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Pricing a defaultable bond with a stochastic recovery rate

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A closed-form formula for the analysis of defaultable bonds is essential for market practitioners and financial researchers. In order to make the model more reasonable without loss of generality, it is necessary to specify the stochastic processes of the default-free short interest rate, the default intensity rate, and the recovery rate as affine functions with multi-dimensional space of the correlated state variables within the models. However, the pricing procedure is more sophisticated when the model includes such specifications. The purpose of our study is to deal with the complicated pricing procedure by utilizing the concept of the moment-generating function and then to derive the closed-form solution of a defaultable bond. Furthermore, we provide an example to illustrate the application of our model and conduct sensitivity analyses of the bond value for changes in the parameters of our model. Our closed-form formula based on more realistic specifications not only enables one to appropriately price the defaultable bond, but also enables portfolio managers to undertake sophisticated portfolio management and hedging analyses.

Keywords: Reduced-form model; Default risk; Stochastic recovery rate

1. Introduction

Default risk affects virtually every financial contract. Thus, both market practitioners and financial economists focus intently on the probability of default and the loss given default. Two methods are usually applied to investigate the value of a risky bond and its termination risk: the structural-form approach and the reduced-form approach. The idea of pricing a defaultable contract first started with the structural-form approach (e.g., Merton 1974, Black and Cox 1976, Leland 1994). This approach models financial contracts with termination risks as American-type options. As the pricing procedure involves solving the second-order partial differential equation, subject to the boundary and termination conditions, researchers usually resort to numerical techniques such as forward pricing (e.g., Monte Carlo simulations) or backward solutions while implementing sophisticated valuation procedures. In recent times, the reduced-form approach has been widely applied to the pricing of defaultable securities and to calculate their termination probabilities (e.g., Jarrow and Turnbull 1995, Jarrow 2001). A reduced-form model treats default as an unpredictable event by taking the default time as an exogenous random variable; the market information on the default intensity rate is the basis of the default probability estimation (e.g., Bielecki and Rutkowski 2002). When using the reduced-form approach for pricing risky securities, researchers cannot only avoid having to decide on the optimal stopping time of an American-type option, which is the complicated part of the structural-form model, but also easily derive a closed-form formula. Moreover, one can accurately value risky bonds with the reduced-form models because the termination function and other parameters can be estimated through historical market data.

When pricing defaultable bonds using a reduced-form model, the following three stochastic processes should be considered: the default-free short interest rate, the default intensity rate, and the recovery rate given default. However, when the valuation models of defaultable bonds include these three correlated stochastic processes,

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the pricing procedure becomes exceedingly difficult. In order to simplify the complicated pricing procedure, current studies usually include only two stochastic processes—the default-free short interest rate and the default intensity rate—in a reduced-form model. In other words, when deriving a closed-form solution for a defaultable security, researchers usually assume the recovery rate to be a constant or an exogenous variable that is estimated using market data (e.g., Jarrow and Turnbull 1995, Duffie and Singleton 1999). For example, Duffie and Singleton (1999) derived an approximate pricing formula for defaultable bonds. Under their evaluation framework, the value of a defaultable bond is calculated by a default-adjusted rate. This rate comprises the default-free short interest rate process and a mixed process, which is the product of the default intensity rate and the loss rate.

Furthermore, in previous studies, the correlation that is found between the recovery rate and the default rate is a negative value (Altman *et al.* 2005). Some researchers report that these two rates depend on the relevant state variables, such as debt-to-equity ratios, bond yield spreads, and macroeconomic variables that are related to the business cycle (e.g., Duffie and Wang 2003, Vassalou and Xing 2004). Therefore, the current literature specifies the correlated stochastic processes as affine functions, which are linear functions of the random state variables in the reduced-form model (e.g., Duffie and Singleton 1999, Duffie *et al.* 2000). Including such specifications in the evaluation model, however, makes the closed-form solution of a defaultable bond difficult to obtain because the expected default value contains the result of the stochastic recovery rate multiplied by the stochastic default intensity rate.

As is known, a closed-form solution is important because it can provide a valuable tool for analysing certain complications associated with portfolio management, such as the hedging analyses. The objective of this paper, therefore, is to evaluate a defaultable zero-coupon bond (hereinafter denoted as ZCB) under a general reduced-form model and to provide an approach to overcome the complex pricing procedure that results from taking the stochastic recovery rate into account. In order to accurately price defaultable bonds, our model includes the specifications of the stochastic processes of the default-free short interest rate, the default intensity rate, and the recovery rate. Furthermore, because the correlation between these three processes needs to be considered, we specify these three processes as affine functions of random and dependent state variables (see Duan and Simonato 1999, Dai and Singleton 2000, Duffie *et al.* 2000, Duffie 2005). We adopt the concept of the moment-generating function to deal with the complex pricing procedure of the defaultable bond and to derive its closed-form solution. One can accurately and efficiently evaluate the defaultable bond through our valuation framework. Compared with the approaches used by Duffie *et al.* (2000), which applied various Laplace and Fourier transforms to derive the valuation models, our derivation of the closed-form formula is technically simpler, as will be apparent in the following pages.

This article is organized as follows. Section 2 illustrates our valuation framework that includes the identification of the risky bond components and our treatment of the default risk of the bond. This section also explains how one can obtain a closed-form valuation formula of a risky ZCB using the concept of the moment-generating function. In section 3, a closed-form solution of a defaultable ZCB will be shown for a case where the state variables are normally distributed. In section 4, we provide a numerical analysis to demonstrate the sensitivity of a defaultable ZCB value to changes in the model's parameters. Finally, the section 5 summarizes our findings.

2. The model

We consider a defaultable ZCB that promises to pay one dollar at maturity date T. If the default occurs at time τ, $\tau \leq T$, the bondholder will receive $\eta(\tau)$, which is the fractional recovery rate. Under the assumptions of the no arbitrage condition, complete probability space (Ω, F, Q), and augmented filtration $\{F_t : t \geq 0\}$, the standard pricing theory implies that a unique risk-neutral probability measure, Q, exists. Therefore, the present value of the defaultable ZCB can be computed by discounting with the default-free short interest rates and then taking the expectation with respect to Q (Jarrow and Turnbull 1995). The valuation formula of the defaultable ZCB at time t, which is denoted as $P(t, T)$, can be expressed as follows:

$$P(t,T) = \int_t^T E_t^Q\left[\eta(u)\exp\left(-\int_t^u r(s)\mathrm{d}s\right)\gamma(t,u)\right]\mathrm{d}u + E_t^Q\left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s\right)G(t,T)\right], \tag{1}$$

where $E_t^Q[\cdot]$ denotes the expectation under Q, conditional on the information available to the investor at time t, $r(u)$ is the instantaneous default-free short interest rate at time u, $G(t, T)$ is the survival distribution function until time T, $\eta(u)$ is the recovery rate at time u, and $\gamma(t, u)$ is the default probability density function at time u.

