A General Framework for Incorporating Stochastic Recovery in Structural Models of Credit Risk

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Abstract: In this work, we introduce a general framework for incorporating stochastic recovery into structural models. The framework extends the approach to recovery modeling developed in Cohen and Costanzino (2015, 2017) and provides for a systematic way to include different recovery processes into a structural credit model. The key observation is a connection between the partial information gap between firm manager and the market that is captured via a distortion of the probability of default. This last feature is computed by what is essentially a Girsanov transformation and reflects untangling of the recovery process from the default probability. Our framework can be thought of as an extension of Ishizaka and Takaoka (2003) and, in the same spirit of their work, we provide several examples of the framework including bounded recovery and a jump-to-zero model. One of the nice features of our framework is that, given prices from any one-factor structural model, we provide a systematic way to compute corresponding prices with stochastic recovery. The framework also provides a way to analyze correlation between Probability of Default (PD) and Loss Given Default (LGD), and term structure of recovery rates.

Keywords: stochastic recovery; partial information; credit risk; jump-to-default

1. Background and Motivation

In his seminal paper, Wang (2002) proposed a transform method to price both liabilities and contingent claims, whether traded or not. One application of the Wang Transform is in option pricing, which for example can be used in the structural modeling of defaultable bonds. In the practice established by actuaries, the time value of money is another important aspect of bond pricing, and in the same issue that Wang's paper appeared, Bühlmann (2002) proposed a paradigm shift in thinking about this technology. Indeed, in his editorial, Bühlmann advocated that actuaries take a *numeraire based approach* to financial risk.

This approach to pricing lines up with that of quantitative finance, where Merton (1974) has laid the groundwork for structural models of corporate bond pricing that provide internally consistent methods to price credit-linked instruments. One major benefit of Merton's model is in its elegant simplicity, where he assumes that the default event, as well as post-default recoverable values, are driven by a common stochastic factor: the firm's asset value. Merton defines equity as a call option on the firm value, and so a change of numeraire also appears in his solution to the bond price via the Miller–Modigliani framework of capital structure. However, a single factor for pricing such bonds assumes that bond and equity returns are perfectly correlated, which need not be true. This also leads to perfect correlation between probability of default and recovery-given-default, which is also not empirically observed. Adding a coupled stochastic recovery driver addresses these issues.

1.1. Recent Extensions

New methods, developed in Cohen and Costanzino (2015, 2017), address the entanglement of default risk (i.e., the risk borne by the holder of the financial instrument that the counterparty will not fulfill its obligations to pay the contract holder) with recovery risk (i.e., the risk of not fully recovering the principal in the event of default) in the pricing of defaultable bonds. The authors show that this mixture can lead to a mis-estimation of the total risk that investors in these bonds have undertaken. We also note here that recovery risk has not received as much attention as default risk, and empirical studies have shown that investors are taking on a significant amount of recovery risk for which they are not being properly compensated (see, for example, Schläfer and Uhrig-Homburg (2014) and references within.)

As we will see, the effect of this extra uncertainty about post-default recovery can be observed through the distortion of the original distance to default in the Merton model, and a distortion of the probability of default in the general structural model of default, when calculating the recoverable value of a bond post-default. In our short note, we develop a transform approach for a multifactor bond-pricing model that represents the partial information available to firm managers about recoverable value. In the spirit of Bühlmann (2002), this approach is dependent on the choice of numeraire as well as the conditions for default. We calculate this probability distortion for the two-factor stochastic recovery Merton model to be a version of the Wang Transform, and present multiple examples to illuminate the extended transform method, including models that incorporate jumps in the asset-recovery values and enforced bound on recovered value. This uncertainty also factors into the calculation of a term structure of expected recovery, which we develop below.

Finally, we establish here that our approach is firmly in the structural model framework pioneered by Merton (1974) , and extended by others such as Black and Cox (1976) and Leland (1994) to name just a few. The reader is directed to Cohen and Costanzino (2015, 2017) for a more robust reference list on the structural approach, as well as the reduced form approach, to credit risk modeling and their differences in application to data.

1.2. Paper Outline

The rest of the paper is developed as follows: in Section 2, we set upon the modeling framework. In Section 3, we revisit the bond pricing techniques presented in Cohen and Costanzino (2015, 2017) and begin their extension via the definition of a Partial Information Transform of the risk-neutral, dollar denominated probability of default. With this definition, we can recast the results in Cohen and Costanzino (2015, 2017) via this technique. Section 4 sees the extension of the models presented in Cohen and Costanzino (2015, 2017) to ones that also incorporate bounded recovery in the the pricing of the bond. In Section 5, we allow for discontinuous information about the firm and so non-predictable defaults are incorporated into our approach to stochastic recovery. Results on short-term credit spreads are also developed, with alternate derivations of these results presented in the Appendix A.

2. Modeling Framework

In order to incorporate recovery risk into classical structural models and understand its effect on the credit spreads, a coupled stochastic recovery risk driver R was added in Cohen and Costanzino (2015, 2017) to the one-dimensional structural models of Merton and Black–Cox that model firm value (assets) A. To enable this extension, the asset process A in these classical models is extended to the joint process of (A, R). This joint process lives in a filtered probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t^{(A,R)}\right\}_{t \geq 0}, \mathbb{P}\right)$ satisfying the usual conditions, where $\mathbb{P}$ is a risk-neutral measure. In this setting, the joint dynamics of (A, R) under $\mathbb{P}$ are given by

$$\frac{dA_t}{A_t} = \mu_A(A_t, t)dt + \sigma_A(A_t, t)dW_t^A,$$

$$\frac{dR_t}{R_t} = \mu_R(R_t, t)dt + \sigma_R(R_t, t)dW_t^R$$

$$= \mu_R(R_t, t)dt + \sigma_R(R_t, t)\left(\rho dW_t^A + \sqrt{1 - \rho^2}dW_t\right),$$

$$\rho\, dt = \langle dW_t^A, dW_t^R \rangle$$

$$0 = \langle dW_t^A, dW_t \rangle$$

(1)

where

- A_t is the firm's asset value at time t.
- R_t is the firm's recoverable value at time t, if default were to happen at t.
- $\mu_A(A_t, t)$ is the rate of return of firm value A_t at time t.
- $\mu_R(R_t, t)$ is the rate of return of recoverable value R_t at time t.
- $\left\{\mathcal{F}_t^A\right\}_{t \geq 0}$ is the filtration generated by A, and
- $\left\{\mathcal{F}_t^{(A,R)}\right\}_{t \geq 0}$ is the filtration generated by (A, R).

Under the risk-neutral measure, we observe that dW_t is proportional to the non-diversifiable portion $d\epsilon_t$ of recovery risk defined via

$$\frac{dR_t}{R_t} = \alpha dt + \beta \frac{dA_t}{A_t} + d\epsilon_t,$$

$$\alpha := \mu_R(R_t, t) - \beta\mu_A(A_t, t),$$

$$\beta := \rho\frac{\sigma_R(R_t, t)}{\sigma_A(A_t, t)},$$

$$d\epsilon_t = \sigma_R(R_t, t)\sqrt{1 - \rho^2}dW_t.$$

(2)

Remark 1. *The constraint of continuous evolution* (1) *is relaxed in Section 5, where a jump-to-0 asset-recovery model is utilized. For a general reference on such processes in credit risk, please consult Zhou (1997).*

In classical structural models of credit risk, recovery is indeed stochastic by the modeling choice $R_t \equiv A_t$. However, this assumes that there is only one driver A for credit risk and so (under the Miller–Modigliani framework assumed in Merton (1974) and subsequent work), equity and bond prices are constrained to be perfectly correlated. We make two further assumptions in the modeling of recovery and firm assets:

Assumption 1. *We assume a constant risk-free rate r, although our results can be generalized to time-varying rate r_t.*

Assumption 2. *Under $\mathbb{P}$, we assume that $\mu_A(A_t, t) = \mu_R(R_t, t) = r$ for all $t \geq 0$, almost surely for risk-free rate r. This allows for us to develop discounted assets and discounted recovery as local martingales.*

Remark 2. *This second assumption is further explored in Cohen and Costanzino (2015), where equity is modeled as a call option on recoverable assets R, rather than pre-default firm value A. The reader is also directed to the excellent resource Korn and Kraft (2004) and references within for further treatment of this assumption of proxies on assets.*

2.1. Classical Bond Pricing

Consider now the case of a zero-coupon bond issued by a firm with notional N and redemption time T. In the classical one-dimensional setting of firm value A and information $\mathcal{F}_T^A$ at time T, default is modeled as a stopping time $\tau \in \mathcal{F}_T^A$. We can also study default and its role in pricing via the set $D := \{\tau \leq T\} \in \mathcal{F}_T^A \subseteq \mathcal{F}$. In this setting, the price $B_{t,T}^{\text{NR}}$ of a bond is the exchange option on underlying asset A for notional N:

$$B_{t,T}^{\text{NR}} = \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(N\mathbf{1}_{\{D^c\}} + A_T\mathbf{1}_{\{D\}}) \mid \mathcal{F}_t^A], \tag{3}$$

where stochastic recovery is simply the firm's assets upon default (NR).

