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Pricing credit derivatives under stochastic recovery in a hybrid model

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SUMMARY

In this article, a framework for the joint modelling of default and recovery risk is presented. The model accounts for typical characteristics known from empirical studies, e.g. negative correlation between recovery-rate process and default intensity, as well as between default intensity and state of the economy, and a positive dependence of recovery rates on the economic environment. Within this framework analytically tractable pricing formulas for credit derivatives are derived. The stochastic model for the recovery process allows for the pricing of credit derivatives with payoffs that are directly linked to the recovery rate at default, e.g. recovery locks. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Over the last decades a great variety of credit risk models has been developed. Whereas most of them address default event risk and default loss risk, there have been only few studies concerning the stochastic behavior of recovery rates. Caused by the rapid growth in the credit derivatives market and the appearance of contingent claims on recoveries, e.g. fixed recovery credit default swap (CDS), recovery locks or recovery swaps (see e.g. [1, 2]), the sound modelling of recovery rates gained in importance lately, not only for pricing purposes but also for portfolio risk management as well as for economic capital requirements.

In the meantime studies on the determinants of historical recovery rates (see e.g. [3–5]) have been conducted, succeeding those on the determinants of default probabilities (see e.g. [6–10]).

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Moreover, after a long history of scoring models for defaults (see e.g. [6–8, 11, 12]), recovery-prediction models (see e.g. [13, 14]) have been developed recently. Nevertheless, pricing models with a dynamic modelling of stochastic recovery rates are still scarce. In the past, most studies and models have treated the recovery rate as a constant or independent of the default process to derive closed-form solutions of corporate bond or CDS prices. While in the first generation of structural-form models based on the seminal work of Merton (see [15]) the recovery rate is an endogenous variable depending on the firm's asset volatility and the firm's leverage, the second generation of structural-form models (first-passage time models) treat the recovery rate as an exogenous variable independent of the firm's asset value as well as of the default process (see e.g. [16]); exceptions are models based on discontinuous processes (see e.g. [17]). Reduced-form models (see e.g. [18, 19]) also assume an exogenous recovery rate independent of the default process. This recovery rate can either be deterministic or stochastic and different recovery rates for different issuers or seniorities are possible.

By contrast, there has been a growing amount of empirical research articles showing that recovery rates vary over time and are negatively correlated with the default process (see e.g. [20, 21]). Furthermore, there is evidence that recovery rates are lower in a distressed economy than in a healthy economy (see e.g. [4, 22]).

Over the last years, different approaches have been made to incorporate the stochastic behavior of recovery rates and their correlation with the default process in credit-pricing models.

One of the first attempts to a joint modelling of recovery and default risk was proposed by Bakshi *et al.* [23] who assume that the recovery rate is exponentially related to the underlying hazard rate. The hazard-rate process itself is assumed to be linear in the short-term interest rate that is driven by a Cox–Ingersoll–Ross (CIR)-process. The main drawback of this modelling approach is that there is only one factor, the short rate, that explains the whole variation. In the appendix of [24] an illustrative multi-factor defaultable bond valuation model is presented. However, this model leads to complex valuation formulas and is hence difficult to implement.

In [25] a three-factor model for the joint evolution of interest rates, intensity, and recovery is introduced where both intensity and recovery rate are affine functions of a common risk factor (short-term interest rate) and an idiosyncratic risk factor. In a simulation-based study, the default intensity and recovery risk components are – under certain conditions (recovery of face value assumption, limited noise in bond yields) – estimated simultaneously from bond yields. Still, there is a fundamental identification problem inherent in the corporate bond yields as soon as measurement noise is added to the true yields. An extension of this model is developed in [26]. The short-term interest rate is an affine function of two interest-rate factors. The default-intensity risk factor is assumed to be a CIR-process and the recovery-rate risk factor is assumed to be Gaussian. Both intensity and recovery rate are driven by the two interest-rate risk factors and their corresponding idiosyncratic risk factors. The author's aim is the separation of default and recovery risk in CDS quotes.

Karoui [27] proposes a discrete-time framework for modelling defaultable instruments under stochastic recovery as the pricing formulas are easier to handle than in a corresponding continuous-time setting. Gaspar and Slinko [28] present a model based on the dynamics of a market index that determines the default intensity as well as the distribution of the loss quota. To be more precise, the loss quota is assumed to be a beta distributed random variable with one constant parameter and the other one driven by the market index. This leads to intractable pricing formulas and requires a simulation-based framework. A completely different approach is used in [29]. Here, a reduced-form calibration method for the joint derivation of market-implied forward hazard rates and forward

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recovery rates is presented, but without a dynamic representation of default and recovery risk components.

The aim of this article is a joint modelling of default and recovery risk accounting for the aforementioned characteristics and the derivation of analytically tractable pricing formulas for different credit derivatives including recovery products. The organization of the article will be as follows: The modelling framework for the short rate, the recovery rate, and the default intensity is introduced in Section 2. Section 3 contains the pricing formulas of all considered credit derivatives. A calibration procedure and a numerical example with market data are presented in Section 4. Section 5 concludes.

2. THE MODEL

The modelling approach presented in this section is based on the framework of the extended Schmid–Zagst defaultable term-structure model (see [30]), which is an extension of the three-factor Schmid–Zagst model (see [31]). This hybrid model directly models the short-rate credit spread in dependence on some unobservable, firm-specific uncertainty index. Under the assumption of fractional recovery of market value, i.e. the recovery payment in case of a default event is assumed to be a fraction of the market value instantaneously before default, closed-form solutions for defaultable bond prices are available without specifying a recovery-rate process (see e.g. [30]). Within the same framework, the pricing of credit derivatives under constant recovery is developed in [32]. In contrast, Jarrow and Turnbull [33] and Madan and Unal [18] used a partial recovery of treasury value approach, i.e. there is a compensation in case of a default event in terms of a fraction of a non-defaultable bond with the same maturity and face value, to price financial securities under default risk. Unlike [30–32], we rather model the default intensity instead of the short-rate credit spread and use a recovery of face value instead of a recovery of market value assumption. Under this recovery of face value assumption, the recovery payment in case of a default event at time t is a fraction $z(t)$, called the recovery rate, of the face value.

2.1. *Modelling framework*

In the following a fixed terminal time horizon T^* is assumed. Uncertainty in the financial market is modelled on a complete probability space $(\Omega, \mathscr{G}, \mathbb{P})$. All random variables and stochastic processes introduced below are defined on this probability space. We assume that $(\Omega, \mathscr{G}, \mathbb{P})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing and right-continuous families of sub-σ-fields of $\mathscr{G}$. The default time τ of an obligor is an arbitrary random time on $(\Omega, \mathscr{G}, \mathbb{P})$. For the sake of convenience, we assume that $\mathbb{P}(\tau=0)=0$ and $\mathbb{P}(\tau>t)>0$ for every $t \in (0, T^*]$. For a given default time τ, consider the associated default indicator or hazard function $H(t)=\mathbf{1}_{\{\tau\leqslant t\}}$ and the survival indicator function $L(t)=1-H(t), t\in(0, T^*]$. Let $\mathbb{H}=(\mathscr{H}_t)_{0\leqslant t\leqslant T^*}$ be the filtration generated by the process H. In addition, let the filtration $\mathbb{F}=(\mathscr{F}_t)_{0\leqslant t\leqslant T^*}$ be defined as the filtration generated by the multi-dimensional standard Brownian motion $W(t)^T$ containing all one-dimensional Brownian motions appearing in the modelled processes. Additionally, let $\mathbb{G}=(\mathscr{G}_t)_{0\leqslant t\leqslant T^*}$ denote the enlarged filtration $\mathbb{G}=\mathbb{H}\vee\mathbb{F}$, i.e. for every t set $\mathscr{G}_t=\mathscr{H}_t\vee\mathscr{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. This subfiltration structure is very common in intensity-based models (see e.g. [30, 34]).

