



OpenGamma Quantitative Research

Inflation caps and floors

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Abstract

The purpose of this document is to present analytical formulas for the inflation caps and floors (year on year and zero-coupon). The framework of the note is the forward price index market model.

This technical note is complementary to [Zine-eddine \(2013b\)](#), and consequently reuses the same notations and definitions, particularly for the price index and the forward price index.

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1 Introduction and notation

The nominal rates refer to the habitual rates, and real rates refer to the rates adjusted with inflation. In the real economy, prices are defined in terms of purchasing power (real price) and not in nominal price.

A relation between nominal rates (n), real rates (r), and inflation (i) is given by the well known Fisher equation:

$$1 + n = (1 + r)(1 + i)$$

An analogy can be made with the foreign currency market: real rates can be interpreted as a foreign asset, nominal as domestic asset and exchange rate as the inflation rate.

We will use the subscript n (respectively r) when a quantity refers to nominal economy (respectively the real economy).

We assume there is a discount curve in both economies $t \mapsto P_x(t_0, t)$ for $x \in \{n, r\}$ and $t_0 \leq t$. For $i = 0 \dots N$, we define a schedule T_i such as $T_0 < T_1 < \dots < T_N$. The nominal and real risk-neutral measures are denoted by $\mathbb{Q}_n, \mathbb{Q}_r$, and the price index is noted $I(t)$.

The forward price index at time t for maturity T_i is defined by

$$\mathcal{I}(t, T_i) = I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)} \quad (1)$$

We notice that for $t \geq T_i$, $\mathcal{I}(t, T_i) = I(T_i)$.

Our purpose in this note is to provide closed formula for classical inflation-linked options.

2 Model description: the forward price index market model

There are two main models to price inflation optional products: the Jarrow Yildirim using a specific HJM three factors ([Jarrow and Yildirim \(2003\)](#)) and the market model ([Belgrade et al. \(2004\)](#)).

There are numerous reasons to choose the market model, for example:

- there seems to be a market consensus on the market model (sometimes with a SABR parametrisation of the smile),
- the Jarrow-Yildirim requires to one to calibrate non observable parameters such as those linked to the real economy,
- the market model allows us to easily incorporate a SABR parametrisation of the smile.

Let's give a quick description of this model. By definition, the forward CPI is a martingale under $\mathbb{Q}_n^{T_i}$ which is the T_i -forward neutral measure (ie the one associated to the numeraire $P_n(t, T_i)$). Observing that, we also make the assumption of a lognormal distribution under the T_i -forward neutral measure:

$$d\mathcal{I}(t, T_i) = \sigma_i^I \mathcal{I}(t, T_i) dW_i^I(t),$$

where the volatility σ_i^I is a positive constant and where $W_i^I(t)$ is a standard brownian motion under $\mathbb{Q}_n^{T_i}$.

For $i, j = 1 \dots N$ we also define $\rho_{i,j}^I$ as the instantaneous correlation between $\mathcal{I}(t, T_i)$ and $\mathcal{I}(t, T_j)$.

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3 Time to maturity for inflation options

3.1 Time to maturity

For a standard option the time to maturity T is usually calculated with the following formulae $T = t - T_{pay}$ where t is the evaluation date (usually today which means $t = 0$) and T_{pay} is payment date.

To be precise, T is always a number, and t and T_{pay} can be either two times or two dates. In the latter case, the formula $T = t - T_{pay}$ implicitly assumes the use of a day counter. Usually for an inflation option, there are three relevant dates: the fixing date T_{fix} which is the date when the price index is fixed; the publishing date T_{pub} which is the date when the price index is published (usually 15 days after the fixing); and the payment (usually at least 2 months after the fixing date). If we use the standard calculation for the time to maturity, then the time to maturity is non null between the publishing date and the payment date. But once the index is published, there is no more optionality so the total variance $\sigma^2 T$ should be null. That's why we use the following formula for the time to maturity

$$T = T_{fix} - T_{last}$$

where T_{last} is date of the last known fixing. Notice that, for price index, fixing dates are always a first of the month, and the last known fixing date is always a date in the past.

3.2 The fixing date

Defining the fixing date T_{fix} in the above formula could be tricky, let's do it precisely. As we know, there are two different standard ways to interpolate a price index.

