag{25}$$

For this reason, rather than define an implied correlation expressing their market value, it is more natural to provide an implied volatility for the cross S_1/S_2. Then to value foreign exchange baskets, or if the correlation smiles are provided with respect to instruments (24), we need to construct a joint density function re-pricing asset smiles and cross smiles.

Austing (2011) constructs a joint density function with this property. We'll call it the triangle density since it re-prices the triangle of smiles. It is not feasible to value baskets with large numbers of assets directly using the triangle density, but we can value 2-asset baskets very fast using the formula (10).

To value N-asset basket options when the correlation instruments are cross vanillas, we begin by following the steps given by equations (16) to (20) for standard baskets. At this stage we have computed the volatilities σ, and have a set of strikes K_{ij} for all the $N(N-1)/2$ possible 2-asset sub-baskets having weights w_i, w_j. We value these sub-baskets using the triangle density, each requiring a single numerical integration. Now that we have the price and volatilities σ_i, σ_j for the(i,j)th 2-asset sub-basket, we can solve to find the correlation $\rho_{ij}^{\text{tri}}(w_i, w_j, K_{ij})$ so that the Black-Scholes price matches the triangle density price.

Then we replace our postulated basket prices (22) with

$$P(w,K) = P_{\text{BS}}(w,K;\sigma_i^{\text{imp}}(K_i), \rho_{ij}^{\text{tri}}(w_i,w_j,K_{ij})). \tag{26}$$

As before we can in principle apply the basket transform (14) to obtain a joint density function. By construction, the density function re-prices the asset smiles and gives prices for all 2-asset sub-baskets matching the triangle density. Since the value of all 2-asset baskets uniquely defines the 2-asset marginal densities (by another application of (14)), this shows that the 2-asset marginals are equal to the triangle density and therefore correctly re-price the cross smiles.

6 Numerical results

The joint density we have constructed is not unique. Indeed, given only the constraints of re-pricing asset and correlation smiles, there is rather a large range of possible prices for a given multi-asset contract (see Piterbarg 2011). Our density is sensible in that it is smooth and reduces to Black-Scholes when all smiles are flat, but it is our duty to test it against a true dynamic model.

This is possible for the fx case in which the correlation instruments are cross vanillas. Let $\sigma_i^{\text{local}}(t,S_i)$ be the Dupire local volatility function for asset i, and let $\sigma_{ij}^{\text{local}}(t,S_i/S_j)$ be the Dupire local volatility function for the ij cross vanilla. Then we can define the fx local correlation

$$\rho_{ij}^{\text{local}}(t,S_i,S_j) = \frac{\sigma_i^{\text{local}}(t,S_i)^2 + \sigma_j^{\text{local}}(t,S_j)^2 - \sigma_{ij}^{\text{local}}(t,S_i/S_j)^2}{2\sigma_i^{\text{local}}(t,S_i)\sigma_j^{\text{local}}(t,S_j)} \tag{27}$$

and allow the spot processes to follow

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i^{\text{local}}(t,S_i)dW_i \tag{28}$$

$$dW_i dW_j = \rho_{ij}^{\text{local}}(t,S_i,S_j)dt. \tag{29}$$

Then asset smiles are re-priced as the assets follow Dupire local volatility, and it is straightforward to check that the implied processes followed by the crosses S_i/S_j are also Dupire local volatility so that their smiles are correctly re-priced also.[2]

[2]In practice, if the local correlation matrix goes negative definite, it must be regularised, e.g. by flooring any negative eigen-value and rescaling (see Austing (2014)). The impact of any regularisation can be assessed by re-valuing cross-vanillas.

We can use the fx local correlation model to value basket options as a comparison for our semi-analytic model developed in section 5. To demonstrate typical results, we have valued baskets containing euros, sterling, Japanese yen and Canadian dollars against US dollars, with expiries of six months and one year. The baskets are equally weighted (with respect to the forward levels) and we used market data on 18th May 2012, including volatility smiles for each of the assets, and each of the crosses (ten volatility smiles in total). Figure 1 shows the results.