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It is assumed throughout that for any $t \in (0, T^*]$, the σ-fields $\mathscr{F}_{T^*}$ and $\mathscr{H}_t$ are conditionally independent given $\mathscr{F}_t$. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale (see [34], p. 167).

2.2. Short-rate model

The short-rate model is specified by a two factor Hull–White model, with stochastic processes r and w describing the non-defaultable short rate and a market factor. The dynamics of the non-defaultable short rate are given by the stochastic differential equation (SDE)

$$\mathrm{d}r(t) = [\theta_r(t) + b_{rw}w(t) - a_r r(t)]\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_r(t), \quad r(0) = r_0, \ 0 \leqslant t \leqslant T^* \tag{1}$$

where r_0, a_r, b_{rw}, $\sigma_r > 0$ are positive constants and $\theta_r(t)$ is a non-negative valued deterministic function.

The dynamics of the market factor are given by the following SDE:

$$\mathrm{d}w(t) = [\theta_w - a_w w(t)]\,\mathrm{d}t + \sigma_w\,\mathrm{d}W_w(t), \quad w(0) = w_0, \ 0 \leqslant t \leqslant T^* \tag{2}$$

where a_w, $\sigma_w > 0$ are positive constants, θ_w is a non-negative constant and $w_0 \in \mathbb{R}$.

2.3. Recovery and intensity model

The recovery rate $z(t)$, or equivalently $\tilde{z}(t) := z(t) - a_z$ with $a_z \geqslant 0$, is given by

$$\tilde{z}(t) = b_z\,\mathrm{e}^{-c_z u(t) + d_z w(t)} \tag{3}$$

with $b_z \geqslant 0$, $a_z + b_z < 1$, and u denoting an (unobservable) idiosyncratic risk factor given by the SDE

$$\mathrm{d}u(t) = [\theta_u - a_u u(t)]\,\mathrm{d}t + \sigma_u\,\mathrm{d}W_u(t), \quad u(0) = u_0, \ 0 \leqslant t \leqslant T^* \tag{4}$$

where a_u, $\sigma_u > 0$ are positive constants, θ_u is a non-negative constant and $u_0 \in \mathbb{R}$.

The dynamics of the default intensity are given by the SDE

$$\mathrm{d}\lambda(t) = [\theta_\lambda + b_{\lambda u}u(t) - b_{\lambda w}w(t) - a_\lambda \lambda(t)]\,\mathrm{d}t + \sigma_\lambda\,\mathrm{d}W_\lambda(t), \quad \lambda(0) = \lambda_0 \tag{5}$$

where λ_0, a_λ, $b_{\lambda u}$, $b_{\lambda w}$, $\sigma_\lambda > 0$ are positive constants, θ_λ is a non-negative constant, and $0 \leqslant t \leqslant T^*$.

In the following, we assume that the Wiener processes W_r, W_w, W_u, and W_λ are uncorrelated.

As mentioned above, it is general consent that default risk and recovery risk are correlated and that recovery rates depend on the state of the economy. The first observation is accounted for in the modelling framework presented above by the impact of u on z and λ and the latter by the positive dependence of z on w.

Note that in this modelling framework, the recovery-rate process can take values greater than 1. Recoveries of more than 100% can indeed be observed in certain situations (see e.g. p. 13 of [4]). Also, short rates as well as default intensities can become negative in this framework. Here, we follow [35] (see p. 74), [36] (see p. 108), and [37] (see p. 166) stating that the computational advantages are worth the approximation error and that small probabilities of negative short rates or default intensities are accepted in practical applications. Using the parameter set from Table I, the (risk-neutral) one-year probability that the default intensity λ is negative is 0.033 and the (risk-neutral) one-year probabilities that the short rate r is negative and that the recovery rate

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z is greater 1 are both below 10^{-10}. One way to overcome the problem of possibly negative short rates and default intensities while preserving the aforementioned dependencies would be to assume CIR-processes with correlated Brownian motions for r, w, u, and λ instead of the dynamics assumed in Equations (1), (2), (4), and (5). However, this would lead to a significant loss of computational tractability of the pricing formulas presented in Section 3 (see also p. 140 of [35] or p. 255 of [38]). In such models with correlated CIR-processes tree- (see e.g. [39]) or simulation-based (see e.g. [40]) methods are required.

2.4. Change of measure

So far, the modelling has taken place under the real-world measure $\mathbb{P}$. For pricing purposes we need a characterization of all processes of Sections 2.2 and 2.3 under an equivalent martingale measure $\mathbb{Q}$, i.e. all discounted security price processes have to be $\mathbb{Q}$-martingales with respect to a suitable numéraire. As numéraire we choose the money-market account $B(t) = e^{\int_0^t r(l)\,dl}$, where $r(t)$ is the non-defaultable short rate from Equation (1).

It is well known that each martingale measure $\mathbb{Q}$ is given by the Radon–Nikodym-derivative

$$L(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t = \exp\left(-\int_0^t \gamma(s)^T\,dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2\,ds\right)$$

where $\gamma(s)^T = (\gamma_r(s),\ \gamma_w(s),\ \gamma_u(s),\ \gamma_\lambda(s))$ is an adapted, measurable four-dimensional process satisfying

$$\int_0^{T^*} \gamma_i(s)^2\,ds < \infty \ \mathbb{P}\text{-a.s.} \quad \text{for } i \in \{r, w, u, \lambda\}$$

Following [41] and [38], we assume $\gamma_i(t) = \eta_i \sigma_i i(t)$ with $t \in [0, T^*]$, $\eta_i \in \mathbb{R}$, and $i \in \{r, w, u, \lambda\}$, such that Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{T^*} \|\gamma(s)\|^2\,ds\right)\right] < \infty$$

holds. This assumption is made in order to preserve the structure of the SDEs (1), (2), (4), and (5) under $\mathbb{Q}$. From Girsanov's theorem (see e.g. p. 159 of [42]), it is known that $\widehat{W}(t)^T = (\widehat{W}_r(t), \widehat{W}_w(t), \widehat{W}_u(t), \widehat{W}_\lambda(t))$ with

$$\widehat{W}_i(t) = W_i(t) + \int_0^t \gamma_i(s)\,ds, \quad i \in \{r, w, u, \lambda\}, \ \ 0 \leqslant t \leqslant T^*$$

is a four-dimensional Brownian motion on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$. Under $\mathbb{Q}$, the dynamics of r, w, u, and λ are given by

$$dr(t) = [\theta_r(t) + b_{rw}w(t) - \widehat{a}_r r(t)]\,dt + \sigma_r\,d\widehat{W}_r(t), \quad r(0) = r_0$$

$$dw(t) = [\theta_w - \widehat{a}_w w(t)]\,dt + \sigma_w\,d\widehat{W}_w(t), \quad w(0) = w_0$$

$$du(t) = [\theta_u - \widehat{a}_u u(t)]\,dt + \sigma_u\,d\widehat{W}_u(t), \quad u(0) = u_0$$

$$d\lambda(t) = [\theta_\lambda + b_{\lambda u}u(t) - b_{\lambda w}w(t) - \widehat{a}_\lambda \lambda(t)]\,dt + \sigma_\lambda\,d\widehat{W}_\lambda(t), \quad \lambda(0) = \lambda_0$$

with $\widehat{a}_i = a_i + \eta_i \sigma_i^2$, $i \in \{r, w, u, \lambda\}$, and $0 \leqslant t \leqslant T^*$.

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2.5. *Valuation of defaultable claims*

Before we continue with the valuation of defaultable claims, we recall an important result for the pricing of non-defaultable zero-coupon bonds. This result will be used later for the calibration of the non-defaultable short-rate process $r(t)$ from Equation (1).