The first one (we can call this way monthly or piecewise constant) is to take the same value of the price index for any date in the same month, for example the value of the price index fixing the 26 of September (or any other day in September) is the value the first of September. In this case, the fixing date used in the time to maturity is the first of September.

The second one (we can call this way interpolated) is to interpolate between the first day of two consecutive months. For example the value of the price index fixing the 26 of September is linearly interpolated between the first of September and the first of October. In this case, the fixing date used in the time to maturity is the first of October, because we know the value only when the value of the first of October is published.

4 Year-on-year caps and floors

4.1 Pricing formulas

Year-on-year caps and floors in the inflation market operate in the same way as their counterparts in the interest rate market. Analogously to Libor caplet, a year-on-year cap (respectively floor) is a string of consecutive and uniform year-on-year caplet (respectively floorlet). A year-on-year caplets (respectively a floorlets) is a call (respectively a put) option on the inflation rate implied by the price index. The payoff at maturity T_{i+1} of this structure is

$$N \delta_i \left[\omega \left(\frac{I(T_{i+1})}{I(T_i)} - 1 - K \right) \right]^+$$

where N is the notional, δ_i is the day count fraction, and $\omega = 1$ for a caplet, $\omega = -1$ for a floorlet. We define the year-on-year forward rate as

$$\mathcal{Y}(t, T_i, T_{i+1}) = \mathbb{E}^{T_{i+1}} \left[\frac{I(T_{i+1})}{I(T_i)} - 1 | \mathcal{F}_t \right]$$

where \mathcal{F}_t is the σ -algebra representing the information until time t and $\mathbb{E}^{T_{i+1}} [\cdot]$ is the expectation under the measure $\mathbb{Q}^{T_{i+1}}$. We assume a normal distribution for this rate, which is coherent with the assumption of a lognormal forward price index (for a complete analysis of this hypothesis see [Kenyon \(2008\)](#)). An intuitive argument is : as the price index forward is lognormal then a quotient of two lognormal distributions minus 1 can be approximated as a normal distribution for low volatilities because if X_1 and X_2 are both close to zero then

$$\frac{e^{X_1}}{e^{X_2}} - 1 \approx X_1 - X_2$$

This practice (assuming a normal distribution for the year-on-year rate) is also a widespread market practice.

So as $\mathcal{Y}(t, T_i, T_{i+1})$ is by construction a martingale under the T_{i+1} -forward neutral measure, so we can write

$$d\mathcal{Y}(t, T_i, T_{i+1}) = \sigma_i^{YoY} dW_i^{YoY}(t),$$

where $W_i^{YoY}(t)$ is a Brownian motion under the T_{i+1} -forward neutral measure and σ_i^{YoY} a positive real constant. Then using the Bachelier formulae, we have the following result for the price at time t

$$P_n(t, T_{i+1}) \left[\omega (\mathcal{Y}(t, T_i, T_{i+1}) - K) \mathcal{N}(\omega d) - \mathcal{Y}(t, T_i, T_{i+1}) \sigma_i^{YoY} \sqrt{T} \mathcal{N}'(d) \right],$$

where \mathcal{N} is the normal cumulative distribution, \mathcal{N}' its derivative and

$$d = \frac{\mathcal{Y}(t, T_i, T_{i+1}) - K}{\sigma_i^{YoY} \sqrt{T}}$$

And T is time to maturity, its calculation is explained in paragraph 3. The fixing date here is the fixing date of $I(T_{i+1})$.

To calculate the forward $\mathcal{Y}(t, T_i, T_{i+1})$, we first need a forward price index curve (see [Zine-eddine \(2013a\)](#) for a methodology to build those kind of curves), and secondly a convexity adjustment (for more details see [Zine-eddine \(2013a\)](#)).