Next, we evaluate the defaultable ZCB using the reduced-form model. Let $h(u)$ be the default intensity rate at time u. The survival function can then be defined as follows:

$$G(t,T) = \exp\left(-\int_t^T h(u)\mathrm{d}u\right). \tag{2}$$

The default distribution function $\Gamma(t, T)$ and its probability density function $\gamma(t, u)$ are described as follows:

$$\Gamma(t,T) = 1 - G(t,T) \quad \text{and} \quad \gamma(t,u) = \frac{\partial\Gamma(t,u)}{\partial u} = h(u)G(t,u). \tag{3}$$

Substituting equations (2) and (3) into equation (1), the valuation formula of a defaultable ZCB can be rewritten as follows:

$$P(t,T)=\int_t^T E_t^Q\left[\eta(u)h(u)\exp\left(-\int_t^u(r(s)+h(s)\mathrm{d}s)\right)\right]\mathrm{d}u + E_t^Q\left[\exp\left(-\int_t^T(r(s)+h(s))\mathrm{d}s\right)\right]. \quad (4)$$

The first term in equation (4) is the expected value of a defaultable ZCB if the default occurs before maturity. The second term represents the expected value of a defaultable ZCB that is not terminated until maturity.

When the processes $r(u)$, $\eta(u)$, and $h(u)$ are independent, the closed-form formula is easily obtained. However, if one considers the correlation between these three processes and simultaneously integrates such specifications into the valuation framework, it is rather difficult to obtain the closed-form solution when solving equation (4). Duffie and Singleton (1999) price a defaultable ZCB based on an approximate valuation formula, as follows:

$$P(t,T)=E_t^Q\left[\exp\left(-\int_t^T R(s)\,\mathrm{d}s\right)\right], \quad (5)$$

where $R(s)\cong r(s)+\xi(s)$, which is defined as a default-adjusted discount rate, and $\xi(s)=h(s)(1-\eta(s))$. Only two random processes, $r(s)$ and $\xi(s)$, are specified to develop the pricing formula. For example, $\xi(s)$ can be directly assumed to be a random process, or a process that comprises a random default intensity rate multiplied by an exogenous expected loss rate. Duffie *et al.* (2000) derived a closed-form expression of the following form:

$$E_t^Q\left[\exp\left(-\int_t^T R(s,X(s))(v_0+v_1X(T)\mathrm{d}s\right)\mathrm{e}^{\theta\cdot X(T)}\right], \quad (6)$$

where v_0 is a scalar and the n elements of each of v_1 and θ are scalars, $R(t,X(t))$ is denoted as the stochastic 'default-adjusted discount rate', and $(v_0+v_1X(T))\,\mathrm{e}^{\theta\cdot X(T)}$ is a generalized terminal payoff function.

Duffie *et al.* (2000) specified $R(t,X(t))$ as an affine function of $X(t)$, where $X(t)$ is a set of state variables at time t. They evaluated the defaultable ZCB under the model with a constant recovery rate. Duffie (2005) argued that the model provided by Duffie *et al.* (2000) could be extended to derive the closed-form solution of a defaultable bond including the stochastic loss rate, if this rate was specified in the form of $\mathrm{e}^{\theta\cdot X(s)}$, with deterministic parameters θ.

In order to accurately price the defaultable ZCB, we construct a model to incorporate the correlated stochastic processes $r(u)$, $\eta(u)$, and $h(u)$, which depend on the state variables, and provide a workable methodology to obtain the closed-form formula. In comparison with previous studies, which used affine models to evaluate risky bonds (e.g., Duffie and Singleton 1999, Duffie *et al.* 2000), our model is significantly different. First, since our pricing procedure is not based on the approximate pricing formula as shown in Duffie and Singleton (1999), a defaultable ZCB is more accurately priced under the more general model proposed in our study. Second, our model does not make the specific assumption with respect to the mixed process $\xi(s)$ (a process with an exogenous expected loss rate), as done by Duffie and Singleton (1999), and does not need to specify the loss rate as a special form as proposed by Duffie (2005). We overcome the most difficult part of the pricing procedure, that is, finding a way to directly solve the expectation that contains the product of two stochastic processes (i.e. $\eta(u)\times h(u)$). Finally, our model only needs to use basic statistical theory, the concept of the moment-generating function, to derive the valuation formula, whereas Duffie *et al.* (2000) derived their valuation model by using various Laplace and Fourier transforms. Therefore, our method is technically simpler than that used by Duffie *et al.* (2000).

While developing the valuation model, we assume the state variable $X_i(s)$ follows the Markov process, for $i=1,\dots,n$, and satisfies the following differential equation:

$$\mathrm{d}X_i(t)=\mu_i(t,X_i)\,\mathrm{d}t+\sigma_i(t,X_i)\,\mathrm{d}W_i(t), \quad (7)$$

where $\mu_i(\cdot)$ is the instantaneous drift of state variable i, $\sigma_i(\cdot)$ is the instantaneous standard deviation of state variable i, and $W_i(t)$ is the standard Brownian motion of state variable i under the risk-neutral measure. The distribution form of $X_i(s)$ depends on the specification of the diffusion term. For example, if we assume that $\sigma_i\,(t,X_i)=\sigma_i$ and σ_i is a constant value, $X_i(s)$ is normally distributed; if we specify that $\sigma_i(t,X_i)=\sigma_i\sqrt{X_i(t)}$, the distribution of $X_i(s)$ is a chi-squared distribution (see Cox *et al.* 1985).

Let $X(s)=[X_1(s)X_2(s)\dots X_n(s)]'$ denote a set of state variables that are random and dependent. Furthermore, we assume that $X(s)$ follows a D multivariate distribution,† denoted as $D(\mu_X(s),\Sigma_X(s))$, where $\mu_X(s)$ and $\Sigma_X(s)$ are the expected value vector and the variance–covariance matrix, respectively, both of which are deterministic. We also assume that the additive property of the distribution holds. That is, if $C(s)\equiv\alpha'X(s)$, where α is a vector of n real numbers, the distribution of $C(s)$ will be denoted as $D(\alpha'\mu_X(s),\alpha'\Sigma_X(s)\alpha)$.