Theorem 1
The time t price of a non-defaultable zero-coupon bond with maturity T is given by

$$P^{\mathrm{nd}}(t,T)=\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r(l)\,\mathrm{d}l}|\mathscr{F}_t]=\mathrm{e}^{A^{\mathrm{nd}}(t,T)-B^{\mathrm{nd}}(t,T)r(t)-E^{\mathrm{nd}}(t,T)w(t)}$$

with

$$B^{\mathrm{nd}}(t,T)=\frac{1}{\widehat{a}_r}(1-\mathrm{e}^{-\widehat{a}_r(T-t)})$$

$$E^{\mathrm{nd}}(t,T)=\frac{b_{rw}}{\widehat{a}_r}\left(\frac{1-\mathrm{e}^{-\widehat{a}_w(T-t)}}{\widehat{a}_w}+\frac{\mathrm{e}^{-\widehat{a}_w(T-t)}-\mathrm{e}^{-\widehat{a}_r(T-t)}}{\widehat{a}_w-\widehat{a}_r}\right)$$

$$A^{\mathrm{nd}}(t,T)=\int_t^T\left[\frac{1}{2}\sigma_r^2B^{\mathrm{nd}}(s,T)^2+\frac{1}{2}\sigma_w^2E^{\mathrm{nd}}(s,T)^2 -\theta_r(s)B^{\mathrm{nd}}(s,T)-\theta_wE^{\mathrm{nd}}(s,T)\right]\mathrm{d}s$$

Proof
This theorem corresponds to a special case of the two-factor Hull–White model (see [39]). □

A defaultable contingent claim is defined as a triplet DCC$=(X,Z,\tau)$ with X denoting the promised payoff at maturity T if no default has taken place up to T, $Z=(Z(t))_{t\in[0,T]}$ the process describing the recovery payoff at the time of default, and τ the default time. If Z is a $\mathbb{G}$-predictable process and X is $\mathscr{G}_T$-measurable, the value process $V(t)$ of the defaultable contingent claim is given by (see e.g. p. 180 of [34])

$$V(t)=\mathbb{E}^{\mathbb{Q}}\left[\int_t^T\mathrm{e}^{-\int_t^s r(l)\,\mathrm{d}l}Z(s)\,\mathrm{d}H(s)+\mathrm{e}^{-\int_t^T r(l)\,\mathrm{d}l}X1_{\{\tau>T\}}\middle|\mathscr{G}_t\right]$$

Under certain assumptions, the value of such a defaultable contingent claim can be expressed by the conditional expectation of the claim's payoffs discounted with a default-risk-adjusted short rate (see e.g. [43] or [34]).

Theorem 2
Assume that the martingale invariance property assumption is fulfilled, Z is an $\mathbb{F}$-predictable process and X is an $\mathscr{F}_T$-measurable random variable. Then, for every $t\in[0,T^*]$, we have

$$V(t)=1_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T\mathrm{e}^{-\int_t^s(r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)Z(s)\,\mathrm{d}s+\mathrm{e}^{-\int_t^T(r(l)+\lambda(l))\,\mathrm{d}l}X\middle|\mathscr{F}_t\right]$$

Proof
Section 8.3 of [34]. □

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While the recovery of market value assumption is suitable for bond-pricing purposes as it leads to analytically tractable formulas, it contains the problem that intensity and recovery risk are not separable. Hence, we will use the recovery of face value assumption in the following. This assumption is generally preferred when contingent claims on recoveries are considered (see e.g. [24]).

Corollary 1
In a model with recovery of face value assumption, i.e. $Z(t) := z(t)X$ with $z(t)$ denoting the recovery-rate process, the price of a defaultable contingent claim under the assumption of no default up to time t is given by

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)z(s)X\,\mathrm{d}s + \mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}X \middle| \mathscr{F}_t\right]$$

3. PRICING OF RECOVERY-DEPENDENT CREDIT DERIVATIVES

In this section pricing equations for credit derivatives under the dynamics assumed in Equations (1)–(5) are derived.

3.1. Building blocks

The main building blocks of the pricing formulas are conditional expectations of the form

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}z(T)|\mathscr{F}_t] \tag{6}$$

and

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(T)z(T)|\mathscr{F}_t] \tag{7}$$

The following theorems show how to calculate the expected values in (6) and (7) under the assumptions (1)–(5).

Theorem 3

$$\begin{aligned} g(r,\lambda,u,w,t,T) :&= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}\mathrm{e}^{-c_z u(T)+d_z w(T)}|\mathscr{F}_t] \\ &= \mathrm{e}^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)} \end{aligned}$$

with

$$B(t,T) = \frac{1}{\widehat{a}_r}(1-\mathrm{e}^{-\widehat{a}_r(T-t)}),$$

$$C(t,T) = \frac{1}{\widehat{a}_\lambda}(1-\mathrm{e}^{-\widehat{a}_\lambda(T-t)}),$$

$$D(t,T) = \frac{b_{\lambda u}}{\widehat{a}_\lambda}\left(\frac{1-\mathrm{e}^{-\widehat{a}_u(T-t)}}{\widehat{a}_u} + \frac{\mathrm{e}^{-\widehat{a}_u(T-t)} - \mathrm{e}^{-\widehat{a}_\lambda(T-t)}}{\widehat{a}_u - \widehat{a}_\lambda}\right) + c_z\mathrm{e}^{-\widehat{a}_u(T-t)}$$

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$$
\begin{aligned}
E(t,T) = & -\frac{b_{\lambda w}}{\widehat{a}_\lambda}\left(\frac{1-\mathrm{e}^{-\widehat{a}_w(T-t)}}{\widehat{a}_w}+\frac{\mathrm{e}^{-\widehat{a}_w(T-t)}-\mathrm{e}^{-\widehat{a}_\lambda(T-t)}}{\widehat{a}_w-\widehat{a}_\lambda}\right)\\
& +b_{rw}\left(\frac{1-\mathrm{e}^{-\widehat{a}_w(T-t)}}{\widehat{a}_w\widehat{a}_r}+\frac{\mathrm{e}^{-\widehat{a}_w(T-t)}-\mathrm{e}^{-\widehat{a}_r(T-t)}}{\widehat{a}_w-\widehat{a}_r}\frac{1}{\widehat{a}_r}\right)\\
& -d_z\mathrm{e}^{-\widehat{a}_w(T-t)}
\end{aligned}
$$

and

$$
\begin{aligned}
A(t,T) = \int_t^T & \left[\frac{1}{2}\sigma_r^2 B^2(s,T)+\frac{1}{2}\sigma_\lambda^2 C^2(s,T)+\frac{1}{2}\sigma_u^2 D^2(s,T)\right.\\
& +\frac{1}{2}\sigma_w^2 E^2(s,T)-\theta_r(s)B(s,T)-\theta_\lambda C(s,T)\\
& \left.-\theta_u D(s,T)-\theta_w E(s,T)\right]\mathrm{d}s
\end{aligned}
$$

Proof
See Appendix A. □

Theorem 4

$$
\begin{aligned}
& \widetilde{g}(r,\lambda,u,w,t,T)\\
& := \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T(r(l)+\lambda(l))\,\mathrm{d}l}\lambda(T)\mathrm{e}^{-c_z u(T)+d_z w(T)}|\mathscr{F}_t]\\
& = g(r,\lambda,u,w,t,T)(G(t,T)+I(t,T)\lambda(t)+J(t,T)u(t)+K(t,T)w(t))\\
& = \mathrm{e}^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)}\\
& \quad \cdot(G(t,T)+I(t,T)\lambda(t)+J(t,T)u(t)+K(t,T)w(t))
\end{aligned}
$$

with $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$ from Theorem 3,

$$
\begin{aligned}
I(t,T) &= \mathrm{e}^{-\widehat{a}_\lambda(T-t)}\\
J(t,T) &= \frac{b_{\lambda u}}{\widehat{a}_u-\widehat{a}_\lambda}(\mathrm{e}^{-\widehat{a}_\lambda(T-t)}-\mathrm{e}^{-\widehat{a}_u(T-t)})\\
K(t,T) &= \frac{b_{\lambda w}}{\widehat{a}_\lambda-\widehat{a}_w}(\mathrm{e}^{-\widehat{a}_\lambda(T-t)}-\mathrm{e}^{-\widehat{a}_w(T-t)})
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,T) = \int_t^T & \left[\theta_\lambda I(s,T)+\theta_u J(s,T)+\theta_w K(s,T)-\sigma_\lambda^2 C(s,T)I(s,T)\right.\\
& \left.-\sigma_u^2 D(s,T)J(s,T)-\sigma_w^2 E(s,T)K(s,T)\right]\mathrm{d}s
\end{aligned}
$$

Proof
See Appendix B. □

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As an immediate consequence of Theorems 3 and 4, the following corollaries can be stated by using Corollary 1.