4.2 Payment delays

Let's consider a year-on-year option with a payment delay. The pay-off of such a structure is the same

$$N \delta_i \left[\omega \left(\frac{I(T_{i+1})}{I(T_i)} - 1 - K \right) \right]^+$$

but it is paid at a time T_p such that $T_p > T_{i+1}$. To price such a structure, we just have to use the diffusion of $\frac{\mathcal{I}(t, T_{i+1})}{\mathcal{I}(t, T_i)} - 1$ under the T_p -forward neutral measure. We then have the following formula (using previous notation)

$$P_n(t, T_p) \left[\omega \left(\mathbb{E}^{T_p} \left[\frac{I(T_{i+1})}{I(T_i)} - 1 | \mathcal{F}_t \right] - K \right) \mathcal{N}(\omega d) - \mathbb{E}^{T_p} \left[\frac{I(T_{i+1})}{I(T_i)} - 1 | \mathcal{F}_t \right] \sigma_i^{YoY} \sqrt{T} \mathcal{N}'(d) \right],$$

where

$$d = \frac{\mathbb{E}^{T_p} \left[\frac{I(T_{i+1})}{I(T_i)} - 1 | \mathcal{F}_t \right] - K}{\sigma_i^{Y \circ Y} \sqrt{T}}$$

For the computation of $\mathbb{E}^{T_p} \left[\frac{I(T_{i+1})}{I(T_i)} - 1 | \mathcal{F}_t \right]$ see [Zine-eddine \(2013a\)](#).

5 Zero-coupon caps and floors

5.1 pricing formulas

A zero-coupon option has only one flow at maturity T_i . The maturity of an inflation zero-coupon option is always a whole number of years. Let's denote n the integer such as $T = n \text{ years}$. It is an option whose underlying is the growth rate of the price index between the start date and the end date of the option

$$\frac{I(T_i)}{I(T_0)} - 1$$

The pay-off at maturity T_i

$$N \delta \left[\omega \left(\frac{I(T_i)}{I(T_0)} - 1 \right) - ((1 + K)^n - 1) \right]^+$$

As the forward price index follow a lognormal, the pricing of this kind of option is straightforward using the Black formula

$$\omega P_n(t, T_i) \left[\frac{\mathcal{I}(t, T_i)}{I(T_0)} \mathcal{N}(\omega d_1) - (1 + K)^n \mathcal{N}(\omega d_2) \right],$$

where

$$d_1 = \frac{\ln \left(\frac{\mathcal{I}(t, T_i)}{I(T_0) K} \right) + \frac{(\sigma_i^I)^2}{2} T}{\sigma_i^I \sqrt{T}}$$

and

$$d_2 = d_1 - \sigma_i^I \sqrt{T}$$

And T is time to maturity. Its calculation is explain in paragraph 3; the fixing date here is the fixing date of $I(T_i)$.

5.2 Volatility adjustment for Zero-coupon options

Let's assume we calibrate our volatility smile with today's market quote. For the maturity T_i , we will obtain the following number σ_i^I by solving the following equation (we are using previous notation here)

$$N \delta \left[\omega \left(\frac{I(T_i)}{I(T_0)} - 1 \right) - ((1 + K)^n - 1) \right]^+ = \text{Quote}_i$$

where Quote_i is the corresponding market quote.

Now suppose we want to price a zero-coupon option in our portfolio. This option has already started in T_{start} , the underlying of the option is not the same, it is $\frac{\mathcal{I}(t, T_i)}{I(T_{start})}$ instead of $\frac{\mathcal{I}(t, T_i)}{I(T_0)}$. Therefore we need to take in account the following volatility adjustment

$$\tilde{\sigma}_i^I = \frac{I(T_0)}{I(T_{start})} \times \sigma_i^I$$

where $\tilde{\sigma}_i^I$ is the volatility we have to use for the option starting in the past.

5.3 Payment delays

Let's consider a zero-coupon option with a payment delay. The pay-off of such a structure is the same

$$N \delta \left[\omega \left(\frac{I(T_i)}{I(T_0)} - 1 \right) - ((1 + K)^n - 1) \right]^+$$

but it is paid at a time T_p such that $T_p > T_i$. We just have to use the diffusion of $\mathcal{I}(t, T_i)$ under the T_p -forward neutral measure. We then we have the following formula (using previous notation)

$$\omega P_n(t, T_p) \left[\frac{\mathbb{E}^{T_p} [I(T_i) | \mathcal{F}_t]}{I(T_0)} \mathcal{N}(\omega d_1) - (1 + K)^n \mathcal{N}(\omega d_2) \right],$$

where

$$d_1 = \frac{\ln \left(\frac{\mathbb{E}^{T_p} [I(T_i) | \mathcal{F}_t]}{I(T_0) K} \right) + \frac{(\sigma_i^I)^2}{2} T}{\sigma_i^I \sqrt{T}}$$

and

$$d_2 = d_1 - \sigma_i^I \sqrt{T}$$

For the computation of $\mathbb{E}^{T_p} [I(T_i) | \mathcal{F}_t]$ see [Zine-eddine \(2013a\)](#).