In order to discuss the correlations between $r(u)$, $\eta(u)$ and $h(u)$, we assume the default-free short interest rate, the default intensity rate, and the recovery rate to be the affine functions of $X(s)$ that follow affine diffusions (Duan and Simonato 1999, Dai and Singleton 2000, Duffie *et al.* 2000, Duffie 2005, Altman *et al.* 2005). These are expressed as follows:

$$r(s)=r_0(s)+A_r'X(s),\quad h(s)=h_0(s)+A_h'X(s)\quad\text{and}$$
$$\eta(s)=\eta_0(s)+A_\eta'X(s), \quad (8)$$

where $r_0(s)$, $h_0(s)$, and $\eta_0(s)$ represent the initial default-free short interest rate, the baseline default intensity rate,

†The D distribution can be set as any form, such as a normal distribution or a chi-squared distribution.

and the baseline recovery rate, respectively, all three of which are deterministic. $A_r = [a_{r,1}\ a_{r,2}\ \cdots\ a_{r,n}]'$ is the vector of coefficients of the affine function of the default-free short interest rate. $A_h = [a_{h,1}\ a_{h,2}\ \cdots\ a_{h,n}]'$ represents the vector of the coefficients of the affine function of the default intensity rate, and $A_\eta = [a_{\eta,1}\ a_{\eta,2}\ \cdots\ a_{\eta,n}]'$ denotes the vector of the coefficients of the affine function of the recovery rate. The values of $a_{r,i}$, $a_{h,i}$, and $a_{\eta,i}$, for $i = 1, 2, \ldots, n$, are the constant magnitudes of the state variable $X_i(u)$, which affect the default-free short interest rate, the default intensity rate, and the recovery rate, respectively. If a state variable does not influence $r(u)$, $\eta(u)$, or $h(u)$, its coefficient can be set as 0 in the equations. According to equation (8), the distributions of the default-free short interest rate, the default intensity rate, and the recovery rate can be expressed as follows:

$$D(r_0(s) + A_r\mu_X(s), A_r'\Sigma_X(s)A_r),$$
$$D(h_0(s) + A_h\mu_X(s), A_h'\Sigma_X(s)A_h)$$

and

$$D(\eta_0(s) + A_\eta\mu_X(s), A_\eta'\Sigma_X(s)A_\eta).$$

It is worth mentioning that, in practical applications, the default-free short interest rate and the default intensity rate should be positive values, and the recovery rate should be limited within the range of 0 to 1. Therefore, when the coefficient vectors are estimated, some restrictions must be put in place in order to obtain reasonable values. For example, when estimating the coefficients of the affine function of the recovery rate, one can add the restrictions— $0 < \eta_0(s) + A_\eta\mu_X(s) < 1$ and a suitable value (e.g. one that is small enough) for $A_\eta'\Sigma_X(s)A_\eta$—in order to satisfy the condition that the probability of $\eta(s) > 1$ and $\eta(s) < 0$, denoted as Prob($\eta(s) > 1$ and $\eta(s) < 0$), be equal to 0. Similarly, when estimating the coefficients of the linear functions of the default-free short interest rate and the default intensity rate, the values of $r_0(s) + A_r\mu_X(s)$, $h_0(s) + A_h\mu_X(s)$, $A_r'\Sigma_X(s)A_r$ and $A_\eta'\Sigma_X(s)A_h$ must each be restricted within an appropriate range in order to satisfy the condition that the probabilities, Prob($r(s) < 0$) and Prob($h(s) < 0$), be equal to 0. However, Duffee (1999) argued that the problem of restrictions can largely be ignored if the model prices the relevant instruments accurately.

In equation (8), the covariance between $r(u), h(u)$, and $\eta(u)$ can be represented by the coefficients and the variance–covariance matrix of the various state variables. For example, the covariance of the default intensity rate and the recovery rate may be expressed as Cov($h(u)$, $\eta(u)) = A_h'\Sigma_X(u)A_\eta$. Furthermore, when the state variables are set orthogonally, the covariance between them will be 0. This will, however, not eliminate the correlations between the three rates because their functions include common state variables. For example, when the state variables are set as mutually independent, the covariance of the default intensity rate and the recovery rate can be obtained as follows:

$$\mathrm{Cov}(h(u), \eta(u)) = \sum_{i=1}^{n} a_{h,i}\ a_{\eta,i}\ \sigma_i^2(u, X_i(u)).$$

Under such a specification, the evaluation of ZCB may be easier in practical applications since the estimated coefficients of the affine function can be efficiently estimated by using the ordinary least squares regression method.

The key step for deriving the closed-form solution of a defaultable ZCB is to solve

$$E_t^Q\left[\exp\left(-\int_t^T (r(s) + h(s))\,\mathrm{d}s\right)\right] \quad \text{and}$$
$$E_t^Q\left[\eta(u)h(u)\exp\left(-\int_t^u (r(s) + h(s))\,\mathrm{d}s\right)\right].$$

The closed-form formulae of these two terms can easily be obtained by using the concept of the moment-generating function. To define $Y(u) = \int_t^u X(s)\,\mathrm{d}s$, we have

$$Y(u) = \left[\int_t^u X_1(s)\,\mathrm{d}s \quad \int_t^u X_2(s)\,\mathrm{d}s \quad \cdots \quad \int_t^u X_n(s)\,\mathrm{d}s\right]'.$$

According to this specification, we have $-\int_t^u (r(s) + h(s))\,\mathrm{d}s \equiv A_I(u) + A'Y(u)$, where $A_I(u) = \int_t^u A_0(s)\,\mathrm{d}s$, $A_0(s) = -(r_0(s) + h_0(s))$ and $A = -(A_r + A_h)$. The distribution of $A_I(u) + A'Y(u)$ is based on the additive property and can be written as follows:

$$A_I(u) + A'Y(u) \sim D(A_I(u) + A'\mu_Y(u), A'\Sigma_Y(u)A), \tag{9}$$

where $\mu_Y(u)$ is the expected value of $Y(u)$ that we denote $\mu_Y(u) = \int_t^u E[X(s)]\,\mathrm{d}s$, and $\Sigma_Y(u)$ is the variance of $Y(u)$ that we denote $\Sigma_Y(u) = \mathrm{Var}(\int_t^u X(s)\,\mathrm{d}s)$. Furthermore, we have $\Sigma_Y(u) = \int_t^u \Sigma_X(s)\,\mathrm{d}s$ if $X(s)$ is time independent.

