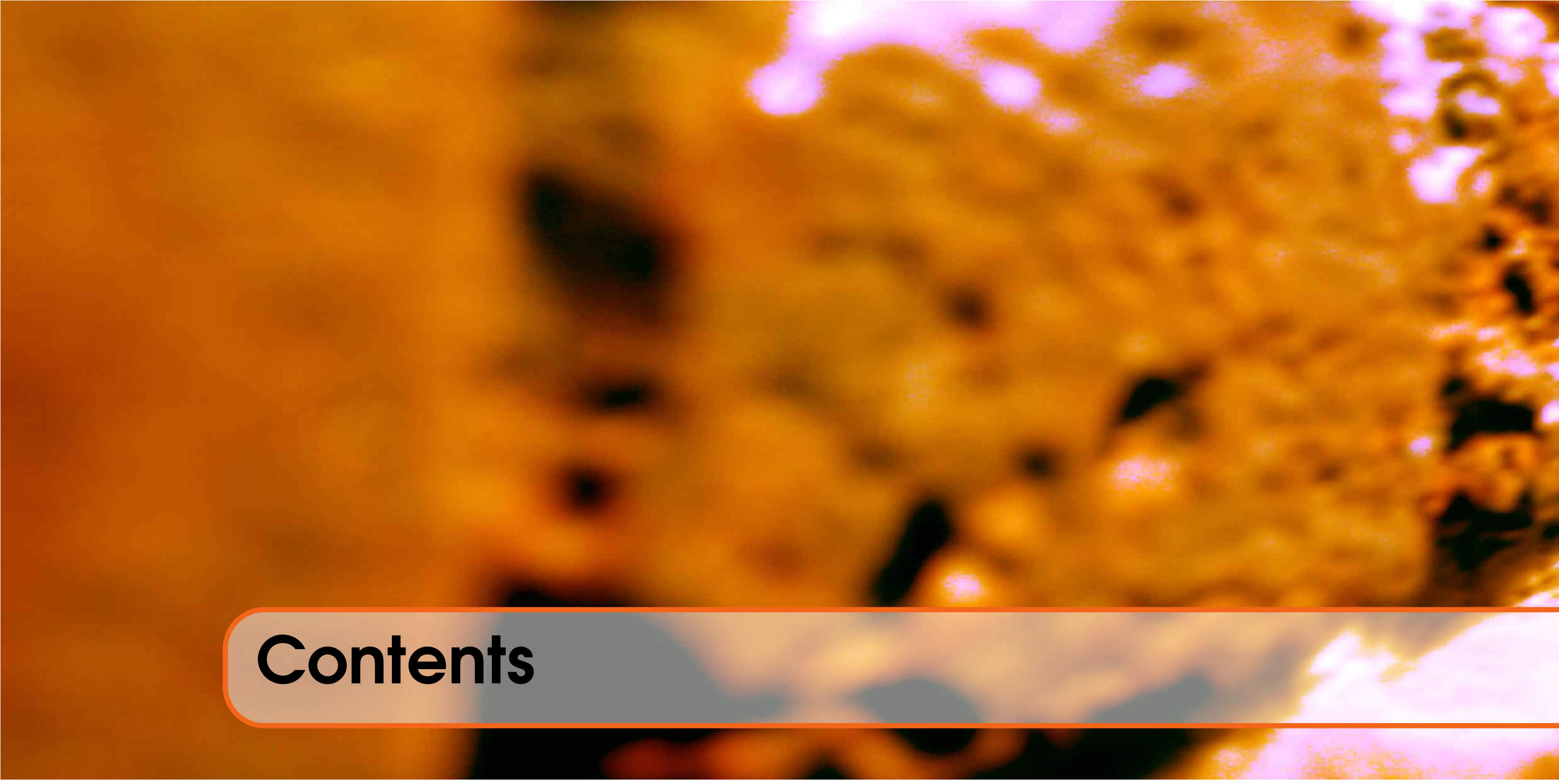
Calculus



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Functions

1. Functions

1.1 Function

Some types of functions: linear, parabolas, ...

The domain of a function f is the set of all valid input values. The range consists of the set of all output values that can be reached using those domain values.

1.2 Rational Function

- basic form: $y = \frac{1}{x}$
- vertical asymptote at $x = 0$
- horizontal asymptote at $y = 0$

1.3 Root Function

- basic form: $y = \sqrt[n]{x}$
- n even: undefined when the root is negative
- n odd: $x \in \mathbb{R}$

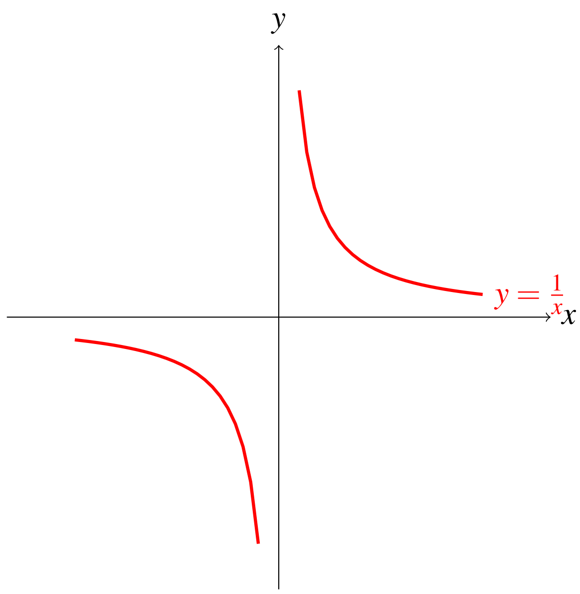


Figure 1.1: rational function

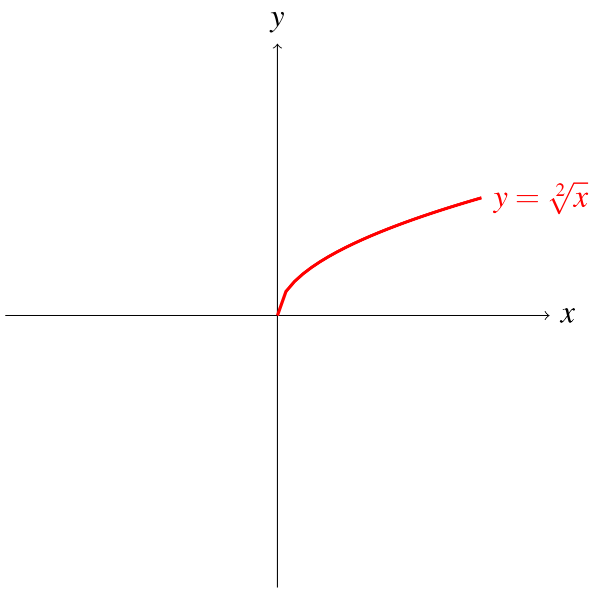


Figure 1.2: root function when n is even

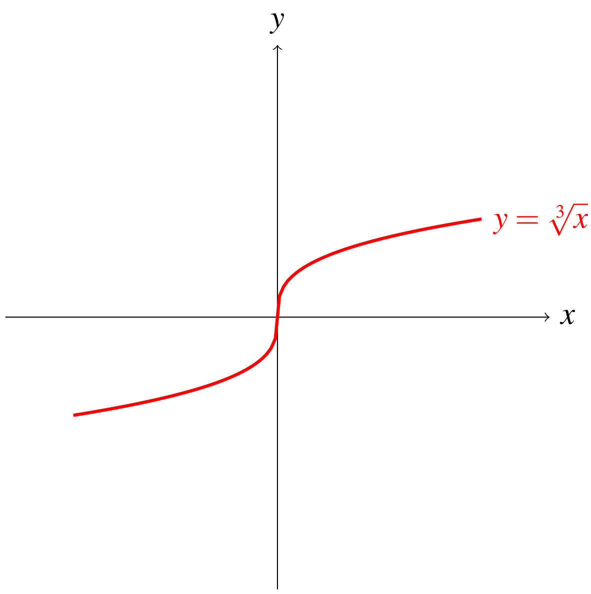


Figure 1.3: root function when n is odd

1.4 Higher-degree of Polynomial Function

- basic form: $y = x^n$
- domain: $x \in \mathbb{R}$
- n even: both ends of the function tend to $+\infty$ or both tend to $-\infty$
- n odd: one end tends to $+\infty$ while the other tends to $-\infty$

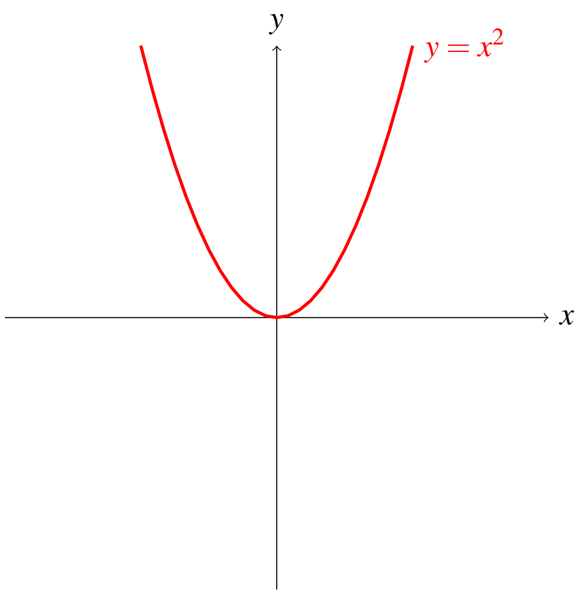


Figure 1.4: polynomial function when n is even

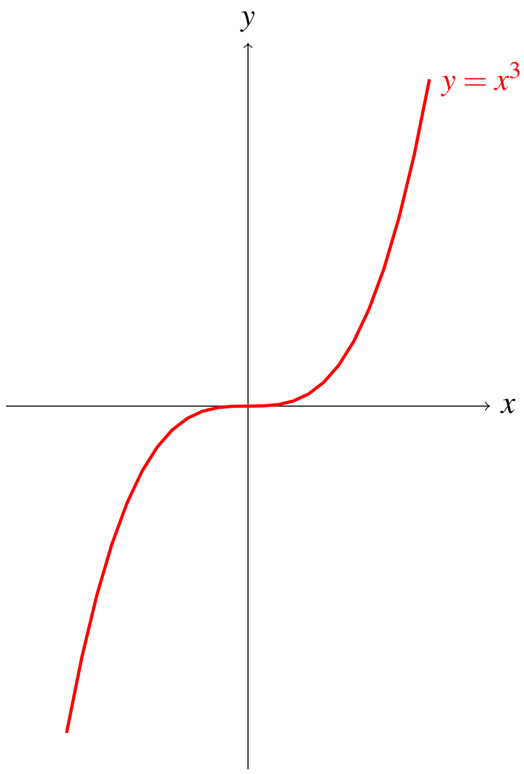


Figure 1.5: polynomial function when n is odd

2. Angles, Degrees, and Radians

2.1 Angle

An angle is created by two rays that intersect at a common endpoint. We use Greek letter θ to denote angles.

An angle that opens counterclockwise from the x-axis is positive.

An angle that opens clockwise from the x-axis is negative.

2.2 Degree and Radian

Angles can be measured in 2 ways:

1. A degree is a measure of the angle formed by $\frac{1}{360}$ of one complete rotation of a circle.

2. A radian is a measure of the angle formed by the arc of a circle whose length is equal to the circle's radius.

Theorem 2.2.1 — Degree.

$$\theta = \frac{s}{r} = \frac{arclength}{radius} \tag{2.1}$$

How are radians and degrees related?

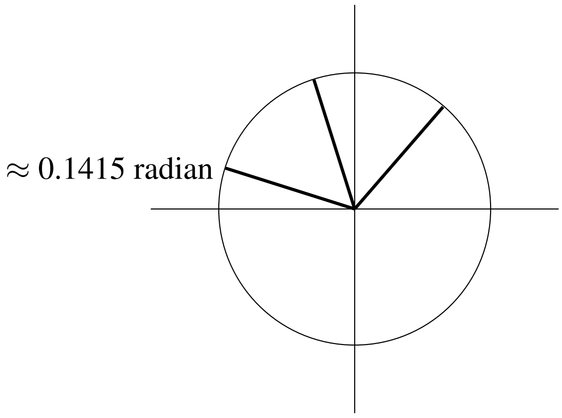


Figure 2.1: radian

Theorem 2.2.2 — Degrees and radians.

$$180^\circ = \pi \; radians \tag{2.2}$$

$$1^\circ = \frac{\pi}{180} \; radians \tag{2.3}$$

$$1 \; radian = \frac{180^\circ}{\pi} \tag{2.4}$$

This relationship provides us with a way to easily convert between the two measures.

Exercise 2.1 Convert from degrees to radians.

(a) $30^\circ =$

(b) $220^\circ =$ ■

Exercise 2.2 Convert from radians to degrees.