Corollary 2
The time t price of a defaultable zero-coupon bond with maturity T and unit notional under the assumption of zero recovery and no default up to time t is given by

$$\begin{aligned} P^{d,\text{zero}}(t,T) &= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}|\mathscr{F}_t] \\ &= \mathrm{e}^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)} \end{aligned}$$

with $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$ from Theorem 3 and $c_z = d_z = 0$.

Corollary 3
The time t price of a defaultable zero-coupon bond with maturity T and unit notional under the assumption of no default up to time t is given by

$$\begin{aligned} P^{d}(t,T) &= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T (r(l)+\lambda(l))\,\mathrm{d}l}|\mathscr{F}_t] \\ &\quad + \int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)z(s)|\mathscr{F}_t]\,\mathrm{d}s \\ &= P^{d,\text{zero}}(t,T) + b_z \int_t^T \tilde{g}(r,\lambda,u,w,t,s)\,\mathrm{d}s \\ &\quad + a_z \int_t^T \tilde{g}^{\text{zero}}(r,\lambda,u,w,t,s)\,\mathrm{d}s \end{aligned}$$

with $\tilde{g}(r,\lambda,u,w,t,s)$ from Theorem 4 and $\tilde{g}^{\text{zero}}(r,\lambda,u,w,t,s)$ denoting $\tilde{g}(r,\lambda,u,w,t,s)$ under the assumption $c_z = d_z = 0$.

Corollary 4
A default digital put option on a defaultable zero-coupon bond with maturity T pays one unit of currency in the case of a default before or at maturity and nothing else. Assuming no default up to time t and that the payoff takes place at default, the time t price of the default digital put is given by

$$\begin{aligned} V^{ddp}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)\,\mathrm{d}s|\mathscr{F}_t\right] \\ &= \int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)|\mathscr{F}_t]\,\mathrm{d}s \\ &= \int_t^T \tilde{g}^{\text{zero}}(r,\lambda,u,w,t,s)\,\mathrm{d}s \end{aligned}$$

with $\tilde{g}^{\text{zero}}(r,\lambda,u,w,t,s)$ denoting $\tilde{g}(r,\lambda,u,w,t,s)$ from Theorem 4 under the assumption $c_z = d_z = 0$.

In the following subsection pricing formulas for credit derivatives based on Equations (6) and (7) are established.

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3.2. Credit default swaps

A credit default swap (CDS) is a swap under which one party (the beneficiary) pays the other party (the guarantor) regular fees, called the CDS spread or the CDS rate. This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default to cover the loss resulting from default. As common for swap products we have to consider two payment streams, the default and the premium leg. We assume, according to the recovery of face value assumption, that in case of a default event the payment on the default leg is one minus the recovery rate times the notional. For ease of notation we assume a unit notional in the following.

The pricing of a CDS consists of two problems. At origination ($t=t_0$) there is no exchange of cash flows and the CDS spread $S^{\mathrm{CDS}}(t_0,T)$ has to be determined such that the market value of the CDS is zero. After origination ($t\in(t_0,T]$), the market value of the CDS will change due to changes in the underlying variable. Therefore, given the CDS spread $S^{\mathrm{CDS}}(t_0,T)$, the current market value of the CDS has to be computed.

We assume throughout that the CDS counterparties (beneficiary and guarantor) are default-free. Furthermore, we assume that the underlying reference credit asset has no coupon payments up to the maturity T^* and that there has been no credit event until time t_0. The scheduled payment dates of the credit swap spread are denoted by $t_i,\ i=1,\dots,m$. The value of the default leg at origination must be the same as paying $S^{\mathrm{CDS}}(t_0,T)$ at some predefined times t_i, $i=1,\dots,m$, with $t_0\leqslant t_1\leqslant\cdots\leqslant t_m=T$ until a default happens. Finally, for ease of notation we assume that in case of a default event the beneficiary receives the compensation at the next premium date rather than right upon default. Under these assumptions the CDS premium is given as follows:

Corollary 5
The swap premium of a CDS is (under the above-mentioned assumptions) given by

$$S^{\mathrm{CDS}}(t_0,T)=\frac{V^{ddp}(t_0)-P^d(t_0,T)+P^{d,\mathrm{zero}}(t_0,T)}{\sum_{i=1}^m (t_i-t_{i-1})P^{d,\mathrm{zero}}(t_0,t_i)} \tag{8}$$

Proof
Under the above-mentioned assumptions, the time t value of the default leg of a CDS is given by (see e.g. [36])

$$\begin{aligned}V_{\mathrm{def}}^{\mathrm{CDS}}(t,T)&=\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)(1-z(s))\,\mathrm{d}s\,\middle|\,\mathscr{F}_t\right]\\&=V^{ddp}(t)-P^d(t,T)+P^{d,\mathrm{zero}}(t,T)\end{aligned}$$

The time t value of the premium leg is given by

$$\begin{aligned}V_{\mathrm{prem}}^{\mathrm{CDS}}(t,T)&=\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^m S^{\mathrm{CDS}}(t_0,T)(t_i-t_{i-1})\,\mathrm{e}^{-\int_t^{t_i}(r(l)+\lambda(l))\,\mathrm{d}l}\,\middle|\,\mathscr{F}_t\right]\\&=S^{\mathrm{CDS}}(t_0,T)\sum_{i=1}^m (t_i-t_{i-1})\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^{t_i}(r(l)+\lambda(l))\,\mathrm{d}l}|\mathscr{F}_t]\\&=S^{\mathrm{CDS}}(t_0,T)\sum_{i=1}^m (t_i-t_{i-1})P^{d,\mathrm{zero}}(t,t_i)\end{aligned}$$

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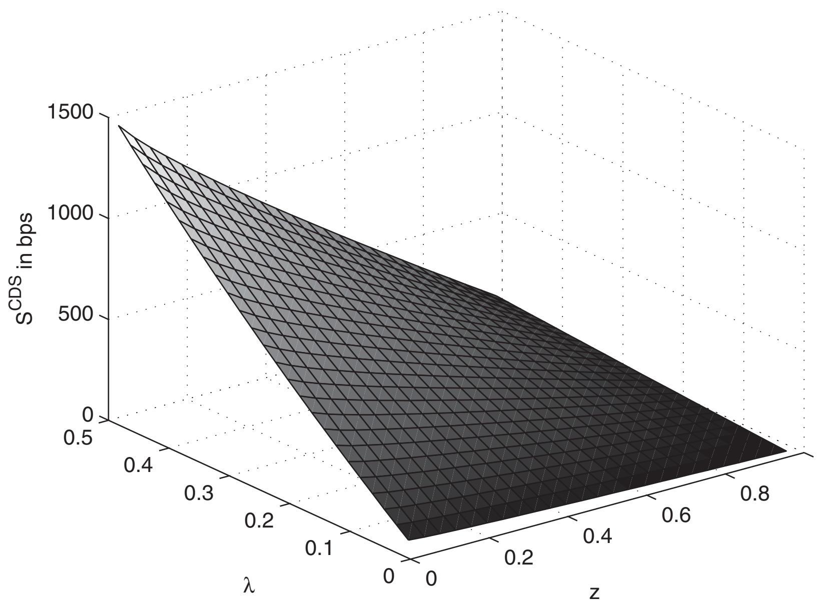

Figure 1. CDS spread in dependence of recovery z and intensity λ.

where $S^{\text{CDS}}(t_0, T)$ is the swap premium of the CDS. To give the contract a value of zero at origination, the relation

$$V_{\text{def}}^{\text{CDS}}(t_0, T) = V_{\text{prem}}^{\text{CDS}}(t_0, T)$$

must hold and hence the swap premium is given by Equation (8). □

Figure 1 shows the impact of different values of λ and z (all other parameters fixed) on CDS spreads in the modelling framework from Section 2.[‡]

While the value of the contract at origination is zero, it changes during the lifetime of the contract. The value of the CDS is then given by the difference between the value of the default leg $V_{\text{def}}^{\text{CDS}}(t, T)$ and the value of the premium leg $V_{\text{prem}}^{\text{CDS}}(t, T)$.