Remark 3. *Although this set D is most commonly generated in the literature as the set $D := \{\tau \leq T\}$ of a **default time** τ, defining default via a set $D \in \mathcal{F}_T^A \subseteq \mathcal{F}$ allows for a slightly more general approach to default modeling.*

3. Bond Pricing in Stochastic Recovery Models

Under the stochastic recovery framework (SR), specifically for Merton and Black–Cox models but extended via the optional sampling theorem, the price of a defaultable bond is shown in Cohen and Costanzino (2015, 2017) to be similar in nature to the traditional structural models without stochastic recovery, where A remains the sole trigger for default: $D := \{\tau \leq T\} \in \mathcal{F}_T^A \subseteq \mathcal{F}_T$:

$$B_{t,T}^{\text{SR}} = \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(N\mathbf{1}_{\{D^c\}} + R_T\mathbf{1}_{\{D\}}) \mid \mathcal{F}_t^{(A,R)}]. \tag{4}$$

We establish here that Equation (4) is true in a general sense when discounted recovery is a local martingale, where the optional sampling theorem allows us to calculate for $\tau^* := \min\{\tau, T\}$

$$
\begin{aligned}
R_t - \mathbb{E}^{\mathbb{P}}[e^{-r(\tau-t)}R_\tau \mathbf{1}_{\{\tau \leq T\}} \mid \mathcal{F}_t^{(A,R)}] &= R_t - \mathbb{E}^{\mathbb{P}}[e^{-r(\min\{\tau,T\}-t)}R_{\min\{\tau,T\}}\mathbf{1}_{\{\tau \leq T\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{-r(\min\{\tau,T\}-t)}R_{\min\{\tau,T\}}\mathbf{1}_{\{\tau > T\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}R_T\mathbf{1}_{\{D^c\}} \mid \mathcal{F}_t^{(A,R)}],
\end{aligned}
\tag{5}
$$

and so

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{-r(\tau-t)}R_\tau\mathbf{1}_{\{\tau \leq T\}} \mid \mathcal{F}_t^{(A,R)}] &= R_t - \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}R_T\mathbf{1}_{\{D^c\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}R_T\mathbf{1}_{\{D\}} \mid \mathcal{F}_t^{(A,R)}].
\end{aligned}
\tag{6}
$$

Depending on the default set D, computation of the bond price in Equation (4) requires the solution of a two-factor PDE with initial and perhaps boundary conditions. See, for example, Cohen and Costanzino (2015, 2017) for futher details on this approach. However, with the strong influence of probabilistic methods in solving such PDEs, it is reasonable to consider parallel solution methods. One such approach is seen in the classical one factor models presented in Ishizaka and Takaoka (2003). By appealing to barrier option theory, the authors in Ishizaka and Takaoka (2003) utilize the measure $\mathbb{Q}$ with the Radon–Nikodym density process defined by the firm value A

$$Z := \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-r(T-t)}\frac{A_T}{A_t}, \tag{7}$$

with the notational conventions of

$$
\begin{aligned}
\mathbb{P}_t[\cdot] &:= \mathbb{P}[\cdot \mid \mathcal{F}_t^A], \\
\mathbb{Q}_t[\cdot] &:= \mathbb{Q}[\cdot \mid \mathcal{F}_t^A],
\end{aligned}
\tag{8}
$$

that the bond price without stochastic recovery can be written as

$$B_{t,T}^{\text{NR}} = e^{-r(T-t)} N \cdot \mathbb{P}_t[D^c] + A_t \cdot \mathbb{Q}_t[D]. \tag{9}$$

This line of reasoning, that of changing numeraire, can be carried over into the case where *recovery* is used as the numeraire, to build a new measure $\mathbb{Q}^*$ to estimate probability of default, with Radon–Nikodym density process and conditional probability defined by

$$Z^* := \frac{d\mathbb{Q}^*}{d\mathbb{P}} = e^{-r(T-t)} \frac{R_T}{R_t},$$

$$\mathbb{Q}_t^*[\cdot] := \mathbb{Q}^*[\cdot \mid \mathcal{F}_t^{(A,R)}]. \tag{10}$$

Armed with this measure, we use recovery R to estimate the probability of default triggered by asset value A. The authors in Cohen and Costanzino (2015, 2017) present closed form solutions in the Merton and Black–Cox cases, which can now be shown to be equal to

$$B_{t,T}^{\text{SR}} = e^{-r(T-t)} N \cdot \mathbb{P}_t[D^c] + R_t \cdot \mathbb{Q}_t^*[D]. \tag{11}$$

Using the idea of changing the base-numeraire to the two-factor model recovery R, from the classical one-factor recovery model numeraire A, we can look to solve the two-factor bond price (11) by solving for the quantity $\mathbb{Q}_t^*[D]$:

Definition 1. *We define the **Partial Information Transform (PIT)** of $\mathbb{P}_t[\tau \leq T]$ for a model with assets A and default set $D \in \mathcal{F}_T^A$, to be the transform which maps $\mathbb{P}_t[\tau \leq T] \to \mathbb{Q}_t^*[\tau \leq T]$.*

For the two-factor model, with Asset–Recovery as a pair of correlated Geometric Brownian Motions, we can compute the **PIT** via the following Lemma:

Lemma 1. *For classical structural models that are driven by our Asset-Recovery pair (1), with constant volatilities (σ_A, σ_R), the **PIT** has the form*

$$\mathbb{Q}_t^*[D] = \mathbb{Q}_t^*[\tau \leq T] := \frac{e^{[\frac{1}{2}\beta(1-\beta)\sigma_A^2 - \beta r](T-t)}}{A_t^\beta} \mathbb{E}^{\mathbb{P}}\left[A_T^\beta 1_{\{D\}} \mid \mathcal{F}_t^A\right]. \tag{12}$$

Proof. Under Assumptions 1 and 2, Ito's formula applied to (1) results in

$$\frac{R_T}{R_t} = \left(\frac{A_T}{A_t}\right)^\beta e^{\left[(r-\frac{1}{2}\sigma_R^2) - \beta(r-\frac{1}{2}\sigma_A^2)\right](T-t) + (\epsilon_T - \epsilon_t)}. \tag{13}$$

Substitution into our pricing formula (11) leads to

$$
\begin{aligned}
B_{t,T}^{\text{SR}} &= \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(N1_{\{D^c\}} + R_T 1_{\{D\}}) \mid \mathcal{F}_t^{(A,R)}] \\
&= Ne^{-r(T-t)} \mathbb{P}[D^c \mid \mathcal{F}_t^A] + \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)} R_T 1_{\{D\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= Ne^{-r(T-t)} \mathbb{P}[D^c \mid \mathcal{F}_t^A] + R_t \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)} \frac{R_T}{R_t} 1_{\{D\}} \mid \mathcal{F}_t^{(A,R)}\right] \\
&= Ne^{-r(T-t)} \mathbb{P}[D^c \mid \mathcal{F}_t^A] + R_t \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)} \left(\frac{A_T}{A_t}\right)^\beta e^{\left[(r-\frac{1}{2}\sigma_R^2) - \beta(r-\frac{1}{2}\sigma_A^2)\right](T-t) + (\epsilon_T - \epsilon_t)} 1_{\{D\}} \mid \mathcal{F}_t^A\right] \\
&= Ne^{-r(T-t)} \mathbb{P}[D^c \mid \mathcal{F}_t^A] + R_t \left(\frac{e^{[\frac{1}{2}\beta(1-\beta)\sigma_A^2 - \beta r](T-t)}}{A_t^\beta} \mathbb{E}^{\mathbb{P}}\left[A_T^\beta 1_{\{D\}} \mid \mathcal{F}_t^A\right]\right).
\end{aligned}
\tag{14}
$$

□

Before using the above for classical structural models, a few points to consider:

Remark 4. *Market Data and Classical Limits:*

- *The quantity (12) also represents the solution to a two-factor bond-pricing PDE, and is highly dependent on the boundary conditions representing default covenants (i.e., the set $D \in \mathcal{F}_T^A$.)*
- *The parameter β, to be estimated from market data, is the only term that relates to post-default recovery R in the Partial Information Transform.*
- *Note that when*

  - *$\beta = 1$, we return the value $\mathbb{Q}_t^*[\tau \leq T] = \mathbb{Q}_t[\tau \leq T]$.*
  - *$\beta = 0$, we return the value $\mathbb{Q}_t^*[\tau \leq T] = \mathbb{P}_t[\tau \leq T]$.*

Remark 5. *Partial Information and the Term Structure of Recovery Rates:*
    We point out here that the distortion also finds its way into the decoupling of recovery at default and probability of default. The result is that the recovery rate is the product of recovery modeled at t (per discounted notional) and the ratio of distorted to risk-neutral probability of default. Indeed, we have

$$\mathbb{E}_t^{\mathbb{P}}\left[e^{-r(T-t)}R_T \mid D\right] = \frac{\mathbb{Q}_t^*[D]}{\mathbb{P}_t[D]}R_t. \tag{15}$$

This decomposition also highlights the effect of distortion of the numeraire-based probability of default on the term structure of recovery rates, and allows for greater flexibility in calibrating to empirical data. This form is a generalization of the one-factor approach to implied recoveries undertaken in the important work of Das and Hanouna (2009). The addition of a term structure of recovery rates addresses the standard assumption of industrial models that pin expected recovery at a constant 40%, only to find that subsequent calculations of quantities such as spreads are far from observed values. For more on the term structure of expected recovery rates, please refer to Doshi et al. (2015) and references within.

Two examples of the Partial Information Transform are presented in this section, where volatilites (σ_A, σ_R) in (1) are constant. In the first case, that of the Merton Model, we recover a version of the Wang Transform as a special case of the Partial Information Transform.

*3.1. Merton Model with Stochastic Recovery (**SRM**)*

In the classical Merton case, the probability of default is known to be

$$\mathbb{P}_t^{\mathrm{Merton},r}[D] = \Phi(-d_0), \tag{16}$$

where Φ and distances-to-default d_0, d_β, d_1 are defined via

$$d_0 := \frac{\ln\left(\frac{A_t}{N}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}},$$
$$d_\beta := d_0 + \beta\sigma_A\sqrt{T-t},$$
$$d_1 := \frac{\ln\left(\frac{A_t}{N}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}}, \tag{17}$$
$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}y^2}dy.$$

This leads to the Partial Information Transform for the Merton model:

Lemma 2.
$$\mathbb{Q}_t^*[D] = \Phi\left(\Phi^{-1}\left(\mathbb{P}_t^{\mathrm{Merton},r}[D]\right) - \beta\sigma_A\sqrt{T-t}\right). \tag{18}$$

Proof. Using the result $e^{-r(T-t)}\mathbb{E}_t\left[R_T 1_{\{A_T<N\}}\right] = R_t\Phi(-d_\beta)$, shown explicitly in Cohen and Costanzino (2015), we can compute the asset and recovery numeraire defined probability of default as $\left(\mathbb{P}_t^{\text{Merton},r}[D], \mathbb{Q}_t^{\text{Merton},r}[D]\right) = (\Phi(-d_0), \Phi(-d_1))$. Furthermore,

$$
\begin{aligned}
\mathbb{Q}_t^*[D] &= \mathbb{Q}_t^*[A_T < N] \\
&= e^{-r(T-t)}\mathbb{E}\left[\frac{R_T}{R_t}1_{\{A_T<N\}} \mid \mathcal{F}_t^{(A,R)}\right] = \frac{1}{R_t}e^{-r(T-t)}\mathbb{E}_t\left[R_T 1_{\{A_T<N\}} \mid \mathcal{F}_t^{(A,R)}\right] \\
&= \frac{1}{R_t}R_t\Phi(-d_\beta) \\
&= \Phi(-d_\beta) \\
&= \Phi\left(\Phi^{-1}\left(\mathbb{P}_t^{\text{Merton},r}[D]\right) - \beta\sigma_A\sqrt{T-t}\right) \\
&= \Phi\left(\Phi^{-1}\left(\mathbb{Q}_t^{\text{Merton},r}[D]\right) + (1-\beta)\sigma_A\sqrt{T-t}\right).
\end{aligned}
\tag{19}
$$

We remark that the distance-to-default d_0 under $\mathbb{P}_t$ has shifted to a new value d_β under $\mathbb{Q}_t^*$. As mentioned above, this new metric requires estimation of only one new parameter, that of the β in our recovery R compared to asset A. From this interpretation, it follows that there are multiple definitions of distance-to-default, depending on the numeraire used. $\square$

3.2. Black–Cox Model with Stochastic Recovery (SRBC)

A similar effect is noted when a level covenant $K \leq N$ on firm assets is introduced, such as in the Black–Cox case, and the probability of default in the Stochastic Recovery Black–Cox case via both asset as well as recovery numeraire are Cohen and Costanzino (2017):

$$
\begin{aligned}
D &:= \{A_T \geq N, \tau_K > T\}^c, \\
\tau_K &:= \inf\{u \geq 0 \mid A_u \leq K\}, \\
\mathbb{P}_t[D] &= \Phi(-d_0) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi(x_0^w), \\
\mathbb{Q}_t[D] &= \Phi(-d_1) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x_1^w), \\
\mathbb{Q}_t^*[D] &= \Phi(-d_\beta) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+2\beta-1}\Phi(x_\beta^w),
\end{aligned}
\tag{20}
$$

where it is assumed $A_t > N$ and we define

$$
\begin{aligned}
x_0^w &:= \frac{\ln\left(\frac{K^2}{NA_t}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}}, \\
x_\beta^w &:= x_0^w + \beta\sigma_A\sqrt{T-t}, \\
x_1^w &:= \frac{\ln\left(\frac{K^2}{NA_t}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}}.
\end{aligned}
\tag{21}
$$

In adding a level barrier K, we are removed from the class of Wang Transforms on standard normal probabilities, but still observe a deformation of the default probability as we pass from firm assets A to recoverable value R as our choice for numeraire.

4. Bounded Stochastic Recovery

In the previous section, we revisited the models developed in Cohen and Costanzino (2015, 2017), where recovery may indeed surpass notional. As expressed in those papers, this approach trades bounded recovery for more tractable calculations, with estimates of the probability of the event that recovery exceeds notional. In this section, we present two models that extend the previous approach, while enforcing bounded recovery and, in the second case, an inclusion of stochastic volatility. We should point out here that bounded recovery is part of the original Merton default model, and, in Cohen and Costanzino (2015, 2017), the recovery is potentially unbounded, but can be rectified by calibrating the model to observed (positive) spreads.

*4.1. Merton Model with Stochastic Recovery and Enforced Positive Credit Spread (**SRM-CR**)*

One of the features of the partial information model proposed in Cohen and Costanzino (2015, 2017) is the non-zero probability attached to the event

$$C := D \cap \{R_T \geq N\}. \tag{22}$$

In this setting, with $\mathbb{P}[C] > 0$, there is correspondingly a chance that, in default, the recovery is greater than notional and spread can be negative. The size of the negative spread depends on model parameters, and the calibrated model will only produce a negative credit spread under very unique circumstances. While negative credit spreads have been observed in the market (c.f. Bhanot and Guo 2011), one can eliminate this possibility with a slight modification of the model. Consider instead the modification of Equation (4) via

$$B_{t,T}^{\text{SRM−CR}} = \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(N 1_{\{D^c\}} + (R_T \wedge N) 1_{\{D\}}) \mid \mathcal{F}_t^{(A,R)}]. \tag{23}$$

It follows that $B_{t,T}^{\text{SRM−CR}} \leq e^{-r(T-t)}N$ and so the corresponding credit spread is non-negative. With a little algebra, we can show the relationship

$$B_{t,T}^{\text{SR}} = B_{t,T}^{\text{SRM−CR}} + \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(R_T - N)_+ 1_{\{D\}} \mid \mathcal{F}_t^{(A,R)}]. \tag{24}$$

In the Merton default setting, with $D = \{A_T < N\}$, we have

$$B_{t,T}^{\text{SR}} = B_{t,T}^{\text{SRM−CR}} + \mathbb{E}^{\mathbb{P}}[e^{-r(T-t)}(R_T - N) 1_{\{A_T < N, R_T \geq N\}} \mid \mathcal{F}_t^{(A,R)}]. \tag{25}$$