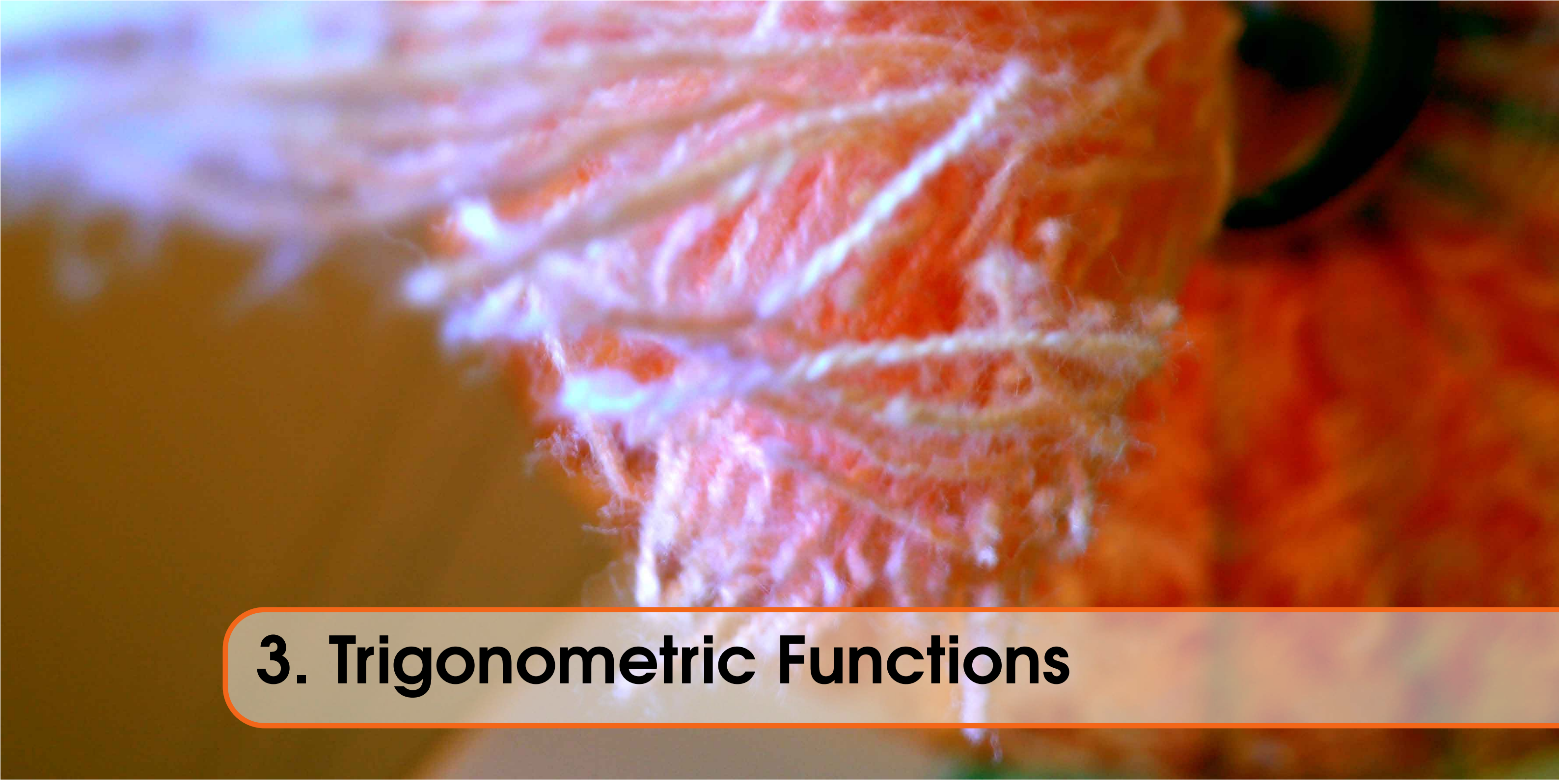
(a) $\frac{\pi}{4} =$

(b) $\frac{5\pi}{6} =$ ■

Given any angle θ, what are these equivalent angles?

Theorem 2.2.3 — Equivalent angles.

$$\theta + 2k\pi \; (k \in \mathbb{Z}) \tag{2.5}$$



3. Trigonometric Functions

3.1 Trigonometric Functions

Let O be the origin and $P(x,y)$ be a point on the unit circle so that the radius OP forms an angle of θ radians with respect to the positive x-axis.

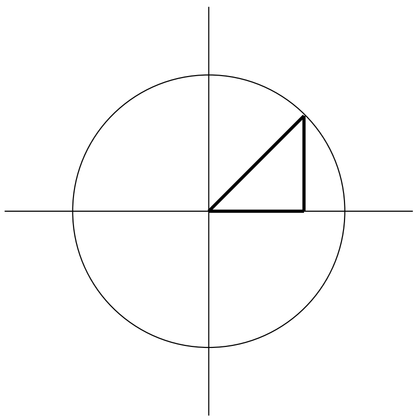
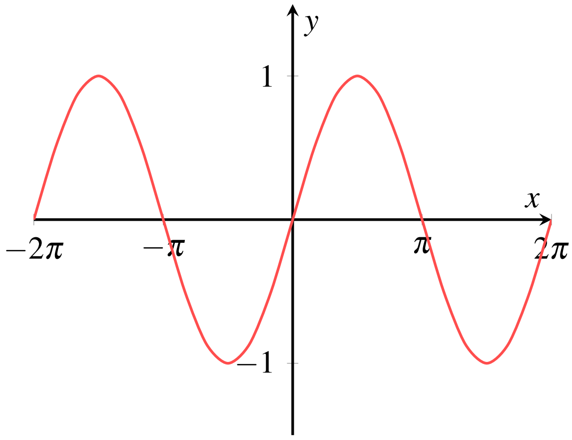


Figure 3.1: radian

Theorem 3.1.1 — sin / cos.

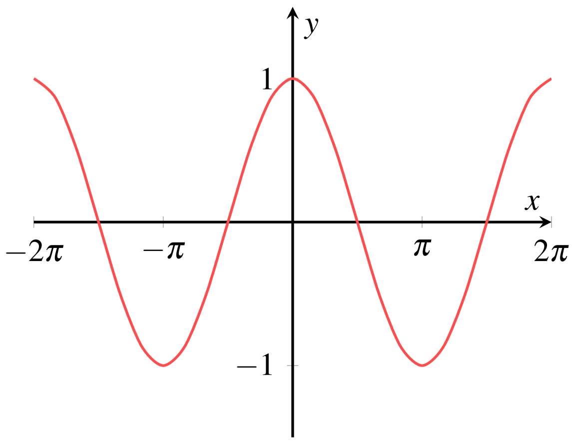
$$x = cos(\theta)$$
$$y = sin(\theta)$$

Here are the three most common trigonometric functions and their reciprocals.



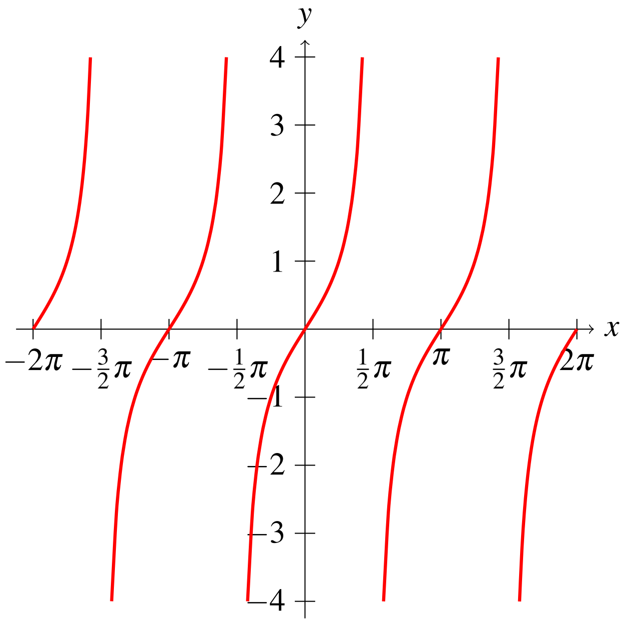
range	$-1 \leq sin(\theta) \geq 1$
doamin	$\theta \in \mathbb{R}$
$sin(\theta) = 0$ when	$\theta = k\pi,\ k \in \mathbb{Z}$

Figure 3.2: $y = sin(\theta)$



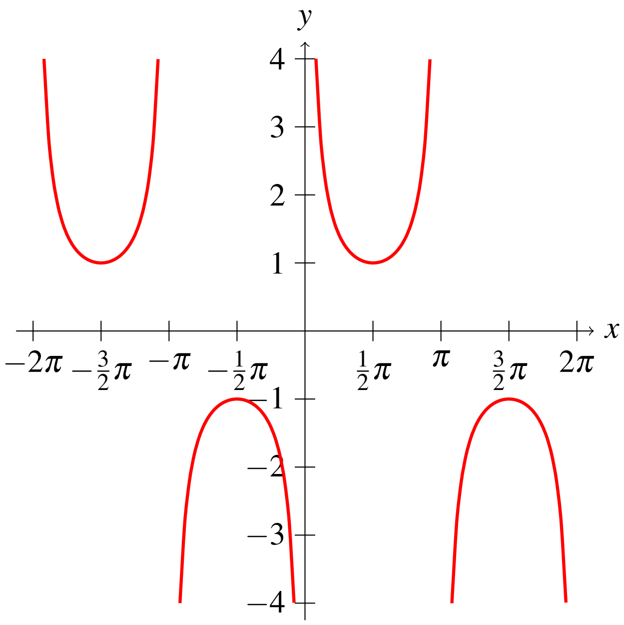
range	$\theta \in \mathbb{R}$
doamin	$-1 \leq cos(\theta) \leq 1$
$cos(\theta) = 0$ when	$\theta = \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$

Figure 3.3: $y = cos(\theta)$



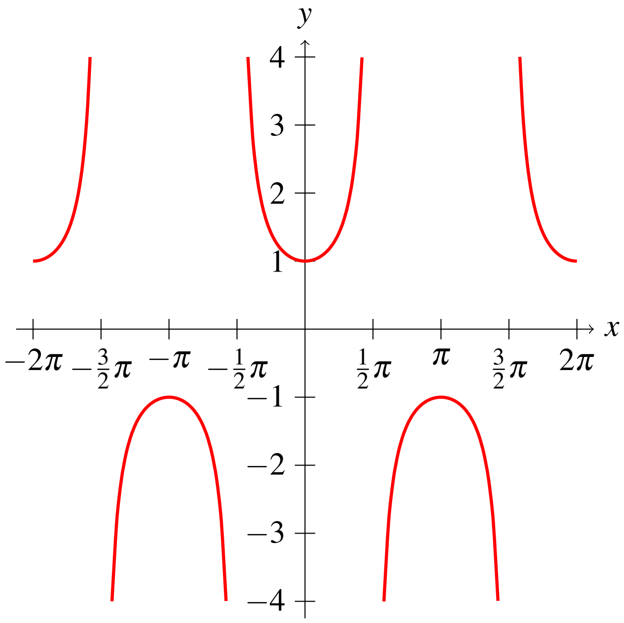
range	$\theta \in \mathbb{R}, \theta \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$
doamin	$tan(\theta) \in \mathbb{R}$
$tan(\theta) = 0$ when	$\theta = k\pi,\ k \in \mathbb{Z}$

Figure 3.4: $y = tan(\theta)$



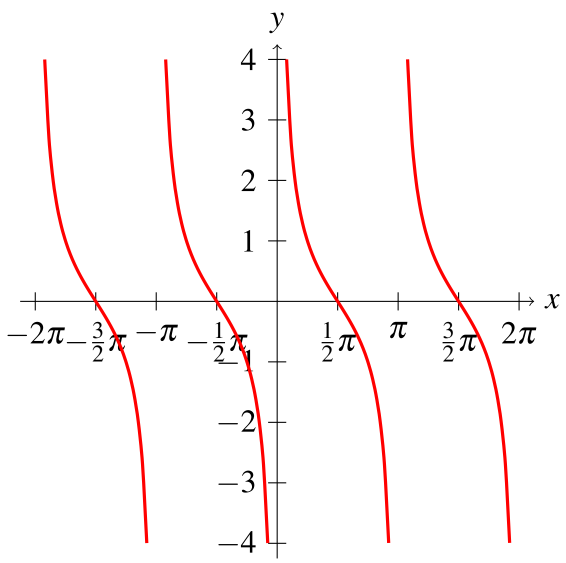
range	$\theta \in \mathbb{R},\ \theta \neq k\pi,\ k \in \mathbb{Z}$
doamin	$csc(\theta) \geq 1$ or $csc(\theta) \leq -1$
$csc(\theta) = 0$ when	*never*

Figure 3.5: $y = csc(\theta)$



range	$\theta \in \mathbb{R},\ \theta \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$
doamin	$sec(\theta) \geq 1$ or $sec(\theta) \leq -1$
$sec(\theta) = 0$ when	*never*

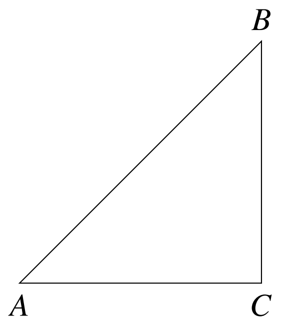
Figure 3.6: $y = sec(\theta)$

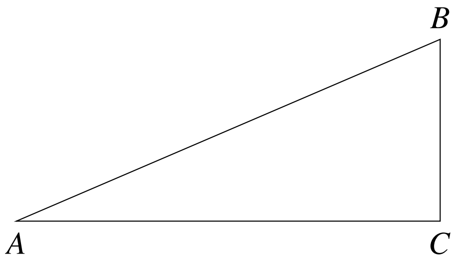


range	$\theta \in \mathbb{R},\ \theta \neq k\pi, k \in \mathbb{Z}$
doamin	$cot(\theta) \in \mathbb{R}$
$cot(\theta) = 0$ when	$\theta = \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$

Figure 3.7: $y = cot(\theta)$

3.2 Special Triangles





Exercise 3.1 Evaluate each of the following.

(a) $sin(\frac{\pi}{4}) =$

(b) $cos(\frac{\pi}{4}) =$

(c) $csc(\frac{\pi}{4}) =$

■

Exercise 3.2 Find all values of θ satisfying the following.

(a) $tan(\theta) = \frac{1}{\sqrt{3}}$

(b) $sec(\theta) = \sqrt{2}$

(c) $cot(\theta) = \sqrt{3}$

■

3.3 Trigonometric Identities

Theorem 3.3.1 — Trigonometric Identities.

$$sin^2(\theta) + cos^2(\theta) = 1 \tag{3.1}$$

$$sin(-\theta) = -sin(\theta) \tag{3.2}$$

$$cos(-\theta) = -cos(\theta) \tag{3.3}$$

$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \tag{3.4}$$

$$cot(\theta) = \frac{cos(\theta)}{sin(\theta)} \tag{3.5}$$

$$tan^2(\theta) + 1 = sec^2(\theta) \tag{3.6}$$

$$cot^2(\theta) + 1 = csc^2(\theta) \tag{3.7}$$

$$sin(\theta) = cos\left(\theta - \frac{\pi}{2}\right) \tag{3.8}$$

$$sin(a \pm b) = sin(a)cos(b) \pm cos(a)sin(b) \tag{3.9}$$

$$cos(a \pm b) = cos(a)cos(b) \mp sin(a)sin(b) \tag{3.10}$$

$$sin(2\theta) = 2sin(\theta)cos(\theta) \tag{3.11}$$

$$cos(2\theta) = cos^2(\theta) - sin^2(\theta) \tag{3.12}$$

$$sin^2(\theta) = \frac{1 - cos(2\theta)}{2} \tag{3.13}$$

$$cos^2(\theta) = \frac{1 + cos(2\theta)}{2} \tag{3.14}$$

Reminder: $trig^n(x)$ is a notation often used to indicate $(trig(x))^n$.