Let the moment-generating function of distribution $D(\mu, \sigma^2)$ be denoted as $M(\mu, \sigma^2)$. The moment-generating function of $A_I(T) + A'Y(T)$, denoted as $M(A_I(T) + A'\mu_Y(T), A'\Sigma_Y(T)A)$, can be used to derive the closed-form formula of $E_t^Q[\exp(-\int_t^T (r(s) + h(s))\,\mathrm{d}s)]$. This can be represented as follows:

$$E_t^Q\left[\exp\left(-\int_t^T (r(s) + h(s))\,\mathrm{d}s\right)\right] = M(A_I(T) + A'\mu_Y(T), A'\Sigma_Y(T)A). \tag{10}$$

In order to solve the closed-form formula of $E_t^Q[\eta(u)h(u)\exp(-\int_t^u (r(s) + h(s))\,\mathrm{d}s)]$, we denote $B = [a_1A'\ \ a_2A_h' + a_3A_\eta']'$ and $Z(u) = [Y'(u)\ \ X'(u)]'$, where a_1, a_2, and a_3 are real values, and then obtain the following result:

$$B'Z(u) = a_1A'Y(u) + (a_2A_h' + a_3A_\eta')X(u) \quad \text{and}$$
$$B_0(u) + B'Z(u) = a_1\left(\int_t^u -(r(s) + h(s))\,\mathrm{d}s) + a_2h(u) + a_3\eta(u)\right), \tag{11}$$

where $B_0(u) = a_1A_I(u) + a_2h_0(u) + a_3\eta_0(u)$. Note that $Z(u)$ also follows a D multivariate distribution, that is

$$Z(u) \sim D(\mu_Z(u), \Sigma_Z(u)),$$

where $\mu_Z(u) = [\,\mu'_Y(u) \quad \mu'_X(u)\,]'$,

$$\Sigma_Z(u) = \begin{bmatrix} \Sigma_Y(u) & \Sigma'_{YX}(u) \\ \Sigma_{YX}(u) & \Sigma_X(u) \end{bmatrix},$$

and $\Sigma_{YX}(u)$ is the covariance of $Y(u)$ and $X(u)$. The moment-generating function of $B_0(u) + B'Z(u)$ can be shown as follows:

$$E_t^Q[\exp(B_0(u) + B'Z(u))] = M(B_0(u) + B'\mu_Z(u), B'\Sigma_Z(u)B). \tag{12}$$

If $E_t^Q[\exp(B_0(u) + B'Z(u)]$ is differentiated with respect to a_2 and a_3, the formula

$$\frac{\partial^2 E_t^Q[\exp(B_0(u) + B'Z(u))]}{\partial a_2\,\partial a_3}$$

can be derived. Moreover, we have the following result:

$$\begin{aligned} &E_t^Q\left[\eta(u)h(u)\exp\left(-\int_t^u (r(s)+h(s))\,\mathrm{d}s\right)\right] \\ &= \frac{\partial^2 M(B_0(u) + B'\mu_Z(u), B'\Sigma_Z(u)B)}{\partial a_2\,\partial a_3}\Big|_{a_1=1,a_2=0,a_3=0}. \end{aligned} \tag{13}$$

Substituting equations (10) and (13) into equation (4), we obtain the closed-form solution of the defaultable ZCB, which includes the correlated stochastic processes of the default-free short interest rate, the default intensity rate, and the recovery rate.

3. The formula of $P(t, T)$ under the assumption of normally distributed state variables

In this section, we provide an illustrative example of the pricing procedure using the method proposed in section 2. Traditional studies usually assume that the stochastic processes of relevant state variables follow a normal distribution in evaluation models. For example, the default-free short interest rate $r(t)$ is assumed to follow the Vasicek model (Vasicek 1977), and the behaviors of stock returns are assumed to follow Brownian motions. Here, we demonstrate the application of our model under the assumption of normally distributed state variables. Let $X(u)$ be a vector of various normally distributed random state variables. We denote this process under the risk-neutral measure as follows (Langetieg 1980):

$$\mathrm{d}X(t) = \mu_X(t, X)\,\mathrm{d}t + \Xi\,\mathrm{d}W(t), \tag{14}$$

where $\mu_X(t, X) = \Theta + \Phi X(t)$, Θ is an $n \times 1$ vector, and Φ is an $n \times n$ matrix, which can be constant or time dependent, $W(t)$ is an $n \times 1$ vector of various standard Brownian motions, which are independent under the risk-neutral measure; and Ξ is an $n \times n$ matrix which is a lower triangular matrix.

Let Σ be a deterministic variance–covariance matrix of state variables. The Ξ for this variance–covariance matrix can be obtained by the Cholesky decomposition (Jorion 1997), that is $\Sigma = \Xi\Xi'$. For example, let us consider the model with two correlated factors; let $\rho_{1,2}$ be the correlation between them, and the standard deviations of factor 1 and 2 be σ_1 and σ_2, respectively. Thus, we have

$$\Xi = \begin{bmatrix} \sigma_1 & 0 \\ \rho_{1,2}\sigma_2 & \sqrt{1-\rho_{1,2}^2}\sigma_2 \end{bmatrix}.$$

Equation (14) represents a general form for various state variables, which follow a multivariate normal distribution. For example, if we specify that $\mu_X(t, X) = -\Phi((-\Phi^{-1}\Theta) - X(t))$, and let $-\Phi^{-1}\Theta$ be the long-term value of $X(t)$ and $-\Phi$ be the adjustment speed rate, equation (14) will become a Vasicek-form model. Additionally, if we specify that $\Phi = 0$, the process for $X(t)$ in equation (14) can be simplified to a random walk. It is worth mentioning here that the above specifications are under the risk-neutral measure. In practical applications, when factors are traded in markets, $\mu_X(t, X)$ must be adjusted to include the risk premium in order to express the factor's processes under the physical measure. As shown by Langetieg (1980), if we denote λ_i to be the market risk premium associated with factor X_i, the risk premium needs to satisfy the condition $\mu^P - r = \Sigma_{i=1}^n \lambda_i \sigma_i^P$, where μ^P denotes the expected return of a risky bond, and $\sigma_i^P = (1/P)(\partial P/\partial X_i)\sigma_i$ represents the unanticipated rate of return due to unexpected changes in factor X_i.

According to equation (14), given the initial value of $X(t)$, the solution of $X(t)$ can be described as follows (Langetieg 1980):

$$\begin{aligned} X(s) &= \varphi(t,s)\left(X(t) + \int_t^s \varphi(t,u)^{-1}\Theta\,\mathrm{d}u + \int_t^s \varphi(t,u)^{-1}\Xi\,\mathrm{d}W(u)\right) \\ &= \varphi(t,s)X(t) + \int_t^s \varphi(u,s)\Theta\,\mathrm{d}u + \int_t^s \varphi(u,s)\Xi\,\mathrm{d}W(u), \end{aligned} \tag{15}$$

where $\varphi(t, s)$ is the matrix solution of $\mathrm{d}\varphi(t, s)/\mathrm{d}t = \Phi\varphi(t, s)$. For example, if we assume that Φ is constant, we have $\varphi(t, s) = \exp(\Phi(s - t))$.