Hence, the partial information approach in Equation (4) that uses R for recovery, under Merton default timing, also allows for robust modeling of credit spreads. In this section, we cap stochastic recovery in the Merton model by notional to restrict to non-negative credit spreads. We present below a closed-formula for both bond and (risk-neutral) credit-spread:

Theorem 1. *The two-factor structural model* (1) *results in bond price and credit spread as follows:*

$$
\begin{aligned}
B_{t,T}^{\text{SRM−CR}} &= Ne^{-r(T-t)} \left[\Phi(d_0) + P_{\rho_{A,R}}(d_0^R, d_0) \right] + R_t \left[\Phi(-d_\beta) - P_{\rho_{A,R}}\left(d_1^R, d_\beta\right) \right] \\
S_{t,T}^{\text{SRM−CR}} &= -\frac{1}{T-t} \ln \left(\left[\Phi(d_0) + P_{\rho_{A,R}}(d_0^R, d_0) \right] + \frac{R_t}{Ne^{-r(T-t)}} \left[\Phi(-d_\beta) - P_{\rho_{A,R}}\left(d_1^R, d_\beta\right) \right] \right),
\end{aligned}
\tag{26}
$$

where $\rho = \rho_{A,R}$ and

$$P_{\rho_{A,R}}(x,y) := \int_{z_1=-x}^{\infty} \int_{z_2=-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho_{A,R}^2}} e^{-\frac{z_1^2 + z_2^2 - 2\rho_{A,R} z_1 z_2}{2(1-\rho_{A,R}^2)}} dz_1 dz_2 \tag{27}$$

is a bivariate normal probability, and we define the distances-to-default for asset and recovery as

$$
\begin{aligned}
d_0 = d_0(A_t, r, \sigma_A, T - t) &:= \frac{\ln(A_t/N) + (r - \frac{1}{2}\sigma_A^2)(T - t)}{\sigma_A\sqrt{T - t}}, \\
d_1 = d_1(A_t, r, \sigma_A, T - t) &:= \frac{\ln(A_t/N) + (r + \frac{1}{2}\sigma_A^2)(T - t)}{\sigma_A\sqrt{T - t}}, \\
d_\beta &:= d_0 + \beta\sigma_A\sqrt{T - t},
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
d_0^R &:= \frac{\ln(R_t/N) + (r - \frac{1}{2}\sigma_R^2)(T - t)}{\sigma_R\sqrt{T - t}}, \\
d_1^R &:= \frac{\ln(R_t/N) + (r + \frac{1}{2}\sigma_R^2)(T - t)}{\sigma_R\sqrt{T - t}}.
\end{aligned}
\tag{29}
$$

Proof. The structural approach to modeling leads to the initial formulation

$$
B_{t,T}^{\text{SRM}-\text{CR}} := \mathbb{E}[e^{-r(T-t)}\left(N\mathbf{1}_{\{D^c\}} + \min\{R_T, N\}\mathbf{1}_{\{D\}}\right) \mid \mathcal{F}_t^{(A,R)}]
\tag{30}
$$

for bond price. By definition (30), and via the fact that $\min\{R_T, N\} = R_T - (R_T - N)_+$, direct computation (also carried out in full detail in Cohen and Costanzino (2015)) leads to the formula for bond price

$$
\begin{aligned}
B_{t,T}^{\text{SRM}-\text{CR}} &= \mathbb{E}[e^{-r(T-t)}\left(N\mathbf{1}_{\{D^c\}} + \min\{R_T, N\}\mathbf{1}_{\{D\}}\right) \mid \mathcal{F}_t^{(A,R)}] \\
&= \mathbb{E}[e^{-r(T-t)}N\mathbf{1}_{\{D^c\}} \mid \mathcal{F}_t] + \mathbb{E}[e^{-r(T-t)}\min\{R_T, N\}\mathbf{1}_{\{D\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= Ne^{-r(T-t)}\Phi(d_0) + \mathbb{E}[e^{-r(T-t)}R_T\mathbf{1}_{\{D\}} \mid \mathcal{F}_t^{(A,R)}] - \mathbb{E}[e^{-r(T-t)}(R_T - N)_+\mathbf{1}_{\{D\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= Ne^{-r(T-t)}\Phi(d_0) + R_t\Phi(-d_\beta) - \mathbb{E}[e^{-r(T-t)}(R_T - N)_+\mathbf{1}_{\{D\}} \mid \mathcal{F}_t^{(A,R)}] \\
&= Ne^{-r(T-t)}\Phi(d_0) + R_t\Phi(-d_\beta) - \left(R_t P_{\rho_{A,R}}\left(d_1^R, d_\beta\right) - Ne^{-r(T-t)}P_{\rho_{A,R}}(d_0^R, d_0)\right) \\
&= Ne^{-r(T-t)}\left[\Phi(d_0) + P_{\rho_{A,R}}(d_0^R, d_0)\right] + R_t\left[\Phi(-d_\beta) - P_{\rho_{A,R}}\left(d_1^R, d_\beta\right)\right].
\end{aligned}
\tag{31}
$$

Correspondingly, we compute the credit spread using its definition

$$
\begin{aligned}
S_{t,T}^{\text{SRM}-\text{CR}} &:= \frac{1}{T - t}\ln\left(\frac{Ne^{-r(T-t)}}{B_{t,T}^{\text{SRM}-\text{CR}}}\right) \\
&= \frac{1}{T - t}\ln\left(\frac{Ne^{-r(T-t)}}{Ne^{-r(T-t)}\left[\Phi(d_0) + P_{\rho_{A,R}}(d_0^R, d_0)\right] + R_t\left[\Phi(-d_\beta) - P_{\rho_{A,R}}\left(d_1^R, d_\beta\right)\right]}\right) \\
&= -\frac{1}{T - t}\ln\left(\left[\Phi(d_0) + P_{\rho_{A,R}}(d_0^R, d_0)\right] + \frac{R_t}{Ne^{-r(T-t)}}\left[\Phi(-d_\beta) - P_{\rho_{A,R}}\left(d_1^R, d_\beta\right)\right]\right).
\end{aligned}
\tag{32}
$$

□

Remark 6. *This extended Stochastic Recovery Merton model that enforces positive credit spreads by strictly capping the payout to bondholders to notional N is not an example of the Partial Information Transform approach, as the triggering event now depends on the* joint *value of* (A_T, R_T)*, and not just* A_T*. It is tempting, of course, to expand the definition of the default event D to a trigger event* $D_* \in \mathcal{F}_T^{(A,R)}$ *for the joint process* (A_T, R_T) *as*

$$
D_* := \{A_T < N, R_T < N\}
\tag{33}
$$

and so

$$
D_*^c = \{A_T \geq N\} \cup \{A_T < N, R_T \geq N\}.
\tag{34}
$$

In this set-up, the bondholder is paid N in full if either firm assets at T are valued to be at least N, or if firm assets A_T are viewed to be less than N, but recovery is valued to be at least N. It follows that

$$B_{t,T}^{\mathrm{SRM-CR}} = e^{-r(T-t)} N \cdot \mathbb{P}\left[D_*^c \mid \mathcal{F}_t^{(A,R)} \right] + R_t \cdot \mathbb{Q}_t^*[D_*]. \tag{35}$$

4.2. A Bounded Discounted Recovery Model with Stochastic Volatility

Consider a recovery R that lives on our probability space, with the same recovery-driving Brownian motion W^R (with same correlation to W^A) as in the previous section, but now the recovery process has a volatility that is driven by R as well:

$$\frac{dR_t}{R_t} = r dt + \sigma_R \left(1 - \frac{R_t}{N e^{-r(T-t)}} \right) dW_t^R. \tag{36}$$

This recovery process (36) can be rewritten in terms of another process π_t:

$$R_t = N e^{-r(T-t)} \pi_t, \tag{37}$$

where

$$d\pi_t = \sigma_R \pi_t (1 - \pi_t) dW_t^R,$$
$$\pi_0 = \frac{R_0}{N e^{-rT}} \in [0,1], \tag{38}$$

and so $\left\{ e^{-rt} R_t \right\}_{\{0 \le t \le T\}}$ is a local martingale with values residing in the interval $[0, N e^{-rT}]$ almost surely, and so is in fact a martingale. We point out that this process π has a deep history in Mathematical Statistics (Shiryaev 2007) and Actuarial Science (Gerber 1997), to name but two important historical resources, and more recent applications of bounded stochastic processes have found their way into election modeling (Taleb 2017) and financial instrument modeling (Carr 2017).