3.3. Fixed-recovery CDS

A fixed-recovery CDS or default digital swap is a CDS with a contractually fixed recovery payment in case of default. Hence, the swap premium for the fixed-recovery CDS can be calculated similar to the swap premium of a standard CDS and is given in the following corollary.

Corollary 6

The swap premium of a fixed-recovery CDS with a contractually fixed recovery rate R_{Fix} is given by

$$S^{\text{FRCDS}}(t_0, T, R_{\text{Fix}}) = \frac{(1 - R_{\text{Fix}}) V^{ddp}(t_0)}{\sum_{i=1}^{m} (t_i - t_{i-1}) P^{d,\text{zero}}(t_0, t_i)} \tag{9}$$

[‡]For this illustration, the parameter set from Table I in Section 4 estimated from market data was used.

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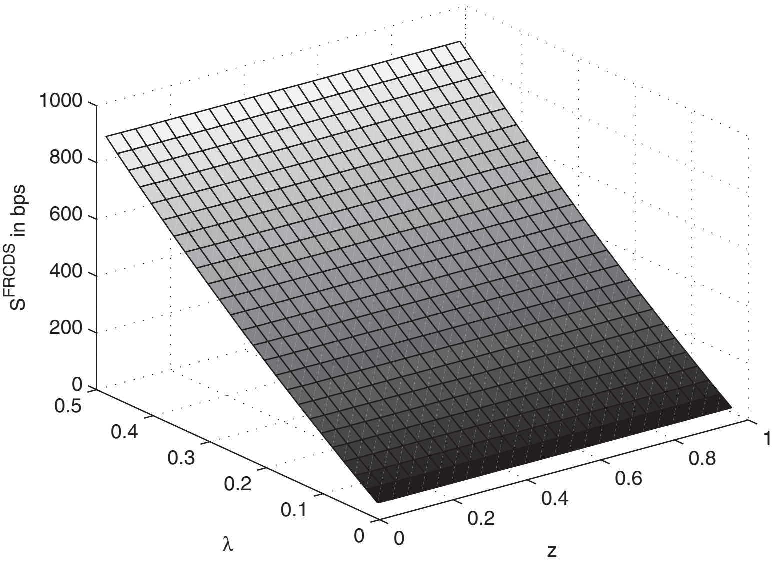

Figure 2. Fixed-recovery CDS spread in dependence of recovery z and intensity λ.

Proof

The time t value of the default leg of a fixed-recovery CDS is given by

$$\begin{aligned} V^{\mathrm{FRCDS}}_{\mathrm{def}}(t,T) &= (1-R_{\mathrm{Fix}})\int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\mathrm{d}l}\lambda(s)|\mathscr{F}_t]\,\mathrm{d}s \\ &= (1-R_{\mathrm{Fix}})V^{ddp}(t) \end{aligned}$$

The time t value of the premium leg of such a fixed-recovery CDS is given by

$$\begin{aligned} V^{\mathrm{FRCDS}}_{\mathrm{prem}}(t,T) &= S^{\mathrm{FRCDS}}(t_0,T,R_{\mathrm{Fix}})\sum_{i=1}^m (t_i-t_{i-1})\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^{t_i}(r(l)+\lambda(l))\,\mathrm{d}l}|\mathscr{F}_t] \\ &= S^{\mathrm{FRCDS}}(t_0,T,R_{\mathrm{Fix}})\sum_{i=1}^m (t_i-t_{i-1})P^{d,\mathrm{zero}}(t,t_i) \end{aligned}$$

where $S^{\mathrm{FRCDS}}(t_0,T,R_{\mathrm{Fix}})$ is the swap premium of the fixed-recovery CDS with the contractually fixed-recovery rate R_{Fix}. By equating the values of default and premium leg at $t=t_0$, the assertion follows immediately. □

Similar to Figure 1, Figure 2 shows the impact of different values of λ and z (all other parameters fixed) on the fixed-recovery CDS spread in the modelling framework of Section 2.[§] By construction the fixed-recovery CDS spread is independent of the dynamics of the recovery-rate process and is therefore a measure of default event risk.

[§]For this illustration, the parameter set from Table I in Section 4 estimated from market data was used.

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As for standard CDS, the value of a fixed-recovery CDS is given by the difference between the value of the default leg $V_{\text{def}}^{\text{FRCDS}}(t,T)$ and the value of the premium leg $V_{\text{prem}}^{\text{FRCDS}}(t,T)$.

3.4. Recovery lock

While standard CDS give protection against default loss risk and fixed-recovery CDS against default event risk, recovery locks give protection against pure recovery risk. Recovery locks, sometimes also called recovery swaps or recovery forwards, allow to purchase or sell the underlying credit instrument at a predetermined price $R_{\text{Lock}}(t_0,T)$ if a credit event occurs. A recovery lock has no upfront or running payments. The only payment stream is the exchange of realized and predetermined recovery in case of a default event. Its payoff can be represented either as a single recovery lock trade or through a recovery swap representation that separates the trade in two legs, a short protection in a standard CDS and a long protection in fixed-recovery CDS (see e.g. [1, 2]). The price of such a recovery lock is given in the following corollary.

Corollary 7
The price of a recovery lock is given by

$$R_{\text{Lock}}(t_0,T)=1-(1-R_{\text{Fix}})\frac{S^{\text{CDS}}(t_0,T)}{S^{\text{FRCDS}}(t_0,T,R_{\text{Fix}})}$$

with $S^{\text{CDS}}(t_0,T)$ denoting the swap premium of a (standard) CDS from Equation (8) and $S^{\text{FRCDS}}(t_0,T,R_{\text{Fix}})$ the premium of a fixed-recovery CDS with contractually fixed recovery rate R_{Fix} from Equation (9).

Proof
Assume we want to replicate the payoff of a long position in a recovery lock by buying φ^{FRCDS} fixed-recovery CDS and selling φ^{CDS} standard CDS. To circumvent arbitrage opportunities, the net cash flows of the two representations have to be equal to zero in all scenarios. As the recovery lock has no running payments, the running payments of the combined position given by

$$\varphi^{\text{CDS}}S^{\text{CDS}}(t_0,T)-\varphi^{\text{FRCDS}}S^{\text{FRCDS}}(t_0,T,R_{\text{Fix}})$$

have to be equal to zero. This leads to a ratio of fixed-recovery CDS to standard CDS of

$$\frac{\varphi^{\text{FRCDS}}}{\varphi^{\text{CDS}}}=\frac{S^{\text{CDS}}(t_0,T)}{S^{\text{FRCDS}}(t_0,T,R_{\text{Fix}})}$$

In case of a default event the payment of the recovery lock equals

$$R_{\text{Lock}}(t_0,T)-z(t)$$

with $z(t)$ denoting the (actual) recovery rate in case of a default event at time t, while the payoff of the combined position is given by

$$\begin{aligned}
&\varphi^{\text{CDS}}(1-z(t))-\varphi^{\text{FRCDS}}(1-R_{\text{Fix}})\\
&\quad=\varphi^{\text{CDS}}(1-z(t))-\varphi^{\text{CDS}}\frac{S^{\text{CDS}}(t_0,T)}{S^{\text{FRCDS}}(t_0,T,R_{\text{Fix}})}(1-R_{\text{Fix}})
\end{aligned}$$