4. Exponential Functions

4.1 Exponential Functions

Exponential functions are of the form $y = a^x$, where a is a positive number and x is any real number. You might see these sorts of functions when studying population growth, economic growth, global temperature, monetary value, etc.

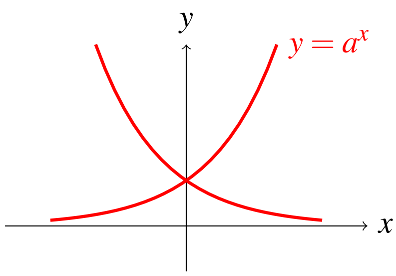


Figure 4.1: exponential function

- Domain: $x \in \mathbb{R}$
- Range: $y > 0$
- The graph $y = a^x$ always passes through $(0,1)$ and $(1,a)$.
- If $a > 1$ then the graph of $y = a^x$ is increasing.
- If $0 < a < 1$ then the graph of $y = a^x$ is decreasing.

- $y = 0$ is always a horizontal asymptote of $y = a^x$.

4.2 Exponent Rules

Theorem 4.2.1 — Exponent Rules.

$$a^{-x} = \frac{1}{a^x} \tag{4.1}$$

$$\frac{1}{a^{-x}} = a^x \tag{4.2}$$

$$(ab)^x = a^x b^x \tag{4.3}$$

$$(\frac{a}{b})^x = \frac{a^x}{b^x} \tag{4.4}$$

$$a^{kx} = (a^k)^x = (a^x)^k \tag{4.5}$$

$$a^m a^n = a^{m+n} \tag{4.6}$$

$$\frac{a^m}{a^n} = a^{m-n} \tag{4.7}$$

$$a^{1/n} = \sqrt[n]{a} \tag{4.8}$$

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \tag{4.9}$$

4.3 The Base e

A very special exponential function is $y = e^x$, where e is just a content with a non-terminating decimal like π.

$$e = 2.718281845...$$

What is so special about an exponential function with base e?

At any point on the graph, the height of the exponential function is equal to the slope of the tangent line to the graph at that point.

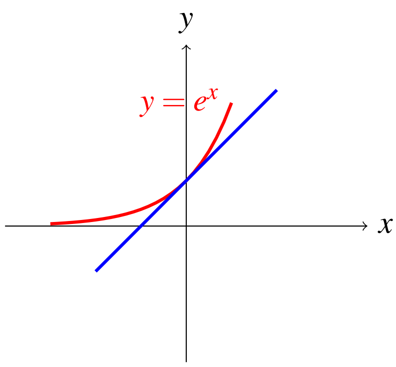
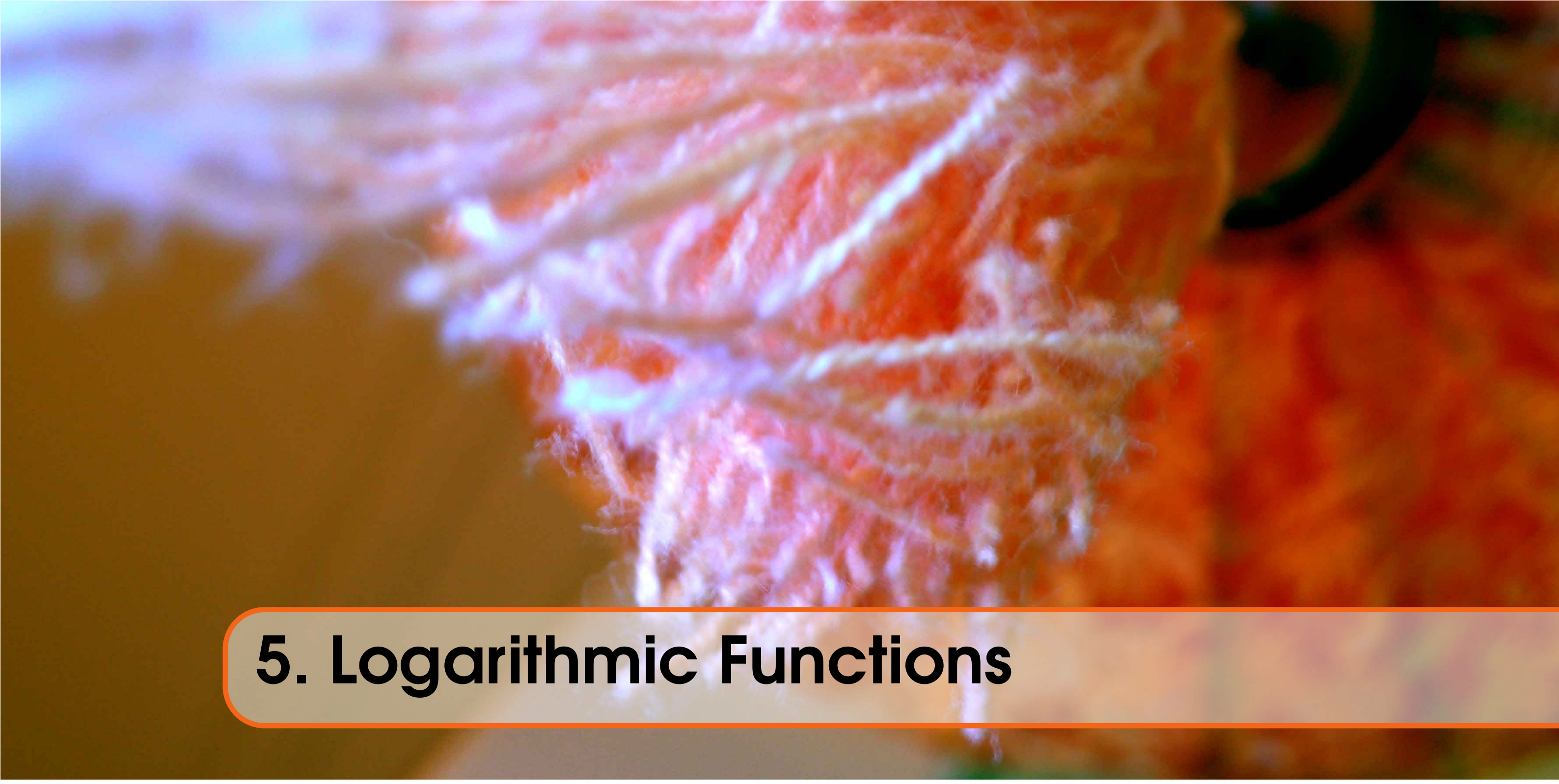


Figure 4.2: $y = e^x$



5. Logarithmic Functions

5.1 Logarithmic Functions

Logarithms are the inverse of exponential functions. Let $a > 0$, then we define a logarithm (log) as follows:

Theorem 5.1.1 — Logarithmic Functions.

$$y = log_a(x) \tag{5.1}$$
$$a^y = x \tag{5.2}$$

If no base a is shown, a base of 10 is assumed.

For example:

$$log(x) = log_{10}(x)$$

Exercise 5.1 Evaluate each of the following.

(a) $log_2 8 =$

(b) $log(100) =$

(c) $log_5 \frac{1}{25} =$

(d) $log_8 1 =$ ■

Since any positive number to the power of 0 is equal to 1, we have the property that

$log_a(1)$, no matter what the base a is.

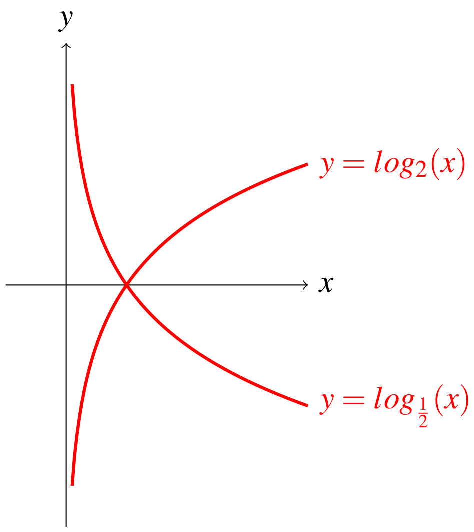


Figure 5.1: logarithmic function

- Domain: $0 < x < \infty$
- Range: $y \in \mathbb{R}$
- $y = log_a(x)$ always passes through $(a, 1)$ and $(1, 0)$.
- If $a > 1$ then the graph of $y = log_a(x)$ is increasing.
- If $0 < a < 1$ then the graph of $y = log_a(x)$ is decreasing.
- $x = 0$ is always a vertical asymptote of $y = log_a(x)$.

5.2 Logarithm Rules

Theorem 5.2.1 — Logarithm Rules.

$$log_a(a^x) = x \tag{5.3}$$
$$a^{log_a(x)} = x \tag{5.4}$$
$$log_a(xy) = log_a(x) + log_a(y) \tag{5.5}$$
$$log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y) \tag{5.6}$$
$$log_a(x^n) = nlog_a(x) \tag{5.7}$$

5.3 Change of Base Formula

We can switch between any two bases easily by using the formula:

Theorem 5.3.1 — Change of Base Formula.

$$log_a(x) = \frac{log_b(x)}{log_b(a)} \tag{5.8}$$

Exercise 5.2 Proof ■

Exercise 5.3 Convert $log_4(x)$ into a logarithm with each of the following bases.

(a) base 3

(b) base 22

■

5.4 The Natural Logarithm

A special logarithm is the natural logarithm, which is the logarithm with a base of e. Rather than write $log_e(x)$, we typically write $ln(x)$.

The natural logarithm has the exact same properties as any other logarithmic function.

Theorem 5.4.1 — Natural Logarithm.

$$ln(e^x) = x \tag{5.9}$$

$$e^{ln(x)} = x \tag{5.10}$$

Exercise 5.4 Solve each of the following for x.

(a)

$$2^x = 2^{1-x}$$

(b)

$$3^{\frac{1}{2}+10} = 27$$

(c)

$$2^x = 10$$

(d)

$$log(x) - 1 = log(x-1)$$

(e)

$$log_2(x) + log_2(x^2) = 6$$

(f)

$$log_2(x^4) + log_2(x^2) = 6$$

II

Inequalities

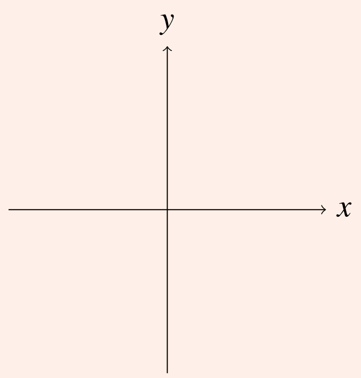
6. Piecewise Functions

6.1 Piecewise Functions

Piecewise functions typically feature one or more points at which the function changes from one form to another. To graph a piecewise function, simply graph each piece and then restrict it to its designated domain. Pay special attention when plotting the breaking point (closed circle includes the point, open circle excludes the point).

Exercise 6.1 Graph the piecewise function given by

$$y = \begin{cases} 1 & x \leq 0 \\ 4^x & x > 0 \end{cases}$$



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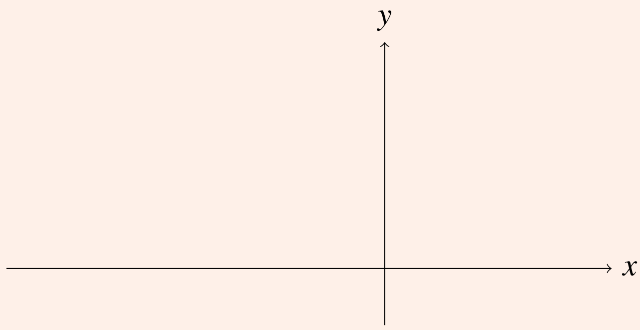
Exercise 6.2 Graph the piecewise function given by

$$y = \begin{cases} \frac{1}{2}x + 3 & x < -2 \\ 0 & -2 \leq x \leq 2 \\ x^2 - 1 & x > 2 \end{cases}$$



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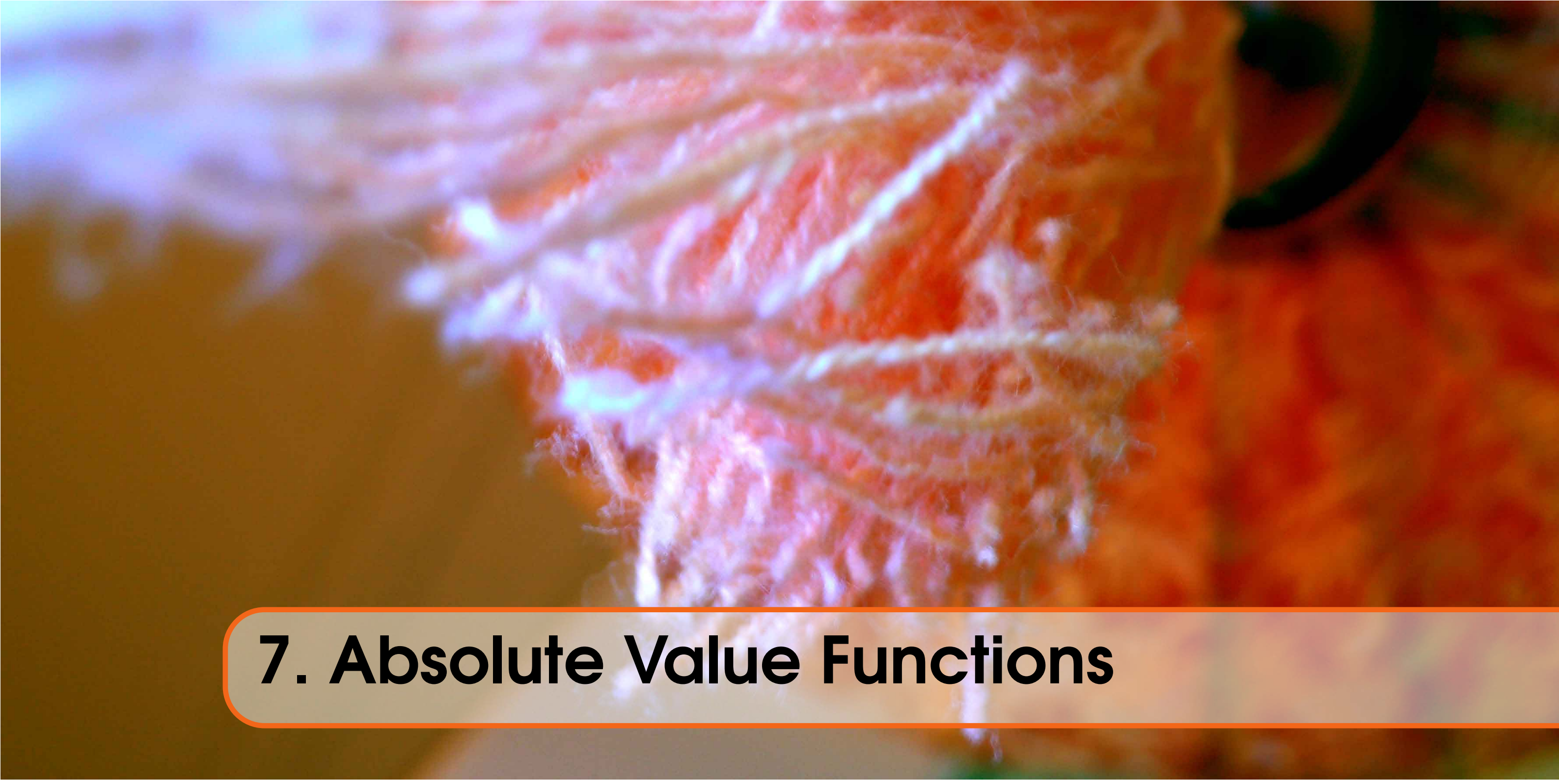


7. Absolute Value Functions

7.1 Absolute Value Functions

A very special and common piecewise function is the absolute value function.