In this setting, the bond price reduces to

$$\begin{aligned} B_{t,T}^{\mathrm{SR}} &= \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)} (N 1_{\{D^c\}} + R_T 1_{\{D\}}) \mid \mathcal{F}_t^{(A,R)} \right] \\ &= e^{-r(T-t)} N \mathbb{P}_t[D^c] + R_t \mathbb{E}^{\mathbb{P}}\left[\frac{\pi_T}{\pi_t} 1_{\{D\}} \mid \mathcal{F}_t^{(A,R)} \right] \\ &= e^{-r(T-t)} N \mathbb{P}_t[D^c] + R_t \mathbb{Q}_t^*[D]. \end{aligned} \tag{39}$$

However, to find a closed form for this transformation $\mathbb{Q}_t^*[D]$, say under a Black–Cox default time with barrier K, would require the solution of a PDE with barrier and initial conditions. A middle path would be to restrict the original Geometric Brownian Motion approach to Recovery to be capped at notional N *when used in pricing*, which we presented in the previous subsection.

4.3. Further Extensions

It should be noted that the examples presented above can be generalized to add extra conditions on the default. Computation of the PIT could involve the joint distribution of A_T with quantities such as the running maximum, minimum, and average of A over the interval $[t, T]$, among other quantities, depending on the new default set D. For example, one could consider the default time corresponding to the cumulative Parisian option, as detailed in Ishizaka and Takaoka (2003) and references within.

5. Stochastic Recovery Merton Model with Jumps

In this last example, we extend the two-factor model in Equation (1) to allow for *abnormal* price variations. These variations are different than those that *arise from new information that causes marginal changes in the stock's value* (Merton 1976). Such a model was proposed in Samuelson (1973) and used to solve for call option prices in Merton (1976).

In the Samuelson (1973) model, the underlying is allowed to suddenly jump to zero value, due to information that relates to the firm specifically or perhaps the larger industry to which it belongs. The jump-to-zero model is reasonable for equity prices as typically equity-holders get very little in the event of default. However, a jump-to-zero model for credit is less intuitive. Indeed, in a one-factor model where the assets A_t drive both default and recovery, a jump-to-zero model is extremely restrictive since it implies that we have zero recovery at default. In a two-factor model, such as Equation (1) whereby there is a separate risk driver for assets and for recovery, a jump-to-zero model for assets is reasonable since the assets define the default risk and not recovery risk. Therefore, even though A_t may jump-to-zero indicating default, the amount recovered at default is driven by R_t, which is not necessarily zero. This is an added benefit of having an additional recovery risk driver.

As we shall see below, the Partial Information Transform in this model is computed directly via integration, and is a modified version of Equation (12) that incorporates jumps as well as diffusion in modeling A.

5.1. The Jump-to-Zero Diffusion Model

The jump-to-zero diffusion model we define allows for a jump to zero in the asset value A_t, and therefore unpredictable default, but not necessarily a jump-to-zero in the recovery value R_t. Unlike in the one-factor model, the decoupling of the default risk driver from the recovery risk driver allows us to treat jump-to-zero of the asset value A_t as a trigger for default, which does not affect the recovery at default.

To define the model, suppose that the asset and its recovery follow the jump-diffusion, modified from Equation (1):

$$
\begin{aligned}
\frac{dA_t^\lambda}{A_{t-}^\lambda} &= (r + \lambda_{AR} + \lambda_A)dt + \sigma_A dW_t^A - dN_t^{AR} - dN_t^A, \\
\frac{dR_t^\lambda}{R_{t-}^\lambda} &= (r + \lambda_{AR})dt + \sigma_R dW_t^R - dN_t^{AR},
\end{aligned}
\tag{40}
$$

where N_t^A and N_t^{AR} are *independent* driving Poisson processes with constant rates λ_A and λ_{AR}, respectively. The process N_t^A models an unpredictable default event with rate λ_A whereby the recovery value upon default is non-zero, whereas the process N_t^{AR} models a (catastrophic) unpredictable default event with rate λ_{AR} whereby both the asset and recovery value upon default are exactly zero.

Consider a filtration $\left\{ \mathcal{F}_t^{A^\lambda} \right\}_{t \geq 0}$ generated by the process $\{A_t^\lambda\}_{t \geq 0}$, and the larger filtration $\{\mathcal{F}_t\}_{t \geq 0} := \left\{ \mathcal{F}_t^{\left(A^\lambda, R^\lambda\right)} \right\}_{t \geq 0}$ generated by the coupled jump-diffusion in Equation (40). This evolution is similar to the one proposed in Martin (2007). We also note here that firm accounted asset's jump to 0 may or not signal that recoverable assets jump to 0.

In this setting, the solution to Equation (40), conditioned on information at time 0, is

$$
\begin{aligned}
A_t^\lambda &= A_0 e^{(r + \lambda_{AR} + \lambda_A - \frac{1}{2}\sigma_A^2)t + \sigma_A W_t^A} 1_{\{N_t^{AR}=0\}} 1_{\{N_t^A=0\}} \\
&= A_t e^{(\lambda_{AR} + \lambda_A)t} 1_{\{N_t^{AR}=0\}} 1_{\{N_t^A=0\}}, \\
R_t^\lambda &= R_0 e^{(r + \lambda_{AR} - \frac{1}{2}\sigma_R^2)t + \sigma_R W_t^R} 1_{\{N_t^{AR}=0\}} \\
&= R_t e^{\lambda_{AR} t} 1_{\{N_t^{AR}=0\}}.
\end{aligned}
\tag{41}
$$

Here, (A_t, R_t) is the solution pair of Equation (1), as before.

5.2. Effect of Jumps on Zero-Recovery Bonds

If we return to the simplest case where $D := \left\{ A_T^\lambda < N \right\}$, then we can see for the zero-recovery, zero-coupon bond

$$
V_{t,T}^{(r, \sigma_A, \lambda_A, \lambda_{AR})} := \mathbb{E}^{\mathbb{P}} \left[e^{-r(T-t)} N 1_{\{D^c\}} \mid \mathcal{F}_t^{A^\lambda} \right],
\tag{42}
$$

which has the solution ([Merton 1976](#))

$$
\begin{aligned}
V_{t,T}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} &= \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}N1_{\left\{A_t e^{(r+\lambda_{AR}+\lambda_A-\frac{1}{2}\sigma_A^2)(T-t)+\sigma_A(W_T^A-W_t^A)}1_{\{N_T^{AR}=0\}}1_{\{N_T^A=0\}}\geq N\right\}} \mid \mathcal{F}_t^{A^\lambda}\right] \\
&= e^{-\lambda_A(T-t)}e^{-\lambda_{AR}(T-t)}\mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}N1_{\left\{A_t e^{(r+\lambda_{AR}+\lambda_A-\frac{1}{2}\sigma_A^2)(T-t)+\sigma_A(W_T^A-W_t^A)}\geq N\right\}} \mid \mathcal{F}_t^{A}\right] \\
&= Ne^{-(r+\lambda_A+\lambda_{AR})(T-t)}\mathbb{P}_t[A_t e^{(r+\lambda_{AR}+\lambda_A-\frac{1}{2}\sigma_A^2)(T-t)+\sigma_A(W_T^A-W_t^A)}\geq N] \\
&= V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)}.
\end{aligned}
\tag{43}
$$

This extra jump due to unexpected, abnormal information, returns a significant modeling advantage, as it adds to the no-jump asset model credit spread:

$$
\begin{aligned}
S_{t,T}^{(r,\sigma_A,\lambda_A+\lambda_{AR})} &= -\frac{1}{T-t}\ln\left(\frac{V_{t,T}^{(r,\sigma_A,\lambda_A,\lambda_{AR})}}{Ne^{-r(T-t)}}\right) \\
&= -\frac{1}{T-t}\ln\left(\frac{V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)}}{Ne^{-r(T-t)}}\right) \\
&= \lambda_A + \lambda_{AR} - \frac{1}{T-t}\ln\left(\frac{V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)}}{Ne^{-(r+\lambda_A)(T-t)}}\right) \\
&= \lambda_A + \lambda_{AR} + S_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0)}.
\end{aligned}
\tag{44}
$$

Thus, in the short-term as $T \to t_+$, formula (44) returns a short-term credit spread of $\lambda_A + \lambda_{AR}$.