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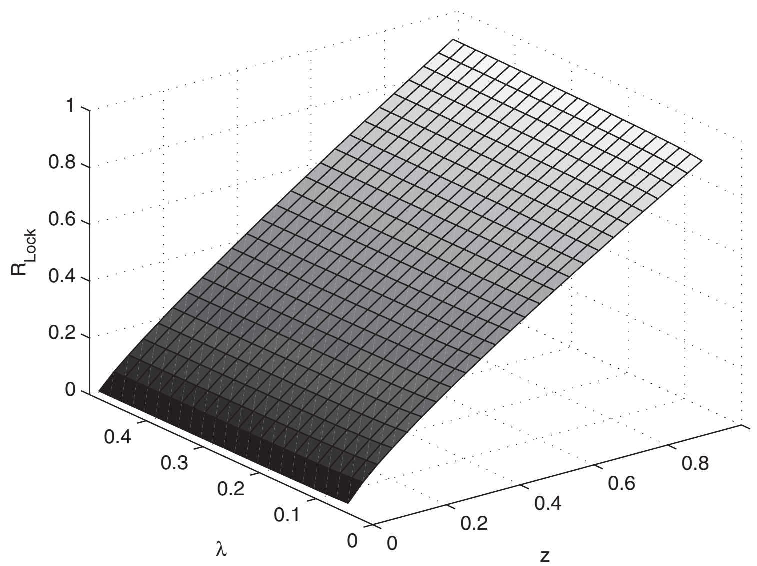

Figure 3. Recovery lock price in dependence of recovery z and intensity λ.

To avoid arbitrage opportunities both payoffs have to be equal. Setting $\varphi^{\mathrm{CDS}}=1$ leads to the proposed recovery price. □

Figure 3 shows the impact of different values of λ and z (all other parameters fixed) on the recovery lock price in the modelling framework from Section 2.[¶] The value of such a recovery lock during its lifetime can be obtained using the values of the two different legs. Owing to its construction the value of the premium leg always equals zero. Hence, the value of the recovery lock is equal to the value of the default leg and is consequently given by

$$\begin{aligned} V^{\mathrm{RL}}(t,T) &= V^{\mathrm{RL}}_{\mathrm{def}}(t,T) \\ &= V^{\mathrm{CDS}}_{\mathrm{def}}(t,T) - \frac{S^{\mathrm{CDS}}(t_0,T)}{S^{\mathrm{FRCDS}}(t_0,T,R_{\mathrm{Fix}})} V^{\mathrm{FRCDS}}_{\mathrm{def}}(t,T) \\ &= \left(1 - \frac{S^{\mathrm{CDS}}(t_0,T)}{S^{\mathrm{FRCDS}}(t_0,T,R_{\mathrm{Fix}})}(1-R_{\mathrm{Fix}})\right) \\ &\quad \cdot \int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)|\mathscr{F}_t]\,\mathrm{d}s \end{aligned} \tag{10}$$

[¶]Again, the parameter set from Table I in Section 4 estimated from market data was used.

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$$
\begin{aligned}
&-\int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)z(s)|\mathscr{F}_t]\,\mathrm{d}s \\
&= R_{\mathrm{Lock}}(t_0,T)\cdot\int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)|\mathscr{F}_t]\,\mathrm{d}s \\
&-\int_t^T \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^s (r(l)+\lambda(l))\,\mathrm{d}l}\lambda(s)z(s)|\mathscr{F}_t]\,\mathrm{d}s \qquad (11)\\
&= R_{\mathrm{Lock}}(t_0,T)V^{ddp}(t)-P^d(t,T)+P^{d,\mathrm{zero}}(t,T)
\end{aligned}
$$

The equivalence of (10) and (11) corresponds to the two different representations of a recovery lock as a short protection in a standard CDS and a long protection in fixed-recovery CDS or as a single recovery lock trade.

4. PARAMETER ESTIMATION, MODEL CALIBRATION, AND EMPIRICAL RESULTS

In this section it is shown how to determine the parameter values for the model introduced in Section 2 from market data by using Kalman filter techniques (see e.g. [44]). As the number of parameters is quite high, the estimation procedure is divided into three steps.

First, the parameters of the short rate r and the market factor w are estimated. Estimating the parameters for w is done by means of Maximum-Likelihood from the market factor, represented, e.g. by GDP growth rates. Then the Kalman filter is applied to time series of non-defaultable zero rates for different maturities to obtain the parameters of the short rate r.

As the markets for digital default swaps and recovery swaps are not very liquid and a reliable joint estimation of default and recovery risk components from CDS quotes is, because of identifiability problems, only possible under some restrictive assumptions, we separate the estimation of default and recovery risk. Therefore, in a second step the obtained estimates from the first step are used to estimate the parameters for the recovery-rate process z and the risk factor u from historical time series of average recovery rates. Comparable applications of Kalman filtering can, e.g. be found in [38, 45].

In the third step the parameters of the default intensity λ are estimated from market quotes of CDS spreads by using the estimates from the first two steps. The approach of using empirical time series and parameters in a risk-neutral valuation framework is commonly used, e.g. in the prepayment modelling for the valuation of mortgage-backed securities (see e.g. [45]). This is similar to our case, where there is little if anything in liquid markets that can be used for a suitable calibration of the recovery-rate process.

Next, the estimation procedure is applied to a sample of market data between September 2004 and March 2007. The market data used in this study are European GDP growth rates, German sovereign yields as a proxy for risk-free interest rates, and iTraxx Europe CDS spreads. In addition to that, aggregated recovery rates of European small and medium-sized enterprises (SMEs) and large corporates are used.

Weekly German sovereign yields with maturities from 3 months to 10 years and quarterly GDP growth rates from Euro countries are used to estimate the parameters of the processes r and w. As the frequency of zero rates is higher than the frequency of the GDP growth rates, a cubic spline

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interpolation is applied to the quarterly GDP data to obtain a time series of the same length as the zero rates. Furthermore, average recovery rates of European SMEs and large corporates are used to estimate the parameters of the recovery-rate process. Finally, iTraxx Europe CDS spreads with a maturity of 5 years (as these are the most liquid ones) are used to estimate the parameters of the default intensity. The parameter estimates of the short-rate, recovery, and intensity model are given in Table I. As the process u is unobservable in this example, we set $c_z = 1$. The estimated standard errors of the parameter estimates are obtained by a moving block bootstrapping procedure (see e.g. [46]). We have chosen a block length of 26 weeks and then concatenated randomly the blocks to obtain series with approximatively the same length as the respective original sample series. The standard error estimates given in Table I are the empirical standard deviations of the respective estimators in a total of 50 bootstrap replications.

Before evaluating the performance of the model, it is checked whether the model assumptions from Sections 2.2 and 2.3 are fulfilled. Using the filtered time series and estimated parameter values,

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}) \quad \text{for } i \in \{r, w, u, \lambda\} \quad \text{and} \quad k = 2, \dots, n$$

are computed. These are supposed to be realizations of independent normally distributed random variables. Hence, each $(\Delta W_i(t_k))_{k=2,\dots,n}$ is tested for autocorrelation and normal distribution.

To test for autocorrelation, a Ljung–Box test (see e.g. [47]) is performed. The null hypothesis of no autocorrelation up to lag $22 \approx 2\sqrt{n-1}$ is not rejected on a 5%-level for ΔW_r and ΔW_λ and rejected for ΔW_w and ΔW_u.

The null hypothesis that ΔW_i for $i \in \{r, w, u, \lambda\}$ are not realizations of normally distributed random variables is tested according to the test proposed by Bera and Jarque [48]. The test indicates

Table I. Estimates of short-rate, recovery, and intensity model.