- $y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- $x \in \mathbb{R}$
- $y \geq 0$

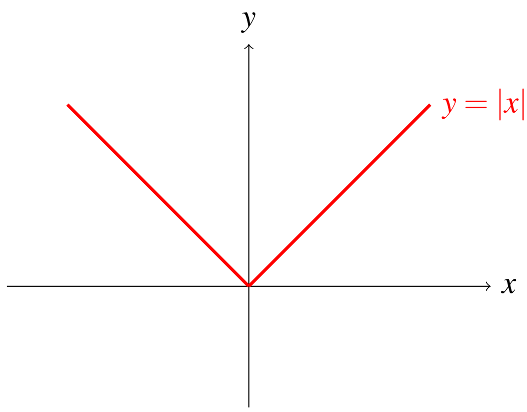


Figure 7.1: absolute value function

Theorem 7.1.1 — Absolute Values.

$$|ab| = |a| \cdot |b| \tag{7.1}$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \tag{7.2}$$

$$if\ |a| \leq b,\ then\ -b \leq a \leq b \tag{7.3}$$

$$if\ |a| \geq b,\ then\ a \geq b\ or\ a \leq -b \tag{7.4}$$

$$(Triangle\ Inequality)\ |a+b| \leq |a| + |b| \tag{7.5}$$

8. Inequalities Notation

8.1 Inequalities Notation

When solving equation we may get a single answer, or a number of answers that satisfy the equation.

Consider $3x-5=1$, only one value satisfies this equation.

But if we consider $x^2-1=3$, more than one value satisfies this equation.

Inequalities notation like $1 \leq x < 3$, where the symbols $\leq$ and $\geq$ indicate inclusion of an endpoint, and $<$ and $>$ indicate exclusion of an endpoint.

A second notation is interval notation, for example, $x \in [1,3)$, where a square (or closed) bracket indicates inclusion of an endpoint, and a round (or open) bracket indicates exclusion of an endpoint.

The infinity symbol ∞ is always accompanied by round brackets.

Exercise 8.1 Write each of the following in interval notation.

(a) $2 \leq x \leq 7$

(b) $x < 9$

(c) $-3 > x > 0$

■

Exercise 8.2 Write each of the following using inequalities.

(a) $x \in [3,6)$

(b) $x \in (-2,4)$

(c) $x \in (-\infty,-1]$

■

It is possible to have ranges of values that are disjoint. We use the union symbol $\cup$ to include all of the values in any of the disjoint ranges. For example, $[-1,4) \cup [7,10)$ meas $-1 \leq x < 4$ or $7 \leq x < 10$.

Exercise 8.3 Express each of the following in interval notation.

(a) $-3 \leq x < \frac{1}{2}$ or $4 < x < 7$

(b) $1 \leq x < 5$ or $3 \leq x < 7$

(c) $x \in [-2,6) \bigcup (0,5)$

■

Intersection symbol $\cap$ allows only the values that are common between intervals. For example, $[-1,6) \cap (2,7)$ means $(2,6)$.

Exercise 8.4 Express each of the following in interval notation.

(a) $-2 < x \leq 6$ and $0 < x < 7$

(b) $x \in [0,5] \bigcap [3,5]$

(c) $-4 < x < 0$ and $3 < x < 7$

■

8.2 Solving Inequalities

When solving inequalities, there are a few rules that we must follow:

1. When it comes to addition, subtraction, multiplication, and division, what you do to one side of the inequality, you must do to the other.
2. If you multiply or divide by a negative quantity, you must flip the inequality.
3. If both sides are positive or both sides are negative, then you can take the reciprocal of both sides, but you must flip the inequality.

Exercise 8.5 Find all values of x that satisfy the following.

(a)

$$-6x + 7 \geq 8x$$

(b)

$$-\frac{5}{2} < 4 - 2x \leq 1$$

(c)

$$5x^3 + 27 > -13$$

(d)

$$3x^2 + 2 < -4$$

(e)

$$\sqrt{x-1} > 4$$

(f)

$$log_2(3x) \leq -3$$

9. The Case Method

9.1 The Case Method

Consider the following example: $\frac{x-3}{x-1} < 10$

You might be tempted to cross multiply, but be careful! The quantity $x-1$ is not always positive. If we multiply by $x-1$, the inequality needs to flip for some values of x. How might we deal with this?

1. Separate into 2 cases

Case 1	**Case 2**
$x-1 > 0$	$x-1 < 0$
$x > 1$	$x < 1$

2. Solve the original problem under each assumption

Case 1	**Case 2**
$\frac{x-3}{x-1} < 10$	$\frac{x-3}{x-1} < 10$
$x-3 < 10(x-1)$	$x-3 > 10(x-1)$
$-9x < -7$	$-9x > -7$
$x > \frac{7}{9}$	$x < \frac{7}{9}$

3. Find all common points between assumption and solution

Case 1	Case 2
$x > 1$ *and* $x > \frac{7}{9}$ $x \in (1, \infty)$	$x < 1$ *and* $x < \frac{7}{9}$ $x \in \left(-\infty, \frac{7}{9}\right)$

4. Consolidate the 2 cases by taking the union

$$x \in (1, \infty) \cup \left(-\infty, \frac{7}{9}\right)$$

Exercise 9.1 Find all values of x such that $\frac{7x-2}{1-2x} \geq 4$.

1. Separate into 2 cases

Case 1	Case 2

2. Solve the original problem under each assumption

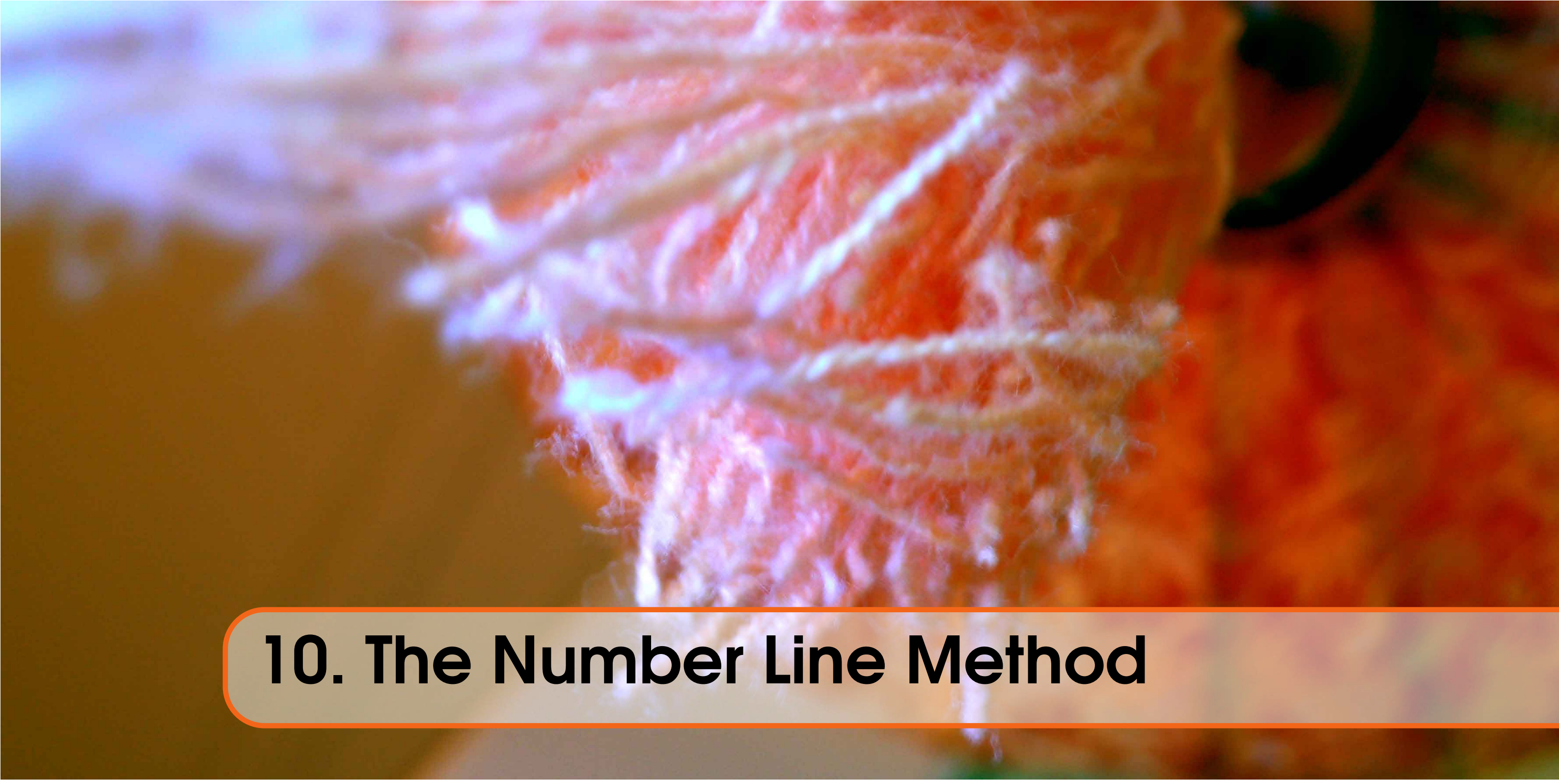
Case 1	Case 2

3. Find all common points between assumption and solution

Case 1	Case 2

4. Consolidate the 2 cases by taking the union

■



10. The Number Line Method

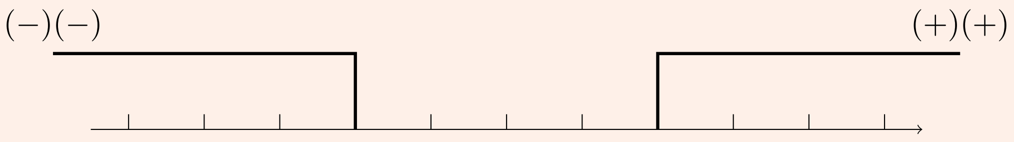
10.1 The Number Line Method

Another method for solving inequalities uses the following basic logic:

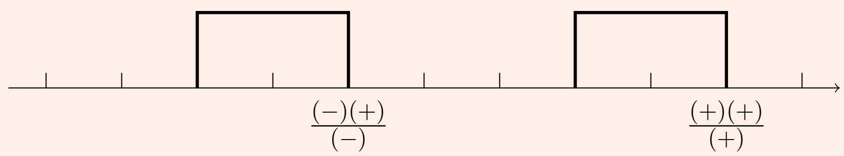
$$
\begin{aligned}
(+)(+) &= + & \qquad \frac{(+)}{(+)} &= + \\
(-)(-) &= + & \qquad \frac{(-)}{(-)} &= + \\
(+)(-) &= - & \qquad \frac{(+)}{(-)} &= - \\
(-)(+) &= - & \qquad \frac{(-)}{(+)} &= -
\end{aligned}
$$

By manipulating expressions into factors that are multiplied and/or divided on one side of the inequality (with a zero appearing on the other side), we can simply consider the combinations of positive and negative factors to draw conclusions.

Exercise 10.1 Find all values of x that satisfy $3x^2 - 13x > -10$.



Exercise 10.2 Find all values of x that satisfy $x-2 \geq \frac{4}{x+1}$.



III Limits and Continuity

11. Basic Limits

11.1 Basic Limits

Suppose we have a function $y = f(x)$. The limit of $f(x)$ as x approaches the number a, denoted $\lim_{x\to a} f(x)$.

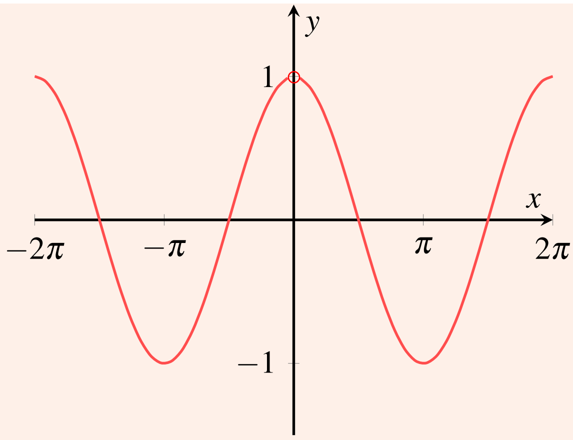
For example, suppose we wanted to determine $\lim_{x\to 3} x^2$.

x **approaches 3 from the left**	x **approaches 3 from the right**
$f(2.9) = 8.41$	$f(3.1) = 9.61$
$f(2.99) = 8.9401$	$f(3.01) = 9.0601$
$f(2.999) = 8.994001$	$f(3.001) = 9.006001$

No matter which way we approach from, as x gets close 3, we can see that x^2 gets very close to 9.