$X(s)$ follows a normal multivariate distribution; hence, we may denote that $X(s) \sim N(\mu_X(s), \Sigma_X(s))$. $\mu_X(s)$ and $\Sigma_X(s)$ can be expressed as follows:

$$\mu_X(s) = \varphi(t,s)X(t) + \int_t^s \varphi(u,s)\Theta\,\mathrm{d}u. \tag{16}$$

Moreover, by using Itô's lemma, we have

$$\Sigma_X(s) = \int_t^s \varphi(u,s)\Sigma\varphi(u,s)'\,\mathrm{d}u. \tag{17}$$

As we have specified, $r(s)$, $h(s)$, and $\eta(s)$ are the affine functions of $X(s)$; these three variables are normally distributed according to the additive property. The distributions of $r(s)$, $h(s)$, and $\eta(s)$ can be described as follows:

$$r(s) \sim N(r_0(s) + A_r\mu_X(s), A'_r\Sigma_X(s)A_r),$$
$$h(s) \sim N(h_0(s) + A_h\mu_X(s), A'_h\Sigma_X(s)A_h),$$

and

$$\eta(s) \sim N(\eta_0(s) + A_\eta\mu_X(s), A'_\eta\Sigma_X(s)A_\eta).$$

Moreover, since $Y(u)=\int_t^u X(s)\,\mathrm{d}s$ and $Z(u)=[\,Y'(u)\quad X'(u)]'$, we have that $Y(u)$ and $Z(u)$ are also normally distributed. The process of $Y(T)$ can be described as follows (Langetieg 1980):

$$\begin{aligned}Y(T)&=\int_t^T(\varphi(t,s)X(t)+\int_t^s\varphi(u,s)\Theta\,\mathrm{d}u\\&\quad+\int_t^s\varphi(u,s)\Xi\,\mathrm{d}W(u))\,\mathrm{d}s\\&=\int_t^T(\varphi(t,s)X(t)+\phi(s,T)\Theta)\,\mathrm{d}s\\&\quad+\int_t^T\phi(s,T)\Xi\,\mathrm{d}W(s),\end{aligned}\tag{18}$$

where $\phi(t,s)=\int_t^s\varphi(u,s)\,\mathrm{d}u$. If Φ is a constant, $\varphi(t,s)=1-\Phi^{-1}\exp(\Phi(s-t))$. The second equality of equation (18) holds on the basis of the lemma of Heath *et al.* (1992). Therefore, using Itô's lemma, the distribution of $Y(T)$ can be described as follows:

$$Y(T)\sim N(\mu_Y(T),\Sigma_Y(T)),\tag{19}$$

where

$$\mu_Y(T)=\int_t^T E[X(s)]\,\mathrm{d}s=\int_t^T(\varphi(t,s)X(t)+\phi(s,T)\Theta)\,\mathrm{d}s,$$

and

$$\Sigma_Y(T)=\mathrm{Var}\left(\int_t^T X(s)\,\mathrm{d}s\right)=\int_t^T\phi(s,T)\Sigma\phi(s,T)'\,\mathrm{d}s.$$

As $Z(u)=[\,Y'(u)\quad X'(u)]'$, we can obtain the distribution of $Z(u)$ based on equations (16), (17), and (19). In addition, according to equations (15) and (18), we have

$$\Sigma_{XY}(u)=\mathrm{Cov}\left(X(u),\int_t^u X(s)\,\mathrm{d}s\right)=\int_t^u\varphi(s,T)\Sigma\phi(s,T)'\,\mathrm{d}s.\tag{20}$$

The closed-form solution of the defaultable ZCB can easily be derived by the moment-generating function of a normal distribution. To begin with, we obtain the expected value of the second term in equation (4) as follows:

$$\begin{aligned}&E_t^Q\left[\exp\left(-\int_t^T(r(s)+h(s))\,\mathrm{d}s\right)\right]\\&=\exp\left(A_I(T)+A'\mu_Y(T)+\frac{1}{2}A'\Sigma_Y(T)A\right).\end{aligned}\tag{21}$$

Moreover, we also obtain the following result:

$$\begin{aligned}&E_t^Q[\exp(B_0(u)+B'Z(u)]\\&=\exp\left(B_0(u)+B'\mu_Z(u)+\frac{1}{2}B'\Sigma_Z(u)B\right).\end{aligned}\tag{22}$$

Therefore, according to equations (13) and (22), the expected value of the first term in equation (4) can be solved as follows:

$$\begin{aligned}&E_t^Q\left[\eta(u)h(u)\exp\left(-\int_t^u(r(s)+h(s))\,\mathrm{d}s\right)\right]\\&=(A_h'\Sigma_X(u)A_\eta+(h_0(u)+A_h'(\mu_X(u)+\Sigma_{YX}(u)A))(\eta_0(u)\\&\quad+A_\eta'(\mu_X(u)+\Sigma_{YX}(u)A)))\\&\quad\times\exp\left(A_I(u)+A'\mu_Y(u)+\frac{1}{2}A'\Sigma_Y(u)A\right).\end{aligned}\tag{23}$$

Substituting equations (22) and (23) into equation (4), we get the closed-form formula of the defaultable ZCB as follows:

$$\begin{aligned}P(t,T)&=\int_t^T\begin{bmatrix}((h_0(u)+A_h'(\mu_X(u)+\Sigma_{XY}(u)A))(\eta_0(u)\\+A_\eta'(\mu_X(u)+\Sigma_{XY}(u)A))\\+A_h'\Sigma_X(u)A_\eta)\exp\Big(A_I(u)+A'\mu_Y(u).\\+\frac{1}{2}A'\Sigma_Y(u)A\Big)\end{bmatrix}\mathrm{d}u\\&\quad+\exp\left(A_I(T)+A'\mu_Y(T)+\frac{1}{2}A'\Sigma_Y(T)A\right).\end{aligned}\tag{24}$$

The right-hand side of equation (24) is the closed-form solution of the defaultable ZCB, which includes correlated stochastic processes of the default-free short interest rate, the default intensity rate, and the recovery rate. The condition that enables the closed-form solution to be obtained is whether the constructed state variables have explicit formulae of moment-generating functions. As a consequence of our assumption that the state variables are normally distributed, the closed-form solution of the defaultable ZCB is derived more easily, by applying the moment-generating function of a normal distribution. The closed-form formula of a defaultable bond can also be obtained using other distribution types, if such distributions have explicit formulae of moment-generating functions. However, the pricing procedure could become more complicated when the moment-generating functions are not of a normal type.

4. Numerical analyses

In this section, we provide numerical analyses to investigate how the model's parameters will influence the value of the defaultable ZCB. The default-free short interest rate and the cumulative excess return on the stock price are assumed as state variables in the valuation model. Let the affine functions of the default-free short interest rate, the default intensity rate, and the recovery rate be specified as follows:

$$r(t)=X_1(t),\tag{25}$$

$$h(t)=h_0(t)+a_{h,1}X_1(t)+a_{h,2}X_2(t),\tag{26}$$

and

$$\eta(t) = \eta_0(t) + a_{\eta,2}X_2(t), \tag{27}$$

where $X_1(t)$ is the process of the default-free short interest rate, and $X_2(t)$ is the cumulative excess return on the stock price.