	Parameter	Estimate	Std. error
Short-rate process	a_r	0.0636	(0.0029)
	$\widehat{a}_r$	0.0635	(0.0027)
	σ_r	0.0053	(7.3e−05)
	b_{rw}	0.1397	(0.0273)
Market-factor process	θ_w	0.0093	(0.0033)
	a_w	0.6146	(0.0802)
	$\widehat{a}_w$	0.6140	(0.0801)
	σ_w	0.0017	(0.0026)
Recovery-rate process	b_z	0.6281	(0.0901)
	d_z	5.1494	(5.6202)
	θ_u	0.0135	(0.0258)
	a_u	0.1318	(0.2025)
	$\widehat{a}_u$	0.1472	(0.2137)
	σ_u	0.0554	(0.0099)
Intensity process	θ_λ	0.0076	(0.0008)
	a_λ	0.8601	(0.1825)
	$\widehat{a}_\lambda$	0.8596	(0.1824)
	σ_λ	0.0127	(0.0030)
	$b_{\lambda u}$	0.0001	(0.0001)
	$b_{\lambda w}$	0.1997	(0.0812)

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that ΔW_w and ΔW_r are normally distributed. If the (in terms of absolute values) highest 5% of ΔW_u and ΔW_λ are removed, the normal distribution assumption can not be rejected anymore. Therefore, one can conclude that the assumption of a normal distribution is justified for ΔW_w and ΔW_r and adequate at least at the center of the distribution for ΔW_u and ΔW_λ.

Furthermore, in Section 2 the Wiener processes were assumed to be uncorrelated. To verify this assumption, the empirical correlations of the processes $(\Delta W_i(t_k))_{k=2,\ldots,n}$ for $i\in\{r,w,u,\lambda\}$ are computed and a t-test for no correlation is performed. The test indicates that not only the processes ΔW_w and ΔW_r, ΔW_w and ΔW_λ, as well as ΔW_r and ΔW_u are uncorrelated, but also the other correlations are on a rather low level.

Finally, it is also tested if the assumption of log-normally distributed recovery rates (Equation (3) with $a_z=0$) is justified. Although the Jarque–Bera test rejects the hypothesis of normal distribution for the logarithm of the recovery rates, a QQ-plot indicates that the distributional assumption fits quite well in most parts of the distribution and that larger deviations appear only in the upper tail.

After having validated the model assumptions, the model performance is investigated. For this, model and market prices/spreads as well as model and market price/spread movements are compared. The first is done by calculating the mean absolute and relative pricing error for each maturity, the latter by regressing the model price/spread movements on the market price/spread movements similar to [49].

The average absolute and relative deviations of the model prices of zero-coupon bonds and CDS spreads from the corresponding market prices are given in Table II. Additionally, Figure 4 shows a comparison of market and model zero rates and CDS spreads for a maturity of 5 years.

Table II. Average pricing errors for risk-free zero-coupon bonds and CDS spreads.

	Risk-free ZCB	CDS
Mean absolute error	0.01052	3.68e−05
Mean relative error	0.01374	0.01031

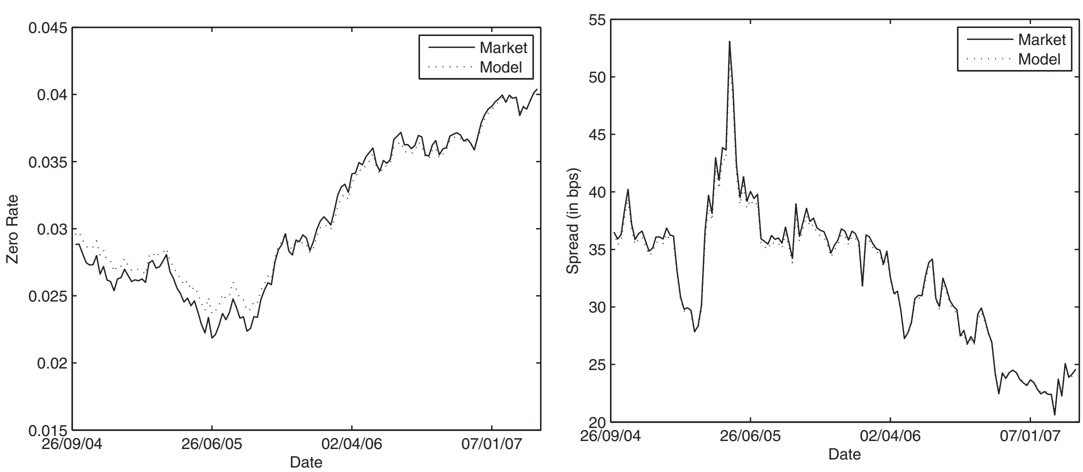

Figure 4. Market and model zero rates and CDS spreads with 5 year maturity.

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For further examination of the model performance, it is tested how well changes in market quotes can be explained. For this, let

$$\Delta R_k(T) := R(t_k, t_k + T) - R(t_{k-1}, t_{k-1} + T)$$

with $R(t,T) = -1/(T-t)\ln P^{\mathrm{nd}}(t,T)$ and

$$\Delta S_k^{\mathrm{CDS}}(T) := S^{\mathrm{CDS}}(t_k, t_k + T) - S^{\mathrm{CDS}}(t_{k-1}, t_{k-1} + T)$$

denote the changes in the zero rate and CDS spread with time to maturity T between t_{k-1} and t_k. We perform the following regressions:

$$\Delta R_k^{\mathrm{market}}(T) = a_{R,T} + b_{R,T} \Delta R_k^{\mathrm{model}}(T) + \varepsilon_{R,T}$$

with $\varepsilon_{R,T} \sim \mathcal{N}(0, h_{R,T}^2)$ and

$$\Delta S_k^{\mathrm{CDS,market}}(T) = a_{S^{\mathrm{CDS}},T} + b_{S^{\mathrm{CDS}},T} \Delta S_k^{\mathrm{CDS,model}}(T) + \varepsilon_{S^{\mathrm{CDS}},T}$$

with $\varepsilon_{S^{\mathrm{CDS}},T} \sim \mathcal{N}(0, h_{S^{\mathrm{CDS}},T}^2)$.

For a good model one would expect $a_{\cdot,T}$ to be around 0, $b_{\cdot,T}$ around 1, and the coefficient of determination R^2 close to 1. For the interest-rate model, the hypothesis $a_{R,T} = 0$ is only rejected for very short maturities, and the hypothesis $a_{R,T} = 0$ and $b_{R,T} = 1$ is rejected for very short and very long maturities. The R^2 for maturities between 1 and 10 years lies between 0.76 and 0.98 with an average R^2 of 0.91. Replacing the zero rate changes by absolute zero rates, even higher degrees of explanation for all maturities (between 0.87 and 0.99) can be achieved. For the CDS spreads the hypothesis $a_{S^{\mathrm{CDS}},T} = 0$ cannot be rejected on a 5%-level. The hypothesis $a_{S^{\mathrm{CDS}},T} = 0$ and $b_{S^{\mathrm{CDS}},T} = 1$ is rejected but the value for $b_{S^{\mathrm{CDS}},T}$ is only slightly higher than 1 and the R^2 is over 0.98.

Finally, the empirical correlations of the historical and filtered time series of the processes w, r, u, and λ are computed (see Table III). The signs of the correlations correspond to what would be expected according to many empirical studies, see e.g. [4, 21, 50].

Table III. Empirical correlations of historical and filtered processes.

	w	r	z	λ
w	1	0.7374	0.6384	−0.6708
r	0.7374	1	0.4989	−0.4143
z	0.6384	0.4989	1	−0.0962
λ	−0.6708	−0.4143	−0.0962	1

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5. CONCLUSION

A joint modelling framework for recovery and default risk accounting for typical characteristics known from empirical studies like a negative correlation of default rates and recovery rates or the positive impact of a healthy macroeconomic environment on recovery rates was presented. Despite its realistic features, the model is still simple enough to obtain closed-form (at least up to one numerically tractable integral) pricing formulas for many (single-name) defaultable assets, like coupon bonds or CDSs. The stochastic nature of the recovery-rate process in this model allows for the pricing of credit derivatives with payoffs directly linked to the recovery rate, e.g. recovery locks. The model parameters are estimated using an (extended) Kalman filter approach. The estimation procedure combines estimation under the real-world measure from historical time series (GDP growth rates and aggregated recovery rates) with calibration to time series of market quotes (zero rates and CDS spreads). In a numerical example, the model is applied to a set of European data and tested for its fitting capability.