Exercise 11.1 Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} cos(x) & x \neq 0 \\ -1 & x = 0 \end{cases}$$



$$\lim_{x\to 0} f(x) =$$
$$f(0) =$$

11.2 Law of Limits

Theorem 11.2.1 — Law of Limits.

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{11.1}$$
$$\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \tag{11.2}$$
$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \tag{11.3}$$
$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{11.4}$$

Exercise 11.2 Evaluate $\lim_{x\to 0} x sin\left(\frac{1}{x}\right)$.

11.3 One-sided Limits

x can approach the value a in two ways:

- left-hand limit: $\lim_{x \to a^-} f(x)$
- right-hand limit: $\lim_{x \to a^+} f(x)$

We can tighten our definition of the limit of a function $f(x)$ as x approaches a. If the left-hand and right-hand limits of $f(x)$ are both equal to a number L, then we say that $\lim_{x \to a} f(x)$ exists and is equal to L.

Exercise 11.3 Consider the piecewise function given by

$$f(x) = \begin{cases} x & x < 1 \\ -1 & x = 1 \\ \sqrt{x-1} & x > 1 \end{cases}$$

(a) $\lim_{x \to 1^-} f(x) =$

(b) $\lim_{x \to 1^+} f(x) =$

(c) $f(1) =$

(d) Does $\lim_{x \to 1} f(x)$ exist? ■

12. Continuity

12.1 Continuity

It seems like sometimes we can evaluate a limit by just plugging in, but sometimes we can't!

Exercise 12.1 Evaluate $\lim_{x \to 1} \left(ln(\sqrt{x}) + \frac{1}{x} \right)$.

A function is continuous at a point a if each of the following conditions hold:

1. $\lim_{x \to a} f(x)$ exists
2. $f(a)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

12.2 Discontinuity

Kinds of discontinuities:

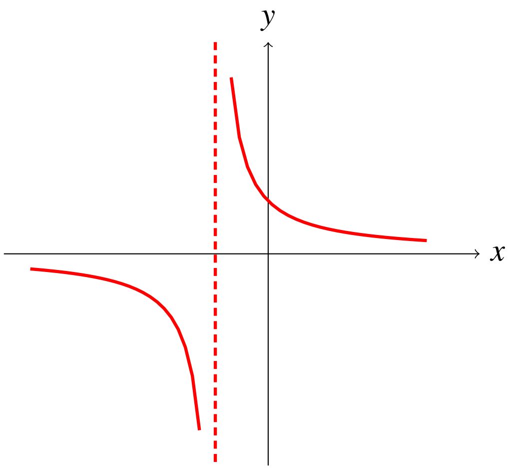


Figure 12.1: essential

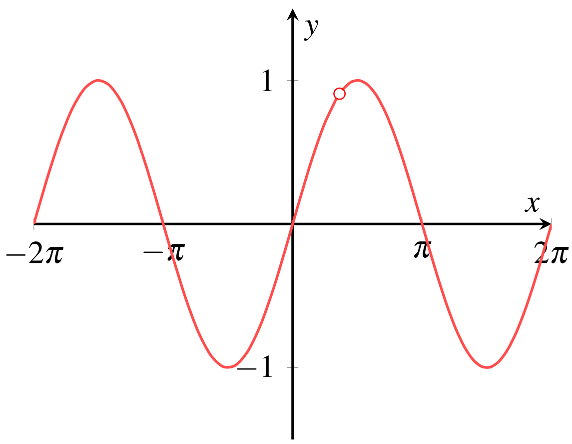


Figure 12.2: removable

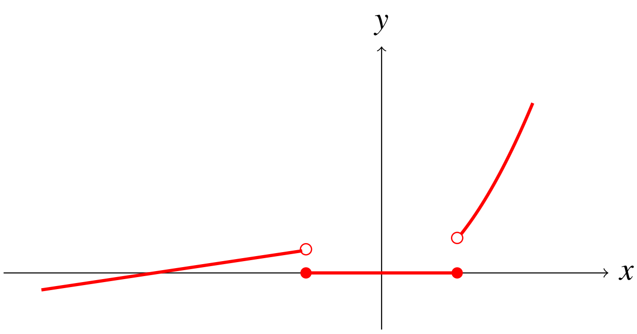


Figure 12.3: jump

So far, we know that evaluating $\lim_{x \to a} f(x)$ is easy if $f(x)$ is continuous at a. But what if it isn't? The limit may still exist.

Exercise 12.2 Evaluate the following limits.

(a) $\lim_{x \to 3^+} \frac{1}{3-x} =$

(b) $\lim_{x \to 3^-} \frac{1}{3-x} =$

(c) $\lim_{x \to 0^+} ln(x) =$

(d) $\lim_{x \to -\frac{\pi}{2}^+} sec(x) =$

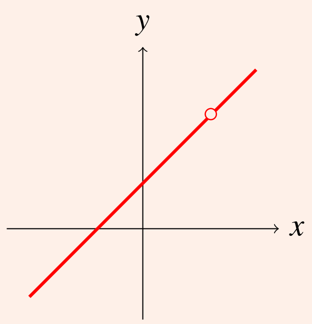
A function doesn't have to go to $\pm\infty$ at a discontinuity though. Consider, for instance, evaluating $\lim_{x \to 3} \frac{x^2-x-6}{x-3}$. Plugging in $x = 3$ gives us "0/0", which is known as an indeterminate form.

A good first step is to factor where possible. Once we cancel the factor $x-3$ on top and bottom, this function is just the function $f(x) = x+2$, but with a hole at $x = 3$. The hole is there because we cannot ignore that the original form of the function had issues at $x = 3$.

Exercise 12.3 Evaluate the following limits.

(a)

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$$



(b)

$$\lim_{x \to -1} \frac{x^3 - 11x^2 + 8x + 20}{x + 1}$$

(c)

$$\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4}$$

(d)

$$\lim_{x \to 2} \frac{\frac{1}{x-4} + \frac{1}{2}}{x - 2}$$

13. The Fundamental Sine Limit

13.1 The Fundamental Sine Limit

Consider the following limit:

$$\lim_{x\to 0} \frac{sin(x)}{x}$$

We can quickly see that this is a "0/0" limit, and therefore is indeterminate. We can't factor anything or otherwise simplify the expression to get rid of the issue.

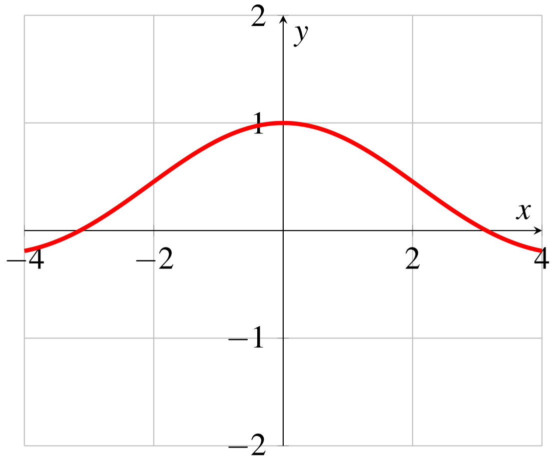


Figure 13.1: $\frac{sin(x)}{x}$

It is pretty clear from the graph that

Theorem 13.1.1 — The Fundamental Sine Limit.

$$\lim_{x\to 0}\frac{sin(x)}{x}=\lim_{x\to 0}\frac{x}{sin(x)}=1 \tag{13.1}$$

Exercise 13.1 Evaluate the following limits.

(a)

$$\lim_{t\to 0}\frac{5sin(t)}{2t}$$

(b)

$$\lim_{\theta\to 0}\frac{\theta}{sin(4\theta)}$$

(c)

$$\lim_{x\to -1}\frac{sin^2(x+1)}{(x+1)^2}$$

(d)

$$\lim_{x \to 0} \frac{sin(3x)}{sin(2x)}$$

(e)

$$\lim_{\theta \to 0} \frac{100\theta}{tan(3\theta)}$$

(f)

$$\lim_{h \to 0} \frac{cos(h) - 1}{h}$$

14. Limits Approaching $\pm\infty$

14.1 Limits Approaching $\pm\infty$

Sometimes, we will be interested in determining what happens as x gets really big, either in the positive or negative direction.

Exercise 14.1 Evaluate the following limits.

(a) $\lim\limits_{x\to\infty} \frac{1}{x} =$

(b) $\lim\limits_{x\to-\infty} \frac{1}{x} =$

(c) $\lim\limits_{x\to-\infty} \frac{401}{x^{203}} =$

(d) $\lim\limits_{x\to-\infty} \frac{-238}{23x^{1/4}} =$ ■

For problems like $\lim\limits_{x\to\infty} \frac{2x^2+3x+1}{x^2-10x+100}$, a good first step is to divide the top and the bottom by the highest power of x appearing in the denominator.

Exercise 14.2 Evaluate the following limits.

(a)

$$\lim_{x\to\infty} \frac{2x^2+3x+1}{x^2-10x+100}$$

(b)

$$\lim_{x\to-\infty} \frac{3-2x^3}{1+x+x^2}$$

(c)

$$\lim_{x\to\infty} \sqrt{x^2-x+1}$$

(d)

$$\lim_{t\to\infty} \sqrt{3t^2+2t+5}+t$$

IV Derivatives

15. The First Principles

15.1 The First Principles

The equation of a line $y = mx + b$ where b is the y-intercept and m is the slope of the line. Calculating the slope is easy using any two points on the line (x_1, y_1) and (x_2, y_2).

Theorem 15.1.1 — Slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h} \tag{15.1}$$

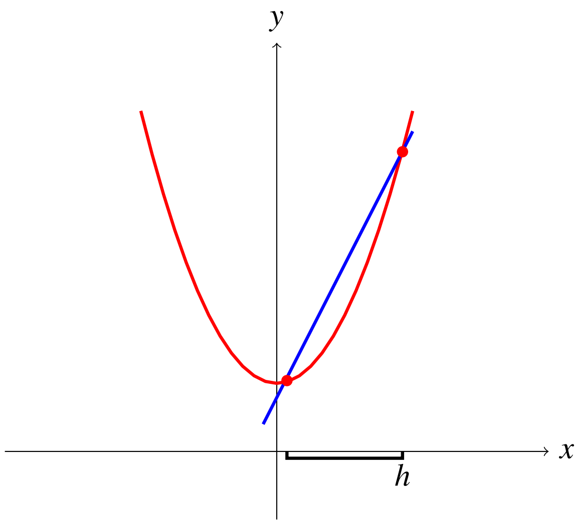


Figure 15.1: secant line

What if we made the point even closer to $(x, f(x))$?

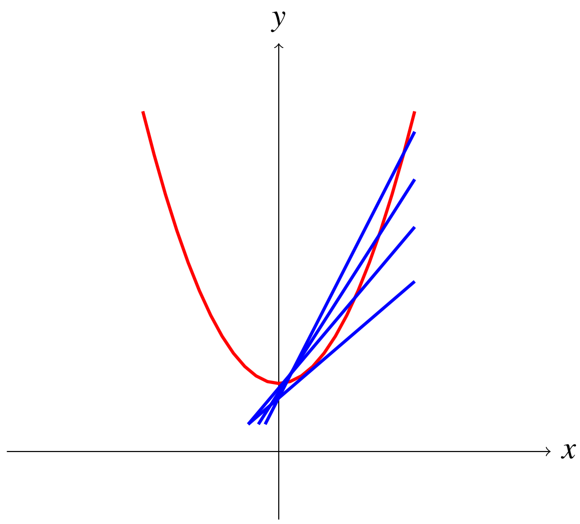


Figure 15.2: tangent line

As h gets smaller and smaller, the point $(x+h, f(x+h))$ approaches the point $(x, f(x))$. As this occurs, the slope of the secant line approaches the slope of the tangent line. We can write this action mathematically as:

Theorem 15.1.2 — The First Principles.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{15.2}$$

If this limit exists as a finite number, the this value is equal to the slope of the tangent line. This is the instantaneous rate of $f(x)$ at the point $(x, f(x))$. This very special limit is also called the derivative of $f(x)$ at $(x, f(x))$.

If $y = f(x)$, then we can denote the derivative in a few different ways:

$$\frac{dy}{dx} \qquad y' \qquad f'(x)$$

When $f(x)$ has a derivative at $(a, f(a))$, we say that $f(x)$ is differentiable at the value $x = a$. This means that

Theorem 15.1.3 — The First Principles.

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \tag{15.3}$$

Exercise 15.1 From The First Principles, find the derivatives.

(a) $f(x) = x^2$

(b) $f(x) = \sqrt{x+2}$ ■

Exercise 15.2 From The First Principles, find the derivative of $f(x) = \frac{1}{x}$ at $x = 5$.