6 Implementation

The implementation of the pricing methods within the Opengamma Analytics Library is done using four main classes (one for each instrument):

- *CapFloorInflationYearOnYearInterpolationBlackNormalSmileMethod*,
- *CapFloorInflationYearOnYearMonthlyBlackNormalSmileMethod*,
- *CapFloorInflationZeroCouponInterpolationBlackSmileMethod*,
- *CapFloorInflationZeroCouponMonthlyBlackSmileMethod*.

For each instruments, greeks are computed using algorithmic differentiation.

Data providers (such as Icap, BGC...) usually provide market quotes for both year-on-year and zero-coupons caps/floors for a range of strikes between -2% and 5% (with steps of $.5\%$), and the available maturities are : 1Y ,2Y ,3Y ,4Y ,5Y ,6Y ,7Y ,8Y ,9Y ,10Y ,12Y ,15Y ,20Y ,25Y ,30Y. There is liquidity for instruments related to the US, UK, Eurozone, France.

The following tables summarise the quotations for the Caps and Floors Zero-Coupon on the European price index (HICPx) the 16th March 2010.

	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%	4.00%	4.50%	5.00%
1Y	75.18	49.31	30.72	18.45	10.88	6.39	3.78	2.27	1.38
3Y	289.14	196.94	127.53	80.76	51.53	33.72	22.77	15.85	11.35
5Y	516.03	360.05	235.23	147.44	91.34	57.30	36.86	24.41	16.62
7Y	773.98	551.12	368.15	234.24	146.05	91.61	58.70	38.64	26.15
10Y	1,127.44	823.55	563.73	365.59	230.66	145.56	93.51	61.60	41.69
12Y	1,311.75	961.99	656.82	421.11	260.90	161.38	101.76	65.99	44.06
15Y	1,513.46	1,119.27	770.33	496.04	306.01	186.36	114.56	71.92	46.30
20Y	1,825.27	1,367.22	951.66	621.59	393.09	248.78	160.67	106.64	72.80
30Y	2,500.60	1,926.13	1,373.61	917.37	596.82	392.65	265.78	185.70	133.64

Table 1: Zero-coupon Cap Prices

	-2.00%	-1.00%	-0.50%	0.00%	0.50%	1.00%	1.50%	2.00%	3.00%
1Y	2.54	6.68	10.97	18.00	29.21	46.22	70.28	101.62	181.65
3Y	9.79	18.99	27.38	40.54	61.61	95.53	148.66	226.03	447.95
5Y	10.24	21.26	31.72	48.54	76.12	121.64	195.14	306.61	649.43
7Y	9.23	19.98	30.48	47.78	76.93	126.59	210.00	342.50	779.91
10Y	8.10	18.24	28.46	45.71	75.64	128.42	220.93	375.51	928.14
12Y	7.28	16.58	26.07	42.27	70.83	122.41	215.84	378.58	997.68
15Y	5.08	13.07	21.89	37.80	67.30	122.93	226.93	411.80	1,131.51
20Y	4.77	11.99	19.96	34.52	62.22	116.74	225.09	431.56	1,305.00
30Y	4.00	10.05	16.83	29.46	54.38	106.37	219.13	459.49	1,649.98

Table 2: Zero-coupon Floor Prices

Note that market quotes are in Bps (Basis points), so bootstrapping is necessary to retrieve volatilities.

One last word about performance: in the OpenGamma Analytics Library build 100 volatility surfaces takes about 5 seconds. Those results are the same for year-on-year or zero-coupon options, and also for interpolated or monthly indices. All tests have been done using a 3.5 GHz Quad-Core Intel Xeon.

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