5.3. Effect of Jumps on Bond Prices with Stochastic Recovery

If we allow for stochastic recovery in this setting, then

$$
B_{t,T}^{(SR,r,\sigma_A,\sigma_R,\lambda_A,\lambda_{AR})} = \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}\left(N1_{\{D^c\}} + R_T^\lambda 1_{\{D\}}\right) \mid \mathcal{F}_t^{\left(A^\lambda,R^\lambda\right)}\right].
\tag{45}
$$

Notice that default is still only driven by A^λ, which now allows for the surprise jump-to-0 value in asset. The correlated jump-diffusion in Equation (40) is still a martingale, in that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[e^{-rt}A_t^\lambda \mid \mathcal{F}_s^{\left(A^\lambda,R^\lambda\right)}\right] &= e^{-rs}A_s^\lambda, \\
\mathbb{E}^{\mathbb{P}}\left[e^{-rt}R_t^\lambda \mid \mathcal{F}_s^{\left(A^\lambda,R^\lambda\right)}\right] &= e^{-rs}R_s^\lambda,
\end{aligned}
\tag{46}
$$

and so, using Equation (40), we can rewrite Equation (45) as

$$
\begin{aligned}
B_{t,T}^{(SR,r,\sigma_A,\sigma_R,\lambda_A,\lambda_{AR})} &= \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}\left(N1_{\{D^c\}} + R_T^\lambda 1_{\{D\}}\right) \mid \mathcal{F}_t^{\left(A^\lambda,R^\lambda\right)}\right] \\
&= V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)} + R_t - \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}R_T^\lambda 1_{\{D^c\}} \mid \mathcal{F}_t^{\left(A^\lambda,R^\lambda\right)}\right] \\
&= V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)} + R_t - e^{-(r+\lambda_A+\lambda_{AR})(T-t)}\mathbb{E}^{\mathbb{P}}\left[R_T e^{\lambda_{AR}(T-t)}1_{\left\{A_T e^{(\lambda_{AR}+\lambda_A)(T-t)}\geq N\right\}} \mid \mathcal{F}_t^{(A,R)}\right] \\
&= V_{t,T}^{(r+\lambda_A+\lambda_{AR},\sigma_A,0,0)} \\
&\quad + R_t - e^{-\lambda_A(T-t)}e^{-(r+\lambda_A+\lambda_{AR})(T-t)}\mathbb{E}^{\mathbb{P}}\left[R_T e^{(\lambda_{AR}+\lambda_A)(T-t)}1_{\left\{A_T e^{(\lambda_{AR}+\lambda_A)(T-t)}\geq N\right\}} \mid \mathcal{F}_t^{(A,R)}\right].
\end{aligned}
\tag{47}
$$

Closed form solutions for the components in the last line of Equation (47) are retained from Lemma 2, but adapted with the parameter change of

$$r' := r + \lambda_A + \lambda_{AR} \tag{48}$$

and corresponding distances-to-default d_0, d_1, and d_β under r' denoted by d'_0, d'_1, and d'_β, respectively. We use these to write the bond price with stochastic recovery and default-to-0 as

$$
\begin{aligned}
B_{t,T}^{(\mathrm{SR},r,\sigma_A,\sigma_R,\lambda_A,\lambda_{AR})} &= Ne^{-r'(T-t)}\Phi\left(d'_0\right) + R_t - e^{-\lambda_A(T-t)}R_t\Phi\left(d'_\beta\right) \\
&= B_{t,T}^{\mathrm{SR},r'} + R_t\Phi\left(d'_\beta\right)\left[1 - e^{-\lambda_A(T-t)}\right].
\end{aligned}
\tag{49}
$$

Here, $B_{t,T}^{\mathrm{SR},r'}$ is the zero-coupon bond price under stochastic recovery, no jumps in asset *or* recovery value, and the new risk-free rate $r' = r + \lambda_A + \lambda_{AR}$, which is also used to calculate the new distance to default d'_0 under r' instead of r.

5.4. Pricing via Change of Numeraire

From the calculations above in Section 5.3, we have the following result:

Lemma 3.

$$
\begin{aligned}
\mathbb{Q}_t^*[D] &= \left(1 - e^{-\lambda_A(T-t)}\right) + e^{-\lambda_A(T-t)}\Phi\left(-d'_\beta\right) \\
&= \left(1 - e^{-\lambda_A(T-t)}\right) + e^{-\lambda_A(T-t)}\Phi\left(\Phi^{-1}\left(\frac{\mathbb{P}_t[D] - \left(1 - e^{-(\lambda_A+\lambda_{AR})(T-t)}\right)}{e^{-(\lambda_A+\lambda_{AR})(T-t)}}\right) - \beta\sigma_A\sqrt{T-t}\right).
\end{aligned}
\tag{50}
$$

Proof. Once again, consider that bond pricing will occur via change of measure. Using the form (40) as well as the results in Equations (43) and (49) above, we compute that, for the Stochastic Recovery Merton Model with short-term credit spread built in, the pair of numeraire measures are

$$
\begin{aligned}
\mathbb{P}_t[D] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{A_T^\lambda < N\}} \mid \mathcal{F}_t^{A^\lambda}\right] \\
&= \left(1 - e^{-(\lambda_A+\lambda_{AR})(T-t)}\right) + e^{-(\lambda_A+\lambda_{AR})(T-t)}\Phi\left(-d'_0\right), \\
\mathbb{Q}_t^*[D] &= \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}\frac{R_T^\lambda}{R_t^\lambda}\mathbb{1}_{\{A_T^\lambda < N\}} \mid \mathcal{F}_t^{(A^\lambda,R^\lambda)}\right] \\
&= 1 - \mathbb{E}^{\mathbb{P}}\left[e^{-r(T-t)}\frac{R_T^\lambda}{R_t^\lambda}\mathbb{1}_{\{A_T^\lambda \geq N\}} \mid \mathcal{F}_t^{(A^\lambda,R^\lambda)}\right] \\
&= 1 - e^{-\lambda_A(T-t)}e^{-(r+\lambda_A+\lambda_{AR})(T-t)}\mathbb{E}^{\mathbb{P}}\left[e^{(r+\lambda_A+\lambda_{AR}-\frac{1}{2}\sigma_R^2)(T-t)+\sigma_R(W_T^R-W_t^R)}\mathbb{1}_{\left\{A_t e^{(r+\lambda_A+\lambda_{AR}-\frac{1}{2}\sigma_A^2)(T-t)+\sigma_A(W_T^A-W_t^A)}\geq N\right\}} \mid \mathcal{F}_t^{(A,R)}\right] \\
&= 1 - e^{-\lambda_A(T-t)}\Phi\left(d'_\beta\right) \\
&= 1 - e^{-\lambda_A(T-t)}\left(1 - \Phi\left(-d'_\beta\right)\right) \\
&= \left(1 - e^{-\lambda_A(T-t)}\right) + e^{-\lambda_A(T-t)}\Phi\left(-d'_\beta\right) \\
&= \left(1 - e^{-\lambda_A(T-t)}\right) + e^{-\lambda_A(T-t)}\Phi\left(\Phi^{-1}\left(\mathbb{P}_t^{\mathrm{Merton},r'}[D]\right) - \beta\sigma_A\sqrt{T-t}\right). \\
&= \left(1 - e^{-\lambda_A(T-t)}\right) + e^{-\lambda_A(T-t)}\Phi\left(\Phi^{-1}\left(\frac{\mathbb{P}_t[D] - \left(1 - e^{-(\lambda_A+\lambda_{AR})(T-t)}\right)}{e^{-(\lambda_A+\lambda_{AR})(T-t)}}\right) - \beta\sigma_A\sqrt{T-t}\right),
\end{aligned}
\tag{51}
$$

where an adjusted risk-free rate $r' = r + \lambda_A + \lambda_{AR}$ is used to calculate

- new distances to default d'_0, d'_1, d'_β under r' instead of r, as well as

- the classical (no-jump) asset value probability of default $\mathbb{P}_t^{\mathrm{Merton},r'}[D] = \Phi\left(-d'_0\right)$ defined in Equation (16), under r'.