16. Differentiability and Continuity

16.1 Differentiability and Continuity

A non-differentiable function has one of the following properties:

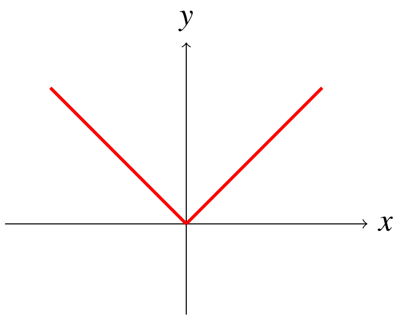


Figure 16.1: corner

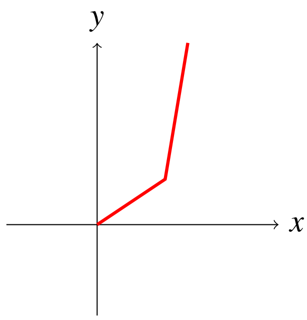


Figure 16.2: sudden turn

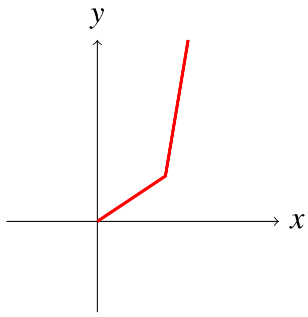


Figure 16.3: asymptote

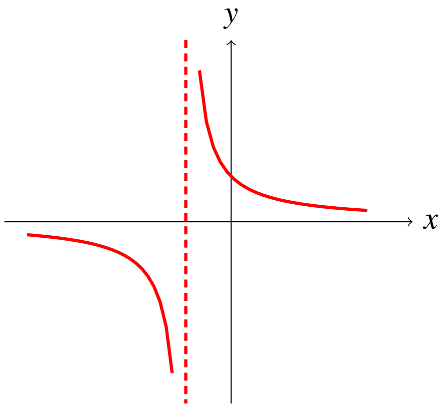


Figure 16.4: asymptote

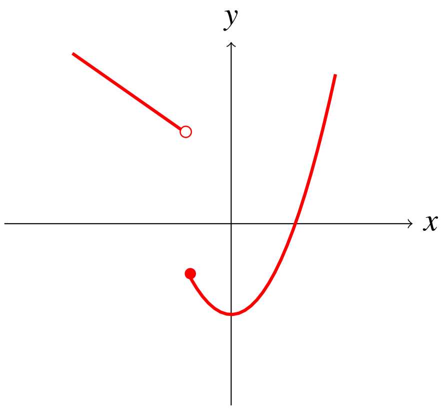
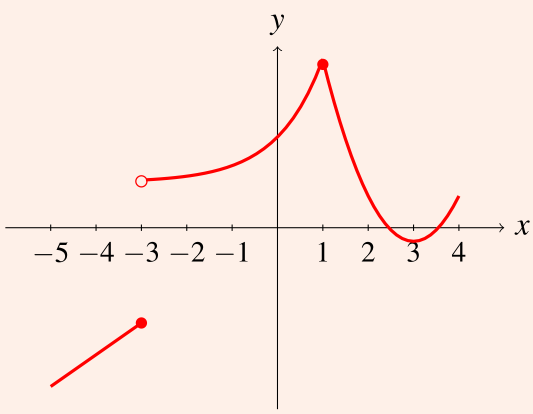


Figure 16.5: left-hand limit $\neq$ right-hand limit

In all of these cases, the limit of the slopes does not exist at a point. Functions are not differentiable at such points.

Differentiability implies continuity, but the reverse is not true. Consider the function $y = |x|$ as a counterexample. This function is continuous for all values of x, but it is not differentiable at $x = 0$, because there is a corner there. The limit of the slopes does not exist at $x = 0$.

Exercise 16.1 Discuss the continuity and differentiability.



(a) $x = 2$:

(b) $x = 1$:

(c) $x = -3$:

17. Derivative Rules

17.1 Derivative Rules

The First Principles definition of the limit can become a cumbersome task when functions get complicated. Luckily, there are rules for finding derivatives more quickly depending on their form. All of these rules are derived from The First Principles.

17.1.1 Power Rule

Theorem 17.1.1 — Power Rule. Let $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Exercise 17.1 Find the derivative of $f(x) = x^7$. ■

17.1.2 Derivative of a Constant

Theorem 17.1.2 — Derivative of a Constant. Let $f(x) = k$, then $f'(x) = 0$. Graphically, since a constant function is a horizontal line, so its slope must be zero.

Exercise 17.2 Find the derivative of $f(x) = 672$. ■

17.1.3 Derivatives of Logarithmic Functions

Theorem 17.1.3 — Derivatives of Logarithmic Functions.

$$\frac{d}{dx}log_a x = \frac{1}{xln(a)},\ where\ a > 0\ and\ a \neq 1 \tag{17.1}$$

$$\frac{d}{dx}ln(x) = \frac{d}{dx}log_e(x) = \frac{1}{xln(e)} = \frac{1}{x} \tag{17.2}$$

Exercise 17.3 Find the derivative of $f(x) = log_4(x)$. ■

17.1.4 Derivatives of Exponential Functions

Theorem 17.1.4 — Derivatives of Exponential Functions.

$$\frac{d}{dx}a^x = a^x ln(a),\ where\ a > 0\ and\ a \neq 1 \tag{17.3}$$

$$\frac{d}{dx}e^x = e^x ln(e) = e^x \tag{17.4}$$

Exercise 17.4 Find the derivative of $f(x) = 5^x$. ■

17.1.5 Derivatives of Trigonometric Functions

Theorem 17.1.5 — Derivatives of Trigonometric Functions.

$$\frac{d}{dx}sin(x) = cos(x) \tag{17.5}$$

$$\frac{d}{dx}csc(x) = -csc(x)cot(x) \tag{17.6}$$

$$\frac{d}{dx}cos(x) = -sin(x) \tag{17.7}$$

$$\frac{d}{dx}sec(x) = sec(x)tan(x) \tag{17.8}$$

$$\frac{d}{dx}tan(x) = sec^2(x) \tag{17.9}$$

$$\frac{d}{dx}cot(x) = -csc^2(x) \tag{17.10}$$

Memory tool: if the trigonometric function starts with "c" then its derivative has a minus sign.

What about combining these basic derivative rules for functions? That is, how do we find the derivatives of the addition, subtraction, multiplication, division, or take the constant of functions? For all of the following rules, we assume that f and g are differentiable on their domains.

17.1.6 Multiplication by a Constant

Theorem 17.1.6 — Multiplication by a Constant.

$$(kf(x))' = kf'(x) \tag{17.11}$$

Exercise 17.5 Find the derivative of $f(x) = 5ln(x)$. ■

17.1.7 Sum / Difference

Theorem 17.1.7 — Sum / Difference.

$$(f \pm g)'(x) = f'(x) \pm g'(x) \tag{17.12}$$

Exercise 17.6 Find the derivative of $f(x) = ln(x) + 4x^3 - 6csc(x)$. ■

17.1.8 Product Rule

Theorem 17.1.8 — Product Rule.

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{17.13}$$

Exercise 17.7 Find the derivative of $f(x) = ln(x)cos(x)$. ■

17.1.9 Quotient Rule

Theorem 17.1.9 — Quotient Rule.

$$\left(\frac{f}{g}\right)'(x) = \frac{f'g - fg'}{g^2}, \; where\; g(x) \neq 0 \tag{17.14}$$

Exercise 17.8 Find the derivative of $f(x) = \frac{5^x}{x^2}$.

Exercise 17.9 Use the Quotient Rule to prove that $\frac{d}{dx}cot(x) = -csc^2(x)$.

17.1.10 Chain Rule

Theorem 17.1.10 — Chain Rule.

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) \tag{17.15}$$

Exercise 17.10 Find the derivative of the following functions.

(a) $f(x) = cos(x^3)$

(b) $f(x) = (x^2 + 3x + 2)^5$

(c) $f(x) = ln(4x^2)$

(d) $f(x) = e^{sec(x)}$

(e) $f(x) = sin^3(x^2 + \frac{1}{x})$

■

18. Higher Order Derivatives

18.1 Higher Order Derivatives

What happens if we take the derivative of a derivative (the second derivative)? What if we do this many times?

Second derivative notations:

$$\frac{d^2y}{dx^2} \qquad\qquad y'' \qquad\qquad f''(x)$$

Third derivative notations:

$$\frac{d^3y}{dx^3} \qquad\qquad y''' \qquad\qquad f'''(x)$$

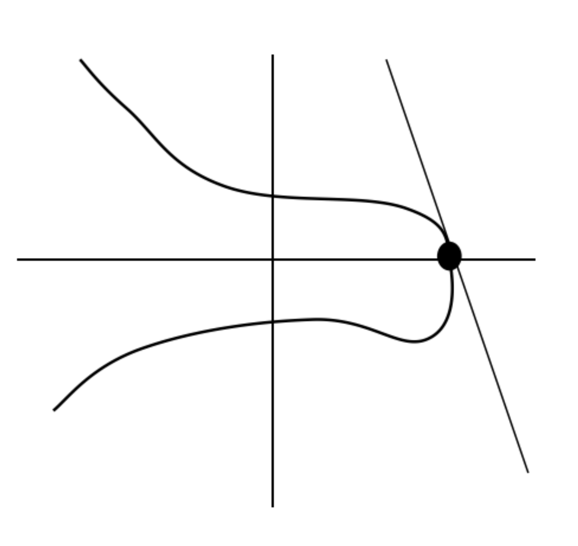
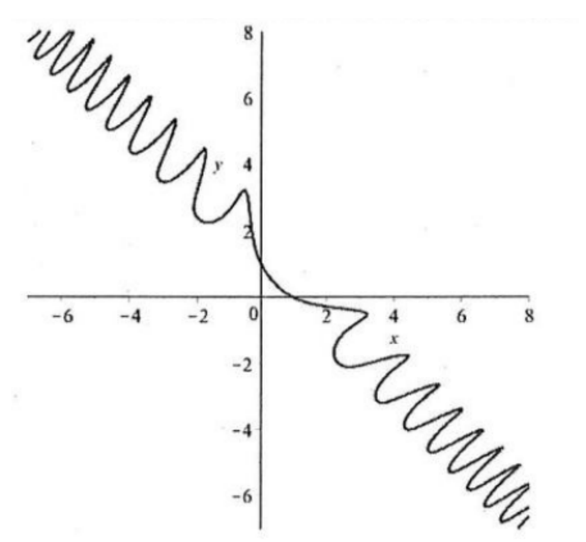
Higher order derivative notations:

$$\frac{d^ny}{dx^n} \qquad\qquad y^{(n)} \qquad\qquad f^{(n)}(x)$$

Exercise 18.1 Let $f(x) = 2x^4$, find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$.

19. Implicit Derivatives

19.1 Implicit Differentiation

$x^3+xy+y^2=1$	$sin(xy)+ln(x+y)=0$
	

For relations like these, there is no way to isolate for either x or y. Either variable is implicitly defined in the equation in both of the above cases.

Exercise 19.1 If $x^3+xy+y^2=1$, find $\frac{dy}{dx}$ at the point $(x,y)=(1,0)$.

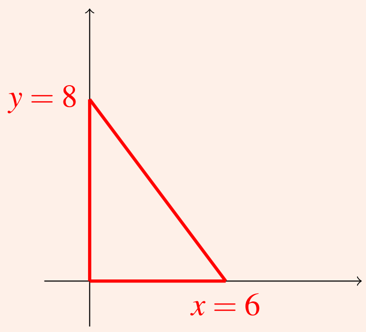
Exercise 19.2 If $sin(tx^3) + ln(t^2 + x) = 0$, find $\frac{dx}{dt}$.

Exercise 19.3 Show that if $e^{2xy} + e^x + e^y = y$ that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

Exercise 19.4 If $x^3 + x = y$, find $\frac{d^2y}{dx^2}$.

Exercise 19.5 A 10m ladder leans against a perpendicular wall.

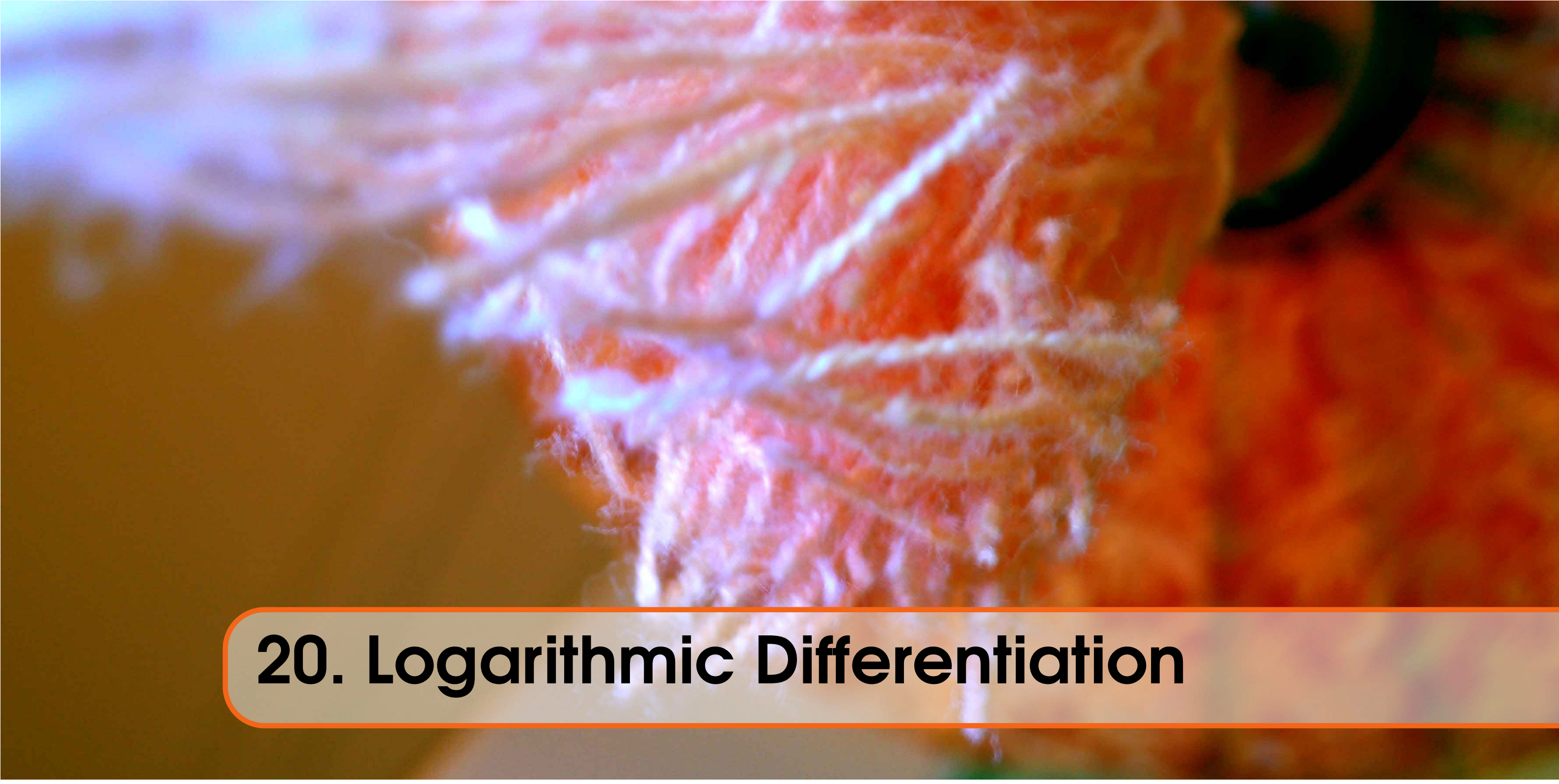
(a) The top of the ladder slides down the wall at 0.5m/s. How fast is the foot of the ladder moving at the moment the top of the ladder touches the wall 8m above the ground?