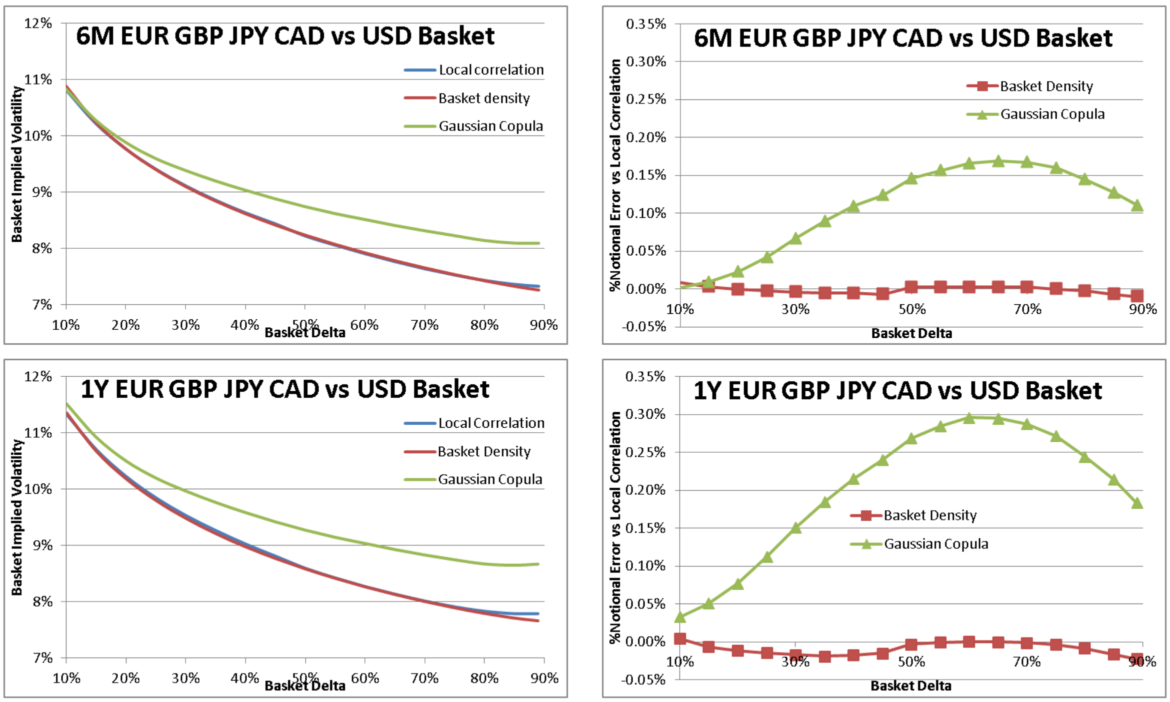

Figure 1: Basket options with expiries 6M and 1Y: equally weighted with respect to forward level EURUSD, GBPUSD, JPYUSD, CADUSD. Implied volatility calculated in model, and %notional error versus fx local correlation

We have plotted the implied volatility of the basket, calculated in our model (labelled basket density), in fx local correlation, and using a Gaussian copula. As is traditional in foreign exchange, we have plotted against absolute value of put delta instead of strike. The put delta is calculated using the Black-Scholes formula with implied volatility from our model.

The local correlation and basket density implied volatilities sit almost on top of each other. In implied volatility terms, they are at most 21 basis points apart. But this is at 90 deltas where vega is small, and corresponds to only a small difference in PV. Indeed the PV difference between the two models measured in percentage of notional is at most 2 basis points for the 1Y option, and 1 basis point for the 6M option. This is within the tolerance of the Monte Carlo simulation used to calculate the local correlation prices.

The Gaussian copula does not fare so well. Being a copula, it re-prices the asset smiles by construction. The correlations have been calibrated so that the at-the-money forward vanillas are correctly re-priced for each of the crosses. The percentage notional error against local correlation is 30 basis points for the 1Y basket at 60% delta. This is significant; the PV of the basket option at that strike is only 2.23%.

In summary, our model has matched fx local correlation for the four asset basket prices to high accuracy.

7 Discussion

We have constructed basket prices with an underlying density function that re-prices implied volatility smiles and pairwise implied correlation smiles, and demonstrated in numerical examples that the prices obtained are close to a true dynamic local correlation model.

The approach of this article is to treat the pairwise correlation smiles as fundamental market quantities. In some cases, notably baskets of equities, it may be that smiles for sub-baskets representing indices of more than two fundamental assets are more readily available. In that case, our approach would be to calibrate the pairwise correlation smiles to match $N(N-1)/2$ of the index smiles.