Under the risk-neutral measure Q, the term-structure evolution is represented by the dynamics of the default-free short interest rate $r(t)$:

$$\mathrm{d}r(t) = a(\bar{r}(t) - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_r(t), \tag{28}$$

where a is the adjustment speed rate, which is a positive constant, σ_r is the instantaneous volatility of the default-free short interest rate, which is a positive constant, $\bar{r}(t)$ is the long-run default-free short interest rate, which is a deterministic function of t, and $W_r(t)$ is a standard Brownian motion of interest rate under the risk-neutral measure Q.

According to the above equation, the default-free short interest rate follows a mean-reverting process under the risk-neutral measure. As mentioned by Heath *et al.* (1992), to match an arbitrary initial forward-rate curve, one can set

$$\bar{r}(u) = f(t,u) + \frac{1}{a}\left(\frac{\partial f(t,u)}{\partial u} + \frac{\sigma_r^2\left(1 - \mathrm{e}^{-2a(u-t)}\right)}{2a}\right). \tag{29}$$

Combining the two equations above, the evolution of the default-free short interest rate based on equation (15) can be shown as follows:

$$r(u) = f(t,u) + \frac{\sigma_r^2\left(\mathrm{e}^{-a(u-t)} - 1\right)^2}{2a^2} + \int_t^u \sigma_r \mathrm{e}^{-a(u-v)}\,\mathrm{d}W_r(v), \tag{30}$$

where $f(t,u)$ is the instantaneous forward rate. From equation (30), one can obtain the mean, $\mu_{X_1}(u)$, and variance, $\Sigma_{X_1}(u)$, of the default-free short interest rate as follows:

$$\mu_{X_1}(u) = f(t,u) + \frac{\sigma_r^2\left(\mathrm{e}^{-a(u-t)} - 1\right)^2}{2a^2},$$

and

$$\Sigma_{X_1}(u) = \frac{\sigma_r^2}{2a}\left(1 - \mathrm{e}^{-2a(u-t)}\right).$$

We denote ρ_{rS} as the correlation between the default-free short interest rate and the cumulative excess return on the stock price. For a practical but realistic empirical specification, the evolution of the stock price process $S(t)$ is assumed to follow geometric Brownian motion under the risk-neutral measure:

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\,\mathrm{d}t + \sigma_S\,\mathrm{d}W_S(t), \tag{31}$$

where σ_S is the instantaneous volatility of the stock return, and $W_S(t)$ is a standard Brownian motion of the stock return under Q.† We assume $X_2(t) = \sigma_S W_S(t)$ to be the state variable, that is, the measure of the cumulative excess return on the stock price (Jarrow 2001, Liao *et al.* 2008). The mean and variance of $X_2(u)$, for $u \geq t$, are $\mu_{X_2}(u) = E_t^Q[X_2(u)] = \sigma_S W_S(t)$ and $\Sigma_{X_2}(u) = \sigma_S^2(u-t)$.

According to the above-mentioned specifications, let $Y_1(s) = \int_t^s r(u)\,\mathrm{d}u$; the mean and variance of $Y_1(s)$ can then be described as follows:

$$\mu_{Y_1}(s) = f(t,s)(s-t) + \frac{\sigma_r^2}{2a^2}\left((s-t) - \frac{2}{a}\left(1 - \mathrm{e}^{-a(s-t)}\right) + \frac{1}{2a}\left(1 - \mathrm{e}^{-2a(s-t)}\right)\right),$$

and

$$\Sigma_{Y_1}(s) = \frac{\sigma_r^2}{a^2}\left((s-t) - \frac{2}{a}\left(1 - \mathrm{e}^{-a(s-t)}\right) + \frac{1}{2a}\left(1 - \mathrm{e}^{-2a(s-t)}\right)\right).$$

Denoting $Y_2(s) = \int_t^s X_2(u)\,\mathrm{d}u$, one can obtain the mean and variance of $Y_2(s)$ as $\mu_{Y_2}(s) = \sigma_S W_S(t)(s-t)$ and $\Sigma_{Y_2} = \sigma_S^2[(s-t)^3/3]$, respectively. In addition, according to equations (16)–(20), the covariance between the variables can be described as follows (Liao *et al.* 2008):

$$\Sigma_{Y_1,Y_2}(s) = \rho_{rS}\sigma_S\sigma_r\left(-\frac{1}{a^3}\left(1 - \mathrm{e}^{-a(s-t)}\right) + \frac{1}{a^2}\mathrm{e}^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2\right),$$

$$\Sigma_{X_1,X_2}(s) = \frac{1}{2a}\rho_{rS}\sigma_r\sigma_S\left(1 - \mathrm{e}^{-2a(s-t)}\right),$$

$$\Sigma_{X_1,Y_1}(s) = \frac{\sigma_r^2}{2a^2}\left(1 - \mathrm{e}^{-a(s-t)}\right)^2,$$

$$\Sigma_{X_1,Y_2}(s) = \rho_{rS}\sigma_S\sigma_r\left(\frac{1}{a^2}\left(1 - \mathrm{e}^{-a(s-t)}\right) - \frac{1}{a}\mathrm{e}^{-a(s-t)}(s-t)\right),$$

$$\Sigma_{Y_1,X_2}(s) = \rho_{rS}\sigma_S\sigma_r\left(\frac{1}{a}(s-t) - \frac{1}{a^2}\left(1 - \mathrm{e}^{-a(s-t)}\right)\right),$$

and

$$\Sigma_{Y_2,X_2}(s) = \frac{1}{2}\sigma_S^2(s-t)^2.$$

While performing the numerical analyses, for the sake of simplicity, we assume that the baseline default intensity rate and the baseline recovery rate are constants. The parameter values that have been adopted by us are as follows:‡ $f(t,s) = 4\%$, $a = 0.2$, $\sigma_r = 0.01$, $\sigma_S = 0.1$, $\rho_{rS} = 0.37$, $h_0 = 0.003526$, $a_{h,1} = 0.1513$, $a_{h,2} = -0.0167$, $\eta_0 = 0.387$, $a_{\eta,1} = 0$, and $a_{\eta,2} = 0.205$. $a_{h,1}$ and $a_{h,2}$ (or $a_{\eta,1}$ and $a_{\eta,2}$) denote the effects of the relative magnitudes of the default-free short interest rate and the cumulative excess return on the stock price, on the default intensity rate (or the recovery rate). By putting these parameter values into the model, we obtain the values of

†According to the method of the Cholesky decomposition, we have $\mathrm{d}W_S(t) = \rho_{rS}\,\mathrm{d}W_r(t) + \sqrt{1-\rho_{rS}^2}\,\mathrm{d}\tilde{W}_S(t)$. $\tilde{W}_S(t)$ is also a standard Brownian motion and is independent of $W_r(t)$.