KNOWN: $\frac{dy}{dt} = -0.5, y = 8$

(b) Consider the angle that the ladder makes with the ground. How fast is this angle changing at that exact moment?

WANT: $\frac{d\theta}{dt}$



20. Logarithmic Differentiation

20.1 Logarithmic Differentiation

Logarithmic differentiation allows us to take the derivatives of more complicated expressions. The derivative rules give us a way to quickly differentiate expression like x^5, e^{4x}, 2^x. But what about x^x?

Exercise 20.1 Use a table of values to sketch the function $f(x) = x^x$.

$x > 0$	$f(x)$	$x < 0$	$f(x)$

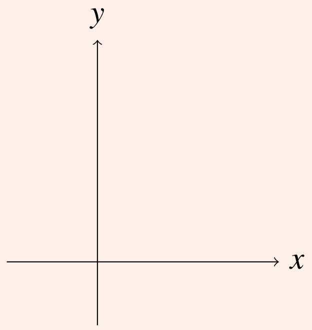


Figure 20.1: $y = x^x$ when $x > 0$

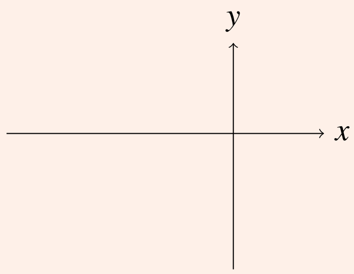


Figure 20.2: $y = x^x$ when $x < 0$

x^x is indeterminate when $x = 0$. Because the domain of x^x is extremely complicated when $x < 0$, this function is often only considered for $x > 0$.

So, how do we determine the derivative of this function? With logarithmic differentiation, we take the ln() of both side. Then we use log rules to simplify what we get before we finish by differentiating implicitly.

Exercise 20.2 If $y = (cos(\pi x))^{2x}$, find $\frac{dy}{dx}$.

Exercise 20.3 If $y = (cos(\pi x))^{2x}$, find $\frac{dy}{dx}$.

It would be tedious to differentiate expression like $y = \frac{(x-2)^2(3x+1)^3\sqrt{x^2+2}}{(x-1)(x+10)^{10}}$ using product and quotient rules, and put pus at a high risk for making mistakes. We can use logarithms to simplify problems. It is technically an important step that we take the absolute value of both sides first that we arenąŕt considering the $ln()$ of a negative number.

Exercise 20.4 If $y = \frac{(x-2)^2(3x+1)^3\sqrt{x^2+2}}{(x-1)(x+10)^{10}}$, find $\frac{dy}{dx}$.

21. Differential Approximation

21.1 Differential Approximation

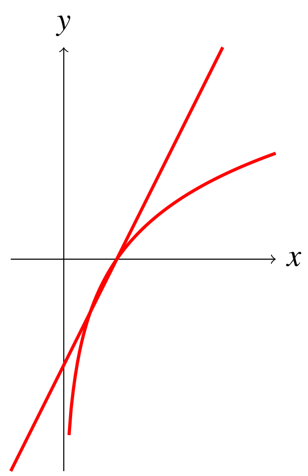
Differential approximation allows us to calculate estimates if numbers that would be impossible to find by hand. For example, we know what $sin(\pi)$ is, but what about $sin(3)$?

The idea is simple. Given a function $f(x)$ and a difficult-to-evaluate value of $x = a$, we want to follow this process to estimate $f(a)$:

1. Find a nearby value a^* at which f is easy to calculate.

2. Find the tangent line there, called $L(x)$, with slope $f'(a^*)$.

3. Estimate $f(a)$ by using the height $L(a)$ instead.



Exercise 21.1 Estimate $sin(3)$ using differential approximation.

Exercise 21.2 Estimate $\sqrt{65}$ using differential approximation.

V Antiderivatives

22. The Indefinite Integral

22.1 The Indefinite Integral

How do we undo a derivative? If we were given the derivative of a function $f'(x)$, how could we find the original function $f(x)$? The answer is called the antiderivative of $f(x)$, which we will denote by the associated capital letter $F(x)$.

Another way to think about this question is: "What function do I have to take the derivative of in order to get the answer?" The antiderivative of $f(x) = 2x$ is x^2.

But what about $F(x) = x^2 + 1$? This works too! In fact, since when we take the derivative of a constant, we get zero. We could have chosen any constant. As a result, we report our antiderivative in its most general form $x^2 + C$. The constant C is an important part of the antiderivative.

Notationally, we denote the operation of take the antiderivative (known as integration) as:

$$\int 2x\,dx = x^2 + C$$

This is also called the indefinite integral of the function $f(x)$, or sometimes just the integral of $f(x)$, where $f(x)$ is called the integrand.

That was a pretty simple example, so how do we find antiderivatives of more complicated expressions? In much the same way as we did with derivatives, we can generate a set of rules for finding antiderivatives, derived simply by thinking of our familiar derivative rules in reverse.

22.2 Basic Antiderivative Rules

The power rule for derivatives multiplies by the power and then subtracts one from the power. Reversing these operations means that we add one to the power and divide by the new power.

22.2.1 Reverse Power Rule

Theorem 22.2.1 — Reverse Power Rule.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \ where\ n \neq -1 \tag{22.1}$$

Exercise 22.1 Find the anitiderivatives.

(a)

$$\int x^6 \, dx$$

(b)

$$\int \sqrt[4]{t} \, dt$$

(c)

$$\int \frac{1}{x^{5/3}} \, dx$$

■

22.2.2 Antiderivative of Zero

Theorem 22.2.2 — Antiderivative of Zero.

$$\int 0 \, dx = C \tag{22.2}$$

22.2.3 Antiderivative of a Constant

Theorem 22.2.3 — Antiderivative of a Constant.

$$\int k \, dx = kx + C, \text{ where } k \text{ is any constant} \tag{22.3}$$

Exercise 22.2 Find the anitiderivatives.

$$\int \pi \, dx$$

■

22.2.4 Multiplicative Constants

Theorem 22.2.4 — Multiplicative Constantst.

$$\int kf(x) \, dx = k \int f(x) \, dx \tag{22.4}$$

Exercise 22.3 Find the anitiderivatives.

(a)

$$\int 4x^7 \, dx$$

(b)

$$\int \frac{\pi}{\sqrt{t}}\,dt$$

22.2.5 Sum / Difference

Theorem 22.2.5 — Sum / Difference.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{22.5}$$

Exercise 22.4 Find the anitiderivatives.

(a)

$$\int (3x^2 + 5)\,dx$$

(b)

$$\int \left(\frac{1}{x^3} - \frac{2}{x^2}\right)\,dx$$

22.2.6 Trigonometric Functions

Theorem 22.2.6 — Trigonometric Functions.

$$\int sin(x)\ dx = -cos(x) + C \tag{22.6}$$

$$\int cos(x)\ dx = sin(x) + C \tag{22.7}$$

$$\int csc^2(x)\ dx = -cot(x) + C \tag{22.8}$$

$$\int sec^2(x)\ dx = tan(x) + C \tag{22.9}$$

$$\int sec(x)tan(x)\ dx = sec(x) + C \tag{22.10}$$

$$\int csc(x)cot(x)\ dx = -csc(x) + C \tag{22.11}$$

22.2.7 Exponential / Logarithmic

Theorem 22.2.7 — Exponential / Logarithmic.

$$\int a^x ln(a)\ dx = a^x + C \tag{22.12}$$

$$\int e^x ln(e)\ dx = e^x + C \tag{22.13}$$

$$\int \frac{1}{xln(a)}\ dx = log_a|x| + C \tag{22.14}$$

$$\int \frac{1}{x}\ dx = ln|x| + C \tag{22.15}$$

Exercise 22.5 Find the antiderivatives.

(a)

$$\int 4^x ln(4) + 5e^x - \frac{6}{x}\ dx$$

(b)

$$\int 3^z\ dz$$

■

Sometimes we need to manipulate the integral a little bit before we can apply the rules.

Exercise 22.6 Find the anitiderivatives.

(a)

$$\int \frac{2s^3 - 5s^4}{3s^2}\, ds$$

(b)

$$\int \left(\frac{1}{x} + \frac{1}{x^2}\right)\left(3 + 2x^2\right)\, dx$$

23. Chain Rule in Reverse

23.1 Chain Rule in Reverse

The derivative of $f(u(x))$ is $f'(u(x))u'(x)$, so

Theorem 23.1.1 — Chain Rule in Reverse.

$$\int f'(u(x))u'(x)\,dx = f(u(x)) + C \tag{23.1}$$

Notice that in the integration, the $u'(x)$ piece disappears, being absorbed back into $f(x)$. The steps for finding the antiderivative of composition functions are as follows:

1. Identify the core layer $u(x)$.
2. Identify the derivative of the core layer $u'(x)$.
3. Identify the outer layer f', and integrate f' leaving $u(x)$ inside.

Exercise 23.1 Find the anitiderivatives.

(a)

$$\int (6x^2+1)sin(2x^3+x)\;dx$$

(b)

$$\int sec^2(4t)\;dt$$

(c)

$$\int 4x^3(3x^4-1)^{14}\,dx$$

(d)

$$\int \frac{e^{\frac{1}{x}}}{4x^2}\,dx$$

Exercise 23.2 Integrate in one step.

$$\int e^{-2t} + sin(3t) + cos\left(\frac{1}{4}t\right)\, dt$$

24. The Method of Substitution

24.1 The Method of Substitution

The idea behind the method of substitution is to change a difficult integral in terms of one variable into an easier integral in terms of some other variable using a substitution.

Theorem 24.1.1 — The Method of Substitution.

$$\int f'(u(x))\,\frac{du}{dx} = \int f'(u)\,du \tag{24.1}$$

Exercise 24.1 The method of substitution.

$$\int (6x+4)(3x^2+4x)^5\,dx$$

1. Identify the core layer $u(x) = 3x^2+4x$.
2. Find the derivative of the core $\frac{du}{dx} = 6x+4$.
3. Transform from an integral in x to an integral in the new variable u using the change of variable theorem.

$$
\begin{aligned}
&\int (6x+4)(3x^2+4x)^5 \, dx \\
&= \int \frac{du}{dx} u^5 \, dx \\
&= \int u^5 \, du \\
&= \frac{u^6}{6} + C
\end{aligned}
$$

4. Convert back to the original variable by substituting $u(x)$ back in.

$$
\begin{aligned}
&\frac{u^6}{6} + C \\
&= \frac{(3x^2+4x)^6}{6} + C
\end{aligned}
$$

■

Exercise 24.2 Calculate using the method of substitution.

(a)

$$\int sin(x)e^{5cos(x)} \, dx$$

(b)

$$\int \frac{x}{(5x+7)^3}\, dx$$

(c)

$$\int x sec(3x^2) tan(3x^2)\, dx$$

(d)

$$\int \frac{-\frac{1}{t^2}+1}{\sqrt{\frac{1}{t}+t}}\,dt$$

(e)

$$\int (1+900x)^{1/15000}\,dx$$

(f)

$$\int sin(\theta)(cos^3\theta - cos^5\theta)\ d\theta$$

(g)

$$\int \frac{7x}{4x^2+9}\ dx$$

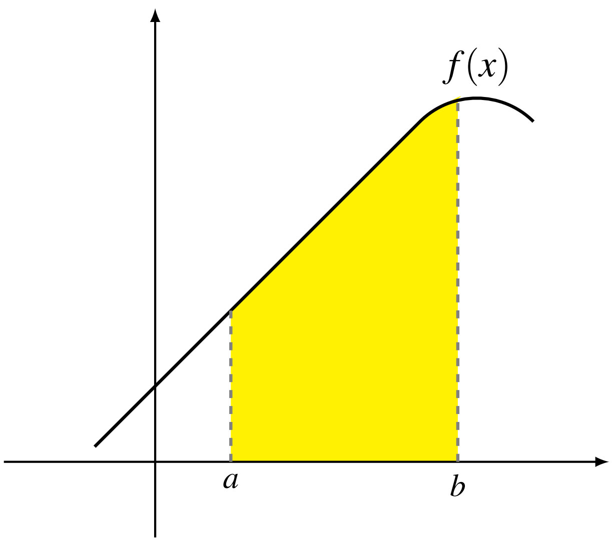
(h)

$$\int (2x+5) \cdot \sqrt[3]{3x+1}\, dx$$

25. Definite Integrals

25.1 Riemann Sum

Suppose we wanted to find the area underneath the graph of a straight line that lies above the x-axis, between $x = a$ and $x = b$. Since we have formulas for finding the area of basic shapes, we can easily figure this out.



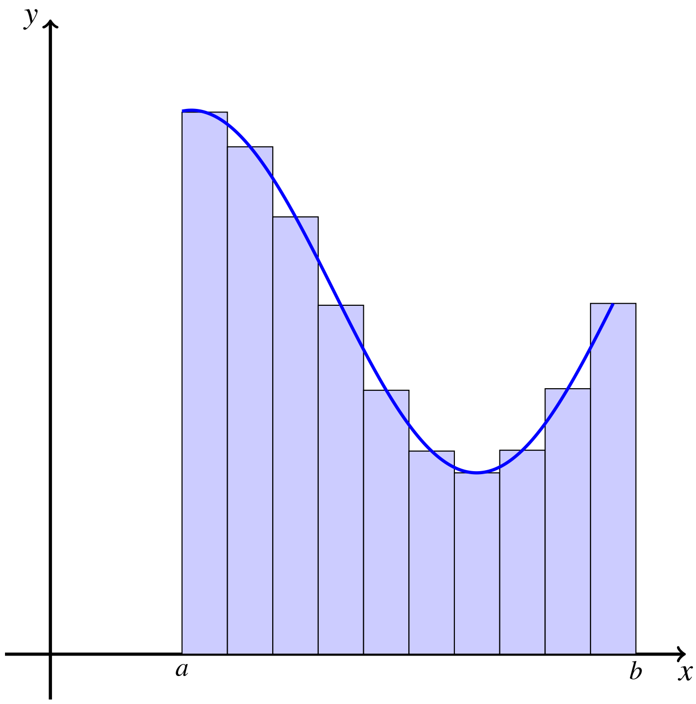
But what about finding the area underneath the graph of a general curve $y = f(x)$ and above the x-axis between $x = a$ and $x = b$?