$\square$

Remark 7. *In analyzing these results, we consider the following points:*

- *The total default probability transformation upon change of numeraire now results in a weighted average of*

  1. *a Wang Transform of the classical Merton default probability $\mathbb{P}_t^{\mathrm{Merton},r'}[D]$, and*
  2. *a probability of default based on a jump-to-0, all under a new distortion of the risk-free rate from r to r'.*

- *This is due to the nature of default now being extended to allow for external triggers to force assets to 0.*
- *Note that the recovery value $R_T^\lambda = 0$ when $N_T^{AR} = 0$, and so recovery, under default triggered by $N_T^{AR} = 0$, does not factor into our bond price. If we distort $\mathbb{P}_t[D]$, then $N_T^{AR} = 0$ does indeed factor into our calculations. Please see the subsections below for more details, and the effect of jumps in this model on short-term credit spreads.*

To conclude this section, we present an analysis of the short-term credit risk in this Merton Model that we have extended to allow for the effect of jumps on the credit spread. We begin by looking at zero-coupon, zero-recovery bonds.

5.5. Widening the Credit Spread

The effect of jumps in the asset value directly carries over into the short-term credit spread:

Theorem 2. *The effect upon short-term credit spreads of adding jumps into both assets and recovery of a firm, in a classical asset model used for structural modeling such as Merton, is reflected in the relationship*

$$\lim_{u \to 0_+} S_{t,t+u}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} := \lambda_{AR} + \lambda_A \cdot \left(1 - \frac{R_t}{N}\right). \tag{52}$$

Proof. Using the result in Equation (49), we repeat the calculation in Equation (44) to compute our credit spread as

$$
\begin{aligned}
S_{t,T}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} &= -\frac{1}{T-t} \ln \left(\frac{B_{t,T}^{(\mathrm{SR},r,\sigma_A,\sigma_R,\lambda_A,\lambda_{AR})}}{N e^{-r(T-t)}} \right) \\
&= -\frac{1}{T-t} \ln \left(\frac{B_{t,T}^{\mathrm{SR},r'} + R_t \Phi\left(d_\beta'\right)\left[1 - e^{-\lambda_A(T-t)}\right]}{N e^{-r(T-t)}} \right) \\
&= \lambda_A + \lambda_{AR} - \frac{1}{T-t} \ln \left(\frac{B_{t,T}^{\mathrm{SR},r'} + R_t \Phi\left(d_\beta'\right)\left[1 - e^{-\lambda_A(T-t)}\right]}{N e^{-r'(T-t)}} \right).
\end{aligned}
\tag{53}
$$

If we let $u := T - t \to 0_+$, holding R_t fixed and assuming $A_t > N$ is also fixed and known, we denote by $d_\beta'(u)$ the distance to default with only $u = T - t$ varying. By L'Hopital's rule,

$$
\begin{aligned}
\lim_{T-t\to 0_+} S_{t,T}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} &= \lambda_A + \lambda_{AR} - \lim_{T-t\to 0_+} \frac{1}{T-t} \ln\left(\frac{B_{t,T}^{\mathrm{SR},r'} + R_t\Phi\left(d_\beta'\right)\left[1 - e^{-\lambda_A(T-t)}\right]}{Ne^{-r'(T-t)}} \right) \\[2mm]
&= \lambda_A + \lambda_{AR} - \lim_{T-t\to 0_+} \frac{1}{T-t} \ln\left(\frac{B_{t,T}^{\mathrm{SR},r'}}{Ne^{-r'(T-t)}} + \frac{R_t\Phi\left(d_\beta'\right)\left[1 - e^{-\lambda_A(T-t)}\right]}{Ne^{-r'(T-t)}} \right) \\[2mm]
&= \lambda_A + \lambda_{AR} - \lim_{u\to 0_+} \frac{1}{u} \ln\left(\frac{B_{t,t+u}^{\mathrm{SR},r'}}{Ne^{-r'u}} + \frac{R_t\Phi\left(d_\beta'\right)\left[1 - e^{-\lambda_A u}\right]}{Ne^{-r'u}} \right) \\[2mm]
&= \lambda_A + \lambda_{AR} - \frac{\lim_{u\to 0_+}\left(\frac{\partial}{\partial u}\left[\frac{B_{t,t+u}^{\mathrm{SR},r'}}{Ne^{-r'u}}\right] + \frac{\partial}{\partial \beta}\left[\frac{(1-e^{-\lambda_A u})R_t\Phi\left(d_\beta'(u)\right)}{Ne^{-r'u}}\right] \right)}{\lim_{u\to 0_+}\left(\frac{B_{t,t+u}^{\mathrm{SR},r'}}{Ne^{-r'u}} + \frac{(1-e^{-\lambda_A u})R_t\Phi\left(d_\beta'(u)\right)}{Ne^{-r'u}} \right)} \\[2mm]
&= \lambda_A + \lambda_{AR} \\[2mm]
&\quad - \frac{\lim_{u\to 0_+}\left(\frac{\partial}{\partial u}\left[\Phi\left(d_0'(u)\right) + \frac{R_t}{Ne^{-r'u}}\Phi\left(-d_\beta'(u)\right)\right] + \frac{\partial}{\partial u}\left[\frac{(1-e^{-\lambda_A u})R_t\Phi\left(d_\beta'(u)\right)}{Ne^{-r'u}}\right] \right)}{1} \\[2mm]
&= \lambda_{AR} + \lambda_A \\[2mm]
&\quad - \lim_{u\to 0_+}\frac{\partial}{\partial u}\left[\Phi\left(d_0'(u)\right)\right] - \lim_{u\to 0_+}\left(\frac{R_t}{Ne^{-r'u}}\frac{\partial}{\partial u}\left[\Phi\left(-d_\beta'(u)\right)\right] \right) \\[2mm]
&\quad - \lim_{u\to 0_+}\left(r'\frac{R_t}{Ne^{-r'u}}\Phi\left(-d_\beta'(u)\right) \right) \\[2mm]
&\quad - \lim_{u\to 0_+}\left(\lambda_A\left[\frac{R_t\Phi\left(d_\beta'(u)\right)}{Ne^{-r'u}}\right] \right) \\[2mm]
&\quad - \lim_{u\to 0_+}\left(\left(1 - e^{-\lambda_A u}\right)\frac{\partial}{\partial u}\left[\frac{R_t\Phi\left(d_\beta'(u)\right)}{Ne^{-r'u}}\right] \right).
\end{aligned}
\tag{54}
$$

To complete this limit calculation, we compute that

$$
\frac{\partial}{\partial u}\left[\Phi\left(d_\beta'(u)\right)\right] = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_\beta'(u)\right)^2}\left[\frac{(r' + (\beta - \frac{1}{2})\sigma_A^2)u - \ln\frac{A_t}{N}}{2\sigma_A u^{\frac{3}{2}}}\right]
\tag{55}
$$

and, since

$$
\begin{aligned}
\lim_{u\to 0_+}\left[\frac{(r' + (\beta - \frac{1}{2})\sigma_A^2)u - \ln\frac{A_t}{N}}{2\sigma_A}\right] &= -\frac{1}{2\sigma_A}\ln\frac{A_t}{N} \\[2mm]
\lim_{u\to 0_+}\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}\left(d_\beta'(u)\right)^2}}{u^{\frac{3}{2}}} &= \lim_{u\to 0_+}\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2u}\left(\ln\frac{A_t}{N} + [r' + (\beta - \frac{1}{2})\sigma_A^2]u\right)^2}}{u^{\frac{3}{2}}} \\[2mm]
&= \lim_{y\to\infty}\frac{1}{\sqrt{2\pi}}y^{\frac{3}{2}}e^{-\frac{1}{2y}\left(y\cdot\ln\frac{A_t}{N} + [r' + (\beta - \frac{1}{2})\sigma_A^2]\right)^2} \\[2mm]
&= 0,
\end{aligned}
\tag{56}
$$

we have the limit

$$0 = \lim_{u \to 0_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d_\beta'(u)\right)^2} \left[\frac{(r' + (\beta - \frac{1}{2})\sigma_A^2)u - \ln \frac{A_t}{N}}{2\sigma_A u^{\frac{3}{2}}} \right]$$

$$= \lim_{u \to 0_+} \frac{\partial}{\partial u} \left[\Phi\left(d_\beta'(u)\right) \right] \tag{57}$$

$$= \lim_{u \to 0_+} \frac{\partial}{\partial u} \left[\Phi\left(-d_\beta'(u)\right) \right].$$