‡The parameter values in the linear functions are taken from Janosi *et al.* (2003) and Altman *et al.* (2005).

Table 1. Defaultable ZCB values for the changes in $f(t,s)$, a, σ_r, σ_S and ρ_{rS}.

	$f(t,s)$			a			σ_r			σ_S			ρ_{rs}		
	1%	4%	8%	0.2	0.4	0.6	0.01	0.05	0.1	0.1	0.3	0.5	–0.37	0	0.37
One-year ZCB values	0.9776	0.9165	0.8409	0.9165	0.9175	0.9192	0.9165	0.9192	0.9235	0.9165	0.9165	0.9164	0.9167	0.9166	0.9165
Five-year ZCB values	0.9226	0.6698	0.6107	0.6721	0.7466	0.8711	0.6698	0.7048	0.797	0.6698	0.6589	0.6496	0.6729	0.6725	0.6698

Note: This table shows the values of one-year defaultable ZCBs and five-year defaultable ZCBs under the change of the initial forward rate $f(0,t)$, the speed of adjustment a, the interest rate volatility σ_r, the stock return volatility σ_S, and the correlation coefficient of the interest rate and the stock return ρ_{rS}. The basic setting parameters are $f(t,s)=4\%$, $a=0.2$, $\sigma_r=0.01$, $\sigma_S=0.1$, $\rho_{rS}=0.37$, $h_0=0.003526$, $a_{h,1}=0.1513$, $a_{h,2}=-0.0167$, $\eta_0=0.387$, $a_{\eta,1}=0$ and $a_{\eta,2}=0.205$.

a defaultable ZCB, which promises to pay one dollar, as 0.9165 and 0.6698 for one-year and five-year maturities, respectively.

To begin with, we conduct sensitivity analyses to investigate how changes in the parameter values, with regard to the term structure and the cumulative excess return on the stock price as well as their correlation, influence the ZCB value. According to the results shown in table 1, we find that if either $f(t,s)$ or σ_S increases, the value of a ZCB decreases. This implies that there is a negative relationship between $f(t,s)$ and σ_S, and the ZCB value. Moreover, the impact of $f(t,s)$ on the ZCB value is larger than that of σ_S. The results also reveal that an increase in a or σ_r will produce an increase in the ZCB value. We can infer that the adjustment speed rate and the volatility of the default-free short interest rate positively influence the ZCB value. In addition, we find that ρ_{rS} does not influence the ZCB value significantly, but does have a small negative effect. Note that a change in the above parameters affects the value of a five-year ZCB to a much greater degree than that of a one-year ZCB.

Next, we discuss the impact of the coefficients of the affine functions on the value of a ZCB with one-year maturity. According to the results displayed in figures 1 and 2, we find that an increase in h_0 or $a_{h,1}$ will produce a decline in the value of the ZCB. On the contrary, an increase in η_0, $a_{\eta,1}$, or $a_{\eta,2}$ will increase the ZCB value. Moreover, we find the existence of a quadratic-form relationship between the ZCB value and $a_{h,2}$. This may be due to the fact that the effect of the variance of the moment-generating function is a squared form. Although our numerical results may be sensitive to the assumed parameter values, we believe that they can enable the reader to understand the influence of state variables and their correlations on the ZCB value.

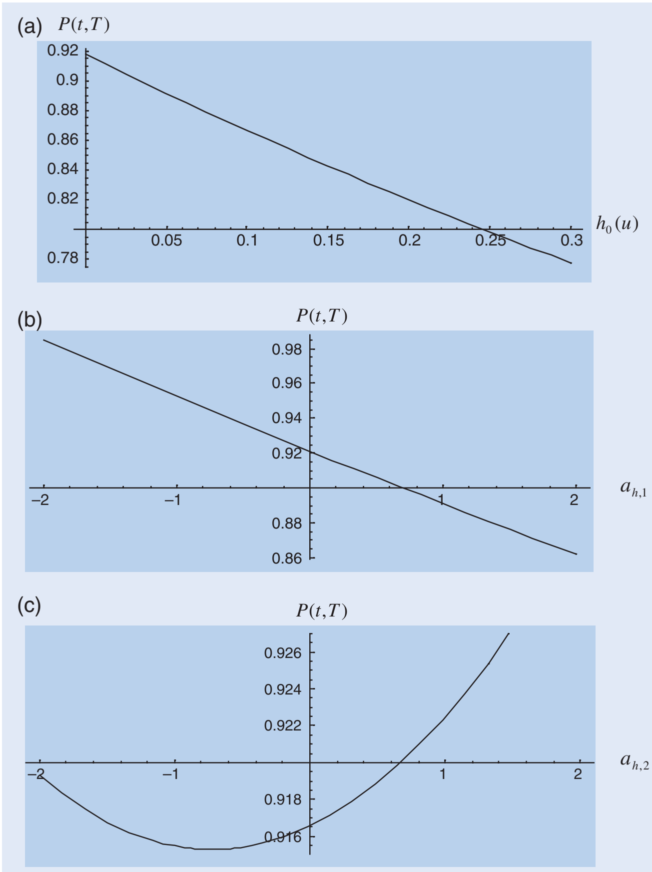

Figure 1. The relations between the ZCB values and different coefficient values in the linear function of the default intensity rate.
Note: This figure shows how the defaultable ZCB values change when the parameter values in the linear function of the default intensity rate, $h_0(u)$, $a_{h,1}$ and $a_{h,2}$, change. The basic parameters are described in the note of table 1.

5. Conclusion

In order to accurately evaluate a risky bond, it is important to simultaneously integrate the correlated stochastic processes of the default-free short interest rate, the default intensity rate, and the recovery rate into the valuation framework. Previous studies have demonstrated that these three processes are correlated and dependent on the relevant state variables. However, specifying these three processes as affine functions in the model will result in difficultly in deriving the closed-form solution of the defaultable bond. In this paper, we provide a feasible approach to deal with this problem.