Bernhard Riemann's idea was to carve up the desired area into rectangle and user their area to estimate the true area. He called this the Riemann Sum and it goes as follows:

1. Create a partition P, dividing up the interval $[a,b]$ into n subintervals $I_1, I_2, \dots, I_n$.

2. Choose an x-value (called it x_k) in each subinterval I_k.

3. For each x_k we choose, draw a rectangle with height $f(x_k)$ and a width spanning I_k (Δx_k).



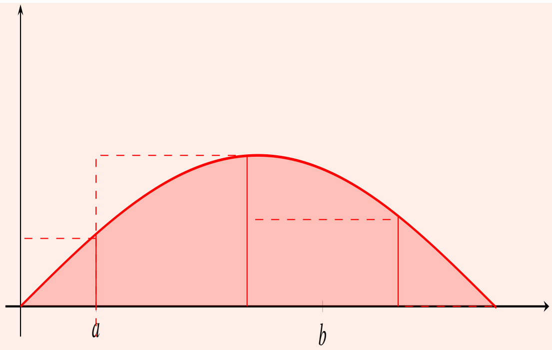
The area of rectangle k is:

$$f(x_k) \cdot \Delta x_k$$

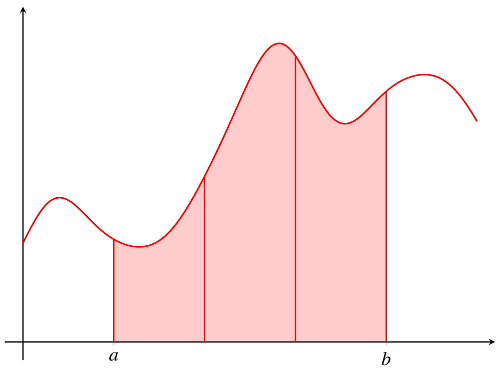
The total area of all the rectangles is:

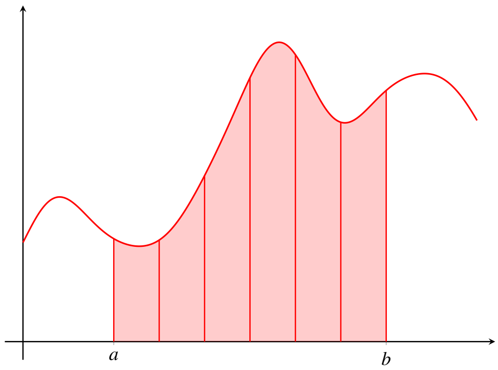
$$f(x_1) \cdot \Delta x_1 + f(x_2) \cdot \Delta x_2 + \cdots + f(x_n) \cdot \Delta x_n = \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k$$

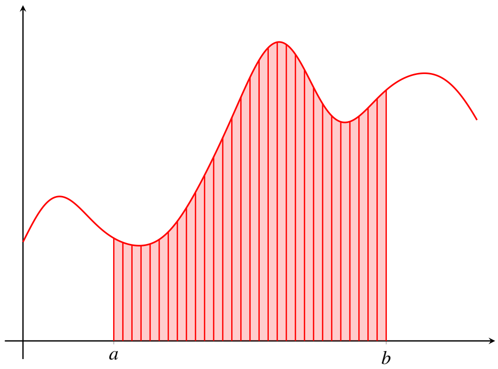
Exercise 25.1 Use the Riemann Sum to estimate the area below $y = sin(\frac{1}{2}x)$ and above the x-axis, between $x = 0$ and $x = 2\pi$. Use a partition of 4 subintervals.



So, what happens as make the rectangles skinnier?







As the width of the rectangles decreases, the accuracy of the estimate increases. So, what happens if we let the width of all the rectangles in the partition approach 0?

Notationally:

$$\lim_{\substack{n\to\infty \\ \|p\|\to 0}} \sum_{k=1}^{n} f(x_k)\cdot \Delta x_k$$

We should get the exact area, that is, our estimate is no longer just an estimate.

Notationally, instead of limit, we write it as:

Theorem 25.1.1 — Definite Integral.

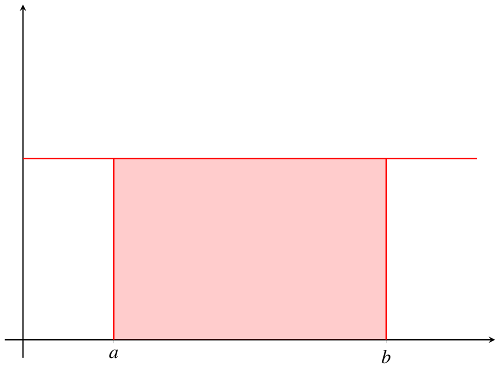
$$\int_{x=a}^{b} f(x)\, dx \tag{25.1}$$

25.2 Definite Integral

For all of the following, suppose that k, a, b, and c are constants with $a < b < c$, and that f and g are integrable functions on the domain of integration.

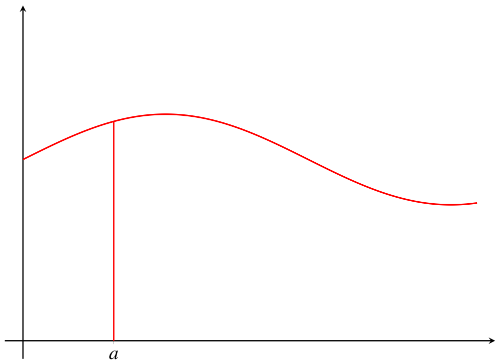
Theorem 25.2.1 — Definite Integral.

$$\int_a^b k\,dx = (b-a)k \tag{25.2}$$



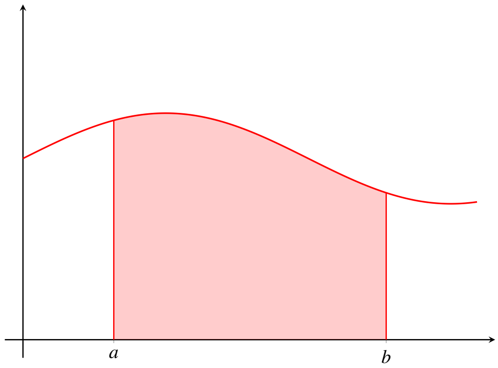
Theorem 25.2.2 — Definite Integral.

$$\int_a^a f(x)\,dx = 0 \tag{25.3}$$



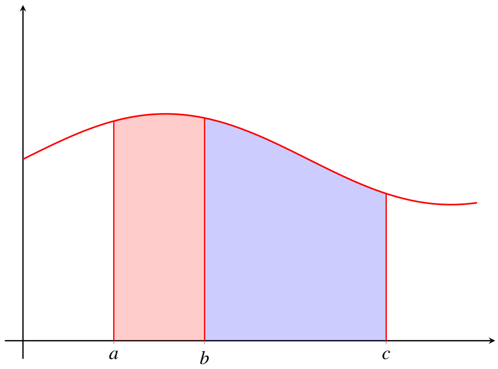
Theorem 25.2.3 — Definite Integral.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{25.4}$$



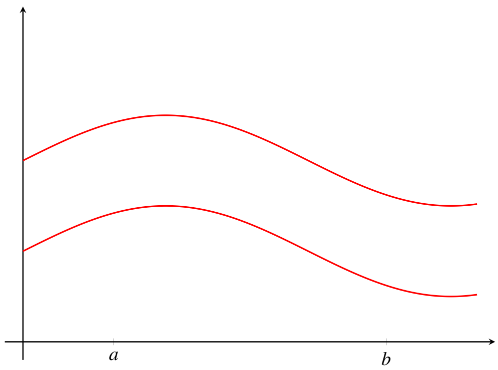
Theorem 25.2.4 — Definite Integral.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \tag{25.5}$$



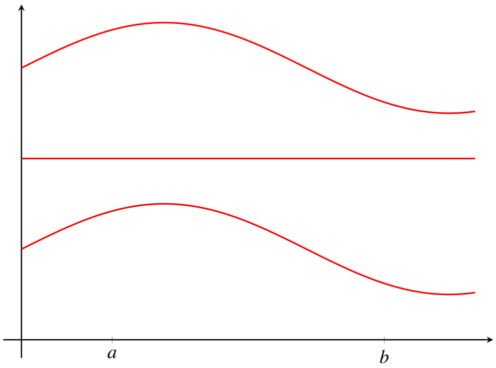
Theorem 25.2.5 — Definite Integral.

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \tag{25.6}$$



Theorem 25.2.6 — Definite Integral.

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \tag{25.7}$$



Exercise 25.2 Evaluate.

(a)

$$\int_1^2 \frac{1}{t}\,dt$$

(b)

$$\int_{-2}^1 x^3\,dx$$

(c)

$$\int_{1}^{5} s(s^2+1)\, ds$$

(d)

$$\int_{\sqrt{ln(2)}}^{\sqrt{ln(4)}} xe^{x^2}\, dx$$

(e)

$$\int_{x}^{x^2} sin(t)\ dt$$

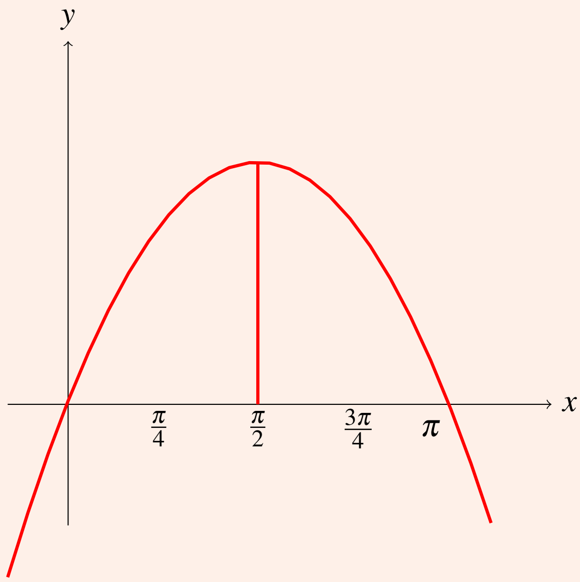
(f)

$$\int_{5}^{13} (x+1)\sqrt{2x-1}\,dx$$

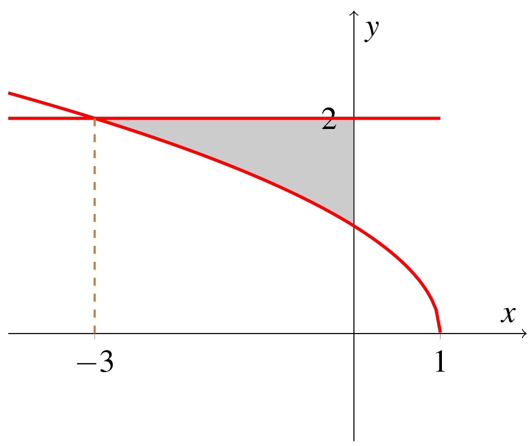
26. Area Under a Curve

26.1 Area Under a Curve

Exercise 26.1 Find the area below the curve $f(x) = sin(x)$ and above the x-axis between $x = \frac{\pi}{2}$ and $x = \pi$.

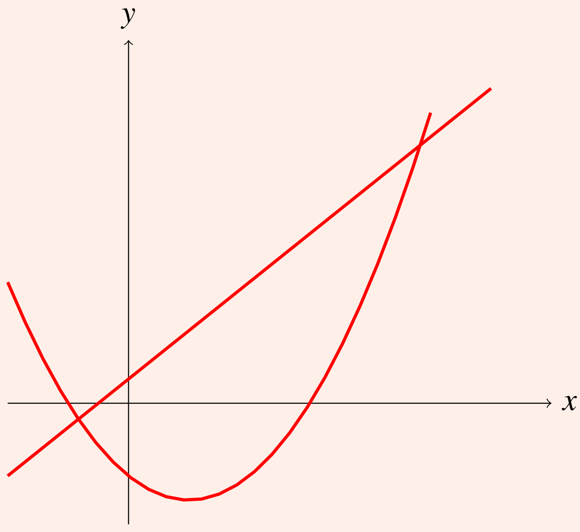


What if we have a more interesting situation where many curves are involved? For instance, how do we find the area between two curves f and g?



$$\text{Area under Upper} - \text{Area under Lower} = \text{Area expected}$$

Exercise 26.2 Calculate the area bounded by $y = 2x + 1$ and $y = x^2 - 2x - 3$.



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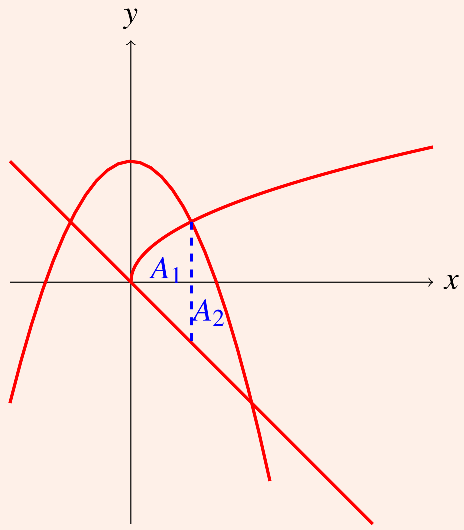
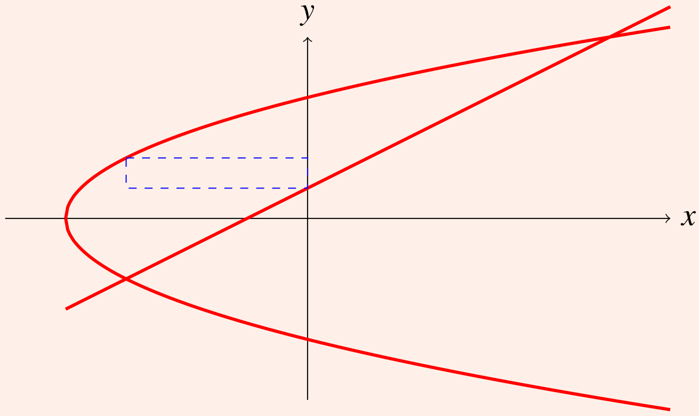
Exercise 26.3 Calculate the area bounded by $y = -x$, $y = \sqrt{x}$, and $y = -x^2 + 2$ between $x = 0$ and $x = 2$.



Exercise 26.4 Find the area bounded by $y^2 = x + 4$ and $y = \frac{1}{2}x + \frac{1}{2}$.


y
x