A similar argument is used to show that

$$\lim_{u \to 0_+} \frac{\partial}{\partial u} \left[\Phi\left(d_0'(u)\right) \right] = 0, \tag{58}$$

and thus our main result:

$$\lim_{T-t \to 0_+} S_{t,T}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} = \lambda_{AR} + \lambda_A - \lim_{u \to 0_+} \left(\lambda_A \left[\frac{R_t \Phi\left(d_\beta'(u)\right)}{N e^{-r'u}} \right] \right)$$

$$= \lambda_{AR} + \lambda_A \left(1 - \frac{R_t}{N} \right). \tag{59}$$

□

Remark 8. *The structure of short-term credit spreads, obtained in this result, suggests another reason for incorporating recovery into structural modeling: to allow for short-term credit spreads that are stochastic in nature. A related viewpoint is that one could start from observed stochastic credit spreads to model recovery in bond pricing. Please see (A8) for the comparison with classical Merton (with jump-to-0) credit spreads. Finally, we offer that the effect of short term credit spreads can also be seen if jump-to-0 is combined with bounded recovery in the pricing of the bond. These calculations follow the same format as those carried out in the sections above, and the interested reader is encouraged to compute the relevant quantities for their own needs.*

6. Conclusions

By incorporating recovery risk into pricing of defaultable bonds, we have shown that the effect mimics default risk by the distortion of default probability estimation. This distortion reflects the extra risk investors take on when bonds recover a non-constant amount, or non-constant fraction of assets, upon default. This technology can be further incorporated into credit default swaps Cohen and Costanzino (2015, 2017) and other financial instruments, as well as used to perhaps better calibrate risk measures arising from observed data. From an industry standpoint, the extra flexibility of allowing for stochastic recovery is also reflected in the ability to model a term structure for recovery rates by understanding the effects the model choice for recovery R has on the distortion probability $\mathbb{Q}_t^*[D]$ that factors directly into these calculations. Furthermore, by allowing jumps to 0 in the asset as a trigger for default, we are able to provide a hybrid model that incorporates stochastic recovery as well as non-zero short term credit spread that could possibly be implied from observed market values.

Appendix A

Appendix A.1. Analysis of Jump-Spread

We provide an alternate proof of the effect of jumps-to-0 on short-term credit spread, and compare it with classical results.

The structural Merton credit spread, when jumps to 0 for asset and recovery are incorporated, is thus widened over a non-jump structural spread. The result is that the credit spread for our jump-diffusion is linear in the loss given default at t, and is the sum of the correlated jump risk λ_{AR} and the product of the asset-specific rate λ_A and the loss given default if it were to happen at t:

$$\lim_{u \to 0_+} S_{t,t+u}^{(r,\sigma_A,\lambda_A,\lambda_{AR})} = \lambda_{AR} + \lambda_A \cdot \left(1 - \frac{R_t}{N}\right). \tag{A1}$$

Note that the short-term credit spread and default rate are exactly the same if $\lambda_A = 0$, meaning that recovery and asset are exactly 0 at the same time. Otherwise, there is a divergence between default risk and credit risk because of the stochastic recovery, and this is a feature of our model. Of course, like the short rate for interest in the Ho–Lee model, our credit spread can become negative if recovery is much larger than the notional. This can be addressed by constraining recovery to be bounded by notional, as in the SRM-CR model above.

Appendix A.2. Comparison with Classical Credit Spread

In calculating (59) directly, we obtain the yield over risk-free rate to compensate investors for investing in a corporate bond. However, we can also compare this result with the classical formula Cohen and Costanzino (2015, 2017) for credit spread:

$$S_{t,T} = \frac{1}{T-t} \ln \left(\frac{1}{1 - \mathbb{P}_t[\textbf{Default}] \cdot \mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}]} \right). \tag{A2}$$

By Taylor expansion, and partitioning the short-term default event into its constituent events, we have for small $\mathbb{P}_t[\textbf{Default}] \cdot \mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}]$,

$$S_{t,T} \approx \frac{1}{T-t} \cdot \mathbb{P}_t[\textbf{Default}] \cdot \mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}], \tag{A3}$$

and, furthermore, if $\Delta t = T - t \ll 1$, it follows that

$$\mathbb{P}_t[\textbf{Default} \mid N_t^A = 0, N_{t+\Delta t}^A > 0] = \lambda_A \Delta t + O(\Delta t),$$
$$\mathbb{P}_t[\textbf{Default} \mid N_t^{AR} = 0, N_{t+\Delta t}^{AR} > 0] = \lambda_{AR} \Delta t + O(\Delta t),$$
$$\mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}, N_t^A = 0, N_{t+\Delta t}^A > 0] = \left(1 - \frac{R_t}{N}\right), \tag{A4}$$
$$\mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}, N_t^{AR} = 0, N_{t+\Delta t}^{AR} > 0] = \left(1 - \frac{0}{N}\right),$$

and so

$$\begin{aligned} S_{t,t+\Delta t} &\approx \frac{1}{\Delta t} \cdot \mathbb{P}_t[\textbf{Default}] \cdot \mathbb{E}_t^{\mathbb{P}}[\textbf{Loss} \mid \textbf{Default}] \\ &= \frac{1}{\Delta t} \cdot \left(\lambda_{AR} \Delta t \cdot \left(1 - \frac{0}{N}\right) + \lambda_A \Delta t \cdot \left(1 - \frac{R_t}{N}\right) + O(\Delta t) \right). \end{aligned} \tag{A5}$$

Therefore,

$$\lim_{\Delta t \to 0} S_{t,t+\Delta t} = \lambda_{AR} + \lambda_A \cdot \left(1 - \frac{R_t}{N}\right), \tag{A6}$$

which is consistent with our result in Equation (59).

Appendix A.2.1. Comparison with 1d Merton (with Jump) Credit Spread

As a final comparison, recall that for a Classical Merton Default model (where non-zero recovery $R_T \equiv A_T$), we reduce the structural model to one factor via $A_t^{\lambda_{AR}} \equiv R_t^{\lambda_{AR}}$ and by setting

- $\lambda_A = 0$.
- $\sigma_A = \sigma_R$.
- $\rho_{A,R} = 1$.
- $\lambda_{AR} = \lambda$.

It follows that the evolution of assets, and hence recovery, are set by

$$\frac{dA_t^\lambda}{A_{t-}^\lambda} = (r + \lambda)dt + \sigma_A dW_t^A - dN_t^\lambda \tag{A7}$$

and so, for $u := T - t$, we have via the calculations used for Equation (54)

$$\lim_{u \to 0_+} S_{t,t+u}^{(r,\sigma_A,\lambda)} = -\lim_{u \to 0_+} \frac{1}{u} \ln\left(\frac{B_{t,t+u}^{(r,\sigma_A,\lambda)}}{Ne^{-ru}}\right) = -\lim_{u \to 0_+} \frac{1}{u} \ln\left(\frac{B_{t,t+u}^{(r+\lambda,\sigma_A,0)}}{Ne^{-ru}}\right)$$

$$= \lambda - \lim_{u \to 0_+} \frac{1}{u} \ln\left(\frac{B_{t,t+u}^{(r+\lambda,\sigma_A,0)}}{Ne^{-(r+\lambda)u}}\right) \tag{A8}$$

$$= \lambda$$

$$= \lim_{(\lambda_A,\lambda_{AR},u) \to (0_+,\lambda,0_+)} S_{t,t+u}^{(r,\sigma_A,\lambda_A,\lambda_{AR})}.$$

Appendix A.2.2. Hedging and Distortion via Stochastic Recovery Merton

When hedging bonds with respect to the numeraire, Δ returns in each case

$$\Delta_{t,T}^{\mathrm{NR}} = \frac{\partial B_{t,T}^{\mathrm{NR}}}{\partial A} = \Phi(-d_1) = \mathbb{Q}_t[D],$$

$$\Delta_{t,T}^{\mathrm{SR}} = \frac{\partial B_{t,T}^{\mathrm{SR}}}{\partial R} = \Phi(-d_\beta) = \mathbb{Q}_t^*[D]. \tag{A9}$$

Once again, we have an interpretation in terms of default risk, but now the measure is explicitly in terms of numeraire. For the Merton model with recovery A, Δ^{NR} is the probability of default from the perspective of asset A as the numeraire, while incorporating recovery R shifts this Greek to Δ^{SR}, the probability of default from the perspective of recovery as numeraire. In estimating both the effect of recovery risk in hedging and in estimation of default probability, we see that the effect of disentangling recovery from firm asset value mimics default risk in the form of the Wang Transform Wang (2002). With the addition of this distortion arises the possibility of better forecasting, and empirical investigation is forthcoming. By assuming some non-zero financial elasticity β in the recovery market, we are able to reflect this effect of partial information directly into default and recovery risk, and gain further insight into the modeling of stochastic recovery.

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