We specify the default-free short interest rate, the default intensity rate, and the recovery rate as affine

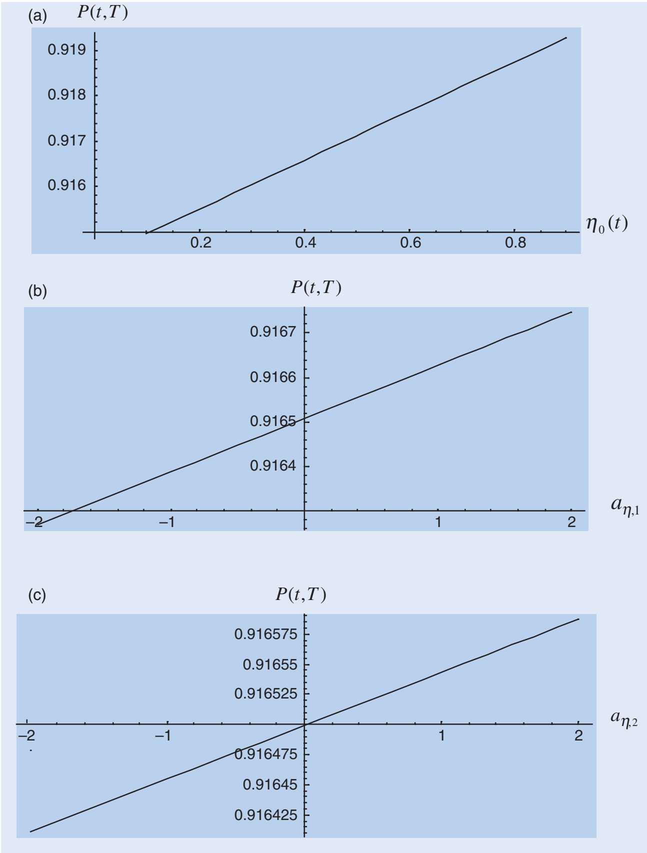

Figure 2. The relations between the ZCB values and different coefficient values in the linear function of the recovery rate. Note: This figure shows how the defaultable ZCB values change when the parameter values in the linear function of the recovery rate, $\eta_0(t)$, $a_{\eta,1}$ and $a_{\eta,2}$, change. The basic parameters are described in the note of table 1.

functions, and then derive the closed-form solution of a defaultable ZCB with the concept of the moment-generating function. Since most of the distributions have moment-generating functions, the closed-form formula of a defaultable security can be solved more easily using our method. In order to illustrate how one can evaluate a defaultable ZCB under our model, we provide an example to show the pricing procedure under the assumption of normally distributed state variables.

We also provide numerical analyses to discuss the sensitivity of a defaultable ZCB value to changes in the model's parameters. According to our results, one can investigate how the parameters of the default-free short interest rate and the cumulative excess return on the stock price, their correlations, and the coefficients in the affine functions, influence the defaultable ZCB value. The values of the basic setting parameters used in our numerical analyses reveal that there is a negative relationship between the ZCB value and the parameters $f(t,s)$ and σ_S. On the contrary, an increase in a or σ_r will produce an increase in the ZCB value. The impact of ρ_{rS} on the ZCB value is negative, but not significant. In addition, an increase in the baseline parameter of default (or recovery) will produce a decline (or increase) in the ZCB value. The value of the ZCB will decrease if the value of $a_{h,1}$ increases. The results reveal that there is a quadratic-form relationship between $a_{h,2}$ and the ZCB value. In previous studies, few scholars have investigated how the parameters in affine functions of the recovery rate influence the ZCB value. According to our results, the value of a ZCB will increase if the values of $a_{\eta,1}$ and $a_{\eta,2}$ increase. We believe that these numerical results can help the reader replicate the technique and verify their results.

In future studies, one can specify the distribution of the state variables as other functional forms, such as the CIR form (Cox *et al.* 1985), or include jump components in the valuation model. Moreover, one can use our closed-form formula to investigate the duration and convexity of a risky bond portfolio and determine optimal diversification strategies.

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References

Altman, E.I., Brady, B., Resti, A. and Sironi, A., The link between default and recovery rates: theory, empirical evidence, and implications. *J. Bus.*, 2005, **78**, 2203–2227.

Bielecki, T.R. and Rutkowski, M., *Credit Risk: Modeling, Valuation and Hedging*, 2002 (Springer: New York).

Black, F. and Cox, J.C., Valuing corporate securities: some effects of bond indenture provisions. *J. Finan.*, 1976, **31**, 351–367.

Cao, M. and Wei, J., Pricing weather derivatives: an equilibrium approach. Working Paper, 1999.

Cox, J.C., Ingersoll, J.E. and Ross, S.A., A theory of the term structures of interest rates. *Econometrica*, 1985, **53**, 385–407.

Dai, Q. and Singleton, K., Specification analysis of affine term structure models. *J. Finan.*, 2000, **55**, 1943–1978.

Duan, J.C. and Simonato, J.G., Estimating and testing exponential-affine term structure models by Kalman filter. *Rev. Quant. Finan. Account.*, 1999, **13**, 111–135.

Duffee, G.R., Estimating the price of default risk. *Rev. Finan. Stud.*, 1999, **12**, 197–226.

Duffie, D. and Singleton, K.J., Modeling term structures of defaultable bonds. *Rev. Finan. Stud.*, 1999, **12**, 687–720.

Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 2000, **68**, 1343–1376.

Duffie, D. and Wang, K., Multi-period corporate failure prediction with stochastic covariates. Working Paper, Stanford University, 2003.

Duffie, D., Credit risk modeling with affine processes. *J. Bank. Finan.*, 2005, **29**, 2751–2802.

Heath, D., Jarrow, R. and Morton, A., Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation. *Econometrica*, 1992, **60**, 77–106.

Janosi, T., Jarrow, R. and Yildirim, Y., Estimating default probabilities implicit in equity prices. *J. Invest. Mgmt.*, 2003, **1**, 1–30.

Jarrow, R.A. and Turnbull, S.M., Pricing derivatives on financial securities subject to credit risk. *J. Finan.*, 1995, **50**, 53–85.

Jarrow, R.A., Default parameter estimation using market prices. *Finan. Anal. J.*, 2001, **57**, 75–92.

Jorion, P., *Value at Risk: The New Benchmark for Managing Financial Risk*, 2nd ed., 1997 (McGraw-Hill: New York).

Langetieg, T.C., A multivariate model of the term structure. *J. Finan.*, 1980, **35**, 71–97.

Leland, H.E., Corporate debt value, bond covenants, and optimal capital structure. *J. Finan.*, 1994, **49**, 1213–1252.

Liao, S.L., Tsai, M.S. and Chiang, S.L., Closed-form mortgage valuation using reduced-form model. *Real Estate Econ.*, 2008, **36**, 313–347.

Merton, R., On the pricing of corporate debt: the risk structure of interest rates. *J. Finan.*, 1974, **29**, 449–470.

Vasicek, O., An equilibrium characterization of the term structure. *J. Finan. Econ.*, 1977, **5**, 177–188.

Vassalou, O. and Xing, Y., Default risk in equity returns. *J. Finan.*, 2004, **59**, 831–868.