APPENDIX A: PROOF OF THEOREM 3

According to the theorem of Feynman–Kac (see e.g. pp. 241–244 of [51] or pp. 38–41 of [52]), g is the solution of the PDE

$$\begin{aligned}0 = &\tfrac{1}{2}(\sigma_r^2 g_{rr} + \sigma_\lambda^2 g_{\lambda\lambda} + \sigma_u^2 g_{uu} + \sigma_w^2 g_{ww})\\ &+(\theta_r(t) + b_{rw}w - \widehat{a}_r r)g_r + (\theta_w - \widehat{a}_w w)g_w + (\theta_u - \widehat{a}_u u)g_u\\ &+(\theta_\lambda + b_{\lambda u}u - b_{\lambda w}w - \widehat{a}_\lambda \lambda)g_\lambda - (r+\lambda)g + g_t\end{aligned}$$

under the condition $g(r,\lambda,u,w,T,T) = \mathrm{e}^{-c_z u(T) + d_z w(T)}$. If

$$g(r,\lambda,u,w,t,T) = \mathrm{e}^{A(t,T) - B(t,T)r(t) - C(t,T)\lambda(t) - D(t,T)u(t) - E(t,T)w(t)}$$

then

$$\begin{aligned}0 = &\tfrac{1}{2}(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2)g\\ &-(\theta_r(t) + b_{rw}w - \widehat{a}_r r)Bg - (\theta_w - \widehat{a}_w w)Eg - (\theta_u - \widehat{a}_u u)Dg\\ &-(\theta_\lambda + b_{\lambda u}u - b_{\lambda w}w - \widehat{a}_\lambda \lambda)Cg - (r+\lambda)g\\ &+(A_t - B_t r - C_t \lambda - D_t u - E_t w)g\end{aligned}$$

This is equivalent to

$$\begin{aligned}0 = &\tfrac{1}{2}(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2)\\ &+r(\widehat{a}_r B - 1 - B_t) + \lambda(\widehat{a}_\lambda C - 1 - C_t) + u(\widehat{a}_u D - b_{\lambda u}C - D_t)\\ &+w(-b_{rw}B + \widehat{a}_w E + b_{\lambda w}C - E_t)\\ &+A_t - \theta_r(t)B - \theta_\lambda C - \theta_u D - \theta_w E\end{aligned}$$

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Therefore, we have to solve the following system of linear equations:

$$B_t = \widehat{a}_r B - 1, \quad C_t = \widehat{a}_\lambda C - 1, \quad D_t = \widehat{a}_u D - b_{\lambda u} C$$

$$E_t = \widehat{a}_w E - b_{rw} B + b_{\lambda w} C$$

$$\begin{aligned} A_t = {} & \theta_r(t) B + \theta_\lambda C + \theta_u D + \theta_w E \\ & - \tfrac{1}{2}(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2) \end{aligned}$$

with boundary conditions

$$A(T,T) = 0, \quad B(T,T) = 0, \quad C(T,T) = 0$$

$$D(T,T) = c_z, \quad E(T,T) = -d_z$$

Using the transformation $s = T - t$ leads to the proposed solutions for $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$.

APPENDIX B: PROOF OF THEOREM 4

According to the theorem of Feynman–Kac (see e.g. pp. 241–244 of [51] or pp. 38–41 of [52]), $\widetilde{g}$ is the solution of the PDE

$$\begin{aligned} 0 = {} & \tfrac{1}{2}(\sigma_r^2 \widetilde{g}_{rr} + \sigma_\lambda^2 \widetilde{g}_{\lambda\lambda} + \sigma_u^2 \widetilde{g}_{uu} + \sigma_w^2 \widetilde{g}_{ww}) \\ & + (\theta_r(t) + b_{rw} w - \widehat{a}_r r)\widetilde{g}_r + (\theta_w - \widehat{a}_w w)\widetilde{g}_w + (\theta_u - \widehat{a}_u u)\widetilde{g}_u \\ & + (\theta_\lambda + b_{\lambda u} u - b_{\lambda w} w - \widehat{a}_\lambda \lambda)\widetilde{g}_\lambda - (r+\lambda)\widetilde{g} + \widetilde{g}_t \end{aligned}$$

under the condition $\widetilde{g}(r,\lambda,u,w,T,T) = \lambda(T)\mathrm{e}^{-c_z u(T) + d_z w(T)}$. If

$$\begin{aligned} \widetilde{g}(r,\lambda,u,w,t,T) = {} & \mathrm{e}^{A(t,T) - B(t,T)r(t) - C(t,T)\lambda(t) - D(t,T)u(t) - E(t,T)w(t)} \\ & \cdot (G(t,T) + H(t,T)r(t) + I(t,T)\lambda(t) \\ & + J(t,T)u(t) + K(t,T)w(t)) \end{aligned}$$

then

$$\begin{aligned} 0 = {} & \tfrac{1}{2}(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2) \\ & \cdot (G + Hr + I\lambda + Ju + Kw) \\ & + (-\sigma_r^2 BH - \sigma_\lambda^2 CI - \sigma_u^2 DJ - \sigma_w^2 EK) \\ & + (\theta_r(t) + b_{rw} w - \widehat{a}_r r)(-B(G + Hr + I\lambda + Ju + Kw) + H) \\ & + (\theta_w - \widehat{a}_w w)(-E(G + Hr + I\lambda + Ju + Kw) + K) \\ & + (\theta_u - \widehat{a}_u u)(-D(G + Hr + I\lambda + Ju + Kw) + J) \\ & + (\theta_\lambda + b_{\lambda u} u - b_{\lambda w} w - \widehat{a}_\lambda \lambda) \end{aligned}$$

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$$
\begin{aligned}
&\cdot(-C(G+Hr+I\lambda+Ju+Kw)+I)\\
&-(r+\lambda)(G+Hr+I\lambda+Ju+Kw)\\
&+(G+Hr+I\lambda+Ju+Kw)\\
&\cdot(A_t-B_tr-C_t\lambda-D_tu-E_tw)\\
&+G_t+H_tr+I_t\lambda+J_tu+K_tw
\end{aligned}
$$

Using the proof of Theorem 3, this reduces to

$$
\begin{aligned}
0=&-\sigma_r^2BH-\sigma_\lambda^2CI-\sigma_u^2DJ-\sigma_w^2EK\\
&+H\theta_r(t)+K\theta_w+J\theta_u+I\theta_\lambda+G_t\\
&+r(-\widehat{a}_rH+H_t)+w(b_{rw}H-\widehat{a}_wK-b_{\lambda w}I+K_t)\\
&+u(-\widehat{a}_uJ+b_{\lambda u}I+J_t)+\lambda(-\widehat{a}_\lambda I+I_t)
\end{aligned}
$$

Therefore, we have to solve the following system of linear equations:

$$
\begin{aligned}
&H_t=\widehat{a}_rH,\quad I_t=\widehat{a}_\lambda I,\quad J_t=\widehat{a}_uJ-b_{\lambda u}I\\
&K_t=\widehat{a}_wK-b_{rw}H+b_{\lambda w}I\\
&G_t=\sigma_r^2BH+\sigma_\lambda^2CI+\sigma_u^2DJ+\sigma_w^2EK\\
&\qquad -H\theta_r(t)-K\theta_w-J\theta_u-I\theta_\lambda
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
&G(T,T)=0,\quad H(T,T)=0,\quad I(T,T)=1\\
&J(T,T)=0,\quad K(T,T)=0
\end{aligned}
$$

Using the transformation $s=T-t$ leads to $H(t,T)\equiv 0$ and the proposed solutions for $G(t,T)$, $I(t,T)$, $J(t,T)$, and $K(t,T)$.

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