The values of multi-asset options including baskets can have significant dependence on the choice of correlation smile aware model. Therefore it is remarkable that our model matches the fx local correlation model so closely. Extensive testing has confirmed this across numerous fx market data sets and expiries. The constructions of both the local correlation model and our semi-analytic model, are driven in a natural way directly from the cross smiles. We suspect that it is this similarity in construction that has lead to such a happy result.

As demonstrated by Qu (2005), a basket book is complex and involves significant second order effects. Therefore, capturing correlation smile with an analytic (or semi-analytic) model represents a significant step forward for risk management. The Greeks that we can obtain by bump and revalue are smooth, and very fast. Furthermore, since our model is an on-smile deformation of Black–Scholes, it is easy for risk managers to explain the meaning of the greeks. For example, if there is concern about the exposure of a book to correlation in the wings, that part of the smile can simply be bumped, and the risk obtained has a very clear interpretation as sensitivity to implied correlation in the region of interest.

With certain configurations of market data, the density function we have constructed may not be everywhere positive, and therefore may not be a true *probability* density. It is not possible to avoid this risk entirely. We are given a market of volatility smiles and correlation smiles, and intend to build a joint density from it. If that market contains arbitrage, then the density we obtain will necessarily be bad.

To mitigate this as far as possible, we set up our basket prices using functions that provide as smooth a deformation from Black-Scholes as possible. Assuming the Black-Scholes value is arbitrage-free then, considering the smiles as small deformations from Black-Scholes, we are on relatively safe ground. However, we emphasise that we have provided no proof that the density function will be everywhere positive even if there is no arbitrage in the market data. Indeed, given the freedom of choice in setting up our model, it would be surprising to learn of such a result. Thus, the comparison of basket prices against the manifestly arbitrage-free local correlation model is a crucial part of this article.

There is more than one choice for the instruments defining implied correlation. If 2-asset baskets are used, our method is fully analytic. If cross vanillas (or composite options) are used, the method is semi-analytic, requiring $N(N-1)/2$ one-dimensional numerical integrations, and it is in this case that we have provided numerical results. In this way, the method provides fast and stable prices for basket options that are fully on-smile. In particular, calculation of Greeks is significantly more stable than can be achieved with a local correlation Monte Carlo simulation. Given that the model is entirely unrelated to fx local correlation, other than that both re-price asset and correlation smiles, it is delightful that they agree to such high accuracy.

References

Austing, Peter. “Repricing the cross smile: an analytic joint density”. *Risk* July (2011).

— *Smile Pricing Explained*. Palgrave Macmillan, 2014.

Avellaneda, Marco, Dash Boyer-Olson, Jerome Busca, and Peter Friz. “Reconstructing Volatility”. *Risk* October (2002).

Baxter, Martin. “Hedging in financial markets”. *Astin Bulletin* 28 (1998), pp. 5–16.

Brigo, Damiano, Camilla Pisani, and Francesco Rapisarda. “The Multivariate Mixture Dynamics Model: Shifted dynamics and correlation skew” (2015).

d’Aspremont, Alexandre and Laurent El Ghaoui. “Static arbitrage bounds on basket option prices”. *Mathematical programming* 106.3 (2006), pp. 467–489.

Guyon, Julien. “Local Correlation Families”. *Risk* February (2014).

Hakala, Jurgen and Uwe Wystup. “FX Basket Options”. *Frankfurt School of Finance and Management* (2008).

Henkin, G and A Shananin. “Bernstein theorems and Radon transform, application to the theory of production functions”. *American Mathematical Society: Translation of mathematical monographs* 81 (1990), pp. 189–223.

Langnau, Alex. “A dynamic model for correlation”. *Risk* April (2010).

Laurence, Peter and Tai-Ho Wang. “What’s a basket worth?” *Risk* (2004).

Lipton, Alexander. *Mathematical Methods for Foreign Exchange*. World Scientific, 2001.

Piterbarg, Vladimir V. “Spread options, Farkas’s lemma and linear programming”. *Risk* September (2011).

Qu, Dong. “Basket Implied Volatility Surface”. *Derivatives Week* June (2001).

— “Pricing basket options with skew”. *Wilmott* July (2005).