Lecture 05 - Gibbs sampling

- Metropolis-Hastings is good, but a bit slow.
- Would be nice to have a way of always accepting the next step ⇒ faster mixing.

If you know the conditionals of your function, you can use Gibbs sampling to achive this.

⇒ need to know $p(x_1 \mid x_2 = x_2^{(i)})$ and
$p(x_2 \mid x_1 = x_1^{(i)})$

We will show later that Gibbs sampling gives us indeed samples from our target distribution, but first:

How does Gibbs sampling work?

- choose a random initialization
  $[x_1^{(0)}, x_2^{(0)}] \sim$ Uniform
- Sample: $x_1^{(i+1)}$ from $p(x_1 \mid x_2 = x_2^{(i)})$
  $x_2^{(i+1)}$ from $p(x_2 \mid x_1 = x_1^{(i+1)})$

Let's look at an example:

$$f(x,y) = x^2 \exp[-xy^2 - y^2 + 2y - 4x]$$

This looks fairly complicated, but the conditionals are actually rather simple:

$$\begin{aligned} f(x,y) &= x^2 \cdot \exp[-xy^2 - 4x] \cdot \underbrace{\exp[-y^2 + 2y]} \\ &= x^2 \cdot \exp[-x(y^2+4)] \cdot g(y) \\ &= g(y) \cdot \text{Gamma}(3, y^2+4) \end{aligned}$$

$$\text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} \exp(-\beta x)$$

$\Rightarrow$ $f(x|y)$ can be sampled from a Gamma distribution

Now the other way around:

$$\begin{aligned} f(x,y) &= x^2 \exp[-y^2(1+x) + 2y] \cdot \exp[-4x] \\ &= \exp[-y^2(1+x) + 2y] \cdot g(x) \end{aligned}$$

Completing the square:

$$= \exp\left[-y^2(1+x) + 2y - \frac{1}{x+1}\right] \cdot \exp\left[\frac{1}{x+1}\right] \cdot g(x)$$

$$= \exp\left[\left(-y^2 + \frac{2y}{x+1} - \frac{1}{(x+1)^2}\right) \cdot (x+1)\right] \cdot g'(x)$$

$$= \exp\left[-\left(y - \frac{1}{x+1}\right)^2 \cdot (x+1)\right] \cdot g'(x)$$

$\Rightarrow f(y|x)$ can be sampled from a normal distribution

$$N\left(\frac{1}{x+1}, \frac{1}{\sqrt{x+1}}\right)$$

PYTHON EXAMPLE